On the Number of Twin Primes less than a Given Quantity: An Alternative Form of Hardy-Littlewood Conjecture

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ABSTRACT. I found an alternative form of Hardy-Littlewood Conjecture using Mertens' 3rd theorem. This new form has a theoretical background and coincides with prime number theorem. It is expected to provide an easier way to prove the conjecture.

1. Introduction

Though it is not proved yet if there are infinitely many twin primes, here is a proposition stating what the number of twin primes would be.

Proposition 1. (Hardy-Littlewood Conjecture) Let $\pi_2(x)$ denote the number of prime numbers p less than or equal to x such that p+2 is also a prime number. Then, this satisfies

$$\pi_2(x) \sim 2C_2 \frac{x}{(\log x)^2} \tag{1}$$

where $C_2 = \prod_{p\geq 3}\left(1-\frac{1}{(p-1)^2}\right) = 0.66016\ldots$ is the twin prime constant.

To make an alternative form of this similarity, the following theorem would be used.

Theorem 1. (Mertens' 3rd Theorem) Let p be prime numbers, then,

$$\lim_{n\to\infty} \log n \prod_{p<n} \left(1 - \frac{1}{p}\right) = e^{-\gamma} \tag{2}$$

where γ is Euler-Mascheroni constant.

2. Estimating the Number of Twin Primes

All prime numbers except 2 and 3 are of form 6k-1 or 6k+1, so all twin primes except (3, 5) are of form $(6k-1, 6k+1)$. Tables below consist of numbers of the form 6k-1, 6k, and 6k+1 with multiples of each prime numbers starting from 5 highlighted with blue.

5	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101	107
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108
7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97	103	109

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6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108
7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97	103	109

47	53	59	65	71	77	83	89	95	101	107	113	119	125	131	137	143	149
48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	150
49	55	61	67	73	79	85	91	97	103	109	115	121	127	133	139	145	151

65	71	77	83	89	95	101	107	113	119	125	131	137	143	149	155	161	167
66	72	78	84	90	96	102	108	114	120	126	132	138	144	150	156	162	168
67	73	79	85	91	97	103	109	115	121	127	133	139	145	151	157	163	169

There exists a pattern where composition numbers appear. This can be examined in two cases.

Case 1 : p is a prime number of form 6k-1

...	6(mp-k)-1	...	6mp-1	...	6(mp+k)-1 = (6m+1)p	...
...	6(mp-k)	...	6mp	...	6(mp+k)	...
...	6(mp-k)+1 = (6m-1)p	...	6mp+1	...	6(mp+k)+1	...

Case 2 : p is a prime number of form 6k+1

…	6(mp-k)-1 = (6m-1)p	…	6mp-1	…	6(mp+k)-1	…
…	6(mp-k)	…	6mp	…	6(mp+k)	…
…	6(mp-k)+1	…	6mp+1	…	6(mp+k)+1 = (6m+1)p	…

Here, m is an arbitrary natural number. Both cases give same conclusion.

Theorem 2. ∀ z ∈ N, a pair of two numbers 6z-1 and 6z+1 are not twin primes if and only if $z = mp \pm k$ for some m ∈ N and prime number p. (k is determined by p as $p = 6k \pm 1$)

Regarding the tables above, the number of columns under a given quantity z is $\frac{z}{6}$ and for all prime number p > 3 (since multiples of 2 and 3 are already excluded as considering only numbers of form $6z \pm 1$), the column with no twin primes appears twice every consecutive p columns. Since this property is independent for two arbitrary prime numbers and it is enough to consider prime numbers less than the square root of x, the number of twin primes under a given quantity x can be estimated by

$$\frac{x}{6}\prod_{p=5}^{p<\sqrt{x}}\left(1-\frac{2}{p}\right) \tag{3}$$

3. An Alternative Form of Hardy-Littlewood Conjecture

Comparing Hardy-Littlewood conjecture and the formula (3), we have following.

$$\lim_{x\to\infty}\frac{2C_2\frac{x}{(\log x)^2}}{\frac{x}{6}\prod_{p=5}^{p<\sqrt{x}}\left(1-\frac{2}{p}\right)} = 12C_2\lim_{x\to\infty}\frac{1}{(\log x)^2\times 3\prod_{p=3}^{p<\sqrt{x}}\left(1-\frac{2}{p}\right)}$$

$$= 4\prod_{p\geq 3}\left(1-\frac{1}{(p-1)^2}\right)\lim_{x\to\infty}\frac{1}{(\log x^2)^2\times\prod_{p=3}^{p<x}\left(1-\frac{2}{p}\right)}$$

$$= 4\lim_{n\to\infty}\prod_{p=3}^{p<n}\left(1-\frac{1}{(p-1)^2}\right)\lim_{x\to\infty}\frac{1}{4(\log x)^2\times\prod_{p=3}^{p<x}\left(1-\frac{2}{p}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{(\log x)^2} \prod_{p=3}^{p<x} \frac{p(p-2)}{(p-1)^2} \frac{1}{\frac{p-2}{p}}$$

$$= \lim_{x\to\infty} \frac{1}{(\log x)^2 \times \prod_{p=3}^{p<x} \left(1-\frac{1}{p}\right)^2}$$

$$= \lim_{x\to\infty} \left(\log x \times 2 \prod_{p=2}^{p<x} \left(1-\frac{1}{p}\right) \right)^{-2}$$

$$= (2e^{-\gamma})^{-2} = \left(\frac{e^{\gamma}}{2}\right)^2$$

Note that their ratio is a constant. This gives an alternative form of Hardy-Littlewood conjecture:

$$\pi_2(\mathrm{x}) \sim 2C_2 \frac{x}{(\log x)^2} \sim \left(\frac{e^{\gamma}}{2}\right)^2 \frac{x}{6} \prod_{p=5}^{p<\sqrt{x}} \left(1-\frac{2}{p}\right) \tag{4}$$

4. Significance of the Alternating Form

If we apply similar method to estimate the number of primes under a given number x, we would have

$$x \prod_{p<\sqrt{x}} \left(1-\frac{1}{p}\right) \tag{5}$$

Then, by Mertens' 3rd Theorem, the prime number theorem also has an alternative form:

$$\pi(\mathrm{x}) \sim \frac{x}{\log x} \sim \frac{e^{\gamma}}{2} x \prod_{p<\sqrt{x}} \left(1-\frac{1}{p}\right) \tag{6}$$

which is a constant multiple of formula (5).

The fact that $\frac{e^\gamma}{2}$ appears without square is noticeable and this coincidence between (4) and (6) provides another circumstantial evidence that Hardy-Littlewood conjecture is true.

5. Conclusion

Here, I suggest a new conjecture stating the number of twin primes less than a given quantity which is equivalent to Hardy-Littlewood Conjecture but more intuitive and convincing. It is expected that finding the meaning of the constant $\frac{e^\gamma}{2}$ might guides to a proof of this conjecture.

References

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