A new type of approximation for the Gamma function based on the Windschitl's Formula

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ABSTRACT

In this paper, we present a new approximate formula based on the Windschitl's type formula, one of the important approximate formulas of the Gamma function. And we introduce interesting double inequality associated with our new formula.

Keywords: Gamma function, Approximation, Windschitl's formula, Stirling's formula

1. Introduction

The Gamma function can be regarded as an extension of the factorial function and has artful applications in statistical physics, probability theory and number theory. The big factorials arise in the research of the pure mathematics and other branches of science. A general method is to find approximations of the factorial function and its extension Gamma function. The Gamma function belongs to the category of the special transcendental functions and we will see that some famous mathematical constants are occurring in its study. It also appears in various area as asymptotic series, definite integration, hypergeometric series, Riemann zeta function and number theory.

It is known that the Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1.1}$$

for a natural number n has various applications in probability theory, statistical physics, number theory and other branches of science.

As an asymptotic expansion of Stirling's formula (1.1), one has the Stirling's series for Gamma function (see [1])

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)x^{2n-1}}\right), x \to \infty \tag{1.2}$$

where B_{2n} is the Bernoulli number.

In [2], authors remarked that Ramanujan type asymptotic expansion and established the

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following asymptotic expansion:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(2^r x^r + \sum_{j=1}^{\infty} p_j x^{r-j}\right)^{1/(2r)}, x \to \infty \tag{1.3}$$

where $p_j \equiv 2^r b_j$, and $b_0 = 1, b_j = -\frac{1}{j}\sum_{k=1}^{j} \frac{2rB_{k+1}}{k+1} b_{j-k}, j \geq 1$.

More asymptotic expansion developed by some closed approximation formulas for the Gamma function can be found in [3-7], [12-14] and the references cited therein.

Windschitl [8] suggested in 2002 the following approximation formula for computing the Gamma function with fair accuracy on calculators with limited program or register memory. Now let us focus on the Windschitl's approximation formula given by

$$\Gamma(x+1) \sim W_0(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2}. \tag{1.4}$$

Up until now, many researchers made great efforts in the area of establishing more accurate approximations for the factorial function, and had a lot of inspiring results.

Recently, Lu, Song and Ma [9] extended Windschitl's formula to an asymptotic expansion that

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(n \sinh\left(\frac{1}{n} + \frac{a_7}{n^7} + \frac{a_9}{n^9} + \cdots\right)\right)^{n/2} \tag{1.5}$$

with $a_7 = \frac{1}{810}, a_9 = -\frac{67}{42525}, a_{11} = \frac{19}{8505}, \cdots$.

In [10], the authors provided a continued fraction approximation for the factorial function starting from the Windschitl's formula as follows,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(n \sinh\left(\frac{1}{n} + \frac{1}{810 n^8}\left(n + \cfrac{b_1}{n + \cfrac{b_2}{n + \cfrac{b_3}{n + \ddots}}}\right)\right)\right)^{n/2}, \tag{1.6}$$

where $b_1 = -\frac{134}{105}, b_2 = \frac{95}{67}, b_3 = \frac{343277909}{529313400}, \cdots$.

Other two asymptotic expansions

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2 + \sum_{j=0}^{\infty} r_j x^{-j}}, \tag{1.7}$$

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x} + \sum_{n=3}^{\infty} \frac{d_n}{x^{2n}}\right)^{x/2}. \tag{1.8}$$

Inspired by the asymptotic expansions (1.7) and (1.8), the aim of this paper is to further present the following two asymptotic expansions related to Windschitl's one as $x \to \infty$,

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\left(1 + \sum_{n=3}^{\infty} \frac{b_n}{x^{2n}}\right)\right)^{x/2}. \tag{1.9}$$

2. Useful Lemma

To obtain the explicit coefficients formula in the asymptotic expansions and to estimate the remainder in (1.9), we need the following lemma.

Lemma. For $|t| < \pi$, we have

$$\ln\frac{\sinh t}{t} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{2n(2n)!} t^{2n}. \tag{2.1}$$

Proof. According to [11]

$$\coth t = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} t^{2n-1}, \ |t| < \pi. \tag{2.2}$$

Then we obtain that for $|t| < \pi$,

$$\ln\frac{\sinh t}{t} = \int_0^t \left(\coth x - \frac{1}{x}\right) dx = \int_0^t \left(\sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} t^{2n-1} - \frac{1}{x}\right) dx = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} t^{2n}. \tag{2.3}$$

The proof is completed.

3. A new type of asymptotic expansion for the Gamma function

Theorem 3.1. As $x \to \infty$, the asymptotic expansion

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\left(1 + \sum_{n=3}^{\infty} \frac{b_n}{x^{2n}}\right)\right)^{x/2}, \tag{3.1}$$

holds with $b_n = \frac{1}{n}\sum_{k=1}^{n} k \frac{4k(2k-2)! - 2^{2k}}{2k(2k)!} B_{2k} b_{n-k}$.

Proof. By the asymptotic expansion (1.1) and Lemma we have that as $x \to \infty$,

$$\ln\Gamma(x+1) - \frac{1}{2}\ln(2\pi) - \left(x + \frac{1}{2}\right)\ln x + x \sim \sum_{n=1}^{\infty} \frac{b_n'}{x^{2n-1}}, \tag{3.2}$$

and

$$\frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{b_n^{''}}{x^{2n-1}}, \tag{3.3}$$

where $b_n^{'}=\frac{B_{2n}}{2n(2n-1)}$ and $b_n^{''}=\frac{2^{2n}B_{2n}}{2n(2n)!}$.

Let $\Gamma(x+1)\sim\sqrt{2\pi x}(x/e)^x(x\sinh(1/x)\exp w_n(x))^{x/2}$ as $x\to\infty$. Then we have that as $x\to\infty$,

$$w_n(x)=\left(\ln\left(\Gamma(x+1)-\ln\sqrt{2\pi}-\left(x+\frac{1}{2}\right)\ln x+x\right)-\frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right)\right)\frac{2}{x}$$
$$=\left(\left(\sum_{n=1}^{\infty}\frac{b_n^{'}}{x^{2n-1}}\right)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{b_n^{''}}{x^{2n-1}}\right)\frac{2}{x}=\sum_{n=1}^{\infty}\left(\frac{b_n^{'}-b_n^{''}/2}{x^{2n-1}}\right)\frac{2}{x}=\sum_{n=1}^{\infty}\frac{4n(2n-2)!-2^{2n}}{2n(2n)!x^{2n}}B_{2n}.$$

It is easy

$$\exp\left(\sum_{n=1}^{\infty}w_nx^{-n}\right)\sim\sum_{n=0}^{\infty}p_nx^{-n} \tag{3.4}$$

with $p_0=1$ and $p_n=\frac{1}{n}\sum_{k=1}^{n}kw_kp_{n-k}$ for $n\geq1$. Thus, we use

$$w_n(x)=\sum_{n=1}^{\infty}\frac{4n(2n-2)!-2^{2n}}{2n(2n)!x^{2n}}B_{2n}=\sum_{n=1}^{\infty}\frac{w_n}{x^{2n}}, \tag{3.5}$$

where $w_1=w_2=0, w_3=\frac{1}{810}, w_4=-\frac{11}{9450}, w_5=\frac{143}{85050},\cdots$.

Then we arrive at

$$p_1=w_1p_0=0,\ p_2=\frac{1}{2}(w_1p_1+2w_2p_0)=0,$$
$$p_3=\frac{1}{3}(w_1p_2+2w_2p_1+3w_3p_0)=\frac{1}{810},$$
$$p_4=\frac{1}{4}(w_1p_3+2w_2p_2+3w_3p_1+4w_4p_0)=-\frac{11}{9450},$$
$$p_5=\frac{1}{5}(w_1p_4+w_2p_3+w_3p_2+w_4p_1+w_5p_0)=\frac{143}{85050}.$$

In [13], result of Theorem 3.1 is the same with our computation.

Next, using Theorem 3.1, we provide the following double inequality for the Gamma function.

Theorem 3.2. For every $x\geq0$, it holds:

$$\left(x\sinh\frac{1}{x}\exp\left(\sum_{k=1}^{n}\frac{w_k}{x^{2k}}-\frac{b_n^{''}}{x^{2n}}\right)\right)^{x/2}<\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}<\left(x\sinh\frac{1}{x}\exp\left(\sum_{k=1}^{n}\frac{w_k}{x^{2k}}+\frac{b_n^{''}}{x^{2n}}\right)\right)^{x/2}. \tag{3.6}$$

Proof. For a natural number $m\geq5$, let

$$f_m(x) = \ln\Gamma(x+1) - \ln\sqrt{2\pi x} - x\ln x + x - \frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right) - \frac{x}{2}\sum_{k=1}^{m}\frac{w_k}{x^{2k}}. \tag{3.7}$$

Using the first inequality of (3.6) with $t = 1/(2x)$ we get

$$\ln\Gamma\left(x+\frac{1}{2}\right) - \ln\sqrt{2\pi x} - x\ln x + x > \sum_{k=1}^{2n}\frac{B_{2k}}{2k(2k-1)x^{2k-1}} = \sum_{k=1}^{2n}\frac{b_k^{'}}{x^{2k-1}}, \tag{3.8}$$

and

$$\sum_{k=1}^{2n}\frac{2^{2k}B_{2k}}{2k(2k)!x^{2k}} < \ln\left(x\sinh\frac{1}{x}\right) < \sum_{k=1}^{2n-1}\frac{2^{2k}B_{2k}}{2k(2k)!x^{2k}} = \sum_{k=1}^{2n-1}\frac{b_k^{''}}{x^{2k}}. \tag{3.9}$$

We get that, from (3.7),

$$\begin{aligned} f_{2n+1}(x) &= \left[\ln\Gamma\left(x+\frac{1}{2}\right) - \ln\sqrt{2\pi} - x\ln x + x\right] - \frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right) - \frac{x}{2}\sum_{k=1}^{2n+1}\frac{w_{\mathrm{k}}}{x^{2k}} \\ &< \sum_{k=1}^{2n+1}\frac{b_k^{'}}{x^{2k-1}} - \sum_{k=1}^{2n}\frac{b_k^{''}/2}{x^{2k-1}} - \sum_{k=1}^{2n+1}\frac{w_{\mathrm{k}}/2}{x^{2k-1}} = \frac{2^{4n+2}B_{4n+2}}{x^{4n+1}(4n+2)(4n+2)!}. \end{aligned} \tag{3.10}$$

Similarly, using $\left(b_k^{'} - \dfrac{b_k^{''}}{2}\right)2 = w_k$, we have

$$\begin{aligned} f_{2n}(x) &= \left[\ln\Gamma\left(x+\frac{1}{2}\right) - \ln\sqrt{2\pi} - x\ln x + x\right] - \frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right) - \frac{x}{2}\sum_{k=1}^{2n}\frac{w_{\mathrm{k}}}{x^{2k}} \\ &> \sum_{k=1}^{2n}\frac{b_k^{'}}{x^{2k-1}} - \sum_{k=1}^{2n-1}\frac{b_k^{''}/2}{x^{2k-1}} - \sum_{k=1}^{2n}\frac{w_{\mathrm{k}}/2}{x^{2k-1}} = \frac{2^{4n-1}B_{4n}}{x^{4n-1}(4n)(4n)!}. \end{aligned} \tag{3.11}$$

Thus, we have $|f_n(x)| < \left|\dfrac{b_n^{''}}{2x^{2n-1}}\right|$. From this result, we can get

$$\left(x\sinh\frac{1}{x}\exp\left(\sum_{k=1}^{n}\frac{w_k}{x^{2k}} - \frac{b_n^{''}}{x^{2n}}\right)\right)^{x/2} < \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} < \left(x\sinh\frac{1}{x}\exp\left(\sum_{k=1}^{n}\frac{w_k}{x^{2k}}\right) + \frac{b_n^{''}}{x^{2n}}\right)^{x/2}. \tag{3.12}$$

4. Conclusions

In this paper, we established new asymptotic expansions starting from Windschitl's formula and the double inequality.

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