Harmonic Graphs Conjecture: Graph-Theoretic Attributes and their Number Theoretic Correlations

Felipe Correa Rodríguez

August 16, 2023

Abstract

The Harmonic Graphs Conjecture states that there exists an asymptotic relation involving the Harmonic Index and the natural logarithm as the order of the graph increases. This conjecture, grounded in the novel context of Prime Graphs, draws upon the Prime Number Theorem and the sum of divisors function to unveil a compelling asymptotic connection. By carefully expanding the definitions of the harmonic index and the sum of divisors function, and leveraging the prime number theorem's approximations, we establish a formula that captures this intricate relationship. This work is an effort to contribute to the advancement of graph theory, introducing a fresh lens through which graph connectivity can be explored. The synthesis of prime numbers and graph properties not only deepens our understanding of structural complexity but also paves the way for innovative research directions.

1 Introduction

Graph theory is a branch of mathematics that studies graphs, which are mathematical structures used to model pairwise relations between objects. Graphs are often used to represent networks, such as social networks, communication networks, and transportation networks.

The harmonic index, first appeared in [4] and as for now is described as a a topological index of a graph that can provide information about the connectivity and complexity of a graph's structure. It is defined as the sum of the reciprocals of the sums of the degrees of pairs of adjacent vertices. In other words, for each edge in the graph, we take the degrees of the two vertices connected by that edge, add them together, and then take the reciprocal of that sum. We then add up all these reciprocals for all the edges in the graph to get the harmonic index.

In this paper, we study the Harmonic Index of a specific type of graph and the distribution of prime numbers as described by the Prime Number Theorem, basically a connection between graph theory and number theory. We show that the harmonic index of a graph converges to a specific value as the number of vertices increases.

The motivation driving this study emanates from the exploration of how numbers and networks interact, could the way we connect prime numbers through graph structures reveal insights into the distribution of these enigmatic figures? Just as roads connect cities, in this study we will try to see if connecting prime numbers in a certain way might give us a new way to look at their patterns in the context of graph theory.

2 Formulation of the Conjecture

2.1 Harmonic Index of a graph

The Harmonic Index (HI) is a graph-theoretic attribute that provides insights into the connectivity and structural properties of a graph. It is defined as follows for a connected graph G:

$$HI(G) = \sum_{(u,v)\in E(G)} \frac{2}{d(u)+d(v)}$$

where $d(u)$ and $d(v)$ are the degrees of vertices u and v in G, respectively.

2.2 Prime Number Theorem:

The Prime Number Theorem is a fundamental result in number theory that describes the distribution of prime numbers. It states that for a large enough x, the number of primes less than or equal to x, denoted as $\pi(x)$, is approximately equal to $\frac{x}{\ln(x)}$, where $\ln(x)$ is the natural logarithm of x.

2.3 The Harmonic Graphs Conjecture:

Consider a family of graphs called "Prime Graphs" denoted by G(n), where n is a positive integer representing the number of vertices. Each vertex in G(n) corresponds to a prime number. Edges between vertices are added according to a specific rule based on number theory.

Rule: Connect vertices u and v in $G(n)$ with an edge if and only if the harmonic mean of their degrees, denoted as $HI(uv)$, satisfies a condition related to the Prime Number Theorem:

$$HI(uv) = \frac{2}{d(u)+d(v)} \approx \frac{2}{\ln(p(u)\cdot p(v))}$$

where $p(u)$ and $p(v)$ are the prime numbers corresponding to vertices u and v.

In other words, $G(n)$ is a graph where vertices represent prime numbers, and edges connect pairs of primes whose sum is also prime. This rule is inspired by the Goldbach Conjecture, which suggests that every even integer greater than 2 can be expressed as the sum of two prime numbers.

Now, let's examine the conjectured relationship between HI(G(n)) and the distribution of primes:

Conjecture Hypothesis: As n becomes large, the Harmonic Index of the Prime Graph G(n), denoted as HI(G(n)), exhibits an asymptotic behavior similar to the distribution of prime numbers described by the Prime Number Theorem.

Basically, as n becomes large, the graph G(n) becomes denser with edges connecting prime numbers whose sums are also prime. This may lead to an increase in the overall harmonic mean of distances between vertices in the graph. The specific pattern of edge connections, based on the sum of prime numbers, could result in a relationship between HI(G(n)) and the distribution of primes.

2.4 Number Theoretic Correlation:

To establish a number theoretic correlation in support of the Harmonic Graphs Conjecture, we consider the prime number theorem and the sum of divisors function.

The prime number theorem states that the n-th prime number $P_{\pi(n)}$ is approximately $\pi(n)\ln(\pi(n))$. Using this approximation and $\pi(n) \approx \frac{n}{\ln(n)}$, we find:

$$P_{\pi(n)} \approx n\ln(n) - n\ln(\ln(n))$$

Moreover, the sum of divisors function $\sigma(n)$, which sums the positive divisors of n, satisfies:

$$\sigma(n) \leq \frac{n(n+1)}{2}$$

2.4.1 Conjecture Tentative Corroboration:

By leveraging the approximation for $P_{\pi(n)}$ and applying the edge connection rule, we derive the Harmonic Index for Prime Graphs $HI(G(n))$:

Initial Harmonic Index Expression:

$$HI(G(n)) = \sum_{(u,v)\in E(G(n))} \frac{2}{d(u)+d(v)}$$

Edge Connection Rule:

$$\approx \sum_{\text{prime } p(u),p(v) \ (u,v)\in E(G(n))} \frac{2}{d(u)+d(v)}$$

Applying Logarithmic Form:

$$\approx \sum_{\text{prime } p(u),p(v)\ (u,v)\in E(G(n))} \frac{2}{\ln(p(u)\cdot p(v))}$$

Separating Logarithmic Terms:

$$\approx \sum_{\text{prime } p(u),p(v)\ (u,v)\in E(G(n))} \frac{2}{\ln(p(u)) + \ln(p(v))}$$

Reciprocal of Sum of Logarithms:

$$\approx \sum_{\text{prime } p(u),p(v)\ (u,v)\in E(G(n))} \frac{1}{\frac{1}{2}\ln(p(u)) + \frac{1}{2}\ln(p(v))}$$

Using Square Root Logarithms:

$$\approx \sum_{\text{prime } p(u),p(v)\ (u,v)\in E(G(n))} \frac{1}{\ln(\sqrt{p(u)\cdot p(v)})}$$

Multiplying and Dividing by Square Root Logarithms:

$$\approx \sum_{\text{prime } p(u),p(v)\ (u,v)\in E(G(n))} \frac{\ln(\sqrt{p(u)\cdot p(v)})}{p(u)\cdot p(v)}$$

Logarithmic Identity:

$$\approx \sum_{\text{prime } p(u),p(v)\ (u,v)\in E(G(n))} \frac{\frac{1}{2}\ln(p(u)) + \frac{1}{2}\ln(p(v))}{p(u)\cdot p(v)}$$

Double Summation:

$$\approx \frac{1}{2} \sum_{\text{prime } p(u)} \sum_{\text{prime } p(v)} \frac{\ln(p(u))}{p(u)\cdot p(v)} + \frac{1}{2} \sum_{\text{prime } p(v)} \sum_{\text{prime } p(u)} \frac{\ln(p(v))}{p(u)\cdot p(v)}$$

Rearranging Double Summations:

$$\approx \frac{1}{2} \sum_{\text{prime } p(u)} \ln(p(u)) \left(\sum_{\text{prime } p(v)} \frac{1}{p(u)\cdot p(v)} \right) + \frac{1}{2} \sum_{\text{prime } p(v)} \ln(p(v)) \left(\sum_{\text{prime } p(u)} \frac{1}{p(u)\cdot p(v)} \right)$$

Simplifying and Combining Terms:

$$\approx \frac{1}{2} \sum_{\text{prime } p(u)} \ln(p(u)) \cdot \frac{1}{p(u)} + \frac{1}{2} \sum_{\text{prime } p(v)} \ln(p(v)) \cdot \frac{1}{p(v)}$$

Further Simplification:

$$\approx \frac{1}{2} \sum_{\text{prime } p(u)} \frac{\ln(p(u))}{p(u)} + \frac{1}{2} \sum_{\text{prime } p(v)} \frac{\ln(p(v))}{p(v)}$$

Combining Double Sums:

$$\approx \frac{1}{2} \sum_{\text{prime } p} \frac{\ln(p)}{p} + \frac{1}{2} \sum_{\text{prime } p} \frac{\ln(p)}{p}$$

Final Approximation:

$$\approx \sum_{\text{prime } p} \frac{\ln(p)}{p}$$

Asymptotic Behavior:

$$\approx h(n) \quad \text{as } n \to \infty$$

Hence, it is evident that as the order of the graph, denoted by n, approaches infinity, the Harmonic Index $HI(G_n)$ converges to a limit represented by the function $h(n)$. This function, $h(n)$, is intricately linked to the underlying distribution of prime numbers, a fundamental aspect of number theory. The emergent asymptotic relationship between $HI(G_n)$ and $h(n)$ draws a remarkable parallel to the well-established behavior delineated by the Prime Number Theorem. This result helps affirming the validity of the Harmonic Graphs Conjecture but more studying, analysis and refinement may be required to do a proper conclusion.

3 Discussion

This investigation into the Harmonic Graphs Conjecture has yielded insights at the intersection of graph theory and number theory, carrying implications for both theoretical understanding and practical applications.

The Harmonic Index, as defined in the conjecture, introduces a captivating avenue for extracting topological information such as diameter of a graph and other information from graph structures. By utilizing the harmonic mean of vertex degrees, this index provides a unique lens through which we can discern the connectivity and structural characteristics of a graph. Expanding on this foundation, we can explore how this innovative perspective intersects with number theory to uncover intriguing patterns and connections.

From a theoretical standpoint, the established prime graph and its distribution could provide a new lens through which graph connectivity can be analyzed. This perspective holds the potential to yield profound implications for algorithmic design, particularly in fields where understanding complex network structures is of paramount importance.

Furthermore, the conjecture's underlying principle of connecting prime numbers based on harmonic mean relationships could inspire the creation of specialized graph structures. These structures may find applications in diverse fields, ranging from social networks and communication systems to biological and ecological networks.

In sum, the harmonic index provides a rich foundation for extracting topological information from graphs, and its fusion with number theory offers a gateway to a realm of unexplored mathematical connections. Through careful investigation and cross-disciplinary collaboration, we have the opportunity to chart new frontiers in both theory and practice, unveiling the intricate tapestry that binds together the seemingly disparate realms of graphs and numbers.

References

[1] G.E. Andrews. *The Theory of Partitions*. Encyclopedia of Mathematics and Its Applications. Addison-Wesley, 1976. URL: https://books.google.cl/books?id=1BtkvwEACAAJ.

[2] Reza Arghandeh et al. "Phasor-based assessment for harmonic sources in distribution networks". In: *Electric Power Systems Research* 116 (2014), pp. 94–105. ISSN: 0378-7796. DOI: https://doi.org/10.1016/j.epsr.2014.05.012. URL: https://www.sciencedirect.com/science/article/pii/S0378779614002004.

[3] J. Derbyshire. *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. New York: Penguin, 2004, pp. 32–47.

[4] Siemion Fajtlowicz. "On conjectures of Graffiti-II". In: *Congr. Numer* 60 (1987), pp. 187–197.

[5] Darij Grinberg. *An introduction to graph theory*. 2023. arXiv: 2308.04512 [math.HO].

[6] X. Li and Y. Shi. "A survey on the Randić index". In: *MATCH Communications in Mathematical and Computer Chemistry* 59 (2008), pp. 127–156.

[7] Abdul Majeed and Ibtisam Rauf. "Graph Theory: A Comprehensive Survey about Graph Theory Applications in Computer Science and Social Networks". In: *Inventions* 5.1 (2020). ISSN: 2411-5134. DOI: 10.3390/inventions5010010. URL: https://www.mdpi.com/2411-5134/5/1/10.

[8] Lingping Zhong. "The harmonic index for graphs". In: *Applied Mathematics Letters* 25.3 (2012), pp. 561–566. ISSN: 0893-9659. DOI: 10.1016/j.aml.2011.09.059. URL: https://www.sciencedirect.com/science/article/pii/S089396591100471X.

[9] B. Zhou and N. Trinajsti´c. "On a novel connectivity index". In: *J. Math. Chem.* 46 (2009), pp. 1252–1270.