# On the numbers ${}_3F_2\left(1, \frac{1-n}{2}, -\frac{n}{2}; \frac{3}{2}, \frac{1}{2}-n; 4\right)$, $n = 0, 1, 2, 3, \ldots$

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*Abstract:In this note we give some formulas related to the numbers ${}_3F_2\left(1, \frac{1-n}{2}, -\frac{n}{2}; \frac{3}{2}, \frac{1}{2}-n; 4\right)$, $n = 0, 1, 2, 3, \ldots$, where ${}_3F_2$ is the generalized hypergeometric function.*

## 1. Introduction

Define $F_n$ by

$$F_n = {}_3F_2\left(1, \frac{1-n}{2}, -\frac{n}{2}; \frac{3}{2}, \frac{1}{2}-n; 4\right) \quad , n = 0, 1, 2, 3, \ldots \tag{1}$$

where ${}_3F_2$ is the generalized hypergeometric function.

In this note we give some formulas related to (1).

Remark 1: The generalized hypergeometric function ${}_pF_q$ is defined by

$${}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!} \tag{2}$$

where $(a)_k$ is the Pochhammer symbol.

Remark 2: ${}_2F_1$ is the Gauss hypergeometric function.

Remark 3: $(a)_0 = 1$, $(a)_k = a(a+1)\ldots(a+k-1)$, $a \in \mathbb{C}$, $k \in \mathbb{N}$.

Remark 4: The Appell (AppellF1 = F1) hypergeometric function is defined by

$$\text{F1}(a, b_1, b_2; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{m!\, n!\, (c)_{m+n}} x^m y^n \tag{3}$$

Remark 5: The number Pi is defined by

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \tag{4}$$

Remark 6: The arctan function is defined by

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad , \ |x| < 1 \tag{5}$$

## 2. Formulas

Entry 1. for $n = 0, 1, 2, 3, \ldots$, we have

$$F_{2n} = {}_3F_2\left(1, \frac{1}{2} - n, -n; \frac{3}{2}, \frac{1}{2} - 2n; 4\right) \tag{6}$$

$$F_{2n+1} = {}_3F_2\left(1, -\frac{1}{2} - n, -n; \frac{3}{2}, -\frac{1}{2} - 2n; 4\right) \tag{7}$$

$$F_{2n} = 5^n \sum_{k=0}^{n} \frac{(-n)_k}{k!} \left(\frac{4}{5}\right)^k {}_3F_2\left(-k, \frac{1}{2} - n, 1; \frac{3}{2}, \frac{1}{2} - 2n; -1\right) \tag{8}$$

$$F_{2n+1} = 5^n \sum_{k=0}^{n} \frac{(-n)_k}{k!} \left(\frac{4}{5}\right)^k {}_3F_2\left(-k, -\frac{1}{2} - n, 1; \frac{3}{2}, -\frac{1}{2} - 2n; -1\right) \tag{9}$$

$$F_{2n} = (-3)^n \sum_{k=0}^{n} \frac{(-n)_k}{k!} \left(\frac{4}{3}\right)^k {}_3F_2\left(-k, \frac{1}{2} - n, 1; \frac{3}{2}, \frac{1}{2} - 2n; 1\right) \tag{10}$$

$$F_{2n+1} = (-3)^n \sum_{k=0}^{n} \frac{(-n)_k}{k!} \left(\frac{4}{3}\right)^k {}_3F_2\left(-k, -\frac{1}{2} - n, 1; \frac{3}{2}, -\frac{1}{2} - 2n; 1\right) \tag{11}$$

$$F_n = \frac{x}{x-4} \sum_{k=0}^{\infty} \left(\frac{4}{4-x}\right)^k {}_3F_2\left(-k, \frac{1-n}{2}, -\frac{n}{2}; \frac{3}{2}, \frac{1}{2} - n; x\right) \quad , x < 0 \lor x > 8 \tag{12}$$

$$F_{2n} = \frac{4^n (-n)_n \left(\frac{1}{2} - n\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{1}{2} - 2n\right)_n} \sum_{k=0}^{n} \frac{\left(-\frac{1}{2} - n\right)_k \left(\frac{1}{2} - n\right)_k}{\left(\frac{1}{2}\right)_k (1)_k 4^k} \tag{13}$$

$$F_{2n+1} = \frac{4^n (-n)_n \left(-\frac{1}{2} - n\right)_n}{\left(\frac{3}{2}\right)_n \left(-\frac{1}{2} - 2n\right)_n} \sum_{k=0}^{n} \frac{\left(-\frac{1}{2} - n\right)_k \left(\frac{3}{2} + n\right)_k}{\left(\frac{3}{2}\right)_k (1)_k 4^k} \tag{14}$$

$$F_{2n} = \frac{4^n (-n)_n \left(\frac{1}{2} - n\right)_n}{\left(\frac{3}{2}\right)_n \left(\frac{1}{2} - 2n\right)_n} \left({}_2F_1\left(\frac{1}{2} + n, -\frac{1}{2} - n; \frac{1}{2}; \frac{1}{4}\right) - \frac{\left(-\frac{1}{2} - n\right)_{n+1} \left(\frac{1}{2} + n\right)_{n+1}}{\left(\frac{1}{2}\right)_{n+1} (1)_{n+1} 4^{n+1}} {}_3F_2\left(\frac{1}{2}, 1, 2n + \frac{3}{2}; n+2, n + \frac{3}{2}; \frac{1}{4}\right)\right) \tag{15}$$

$$F_{2n+1} = \frac{4^n (-n)_n \left(-\frac{1}{2} - n\right)_n}{\left(\frac{3}{2}\right)_n \left(-\frac{1}{2} - 2n\right)_n} \left({}_2F_1\left(-\frac{1}{2} - n, \frac{3}{2} + n; \frac{3}{2}; \frac{1}{4}\right) - \frac{\left(-\frac{1}{2} - n\right)_{n+1} \left(\frac{3}{2} + n\right)_{n+1}}{\left(\frac{3}{2}\right)_{n+1} (1)_{n+1} 4^{n+1}} {}_3F_2\left(\frac{1}{2}, 1, 2n + \frac{5}{2}; n+2, n + \frac{5}{2}; \frac{1}{4}\right)\right) \tag{16}$$

$$F_{2n} = \left(\frac{1 + i\sqrt{3}}{2}\right)^{2n} {}_3F_2\left(-2n, \frac{1}{2}, -\frac{1}{2} - 2n; \frac{3}{2}, \frac{1}{2} - 2n; \frac{1 + i\sqrt{3}}{2}\right) \quad , i = \sqrt{-1} \tag{17}$$

Entry 2. for $n = 0, 1, 2, 3, \ldots$, we have

$$F_n = \binom{2n}{n}^{-1} \sum_{k=0}^{[n/2]} \frac{(-1)^k 2^{4k}}{2k+1} \binom{2n - 2k}{n - k} \binom{n - k}{k} \binom{2k}{k}^{-1} \tag{18}$$

Entry 3.

$$\{F_n : n \geq 0\} = \left\{1, 1, \frac{1}{9}, -\frac{3}{5}, -\frac{97}{175}, \frac{13}{189}, \frac{985}{1617}, \frac{245}{429}, \ldots\right\} \tag{19}$$

$$F_n = {}_3F_2\left(1, \frac{1-n}{2}, -\frac{n}{2}; \frac{3}{2}, \frac{1}{2} - n; 4\right) = \frac{1}{2} \int_0^1 \frac{1}{\sqrt{1-x}} {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} - n; 4x\right) dx \tag{20}$$

Entry 4. for $-\infty < x < \frac{1}{\sqrt{2}}$, we have

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n 2^{-2n} \binom{2n}{n} \left( \frac{x^2 + x\sqrt{4 + x^2}}{2} \right)^{n+1} F_n \tag{21}$$

Entry 5.

$$\pi = 6 \sum_{n=0}^{\infty} (-1)^n 2^{-2n} \binom{2n}{n} \left( \frac{\sqrt{13} + 1}{6} \right)^{n+1} F_n \tag{22}$$

$$\pi = 6 \times \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \left( \frac{\sqrt{13} - 1}{6} \right)^{n+1} F_n \tag{23}$$

$$\pi = 8 \sum_{n=0}^{\infty} (-1)^n 2^{-2n} \binom{2n}{n} \left( \frac{\left(\sqrt{2} - 1\right)^2 \left(\sqrt{13 + 8\sqrt{2}} + 1\right)}{2} \right)^{n+1} F_n \tag{24}$$

$$\pi = 8 \times \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \left( \frac{\left(\sqrt{2} - 1\right)^2 \left(\sqrt{13 + 8\sqrt{2}} - 1\right)}{2} \right)^{n+1} F_n \tag{25}$$

$$\pi = 3 \times \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \left( \frac{\sqrt{21} - 3}{2} \right)^{n+1} F_n \tag{26}$$

$$\pi = 4 \times \sum_{n=0}^{\infty} 2^{-2n} \binom{2n}{n} \left( \frac{\sqrt{5} - 1}{2} \right)^{n+1} F_n \tag{27}$$

Entry 6. for $|x| < \frac{1}{\sqrt{2}}$, we have

$$\tan^{-1}(x) = x\sqrt{4 + x^2} \sum_{n=0}^{\infty} (-1)^n 2^{-3n-1} x^{2n} \binom{2n}{n} F_n \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} \left(1 + 4x^{-2}\right)^k \tag{28}$$

$$\pi = \sqrt{13} \sum_{n=0}^{\infty} (-1)^n 24^{-n} \binom{2n}{n} F_n \sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} 13^k \tag{29}$$

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## 3. Catalan’s constant

The Catalan’s constant is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \tag{30}$$

Entry 7.

$$G = \sum_{n=0}^{\infty} (-1)^n 2^{-3n-1} \binom{2n}{n} F_n \sum_{k=9}^{n+1} \binom{n+1}{k} \frac{(-1)^{k+1} 2^k}{2n - k + 2} {}_2F_1\left(-\frac{k}{2}, n + 1 - \frac{k}{2}; n + 2 - \frac{k}{2}; -\frac{1}{4}\right) \tag{31}$$

$$G = \sum_{n=0}^{\infty} (-1)^n 2^{-3n-1} \binom{2n}{n} F_n \sum_{k=9}^{n+1} \binom{n+1}{k} \frac{(-1)^{k+1} 5^{k/2}}{2n - k + 2} {}_2F_1\left(-\frac{k}{2}, 1; n + 2 - \frac{k}{2}; \frac{1}{5}\right) \tag{32}$$

$$G = \frac{2}{5}\sum_{n=0}^{\infty}(-1)^n\, 10^{-n}\binom{2n}{n}F_n\sum_{k=9}^{n+1}\binom{n+1}{k}\frac{(-1)^{k+1}\,5^{k/2}}{2n-k+2}\,{}_2F_1\left(n+1-\frac{k}{2}, n+2\,;n+2-\frac{k}{2}\,;\frac{1}{5}\right) \tag{33}$$

$$G = \frac{5}{4}\sum_{n=0}^{\infty}(-1)^n\, 2^{-3n-1}\binom{2n}{n}F_n\sum_{k=9}^{n+1}\binom{n+1}{k}\frac{(-1)^{k+1}\,5^{k/2}}{2n-k+2}\,{}_2F_1\left(1, n+2\,;n+2-\frac{k}{2}\,;-\frac{1}{4}\right) \tag{34}$$

Entry 8. for $i = \sqrt{-1}$ , we have

$$G = \sum_{n=0}^{\infty}(-1)^n\frac{2^{-2n}}{n+1}\binom{2n}{n}F_n\,\text{AppellF1}\left(n+1, -\frac{n+1}{2}, -\frac{n+1}{2}; n+2\,; \frac{2i}{\sqrt{15}+i}, -\frac{2i}{\sqrt{15}-i}\right) \tag{35}$$

where AppellF1 is the Appell hypergeometric function.

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## 4. References


[1] Andrews, G.E., Askey, R., Roy, R.: Special Functions. Cambridge University Press, Cambridge, 2000.

[2] Bailey, W.N.: Generalized Hypergeometric Series, Cambridge University Press, 1935.

[3] Buhring, W.: An analytic continuation of the hypergeometric series, SIAM J. Math. Anal., 18, 1987.

[4] Carlitz, L.: A class of generating functions. SIAM J. Math. Anal., 8, 1977.

[5] Fields, J.L., Wimp, J.: Expansions of hypergeometric functions in hypergeometric functions, Mathematics of Computation 15(76), 1961.

[6] Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products, 7th edn. Academic Press, Elsevier Inc, 2007.

[7] Hansen, E.R.: A Table of Series and Products, Prentice-Hall, Englewood Cliffs,NJ,USA, 1975.

[8] Koepf, W.: Power series in computer algebra. J. Symb. Comput. 11, 1992.

[9] Norlund, N.E.: Hypergeometric functions, Acta Math. 94, 1955.

[10] Rainville, E.D.: Special Functions, Macmillan Company, New York, 1960.

[11] Roy, R.: Binomial identities and hypergeometric series, The American Mathematical Monthly, 94(1) , 1987.

[12] Seaborn, J.B.: Hypergeometric Functions and Their Applications, Springer,New York, 1991.

[13] Slater, L.J.: Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, 1966.

[14] Spanier, J., Oldham, K.: An Atlas of Functions, Hemisphere Publ. Corp., Washington, 1987.

[15] Wilf, H.S.: Generatingfunctionology, 2nd ed., Academic Press: New York, NY, USA, London, UK, 1994.