An Elementary Proof of Goldbach's Conjecture v. 2.2

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Abstract

In this present paper we will show you an elementary proof of the Goldbach's Conjecture based on probabilities.

Keywords: prime, $\pi(x)$, prime counting function , Goldbach's Conjecture, probability, proof

1 INTRODUCTION

On the year 1742, professor Christian Goldbach had some correspondence with the famous mathematician Leonhard Euler establishing, in your comments, the basis of the problem that we know in modern times as "Goldbach's Conjecture", that says "EVERY EVEN NUMBER GREATER OR EQUAL TO 4 IS THE SUM OF TWO PRIMES" [1] [2].

In the dawn of January 9 in 2022 we was thinking, relaxed, at the moment of almost sleeping, about how to solve the problem, that arose by a random though, and suddenly became in an illuminated key idea: PROBABILITIES. We have an even number greater or equal to 4 that can be expressed as the sum of two numbers. Some combinations are: not prime + not prime, prime + not prime, not prime + prime and prime + prime. We mean: not prime "and" not prime, prime "and" not prime, not prime "and" prime and prime "and" prime. We have a set of pairs and like the set of poker all the possibilities of its combinations can be calculated as probabilities and all of them exists actually as events. In two hours of strong thinking we came to the solution of the theorem as an sketch. In the next afternoon we proceeded to write the first proof and calculate its correctness. Later, we discovered the second proof in another way. We show you the results for your enjoyment.

2 PRELIMINARY THEOREM

Theorem 1. (Christian Goldbach 1742, Danilo Chávez 2022-01-17)

Let be $N \geq 14$ EVEN NUMBERS. The case of $4 \leq N < 14$ is very known by simple counting. Let be E: $\{1, 2, 3, ...N-1\}$ a set of numbers smaller than N. Let be $E \times E$: $\{(1,1), (1,2), (1,3), ...(N-1, N-2), (N-1, N-1)\}$ the cartesian product of every number smaller than N which represents the pairs of sums of the numbers. The cardinality of the set $E \times E$ is

$$\#(E \times E) = (N-1)^2$$

which represents the total quantity of sums between the numbers.
Let be G: $\{(1, N-1), (2, N-2), (3, N-3), ...(N-2, 2), (N-1, 1)\}$ a subset of $E \times E$ which REPRESENTS the set of PAIRS whose sum is equal to N.

The cardinality of the set G is

$$\#G = N - 1$$

If we consider INDEPENDENT EVENTS in the probability of G to find $N = prime + prime$

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"EVERY EVEN NUMBER GREATER OR EQUAL TO 4 IS THE SUM OF TWO PRIMES"

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Proof. Proof by contradiction.

When we take a pair whose sum is equal to N (an even number), we can see the event of taking two numbers whose possible combinations are: not prime + not prime, prime + not prime, not prime + prime, prime + prime. That means: not prime AND not prime, prime AND not prime, not prime AND prime, prime AND prime. We can calculate the probability of each one of that events. If the probability of an event exists is because the event actually exists (the pairs of numbers we are looking for) like in a set of poker. We are looking for the event where we have a prime + prime, that means prime AND prime, simultaneously, in the subset G (G by Goldbach).

DEFINITION OF COUNTEREXAMPLE TO TEST. Suppose an hypothetical even number N that CAN NOT be expressed as the sum of two prime numbers. If we suppose that the event to find one number simultaneously with another number whose sum is equal to N are totally INDEPENDENT events, we have that the probabilities of the numbers given its sums equal to N are as follows

$$P(\text{not prime + not prime}) = \left(\frac{(N-1) - \pi(N-1)}{N-1}\right)\left(\frac{(N-1) - \pi(N-1)}{N-1}\right)$$
$$= \frac{((N-1) - \pi(N-1))^2}{(N-1)^2}$$

$$P(\text{prime + not prime}) = \left(\frac{\pi(N-1)}{N-1}\right)\left(\frac{(N-1) - \pi(N-1)}{N-1}\right)$$
$$= \frac{(\pi(N-1))((N-1) - \pi(N-1))}{(N-1)^2}$$

$$P(\text{not prime} + \text{prime}) = \left(\frac{(N-1)-\pi(N-1)}{N-1}\right)\left(\frac{\pi(N-1)}{N-1}\right)$$
$$= \frac{(\pi(N-1))((N-1)-\pi(N-1))}{(N-1)^2}$$

Because the hypothetical number we choose CAN NOT be expressed as the sum of two prime numbers

$$P(\text{prime} + \text{prime}) = 0$$

The probability of all its possibilities are as follows

$$P(\text{not prime} + \text{not prime}) + P(\text{prime} + \text{not prime}) + P(\text{not prime} + \text{prime}) + P(\text{prime} + \text{prime})$$

$$= \frac{((N-1)-\pi(N-1))^2}{(N-1)^2} + \frac{(\pi(N-1))((N-1)-\pi(N-1))}{(N-1)^2} + \frac{(\pi(N-1))((N-1)-\pi(N-1))}{(N-1)^2} + 0$$

$$= \frac{(N-1)^2-(\pi(N-1))^2}{(N-1)^2} < 1$$

An ABSURD because we have considered all the possibilities of such an hypothetical number N, the sum must be equal to 1!!, the fraction of pairs of numbers whose sum is equal to N. WE FOUND A CONTRADICTION!! DOES NOT EXIST such a number whose sum never is a prime plus another prime if we consider INDEPENDENT EVENTS.

We conclude that "EVERY EVEN NUMBER GREATER OR EQUAL TO 4 IS THE SUM OF TWO PRIME NUMBERS" if we consider INDEPENDENT EVENTS.

Observation: To reaffirm our result, we can see that assigning a probability to the two prime numbers combination we have

$$P(\text{prime} + \text{prime}) = \left(\frac{\pi(N-1)}{N-1}\right)\left(\frac{\pi(N-1)}{N-1}\right)$$
$$= \frac{(\pi(N-1))^2}{(N-1)^2}$$

The probability of all the possibilities are as follows

$$P(\text{not prime} + \text{not prime}) + P(\text{prime} + \text{not prime}) + P(\text{not prime} + \text{prime}) + P(\text{prime} + \text{prime})$$

$$= \frac{((N-1)-\pi(N-1))^2}{(N-1)^2} + \frac{(\pi(N-1))((N-1)-\pi(N-1))}{(N-1)^2}$$
$$+ \frac{(\pi(N-1))((N-1)-\pi(N-1))}{(N-1)^2} + \frac{(\pi(N-1))^2}{(N-1)^2}$$
$$= 1$$

This is the probability of the set G of numbers whose sum is equal to N.

We finally conclude again that "EVERY EVEN NUMBER GREATER OR EQUAL TO 4 IS THE SUM OF TWO PRIME NUMBERS" if we consider INDEPENDENT EVENTS.

Quod erat demonstrandum (Q.E.D). □

We see that the function

$$\frac{(\pi(N-1))^2}{(N-1)^2}$$

which represents the probability to find $N = prime + prime$, if we SUPPOSE INDEPENDENT EVENTS, is always greater than zero for finite numbers and tends to zero in the infinite. This result is necessary to understand the proof of Goldbach's Conjecture.

3 PRELIMINARY LEMMAS ON $(\pi(x))^2 > x$

Theorem 2. (Danilo Chávez 2023-08-08)

Let be $x > 0$. If $\sqrt{x} > ln(x)$ then

$$e^{\sqrt{x}} > x$$

Proof. Let $f(x) = e^x$ and $g(x) = x^2$. We know that, if $x \geq 0$

$$e^x > x^2$$

Taking the inverse functions of $f(x)$ and $g(x)$, $f^{-1}(x) = ln(x)$ and $g^{-1}(x) = \sqrt{x}$, we have

$$\sqrt{x} > ln(x)$$

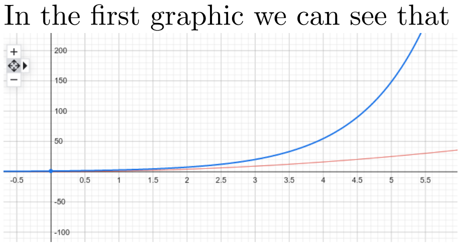
Now developing it's consequences we have

$$\sqrt{x} > ln(x)$$
$$e^{\sqrt{x}} > x$$

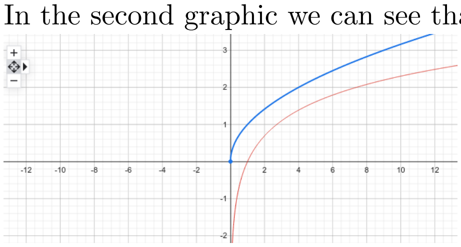
Quod erat demonstrandum (Q.E.D).

□

In the first graphic we can see that $e^x > x^2$



In the second graphic we can see that $\sqrt{x} > ln(x)$



Here we show three different approaches to show that $(\pi(x))^2 > x$.

Lemma 1. (Danilo Chávez 2023-02-10)
Let be $x \geq 5393$. If $e^{\sqrt{x}+1} > x$ then

$$(\pi(x))^2 > x$$

Proof. First we begin with an inequality (please see the graphics of the lemmas at the end)

$$e^{\sqrt{x}+1} > e^{\sqrt{x}} > x$$

$$e^{\sqrt{x}+1} > x$$

Rearranging we have

$$\sqrt{x} + 1 > ln(x)$$

$$\sqrt{x} > ln(x) - 1$$

$$\frac{\sqrt{x}}{ln(x) - 1} > 1$$

$$\frac{x}{ln(x) - 1} > \sqrt{x}$$

In 2010, Pierre Dusart [3] proved that

$$\pi(x) >= \frac{x}{ln(x) - 1}$$

if

$$x >= 5393$$

So

$$\pi(x) >= \frac{x}{ln(x) - 1} > \sqrt{x}$$

$$\pi(x) > \sqrt{x}$$

and it follows that

$$(\pi(x))^2 > x$$

Quod erat demonstrandum (Q.E.D). $\square$

Lemma 2. (Danilo Chávez 2023-02-15)
Let be $x \geq 17$. If $e^{\sqrt{x}} > x$ then

$$(\pi(x))^2 \geq x$$

Proof. First we begin with an inequality (please see the graphics of the lemmas at the end)

$$e^{\sqrt{x}} > x$$

Rearranging we have

$$\sqrt{x} > ln(x)$$
$$x > (ln(x))^2$$
$$\frac{x}{(ln(x))^2} > 1$$
$$\frac{x^2}{(ln(x))^2} > x$$
$$\left(\frac{x}{(ln(x))})\right)^2 > x$$

In 1962, J. Barkley Rosser and Lowell Schoenfeld [4] proved that

$$\pi(x) > \frac{x}{ln(x)}$$

if

$$x >= 17$$

So

$$(\pi(x))^2 > \left(\frac{x}{ln(x)}\right)^2 > x$$

and it follows that

$$(\pi(x))^2 > x$$

Quod erat demonstrandum (Q.E.D). □

Lemma 3. (Danilo Chávez 2023-02-15)
Let be $x \geq 88783$. If $e^{\sqrt{x}} > x$ then

$$(\pi(x))^2 > x$$

Proof. First we begin with an inequality (please see the graphics of the lemmas at the end)

$$e^{\sqrt{x}} > x$$

Rearranging we have

$$\sqrt{x} > ln(x)$$
$$x > (ln(x))^2$$
$$\frac{x}{(ln(x))^2} > 1$$
$$\frac{x^2}{(ln(x))^2} > x$$

$$\left(\frac{x}{(ln(x))})\right)^2 > x$$

In 2010, Pierre Dusart [3], in page 9, proved that if $x \geq 88783$

$$\pi(x) \geq \frac{x}{ln(x)} \left(1 + \frac{1}{ln(x)} + \frac{2}{(ln(x))^2}\right)$$

we see that

$$\pi(x) > \frac{x}{ln(x)}$$

So

$$(\pi(x))^2 > \left(\frac{x}{ln(x)}\right)^2 > x$$

and it follows that

$$(\pi(x))^2 > x$$

Quod erat demonstrandum (Q.E.D). □

4 PROOF OF THE GOLDBACH's CONJECTURE

The key idea to prove the Goldbach's Conjecture is to use the Set G and its probabilities. We make a function that describes the TRUE PROBABILITY of finding $N = prime + prime$ and is directly proportional to the probability of finding $N = prime + prime$, if we assume INDEPENDENT EVENTS, that we saw in the preliminary theorem. When we have the definition of the TRUE PROBABILITY, we can set the proportional function to be zero (as an argument of nullification of the TRUE PROBABILITY) but it fails in the main inequation that we found, excluding the zero as a solution of the TRUE PROBABILITY. So, always there is a probability to have $N = prime + prime$ if $N \geq 88783$ as even numbers.

Theorem 3. (Christian Goldbach 1742, Danilo Chávez 2023-02-22)

Let be $N \geq 4$ EVEN NUMBERS.

Let be E: $\{1, 2, 3, ...N - 1\}$ a set of numbers smaller than N.

Let be $E \times E$: $\{(1, 1), (1, 2), (1, 3), ...(N - 1, N - 2), (N - 1, N - 1)\}$ the Cartesian product of every number smaller than N which represents the pairs of sums of the numbers.

The cardinality of $E \times E$ is

$$\#(E \times E) = (N - 1)^2$$

which represents the total quantity of sums between the numbers.

Let be G: $\{(1, N - 1), (2, N - 2), (3, N - 3), ...(N - 2, 2), (N - 1, 1)\}$ a subset of $E \times E$ which REPRESENTS the set of PAIRS whose sum is equal to N.

The cardinality of the set G is

$$\#G = N - 1$$

Let be $E_{Npp}(N-1)$ the event to find $N = \text{prime} + \text{prime}$, actually it is a function of $N-1$, its domain is the set of even numbers $N \geq 88783$ and its codomain is the set of integers.

Let be $\frac{E_{Npp}(N-1)}{N-1}$ the TRUE PROBABILITY to find $N = \text{prime} + \text{prime}$ if we NOT ASSUME INDEPENDENT EVENTS, actually it is a function of $N-1$, its domain is the set of even numbers $N \geq 88783$ and its codomain is the set of rational numbers.

Let be $\frac{(\pi(N-1))^2}{(N-1)^2}$ the probability to find $N = \text{prime} + \text{prime}$ if we assume INDEPENDENT EVENTS, actually it is a function of $N-1$, its domain is the set of even numbers $N \geq 88783$ and its codomain is the set of rational numbers.

Let be $c(N-1)$ the proportional function that we will use between $\frac{E_{Npp}(N-1)}{N-1}$ and $\frac{(\pi(N-1))^2}{(N-1)^2}$, its domain is the set of even numbers $N \geq 88783$ and its codomain is the set of rational numbers.

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"EVERY EVEN NUMBER GREATER OR EQUAL TO 4 IS THE SUM OF TWO PRIMES".

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Proof. The case of $4 \leq N < 88783$ is very known to be true by intensive computation by Matti K. Sinisalo [5], or by Jörg Richstein [6], or by Tomás Oliveira e Silva, Sigfried Herzog and Silvio Pardi [7].

We will take the case of $N \geq 88783$ as even numbers, the limit given by Pierre Dusart [3] in 2010, in page 9.

If we pull apart the number N into two numbers

$$N = number1 + number2$$

being elements of the set G, The TRUE PROBABILITY to find two prime numbers, SIMULTANEOUSLY, given its sum equal to N in the set G is

$$P(Prime + Prime) = \frac{E_{Npp}(N-1)}{(N-1)}$$

We do not know for sure what is the complete expression for $E_{Npp}(N-1)$ but we can work with it in this way. The event $E_{Npp}(N-1)$ is a random integer (in appearance) greater or equal than zero.

We will show that $E_{Npp}(N-1) \neq 0$ which means that always there is $N = prime + prime$.

****

As

$$\frac{E_{Npp}(N-1)}{(N-1)}$$

is the TRUE PROBABILITY to have $N = prime + prime$ and is directly proportional to

$$\frac{(\pi(N-1))^2}{(N-1)^2}$$

the TRUE PROBABILITY to have $N = prime + prime$ is

$$\frac{E_{Npp}(N-1)}{(N-1)} \propto \frac{(\pi(N-1))^2}{(N-1)^2}$$

so we have

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$$\frac{E_{Npp}(N-1)}{(N-1)} = \frac{c(N-1)(\pi(N-1))^2}{(N-1)^2}$$

This is our MAIN EQUATION, remember that.

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By lemma 1, lemma 2 and lemma 3, above this proof, we know that if $N-1 >= 88783$

$$(\pi(N-1))^2 > N-1$$

so

$$(\pi(N-1))^4 > (N-1)^2$$

Returning to our main equation, we have

$$\frac{E_{Npp}(N-1)}{(N-1)} = \frac{c(N-1)(\pi(N-1))^2}{(N-1)^2} > \frac{c(N-1)(\pi(N-1))^2}{(\pi(N-1))^4} = \frac{c(N-1)}{(\pi(N-1))^2}$$

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$$\frac{E_{Npp}(N-1)}{(N-1)} = \frac{c(N-1)(\pi(N-1))^2}{(N-1)^2} > \frac{c(N-1)}{(\pi(N-1))^2}$$

This is our MAIN INEQUATION, remember that.

****

so

$$\frac{E_{Npp}(N-1)}{(N-1)} > \frac{c(N-1)}{(\pi(N-1))^2}$$

If we set $c(N-1) = 0$

$$\frac{E_{Npp}(N-1)}{N-1} > 0$$

but in our main equation

$$\frac{E_{Npp}(N-1)}{N-1} = 0$$

AN ABSURD!! A CONTRADICTION!!

In our main inequation we see that

$$\frac{E_{Npp}(N-1)}{(N-1)} = 0 > 0$$

AN ABSURD!! A CONTRADICTION!!

We note that there is no loss of solutions because we never altered the main equation and the main inequation.

We conclude that

$$c(N-1) \neq 0$$

which means that

$$E_{Npp}(N-1) \neq 0$$

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By lemma 1, lemma 2 and lemma 3, above this proof, we know that, if $N-1 >= 88783$

$$(\pi(N-1))^2 > N-1$$

rearranging we have

$$\frac{(\pi(N-1))^2}{(N-1)^2} > \frac{1}{(N-1)}$$

So, because

$$\frac{E_{Npp}(N-1)}{(N-1)} \propto \frac{(\pi(N-1))^2}{(N-1)^2}$$

and

$$E_{Npp}(N-1) \neq 0$$

and

$$\frac{(\pi(N-1))^2}{(N-1)^2} > \frac{1}{(N-1)}$$

then

$$\frac{E_{Npp}(N-1)}{(N-1)} > \frac{1}{(N-1)}$$

which shows that the true probability to find $N = prime + prime$ is greater than the minimal probability to find the sum of only one pair of numbers, assuring that ALWAYS THERE IS A SUM OF TWO PRIMES EQUAL TO N.

Because

$$E_{Npp}(N-1) \propto \frac{(\pi(N-1))^2}{(N-1)}$$

and

$$E_{Npp}(N-1) \neq 0$$

and

$$\frac{(\pi(N-1))^2}{(N-1)} > 1$$

then

$$E_{Npp}(N-1) > 1$$

Assuring that the event $E_{Npp}(N-1)$ is always greater to 1. Always there is $N = prime + prime$.

We conclude that EVERY EVEN NUMBER GREATER OR EQUAL TO 4 IS THE SUM OF TWO PRIME NUMBERS.

Quod erat demonstrandum (Q.E.D). $\square$

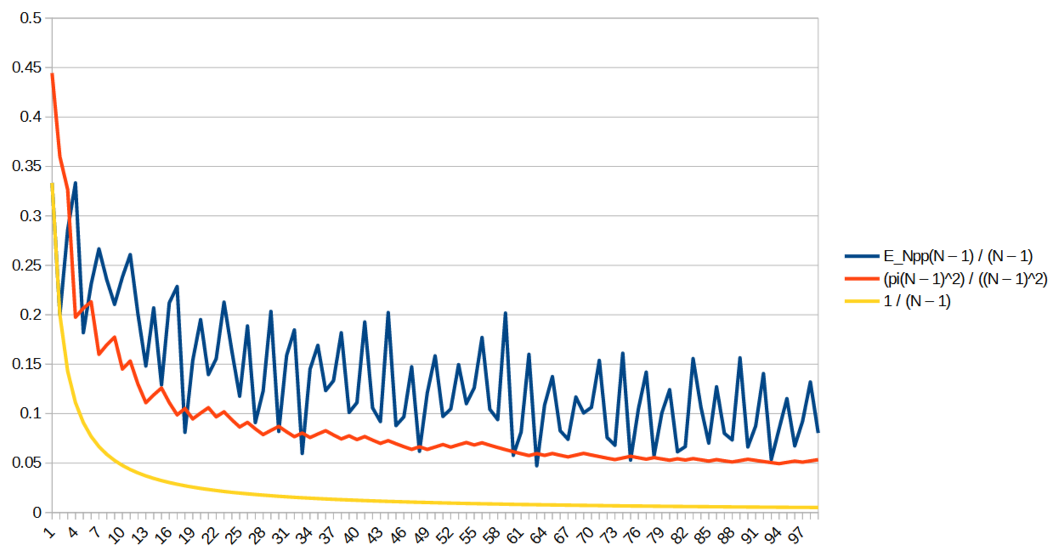
5 TABLES AND GRAPHICS OF THE THEOREM

In this section we present the tables and related graphics that shows the behaviour of the Goldbach's Conjecture.

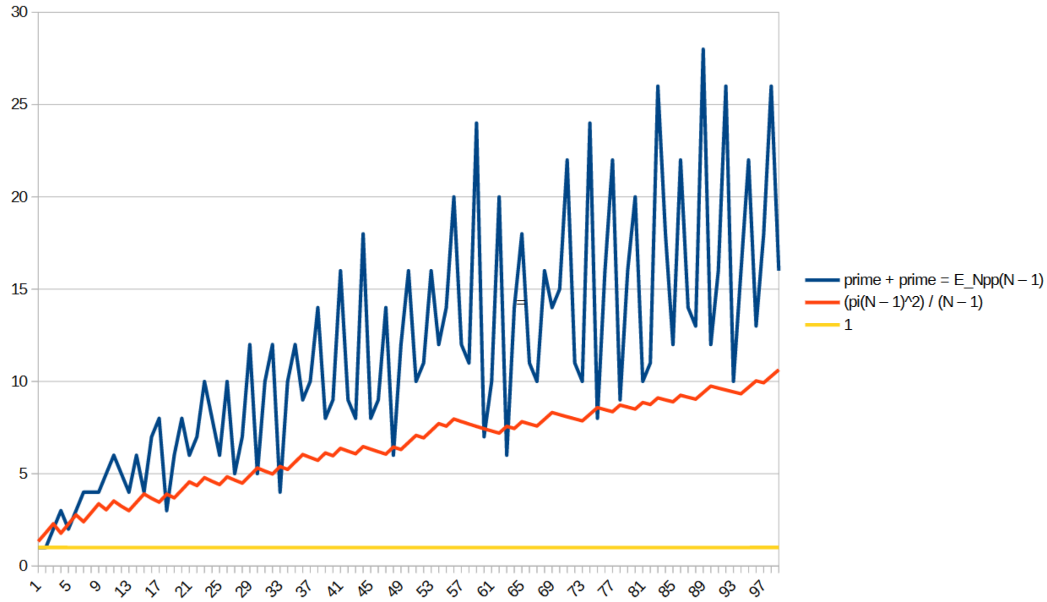
We plotted the even numbers $4 \leq N \leq 200$. $\pi(N-1)$ is taken from N. J .A. Sloane OEIS A000720 [8].

N	pi(N-1)	N – 1	prime + prime = E_Npp(N – 1)	E_Npp(N – 1) / (N – 1)	(pi(N – 1)^2) / ((N – 1)^2)	1 / (N – 1)	(pi(N – 1)^2) / (N – 1)
4	2	3	1	0.333333333333333	0.444444444444444	0.333333333333333	1.33333333333333
6	3	5	1	0.2	0.36	0.2	1.8
8	4	7	2	0.285714285714286	0.326530612244898	0.142857142857143	2.28571428571429
10	4	9	3	0.333333333333333	0.197530864197531	0.111111111111111	1.77777777777778
12	5	11	2	0.181818181818182	0.206611570247934	0.0909090909090909	2.27272727272727
14	6	13	3	0.230769230769231	0.21301775147929	0.0769230769230769	2.76923076923077
16	6	15	4	0.266666666666667	0.16	0.0666666666666667	2.4
18	7	17	4	0.235294117647059	0.169550173010381	0.0588235294117647	2.88235294117647
20	8	19	4	0.210526315789474	0.177285318559557	0.0526315789473684	3.36842105263158
22	8	21	5	0.238095238095238	0.145124716553288	0.0476190476190476	3.04761904761905
24	9	23	6	0.260869565217391	0.153119092627599	0.0434782608695652	3.52173913043478
26	9	25	5	0.2	0.1296	0.04	3.24
28	9	27	4	0.148148148148148	0.111111111111111	0.037037037037037	3
30	10	29	6	0.206896551724138	0.118906064209275	0.0344827586206897	3.44827586206897
32	11	31	4	0.129032258064516	0.125910509885536	0.032258064516129	3.90322580645161
34	11	33	7	0.212121212121212	0.111111111111111	0.0303030303030303	3.66666666666667
36	11	35	8	0.228571428571429	0.0987755102040816	0.0285714285714286	3.45714285714286
38	12	37	3	0.0810810810810811	0.10518626734843	0.027027027027027	3.89189189189189
40	12	39	6	0.153846153846154	0.0946745562130178	0.0256410256410256	3.69230769230769
42	13	41	8	0.195121951219512	0.100535395597858	0.024390243902439	4.1219512195122
44	14	43	6	0.13953488372093	0.106003244997296	0.0232558139534884	4.55813953488372
46	14	45	7	0.155555555555556	0.0967901234567901	0.0222222222222222	4.35555555555556
48	15	47	10	0.212765957446809	0.101856043458579	0.0212765957446809	4.78723404255319
50	15	49	8	0.163265306122449	0.0937109537692628	0.0204081632653061	4.59183673469388
52	15	51	6	0.117647058823529	0.0865051903114187	0.0196078431372549	4.41176470588235
54	16	53	10	0.188679245283019	0.0911356354574582	0.0188679245283019	4.83018867924528
56	16	55	5	0.0909090909090909	0.0846280991735537	0.0181818181818182	4.65454545454545
58	16	57	7	0.12280701754386	0.0787934749153586	0.0175438596491228	4.49122807017544
60	17	59	12	0.203389830508475	0.0830221200804367	0.0169491525423729	4.89830508474576
62	18	61	5	0.0819672131147541	0.0870733673743617	0.0163934426229508	5.31147540983607
64	18	63	10	0.158730158730159	0.0816326530612245	0.0158730158730159	5.14285714285714
66	18	65	12	0.184615384615385	0.0766863905325444	0.0153846153846154	4.98461538461539
68	19	67	4	0.0597014925373134	0.080418801514814	0.0149253731343284	5.38805970149254
70	19	69	10	0.144927536231884	0.0758244066372611	0.0144927536231884	5.23188405797102
72	20	71	12	0.169014084507042	0.0793493354493156	0.0140845070422535	5.63380281690141
74	21	73	9	0.123287671232877	0.0827547382248077	0.0136986301369863	6.04109589041096
76	21	75	10	0.133333333333333	0.0784	0.0133333333333333	5.88
78	21	77	14	0.181818181818182	0.0743801652892562	0.012987012987013	5.72727272727273
80	22	79	8	0.10126582278481	0.0775516744111521	0.0126582278481013	6.12658227848101
82	22	81	9	0.111111111111111	0.0737692424935223	0.0123456790123457	5.97530864197531
84	23	83	16	0.192771084337349	0.0767890840470315	0.0120481927710843	6.37349397590362
86	23	85	9	0.105882352941176	0.0732179930795848	0.0117647058823529	6.22352941176471
88	23	87	8	0.0919540229885058	0.0698903421852292	0.0114942528735632	6.08045977011494
90	24	89	18	0.202247191011236	0.0727180911501073	0.0112359550561798	6.47191011235955
92	24	91	8	0.0879120879120879	0.0695568168095641	0.010989010989011	6.32967032967033
94	24	93	9	0.0967741935483871	0.0665972944849116	0.010752688172043	6.19354838709677
96	24	95	14	0.147368421052632	0.0638227146814405	0.0105263157894737	6.06315789473684
98	25	97	6	0.0618556701030928	0.0664257625677543	0.0103092783505155	6.44329896907217
100	25	99	12	0.121212121212121	0.0637690031629426	0.0101010101010101	6.31313131313131

102	26	101	16	0.158415841584158	0.0662680129399079	0.0099009900990099	6.69306930693069
104	27	103	10	0.0970873786407767	0.0687152417758507	0.00970873786407767	7.07766990291262
106	27	105	11	0.104761904761905	0.0661224489795918	0.00952380952380953	6.94285714285714
108	28	107	16	0.149532710280374	0.0684775962966198	0.00934579439252336	7.32710280373832
110	29	109	12	0.110091743119266	0.0707852874337177	0.00917431192660551	7.71559633027523
112	29	111	14	0.126126126126126	0.068257446635825	0.00900900900900901	7.57657657657658
114	30	113	20	0.176991150442478	0.0704832015036416	0.00884955752212389	7.9646017699115
116	30	115	12	0.104347826086957	0.0680529300567108	0.00869565217391304	7.82608695652174
118	30	117	11	0.094017094017094	0.0657462195923734	0.00854700854700855	7.69230769230769
120	30	119	24	0.201680672268908	0.0635548336981852	0.00840336134453782	7.56302521008403
122	30	121	7	0.0578512396694215	0.0614712109828564	0.00826446280991736	7.43801652892562
124	30	123	10	0.0813008130081301	0.0594883997620464	0.00813008130081301	7.31707317073171
126	30	125	20	0.16	0.0576	0.008	7.2
128	31	127	6	0.047244094488189	0.0595821191642383	0.0078740157480315	7.56692913385827
130	31	129	14	0.108527131782946	0.0577489333573704	0.00775193798449612	7.44961240310078
132	32	131	18	0.137404580152672	0.0596701823903036	0.00763358778625954	7.81679389312977
134	32	133	11	0.0827067669172932	0.0578890836112839	0.0075187969924812	7.69924812030075
136	32	135	10	0.0740740740740741	0.0561865569272977	0.00740740740740741	7.58518518518519
138	33	137	16	0.116788321167883	0.0580212051787522	0.0072992700729927	7.94890510948905
140	34	139	14	0.100719424460432	0.0598312716733088	0.00719424460431655	8.31654676258993
142	34	141	15	0.106382978723404	0.0581459685126503	0.00709219858156028	8.19858156028369
144	34	143	22	0.153846153846154	0.0565308817057069	0.00699300699300699	8.08391608391608
146	34	145	11	0.0758620689655172	0.0549821640903686	0.00689655172413793	7.97241379310345
148	34	147	10	0.0680272108843538	0.0534962284233421	0.00680272108843537	7.86394557823129
150	35	149	24	0.161073825503356	0.055177694698437	0.00671140939597315	8.22147651006712
152	36	151	8	0.0529801324503311	0.0568396122977062	0.00662251655629139	8.58278145695364
154	36	153	16	0.104575163398693	0.055363321799308	0.0065359477124183	8.47058823529412
156	36	155	22	0.141935483870968	0.0539438085327784	0.00645161290322581	8.36129032258065
158	37	157	9	0.0573248407643312	0.0555397784899996	0.00636942675159236	8.71974522292994
160	37	159	16	0.10062893081761	0.0541513389501998	0.00628930817610063	8.61006289308176
162	37	161	20	0.124223602484472	0.0528143204351684	0.0062111801242236	8.50310559006211
164	38	163	10	0.0613496932515337	0.0543490534081072	0.00613496932515337	8.85889570552147
166	38	165	11	0.0666666666666667	0.0530394857667585	0.00606060606060606	8.75151515151515
168	39	167	26	0.155688622754491	0.054537631324178	0.00598802395209581	9.10778443113773
170	39	169	18	0.106508875739645	0.0532544378698225	0.00591715976331361	9
172	39	171	12	0.0701754385964912	0.0520160049245922	0.00584795321637427	8.89473684210526
174	40	173	22	0.127167630057803	0.0534598549901433	0.00578034682080925	9.2485549132948
176	40	175	14	0.08	0.0522448979591837	0.00571428571428571	9.14285714285714
178	40	177	13	0.0734463276836158	0.0510708927830445	0.00564971751412429	9.03954802259887
180	41	179	28	0.156424581005587	0.0524640304609719	0.00558659217877095	9.39106145251397
182	42	181	12	0.0662983425414365	0.0538445102408351	0.00552486187845304	9.74585635359116
184	42	183	16	0.087431693989071	0.0526740123622682	0.00546448087431694	9.63934426229508
186	42	185	26	0.140540540540541	0.0515412710007305	0.00540540540540541	9.53513513513514
188	42	187	10	0.053475935828877	0.0504446795733364	0.0053475935828877	9.4331550802139
190	42	189	16	0.0846560846560847	0.0493827160493827	0.00529100529100529	9.33333333333333
192	43	191	22	0.115183246073298	0.050683917655766	0.00523560209424084	9.68062827225131
194	44	193	13	0.0673575129533679	0.0519745496523397	0.00518134715025907	10.0310880829016
196	44	195	18	0.0923076923076923	0.050913872452334	0.00512820512820513	9.92820512820513
198	45	197	26	0.131979695431472	0.0521786183617202	0.0050761421319797	10.2791878172589
200	46	199	16	0.0804020100502513	0.0534329941163102	0.0050251256281407	10.6331658291457



We can see that $\frac{E_{Npp}(N-1)}{N-1}$ is about the order of $\frac{(\pi(N-1))^2}{(N-1)^2}$ and guided by it, both of them are greater than $\frac{1}{N-1}$



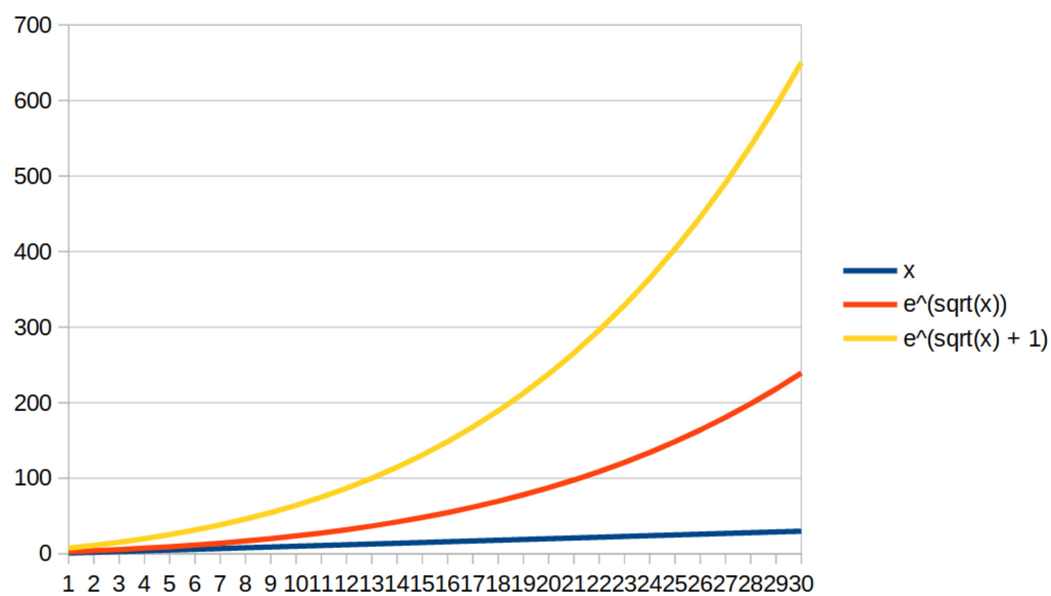
We can see that $E_{Npp}(N-1)$ is about the order of $\frac{(\pi(N-1))^2}{(N-1)}$ and guided by it, both of them are greater than 1

6 TABLES AND GRAPHICS OF THE LEMMAS

In this section we present the tables and related graphics that shows the behaviour of $\pi(N-1)^2 > N-1$.

We plotted the even numbers $4 \leq N \leq 200$. $\pi(N-1)$ is taken from N. J .A. Sloane OEIS A000720 [8].

x	e^(sqrt(x))	e^(sqrt(x) + 1)
1	2.7182818285	7.3890560989
2	4.1132503788	11.180973761
3	5.652233674	15.364364086
4	7.3890560989	20.085536923
5	9.3564690166	25.433519706
6	11.58243519	31.484323106
7	14.094030107	38.31154593
8	16.918828679	45.990144556
9	20.085536923	54.598150033
10	23.624342922	64.217622074
11	27.567148453	74.935278703
12	31.947745506	86.842976069
13	36.801966287	100.03811621
14	42.167820669	114.62402067
15	48.085628381	130.71028984
16	54.598150033	148.4131591
17	61.750719398	167.85585844
18	69.591378471	189.16897951
19	78.171016319	212.49085317
20	87.543512459	237.96793912
21	97.765885283	265.75522941
22	108.898446	296.0166669
23	121.00495841	328.92557959
24	134.15280493	364.66513188
25	148.4131591	403.42879349
26	163.86116487	445.42082684
27	180.57612296	490.85679369
28	198.64168466	539.96408178
29	218.14605317	592.98245228
30	239.18219293	650.16460874

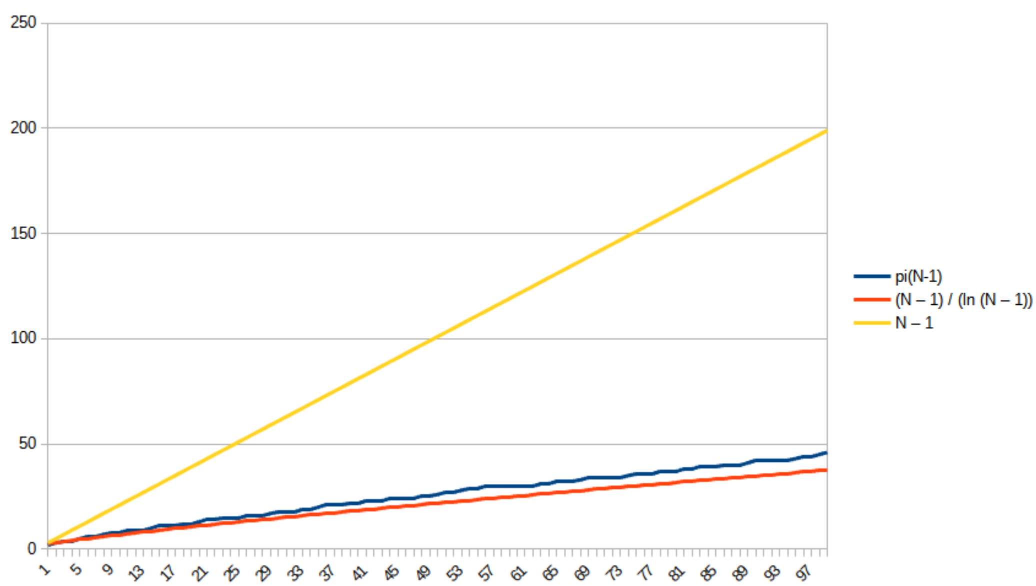


We can see that

$$e^{\sqrt{x}+1} > e^{\sqrt{x}} > x$$

N	pi(N-1)	(N – 1) / (ln (N – 1))	pi(N – 1)^2	(N – 1)^2 / (ln (N – 1)^2)	N – 1
4	2	2.73071767988051	4	7.45681904721201	3
6	3	3.10667467279806	9	9.65142752260493	5
8	4	3.59728839658826	16	12.9404838082285	7
10	4	4.09607651982077	16	16.777842856227	9
12	5	4.58735630566671	25	21.0438378751401	11
14	6	5.06832618826664	36	25.6879303506695	13
16	6	5.53904059603283	36	30.6809707244997	15
18	7	6.00025410570094	49	36.003049332981	17
20	8	6.45284216600706	64	41.6391720193987	19
22	8	6.89763351381407	64	47.5773480908911	21
24	9	7.33536674478742	81	53.8076052805332	23
26	9	7.76668668199515	81	60.3214220162808	25
28	9	8.19215303964154	81	67.1113714249081	27
30	10	8.61225192682773	100	74.170883251148	29
32	11	9.02740696281884	121	81.49407647235	31
34	11	9.43798902758825	121	89.0756368848763	33
36	11	9.84432449219549	121	96.91072470764	35
38	12	10.2467020561277	144	104.994903027052	37
40	12	10.6453783959665	144	113.32408119331	39
42	13	11.04058283064	169	121.894469240223	41
44	14	11.4325211840186	196	130.702540623035	43
46	14	11.8213789956238	196	139.745001358176	45
48	15	12.2073242020968	225	149.018764175097	47
50	15	12.5905093880589	225	158.520926650799	49
52	15	12.9710736853462	225	168.24875255068	51
54	16	13.3491443838401	256	178.199655780609	53
56	16	13.7248383046067	256	188.371186487598	55
58	16	14.0982629761529	256	198.761018944764	57
60	17	14.46951764678	289	209.366940930478	59
62	18	14.8386941598041	324	220.186844368206	61
64	18	15.2058777134843	324	231.218717037439	63
66	18	15.5711475235562	324	242.460635200351	65
68	19	15.934577403117	361	253.910757015926	67
70	19	16.296236272064	361	265.567316634935	69
72	20	16.6561886062344	400	277.428618886451	71
74	21	17.0144948347212	441	289.493034480754	73
76	21	17.3712116924788	441	301.758995664911	75
78	21	17.7263925342081	441	314.22499227683	77
80	22	18.0800876145932	484	326.889568151368	79
82	22	18.4323443391935	484	339.751317838597	81
84	23	18.7832074896648	529	352.8088835998	83
86	23	19.1327194264512	529	366.060952651302	85
88	23	19.4809202716445	529	379.506254630171	87
90	24	19.8278480743387	576	393.143559259056	89
92	24	20.1735389604868	576	406.971674190279	91
94	24	20.5180272690057	576	420.989443011662	93
96	24	20.8613456756427	576	435.195743398655	95
98	25	21.2035253059273	625	449.5894853991	97
100	25	21.5445958383654	625	464.169609838513	99

102	26	21.8845855988887	676	478.935086835087	101
104	27	22.2235216474532	729	493.884914414819	103
106	27	22.561429857572	729	509.01811721814	105
108	28	22.8983349894778	784	524.333745290345	107
110	29	23.2342607575304	841	539.830872948918	109
112	29	23.5692298924146	841	555.50859772149	111
114	30	23.903264198616	900	571.366039348837	113
116	30	24.2363846076064	900	587.402338847819	115
118	30	24.5686112271262	900	603.616657629673	117
120	30	24.8999633869103	900	620.008176669474	119
122	30	25.2304596811669	900	636.576095722989	121
124	30	25.5601180080893	900	653.31963258745	123
126	30	25.8889556066505	900	670.23802240312	125
128	31	26.2169890909075	961	687.330516992764	127
130	31	26.5442344820188	961	704.596384236398	129
132	32	26.8707072381601	1024	722.034907478911	131
134	32	27.1964222825052	1024	739.645384968346	133
136	32	27.5213940294241	1024	757.42712932282	135
138	33	27.8456364090355	1089	775.379467024205	137
140	34	28.1691628902397	1156	793.501737936855	139
142	34	28.4919865023448	1156	811.793294849797	141
144	34	28.8141198553925	1156	830.253503040924	143
146	34	29.1355751592761	1156	848.881739861826	145
148	34	29.4563642417394	1156	867.677394342021	147
150	35	29.7764985653353	1225	886.639866811417	149
152	36	30.095989243418	1296	905.768568539932	151
154	36	30.414847055234	1296	925.062921393275	153
156	36	30.7330824601756	1296	944.522357503953	155
158	37	31.0507056112524	1369	964.14631895666	157
160	37	31.3677263678324	1369	983.934257487208	159
162	37	31.6841543077024	1369	1003.88563419429	161
164	38	31.9999987384901	1444	1023.99991926337	163
166	38	32.3152687084909	1444	1044.27659170197	165
168	39	32.6299730169352	1521	1064.71513908592	167
170	39	32.9441202237332	1521	1085.31505731578	169
172	39	33.2577186587279	1521	1106.0758503831	171
174	40	33.5707764304884	1600	1126.99703014584	173
176	40	33.8833014346688	1600	1148.07811611263	175
178	40	34.1953013619618	1600	1169.31863523539	177
180	41	34.5067837056682	1681	1190.71812170976	179
182	42	34.8177557689067	1764	1212.27611678323	181
184	42	35.1282246714843	1764	1233.99216857028	183
186	42	35.4381973564464	1764	1255.86583187444	185
188	42	35.7476805963248	1764	1277.89666801685	187
190	42	36.0566809991019	1764	1300.08424467099	189
192	43	36.3652050139048	1849	1322.42813570332	191
194	44	36.6732589364455	1936	1344.92792101958	193
196	44	36.980848914221	1936	1367.58318641644	195
198	45	37.2879809514856	2025	1390.39352343836	197
200	46	37.594660914008	2116	1413.35852923924	199

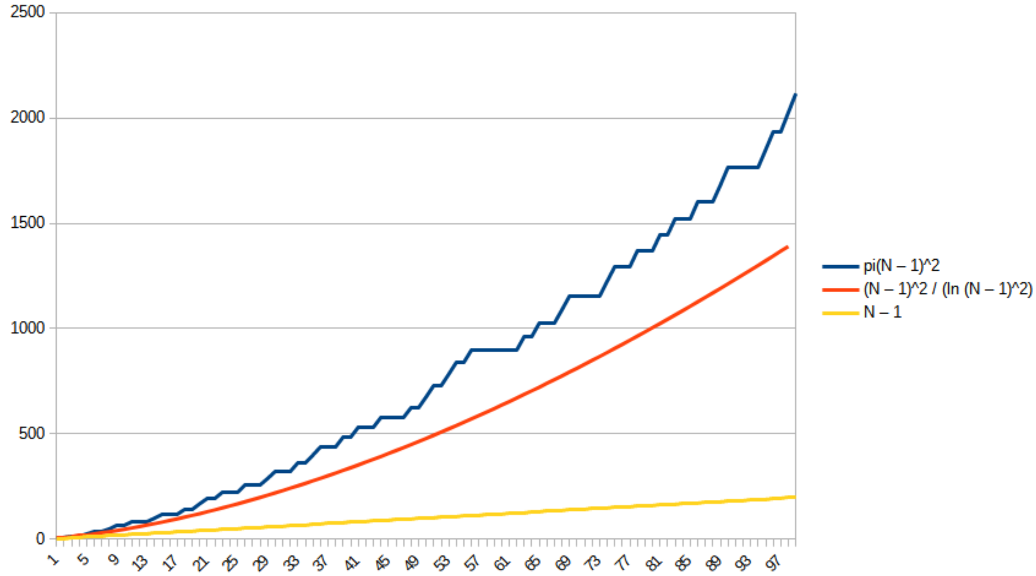


We can see that

$$N - 1 > \pi(N-1) > \frac{N-1}{ln(N-1)}$$

if

$$N - 1 \geq 11$$



We can see that

$$(\pi(N-1))^2 > \frac{(N-1)^2}{(ln(N-1))^2} > N - 1$$

if

$$N - 1 \geq 11$$

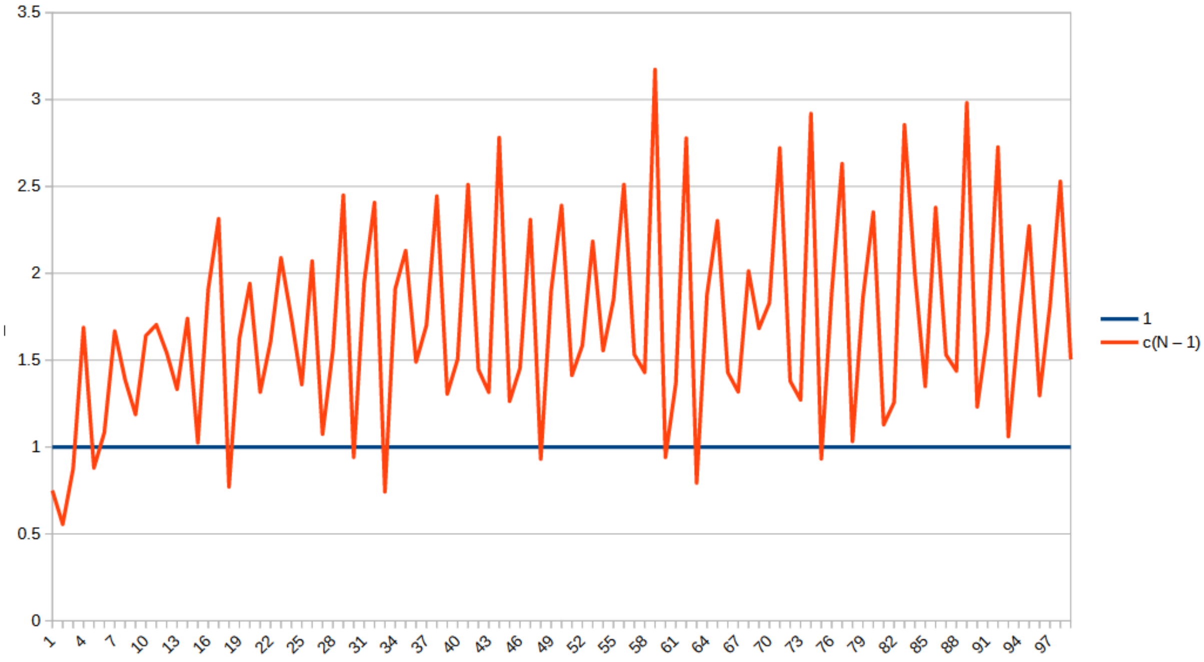
7 TABLES AND GRAPHICS OF THE FUNCTION c(N - 1)

In this section we present the tables and related graphics that shows the behaviour of the function $c(N-1)$.

We plotted the even numbers $4 \leq N \leq 200$, here it is the function $c(N-1)$. $\pi(N-1)$ is taken from N. J .A. Sloane OEIS A000720 [8].

N	c(N – 1)
4	0.75
6	0.555555555555556
8	0.875
10	1.6875
12	0.88
14	1.08333333333333
16	1.66666666666667
18	1.38775510204082
20	1.1875
22	1.640625
24	1.7037037037037
26	1.54320987654321
28	1.33333333333333
30	1.74
32	1.02479338842975
34	1.90909090909091
36	2.31404958677686
38	0.770833333333333
40	1.625
42	1.94082840236686
44	1.31632653061225
46	1.60714285714286
48	2.08888888888889
50	1.74222222222222
52	1.36
54	2.0703125
56	1.07421875
58	1.55859375
60	2.44982698961938
62	0.941358024691358
64	1.94444444444444
66	2.40740740740741
68	0.742382271468144
70	1.91135734072022
72	2.13
74	1.48979591836735
76	1.70068027210884
78	2.44444444444444
80	1.30578512396694
82	1.50619834710744
84	2.51039697542533
86	1.4461247637051
88	1.31568998109641
90	2.78125
92	1.26388888888889
94	1.453125
96	2.30902777777778
98	0.9312
100	1.9008

102	2.3905325443787
104	1.41289437585734
106	1.5843621399177
108	2.18367346938776
110	1.55529131985731
112	1.84780023781213
114	2.51111111111111
116	1.53333333333333
118	1.43
120	3.17333333333333
122	0.941111111111111
124	1.36666666666667
126	2.77777777777778
128	0.792924037460978
130	1.8792924037461
132	2.302734375
134	1.4287109375
136	1.318359375
138	2.01285583103765
140	1.68339100346021
142	1.82958477508651
144	2.72145328719723
146	1.37975778546713
148	1.27162629757785
150	2.91918367346939
152	0.932098765432099
154	1.88888888888889
156	2.63117283950617
158	1.03214024835646
160	1.85829072315559
162	2.35208181154127
164	1.12880886426593
166	1.25692520775623
168	2.85470085470085
170	2
172	1.3491124260355
174	2.37875
176	1.53125
178	1.438125
180	2.98155859607377
182	1.2312925170068
184	1.65986394557823
186	2.72675736961451
188	1.06009070294785
190	1.71428571428571
192	2.27257977285019
194	1.29597107438017
196	1.81301652892562
198	2.52938271604938
200	1.5047258979206



As we can see, the inferior limit of the function $c(N-1)$ tends to 1.

$$\lim_{N\to\infty} \inf \frac{(N-1)E_{Npp}(N-1)}{(\pi(N-1))^2} = \lim_{N\to\infty} \inf c(N-1) = 1$$

8 TABLES AND GRAPHICS OF THE INEQUALITY

In this section we present the tables and related graphics that shows the behaviour of the main inequality.

We plotted the even numbers $4 \leq N \leq 200$. Here it is the main inequality. $\pi(N-1)$ is taken from N. J .A. Sloane OEIS A000720 [8].

E_Npp(N – 1) / (N – 1)	(c(N – 1) / (pi(N – 1))^2
0.333333333333333	0.1875
0.2	0.0617283950617285
0.285714285714286	0.0546875
0.333333333333333	0.10546875
0.181818181818182	0.0352
0.230769230769231	0.0300925925925925
0.266666666666667	0.0462962962962964
0.235294117647059	0.0283215326947106
0.210526315789474	0.0185546875
0.238095238095238	0.025634765625
0.260869565217391	0.0210333790580704
0.2	0.0190519737844841
0.148148148148148	0.0164609053497942
0.206896551724138	0.0174
0.129032258064516	0.00846936684652686
0.212121212121212	0.0157776108189331
0.228571428571429	0.019124376750222
0.0810810810810811	0.00535300925925926
0.153846153846154	0.0112847222222222
0.195121951219512	0.0114841917299814
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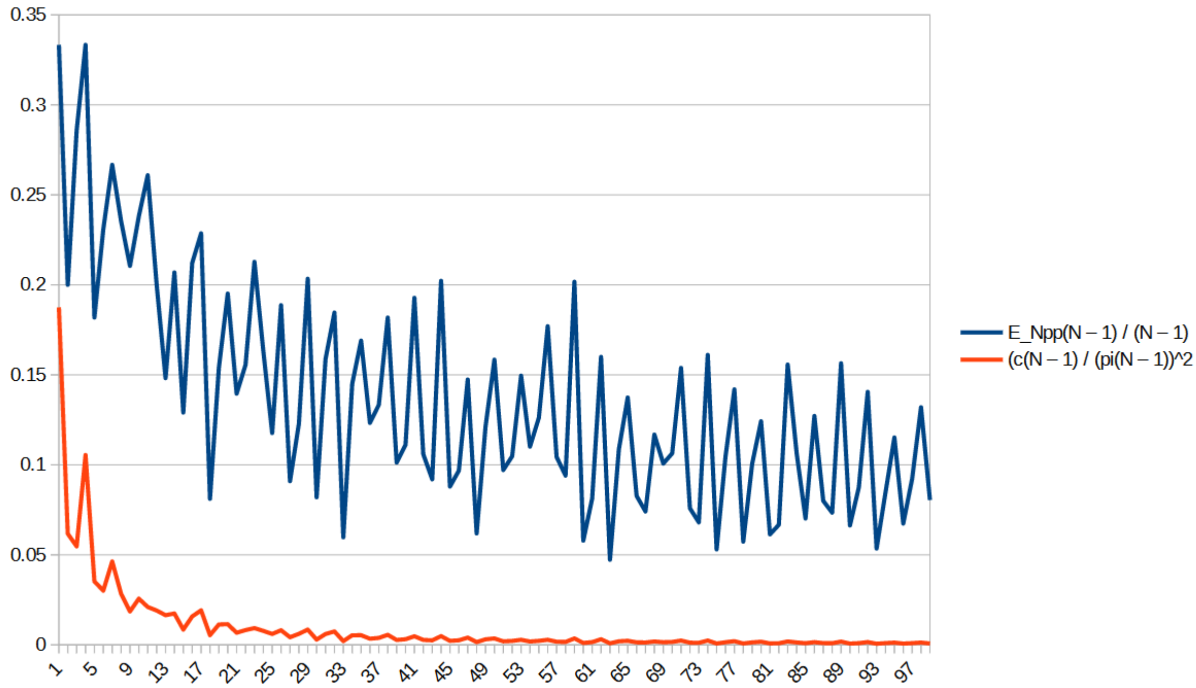

As we can see

$$\frac{E_{Npp}(N-1)}{(N-1)} > \frac{c(N-1)}{(\pi(N-1))^2}$$

the right expression never is zero, but tends to zero in the infinite

$$\lim_{N\to\infty} \frac{c(N-1)}{(\pi(N-1))^2} = 0$$

E_Npp(N – 1)	(c(N – 1) (N – 1)) / (pi(N – 1))^2
1	0.5625
1	0.308641975308642
2	0.3828125
3	0.94921875
2	0.3872
3	0.391203703703703
4	0.694444444444446
4	0.48146605581008
4	0.3525390625
5	0.538330078125
6	0.483767718335619
5	0.476299344612102
4	0.444444444444443
6	0.5046
4	0.262550372242333
7	0.520661157024794
8	0.669353186257769
3	0.198061342592593
6	0.440104166666667
8	0.470851860929238
6	0.288785922532279
7	0.368986880466473
10	0.436345679012346
8	0.379417283950617
6	0.308266666666667
10	0.428619384765625
5	0.230789184570313
7	0.347030639648438
12	0.500137689922296
5	0.177230986130163
10	0.378086419753086
12	0.482967535436672
4	0.13778285924755
10	0.365328688392507
12	0.378075
9	0.246610208709335
10	0.28923133879402
14	0.426807760141093
8	0.21313434874667
9	0.252070384536576
16	0.39388081088904
9	0.232364092466793
8	0.216380015794684
18	0.429741753472222
8	0.199676890432099
9	0.234619140625
14	0.380829234182099
6	0.14452224
12	0.30108672

16	0.357165365358356
10	0.19962705173293
11	0.228200308218599
16	0.298026863806748
12	0.201577590801958
14	0.243883265632754
20	0.315283950617284
12	0.195925925925926
11	0.1859
24	0.419585185185185
7	0.126527160493827
10	0.186777777777778
20	0.385802469135803
6	0.104788088197236
14	0.252267138484128
18	0.294588088989258
11	0.185564994812012
10	0.173807144165039
16	0.253224287283892
14	0.202414662180769
15	0.223158696615223
22	0.336650363381664
11	0.173066504232469
10	0.161703344069156
24	0.355068054977093
8	0.108601013565005
16	0.222993827160494
22	0.314685023243408
9	0.118368165808593
16	0.215827775735383
20	0.276614442409163
10	0.127420945204534
11	0.143623725263004
26	0.313435268070376
18	0.222222222222222
12	0.15167536150695
22	0.25720234375
14	0.16748046875
13	0.159092578125
28	0.317488987922192
12	0.126340105203079
16	0.172196769864408
26	0.285969452028733
10	0.112379229847646
16	0.183673469387755
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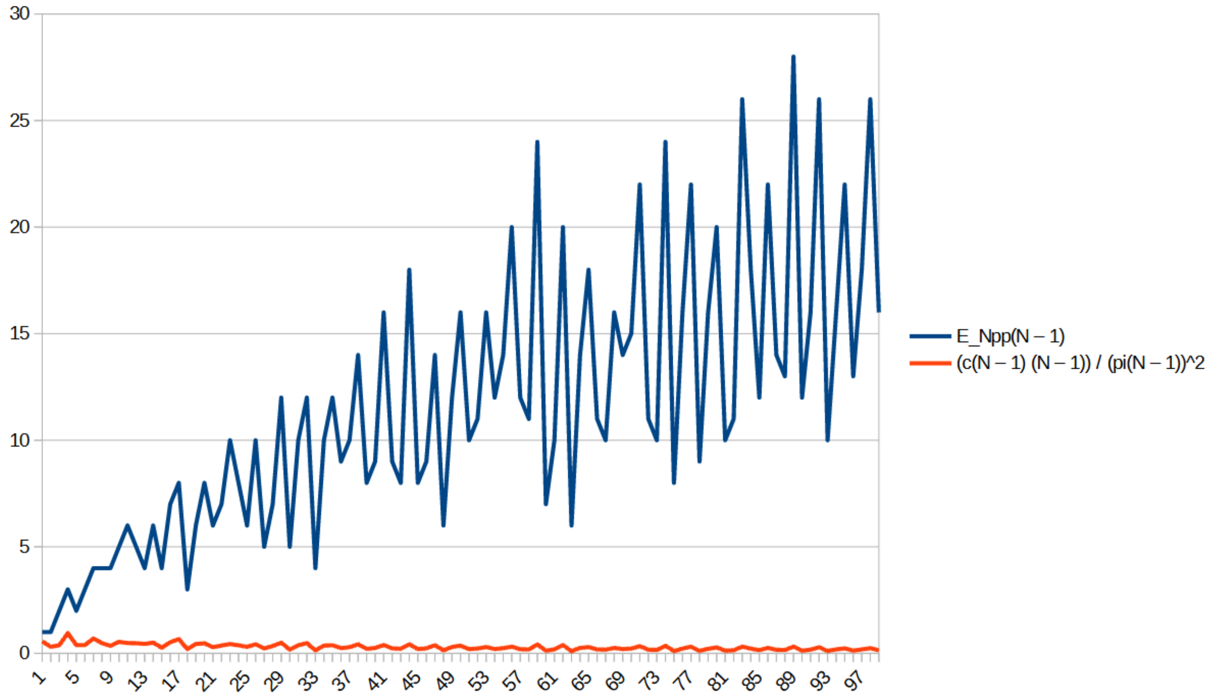
16	0.14151250174206



As we can see

$$E_{Npp}(N-1) > \frac{c(N-1)(N-1)}{(\pi(N-1))^2}$$

the right expression never is zero, but tends to zero in the infinite

$$\lim_{N\to\infty} \frac{c(N-1)(N-1)}{(\pi(N-1))^2} = 0$$

9 ACKNOWLEDGEMENTS

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