# Analytic Proof of The Prime Number Theorem

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**Abstract**

In this paper, we shall prove the *Prime Number Theorem* by providing a brief introduction about the famous *Riemann Zeta Function* and using its properties.

***Keywords :*** Prime Number Theorem, Riemann Zeta Function, Holomorphic, Contour Integral, Critical Line.

## Contents

# 1 Introduction

In Number Theory, the **Prime Number Theorem (PNT)** describes the asymptotic distribution of the prime numbers among the positive integers. It formalizes the intuitive idea that primes become less common as they become larger by precisely quantifying the rate at which this occurs. The theorem was proved independently by *Jacques Hadamard* and *Charles Jean de la Vallée-Poussin* in 1896 using ideas introduced by *Bernhard Riemann* (in particular, the *Riemann zeta function*).

The first such distribution found is $\pi(x) \sim \frac{x}{log(x)}$, where $\pi(x)$ is the prime-counting function and $log(x)$ is the natural logarithm of $x$. This means that for large enough $x$, the probability that a random integer $n$ not greater than $x$ is prime is very close to $\frac{1}{log(x)}$. Consequently, a random integer with at most $2k$ digits (for large enough $k$) is about half as likely to be prime as a random integer with at most $k$ digits. For example, among the positive integers of at most 1000 digits, about one in 2300 is prime ($log(101000) \approx 2302.6$), whereas among positive integers of at most 2000 digits, about one in 4600 is prime ($log(102000) \approx 4605.2$). In other words, the average gap between consecutive prime numbers among the first $n$ integers is roughly $log(n)$.

# 2 Statement of the Theorem

We first introduce some special arithmetic functions and notations before stating the *Prime Number Theorem* and later proving it:

**Definition 2.0.1.** For each $x \geq 0$ ,we define,

$$\pi(x) = \text{The number of primes} \leq x.$$

**Definition 2.0.2.** For each $x \geq 0$ , we define,

$$\psi(x) = \sum_{n \leq x} \Lambda(n) \ ,$$

Where ,

$$\Lambda(n) = \begin{cases} log(p) \ , & \text{if } n = p^m, \ p^m \leq x, \ m \in \mathbb{N} \\ 0 \ , & \text{otherwise} \ . \end{cases} \tag{2.1}$$

$\Lambda(n)$ is said to be the "*Mangoldt Function*" .
Therefore,

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{m=1}^{\infty} \sum_{p, p^m \leq x} \Lambda(p^m) = \sum_{m=1}^{\infty} \sum_{p \leq x^{\frac{1}{m}}} log(p) \tag{2.2}$$

**Definition 2.0.3.** (Chebyshev's Theta Function) For each $x \geq 0$, we define,

$$\vartheta(x) = \sum_{p \leq x} log(p).$$

***Remark*** 2.0.1. From the two definitions, it can be deduced that,

$$\psi(x) = \sum_{m \leq log_2 x} \vartheta(x^{\frac{1}{m}})$$

**Definition 2.0.4.** (Möbius Function) The Möbius Function $\mu$ is defined as follows :

$$\mu(1) = 1.$$

If $n > 1$, such that suppose, $n = {p_1}^{a_1} {p_2}^{a_2} {p_3}^{a_3} \cdots {p_k}^{a_k}$ . Then,

$$\mu(n) = \begin{cases} (-1)^k \ , & \text{if } a_1 = a_2 = \cdots\cdots = a_k = 1 \\ 0 \ , & \text{otherwise .} \end{cases} \tag{2.3}$$

**Definition 2.0.5.** (Big $O$ Notation) If $g(x) > 0$ for all $x \geq a$, we define, $f(x) = O(g(x))$ to mean that, the quotient, $\frac{f(x)}{g(x)}$ is bounded for all $x \geq a$; i.e., there exists a constant $M > 0$ such that,

$$|f(x)| \leq M.g(x), \text{ for all } x \geq a \ .$$

**Definition 2.0.6.** We say g(x) $\sim$ f(x) if, $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.

Therefore, we give the following statement of the *"Prime Number Theorem"* as,

**Theorem 2.0.2.** *(Prime Number Theorem)*

$$\pi(x) \sim \frac{x}{log(x)}$$

*In other words,*

$$\lim_{x \to \infty} \frac{\pi(x) log(x)}{x} = 1$$

.

Now we prove an important result :

**Theorem 2.0.3.** *We have,*

$$\vartheta(x) \sim \pi(x) log(x)$$

*Proof.* Using definition,

$$\vartheta(x) = \sum_{x^{1-\varepsilon} \leq p \leq x} log(p) \lfloor \frac{log(x)}{log(p)} \rfloor \leq \sum_{p \leq x} log(x) \leq \pi(x) log(x)$$

(by Theorem of Partial Sums of Dirichlet Product )

Therefore, for any $\varepsilon > 0$ ,

$$\vartheta(x) \geq \sum_{x^{1-\varepsilon} \leq p \leq x} log(p) \geq \sum_{x^{1-\varepsilon} \leq p \leq x} (1-\varepsilon) log(x) = (1-\varepsilon)(\pi(x) + O(x^{1-\varepsilon})) log(x)$$

Since we can choose $\varepsilon > 0$ arbitrarily, hence,

$$\Rightarrow \vartheta(x) \to \pi(x) log(x), \text{ as } x \to \infty$$
$$\Rightarrow \vartheta(x) \sim \pi(x) log(x) \ .$$

□

Using *Theorem* (2.0.3) along with properties of the functions $\vartheta(x)$ and $\psi(x)$ we obtain the following result also known as an *alternative statement of Prime Number Theorem* :

**Theorem 2.0.4.** *(Alternative Statement of Prime Number Theorem)*

$$\psi(x) \sim x, \quad for\ x \to \infty.$$

Hence, in order to provide an analytic proof of the **Prime Number Theorem**, it is enough to establish that alternative statement mentioned above holds true.

Firstly, we state two very important results used in Complex Analysis, that will be required later during the proof of the theorem:

**Theorem 2.0.5.** *(Riemann-Lebesgue Lemma) If $f$ is $L^1$ integrable on $\mathbb{R}^d$, that is to say, if the Lebesgue integral of $|f|$ is finite, then the Fourier transform of $f$ satisfies,*

$$\hat{f}(z) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i z.x} dx \to 0 \quad as, \quad |z| \to \infty.$$

**Theorem 2.0.6.** *(Cauchy Integral Theorem) In mathematics, the* ***Cauchy Integral Theorem*** *(also known as the Cauchy-Goursat theorem) in complex analysis, named after Augustin-Louis Cauchy, is an important statement about line integrals for holomorphic functions in the complex plane. Essentially, it says that if two different paths connect the same two points, and a function is holomorphic everywhere "in between" the two paths, then the two path integrals of the function will be the same.*
*The theorem is usually formulated for closed paths as follows:*

*Let $U$ be an open subset of $\mathbb{C}$ which is simply connected, let $f : U \to \mathbb{C}$ be a holomorphic function, and let $\gamma$ be a rectifiable path in $U$ whose start point is equal to its end point. Then,*

$$\oint_\gamma f(z)dz = 0.$$

A precise *(homology)* version can be stated using *winding numbers*. The winding number of a closed curve around a point $a$ not on the curve is the integral of $\frac{f(z)}{2\pi i}$, where $f(z) = \frac{1}{(z-a)}$ around the curve. It is an integer. Briefly, the path integral along a Jordan curve of a function holomorphic in the interior of the curve, is 0. Instead of a single closed path we can consider a linear combination of closed paths, where the scalars are integers. Such a combination is called a *closed chain*, and one defines an integral along the chain as a linear combination of integrals over individual paths. A closed chain is called a *cycle* in a region, if it is homologous to 0 in the region; that is, the winding number, expressed by the integral of $\frac{1}{(z-a)}$ over the closed chain, is 0 for each point $a$ not in the region. This means that the closed chain does not wind around points outside the region. Then **Cauchy's Theorem** can be stated as the integral of a function holomorphic in an open set taken around any cycle in the open set is 0. An example is furnished by the **ring-shaped region**. This version is crucial for rigorous derivation of *Laurent Series* and *Cauchy's Residue Formula* without involving any physical notions such as cross cuts or deformations. This version enables the extension of Cauchy's theorem to multiply-connected regions analytically.

**Theorem 2.0.7.** *(Cauchy Residue Theorem) In complex analysis, the* ***Residue Theorem****, sometimes called* ***Cauchy's Residue Theorem****, is a powerful tool to evaluate line integrals of analytic functions over closed curves; it can often be used to compute real integrals as well. It generalizes the Cauchy Integral Theorem and Cauchy's Integral Formula. From a geometrical perspective, it is a special case of the generalized* ***Stokes' Theorem****. The statement goes as follows:*

*Suppose $U$ is a simply connected open subset of the complex plane, and $a_1, a_2, \cdots a_n$ are finitely many points of $U$ and $f$ is a function which is defined and holomorphic on $U/a_1, a_2, \cdots a_n$. If $\gamma$ is a closed rectifiable curve in $U$ which does not meet any of the $a_k$,*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \eta(\gamma, a_k) Res(f; a_k).$$

*If $\gamma$ is a positively oriented simple closed curve, $\eta(\gamma, a_k) = 1$ if $a_k$ is in the interior of $\gamma$, and $0$ otherwise, so,*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f; a_k).$$

*with the sum over those $k$ for which $a_k$ is inside $\gamma$.*

Here, $Res(f; a_k)$ denotes the residue of $f$ at $a_k$, and $\eta(\gamma, a_k)$ is the winding number of the curve $\gamma$ about the point $a_k$. This *winding number* is an integer which intuitively measures how many times the curve $\gamma$ winds around the point $a_k$; it is positive if $\gamma$ moves in a counter clockwise (mathematically positive) manner around $a_k$ and 0 if $\gamma$ doesn't move around $a_k$ at all.

# 3 Some Important Results

**Lemma 3.0.1.** *For any arithmetical function $a(n)$, suppose,*

$$A(x) = \sum_{n \leq x} a(n),$$

*where, $A(x) = 0$, if $x < 1$. Then,*

$$\sum_{n \leq x} (x - a)a(n) = \int_1^x A(t)dt. \tag{3.1}$$

*Proof.* We apply *Abel's Identity*, which states that,

$$\sum_{n \leq x} a(n)f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt \tag{3.2}$$

provided $f$ has a continuous derivative on $[1, x]$. Taking $f(t) = t$, we obtain,

$$\sum_{n \leq x} a(n)f(n) = \sum_{n \leq x} na(n) \text{ and, } A(x)f(x) = n \sum_{n \leq x} a(n).$$

Applying the above results in (3.2), we obtain the result. □

The next lemma can be treated as a form of *L'Hòspital's* rule for increasing piece-wise linear functions.

**Lemma 3.0.2.** *Let $A(x) = \sum_{n \leq x} a(n)$ and let $A_1(x) = \int_1^x A(t)dt$. Assume also that, $a(n) \geq 0$, $\forall$ $n \in \mathbb{N}$. Given the asymptotic formula ,*

$$A_1(x) \sim Lx^c \text{ as } x \to \infty \tag{3.3}$$

*For some $c > 0$ and $L > 0$, we shall have,*

$$A(x) \sim cLx^{c-1} \text{ as } x \to \infty \tag{3.4}$$

*In other words, formal differentiation of* (3.3) *gives the result* (3.4).

*Proof.* The function $A(x)$ is increasing, since the $a(n)$ 's are non-negative. Next, we choose any $\beta > 1$ and consider the difference, $A_1(\beta x) - A_1(x)$. Thus, we get,

$$A_1(\beta x) - A_1(x) = \int_x^{\beta x} A(u)du \geq \int_x^{\beta x} A(x)du = A(x)(\beta(x) - x)$$

$$= x(\beta - 1)A(x).$$

This gives us,

$$xA(x) \leq \tfrac{1}{(\beta-1)} A_1(\beta x) - A_1(x),$$

or,

$$\tfrac{A(x)}{x^{c-1}} \leq \tfrac{1}{(\beta-1)} \{ \tfrac{A_1(\beta x)}{(\beta x)^c} \beta^c - \tfrac{A_1(x)}{x^c} \}.$$

Keeping $\beta$ fixed and taking $x \to \infty$ in the above inequality, we get,

$$\limsup_{x\to\infty} \tfrac{A(x)}{x^{c-1}} \leq \tfrac{1}{\beta-1}(L\beta^c - L) = L\tfrac{\beta^c - 1}{\beta - 1}.$$

Now, let us take, $\beta \to 1^+$. The quotient on the right is the difference quotient for the derivative of $x^c$ at $x = 1$ and has the limit $c$. Therefore,

$$\limsup_{x\to\infty} \frac{A(x)}{x^{c-1}} \leq cL. \tag{3.5}$$

Assume any $\alpha$ with $0 < \alpha < 1$ and consider the difference, $\{A_1(x) - A_1(\alpha x)\}$. By similar arguments done previously during the proof of this lemma, we get,

$$\liminf_{x\to\infty} \tfrac{A(x)}{x^{c-1}} \geq L\tfrac{1-\alpha^c}{1-\alpha}.$$

Now, as $\alpha \to 1^-$, the term in the R.H.S. tends to $cL$. This, together with (3.5), shows that,

$$\tfrac{A(x)}{x^{c-1}} \to cL \text{ as } x \to \infty.$$

And thus the lemma is proved. □

When $a(n) = \Lambda(n)$, we have, $A(x) = \psi(x)$, $A_1(x) = \psi_1(x)$, and $a_n \geq 0$.
Therefore, with the help of lemmas (3.0.1) and (3.0.2) we obtain:

**Theorem 3.0.3.** *We have,*

$$\psi_1(x) = \sum_{n\leq x}(x - n)\lambda(n). \tag{3.6}$$

*Where, the Liouville's Function $\lambda(n)$ is defined as follows:*

$$\lambda(n) = \begin{cases} 1 \ , & \text{if } n = 1 \\ (-1)^{\sum_{i=1}^{k} \alpha_i} \ , & \text{if } \ n = \prod_{i=1}^{k} {p_i}^{\alpha_i}. \end{cases}$$

*Also the asymptotic relation,* $\psi_1(x) \sim \frac{x^2}{2}$ *implies,* $\psi(x) \sim x$ *as* $x \to \infty$.

Our next goal is to express $\frac{\psi_1(x)}{x^2}$ as a *Contour Integral* involving the *zeta function* ( We shall provide an overview of the *Riemann Zeta Function* in the next section ). For this, we will require the special cases, $k = 1$ and $k = 2$ of the following lemma on contour integrals.

**Lemma 3.0.4.** *If* $c > 0$ *and* $u > 0$*, then for every integer* $k \geq 1$*, we have,*

$$\frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{u^{-z}}{z(z+1)(z+2)\cdots(z+k)} dz = \begin{cases} \frac{1}{k!}(1-u)^k \ , & \text{if } 0 < u \leq 1, \\ 0 \ , & \text{if } u > 1. \end{cases}$$

*the integral being absolutely convergent.*

*Proof.* Follows from the observation that, the integrand is $= \frac{u^{-z}\Gamma(z)}{\Gamma(z+k+1)}$ ( This follows by repeatedly using the functional equation, $\Gamma(z+1) = z.\Gamma(z)$ ), and application of *Cauchy's Residue Theorem*. ( For detailed proof of the lemma, readers can see [1, p. 281-282] ) □

# 4 A Brief Introduction to the "Riemann Zeta Function"

## 4.1 Historical Significance

The *Riemann Zeta Functions* is an integral part of *Analytic Number Theory*, often treated as a special case of the *Hurwitz Zeta Function* $\zeta(s, a)$, defined for $Re(s) > 1$, as the series,

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \text{ where, } a \in \mathbb{R},\ 0 < a \leq 1 \text{ is fixed.}$$

*Riemann Zeta Function* was first introduced by *Leonhard Euler* in the first half of eighteenth century, using only *Real numbers*. Also, he even computed the values of the zeta function at even positive integers.

Later on, famous mathematician *Bernhard Riemann* extended Euler's definition of the Riemann Zeta Function on $\mathbb{R}$ to the field of *Complex Numbers*, also deriving the *meromorphic continuation* and *functional equation*, and gave us an idea about the relationship between the zeroes of the Riemann Zeta Function and the *distribution of prime numbers*. The details can be found in his article titled *On the Number of Primes less than a given Magnitude* published in 1859. That's the reason why, afterwards, this function was named after him.

*Riemann Zeta Function* is hugely significant in the field of *Number Theory*. For example, $\zeta(2)$ provides solution to the famous *"Basel Problem"*. Also, famous Greek-French mathematician *Roger Apéry* proved in the year 1979 that, $\zeta(3)$ is *irrational*. *Euler* established the fact that, the Riemann Zeta Function yields *rational values* at the negative integer points and moreover, these values are incredibly useful in the field of *Modular Forms*. Functions like, *Dirichlet Series*, *Dirichlet L-functions*, and *L-functions* are often considered to be generalisation of the *Riemann Zeta Function*.

## 4.2 Definition and Properties of $\zeta(s)$

Enough with its historical significance, let us give a proper definition of the *Riemann Zeta Function* below.

**Definition 4.2.1.** (Riemann Zeta Function) The *Riemann Zeta Function* $\zeta(s)$ is defined as,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ where, } s \in \mathbb{C}, \text{ with } Re(s) > 1.$$

And we can further extend the *Riemann Zeta Function* to the whole complex plane using the *Analytic Continuation Property* of the function defined for $Re(s) > 1$.

Using the definition of the *Gamma Function* $\Gamma(s)$ given as,

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx, \text{ where, } s \in \mathbb{C},\ Re(s) > 0,$$

we can thus provide an alternative definition of the *Riemann Zeta Function* as,

**Definition 4.2.2.** (Alternative Definition of Riemann Zeta Function) We have,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx, \text{ where, } s \in \mathbb{C} \text{ and, } Re(s) > 1.$$

As for some important properties of $\zeta(s)$, they can be summarised as follows:

**Proposition 4.2.1.** *$\zeta(s)$ is meromorphic on $\mathbb{C}$, i.e., $\zeta(s)$ is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1.*

*Proof.* Follows from the deduction that,

$$\lim_{s \to 1} (s-1).\zeta(s) = 1.$$

□

**Proposition 4.2.2.** *For $s = 1$, $\zeta(s)$ is the harmonic series that diverges to $+\infty$.*

**Proposition 4.2.3.** $\zeta(2) = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

[ Establishing this equality is known as the *Basel Problem*]

**Proposition 4.2.4.** *(Trivial Zero-Free Region) The Riemann Zeta Function $\zeta(s)$ has no zeroes in the region , $\{s \in \mathbb{C} : Re(s) > 1\}$ .*

### 4.3 Euler Product Formula

Deduced by famous mathematician *Euler*, in the year 1737, the following identity establishes a relation between the *Riemann Zeta Function* and *prime numbers.* The result is as follows :

**Theorem 4.3.1.** *(Euler Product Formula) We have,*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=prime} \frac{1}{(1-p^{-s})}$$

*Where, the product on the R.H.S. is taken over all primes p and, converges for $Re(s) > 1$.*

*Proof.* First, we shall state a lemma useful in proving the above result.

**Lemma 4.3.2.** *Suppose $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers with $a_n \neq -1$, $\forall\ n \in \mathbb{N}$. Then,*

$$\sum_{n=1}^{\infty} |a_n| \text{ is convergent, implies, } \prod_{n=1}^{\infty} (1 + a_n) \text{ is convergent.}$$

Assume, $\sigma = Re(s)$. Then for natural numbers $N_1$ and $N_2$ with, $N_1 < N_2$ (without loss of generality),

$$|\sum_{N_1+1}^{N_2} \frac{1}{n^s}| \leq |\sum_{N_1+1}^{N_2} \frac{1}{n^\sigma}| \qquad [\text{ Using the result, } |n^{-\sigma}| = |n^{-s}| \text{ ]}$$

Hence, using the inequality,

$$n^{-\sigma} \leq \int_{n-1}^{n} x^{-\sigma} dx,$$

after performing summation on both sides, we obtain,

$$\sum_{N_1+1}^{N_2} n^{-\sigma} \leq \int_{N_1}^{N_2} x^{-\sigma} dx = \frac{1}{\sigma}(N_1^{-\sigma} - N_2^{-\sigma}) \longrightarrow 0 \text{ as, } N_1, N_2 \to \infty$$

Hence, $\sum\limits_{n=1}^{\infty} n^{-s}$ is convergent . [Applying *Cauchy's Criterion* for convergence of series]

Also, $\sum\limits_{p=prime} |p^{-s}| = \sum\limits_{p=prime} p^{-\sigma} \leq \sum\limits_{n=1}^{\infty} n^{-\sigma}.$

Which implies that, $\sum_{p=prime} p^{-s}$ is *absolutely convergent*.

Applying Lemma (4.3.2), we obtain that, the product, $\prod_{p=prime}(1-p^{-s})$ is absolutely convergent, hence convergent. Thus,

$$\prod_{p=prime} \frac{1}{(1-p^{-s})} \text{ is also convergent.}$$

To prove that, both sides of the given identity in this theorem are equal, we observe that,

$$\prod_{p\leq N_2} \frac{1}{(1-\chi(p)p^{-s})} = \sum_{n>N_2} \chi(n)n^{-s} + \sum_{n\leq N_2} \chi(n)n^{-s}$$

As, $N_2 \longrightarrow \infty$,

$$\text{R.H.S.} = \sum_{n\in\mathbb{N}} n^{-s} \qquad \text{[The sum being } \textit{absolutely convergent}\text{]}$$

So, using definition, we conclude that, the product on the L.H.S. tends to $\prod_{p=prime} \frac{1}{(1-p^{-s})}$, therefore, the identity holds true, and hence, *Euler's Formula* is established. □

## 4.4 Riemann's Functional Equation for $\zeta(s)$

We define a particular form of *Dirichlet Series* $F(x,s)$ as ,

$$F(x,s) = \sum_{n=1}^{\infty} \frac{e^{2\pi inx}}{n^s} \ , \quad \text{where , } x \in \mathbb{R} \quad \text{and , } \quad Re(s) > 1 \ .$$

Important to observe that, $F(x,s)$ is a periodic function of $x$ ( also known as the *periodic zeta function* ) with period 1 and , $F(1,s) = \zeta(s)$ .

**Theorem 4.4.1.** *(Hurwitz's Formula) We have,*

$$\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-\pi is/2}F(a,s) + e^{\pi is/2}F(-a,s)\} \ .$$

*For* , $0 < a \leq 1$ *and* , $Re(s) > 1$ .

***Remark*** 4.4.2. For $a \neq 1$ , *Hurwitz's Formula* is valid for $Re(s) > 0$ .

Using *Hurwitz's Formula* , we shall establish the functional equation for $\zeta(s)$ .

**Theorem 4.4.3.** *The functional equation for the Riemann Zeta Function* $\zeta(s)$ *is given by* ,

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)cos(\tfrac{\pi s}{2})\zeta(s)$$

*In other words* ,

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)cos(\tfrac{\pi s}{2})\zeta(1-s)$$

*Proof.* Putting $a = 1$ in *Hurwitz's Formula* ,

$$\zeta(1-s) = \frac{\Gamma(s)}{(2\pi)^s}\{e^{-\pi is/2}\zeta(s) + e^{\pi is/2}\zeta(s)\} = \frac{\Gamma(s)}{(2\pi)^s}2cos(\frac{\pi s}{2})\zeta(s)\ .$$

Which proves the first part of the theorem . Replacing $s$ by $1-s$ , we establish the equivalent definition of the functional equation for $\zeta(s)$ . ☐

***Remark*** 4.4.4. Putting , $s = 2n+1$ , $\forall\ n \in \mathbb{N}$ , we obtain the *trivial zeroes* of $\zeta(s)$ , therefore,

$$\zeta(-2n) = 0 \qquad \forall\ n \in \mathbb{N}\ .$$

# 5 Properties of $\zeta(s)$

## 5.1 A Contour Integral representation of $\frac{\psi_1(x)}{x^2}$

**Theorem 5.1.1.** *If $c > 1$ and $x \geq 1$, then,*

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^{s-1}}{s(s+1)}(-\frac{\zeta'(s)}{\zeta(s)})ds. \tag{5.1}$$

*Proof.* From equation (3.6), we get,

$$\frac{\psi_1(x)}{x} = \sum_{n\leq x}(1-\frac{n}{x})\Lambda(n).$$

Now, we use Lemma (3.0.4) with putting $k = 1$, and $u = \frac{n}{x}$. If $n \leq x$, we obtain,

$$(1-\frac{n}{x}) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{(\frac{x}{n})^s}{s(s+1)}ds.$$

Multiplying this relation above by $\Lambda(n)$ and summing over all $n \leq x$,

$$\frac{\psi_1(x)}{x} = \sum_{n\leq x}\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{\Lambda(n)(\frac{x}{n})^s}{s(s+1)}ds = \sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{\Lambda(n)(\frac{x}{n})^s}{s(s+1)}ds.$$

Since the integral above vanishes if, $n > x$, this can also be written as,

$$\frac{\psi_1(x)}{x} = \sum_{n=1}^{\infty}\int_{c-\infty i}^{c+\infty i}f_n(s)ds, \qquad \text{where,}\quad 2\pi i f_n(x) = \Lambda(n)\frac{(\frac{x}{n})^s}{s(s+1)}. \tag{5.2}$$

Suppose, we wish to interchange the sum and the integral in (5.2). For this it suffices to prove that, the series,

$$\sum_{n=1}^{\infty}\int_{c-\infty i}^{c+\infty i}|f_n(s)|ds \tag{5.3}$$

is convergent. Important to observe that, the partial sum of the series given in (5.3) saitsfy the inequality,

$$\sum_{n=1}^{N}\int_{c-\infty i}^{c+\infty i}\frac{\Lambda(n)(\frac{x}{n})^c}{|s||s+1|}ds=\sum_{n=1}^{N}\frac{\Lambda(n)}{n^c}\int_{c-\infty i}^{c+\infty i}\frac{x^c}{|s||s+1|}ds\le A\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^c},$$

Where, $A$ is a constant. Therefore, we can conclude that, (5.3) converges. Therefore, we can interchange the sum and the integral in (5.2) to obtain,

$$\frac{\psi_1(x)}{x}=\int_{c-\infty i}^{c+\infty i}\sum_{n=1}^{\infty}f_n(s)ds=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^s}{s(s+1)}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}ds$$

$$=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^s}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds.$$

Now, we divide both sides of the above identity to obtain the desired result. □

**Theorem 5.1.2.** *If $c>1$ and $x\ge 1$ we have,*

$$\frac{\psi_1(x)}{x^2}-\frac{1}{2}\left(1-\frac{1}{x}\right)^2=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}x^{s-1}h(s)ds \tag{5.4}$$

*where,*

$$h(s)=\frac{1}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}-\frac{1}{s-1}\right). \tag{5.5}$$

*Proof.* We use Lemma (3.0.4) with $k=2$ to get,

$$\tfrac{1}{2}(1-\tfrac{1}{x})^2=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^s}{s(s+1)(s+2)}ds,\qquad \text{where, } c>0.$$

Replacing $s$ by $s-1$ in the integral ( keeping $c>1$ ),

$$\tfrac{1}{2}(1-\tfrac{1}{x})^2=\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{x^{s-1}}{(s-1)(s)(s+1)}ds,$$

Subtracting the above identity from the identity in Theorem (5.1.1), we get the desired result. □

***Remark*** 5.1.3. If we parametrize the path of integration by writing $s=c+it$, we obtain, $x^{s-1}=x^{c-1}x^{it}=x^{c-1}e^{itlog(x)}$. As a result, equation (5.4) becomes,

$$\frac{\psi_1(x)}{x^2}-\frac{1}{2}\left(1-\frac{1}{x}\right)^2=\frac{x^{c-1}}{2}\int_{c-\infty i}^{c+\infty i}h(c+it)e^{itlog(x)}dt. \tag{5.6}$$

Our next task is to show that the R.H.S. of the identity (5.6) $\to 0$ as $x\to\infty$. As mentioned earlier, we first have to establish that, we can put $c=1$ in (5.6). For this purpose, we need to study $\zeta(s)$ in the neighbourhood of the line, $\sigma=1$ ( taking, $s=\sigma+it$ in complex plane ).

## 5.2 Upper Bounds for $|\zeta(s)|$ and $|\zeta'(s)|$ near the line $\sigma = 1$

In order to study $\zeta(s)$ near the line, $\sigma = 1$, we use a particular representation of $\zeta(s)$ obtained from the theorem given below:

**Theorem 5.2.1.** *For any integer $N \geq 0$ and $\sigma > 0$, we have,*

$$\zeta(s) = \sum_{n=0}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s \int_{N}^{\infty} \frac{x-[x]}{x^{s+1}} dx \tag{5.7}$$

*Proof.* Apply *Euler's Summation Formula* with $f(t) = \frac{1}{t^s}$ and with integers $x$ and $y$ to obtain,

$$\sum_{y<n\leq x} \frac{1}{n^s} = \int_{y}^{x} \frac{dt}{t^s} - s \int_{y}^{x} \frac{t-[t]}{t^{s+1}} dt$$

Taking $y = N$ and, letting $x \to \infty$, keeping $\sigma > 1$, this yields,

$$\sum_{n=N+1}^{\infty} \frac{1}{n^s} = \int_{N}^{\infty} \frac{dt}{t^s} - s \int_{N}^{\infty} \frac{t-[t]}{t^{s+1}} dt$$

or,

$$\zeta(s) - \sum_{n=0}^{N} \frac{1}{n^s} = \frac{N^{1-s}}{s-1} - s \int_{N}^{\infty} \frac{x-[x]}{x^{s+1}} dx.$$

This proves (5.7) for $\sigma > 1$. For the case when $0 < \delta \leq \sigma$, the integral is dominated by, $\int_{N}^{\infty} \frac{1}{t^{\delta+1}} dt$, so it converges uniformly for $\sigma \geq \delta$ and hence represents an analytic function in the half-plane $\sigma > 0$. Therefore, (5.7) holds for $\sigma > 0$ by analytic continuation. □

***Remark*** 5.2.2. We also deduce the formula of $\zeta'(s)$ obtained by differentiating each member of (5.7),

$$\zeta'(s) = -\sum_{n=1}^{N} \frac{log(n)}{n^s} + s \int_{N}^{\infty} \frac{(x-[x])log(x)}{x^{s+1}} dx - \int_{N}^{\infty} \frac{(x-[x])}{x^{s+1}} dx - \frac{N^{1-s}log(N)}{s-1} - \frac{N^{1-s}}{(s-1)^2}. \tag{5.8}$$

The next theorem uses the relations derived to obtain an upper bound for both $|\zeta(s)|$ and $|\zeta'(s)|$.

**Theorem 5.2.3.** *For every $A > 0$, $\exists$ a constant $M$ ( depending on $A$ ) such that,*

$$|\zeta(s)| \le Mlog(t) \qquad and, |\zeta'(s)| \le Mlog^2(t) \tag{5.9}$$

*For all $s$ with $\sigma \ge \frac{1}{2}$ satisfying,*

$$\sigma > 1 - \frac{A}{log(t)} \qquad and,\ t \ge e. \tag{5.10}$$

*Proof.* If $\sigma \ge 2$, we have, $|\zeta(s)| \le \zeta(2)$ and $|\zeta'(s)| \le |\zeta'(2)|$ . Hence, the inequalities in (5.9) are trivially satisfied.
Therefore, we can assume, $\sigma < 2$ and $t \ge e$. Consequently, we get,

$$|s| \le \sigma + t \le 2 + t < 2t \text{ and } |s-1| \ge t.$$

Hence, $\frac{1}{|s-1|} \le \frac{1}{t}$. Estimating $|\zeta(s)|$ by using (5.7), we find,

$$|\zeta(s)| \le \sum_{n=1}^{N} \frac{1}{n^\sigma} + 2t \int_{N}^{\infty} \frac{1}{x^{\sigma+1}} dx + \frac{N^{1-\sigma}}{t} = \sum_{n=1}^{N} \frac{1}{n^\sigma} + \frac{2t}{\sigma N^\alpha} + \frac{N^{1-\sigma}}{t}.$$

Now, we make $N$ depend on $t$ by choosing, $N = [t]$. Then, $N \le t < N+1$ and $log(n) \le log(t)$ if $n \le N$.
The inequality (5.10) implies, $1 - \sigma < \frac{A}{log(t)}$, so,

$$\frac{1}{n^\sigma} = \frac{n^{1-\sigma}}{n} = \frac{1}{n} e^{(1-\sigma)log(n)} < \frac{1}{n} e^{A \frac{logn}{logt}} \le \frac{1}{n} e^A = O(\frac{1}{n}).$$

Therefore,

$$\frac{2t}{\sigma N^\sigma} \le \frac{N+1}{N} \text{ and, } \frac{N^{1-\sigma}}{t} = \frac{N}{t} . \frac{1}{N^\sigma} = O(\frac{1}{N}) = O(1),$$

So,

$$|\zeta(s)| = O\left( \sum_{n=1}^{N} \frac{1}{n} \right) + O(1) = O(log(N)) + O(1) = O(log(t)).$$

This proves the inequality for $|\zeta(s)|$ in (5.9).

To obtain the inequality for $|\zeta'(s)|$ we apply the same type of argument to the equation (5.8). The only essential factor is that, an extra factor $log(N)$ apears on the right. Although the following estimate, $log(N) = O(log(t))$ gives, $|\zeta'(s)| = O(log^2(t))$ in the specified region.

$\square$

### 5.3 The Non-Vanishing of $\zeta(s)$ on the line, $\sigma = 1$

In this section, our main objective is to prove that, $\zeta(1+it) \neq 0$, $\forall t \in \mathbb{R}$. The proof is mainly based on an inequality, which will also be needed in the next section.

**Theorem 5.3.1.** *Let the Dirichlet Series $F(s) = \sum \frac{f(n)}{n^s}$ ( $f(n)$ be an arithmetic function ) be absolutely convergent for $\sigma > \sigma_a$, and we asume that, $f(1) \not\equiv 0$ If $F(s) \not\equiv 0$ for $\sigma > \sigma_0 > \sigma_a$, then, for $\sigma > \sigma_0$, we have,*

$$F(s) = e^{G(s)}$$

*with,*

$$G(s) = log(f(1)) + \sum_{n=2}^{\infty} \frac{(f' * f^{-1})(n)}{log(n)} \frac{1}{n^s};$$

*where, $f^{-1}$ is the Dirichlet Inverse of n and, $f'(n) = f(n)log(n)$.*

***Remark*** 5.3.2. For complex $z \neq 0$, $log(z)$ denotes that branch of the logarithm which is real when $z > 0$.

*Proof.* Since, $F(s) \neq 0$, we can write, $F(s) = e^{G(s)}$ for some function $G(s)$, which is analytic for $\sigma > \sigma_0$. Differentiation gives us,

$$F'(s) = e^{G(s)} G'(s) = F(s) G'(s).$$

Therefore, we get, $G'(s) = \frac{F'(s)}{F(s)}$. Although, we have,

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n)log(n)}{n^s} = -\sum_{n=1}^{\infty} \frac{f'(n)}{n^s} \qquad \text{and,} \qquad \frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{f^{-1}(n)}{n^s}$$

hence, $G'(s) = \frac{F'(s)}{F(s)} = -\sum_{n=2}^{\infty} \frac{(f' * f^{-1})(n)}{n^s}$.

*Integration* gives,

$$G(s) = C + \sum_{n=2}^{\infty} \frac{(f' * f^{-1})(n)}{n^s log(n)}$$

Where, $C$ is a constant. As, $\sigma \to +\infty$, we obtain,

$$\lim_{\sigma \to +\infty} G(\sigma + it) = C.$$

Therefore,

$$f(1) = \lim_{\sigma \to +\infty} F(\sigma + it) = e^C.$$

Hence, $C = log f(1)$ and this completes the proof . As an important observation, it can also be deduced from the proof that, the series for $G(s)$ converges absolutely if $\sigma > \sigma_0$. □

**Corollary 5.3.3.** *For the Riemann Zeta Function $\zeta(s)$, we have,*

$$\zeta(s) = e^{G(s)}, \qquad for, \quad \sigma > 1$$

*where,*

$$G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{log(n)} n^{-s}.$$

*Proof.* Using, $f(n) = 1$, and applying Theorem (5.3.1), we obtain, $f'(n) = log(n)$ and $f^{-1}(n) = \mu(n)$,
Therefore,

$$(f' * f^{-1})(n) = \sum_{d|n} log(d)\mu(\tfrac{n}{d}) = \Lambda(n).$$

Hence, for $\sigma > 1$, we obtain the result. □

As a direct application of *Corollary* (5.3.3), we prove our main result in this section.

**Theorem 5.3.4.** *For $\sigma > 1$,*

$$\zeta^3(\sigma)|\zeta(\sigma + it)|^4|\zeta(\sigma + 2it)| \geq 1. \tag{5.11}$$

*Proof.* From Corollary (5.3.3), we have, for $\sigma > 1$,

$$\zeta(s) = e^{G(s)}$$

where,

$$G(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{log(n)n^s} = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \ .$$

In other words,

$$\zeta(s) = e^{\{\sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}\}} = e^{\{\sum_p \sum_{m=1}^{\infty} \frac{e^{-imtlog(p)}}{mp^{m\sigma}}\}}.$$

from which, we obtain,

$$|\zeta(s)| = e^{\{\sum_p \sum_{m=1}^{\infty} \frac{cos(mtlog(p))}{mp^{m\sigma}}\}}.$$

We apply this formula repeatedly with $s = \sigma$, $s = \sigma + it$, $s = \sigma + 2it$ to obtain,

$$\zeta^3(\sigma)|\zeta(\sigma + it)|^4|\zeta(\sigma + 2it)| = e^{\{\sum_p \sum_{m=1}^{\infty} \frac{3+4cos(mtlog(p))+cos(2mtlog(p))}{mp^{m\sigma}}\}}$$

Now the following identity,

$$3 + 4cos\theta + cos2\theta = 3 + 4cos\theta + 4cos^2 2\theta - 1 = 2(1 + cos\theta)^2 \geq 0.$$

helps us conclude that, each term in the last infinite series is non-negative, so, we obtain the result (5.11). ☐

Hence, we can conclude,

**Theorem 5.3.5.** *$\zeta(1 + it) \neq 0$ for every real $t$.*

*Proof.* It is sufficient to consider $t \neq 0$. Rewriting (5.11) ,

$$(\sigma - 1)^3 \zeta(\sigma)^3 |\frac{\zeta(\sigma + it)}{\sigma - 1}|^4 |\zeta(\sigma + 2it)| \geq \frac{1}{\sigma - 1} \tag{5.12}$$

This identity is valid for $\sigma > 1$. Now, as $\sigma \to 1^+$ in (5.12), we observe that, the first factor approaches 1 [ Since, $\zeta(s)$ has residue 1 at the pole $s = 1$ ]. The third factor tends to $|\zeta(1 + 2it)|$. If, $|\zeta(1 + it)| = 0$, then we could have written the middle factor as,

$$|\tfrac{\zeta(\sigma+it)-\zeta(1+it)}{\sigma-1}|^4 \to |\zeta(1 + it)|^4 \text{ as } \sigma \to 1^+.$$

Thus, for some, $t \neq 0$, if we had, $\zeta(1 + it) = 0$, the L.H.S. of the equation(5.12) would approach the limit,

$$|\zeta'(1 + it)|^4 |\zeta(1 + 2it)| \qquad \text{as} \qquad \sigma \to 1^+.$$

But, the R.H.S. tends to $\infty$ as $\sigma \to 1^+$ and this gives a contradiction. ☐

## 5.4 Inequalities for $|\frac{1}{\zeta(s)}|$ and $|\frac{\zeta'(s)}{\zeta(s)}|$

Applying Theorem (5.3.4), we deduce the following inequalities regarding $|\frac{1}{\zeta(s)}|$ and $|\frac{\zeta'(s)}{\zeta(s)}|$.

**Theorem 5.4.1.** *There is a constant $M > 0$ such that,*

$$|\tfrac{1}{\zeta(s)}| < Mlog^7 t \text{ and,} |\tfrac{\zeta'(s)}{\zeta(s)}| < Mlog^9 t$$

*whenever, $\sigma \geq 1$ and, $t \geq e$.*

*Proof.* For $\sigma \geq 2$, we have,

$$|\tfrac{1}{\zeta(s)}| = |\sum_{n=1}^{\infty} \tfrac{\mu(n)}{n^s}| \leq \sum_{n=1}^{\infty} \tfrac{1}{n^2} \leq \zeta(2)$$

and,

$$|\tfrac{\zeta'(s)}{\zeta(s)}| \leq \sum_{n=1}^{\infty} \tfrac{\Lambda(n)}{n^2},$$

Hence, the inequalities hold trivially, if $\sigma \geq 2$. Suppose, then, we have, $1 \leq \sigma \leq 2$ and $t \geq e$. Rewriting inequality (5.11), we get,

$$\frac{1}{|\zeta(\sigma+it)|} \leq (\zeta(\sigma))^{\frac{3}{4}}|\zeta(\sigma+2it)|^{\frac{1}{4}}.$$

Now, observe that, $(\sigma-1)\zeta(\sigma)$ is bounded in the interval $1 \leq \sigma \leq 2$, say, $(\sigma-1)\zeta(\sigma) \leq M$, where, $M$ is an absolute constant. Then,

$$\zeta(\sigma) \leq \frac{M}{\sigma-1} \qquad \text{if,} \quad 1 < \sigma \leq 2 \ .$$

Also, $\zeta(\sigma+2it) = O(logt)$ if, $1 \leq \sigma \leq 2$ [ By Theorem (5.2.3) ]. Hence, for $1 < \sigma \leq 2$, we have,

$$\frac{1}{|\zeta(\sigma+it)|} \leq \frac{M^{\frac{3}{4}}(logt)^{\frac{1}{4}}}{(\sigma-1)^{\frac{3}{4}}} = \frac{A(logt)^{\frac{1}{4}}}{(\sigma-1)^{\frac{3}{4}}},$$

where, $A$ is an absolute constant. Therefore, for some constant $B > 0$,

$$|\zeta(\sigma+it)| > \frac{B(\sigma-1)^{\frac{3}{4}}}{(logt)^{\frac{1}{4}}} \qquad \text{if,} 1 < \sigma \leq 2 \quad \text{and} t \geq e. \tag{5.13}$$

This also holds trivially for $\sigma = 1$. Let, $\alpha$ be any number satisfying, $1 < \alpha < 2$. Then, if $1 \leq \sigma \leq \alpha$, $t \geq e$, we may use Theorem (5.2.3) to write,

$$|\zeta(\sigma+it) - \zeta(\alpha+it)| \leq \int_{\sigma}^{\alpha} |\zeta'(u+it)|du \leq (\alpha-\sigma)Mlog^2 t$$
$$\leq (\alpha-1)Mlog^2 t.$$

Hence, by *triangle inequality*,

$$|\zeta(\sigma+it)| \geq |\zeta(\alpha+it)| - |\zeta(\sigma+it) - \zeta(\alpha+it)| \geq |\zeta(\alpha+it)| - (\alpha-1)Mlog^2 t \geq \frac{B(\sigma-1)^{\frac{3}{4}}}{(logt)^{\frac{1}{4}}} - (\alpha-1)Mlog^2 t.$$

provided, $1 \leq \sigma \leq \alpha$, and using (5.13), it also holds for $\alpha \leq \sigma \leq 2$, since, $(\sigma-1)^{\frac{3}{4}} \geq (\alpha-1)^{\frac{3}{4}}$. In other words, $1 \leq \sigma \leq 2$ and, $t \geq e$, we have the inequality,

$$|\zeta(\sigma+it)| \geq \frac{B(\sigma-1)^{\frac{3}{4}}}{(logt)^{\frac{1}{4}}} - (\alpha-1)Mlog^2 t,$$

for any $\alpha$ satisfying $1 < \alpha < 2$. Now, we make $\alpha$ depend on $t$ and choose $\alpha$ so the first term on R.H.S. is twice the second. This requires,

$$\alpha = 1 + (\frac{B}{2M})^4 \frac{1}{(logt)^9}.$$

Clearly, $\alpha > 1$ and also $\alpha < 2$ if, $t \geq t_0$ for some $t_0$. Thus, if , $t \geq t_0$ and $1 \leq \sigma \leq 2$, we then have,

$$|\zeta(\sigma+it)| \geq (\alpha-1)Mlog^2 t = \frac{C}{(logt)^7}.$$

This inequality should also be true perhaps for a different value of $C$ if, $e \leq t \leq t_0$.

Hence,

$$|\zeta(s)| \geq \frac{C}{(logt)^7}, \qquad \forall\ \sigma \geq 1 \text{ and, } t \geq e,$$

which gives us a corresponding upper bound for $|\frac{1}{\zeta(s)}|$.

To get the inequality for $|\frac{\zeta'(s)}{\zeta(s)}|$, we apply Theorem (5.2.3) to obtain an extra factor $log^2 t$. □

# 6 Analytic Proof of PNT

Before giving our readers the detailed proof of the *"Prime Number Theorem"*, we shall introduce the following result from complex function theory and another theorem which will be required later on in the proof of the main theorem.

**Lemma 6.0.1.** *If $f(s)$ has a pole of order $k$ at $s = \alpha$, then the quotient, $\frac{f'(s)}{f(s)}$ has a first order pole at $s = \alpha$ with residue $-k$.*

*Proof.* We have, $f(s) = \frac{g(s)}{(s-\alpha)^k}$, where, $g$ is analytic at $\alpha$ and $g(\alpha) \neq 0$. Hence, for all $s$ in a neighbourhood of $\alpha$, we get,

$$f'(s) = \frac{g'(s)}{(s-\alpha)^k} - \frac{kg(s)}{(s-\alpha)^{k+1}} = \frac{g(s)}{(s-\alpha)^k}\left\{\frac{-k}{(s-\alpha)} + \frac{g'(s)}{g(s)}\right\}.$$

Thus, we get,

$$\frac{f'(s)}{f(s)} = \left\{\frac{-k}{(s-\alpha)} + \frac{g'(s)}{g(s)}\right\}.$$

Which proves the lemma, since $\frac{g'(s)}{g(s)}$ is analytic at $\alpha$. □

**Theorem 6.0.2.** *The function,*

$$F(s) = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

*is analytic at $s = 1$.*

*Proof.* Using Lemma (6.0.1), we can say that, $-\frac{\zeta'(s)}{\zeta(s)}$ has a first order pole at 1 with residue 1, as does $\frac{1}{s-1}$. Hence their difference is analytic at $s = 1$, and the theorem is proved. □

Now, having proven all the neccessary results required, we shall prove the *Prime Number Theorem.*

**Theorem 6.0.3.** *For, $x \geq 1$, we have,*

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}(1-\frac{1}{x})^2 = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} h(1+it)e^{it.logx}dt,$$

*where, the integral,* $\int\limits_{-\infty}^{\infty} |h(1+it)|dt$ *converges.*

*Therefore, by the* ***Riemann-Lebesgue Lemma*** *[ Theorem* (2.0.5) *], we have,*

$$\psi_1(x) \sim \frac{x^2}{2} \tag{6.1}$$

*and hence,*

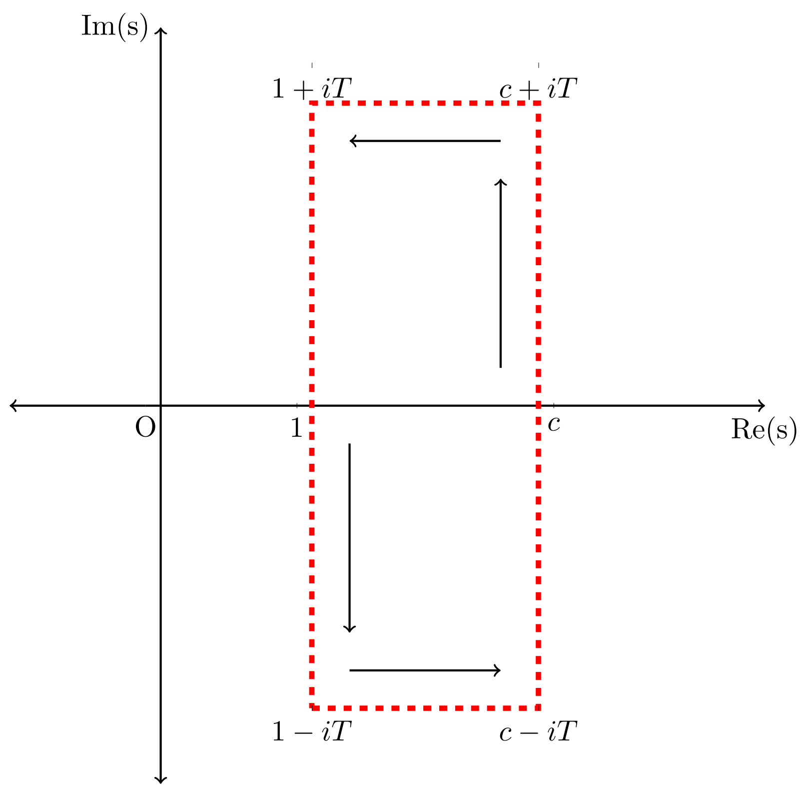
$$\psi(x) \sim x \qquad \text{as, } x \to \infty. \qquad \text{[ the statement of "Prime Number Theorem" ]}$$

*Proof.* In Theorem (5.1.2), we have proved that, if $c > 1$ and $x \geq 1$, then,

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}(1-\frac{1}{x})^2 = \frac{1}{2\pi i}\int\limits_{c-\infty i}^{c+\infty i} x^{s-1}h(s)ds,$$

where,

$$h(s) = \frac{1}{s(s+1)}\{-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}\}.$$




**Figure**

Our first objective is to show that, we can shift the path of integration to the line, $\sigma = 1$. To do this, we apply **Cauchy's Theorem** [ Theorem (2.0.6) ] to the rectangle $R$ shown in the *Figure* (6).

The integral of $x^{s-1}h(s)$ around $R$ is 0, since the integrand is analytic inside and on $R$. Next, we establish that, the integrals along the horizontal segments tends to 0 as $T \to \infty$. Since the integrand has the same absolute values at the conjugate points, it suffices to consider only the upper segment, $t = T$. On this segment, we have, the estimates,

$$|\tfrac{1}{s(s+1)}| \leq \tfrac{1}{T^2} \text{ and, } |\tfrac{1}{s(s+1)(s-1)}| \leq \tfrac{1}{T^3} \leq \tfrac{1}{T^2}.$$

Therefore, there is a constant $M$ such that, $|\frac{\zeta'(s)}{\zeta(s)}| \leq M(logT)^9$ if, $\sigma \geq 1$ and $t \geq e$. Hence, if $T \geq e$, then, we shall have,

$$|h(s)| \leq \tfrac{M(logT)^9}{T^2}$$

so that,

$$|\int_{1+iT}^{c+iT} x^{s-1}h(s)ds| \leq \int_{1}^{c} x^{c-1}\tfrac{M(logT)^9}{T^2}d\sigma = Mx^{c-1}\tfrac{(logT)^9}{T^2}(c-1).$$

And, similarly, for the lower segment , $t = -T$ , we obtain similar type of estimate ,

$$|\int_{1-iT}^{c-iT} x^{s-1}h(s)ds| \leq \int_{1}^{c} x^{c-1}\tfrac{M(logT)^9}{T^2}d\sigma = Mx^{c-1}\tfrac{(logT)^9}{T^2}(c-1).$$

Therefore, the integrals along the horizontal segments tend to 0 as $T \to \infty$ , and hence,

$$\int_{c-\infty i}^{c+\infty i} x^{s-1}h(s)ds = \int_{1-\infty i}^{1+\infty i} x^{s-1}h(s)ds.$$

On the line $\sigma = 1$, we put $s = 1 + it$ to get,

$$\tfrac{1}{2\pi i}\int_{1-\infty i}^{1+\infty i} x^{s-1}h(s)ds = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} h(1+it)e^{itlogx}dt.$$

Note that,

$$\int_{-\infty}^{\infty} |h(1+it)|dt = \int_{-e}^{e} |h(1+it)|dt + \int_{e}^{\infty} |h(1+it)|dt + \int_{-\infty}^{-e} |h(1+it)|dt.$$

Now, in the integral, $\int_{e}^{\infty} |h(1+it)|dt$, we observe that,

$$|h(1+it)| \leq \tfrac{M(logt)^9}{t^2}$$

Hence, $\int\limits_{e}^{\infty} |h(1+it)|dt$ converges. Similarly, $\int\limits_{-\infty}^{-e} |h(1+it)|dt$ converges, therefore, $\int\limits_{-\infty}^{\infty} |h(1+it)|dt$ converges. Thus, we may apply the **Riemann-Lebesgue Lemma** [ Theorem (2.0.5) ] to obtain,

$$\psi_1(x) \sim \tfrac{x^2}{2}.$$

By Theorem (3.0.3), the above result implies that,

$$\psi(x) \sim x \text{ as, } x \to \infty.$$

This proves the ***Prime Number Theorem***.

$\square$

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