# On the calculation of the corner frequency in Bode plots

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July 27, 2023

**Abstract**

In this paper we prove that the relationship $\omega\tau = 1$ is a property of logarithmic scale of the horizontal axis in Bode plots. We illustrate the results of our derivations and mathematical conclusions by calculating the point of intersection of the horizontal and the oblique asymptote in a magnitude plot and the point of symmetry in a phase plot.

## 1 Bode plot

Bode plots are used in the field of electrical and control engineering to display the gain or magnitude and the phase shift of a linear, time–invariant system as a function of frequency. The exact behaviour of a linear, time–invariant system as a function of frequency is graphically and mathematically simplified by horizontal and oblique asymptotes intersecting at a frequency $\omega := \omega_0$, the so called corner–frequency. In the following, we limit the discussion of the calculation of the corner frequency to two cases.

### 1.1 Case 1

Refering Example 1, Figure 1, consider the first order transfer function

$$
\begin{aligned}
&H_1(p) := \frac{1}{1 + \jmath \cdot p}, \quad p := \frac{\omega}{\omega_0} \\
&\rho_1(p) := |H_1(p)| \\
&\varphi_1(p) := \arg(H_1(p))
\end{aligned}
$$

$$
\begin{cases}
\tilde{x}(p) := {}^{g}\log(p) \\
\tilde{y}(p) := {}^{g}\log(\rho_1(p))
\end{cases}
$$

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$$\tilde{\rho}_1(\tilde{x}) \coloneqq \rho_1(p) = \rho_1\left(g^{\tilde{x}}\right)$$

$$\rho_1(p) = |H_1(p)| = \frac{1}{\sqrt{1+p^2}} = \frac{1}{p} \cdot \frac{1}{\sqrt{1+\left(\frac{1}{p}\right)^2}}$$

$$\begin{aligned}
\tilde{y}(p) &= {}^g\log\left(H_1(p)\right) = {}^g\log\left(\frac{1}{p} \cdot \frac{1}{\sqrt{1+\left(\frac{1}{p}\right)^2}}\right) \\
&= {}^g\log\left(\frac{1}{p}\right) + {}^g\log\left(\frac{1}{\sqrt{1+\left(\frac{1}{p}\right)^2}}\right) \\
&= -{}^g\log(p) - {}^g\log\left(\sqrt{1+\left(\frac{1}{p}\right)^2}\right)
\end{aligned}$$

$$\tilde{y}(p) = -\tilde{x}(p) - {}^g\log\left(\sqrt{1+\left(\frac{1}{p}\right)^2}\right)$$

$$\tilde{y}(p) + \tilde{x}(p) = -{}^g\log\left(\sqrt{1+\left(\frac{1}{p}\right)^2}\right)$$

$$p \to \infty \Leftrightarrow \tilde{x}(p) \to +\infty$$

$$\lim_{p\to\infty}\left(\tilde{y}(p) + \tilde{x}(p)\right) = \lim_{p\to\infty}\left(-{}^g\log\left(\sqrt{1+\left(\frac{1}{p}\right)^2}\right)\right) = -{}^g\log(1) = 0$$

Oblique asymptote of the graph of the function $\tilde{\rho}_1(\tilde{x})$:

$$\tilde{y} + \tilde{x} = 0 \Leftrightarrow \tilde{y} = -\tilde{x},\ \tilde{x} > 0$$

$$p \downarrow 0 \Leftrightarrow \tilde{x}(p) =\to -\infty$$

$$\lim_{p\downarrow 0} \tilde{y}(p) = \lim_{p\downarrow 0} {}^g\log\left(\rho_1(p)\right) = \lim_{p\downarrow 0} {}^g\log\left(\frac{1}{\sqrt{1+(p^2)}}\right) = {}^g\log(1) = 0$$

Horizontal asymptote of the graph of the function $\tilde{\rho}_1(\tilde{x}) :\ \tilde{y} = 0,\ \tilde{x} < 0.$

$$\varphi_1(p) := \arg(H_1(p))$$
$$\tilde{\varphi}_1(\tilde{x}) := \varphi_1(p) = \varphi_1\left(g^{\tilde{x}}\right)$$

Point symmetry of the function $\tilde{\varphi}_1(\tilde{x})$, symmetrically with respect to the point: $(\tilde{x}, \tilde{\varphi}) = \left(0, -\frac{\pi}{4}\right)$

$$\varphi_1(p) = \arg\left(\frac{1}{1 + \jmath \cdot p}\right) = \arg\left(\frac{1 - \jmath \cdot p}{1 + p^2}\right) = \arg(1 - \jmath \cdot p) = \arctan(-p)$$
$$\varphi_1(p) = -\arctan(p) \in \left\langle -\frac{\pi}{2}, 0\right\rangle$$
$$\tilde{\varphi}_1(\tilde{x}) := -\arctan(p) = -\arctan\left(g^{\tilde{x}}\right)$$
$$\tilde{\varphi}_1(-\tilde{x}) = -\arctan\left(g^{-\tilde{x}}\right) = -\arctan\left(\frac{1}{g^{\tilde{x}}}\right) = -\left(\frac{\pi}{2} - \arctan\left(g^{\tilde{x}}\right)\right)$$
$$\tilde{\varphi}_1(-\tilde{x}) = -\left(\frac{\pi}{2} + \varphi_1(\tilde{x})\right) = -\frac{\pi}{2} - \tilde{\varphi}_1(\tilde{x})$$
$$\tilde{\varphi}_1(0 - \tilde{x}) - \left(-\frac{\pi}{4}\right) = -\left(\tilde{\varphi}_1(0 + \tilde{x}) - \left(-\frac{\pi}{4}\right)\right)$$

Translation

$$\begin{cases} x := x_0 + \tilde{x} \\ y := y_0 + \tilde{y} \end{cases}$$

Oblique asymptote of the function $\tilde{\rho}_1(x - x_0)$:

$$\tilde{y} = -\tilde{x}, \tilde{x} > 0, \Leftrightarrow y - y_0 = -(x - x_0), x > x_0$$

Horizontal asymptote of the function $\tilde{\rho}_1(x - x_0)$:

$$\tilde{y} = 0, \tilde{x} < 0 \Leftrightarrow y = y_0, x < x_0$$

Intersection of the horizontal asymptote and the oblique asymptote of the function $\tilde{\rho}_1(x - x_0)$:

$$(\tilde{x}, \tilde{y}) = (0, 0) \Leftrightarrow (x, y) = (x_0, y_0)$$

Point symmetry of the function $\tilde{\rho}_1 = (x - x_0)$, symmetrically with respect to the point:

$$(\tilde{x}, \tilde{\varphi}) = \left(0, -\frac{\pi}{4}\right) \Leftrightarrow (x, y) = \left(x_0, -\frac{\pi}{4}\right)$$

## 1.2 Case 2

Refering to Example 2, Figure 4, consider the first order transfer function

$$H_2(p) := \frac{1}{1 - \jmath \cdot \frac{1}{p}}, \quad p := \frac{\omega}{\omega_0}$$
$$\rho_2(p) := |H_2(p)|$$
$$\varphi_2(p) := \arg(H_2(p))$$
$$\tilde{\rho}_2(\tilde{x}) := \rho_2(p) = \rho_2\left(g^{\tilde{x}}\right)$$
$$\tilde{\varphi}_2(\tilde{x}) := \varphi_2(p)$$

## 2 Lemma

$$H_2\left(\frac{1}{p}\right) = \frac{1}{1 - \jmath \cdot \frac{1}{\frac{1}{p}}} = \frac{1}{1 - \jmath \cdot p} = \left(\frac{1}{1 + \jmath \cdot p}\right)^* = H_1\left(p\right)^* \Leftrightarrow H_2\left(p\right) = H_1\left(\frac{1}{p}\right)^*$$

$$\rho_2\left(p\right) = \left|H_1\left(\frac{1}{p}\right)^*\right| = \left|H_1\left(\frac{1}{p}\right)\right| = \rho_1\left(\frac{1}{p}\right)$$

$$\varphi_2\left(p\right) = \arg\left(\frac{1}{1 - \jmath \cdot \frac{1}{p}}\right) = \arg\left(\frac{1}{1 + \jmath \cdot p} \cdot \jmath \cdot p\right) = \arg\left(H_1\left(p\right) \cdot \jmath \cdot p\right)$$

$$= \arg\left(H_1\left(p\right)\right) + \arg\left(\jmath \cdot p\right)$$

$$\varphi_2\left(p\right) = \varphi_1\left(p\right) + \frac{\pi}{2} \in \langle 0, \frac{\pi}{2} \rangle$$

## 3 Theorem

From the lemma it follows, that

$$\tilde{\rho}_2\left(-\tilde{x}\right) = \rho_2\left(g^{-\tilde{x}}\right) = \rho_2\left(\frac{1}{g^{\tilde{x}}}\right) = \rho_2\left(\frac{1}{p}\right) = \rho_1\left(p\right) = \tilde{\rho}_1\left(\tilde{x}\right)$$

$$\tilde{\varphi}_2\left(\tilde{x}\right) = \varphi_2\left(p\right) = \varphi_1\left(p\right) + \frac{\pi}{2} = \tilde{\varphi}_1\left(\tilde{x}\right) + \frac{\pi}{2}$$

Translation

$$\begin{cases} x \coloneqq x_0 + \tilde{x} \\ y \coloneqq y_0 + \tilde{y} \end{cases}$$

Oblique asymptote of the function $\tilde{\rho}_2\left(x - x_0\right) = \tilde{\rho}_2\left(\tilde{x}\right) = \tilde{\rho}_1\left(-\tilde{x}\right) = \tilde{\rho}_1\left(x_0 - x\right)$:

$$\tilde{y} = -\left(-\tilde{x}\right) = \tilde{x},\ \tilde{x} < 0 \Leftrightarrow y - y_0 = x - x_0,\ x < x_0$$

Horizontal asymptote of the function
$\tilde{\rho}_2\left(x - x_0\right) = \tilde{\rho}_2\left(\tilde{x}\right) = \tilde{\rho}_1\left(-\tilde{x}\right) = \tilde{\rho}_1\left(x_0 - x\right)$:

$$\tilde{y} = 0, \tilde{x} > 0 \Leftrightarrow y = y_0, x > x_0$$

Intersection of the horizontal asymptote and the oblique asymptote of the function $\tilde{\rho}_2\left(x - x_0\right)$:

$$\left(\tilde{x}, \tilde{y}\right) = \left(0, 0\right) \Leftrightarrow \left(x, y\right) = \left(x_0, y_0\right)$$

Point symmetry of the function $\tilde{\varphi}_2\left(x - x_0\right) = \tilde{\varphi}_2\left(\tilde{x}\right) = \tilde{\varphi}_1\left(\tilde{x}\right) + \frac{\pi}{2} = \tilde{\varphi}_1\left(x - x_0\right) + \frac{\pi}{2}$ with respect to the point:

$$\left(\tilde{x}, \tilde{\varphi}\right) = \left(0, \frac{\pi}{4}\right) \Leftrightarrow \left(x, \varphi\right) = \left(x_0, \frac{\pi}{4}\right)$$

# 4 Example 1

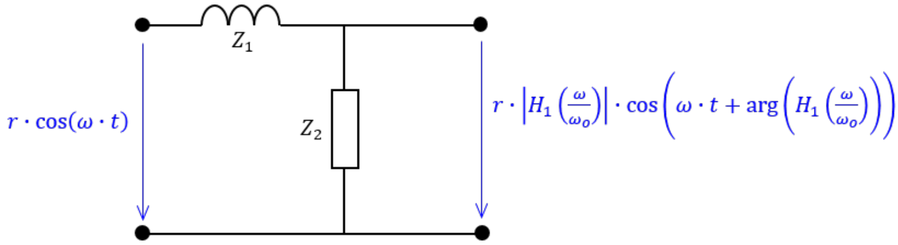


Figure 1

Given $(x_0, y_0) = (2, 3), \omega_0 = 1\ 000$ rad/s, $Z_1 = \jmath \cdot \frac{\omega}{\omega_0} \cdot R, Z_2 = R$

$$H_1(p) := \frac{Z_2}{Z_1 + Z_2} = \frac{R}{\jmath \cdot \frac{\omega}{\omega_0} \cdot R + R} = \frac{1}{1 + \jmath \cdot \frac{\omega}{\omega_0}} = \frac{1}{1 + \jmath \cdot p}, p := \frac{\omega}{\omega_0}$$

Oblique asymptote of the function $\tilde{\rho}_1(x - x_0) : y = x_0 + y_0 - x = 2 + 3 - x = 5 - x, x > x_0 = 2$
Horizontal asymptote of the function $\tilde{\rho}_1(x - x_0) : y = y_0 = 3, x < x_0 = 2$
Intersection of the horizontal asymptote and the oblique asymptote of the function $\tilde{\rho}_1(x - x_0)$ at the point $(x_0, y_0) = (2, 3)$

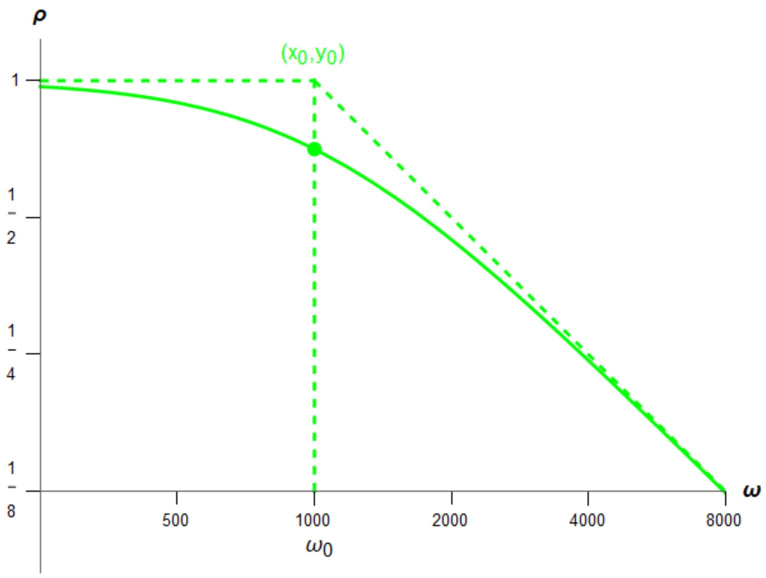


Figure 2

Point symmetry of the function $\tilde{\varphi}_1(x - x_0)$, symmetry with respect to the point $(x, \varphi) = \left(x_0, -\frac{\pi}{4}\right) = \left(2, -\frac{\pi}{4}\right)$

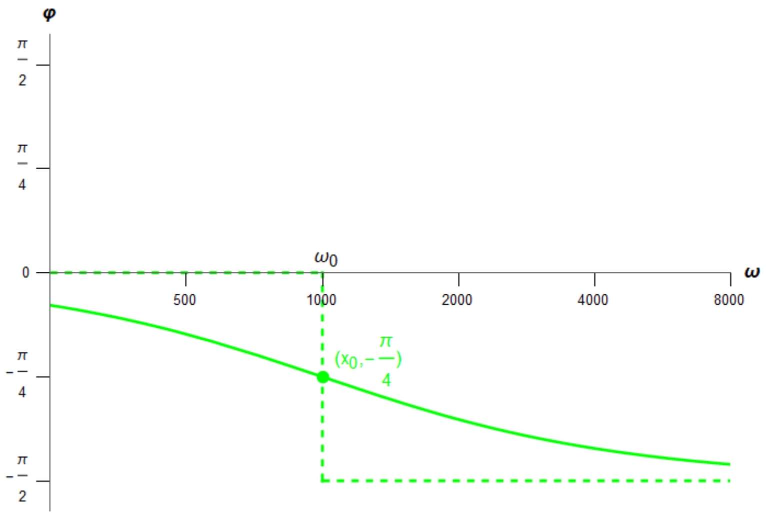


Figure 3

# 5 Example 2

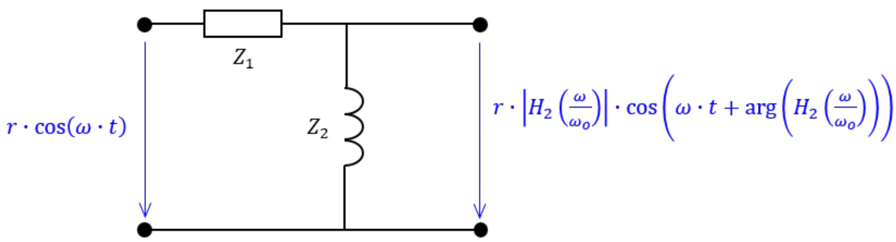


Figure 4

Given $(x_0, y_0) = (2, 3), \omega_0 = 1\ 000$ rad/s, $Z_1 = R, Z_2 = \jmath \cdot \frac{\omega}{\omega_0} \cdot R$

$$H_2(p) := \frac{Z_2}{Z_1 + Z_2} = \frac{\jmath \cdot \frac{\omega}{\omega_0} \cdot R}{\jmath \cdot \frac{\omega}{\omega_0} \cdot R + R} = \frac{\jmath \cdot \frac{\omega}{\omega_0}}{1 + \jmath \cdot \frac{\omega}{\omega_0}} = \frac{1}{1 - \jmath \cdot \frac{\omega_0}{\omega}} =$$
$$= \frac{1}{1 - \jmath \cdot \frac{1}{p}},\ p := \frac{\omega}{\omega_0}$$

Oblique asymptote of the function $\tilde{\rho}_2(x - x_0)\colon y = x_0 + y_0 - x = 3 - 2 + x = 1 + x,\ x < x_0 = 2$
Horizontal asymptote of the function $\tilde{\rho}_2(x - x_0)\colon y = y_0 = 3, x > x_0 = 2$
Intersection of the horizontal asymptote and the oblique asymptote of the function $\tilde{\rho}_2(x - x_0)$ at the point $(x_0, y_0) = (2, 3)$

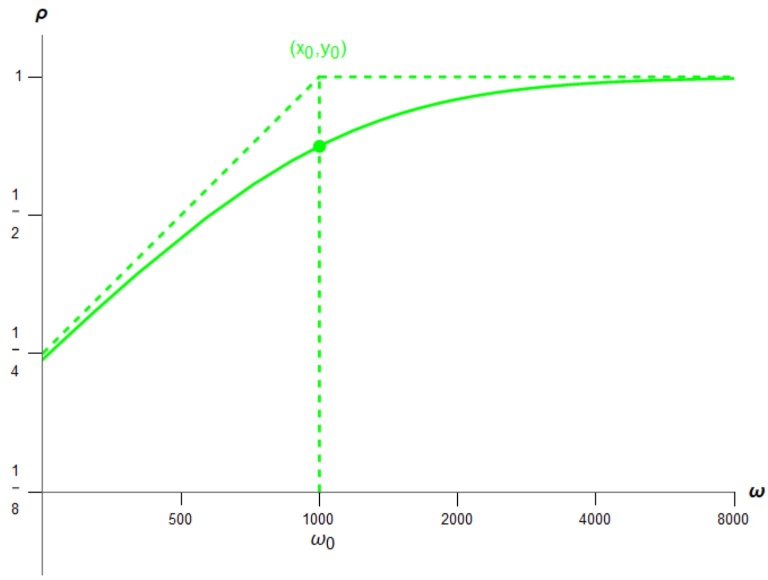


Figure 5

Point symmetry of the function $\tilde{\varphi}_2\,(x - x_0)$, symmetry with respect to the point $(x, \varphi) = \left(x_0, \dfrac{\pi}{4}\right) = \left(2, \dfrac{\pi}{4}\right)$

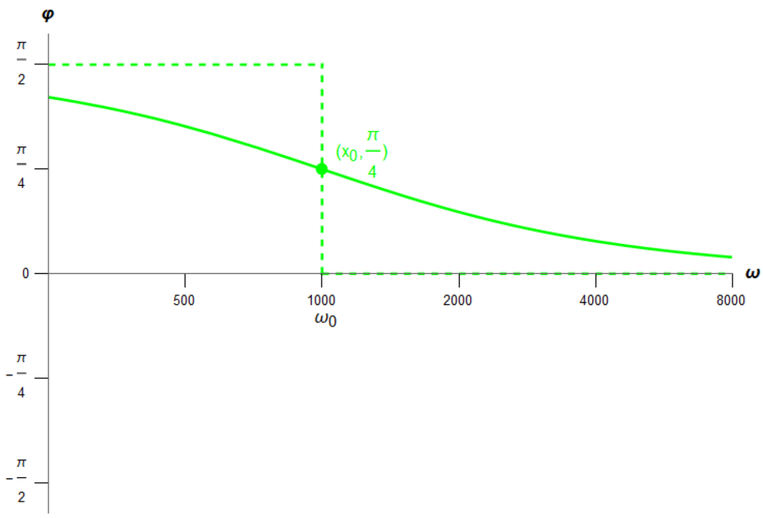


Figure 6

# 6 Acknowledgement

The authors acknowledge the support of Charlotte Creusen and Ad Klein both affiliated with the Department of Engineering of Zuyd University of Applied Sciences.