Mathematical Proof of $1^i = 1$

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Abstract

This paper details the expression 1^i=1. It uses a known trigonometric substitutions in a mathematical proof showing the accuracy of the expression.

1.1—Aspect of 1^{i}:

Part of the problem with analyzing complex values is that they are bound to the complex plane. On some occasions we are able to use various equalities and relations to show that an expression which contains the complex number *i* is in fact equal to a completely real number.

1.1.a:

I here posit the following definition—

$$1^{i} = 1$$

2.0—Proof of $1^{i} = 1$:

We will begin this proof by examining two known substitutions using the complex number *i*.

2.1.a— $i = \mathbf{e}^{i\frac{\pi}{2}}$:

The trigonometric expression $i = \mathbf{e}^{i\frac{\pi}{2}}$ is known and easily verifiable.

The Euler Formula is defined as: $\mathbf{e}^{ix} = \cos x + i \sin x$

It results from the Maclaurin series expansion of $\mathbf{e}^{ix}$.

When $x = \frac{\pi}{2}$, equivalent to a 90 degree rotation on the complex plane, the Euler Identity expression will provide:

$$\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \quad = \quad 0 + i\,(1) \quad = \quad i$$

2.1.b— $i^{i} \approx 0.20788\ldots$

This value obtained from traditional algebra relies on the result obtained from the Euler Formula in 2.1.a.

$$i^{i} \;=\; \left(\mathbf{e}^{i\frac{\pi}{2}}\right)^{i} \;=\; \mathbf{e}^{i^{2}\frac{\pi}{2}} \;= \mathbf{e}^{-\frac{\pi}{2}} \;\approx\; 0.20788\ldots$$

2.1.b.i:

This is easily verifiable as well from two separate methods.

Direct Computation: $\mathbf{e}^{-\frac{\pi}{2}} = (2.71828\ldots)^{(-1.570796\ldots)} \approx 0.20788$

Hyperbolic Euler Formula: $\cosh\left(-\frac{\pi}{2}\right) + \sinh\left(-\frac{\pi}{2}\right) \approx 0.20788$

*** The formula here is used due to the incident of i^{2} shown in 2.1.b which effectively removes i from the Euler Formula, resulting in hyperbolic trigonometric relations.*

2.2—Generalizing the trigonometric substitution for i^i:

We can generalize the expression $i^i = \left(\mathbf{e}^{i\frac{\pi}{2}}\right)^i$ so that it becomes an equation in terms of x for the domain of $-\infty < x \leq 0$, using the general function $y = \sqrt{x}^{\sqrt{x}}$.

2.2.a:

We use the expression: $y = \sqrt{x}^{\sqrt{x}}$ with $b = \sqrt{|x|}$

This expression provides that b will be the magnitude of a complex number bi, which shall be generated by values of the x-variable input. We use this equation and the trigonometric substitution as follows:

2.2.a.i:

For $-\infty < x \leq 0$

$$\sqrt{x}^{\sqrt{x}} \rightarrow bi^{bi} = \left(b \cdot \mathbf{e}^{i\frac{\pi}{2}}\right)^{bi} = b^{bi} \cdot \mathbf{e}^{-b\frac{\pi}{2}}$$

2.b:

When $x = -1$ the expression in 2.2.a.i becomes $\mathbf{e}^{-\frac{\pi}{2}}$

For $x = -1$

$$\sqrt{x}^{\sqrt{x}} \rightarrow bi^{bi} = \left(b \cdot \mathbf{e}^{i\frac{\pi}{2}}\right)^{bi} = b^{bi} \cdot \mathbf{e}^{-b\frac{\pi}{2}}$$

$$\sqrt{-1}^{\sqrt{-1}} = 1i^{1i} = \left(1 \cdot \mathbf{e}^{i\frac{\pi}{2}}\right)^{1i} = \mathbf{e}^{-\frac{\pi}{2}} \approx 0.20788$$

3.0—Exponent Power Rule and Multiplication:

A given number can be re-written to be a product of two other numbers. Consider the below example:

3.1:

$$15 = (3 \cdot 5)$$

3.1.a:

The number 15 and the Product $(3 \cdot 5)$ are identical. The Product 15 is identical to the expression of the multiplicand of 3 times the multiplier 5.

3.2:
If we raise the values in section 3.1 to a given power the result will be identical.

$$(15)^3 = 3{,}375 \qquad\qquad (3 \cdot 5)^3 = 3^3 \cdot 5^3 = 27 \cdot 125 = 3{,}375$$

More generally:

Let $a = b \cdot c$ and Let $a^n = d$

Then:

$$a^n = d \quad \equiv \quad (b \cdot c)^n = b^n \cdot c^n = d$$

This feature can now be applied to the situation in section 2.b

3.2.a:
For the domain of $-\infty < x \leq 0$, using the equation $y = \sqrt{x}^{\sqrt{x}}$. x inputs will result in complex numbers of the form *bi* with the magnitude of each complex number defined by $b = \sqrt{|x|}$.

We have the general trigonometric equation substitution:

$$\sqrt{x}^{\sqrt{x}} \;\rightarrow\; bi^{bi} = \left(b \cdot \mathbf{e}^{i\frac{\pi}{2}}\right)^{bi} \;=\; b^{bi} \cdot \mathbf{e}^{-b\frac{\pi}{2}}$$

Because a product raised to power will equal the same value as its multiplicand and multiplier raised the same power, we have the following:

3.2.b:
When $x = -1$

General Equation	$\sqrt{x}^{\sqrt{x}} = bi^{bi} = \left(\boldsymbol{b} \cdot \mathbf{e}^{\boldsymbol{i}\frac{\boldsymbol{\pi}}{\mathbf{2}}}\right)^{\boldsymbol{bi}} = \boldsymbol{b}^{\boldsymbol{bi}} \cdot \mathbf{e}^{-b\frac{\pi}{2}}$
Specific Value	$\sqrt{-1}^{\sqrt{-1}} = 1i^{1i} = \left(\mathbf{1} \cdot \mathbf{e}^{\boldsymbol{i}\frac{\boldsymbol{\pi}}{\mathbf{2}}}\right)^{\mathbf{1}\boldsymbol{i}} = \mathbf{1}^{\mathbf{1}\boldsymbol{i}} \cdot \mathbf{e}^{-1\frac{\pi}{2}}$

3.2.b.i:

In the third step of the specific value equation of 3.2.b, $\left(1 \cdot \mathbf{e}^{i\frac{\pi}{2}}\right)$, we could choose to simply multiply $\mathbf{e}^{i\frac{\pi}{2}}$ with the +1.

$$1 \cdot \mathbf{e}^{i\frac{\pi}{2}} = \mathbf{e}^{i\frac{\pi}{2}}$$

If we do that step first we get:

$$\left(\mathbf{1} \cdot \mathbf{e}^{i\frac{\pi}{2}}\right)^{\mathbf{1}i} = \left(\mathrm{e}^{i\frac{\pi}{2}}\right)^{i} = \mathbf{e}^{-\frac{\pi}{2}}$$

Or we could choose to distribute the exponent to both components of the product.

$$\left(\mathbf{1} \cdot \mathbf{e}^{i\frac{\pi}{2}}\right)^{\mathbf{1}i} = 1^{1i} \cdot \mathrm{e}^{(i^2)\cdot(1)\frac{\pi}{2}i} = \mathbf{1}^{\mathbf{1}i} \cdot \mathbf{e}^{-1\frac{\pi}{2}} = \mathbf{e}^{-\frac{\pi}{2}}$$

In keeping with the with the example shown in section 3.2 and 3.2.a, we know that the product of two numbers raised to a power is identical to both the multiplicand and multiplier both raised to the same power. Thus both of these methods are evaluating are identical and must result in the same value, $\mathbf{e}^{-\frac{\pi}{2}}$.

Thus we must conclude that $1^{1i} = 1^{i} = 1$

Because $\left(\mathbf{1} \cdot \mathbf{e}^{i\frac{\pi}{2}}\right)^{\mathbf{1}i} = \left(\mathrm{e}^{i\frac{\pi}{2}}\right)^{1i} = \mathbf{1}^{\mathbf{1}i} \cdot \mathbf{e}^{-1\frac{\pi}{2}}$.

Then $\mathbf{1} = \mathbf{1}^{\mathbf{1}i}$

Thus: $\mathbf{1}^{i} = \mathbf{1}$