Quaternionic generalization of telegraph equations

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Abstract

Using non-commutative space-time quaternion algebra, we represent the generalization of one-dimensional and three-dimensional telegraph equations, which are widely applied to consider the propagation of an electromagnetic signal in communication lines, as well as to describe particle diffusion and heat transfer. It is shown that the system of telegraph equations can be represented in compact form as a single quaternion equation taking into account the space-time properties of physical quantities. The distinctive features of the one-dimensional and three-dimensional telegraph equations are discussed.

1 Introduction

In the last few decades, significant progress has been achieved in the application of non-commutative algebras of hypercomplex numbers (quaternions, octonions, sedenions) [1]-[4] to generalize the differential equations of mathematical physics. In particular, these algebras were used to describe electromagnetic [5]-[8] and weak gravitational [9]-[15] fields, fluid flows [16]-[18], and plasma motion [19]-[23]. In addition, hypercomplex algebras have been applied to reformulate the equations of relativistic quantum mechanics in terms of fields with non-zero mass of quantum [24]-[33]. Using the algebra of hypercomplex numbers, one can write complicated systems of differential equations of mathematical physics as a single compact hypercomplex equation. On the one hand, this enables the easy generalization of these equations to more complex models of physical processes, and on the other hand, it makes it possible to naturally take into account the properties of physical quantities with respect to operations of rotation and space-time inversion [4]. In present article, we consider another large class of equations, the so-called telegraph equations. Telegraph equations were originally written by Heaviside [34]-[37] to describe signal propagation in a two-wire electrical communication line. Subsequently, equations of the same type were proposed to describe the processes of heat propagation [38]-[41] and diffusion of particles, taking into account ballistic transport [42]-[45]. In the present paper, we generalize telegraph equations using space-time algebra, which takes into account the properties of physical values in relation to spatial and temporal inversions [4]. Such approach to the generalization of telegraph equations has not been previously considered in the literature.

2 Space-time algebra based on Macfarlane quaternions

For a compact representation of the equations, we use the space-time algebra [4] based on Macfarlane hyperbolic quaternions [46]. The Macfarlane quaternion $\tilde{\mathbf{M}}$ consists of scalar m and vector $\mathbf{M}$ part

$$\tilde{\mathbf{M}} = m + \mathbf{M} = m + M_1\mathbf{a}_1 + M_2\mathbf{a}_2 + M_3\mathbf{a}_3. \tag{1}$$

Here values $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the unit vectors, which form the basis of the quaternion. Thus, in this algebra any vector $\mathbf{A}$ can be presented as a linear combination of unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$:

$$\mathbf{A} = A_1\mathbf{a}_1 + A_2\mathbf{a}_2 + A_3\mathbf{a}_3. \tag{2}$$

The unit vectors have the following rules of multiplication and commutation

$$\mathbf{a}_n\mathbf{a}_m = \delta_{nm} + \lambda_{nmk} i\mathbf{a}_k, \tag{3}$$

where δ_{nm} is Kronecker delta, λ_{nmk} is Levi-Civita symbol $(n, m, k \in \{1, 2, 3\})$ and i is the imaginary unit $(i^2 = -1)$. For clarity, these rules are also presented in the form of Table 1.

The main advantage of this algebra is the Clifford product of vectors. For example, the Clifford product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is represented as the sum of scalar and vector products

$$\mathbf{AB} = (\mathbf{A} \cdot \mathbf{B}) + i\,[\mathbf{A} \times \mathbf{B}]. \tag{4}$$

Here scalar and vector products are defined as usually

$$(\mathbf{A} \cdot \mathbf{B}) = A_1B_1 + A_2B_2 + A_3B_3, \tag{5}$$

$$[\mathbf{A} \times \mathbf{B}] = (A_2B_3 - A_3B_2)\,\mathbf{a}_1 + (A_3B_1 - A_1B_3)\,\mathbf{a}_2 + (A_1B_2 - A_2B_1)\,\mathbf{a}_3. \tag{6}$$

The Clifford multiplication (4) allows writing the equations in a very compact form.

In addition, the basis $\mathbf{a}_n$ is associated with spatial rotation of vector values. The spatial rotation of the vector $\mathbf{A}$ on the angle θ around axis $\mathbf{n}$ is described as

$$\mathbf{A}' = \tilde{\mathbf{U}}^*\mathbf{A}\,\tilde{\mathbf{U}}, \tag{7}$$

where quaternions $\tilde{\mathbf{U}}^*$ and $\tilde{\mathbf{U}}$ are

$$\tilde{\mathbf{U}} = \cos(\theta/2) + i\,\mathbf{n}\sin(\theta/2)\,, \tag{8}$$

$$\tilde{\mathbf{U}}^* = \cos(\theta/2) - i\,\mathbf{n}\sin(\theta/2)\,. \tag{9}$$

To take into account the space-time properties of physical quantities, we use another special type of quaternions. The space-time (ST) quaternion $\tilde{\mathbf{Q}}$ is a four-component value, which is formally written as follows:

$$\tilde{\mathbf{Q}} = a + \mathbf{e_t}b + \mathbf{e_r}c + \mathbf{e_{tr}}d. \tag{10}$$

The components of a quaternion a, b, c, d are real numbers. Three elements $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ form the basis of the ST quaternion, which allows one to take into account the symmetry of physical quantities with respect to the operations of spatial and temporal inversion. The value $\mathbf{e_t}$ is the time scalar unit; $\mathbf{e_r}$ is the spatial scalar unit; $\mathbf{e_{tr}}$ is the space-time scalar unit. The rules of multiplication and commutation for space-time units $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$ are presented in Table 2.

Table 1: The rules of multiplication for unit vectors $\mathbf{a}_n$

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{a}_1$	1	$i\mathbf{a}_3$	$-i\mathbf{a}_2$
$\mathbf{a}_2$	$-i\mathbf{a}_3$	1	$i\mathbf{a}_1$
$\mathbf{a}_3$	$i\mathbf{a}_2$	$-i\mathbf{a}_1$	1

Table 2: The rules of multiplication for space-time units $\mathbf{e_t}$, $\mathbf{e_r}$, $\mathbf{e_{tr}}$

	$\mathbf{e_t}$	$\mathbf{e_r}$	$\mathbf{e_{tr}}$
$\mathbf{e_t}$	1	$i\mathbf{e_{tr}}$	$-i\mathbf{e_r}$
$\mathbf{e_r}$	$-i\mathbf{e_{tr}}$	1	$i\mathbf{e_t}$
$\mathbf{e_{tr}}$	$i\mathbf{e_r}$	$-i\mathbf{e_t}$	1

Also, to designate basis elements, we enter digital indices $\mathbf{e_t} \equiv \mathbf{e}_1$, $\mathbf{e_r} \equiv \mathbf{e}_2$, $\mathbf{e_{tr}} \equiv \mathbf{e}_3$. Then the multiplication and commutation rules for ST basis elements are written in the following form

$$\mathbf{e}_n\mathbf{e}_m = \delta_{nm} + \lambda_{nmk} i\mathbf{e}_k. \tag{11}$$

The unit vectors $\mathbf{a}_n$ commute with units $\mathbf{e}_m$

$$\mathbf{a}_n\mathbf{e}_m = \mathbf{e}_m\mathbf{a}_n. \tag{12}$$

The basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is related to the operations of space-time inversion. The operation of spatial inversion ($\hat{I}_\mathbf{r}$) is carried out using the basis element $\mathbf{e_t}$

$$\hat{I}_\mathbf{r}\tilde{\mathbf{Q}} = \mathbf{e_t}\tilde{\mathbf{Q}}\mathbf{e_t} = a + \mathbf{e_t}b - \mathbf{e_r}c - \mathbf{e_{tr}}d. \tag{13}$$

This changes the sign of the spatial and space-time parts of the quaternion. Accordingly, the operation of time inversion ($\hat{I}_\mathbf{t}$) is carried out using the basis element $\mathbf{e_r}$

$$\hat{I}_\mathbf{t}\tilde{\mathbf{Q}} = \mathbf{e_r}\tilde{\mathbf{Q}}\mathbf{e_r} = a - \mathbf{e_t}b + \mathbf{e_r}c - \mathbf{e_{tr}}d. \tag{14}$$

Similarly, the space-time inversion (($\hat{I}_\mathbf{tr}$) is carried out using the basis element $\mathbf{e_{tr}}$

$$\hat{I}_\mathbf{tr}\tilde{\mathbf{Q}} = \mathbf{e_r}\tilde{\mathbf{Q}}\mathbf{e_r} = a - \mathbf{e_t}b - \mathbf{e_r}c + \mathbf{e_{tr}}d. \tag{15}$$

Further we use this formalism of Macfarlane quaternions to describe the telegraph equations.

3 Telegraph equations for electric lines

Telegraph equations describe the propagation of electromagnetic signal in a two-wire electrical communication line. Conventionally, such line can be represented as an infinite chain of series-connected cells, which are infinitely short sections of the line. The electrical circuit of an elementary section of two-wire line is shown in Fig. 1.

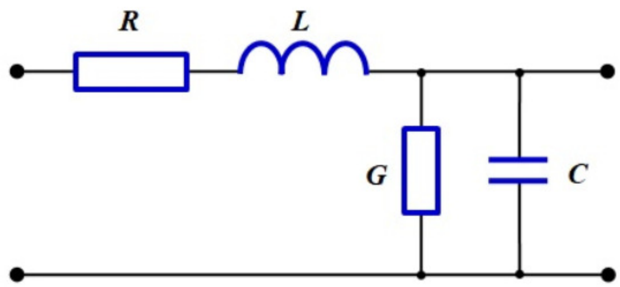

Figure 1: Sketch of an elementary section of two-wire electric line.

For an infinitely small section of the two-wire line (Fig. 1), the equations are written in the following form:

$$\begin{gathered} C\frac{\partial U}{\partial t} + GU + \frac{\partial I}{\partial x} = 0, \\ L\frac{\partial I}{\partial t} + RI + \frac{\partial U}{\partial x} = 0. \end{gathered} \tag{16}$$

Here $U = U(x,t)$ is the voltage distribution and $I = I(x,t)$ is the electric current; R is the resistance corresponding to resistance interior to the two wires; L is the line inductance; C is the capacitance and G is conductance between the wires. All parameters refer to the unit of line length.

3.1 Quaternion form of telegraph equations without damping

Let us first consider the case of an ideal line without attenuation ($R = 0, G = 0$). We assume that all line parameters are constant. In this case, the system of equations (16) can be represented in the following symmetrical form:

$$\begin{gathered} \frac{1}{s}\frac{\partial U}{\partial t} + \frac{\partial J}{\partial x} = 0, \\ \frac{1}{s}\frac{\partial J}{\partial t} + \frac{\partial U}{\partial x} = 0. \end{gathered} \tag{17}$$

Here s is the phase velocity of electromagnetic wave propagation ($s = 1/\sqrt{LC}$), $J = zI$ (where z is the line impedance $z = \sqrt{L/C}$). The equations (17) can be represented as the single quaternion equation

$$\left(i\mathbf{e_1}\frac{1}{s}\frac{\partial}{\partial t} - \mathbf{e_2}\frac{\partial}{\partial x}\right)(i\mathbf{e_1}U + \mathbf{e_2}J) = 0. \tag{18}$$

Indeed, after the action of the operator in equation (18), we have

$$-\frac{1}{s}\frac{\partial U}{\partial t} - \frac{\partial J}{\partial x} - \mathbf{e_3}\frac{1}{s}\frac{\partial J}{\partial t} - \mathbf{e_3}\frac{\partial U}{\partial x} = 0. \tag{19}$$

Hence, separating the components of the quaternion in equation (19), we obtain the system of equations (17).

On the other hand, acting by the operator

$$\left(i\mathbf{e_1}\frac{1}{s}\frac{\partial}{\partial t}-\mathbf{e_2}\frac{\partial}{\partial x}\right) \tag{20}$$

on equation (18), we get the following wave equation in quaternion form

$$\left(i\mathbf{e_1}\frac{1}{s}\frac{\partial}{\partial t}-\mathbf{e_2}\frac{\partial}{\partial x}\right)\left(i\mathbf{e_1}\frac{1}{s}\frac{\partial}{\partial t}-\mathbf{e_2}\frac{\partial}{\partial x}\right)(i\mathbf{e_1}U+\mathbf{e_2}J)=0. \tag{21}$$

Multiplying the operators on the left side of equation (21) and separating the quaternion components, we obtain the following wave equations for the quantities U and J:

$$\begin{aligned}\left(-\frac{1}{s^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)U=0,\\ \left(-\frac{1}{s^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)J=0.\end{aligned} \tag{22}$$

3.2 Quaternion form of telegraph equations with damping

Telegraph equations (16), which take into account attenuation in the line, can also be reduced to the following symmetrical form:

$$\begin{aligned}\frac{1}{s}\frac{\partial U}{\partial t}+(\alpha-\beta)\,U+\frac{\partial J}{\partial x}=0,\\ \frac{1}{s}\frac{\partial J}{\partial t}+(\alpha+\beta)\,J+\frac{\partial U}{\partial x}=0,\end{aligned} \tag{23}$$

where the quantities α and β are expressed in terms of the line parameters as follows:

$$\begin{aligned}\alpha&=\frac{RC+GL}{2\sqrt{LC}},\\ \beta&=\frac{RC-GL}{2\sqrt{LC}}.\end{aligned} \tag{24}$$

The system of equations (23) can be represented as a single quaternion equation

$$\left\{i\mathbf{e_1}\left(\frac{1}{s}\frac{\partial}{\partial t}+\alpha\right)-\mathbf{e_2}\frac{\partial}{\partial x}+\mathbf{e_3}\beta\right\}\{(i\mathbf{e_1}+\mathbf{e_3})\,U-(1-\mathbf{e_2})\,J\}=0. \tag{25}$$

Indeed, after the action of the operator on the left side of equation (25), we obtain the following quaternion expression:

$$\begin{aligned}&-(1-\mathbf{e_2})\left(\frac{1}{s}\frac{\partial}{\partial t}+\alpha\right)U-(1-\mathbf{e_2})\,\frac{\partial}{\partial x}J+(1-\mathbf{e_2})\,\beta U\\ &-(\mathbf{e_3}+i\mathbf{e_1})\,z\left(\frac{1}{s}\frac{\partial}{\partial t}+\alpha\right)I-(\mathbf{e_3}+i\mathbf{e_1})\,\frac{\partial}{\partial x}U-(\mathbf{e_3}+i\mathbf{e_1})\,\beta J=0.\end{aligned} \tag{26}$$

Hence, separating the quaternion components in (26), we obtain the system of equations (23).

On the other hand, applying to (25) the operator

$$\left\{i\mathbf{e_1}\left(\frac{1}{s}\frac{\partial}{\partial t}+\alpha\right)-\mathbf{e_2}\frac{\partial}{\partial x}+\mathbf{e_3}\beta\right\} \tag{27}$$

and multiplying the operators on the left side of equation we obtain the following quaternion expression:

$$\left(-\frac{1}{s^2}\frac{\partial^2}{\partial t^2}-\frac{2\alpha}{s}\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}-\alpha^2+\beta^2\right)\{(i\mathbf{e_1}+\mathbf{e_3})\,U-(1-\mathbf{e_2})\,J\}=0. \tag{28}$$

Hence, separating the quaternion components and taking into account that

$$\frac{2\alpha}{s}=RC+GL, \tag{29}$$

$$-\alpha^2+\beta^2=-RG, \tag{30}$$

we obtain the following well-known wave equations for the quantities U and I:

$$\left(-LC\frac{\partial^2}{\partial t^2}-(RC+GL)\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}-RG\right)U=0, \tag{31}$$

$$\left(-LC\frac{\partial^2}{\partial t^2}-(RC+GL)\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}-RG\right)I=0. \tag{32}$$

4 Three-dimensional telegraph equations

Three-dimensional telegraph equations are used to describe the heat transfer in the framework of the Cattaneo-Vernotte (CV) model [38]-[41] and to describe the diffusion of particles taking into account ballistic transport [42]-[45]. The corresponding equations are written as follows:

$$\left(\frac{\partial^2}{\partial t^2}+\frac{1}{\tau_T}\frac{\partial}{\partial t}-s_T^2\Delta\right)T=0, \tag{33}$$

$$\left(\frac{\partial^2}{\partial t^2}+\frac{1}{\tau_n}\frac{\partial}{\partial t}-s_n^2\Delta\right)n=0. \tag{34}$$

Here T is the temperature, s_T is the heat propagation rate, τ_T is the characteristic time of temperature relaxation, n is the particle concentration, s_n is the characteristic propagation velocity of the concentration profile, τ_n is the characteristic relaxation time of the nonequilibrium concentration.

4.1 Symmetric form of CV heat conduction equations

Let us consider the heat conduction equations within the framework of the CV model. The continuity equation for the heat flux is written as follows:

$$\frac{\partial\varepsilon}{\partial t}+\nabla\cdot\mathbf{q}_T=0, \tag{35}$$

where ε is the density of internal energy, $\mathbf{q}_T$ is the heat flux. According to the thermodynamic relation, we have

$$d\varepsilon = c\rho dT, \tag{36}$$

where c is the specific heat capacity, ρ is the density of the medium. Then equation (35) is rewritten as

$$c\rho \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q}_T = 0. \tag{37}$$

Instead of Newton's law of heat conduction[45], the following relationship is used in the CV model

$$\tau_T \frac{\partial \mathbf{q}_T}{\partial t} + \mathbf{q}_T = -\kappa_T \nabla T \tag{38}$$

where κ_T is the coefficient of thermal conductivity. The relation (38) transforms into Newton's law at $\tau_T \to 0$.

From equations (37) and (38) we obtain the following telegraph equations for temperature and heat flux:

$$\left(\frac{\partial^2}{\partial t^2} + \frac{1}{\tau_T} \frac{\partial}{\partial t} - s_T^2 \Delta \right) T = 0, \tag{39}$$

$$\left(\frac{\partial^2}{\partial t^2} + \frac{1}{\tau_T} \frac{\partial}{\partial t} - s_T^2 \Delta \right) \mathbf{q}_T = 0. \tag{40}$$

The rate of heat propagation is expressed in terms of the medium parameters as follows:

$$s_T = \sqrt{\frac{\kappa_T}{c\rho\tau_T}}. \tag{41}$$

Equations (37) and (38) can be rewritten in the following symmetrical form:

$$\frac{1}{s_T} \frac{\partial \theta_T}{\partial t} + \nabla \cdot \mathbf{q}_T = 0, \tag{42}$$

$$\frac{1}{s_T} \frac{\partial \mathbf{q}_T}{\partial t} + 2\alpha_T \, \mathbf{q}_T + \nabla \theta_T = 0. \tag{43}$$

Here, θ_T is the renormalized temperature ($\theta_T = T/\xi$). We have also entered the following parameters

$$\xi = \sqrt{\frac{\tau_T}{c\rho\kappa_T}}, \tag{44}$$

$$2\alpha_T = \sqrt{\frac{c\rho}{\tau_T}} \tag{45}$$

4.2 Symmetric form of telegraph diffusion equations

The continuity equation for the particle diffusion is written as

$$\frac{\partial n}{\partial t} + (\nabla \cdot \mathbf{q}_n) = 0, \tag{46}$$

where n is the particle concentration, $\mathbf{q}_n$ is the density of the particle flux. By analogy with (38), the equation for the particle flux density is written as follows:

$$\tau_n \frac{\partial \mathbf{q}_n}{\partial t} + \mathbf{q}_n = -\kappa_n \nabla n. \tag{47}$$

Here κ_n is the diffusion coefficient. Under the condition $\tau_n \to 0$, the relation (47) transforms into Fick's law [48] for the diffusion flux.

Equations (46) and (47) can be rewritten in the following symmetrical form:

$$\frac{1}{s_n} \frac{\partial \theta_n}{\partial t} + \nabla \cdot \mathbf{q}_n = 0, \tag{48}$$

$$\frac{1}{s_n} \frac{\partial \mathbf{q}_n}{\partial t} + 2\alpha_n \, \mathbf{q}_n + \nabla \theta_n = 0. \tag{49}$$

Here, s_n is the characteristic diffusion rate ($s_n^2 = \kappa_n / \tau_n$), θ_n is the renormalized concentration ($\theta_n = s_n n$). We also introduce the following parameter

$$2\alpha_n = \frac{1}{\tau_n s_n}. \tag{50}$$

From the equations (48) and (49) we obtain the following telegraph wave equations for concentration and particle flux:

$$\left(\frac{\partial^2}{\partial t^2} + \frac{1}{\tau_n} \frac{\partial}{\partial t} - s_n^2 \Delta \right) n = 0, \tag{51}$$

$$\left(\frac{\partial^2}{\partial t^2} + \frac{1}{\tau_n} \frac{\partial}{\partial t} - s_n^2 \Delta \right) \mathbf{q}_n = 0. \tag{52}$$

4.3 Quaternionic form of the telegraph equations of heat conduction and diffusion

Heat transfer equations (42)-(43) and diffusion equations (48)-(49) can be written in the following generalized form

$$\frac{1}{s_\nu} \frac{\partial \theta_\nu}{\partial t} + \nabla \cdot \mathbf{q}_\nu = 0, \tag{53}$$

$$\frac{1}{s_\nu} \frac{\partial \mathbf{q}_\nu}{\partial t} + 2\alpha_\nu \, \mathbf{q}_\nu + \nabla \theta_\nu = 0, \tag{54}$$

where the index ν takes the value T for the heat conduction equations and n for the diffusion equations ($\nu \in \{T, n\}$). Equations (53) and (54) can be written as one generalized quaternion equation in the following form:

$$\left\{ i\mathbf{e}_1 \left(\frac{1}{s_\nu} \frac{\partial}{\partial t} + \alpha_\nu \right) - \mathbf{e}_2 \nabla + \mathbf{e}_3 \alpha_\nu \right\} \left\{ (i\mathbf{e}_1 + \mathbf{e}_3)\, \theta_\nu - (1 - \mathbf{e}_2)\, \mathbf{q}_\nu \right\} = 0. \tag{55}$$

Indeed, after the action of the operator on the left side of equation (55), we have

$$\begin{aligned} & -(1 - \mathbf{e}_2) \left\{ \frac{1}{s_\nu} \frac{\partial \theta_\nu}{\partial t} + \nabla \cdot \mathbf{q}_\nu \right\} - (\mathbf{e}_3 + i\mathbf{e}_1) \left\{ \frac{1}{s_\nu} \frac{\partial \mathbf{q}_\nu}{\partial t} + 2\alpha_\nu \mathbf{q}_\nu + \nabla \theta_\nu \right\} \\ & -i\, (1 - \mathbf{e}_2)\, \nabla \times \mathbf{q}_\nu = 0. \end{aligned} \tag{56}$$

Hence, separating the quaternion components with different space-time properties, we obtain a system of three equations

$$
\begin{aligned}
&\frac{1}{s_\nu}\frac{\partial \theta_\nu}{\partial t} + \nabla \cdot \mathbf{q}_\nu = 0, \\
&\frac{1}{s_\nu}\frac{\partial \mathbf{q}_\nu}{\partial t} + 2\alpha_\nu \mathbf{q}_\nu + \nabla \theta_\nu = 0, \\
&\nabla \times \mathbf{q}_\nu = 0.
\end{aligned} \tag{57}
$$

The first and second equations in (57) coincide with equations (53) and (54). The third equation in (57) means that there are no eddy heat and diffusion flows.

On the other hand, acting by the operator

$$
i\mathbf{e}_1 \left(\frac{1}{s_\nu}\frac{\partial}{\partial t} + \alpha_\nu \right) - \mathbf{e}_2 \nabla + \mathbf{e}_3 \alpha_\nu \tag{58}
$$

to the equation (55) and multiplying the operators on the left side we obtain the following quaternion wave telegraph equation

$$
\left(-\frac{1}{s_\nu^2}\frac{\partial^2}{\partial t^2} - \frac{2\alpha_\nu}{s_\nu}\frac{\partial}{\partial t} + \Delta \right) \left\{ \left(i\mathbf{e}_1 + \mathbf{e}_3 \right) \theta_\nu - \left(1 - \mathbf{e}_2 \right) \mathbf{q}_\nu \right\} = 0. \tag{59}
$$

Separating in (59) the quantities with different space-time properties, we obtain

$$
\left(-\frac{1}{s_\nu^2}\frac{\partial^2}{\partial t^2} - \frac{2\alpha_\nu}{s_\nu}\frac{\partial}{\partial t} + \Delta \right) \theta_\nu = 0, \tag{60}
$$

$$
\left(-\frac{1}{s_\nu^2}\frac{\partial^2}{\partial t^2} - \frac{2\alpha_\nu}{s_\nu}\frac{\partial}{\partial t} + \Delta \right) \mathbf{q}_\nu = 0. \tag{61}
$$

The generalized equation (60) coincides with previously given equations (33) and (34).

5 Concluding remarks

Thus, we have considered a generalization of one-dimensional (1D) and three-dimensional (3D) telegraph equations using space-time quaternion algebra. The advantage of Macfarlane quaternions is that the squares of the basis elements $\mathbf{a}_n$ are positive definite quantities ($\mathbf{a}_n^2 = 1$). This allows us to interpret them as the unit vectors, in contrast to Hamiltonian quaternions, where the basis elements are imaginary units. On the other hand, in contrast to the usual Gibbs-Heaviside vector algebra in Macfarlane quaternions the Clifford product of vectors is defined, which allows one to write the equations in a compact form.

In contrast to other hypercomplex equations of mathematical physics, the telegraph quaternion equation (59) has an unusual space-time structure of the wave function

$$
\tilde{\mathbf{W}} = \left(i\mathbf{e}_1 + \mathbf{e}_3 \right) \theta_\nu - \left(1 - \mathbf{e}_2 \right) \mathbf{q}_\nu. \tag{62}
$$

It is easy to see that in this expression the members $(i\mathbf{e}_1 + \mathbf{e}_3)$ and $(1 - \mathbf{e}_2)$ are elementary eigenfunctions of the operator $\mathbf{e}_2$ [4]. At the same time, the action of the other space-time operators $\mathbf{e}_1$ and $\mathbf{e}_3$ cause only a permutation of these elementary functions. The expansion in terms of eigenfunctions of the operator $\mathbf{e}_2$

(62) is not the only possible one. The similar combinations of eigenfunctions of any operators $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ can be used as the wave functions.

The difference between 1D and 3D equations is that the generalized quaternionic equations contain different operators. The 1D equation has the operator

$$\left\{ i\mathbf{e}_1 \left(\frac{1}{s} \frac{\partial}{\partial t} + \alpha \right) - \mathbf{e}_2 \frac{\partial}{\partial x} + \mathbf{e}_3 \beta \right\}, \tag{63}$$

while 3D equation has the operator

$$\left\{ i\mathbf{e}_1 \left(\frac{1}{s_\nu} \frac{\partial}{\partial t} + \alpha_\nu \right) - \mathbf{e}_2 \nabla + \mathbf{e}_3 \alpha_\nu \right\}. \tag{64}$$

This shows that 1D equations pass in form to 3D equations under the condition $\alpha = \beta$. This condition is equivalent to the condition $G = 0$, i.e. matches the line with perfect wire insulation.

In principle, by analogy with 1D equations, the relaxation term of the form n/τ_r (where τ_r is the time of recombination) can also be included in the continuity equation (46). This corresponds, for example, to taking into account the processes of recombination of cosmic ray particles [44],[45]. In this case, if the relaxation time τ_n differs from the recombination time τ_r, then the telrgraph equations describing 3D diffusion will have the form of the corresponding 1D telegraph equations.

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