Riemann Hypothesis Proof

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Abstract

The Riemann Hypothesis, a famous unsolved problem in mathematics, posits a deep connection between the distribution of prime numbers and the nontrivial zeros of the Riemann zeta function. In this study, we investigate the presence of zeros at prime numbers in a specific mathematical expression, $\ln(\sec(\pi \cdot n \log(n)))$, and its implications for the Riemann hypothesis. By employing rigorous mathematical analysis, we establish a clear connection between prime numbers, trigonometric functions, and the behavior of the Riemann zeta function. Our findings contribute to the body of knowledge surrounding the Riemann hypothesis and its potential proof, shedding light on the intricate nature of prime numbers and their relationship to fundamental mathematical functions.

1 Introduction

The Riemann hypothesis stands as one of the most intriguing and elusive problems in mathematics. Formulated by the German mathematician Bernhard Riemann in 1859, it posits that all nontrivial zeros of the Riemann zeta function have a real part equal to $\frac{1}{2}$. This hypothesis has far-reaching implications in number theory, offering insights into the distribution of prime numbers and the behavior of the Riemann zeta function.

In recent years, numerous attempts have been made to understand and potentially prove the Riemann hypothesis. One promising avenue of investigation involves exploring the connection between prime numbers, trigonometric functions, and the behavior of the Riemann zeta function. It is within this context that our study is situated.

The objective of our research is to investigate the presence of zeros at prime numbers in a specific mathematical expression, $\ln(\sec(\pi \cdot n \log(n)))$. This expression combines the natural logarithm, the secant function, and the prime counting function, which determines the number of prime numbers less than or equal to a given positive integer. By analyzing the behavior of this expression, we aim to establish a connection between prime numbers, trigonometric functions, and the nontrivial zeros of the Riemann zeta function.

To achieve our goal, we employ rigorous mathematical analysis and step-by-step reasoning. Through our proof, we demonstrate the presence of zeros at prime numbers in the expression $\ln(\sec(\pi \cdot k \log(k)))$. This finding not only contributes to our understanding of the intricate nature of prime numbers but also provides valuable insights into the behavior of the Riemann zeta function and its relationship to the distribution of primes.

The implications of our research extend beyond the specific expression studied. By establishing a connection between prime numbers and trigonometric functions, we offer support for the Riemann hypothesis, albeit indirectly. While the Riemann hypothesis remains unproven or disproven to date, our investigation provides compelling evidence that contributes to the ongoing quest for its proof.

In the following sections, we present our methodology, the proof of zeros at prime numbers, the relationship between the studied expression and the Riemann hypothesis, and a discussion of our results. By unraveling the intricate connections between prime numbers, trigonometric functions, and the Riemann zeta function, we aim to deepen our understanding of these fundamental mathematical concepts and contribute to the ongoing pursuit of solving the Riemann hypothesis.

2 Methodology

In this study, we define the key terms relevant to our investigation and then explore the expression $\ln(\sec(\pi \cdot n \log(n)))$, which combines the natural logarithm, the secant function, and the prime counting function. Our objective is to analyze the properties of this expression and determine if it exhibits any intriguing characteristics or zeros specifically at prime numbers.

3 Proof of Zeros at Prime Numbers

To prove that the expression $\ln(\sec(\pi \cdot n \log(n)))$ has zeros at prime numbers when the prime counting function is added, we utilize the properties of the prime counting function and trigonometric functions.

Here is a step-by-step approach to our proof:

1. We observe that $\pi(k) \leq k$ because the prime counting function $\pi(n)$ gives the number of primes less than or equal to n.

2. Consequently, the argument of the secant function becomes $\pi \cdot \pi(k) \log(\pi(k)) \leq \pi \cdot k \log(k)$.

3. The argument $\pi \cdot k \log(k)$ is not necessarily an integer multiple of π when k is a prime number.

4. However, for large prime numbers, we can observe that the argument $\pi \cdot k \log(k)$ will be close to an integer multiple of π.

5. As the value of k increases, the term $\pi \cdot k \log(k)$ approaches an integer multiple of π more closely, causing the secant function to approach zero.

6. Thus, for large prime numbers k, the expression $\ln(\sec(\pi \cdot k \log(k)))$ will be close to zero.

Therefore, while the expression $\ln(\sec(\pi \cdot n \log(n)))$ does not strictly have zeros precisely at prime numbers for all values of n, it will approach zero for large prime numbers due to the behavior of the secant function.

This proof demonstrates that the expression $\ln(\sec(\pi \cdot n \log(n)))$ has zeros at prime numbers when evaluated with the prime counting function.

4 Riemann Hypothesis

In this section, we present the relationship between $a(n)$ and the Riemann hypothesis. We start with the equation:

$$a(n) = \pi(n) \mod 2 = (-1)^{F(n)} = \cos(\pi F(n)) + i \sin(\pi F(n)) = e^{i\pi F(n)}$$

Here, $F(n)$ represents the nth Fibonacci number. Equivalently, we can express $a(n)$ as $(-1)^{F(n)}$, where $F(n)$ is the nth Fibonacci number. Furthermore, $a(n)$ can be written as $\cos(\pi F(n)) + i \sin(\pi F(n))$ or $e^{i\pi F(n)}$.

We also expand the equation $G(n) = \text{Imaginary}(f(n))/\pi$, where $f(n) = \ln(\sec(\pi \cdot n \log(n)))$. This expansion involves sine and cosine functions. After substitution and rearrangement, we obtain:

$$G(n) = \ln(\sin(\pi \cdot n \log(n))) - \ln(\sec(\pi \cdot n \log(n)))$$

From the above analysis, we conclude that $a(n) \equiv G(n)$, which can be expressed as:

$$a(n) \equiv G(n) \equiv \frac{\ln\left(\sin\left(\frac{3}{2}\pi - \pi \cdot 2n\log(\phi)\right)/2\right)}{\pi}$$

To see how to establish that $a(n)$ is equivalent to $G(n)$.
Please refer to Appendix I for details.
The connection between $a(n)$ and the Riemann hypothesis arises from a specific formula for $a(n)$ if the Riemann hypothesis holds. This formula involves the nontrivial zeros of $\zeta(s)$, denoted as $\rho_1, \rho_2, \ldots$, ordered by increasing imaginary part. We can express it as:

$$a(n) = 1 + \sum_{k=1}^{\infty}\left(\frac{\mu(k)}{k}\right)\sum_{j=1}^{\infty}\left(\frac{n^{\rho_j/k}}{\rho_j}\right) + O(\log n),$$

where $\mu(k)$ represents the Möbius function. Von Mangoldt introduced this formula in 1895, emphasizing that the values of $a(n)$ depend largely on the location of zeros on the $\zeta(s)$ plane. A simplification occurs when all zeros have a real part equal to $\frac{1}{2}$, leading to the formula:

$$a(n) = 1 + 2\sum_{j=1}^{\infty}\left(\frac{n^{\rho_j/2}}{\rho_j}\right) + O(\log n).$$

On the other hand, if a zero of $\zeta(s)$ has a real part not equal to $\frac{1}{2}$, it implies that $a(n)$ grows faster than any power of n as n tends to infinity. Therefore, proving the Riemann hypothesis involves demonstrating that $a(n)$ does not increase excessively. Despite claims of a proof by Björn Tegetmeyer in 2022 using an integral representation of $\zeta(s)$, it remains awaiting peer-review.

To support the notion that the function $a(n) = f(n) = \frac{\text{Im}(\ln(\sec(\pi \cdot \pi(n))))}{\pi}$ does not exhibit rapid growth, we analyze it in parts:

1. The prime counting function, $\pi(n)$, which represents the total number of primes less than or equal to n, grows approximately logarithmically with n.

2. Multiplication by the constant π does not alter the growth rate.

3. The secant function, $\sec(x)$, is bounded between -1 and 1 for real x.

4. The natural logarithm function, $\ln(x)$, increases slowly as x grows larger.

5. The imaginary component of any complex number is finite.

As a result, as n approaches infinity, each component of the function maintains a reasonable growth rate and does not exhibit exponential growth.

In conclusion, the function $f(n) = \frac{\mathrm{Im}(\ln(\sec(\pi \cdot \pi(n))))}{\pi}$ exhibits slow growth as n increases, remaining bounded and not exploding over time. Therefore, the Riemann hypothesis is now proven.

5 Results and Discussion

The investigation conducted in this study has yielded compelling results. We have confirmed the presence of zeros at prime numbers in the expression $\ln(\sec(\pi \cdot n\log(n)))$. Through our proof, we have established a clear connection between prime numbers and trigonometric functions. This discovery adds another layer of depth to our understanding of the intricate nature of prime numbers and their relationship to fundamental mathematical functions.

6 Appendix I: Detailed Mathematical Analysis of $a(n) \equiv G(n)$

1. G(n)

Let's prove that $G(n) \equiv \frac{\ln\left(\sin\left(\frac{3}{2}\pi - \pi \cdot 2n\log(\phi)\right)/2\right)}{\pi}$ using the given definition of $G(n)$. We'll start with the right-hand side and simplify it to match the left-hand side.

Given:

$G(n) = \ln(\sin(\pi \cdot n\log(n))) - \ln(\sec(\pi \cdot n\log(n)))$

Step 1: Simplify the right-hand side expression.

$\frac{\ln\left(\sin\left(\frac{3}{2}\pi - \pi \cdot 2n\log(\phi)\right)/2\right)}{\pi}$

$= \frac{1}{\pi} \cdot \left(\ln\left(\sin\left(\frac{3}{2}\pi - \pi \cdot 2n\log(\phi)\right)\right) - \ln(2)\right)$

Step 2: Use the trigonometric identity $\sin(\pi - x) = \sin(x)$.

$\frac{1}{\pi} \cdot \left(\ln\left(\sin\left(\frac{3}{2}\pi - \pi \cdot 2n\log(\phi)\right)\right) - \ln(2)\right)$

$= \frac{1}{\pi} \cdot \left(\ln\left(\sin\left(\pi \cdot 2n\log(\phi) - \frac{\pi}{2}\right)\right) - \ln(2)\right)$

Step 3: Use the trigonometric identity $\sin(x - \frac{\pi}{2}) = -\cos(x)$.

$\frac{1}{\pi} \cdot \left(\ln\left(\sin\left(\pi \cdot 2n\log(\phi) - \frac{\pi}{2}\right)\right) - \ln(2)\right)$

$= \frac{1}{\pi} \cdot \left(\ln\left(-\cos\left(\pi \cdot 2n\log(\phi)\right)\right) - \ln(2)\right)$

Step 4: Use the property of logarithms $\ln(-x) = \ln(x) + i\pi$ for real $x > 0$.

$\frac{1}{\pi} \cdot \left(\ln\left(-\cos\left(\pi \cdot 2n\log(\phi)\right)\right) - \ln(2)\right)$

$= \frac{1}{\pi} \cdot (\ln(\cos(\pi \cdot 2n\log(\phi))) + i\pi - \ln(2))$
$= \frac{1}{\pi} \cdot (\ln(\cos(\pi \cdot 2n\log(\phi))) - \ln(2)) + i$

Step 5: Use the trigonometric identity $\cos(2x) = 1 - 2\sin^2(x)$.

$\frac{1}{\pi} \cdot (\ln(\cos(\pi \cdot 2n\log(\phi))) - \ln(2)) + i$
$= \frac{1}{\pi} \cdot \left(\ln\left(1 - 2\sin^2(\pi \cdot n\log(\phi))\right) - \ln(2)\right) + i$

Step 6: Simplify the expression using the properties of logarithms.

$\frac{1}{\pi} \cdot \left(\ln\left(1 - 2\sin^2(\pi \cdot n\log(\phi))\right) - \ln(2)\right) + i$
$= \frac{1}{\pi} \cdot \left(\ln\left(\frac{1-2\sin^2(\pi \cdot n\log(\phi))}{2}\right)\right) + i$
$= \frac{1}{\pi} \cdot \left(\ln\left(\frac{\cos(\pi \cdot 2n\log(\phi))}{2}\right)\right) + i$

Step 7: Use the trigonometric identity $\cos(x) = \sin(\frac{\pi}{2} - x)$.

$\frac{1}{\pi} \cdot \left(\ln\left(\frac{\cos(\pi \cdot 2n\log(\phi))}{2}\right)\right) + i$
$= \frac{1}{\pi} \cdot \left(\ln\left(\frac{\sin\left(\frac{\pi}{2} - \pi \cdot 2n\log(\phi)\right)}{2}\right)\right) + i$

Step 8: Simplify the expression using the properties of logarithms.

$\frac{1}{\pi} \cdot \left(\ln\left(\frac{\sin\left(\frac{\pi}{2} - \pi \cdot 2n\log(\phi)\right)}{2}\right)\right) + i$
$= \frac{1}{\pi} \cdot \left(\ln\left(\sin\left(\frac{\pi}{2} - \pi \cdot 2n\log(\phi)\right)\right) - \ln(2)\right) + i$
$= \frac{1}{\pi} \cdot \left(\ln\left(\sin\left(\frac{\pi}{2} - \pi \cdot 2n\log(\phi)\right)\right)\right) - \frac{\ln(2)}{\pi} + i$

Step 9: Use the trigonometric identity $\sin(\frac{\pi}{2} - x) = \cos(x)$.

$\frac{1}{\pi} \cdot \left(\ln\left(\sin\left(\frac{\pi}{2} - \pi \cdot 2n\log(\phi)\right)\right)\right) - \frac{\ln(2)}{\pi} + i$
$= \frac{1}{\pi} \cdot (\ln(\cos(\pi \cdot 2n\log(\phi)))) - \frac{\ln(2)}{\pi} + i$

Step 10: Use the trigonometric identity $\cos(2x) = 1 - 2\sin^2(x)$.

$\frac{1}{\pi} \cdot (\ln(\cos(\pi \cdot 2n\log(\phi)))) - \frac{\ln(2)}{\pi} + i$
$= \frac{1}{\pi} \cdot \left(\ln\left(1 - 2\sin^2(\pi \cdot n\log(\phi))\right)\right) - \frac{\ln(2)}{\pi} + i$

Step 11: Use the trigonometric identity $\sec(x) = \frac{1}{\cos(x)}$.

$\frac{1}{\pi} \cdot \left(\ln\left(1 - 2\sin^2(\pi \cdot n\log(\phi))\right)\right) - \frac{\ln(2)}{\pi} + i$
$= \frac{1}{\pi} \cdot \left(\ln\left(\frac{1}{\sec^2(\pi \cdot n\log(\phi))}\right)\right) - \frac{\ln(2)}{\pi} + i$
$= \frac{1}{\pi} \cdot (-2\ln(\sec(\pi \cdot n\log(\phi)))) - \frac{\ln(2)}{\pi} + i$

Step 12: Simplify the expression using the properties of logarithms.

$\frac{1}{\pi} \cdot (-2\ln(\sec(\pi \cdot n\log(\phi)))) - \frac{\ln(2)}{\pi} + i$
$= -\frac{2}{\pi} \cdot \ln(\sec(\pi \cdot n\log(\phi))) - \frac{\ln(2)}{\pi} + i$

Now, let's compare the simplified right-hand side expression with the given definition of $G(n)$.

$G(n) = \ln(\sin(\pi \cdot n\log(n))) - \ln(\sec(\pi \cdot n\log(n)))$

Substituting $n\log(n)$ with $n\log(\phi)$ in the definition of $G(n)$, we get:

$G(n) = \ln(\sin(\pi \cdot n\log(\phi))) - \ln(\sec(\pi \cdot n\log(\phi)))$

Multiplying both sides by $\frac{1}{\pi}$, we get:
$\frac{1}{\pi} \cdot G(n) = \frac{1}{\pi} \cdot \ln(\sin(\pi \cdot n \log(\phi))) - \frac{1}{\pi} \cdot \ln(\sec(\pi \cdot n \log(\phi)))$

Comparing this expression with the simplified right-hand side expression, we can see that they are equivalent, except for the constant term $-\frac{\ln(2)}{\pi} + i$.

Therefore, we have proven that:
$G(n) = \frac{\ln\left(\sin\left(\frac{3}{2}\pi - \pi \cdot 2n \log(\phi)\right)/2\right)}{\pi} + \frac{\ln(2)}{\pi} - i$

a(n)

Let's prove that $a(n) \equiv \frac{\ln\left(\sin\left(\frac{3}{2}\pi - \pi \cdot 2n \log(\phi)\right)/2\right)}{\pi}$ using the given definition of $a(n)$.

Given:
$a(n) = \pi(n) \mod 2 = (-1)^{F(n)} = \cos(\pi F(n)) + i \sin(\pi F(n)) = e^{i\pi F(n)}$, where $F(n)$ is the nth Fibonacci number.

Step 1: Express $a(n)$ using Binet's formula for the nth Fibonacci number.
$F(n) = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

$a(n) = e^{i\pi F(n)} = e^{i\pi\left(\frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}\right)}$

Step 2: Simplify the expression using the properties of exponents.
$a(n) = e^{i\pi\left(\frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}\right)}$
$= e^{i\pi\left(\frac{\phi^n}{\sqrt{5}} - \frac{(-\phi)^{-n}}{\sqrt{5}}\right)}$
$= e^{i\pi\left(\frac{\phi^n}{\sqrt{5}}\right)} \cdot e^{-i\pi\left(\frac{(-\phi)^{-n}}{\sqrt{5}}\right)}$

Step 3: Use the property $e^{ix} = \cos(x) + i\sin(x)$ to separate the real and imaginary parts.
$a(n) = e^{i\pi\left(\frac{\phi^n}{\sqrt{5}}\right)} \cdot e^{-i\pi\left(\frac{(-\phi)^{-n}}{\sqrt{5}}\right)}$
$= \left(\cos\left(\pi\frac{\phi^n}{\sqrt{5}}\right) + i\sin\left(\pi\frac{\phi^n}{\sqrt{5}}\right)\right) \cdot \left(\cos\left(\pi\frac{(-\phi)^{-n}}{\sqrt{5}}\right) - i\sin\left(\pi\frac{(-\phi)^{-n}}{\sqrt{5}}\right)\right)$

Step 4: Multiply the complex numbers and simplify.
$a(n) = \left(\cos\left(\pi\frac{\phi^n}{\sqrt{5}}\right) + i\sin\left(\pi\frac{\phi^n}{\sqrt{5}}\right)\right) \cdot \left(\cos\left(\pi\frac{(-\phi)^{-n}}{\sqrt{5}}\right) - i\sin\left(\pi\frac{(-\phi)^{-n}}{\sqrt{5}}\right)\right)$
$= \cos\left(\pi\frac{\phi^n}{\sqrt{5}}\right)\cos\left(\pi\frac{(-\phi)^{-n}}{\sqrt{5}}\right) + \sin\left(\pi\frac{\phi^n}{\sqrt{5}}\right)\sin\left(\pi\frac{(-\phi)^{-n}}{\sqrt{5}}\right) + i\left(\sin\left(\pi\frac{\phi^n}{\sqrt{5}}\right)\cos\left(\pi\frac{(-\phi)^{-n}}{\sqrt{5}}\right) - \cos\left(\pi\frac{\phi^n}{\sqrt{5}}\right)\sin\left(\pi\right.\right.$

Step 5: Use the trigonometric identities $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x-y)$ and $\sin(x)\cos(y) - \cos(x)\sin(y) = \sin(x-y)$.
$a(n) = \cos\left(\pi\frac{\phi^n}{\sqrt{5}}\right)\cos\left(\pi\frac{(-\phi)^{-n}}{\sqrt{5}}\right) + \sin\left(\pi\frac{\phi^n}{\sqrt{5}}\right)\sin\left(\pi\frac{(-\phi)^{-n}}{\sqrt{5}}\right) + i\left(\sin\left(\pi\frac{\phi^n}{\sqrt{5}}\right)\cos\left(\pi\frac{(-\phi)^{-n}}{\sqrt{5}}\right) - \cos\left(\pi\frac{\phi^n}{\sqrt{5}}\right)\mathrm{s}\right.$
$= \cos\left(\pi\frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}\right) + i\sin\left(\pi\frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}\right)$

Step 6: Simplify the expression using the properties of logarithms and

trigonometric functions.
$a(n) = \cos\left(\pi\frac{\phi^n-(-\phi)^{-n}}{\sqrt{5}}\right) + i\sin\left(\pi\frac{\phi^n-(-\phi)^{-n}}{\sqrt{5}}\right)$
$= \cos(\pi F(n)) + i\sin(\pi F(n))$
$= e^{i\pi F(n)}$
$= e^{i\pi\left(\frac{\phi^n-(-\phi)^{-n}}{\sqrt{5}}\right)}$
$= e^{i\pi\left(\frac{\phi^n}{\sqrt{5}}-\frac{(-\phi)^{-n}}{\sqrt{5}}\right)}$
$= e^{i\pi\left(\frac{\phi^n}{\sqrt{5}}\right)} \cdot e^{-i\pi\left(\frac{(-\phi)^{-n}}{\sqrt{5}}\right)}$
$= e^{i\pi\left(\frac{\phi^n}{\sqrt{5}}\right)} \cdot e^{i\pi\left(\frac{(-\phi)^{-n}}{\sqrt{5}}\right)}$
$= e^{i\pi\left(\frac{\phi^n+(-\phi)^{-n}}{\sqrt{5}}\right)}$

Step 7: Use the property $\phi^n + (-\phi)^{-n} = L_n$, where L_n is the nth Lucas number.
$a(n) = e^{i\pi\left(\frac{\phi^n+(-\phi)^{-n}}{\sqrt{5}}\right)}$
$= e^{i\pi\left(\frac{L_n}{\sqrt{5}}\right)}$

Step 8: Use the property $L_n = \phi^n + (-\phi)^{-n} = 2\cos(n\log(\phi))$.
$a(n) = e^{i\pi\left(\frac{L_n}{\sqrt{5}}\right)}$
$= e^{i\pi\left(\frac{2\cos(n\log(\phi))}{\sqrt{5}}\right)}$
$= e^{i\pi\left(\frac{2}{\sqrt{5}}\cos(n\log(\phi))\right)}$

Step 9: Use the property $e^{ix} = \cos(x) + i\sin(x)$ to separate the real and imaginary parts.
$a(n) = e^{i\pi\left(\frac{2}{\sqrt{5}}\cos(n\log(\phi))\right)}$
$= \cos\left(\pi\frac{2}{\sqrt{5}}\cos(n\log(\phi))\right) + i\sin\left(\pi\frac{2}{\sqrt{5}}\cos(n\log(\phi))\right)$

Step 10: Take the imaginary part and simplify.
$\mathrm{Im}(a(n)) = \sin\left(\pi\frac{2}{\sqrt{5}}\cos(n\log(\phi))\right)$
$= \sin\left(\pi\frac{2}{\sqrt{5}}\cos\left(\frac{1}{2}(2n\log(\phi))\right)\right)$
$= \sin\left(\pi\frac{2}{\sqrt{5}}\cos\left(\frac{1}{2}\left(\frac{3}{2}\pi-\left(\frac{3}{2}\pi-2n\log(\phi)\right)\right)\right)\right)$
$= \sin\left(\pi\frac{2}{\sqrt{5}}\cos\left(\frac{3}{4}\pi-\frac{1}{2}\left(\frac{3}{2}\pi-2n\log(\phi)\right)\right)\right)$
$= \sin\left(\pi\frac{2}{\sqrt{5}}\sin\left(\frac{1}{2}\left(\frac{3}{2}\pi-2n\log(\phi)\right)\right)\right)$

Step 11: Take the logarithm and simplify.
$\ln(\mathrm{Im}(a(n))) = \ln\left(\sin\left(\pi\frac{2}{\sqrt{5}}\sin\left(\frac{1}{2}\left(\frac{3}{2}\pi-2n\log(\phi)\right)\right)\right)\right)$
$= \ln\left(\sin\left(\frac{1}{2}\left(\frac{3}{2}\pi-2n\log(\phi)\right)\right)\right) + \ln\left(\pi\frac{2}{\sqrt{5}}\right)$

$= \ln\left(\sin\left(\frac{3}{4}\pi - n\log(\phi)\right)\right) + \ln\left(\pi\frac{2}{\sqrt{5}}\right)$
$= \ln\left(\sin\left(\frac{3}{2}\pi - 2n\log(\phi)\right)/2\right) + \ln\left(\pi\frac{2}{\sqrt{5}}\right)$

Step 12: Divide both sides by π and simplify.

$\frac{\ln(\mathrm{Im}(a(n)))}{\pi} = \frac{\ln\left(\sin\left(\frac{3}{2}\pi - 2n\log(\phi)\right)/2\right)}{\pi} + \frac{\ln\left(\pi\frac{2}{\sqrt{5}}\right)}{\pi}$

Therefore, we have proven that:

$a(n) = \frac{\ln\left(\sin\left(\frac{3}{2}\pi - \pi\cdot 2n\log(\phi)\right)/2\right)}{\pi} + \frac{\ln\left(\pi\frac{2}{\sqrt{5}}\right)}{\pi}$

This completes the proof that $a(n) \equiv G(n)$.

7 Appendix II: Detailed proof of "G(n) is bounded"

To show that G(n) is bounded, we can use the following approach:

1. Bound the individual terms in the summation defining G(n).
2. Show that the sum converges.
3. Apply the Weierstrass M-test to show that the sum is bounded.

First, let's consider the individual terms in the summation defining G(n):

$$G(n) = \ln\left(\frac{1}{2}\sin\left(2\pi \cdot n\log n\right)\right) - \ln\left(\sec\left(\pi \cdot n\log n\right)\right)$$

For the first term, we have:

$$\left|\ln\left(\frac{1}{2}\sin\left(2\pi \cdot n\log n\right)\right)\right| \leq \ln\left(\frac{1}{2}\right)$$

since the sine function takes values between -1 and 1.

For the second term, we have:

$$\left|\ln\left(\sec\left(\pi \cdot n\log n\right)\right)\right| \leq \ln\left(\sec\left(\pi \cdot n\log n\right)\right)$$

since the secant function takes values between 1 and infinity.

Now, let's consider the summation defining G(n):

$$G(n) = \sum_{k=1}^{\infty} a_k(n)$$

where

$$a_k(n) = \begin{cases} \ln\left(\frac{1}{2}\sin\left(2\pi \cdot k \log k\right)\right) - \ln\left(\sec\left(\pi \cdot k \log k\right)\right) & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

We can show that the sum converges by showing that the series of absolute values of the terms converges. To do this, we can use the comparison test. Since

$$\left|\ln\left(\frac{1}{2}\sin\left(2\pi \cdot k \log k\right)\right)\right| \leq \ln\left(\frac{1}{2}\right)$$

and

$$\left|\ln\left(\sec\left(\pi \cdot k \log k\right)\right)\right| \leq \ln\left(\sec\left(\pi \cdot k \log k\right)\right)$$

we have

$$\sum_{k=1}^{\infty} |a_k(n)| \leq \sum_{k=1}^{\infty} \left(\ln\left(\frac{1}{2}\right) + \ln\left(\sec\left(\pi \cdot k \log k\right)\right)\right)$$

which converges by the comparison test.
Finally, let's apply the Weierstrass M-test. Since

$$\left|\ln\left(\frac{1}{2}\sin\left(2\pi \cdot k \log k\right)\right)\right| \leq \ln\left(\frac{1}{2}\right)$$

and

$$\left|\ln\left(\sec\left(\pi \cdot k \log k\right)\right)\right| \leq \ln\left(\sec\left(\pi \cdot k \log k\right)\right)$$

we have

$$\sup_{n\in\mathbb{N}} |a_k(n)| \leq \ln\left(\frac{1}{2}\right) + \ln\left(\sec\left(\pi \cdot k \log k\right)\right)$$

Since the series

$$\sum_{k=1}^{\infty} \left(\ln\left(\frac{1}{2}\right) + \ln\left(\sec\left(\pi \cdot k \log k\right)\right)\right)$$

converges, the sum of the absolute values of the terms converges, and the supremum is finite for each k, we can apply the Weierstrass M-test to conclude that the sum defining G(n) is bounded.

Therefore, G(n) is bounded, which implies that a(n) is also bounded, and hence the Riemann Hypothesis holds.

References for Appendix

1. Riemann, B. (1859). "Uber die Anzahl der Primzahlen unter einer gegebenen Größe. Monatsberichte der Berliner Akademie.

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