New improved classes converging towards the generalized Euler-Mascheroni constant

YunJong Kang, HyonChol Kim[1], JinSong Kim, JinSong Yu

Faculty of Mathematics, **Kim Il Sung** University,

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ABSTRACT

In this paper, we provide new quicker sequences convergent to the generalized Euler-Mascheroni constant, which is a generalization of the Euler-Mascheroni constant.

Keywords: Euler-Mascheroni constant**,** Generalized Euler-Mascheroni constant, Rate of convergence, Power series

1. Introduction

Mathematical constants play a key role in several branches of mathematics, as number theory, special functions, analysis or probability. In this theory, an important concern is the definition of new sequences convergent to the mathematical constants with quicker rate of convergence.

The Euler-Mascheroni constant is the most important mathematical constant after π and the Napier constant e, and widely used in mathematics and engineering.

Many mathematicians made great efforts in this area of concerning the properties of generalized Euler-Mascheroni constant. In particular, they considered convergent sequences to the generalized Euler- Mascheroni constant and present the effective methods to estimate their rates of convergence.

For $a>0$, the generalized Euler–Mascheroni constant $\gamma(a)$ is given by

$$\gamma(a)=\lim_{n\to\infty}\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right). \tag{1.1}$$

We can see that the generalized Euler–Mascheroni constant $\gamma(a)$ is the natural generalization of the Euler–Mascheroni constant

$$\gamma=\gamma(1)=\lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)=0.57721566490115328\cdots.$$

Recently, many researchers are preoccupied to improve the rates of convergence of remarkable sequences convergent towards γ.

[1] The corresponding author. Email: HC.Kim@star-co.net.kp

You [1] provided new classes of convergent sequences for the Euler–Mascheroni constant as follows

$$r_m(n) = \sum_{k=1}^{n} \frac{1}{k} - \ln n - \sum_{k=1}^{m} \ln\left(1 + \frac{a_k}{n^k}\right), \tag{1.2}$$

where

$$a_1 = \frac{1}{2}, a_2 = \frac{1}{24}, a_3 = -\frac{1}{24}, a_4 = \frac{143}{5760}, a_5 = -\frac{1}{160}, a_6 = -\frac{151}{290304}, a_7 = -\frac{1}{896}, \cdots.$$

In [2-4], some convergent sequences towards the generalized Euler-Mascheroni constant was introduced.

$$\alpha_n(a) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-2} + \frac{1}{2(a+n-1)} - \ln\left(\frac{a+n-1}{a}\right) \tag{1.3}$$

$$\beta_n(a) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1/2}{a}\right) \tag{1.4}$$

$$\lambda_n(a) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{2(a+n-1)} - \ln\left(\frac{a+n-1/2}{a} - \frac{1}{24(a+n-1)}\right) \tag{1.5}$$

$$\eta_n(a) = \gamma_n(a) - \frac{1}{2(a+n-1)} + \frac{1}{12(a+n-1)^2} - \frac{1}{120(a+n-1)^4} \tag{1.6}$$

In this paper, we provide new classes of sequence convergent to the generalized Euler-Mascheroni constant.

2. Approximations for the generalized Euler-Mascheroni constant

For our consideration, the following lemma is necessary.

Lemma([5, 6]). If $(x_n)_{n\geq 1}$ is convergent to zero and there exists the limit

$$\lim_{n\to\infty} n^s (x_n - x_{n+1}) = L \in [-\infty, +\infty] \tag{2.1}$$

with $s > 1$, then

$$\lim_{n\to\infty} n^{s-1} x_n = \frac{L}{s-1}. \tag{2.2}$$

Using **Lemma**, we can see that the rate of convergence of the sequence $(x_n)_{n\geq 1}$ increases together with the value s satisfying (2.1).

We give new classes of sequence convergent for the generalized Euler-Mascheroni constant.

Our aim is to find the values of the parameters such that new sequence is the fastest sequence which would approximate for the generalized Euler-Mascheroni constant.

Theorem. For the generalized Euler-Mascheroni constant, we have the following convergent sequence,

$$\gamma_n^p(a) = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-1} - \ln\frac{a+n-1}{a} - \ln\left(1 + \frac{B_1}{a+n-1}\right) - \ln\left(1 + \frac{B_2}{(a+n-1)n}\right) - \cdots - \ln\left(1 + \frac{B_p}{(a+n-1)n^{p-1}}\right), \tag{2.3}$$

where

$$B_1 = \frac{1}{2}\ , B_2 = \frac{1}{24}\ ,\ B_3 = -\frac{a}{24}\ ,\cdots. \tag{2.4}$$

Furthermore, for any natural number p, we have

$$\lim_{n\to\infty} n^{p+1}(\gamma_n^p(a) - \gamma(a)) = C_p, \tag{2.5}$$

where

$$C_1 = \frac{1}{24}\ ,\ C_2 = -\frac{a}{24}\ ,\cdots.$$

Also we have $B_{p+1} = C_p$.

Proof. We need to give the value $B_1, B_2, \cdots, B_p \in (-\infty, +\infty)$ that produces the best approximation of (2.4).

The method to measure the accuracy of approximation (2.3) is to say that an approximation is better if $\gamma_n^p(a) - \gamma(a)$ converges to zero quicker. From (2.3), we have

$$\begin{aligned}\gamma_n^p(a) - \gamma_{n+1}^p(a) &= -\frac{1}{a+n} - \ln\left(1 - \frac{1}{a+n}\right) \\ &\quad + \sum_{i=0}^{p-1} \ln\left(1 + \frac{B_{i+1}}{(a+n)(n+1)^i}\right) - \sum_{i=0}^{p-1} \ln\left(1 + \frac{B_{i+1}}{(a+n-1)n^i}\right)\end{aligned} \tag{2.6}$$

and develop the power series in $1/n$. To obtain the power series of (2.6), we compute respectively as follows:

$$\begin{aligned}\ln\left(1 - \frac{1}{n+a}\right) + \frac{1}{n+a} &= -\sum_{i=2}^{\infty} \frac{1}{i}\left(\frac{1}{n}\right)^i \sum_{j=0}^{\infty} (-1)^j C_{i+j-1}^j \left(\frac{a}{n}\right)^j \\ &= -\sum_{k=2}^{\infty} \left(\frac{1}{n}\right)^k \sum_{i=2}^{k} \frac{1}{i} C_{k-1}^{k-i} (-a)^{k-i} \\ &= -\sum_{k=2}^{\infty} \frac{1}{k}\left(\frac{1}{n}\right)^k \sum_{i=2}^{k} C_k^i (-a)^{k-i},\end{aligned} \tag{2.7}$$

$$\begin{aligned}\ln\left(1 + \frac{B_{i+1}}{(a+n-1)n^i}\right) &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} B_{i+1}^j \left(\frac{1}{n}\right)^{j(i+1)} \sum_{m=0}^{\infty} (-1)^m C_{j+m-1}^m \left(\frac{a-1}{n}\right)^m \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{n}\right)^k \sum_{j=1}^{\left[\frac{k}{i+1}\right]} \frac{(-1)^{j+1}}{j} B_{i+1}^j C_{k-i\cdot j-1}^{k-(i+1)j} (1-a)^{k-(i+1)j},\end{aligned} \tag{2.8}$$

and

$$\left(\frac{1}{n}\right)^k - \left(\frac{1}{n+1}\right)^k = \left(\frac{1}{n}\right)^k \sum_{i=1}^{\infty} (-1)^{i+1} C_{k+i-1}^i \left(\frac{1}{n}\right)^i. \tag{2.9}$$

Substituting (2.7)-(2.9) into (2.6), we have

$$\gamma_n^p(a) - \gamma_{n+1}^p(a) = \sum_{k=2}^{p+2} Q(k)\left(\frac{1}{n}\right)^k + O\left(\frac{1}{n^{p+2}}\right), \tag{2.10}$$

where

$$Q(k)=\frac{1}{k}\sum_{j=0}^{k-2}C_k^j(-a)^j+\sum_{j=1}^{k-1}(-1)^{k-j}C_{k-1}^{j-1}\sum_{i=0}^{j-1}\sum_{m=1}^{\left[\frac{j}{i+1}\right]}\frac{(-1)^{m+1}B_{i+1}^m}{m}C_{j-mi-1}^{j-(i+1)m}(1-a)^{j-(i+1)m}. \qquad (2.11)$$

Using (2.11), we have

$$\begin{aligned}
Q(2)&=-B_1+\frac{1}{2},\\
Q(3)&=\frac{1}{3}(1-3a)-B_1(1-2a)+B_1^2-2B_2,\\
Q(4)&=\frac{1}{4}(6a^2-4a+1)-B_1^{\,3}-3\left(a-\frac{1}{2}\right)B_1^{\,2}-(3a^2-3a+1)B_1+3aB_2-3B_3,
\end{aligned} \qquad (2.12)$$

… … …

From **Lemma**, we know that the rates of convergence of the sequence $(\gamma_n^p(a))_{n\ge1}$ is even higher when the value s ($s\le p+1$) satisfies (2.1). Thus, combining **Lemma** and (2.12), we have that

(i) if $B_1\ne\frac{1}{2}$, then the rate of convergence of the $(\gamma_n^p(a)-\gamma(a))_{n\ge1}$ is n^{-1}, since

$$\lim_{n\to\infty}n^2(\gamma_n^p-\gamma_{n+1}^p)=-B_1+\frac{1}{2},$$

(ii) if $B_1=\frac{1}{2}, B_2\ne\frac{1}{12}$, then the rate of convergence of the $(\gamma_n^p(a)-\gamma(a))_{n\ge1}$ is n^{-2}, since

$$\lim_{n\to\infty}n^3(\gamma_n^p-\gamma_{n+1}^p)=-2B_2+\frac{1}{12},$$

(iii) if $B_1=\frac{1}{2}, B_2=\frac{1}{24}, B_3\ne-\frac{a}{24}$, then the rate of convergence of the $(\gamma_n^p(a)-\gamma(a))_{n\ge1}$ is n^{-3}, since

$$\lim_{n\to\infty}n^4(\gamma_n^p-\gamma_{n+1}^p)=-3B_3-\frac{a}{8},$$

… …

We continue our approach to determine the coefficients $B_1,B_2,\cdots,B_p\in(-\infty,+\infty)$.

In the case of $k>3$, $Q(k)$ consists of the linear combination of $B_i, i=\overline{1,k-1}$.

So only when $j=k-1$, $i=j-1$, does the term of $-(k-1)B_{k-1}$ remain and $m=1$.

Now, let

$$E_k=Q(k)+(k-1)B_{k-1},$$

and then it consists of the linear combination of $B_i, i=\overline{1,k-2}$.

If $(k-1)B_{k-1}\ne E_k$, then rates of convergence is $n^{-(k-1)}$ and, if $(k-1)B_{k-1}=E_k$, then rates of convergence is at least n^{-k}. So, by computing B_i ($i=\overline{1,p}$) according to $(k-1)B_{k-1}=E_k$, we can obtain sequences with the rates of convergence of $n^{-(p+1)}$.

In other words, let

$$\gamma_n^p(a)-\gamma_{n+1}^p(a)=\frac{(p+1)C_p}{n^{p+2}}+O\left(\frac{1}{n^{p+3}}\right),$$

and

$$\lim_{n\to\infty} n^{p+1}(\gamma(a)-\gamma_n^p(a))=C_p.$$

We can obtain

$$\gamma_n^{p+1}(a)-\gamma_{n+1}^{p+1}(a)=\gamma_n^p(a)-\gamma_{n+1}^p(a)+\ln\left(1+\frac{B_{p+1}}{(a+n)(n+1)^p}\right)-\ln\left(1+\frac{B_{p+1}}{(a+n-1)n^p}\right)$$

$$=\left((p+1)C_p-(p+1)C_{p+1}\right)\frac{1}{n^{p+2}}+O\left(\frac{1}{n^{p+3}}\right).$$

Then, we have $B_{p+1}=C_p$.

The proof of **Theorem** is completed. □

Remark. (2.11) gives the coefficients which offer the best approximations of $\gamma(a)$.

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