Any Formal System That Contains Sets Arithmetic and Rational Numbers is Inconsistent

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Abstract: Gödel proved that any formal system containing arithmetic is incomplete. We show that any such formal system is inconsistent. We establish a collection of nested sets of rational numbers in a descending hierarchy. The sets higher in the descending hierarchy contain element(s) that are not in the sets below them in the hierarchy. Given such a descending set hierarchy, it is easy to develop two arguments that contradict each other. The conclusion of Argument#2 is false. But, Argument#2 is a valid argument.

For rational numbers *a* in [*0, 1*] let the collection of R*a* sets be { *y* is a rational number | $0 \leq y < a$ }

Argument #1: No R***a*** **contains a largest element.**

1) Suppose there is a largest element a' in some individual R***a***.

2) $a' < (a' + a)/2 < a$.

3) Let $b = (a' + a)/2$.

4) Then b is in R***a*** and $a' < b$.

5) Therefore, **no** R***a*** **contains a largest element.**

When a largest element is assumed in Argument#1, it leads to a contradiction; so there is no largest element. Every R***a*** set element is in one of the proper subsets below R***a*** in the set hierarchy. It is a valid proof by contradiction.

Argument #2: Each R***a*** **contains a largest element.**

1) Below each R***a*** for all rationals $x < a$ is a collection of R***x*** subsets{ y is a rational number | $0 \leq y < x$ }.

2) Each R***a*** and its collection of R***x*** subsets comprise a descending set hierarchy.

3) Each R***x*** is missing its index ***"x".*** R***a*** contains all the ***"x"*** indices.

4) Since the union of the collection of R***x*** sets does not contain any element greater than the elements in all the individual R***x*** sets, the union of the collection of R***x*** sets does not equal R***a***.

5) There exists at least one R*a* set element *s* ≥ (all values of) ***x*.**

6) Let ***c*** and ***d*** be two elements of a single R***a*** set with ***c > d***.

7) ***d*** is an element of R***c***, which is a proper subset of R***a***.

8) For any two elements in R***a*** the smaller element is contained in a R***x*** subset of R***a.***

9) By steps 6) 7) and 8), **there is at most one R*a* set element** missing from all the R***x*** subsets.

10) By steps 5) 9), **each R*a* set contains a largest element *a′*** not in a R***x*** set below in the hierarchy.

11) There is no $b = (a' + a)/2$. It would be a second element not in a R***x*** set below **R*a*** in the hierarchy. We know by step 8) that isn't possible.

Argument #1 is generally considered correct and its conclusion is true. The first three statements of **Argument #2** are generally uncontested. The first part of **Statement #4** stating that the union of the collection of **R*x*** sets doesn't contain any element greater than the elements in the individual **R*x*** sets is not an issue. It is the latter part of **Statement *#4*** that states "the union of the collection of **R*x*** sets does not equal **R*a*"** that is a false statement. This causes most people to dismiss **Argument #2.**

Containing a false statement does not keep the second part **Statement #4** from being a valid logical deduction from **Statement #3** and the first part of **Statement #4**, which are true. It simply means that in **all formal systems containing sets, arithmetic, and rational numbers** this false statement can be deduced and such formal systems are therefore inconsistent. Likewise, false statements **#5, #9, #10,** and **#11** are valid logical deductions from previous statements.