Bertrand-Chebyshev Theorem and the Sum of Primes

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Abstract
It is deduce from Bertrand-Chebyshev theorem that every even integer *2n* greater than four can be written as the sum of two primes. Thus, Goldbach conjecture is true.

Keywords Bertrand-Chebyshev theorem, Goldbach Conjecture.

MSC2020 11P32, 11A41

1 Introduction

One of the most famous problems in mathematics is Goldbach conjecture. The best progress occurred in 1973, when Jingrun Chen proved that every sufficiently large even number is the sum of a prime and a number with at most two prime factors [1]. Unfortunately, there is a key limitations inherent in standard sieve methods. The conjecture remains unsolved and very significant new ideas are required for the proof.

In fact, the solution lies in Bertrand-Chebyshev theorem.

2 The Sum of Primes

Bertrand-Chebyshev theorem *For all integers $n \geq 2$, there is always at least one prime P such that $n < P < 2n$*

For all integers n$\geq$ 3, we construct the following interval:

$$(n,\ 2n)$$

In this interval, there is at least one prime, and let P_n denote the prime.

We construct the following interval:

$$(0,\quad 2n - P_n + 2)$$

Let $P_n^{'}$ denote the largest prime in this interval. Then, by Bertrand-Chebyshev theorem,

$$\frac{(2n - P_n + 2)}{2} < P_n^{'} < (2n - P_n + 2)$$

namely,

$$P_n^{'} \leq (2n - P_n)$$

There are two possibilities:

(1) $$P_n^{'} = (2n - P_n)$$

(2) $$P_n^{'} < (2n - P_n)$$

If it is case (1), then

$$2n = P_n + P_n^{'}$$

It indicates that, for any integer n ≥ 3, even integer $2n$ can be written as the sum of two primes. Thus, Goldbach conjecture is true.

If it is case (2), we go on to construct the following interval:

$$(0, \quad 2n - P_n + 4 \ )$$

By Bertrand-Chebyshev theorem,

$$\frac{(2n - P_n + 4)}{2} < P_n^{'} < (2n - P_n + 4)$$

namely,

$$P_n^{'} \leq (2n - P_n + 2)$$

There are two possibilities:

(1) $$P_n^{'} = (2n - P_n + 2)$$

(2) $$P_n^{'} < (2n - P_n)$$

If it is case (1), then

$$2(n + 1) = P_n + P_n^{'}$$

It indicates that, for any integer n ≥ 3, even integer $2(n + 1)$ can be written as the sum of two primes. Thus, Goldbach conjecture is true.

If it is case (2), we go on to construct the following interval:

$$(0, \quad 2n - P_n + 6 \ )$$

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$$(0, \quad 2n - P_n + 2k \ )$$

It can be proved that: $2(n + k) = P_n + P_n^{'}$, where $1 \leq$ integer $k \leq \frac{(P_n - n)}{2}$. When $k = \frac{(P_n - n)}{2}$, $n = P_n = P_n^{'}$.

Similarly, it can be proved that: $2(n - k)$ is the sum of two primes, where $1 \leq$ integer $k \leq \frac{(n - P_n)}{2}$. Thus, Goldbach conjecture is true.

3 Conclusion

Wir müssen wissen
Wir werden wissen *(D. Hilbert)*

In 1900, D. Hilbert listed Goldbach conjecture as one of the 8th mathematical problems at the International Mathematical Conference. Today, it is proved with Bertrand-Chebyshev theorem.

References

[1] J. CHEN, On the representation of a large even integer as the sum of a prime and the product of at most two primes. *Science in China, Series A.* 2(1973) 111-128