Dimensional Regularization and Fractal Spacetime

Ervin Goldfain

Ronin Institute, Montclair, New Jersey 07043, USA

E-mail ervin.goldfain@ronininstitute.org

Abstract

This report is about dimensional regularization and fractal spacetime.

Key words: Dimensional Regularization, Renormalization Group, scalar field theory, fractal spacetime, mass generating mechanism, Higgs mechanism, Beyond the Physics of the Standard Model.

1. Introduction and Motivation

Quantum Field Theory (QFT) lies at the foundation of the Standard Model for particle physics (SM) and is built in compliance with several postulates

Note: This report expands https://doi.org/10.32388/DW6ZZS

called *consistency conditions*. The remarkable success of SM stems from a unitary, local, renormalizable, gauge invariant and anomaly-free formulation of QFT. Notwithstanding this impressive achievement, many nagging puzzles confronting SM still exist and continue to defy explanation. Over the years, particle theory has seen a steady overflow of proposals targeting physics beyond the Standard Model (BSM). The bulk of these proposals postulate new objects (elementary fields or bound states) or hidden symmetries that allegedly break down somewhere above the SM scale. Unfortunately, the majority of BSM scenarios resolve some unsatisfactory aspects of the theory while introducing new unknowns.

What mainstream research seems to be overlooking is that there are compelling arguments for the onset of *non-integrability* in the high-energy sector of field theory. This globally unstable setting prevents thermalization of quantum fluctuations and favors the onset of *chaotic dynamics* and *fractal spacetime.* The underlying principles of classical statistical physics and perturbative QFT are likely to break down in this far-of-equilibrium regime.

In particular, the ergodic theorem, the fluctuation-dissipation theorem, analyticity, unitarity, locality, finiteness in all orders of perturbation theory and renormalizability are either violated or lose their conventional meaning [4-13].

It was conjectured in [1-5] that the transition from perturbative QFT to chaotic dynamics occurs via a spacetime having arbitrarily small and scale-dependent deviations from four space dimensions, dubbed *minimal fractal manifold* (MFM). The expectation is that the MFM becomes increasingly relevant in far-from-equilibrium and nonintegrable conditions, prone to develop far above the Fermi scale. Based on these considerations, ref. [1] argues that the technique of Dimensional Regularization (DR) of QFT provides an alternative mechanism for mass generation in particle physics. This mechanism can potentially reconcile the Higgs model of electroweak symmetry breaking with the minimal fractal topology of spacetime above the Fermi scale. The goal of this work is to bring additional clarifications in support of [1].

The paper is organized as follows: the necessary background is introduced in the first section. The second and third sections cover momentum integration in non-integer dimensions and the emergence of non-trivial fixed points in Statistical Physics.

Being aware of the controversial nature of our approach, we encourage the reader to keep in mind that these ideas are in their infancy. Readers unfamiliar with the topic are urged to carefully study the references prior to drawing premature conclusions.

2. Theoretical background

Many Feynman diagrams of Quantum Field Theory (QFT) are plagued by *ultraviolet (UV) divergences* – the integrals over loop momenta k^{μ} diverge for $k^{\mu} \to \infty$. The goal of the Regularization program is to suppress the UV regime of high momenta by decoupling the divergences from the observable physics at low energies ($E << \Lambda_{UV}$). By contrast, the Renormalization Group (RG) is based on a conceptually different program, which consists of absorbing the UV momenta through an iterative process of *coarse graining* and *parameter rescaling*. The endpoint of either Regularization or RG programs is an "*effective field theory*" whereby low energy physics (called infrared or IR physics) is completely shielded away from the effects produced by the UV regime.

Tab. 1 of [1] summarizes the divergent parts of the 4-point and 2-point functions of scalar field theory following the Pauli-Villars (PV) and Dimensional Regularization (DR) methods, respectively.

	Pauli-Villars Regularization	Dimensional Regularization
$\Gamma(0)$	$\frac{ig^2}{32\pi^2}\ln(\Lambda_{UV}^2/m^2)$	$\frac{ig^2}{32\pi^2}(2/\varepsilon)$
$\Sigma(0)$	$\frac{g}{32\pi^2}\Lambda_{UV}^2$	$\frac{g}{32\pi^2}(-2m^2/\varepsilon)$

Tab. 1: Pauli-Villars versus Dimensional Regularization of φ^4 theory

Side-by-side evaluation of entries in Tab. 1 hints that, in the limit $m << \Lambda_{UV}$,

$$\boxed{\varepsilon(\mu) = 4 - d(\mu) = O(m^2/\Lambda_{UV}^2) << 1} \tag{1}$$

in which μ is the running scale and Λ_{UV} the ultraviolet cutoff of the theory.

The object of this report is to show that (1) is also supported by a couple of observations pertaining to both Regularization and RG programs, namely,

1) evaluation of Euclidean momentum integrals in non-integer dimensions,

2) existence of nontrivial fixed points in Statistical Physics.

2.1) Momentum integration in non-integer dimensions

Dimensional Regularization (DR) involves analytic continuation of the Euclidean momentum integrals to non-integer dimensions $d<4$, which renders the integrals finite, followed by taking the limit $d \to 4$. Momentum integrals assume the form,

$$\int_{reg} \frac{d^4 k_E}{(2\pi)^4} f(k_E) = \int \frac{\mu^{4-d} d^d k_E}{(2\pi)^d} f(k_E) \tag{2}$$

in which μ is the reference energy scale at which the spherical shell in momentum space dk^{rad} has the same volume in d dimensions as in 4 dimensions [14-15]. At large loop momenta $k >> 1$, the shell volume becomes smaller in $d < 4$ dimensions than in 4 dimensions, namely,

$$d^4 k_E \sim (k^{rad})^3 dk^{rad} \to \mu^{4-d} (k^{rad})^{d-1} dk^{rad} = (\frac{\mu}{k^{rad}})^{4-d} (k^{rad})^3 dk^{rad} \tag{3}$$

in which the UV regulating factor (highlighted in red) drops the shell volume in $d < 4$ dimensions according to

$$(\frac{\mu}{k^{rad}})^{4-d} (k^{rad})^3 dk^{rad} << (k^{rad})^3 dk^{rad} \tag{4}$$

Setting $d = 4 - 2\varepsilon$ leads to

$$(\frac{\mu}{k^{rad}})^{2\varepsilon} = (\frac{k^2}{\mu^2})^{-\varepsilon} = \exp(-\varepsilon \log \frac{k^2}{\mu^2}) \tag{5}$$

It is apparent that the UV regulating factor becomes arbitrarily small when,

$$\log \frac{k^2}{\mu^2} \sim \frac{1}{\varepsilon} \to k^2 \sim \mu^2 \exp(1/\varepsilon) \tag{6}$$

from which one concludes that the UV cutoff scale in dimensional regularization is given by

$$\boxed{(\Lambda_{UV}^{DR})^2 = \mu^2 \exp(1/\varepsilon) >> \mu^2} \tag{7}$$

Similar considerations lead to the following relationship between the UV cutoff scale of the DR and PV regularization methods,

$$(\Lambda_{UV}^{DR})^2 = \frac{\exp(\gamma_E)}{4\pi} (\Lambda_{UV}^{PV})^2 \tag{8}$$

in which γ_E represents the Euler-Mascheroni constant.

For all practical purposes, μ is set to match the characteristic mass scale of the theory [14-15], $\mu = O(m)$. A convenient (although coarse) approximation of (7) is obtained in the far limit $m^2 << \Lambda_{UV}^2$. *Under these conditions, it is seen that (1) emerges as an approximation of (7).*

There are two distinct interpretations of (1) or (7):

a) If masses are nonvanishing $(m \neq 0)$ and the cutoffs are set to infinity $(\Lambda_{UV}^{DR} \to \infty;\ \Lambda_{UV}^{PR} \to \infty)$, (1) or (7) indicate that the spacetime fractality described by ε vanishes away in the continuum spacetime limit of classical spacetime.

b) If the cutoffs are finite and on the same order of magnitude as the Planck scale $[\Lambda_{UV} = O(M_{Pl})]$, vanishing fractality ($\varepsilon = 0$) implies vanishing masses ($m = 0$), in line with the basis of *conformal field theory*. From a Statistical Physics viewpoint, conformality is akin to the onset of criticality, whereby vanishing masses correspond to diverging correlation lengths [$\xi = O(m^{-1})$]. *In this interpretation, mass develops from the fractality of spacetime in the UV sector of field theory as deviations from ideal conformal behavior*.

A key observation is now in order: as alluded to in the Introduction, relations (1) and (7) reflect a dynamic regime that is *non-integrable* and falls entirely outside the boundaries of QFT. Indeed, DR cannot be extrapolated beyond perturbation theory, as fractal spacetime leads to a manifest violation of all consistency requirements mandated by QFT [2, 16]. *In line* *with the Decoupling Theorem of QFT, the non-integrable regime of (1) or (7) asymptotically match QFT in the conventional limit* $\varepsilon \to 0,\ \Lambda_{UV} \to \infty$.

Ref. [1, 17-18] indicate that the mass and coupling generation mechanism embodied in (1) or (7) are compatible with the standard Higgs mechanism of electroweak symmetry breaking. Taking complex-scalar field theory as baseline model, refs. [17-18] point out that the SM symmetry group unfolds sequentially from bifurcations driven by the running scale μ .

It is also instructive to emphasize that the UV/IR mixing generated by (1) or (7) echoes the bounds derived from non-commutative field theory and quantum gravity, see e.g. [19-22].

<u>2.2) The Wilson-Fisher point of Statistical Physics</u>

Consider the Higgs potential of field theory written as,

$$V(\varphi) = \lambda(|\varphi|^2 - \frac{1}{2}\mathrm{v}^2)^2 \qquad (9)$$

where v stands for the vacuum expectation value of the Higgs boson and φ is considered a real scalar field for simplicity. (9) can be cast in the form

$$V_H(\varphi) = V(\varphi) - \frac{1}{4}\lambda v^4 = -\lambda v^2 \varphi^2 + \lambda \varphi^4 \tag{10a}$$

(10) can be associated with the partition function of Statistical Physics based upon the functional integral [see e.g., 23]

$$Z[j] = \int D\varphi \exp(-S[\varphi]) \tag{10b}$$

where the action has the form,

$$S[\varphi] = \frac{1}{2}\int d\vec{x}\,\varphi(\vec{x})[r - \nabla^2]\,\varphi(\vec{x}) + \frac{1}{4}u\int d\vec{x}\,\varphi^4(\vec{x}) - \int d\vec{x}\, j(\vec{x})\varphi(\vec{x}) \tag{11}$$

Here, $j(\vec{x})$ plays the role of an external current and the coefficient r has the dimensions of [mass] ^ 2. Side by side comparison of (10a) and (11) leads to the identification,

$$r = m^2 \Leftrightarrow 2\lambda v^2 \tag{12}$$

$$\lambda \Leftrightarrow -\frac{1}{4}u \tag{13}$$

The RG analysis of (11) stars by splitting the field into its long and short wavelengths components according to [23]

$$\varphi(\vec{k}) = \varphi_{long}(\vec{k}) + \varphi_{short}(\vec{k}) \tag{14}$$

$$\varphi_{long}(\vec{k}) = \begin{cases} \varphi(\vec{k}), \; 0 < k < \Lambda/b \\ 0, \quad \Lambda/b < k < \Lambda \end{cases} \tag{15}$$

$$\varphi_{short}(\vec{k}) = \begin{cases} 0 \;, \; 0 < k < \Lambda/b \\ \varphi(\vec{k}), \quad \Lambda/b < k < \Lambda \end{cases} \tag{16}$$

where b is a scaling factor and $\Lambda < \Lambda_{UV}$ denotes the upper energy scale of RG calculations. The RG flow of parameters (r, λ, j) is described by the so-called β-functions of the theory. To compute the β-functions, one considers an infinitesimal momentum shell integration defined by

$$b = \exp(\delta l) \approx 1 + \delta l \tag{17}$$

and the RG flow equations in near 4-dimensional spacetime read,

$$\beta_r = \frac{dr}{dl} = 2r + 3K_4\Lambda^4 \frac{u}{r+\Lambda^2} + O(u^2) \tag{18}$$

$$\beta_u = \frac{du}{dl} = \varepsilon u - 9K_4\Lambda^4 \frac{u^2}{(r+\Lambda^2)^2} + O(u^2) \tag{19}$$

in which $K_4 = (8\pi^2)^{-1}$[24]. Equations (18) and (19) have a trivial (Gaussian) fixed point solution defined as

$$\beta_r = \beta_u = 0 \Rightarrow r^* = u^* = 0 \tag{20}$$

To analyze the behavior of RG flows near (20), one proceeds by linearizing (18) and (19) and solving the corresponding eigenvalue equation. The pair of solutions satisfying the eigenvalue equation is given by,

$$\lambda_1^a = 2 > 0 \tag{21}$$

$$\lambda_2^a = \varepsilon = 4 - d \tag{22}$$

It is seen that (22) is positive (relevant) for $d < 4$ but negative (irrelevant) for $d > 4$. Since the Gaussian fixed point (20) corresponds to a vanishing quartic coefficient (12)-(13), it follows from this analysis that Higgs sector is *unstable* in less than $d=4$ dimensions but turns *stable* in $d>4$ dimensions. A non-trivial fixed-point (called the Wilson-Fisher or WF point) of RG equations (18) - (19) emerges if one considers the small dimensional deviation $\varepsilon=4-d<<1$ as a *tunable parameter*. Expanding the RG equations to quadratic order yields

$$\frac{dr}{dl}\approx 2r+au-bur \tag{23}$$

$$\frac{du}{dl}\approx \varepsilon u-3bu^2 \tag{24}$$

with $a=3K_4\Lambda^2$ and $b=3K_4$. The WF point derived from (23) - (24) is located at,

$$\boxed{u^*=-4\lambda^*=\frac{1}{3b}\varepsilon} \tag{25}$$

$$\boxed{r^*=(m^*)^2=-\frac{a}{6b}\varepsilon} \tag{26}$$

The pair of eigenvalues associated with the WF point are found to be,

$$\lambda_1^{WF} = 2 - \frac{\varepsilon}{3} > 0 \tag{27}$$

$$\lambda_2^{WF} = -\varepsilon < 0 \tag{28}$$

which makes the WF point *stable* in less than four dimensions ($d < 4$). The flows corresponding to the Gaussian and WF fixed points are displayed in Figs. 1 - 2 below. The key point of this analysis is that, according to (25) – (26), both mass ^ 2 term and coupling parameter of scalar field *arise from the continuous and nonvanishing dimensional deviation* ε. In this sense, it is apparent that (25) - (26) are a replica of (1).

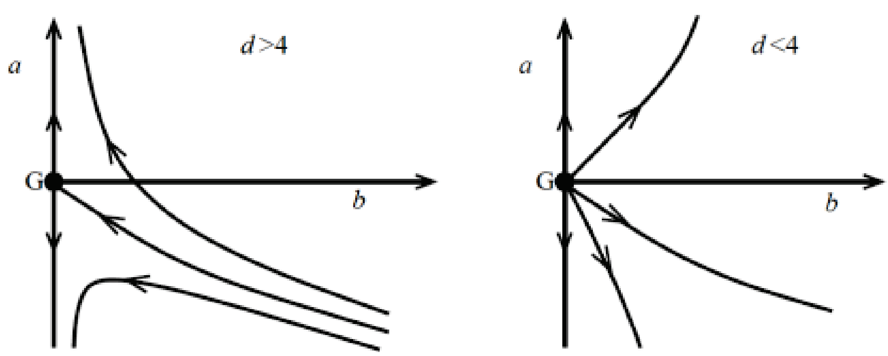

Fig.1: The Gaussian fixed point of scalar field theory

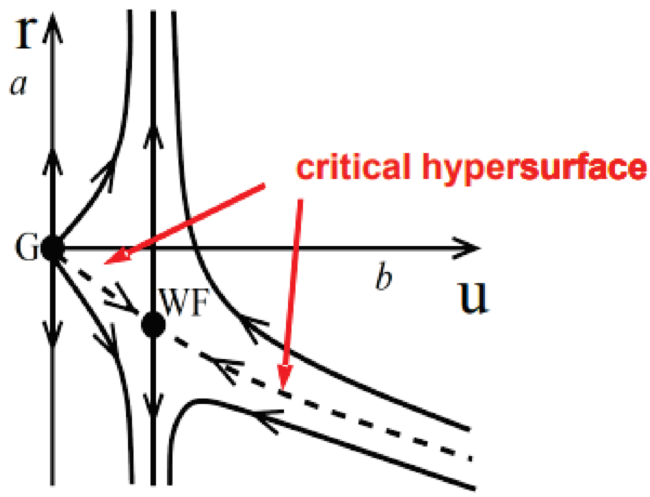

Fig. **2**: The Wilson-Fisher fixed point of scalar field theory

We close our report with several observations which may be relevant for the further development of ideas detailed here,

a) Ref. [25] asserts that SM represents a *self-contained multifractal set*, whose flavor and mass composition follows the bifurcation scenario of transition to chaos.

b) Refs. [26, 28-30] list some of the early contributions devoted to the study of fractal spacetime in foundational physics.

c) Ref. [27] explores the potential benefit of fractal spacetime in explaining some of the challenges of the Higgs sector (gauge hierarchy problem, triviality problem, and tachyonic mass term).

References

1. https://doi.org/10.32388/DW6ZZS

2. Goldfain E., "Introduction to Fractional Field Theory", Aracne Editrice, 2015, ISBN-10: 8854889970.

3. Goldfain E., "Fractional Field Theory and High-Energy Physics: New Developments", International Journal of Theoretical Physics, Group Theory, and Nonlinear Optics; Vol. 16, Issue 3/4, 2012, pp. 239-262.

4. https://www.researchgate.net/publication/369584882

5. https://www.researchgate.net/publication/369814408

6. Nicolis, G., “Physics of far-from-equilibrium systems and self-organization”, in “New Physics” pp. 316-347, Ed. Paul Davies, Cambridge Univ. Press, 1992.

7. Gutzwiller M. C., Chaos in Classical and Quantum Mechanics, Springer-Verlag, 1990.

8. Zaslavsky G. M., Hamiltonian Chaos and Fractional Dynamics, Oxford Univ. Press, 2006.

9. https://arxiv.org/pdf/cond-mat/0001198.pdf

10. https://arxiv.org/pdf/2202.09846.pdf

11. https://www.researchgate.net/publication/361644297

12. https://www.researchgate.net/publication/349476440

13. https://www.researchgate.net/publication/366580553

14. https://web2.ph.utexas.edu/~vadim/Classes/2022f/DR.pdf

15. https://web2.ph.utexas.edu/~vadim/Classes/2022f/cutoffs.pdf

16. Duncan A., The Conceptual Framework of Quantum Field Theory, Oxford Univ. Press, 2012.

17. https://www.researchgate.net/publication/357093456

18. https://www.researchgate.net/publication/357093467

19. https://arxiv.org/abs/2112.10796

20. https://arxiv.org/abs/hep-th/9912072

21. https://arxiv.org/pdf/1909.01365.pdf

22. https://arxiv.org/pdf/2202.06890.pdf

23. http://courses.physics.fsu.edu/~phasetrans/4-epsilon.pdf

24. Kaku M., Quantum Field Theory: A Modern Introduction, Oxford Univ. Press, 1993.

25. https://www.researchgate.net/publication/252594638

26. https://www.researchgate.net/publication/266393623

27. https://www.researchgate.net/publication/364294648

28. https://arxiv.org/pdf/0806.2675.pdf

29. http://tph.tuwien.ac.at/~svozil/publ/1987-dimensionalregularisation.pdf

30. https://arxiv.org/pdf/1805.08566.pdf