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Numeric Representation and the Boundary of Pure and Applied Mathematics

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Abstract:
Being so basic to mathematics, numeric representational systems touch many areas of the discipline. Numeric systems could be said to lie on the boundary of pure and applied mathematics, providing the means to apply mathematics to the real world and to define how we manipulate numbers.
The numeric system we use today is the decimal numeric system, which is a key system underlying current science and technology. This system is over one thousand years old, predating much of our science and technology. It can be argued that this mathematical tool, missing for the ancients, prevented them from attaining technology close to ours.
In an analogous manner, discovering an expanded, more powerful numeric system would affect both sides of the pure/applied boundary. Being so basic to measurements, expanding the power of our numeric system could expand what we are able to measure, and hence provide a quantum leap in many areas of science. Being so basic to the concept of number, the expansion could provide new areas for theoretic mathematical investigation, potentially expanding what we think of as a number.
This talk will consider characteristics of numeric representation and the potential for expanding our current systems, considering directions to work on and where this might lead.

Discussion:
I will suggest there is a space on the boundary between Pure and Applied Mathematics, which might be an area to advance mathematics. This space involves the representation of number values and the ability to represent systems of numbers – in particular complex numbers.

There is an intriguing interplay between numbers and their representations. We have numbers and we have representations of numbers – which I shall call 'numerals'. It is simple to state that the representation of a number is not the same as that number. However, the two are intimately connected and we could argue that only by some representation of a number (even if verbally or by show of fingers) could there be the concept of number in the first place. Early humans had marks on a stick or on a papyrus pad. Romans invented Roman numerals, and Greeks had ratios. All of these involved representations for numbers in order for them to be utilized and communicated. So a simple conclusion is that numerals are critical to the use of numbers. Since this session is on Mathematical Notation, we can state that: mathematical notation, in the form of numerals, affects the development of mathematical concepts at its' most basic level. End of question on this topic.

Simple answers aside, we do need to understand the relationship of a numeral system to the theoretical number system it represents. In particular, we should consider if and how representations of numbers impact mathematical thinking today.

While simple numerals, primarily for small counting numbers, have unique representations for each, fractions are not the same as there can be an infinite number of ratios, and therefore of

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numerals, to represent the same Rational number (eg 1/3, 2/6, 3/9,...). Even with decimals, we have the situation of 1.000… and 0.999… needing to be considered the same number value. Consider that shifting the symbol set from decimal to fractions or changing the base of a positional numeral system can change the characteristics of the numeral representation. Pi, in base pi, is 10.0 – an exact representation without an infinite decimal expansion. This demonstrates an important distinction between the theoretic 'value' of a number and the representation of that number: Characteristics of numerals do not always match those of the numbers they represent.

This distinction can become an issue when developing theorems about number systems based upon numerals. Such an issue would be Cantor's diagonal method, since it uses the rules of numerals (infinite decimal expansions) to attempt to prove something about the numbers those numerals represent – that Real numbers cannot be put 1-1 with Integers. I do not mean to imply anything about Real numbers mapping 1-1 with Integers, I do mean to state that such a proof should be very careful when using properties of decimal numerals in place of properties of Real numbers.

In a sense, Cantor's diagonal method inappropriately mixes pure with applied mathematics. We might think that pure mathematics should be able to step away from the representation of numbers and just deal with Integers, Rationals, Reals, Complex numbers and beyond. However, even pure mathematicians need methods of representing these numbers in order to work with them and calculate with them, thus moving into the applied mathematics realm. Inventing a new numeral system could, therefore, impact both applied and pure mathematics and such a numeral system can be said to lie on the boundary between these realms of mathematics.

Applied mathematics must use numerals to represent measurements and quantities. We cannot manipulate nor measure without numeral systems – so science depends upon our numeral systems in a very foundational way. If we consider that limitations of our representational systems can impact science and technology in such a potentially significant way, the question needs to be asked: Why should we believe the decimal numeral system, or positional numeral systems in general, are the end all of numeral systems?

We cannot use theoretic values for a measurement; and manipulation of measurements has to conform to operations that are manageable by humans or human inventions (such as computers). Although theoretically possible, it can be difficult to multiply by 'pi' if 'pi' is represented by an infinite decimal (or other positional base) expansion.

So, let us consider complex numbers. We have a numeral system for representing complex numbers, which theoretically appears to produce unique values for all complex numbers. Yet there are always two parts to that complex value, one part of which we cannot resolve into a useable numeral value. We found a 'work-around' to representing complex values through defining them as 'x + iy'. However this is not a full representation of a complex value precisely because it includes an undefined term i = sqrt(-1).

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From a practical standpoint, we can use complex numbers for all sorts of calculations, but because we cannot resolve the imaginary part into an actual value, we ‘toss out’ this part when we use complex values for quantities or measurements. This allows many theoretic calculations involving measurements to produce different complex values, yet result in the same real quantifiable value. This is a logical problem for physical theories, since calculations in a theory could produce different complex values, yet the theory would predict all these different values to be considered the same real quantity or measurement. If the scientific philosophy is that all we can know of the physical world is through measurements, and we realize that our numeral systems are not capable of entirely specifying all practical quantities, then we have a direction to look into before any scientific theories can be considered complete.

On the theoretic mathematics side, we have become used to understanding complex numbers as 2-dimensional numbers. This situation appears to have ‘gelled’ into the idea that this is a property of the complex numbers. However, it is really the result of the numeral methods we use to represent complex numbers – with 2-part numerals that involve an ‘always unknown’ value. We are unable to resolve the imaginary part into an actual numeral value and so we leave it apart – unresolved. Why should this be the case – forever?

To resolve this theoretic issue with complex numbers, we need a method of representing them as singular numeral values – as a single representation of a complex number value. This requires a representational solution to the question ‘what is the square root of -2?’. With such a solution we would be able to measure and manipulate this value - as a single value. We need to break it into parts when needed, but there is no reason why we must always represent it by two components - except due to the limitations of our current mathematical tools. Without such a numeral solution, we will continue to work with complex numbers using incomplete numeral representations, will continue to mistake properties of our complex numeral system with the number system being represented, and will leave science without a solution for measuring fully complex quantities.

What if we could find a means of fully representing a value for that pesky 'i' (or of any value that, when squared, produces a negative real number)? This value certainly does not fall into the mathematical notations of today. So maybe mathematics needs to take a new step here. Maybe the 1500-year-old numerals we use today are not sufficient to represent 'modern' complex values. The symbols used to define ‘imaginary’ values could be consolidated with the ‘real’ part of a complex number and be reduced to a single value. What might we find, through the simplification of many equations made complicated due to 2-part complex values?

This representational discovery or invention could open up a new universe of possibilities for mathematics. It might also alter the interpretation of physical equations that ‘toss out’ the imaginary value for quantities and measurements given that we can only use real numeral values. Now we could have a value or measure, that included the imaginary part. Now a complex value could be handled in its complete form, without an infinite number of complex values being reduced or equated to a single real quantity. There would be complex

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measurements, not real measurements + imaginary placeholders – potentially identifying measurements we cannot make today.

Integer numerals can be defined using a basic unit and the reversing operations of addition and subtraction. Rational numerals can be defined using integers plus the reversing operations of multiplication and division. Positional base numerals, like decimals, that can represent Real numbers, add the reversing operations of exponentials and logarithms. Using these three pairs of reversing operations, we are able to represent all Real numbers, even if only theoretically.

Now, what about complex numbers? Maybe we need a fourth pair of reversing operations in order to fully represent, as single values without any unknown placeholder, complex numbers. A possibility would be integration and differentiation added into the definition of a complex numeral value. In order to represent negative square roots, we might need to define an undefined area of mathematics – that of negative bases. As the ability to represent negative base numbers is currently undefined in mathematics, there is a bit of theoretical work to perform here – maybe even a little inventing.

A thought for another time: Is it just a coincidence that our current world view involves three physical dimensions and our representational number system involves three pairs of reversing operations? Might we be limited to measurements in three dimensions due to our numeral representational system?

As Donald Knuth worked on more than 60 years ago, maybe we need to develop – to invent – numerals using negative bases, which can represent negative square roots. Euler's great equation might provide a clue to how to construct complex numerals, using 'e' as a base, which also provides a continuous integration and differentiation component. A complex numeral might involve the positional placement of integrated and/or differentiated 'digits' in some similar way as exponential digits are used for decimal and positional base numerals. It is very possible that such a numeral system may not be representable using traditional paper and pencil methods – requiring the use of computers.

It is not a huge step to consider systems beyond a complex numeral system - beyond where we do not quite see yet. So we may still be in the early stages of understanding the extend of what mathematics can provide and science can utilize. Where mathematics needs to go could be well outside the 'standard model' of current mathematics (with only a real line continuum) - and there might be tremendous dividends for science as well...

I, for one, believe there is much more to Mathematics that we do not even have a clue about. And that our current numeral systems, while very powerful and capable of producing our current tremendous technology, are not close to being the end-all of what can be represented. We have achieved a numeral system involving only 3 reversing pair operations and we are likely a long way from being able to represent all possible numbers or all possible measureable quantities.
Thank you for your time and patience.

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