Euler-gamma function

Edgar Valdebenito

July 1, 2023

abstract

The gamma function was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial to non integer values.

keywords: Gamma function, number pi, integrals

1. Introduction

Historically, the idea of extending the factorial to non-integers was considered by Daniel Bernoulli and Christian Goldbach in the 1720s. It was solved by Leonhard Euler at the end of the same decade. James Stirling, contemporary of Euler, also tried to extend the factorial and came up with the Stirlingg formula, which gives a good approximation of n! but it is not exact. Later on, Carl Gauss, introduced the gamma function for complex numbers using the Pochhammer factorial. In the early 1810s, it was Adrien Legendre who first used the Γ symbol and named the Gamma function.

2. The Gamma Function

Theorem: There exists a unique function Γ on $\mathbb{C}$ such that:

(a) Γ is meromorphic on $\mathbb{C}$

(b) $\forall\, n \in \mathbb{N},\ \ \Gamma(n+1) = n\,!$

(c) $\forall\, s \in \mathbb{C}$ such that $R(s) > 0$

$$\Gamma(s) = \int_0^\infty e^{-x}\, x^{s-1}\, d\,x \tag{1}$$

(d) $\forall\, s \in \mathbb{C} - \{0, -1, -2, -3, \ldots\}$

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+s)\, n\,!} + \int_1^\infty e^{-x}\, x^{s-1}\, d\,x \tag{2}$$

(e) $\forall\, s \in \mathbb{C}$

$$\frac{1}{\Gamma(s)} = s\, e^{\gamma\, s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-x/n} \tag{3}$$

where

$$\gamma = \lim_{n \longrightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) \tag{4}$$

is called the Euler constant

(f) $\forall\, s \in \mathbb{C} - \{0, -1, -2, -3, \ldots\}$

$$\Gamma(s) = \lim_{n \longrightarrow \infty} \frac{n\,!\, n^s}{s\,(s+1)\ldots(s+n)} \tag{5}$$

(g) $\Gamma(s+1) = s\,\Gamma(s)$, $s \neq 0, -1, -2, -3, \ldots$

(h) $\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, $s \neq 0, -1, -2, -3, \ldots$

Proof. See Ref. [B], Andrews et al.

3. Elementary formulas

Entry1.

$$\frac{2\,(\Gamma(1/4))^2}{3\,\sqrt{2-\sqrt{2}}\,\sqrt{\pi}\,3^{1/8}} = \frac{4\,\pi\,\sqrt{\pi}}{3\,\sqrt{2-\sqrt{2}}\,3^{1/8}\,(\Gamma(3/4))^2} \tag{6}$$

Entry 2.

$$\frac{2\,(\Gamma(1/4))^2}{3\,\sqrt{2-\sqrt{2}}\,\sqrt{\pi}\,3^{1/8}} = \int_0^1 \sec\left(\frac{\pi}{6} - \frac{1}{3}\tan^{-1}\left(\frac{x^4}{\sqrt{1-x^8}}\right)\right) dx + 2\int_1^\infty \frac{\left(x^4+\sqrt{x^8-1}\right)^{1/3}}{1+\left(x^4+\sqrt{x^8-1}\right)^{2/3}}\,dx \tag{7}$$

$$\frac{2\,(\Gamma(1/4))^2}{3\,\sqrt{2-\sqrt{2}}\,\sqrt{\pi}\,3^{1/8}} = \int_0^1 \sqrt[4]{\cosh\left(3\cosh^{-1}\left(\frac{1}{x}\right)\right)}\,dx + \int_1^{2/\sqrt{3}} \sqrt[4]{\cos\left(3\cos^{-1}\left(\frac{1}{x}\right)\right)}\,dx \tag{8}$$

References

A. Abramowitz, M, and Stegun, I.A., (Eds.), Handbook of Mathematical Functions, Dover, New York, 1972.

B. Andrews, G.E., Askey, R. and Roy, R., Special Functions, Cambridge University Press, 1999.

C. Artin, E., The Gamma Function, Holt, Rinehart, and Winston, New York, 1964.

D. Boros, G, and Moll, V., Irresistible Integrals, Cambridge University Press, 2004.

E. Gradshteyn, I.S., and Ryzhik, I.M., Table of Integrals, Series, and Products. Edited by A. Jaffrey and D. Zwillinger. Academic Press, 2000.