On Non-Trivial Zeros and Riemann Zeta Function

Bertrand Wong
Eurotech, S'pore Branch
Email: bwong8@singnet.com.sg

Abstract

This paper examines the mysterious non-trivial zeros of the Riemann zeta function ζ and explains their role, e.g., in the computation of the error term in Riemann's *J* function for estimating the quantity of primes less than a given number. The paper also explains the close connection between the Riemann zeta function ζ and the prime numbers. [Version edited and typeset in Latex by Editor-in-Chief is published in Bulletin of Pure and Applied Sciences Section E - Mathematics and Statistics, January-June 2023, Volume 42E, Number 1, P.57-60.]

MSC: 11A41, 11A99

Keywords: Complex numbers; complex plane; primes; Riemann zeta function; non-trivial zeros; error; estimate of quantity of primes.

1 Introduction

What is the part played by the non-trivial zeros of the Riemann zeta function ζ, which are mysterious and evidently not much understood? To understand what Riemann wanted to achieve with the non-trivial zeros, we have to understand the role of the complex numbers and the complex plane, which is explained below.

There is also a close link between the Riemann zeta function ζ and the prime numbers, which is also explained below.

2 Main Results: Part Played by Non-Trivial Zeros of Riemann Zeta Function ζ and Close Link between Riemann Zeta Function ζ and Prime Numbers

First, we look at the terms in the Riemann zeta function ζ:-

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \ldots \qquad (2.1)$$

where s is the complex number $1/2 + bi$

For the term $1/2^{1/2 + bi}$ above, e.g., whether it would be positive or negative in value would depend on which part of the complex plane this term $1/2^{1/2 + bi}$ would be found in, which depends on $2(n)$ and b (it does not depend on 1/2 - 1/2 and $2(n)$ only determine how far the term is from zero in the complex plane). This term could be in the positive half (whereby the term would have a positive value) or the negative half (whereby the term would have a negative value) of the complex plane. Hence, some of the terms in the Riemann zeta function

ζ would have positive values while the rest have negative values (depending on the values of n and b). The sum of the series in the Riemann zeta function ζ is obtained with a formula, e.g., the Riemann-Siegel formula, or, the Euler-Maclaurin summation formula.

Riemann evidently thought that there would be an equal, or, almost equal quantity of primes among the terms in the positive half and the negative half of the complex plane when there is a zero. That is, he thought that the distribution of the primes would be statistically fair, the more terms are added to the Riemann zeta function ζ, the fairer or "more equal" would be the distribution of the primes in the positive half and the negative half of the complex plane when there is a zero. This is like the tossing of a coin whereby the more tosses there are the "more equal" would be the number of heads and the number of tails. In other words, in the longer term, with more and more terms added to the Riemann zeta function ζ, more or less 50% of the primes should be found in the positive half of the complex plane and the balance 50% should be found in the negative half of the complex plane, the more terms there are the fairer or "more equal" would be this distribution, when there is a zero.

A non-trivial zero indicates the point in the Riemann zeta function ζ whereby the total value of the positive terms equals the total value of the negative terms. There is an infinitude of such points, i.e., non-trivial zeros, as had been proved by G. H. Hardy. Riemann evidently thought that for the case of a zero the quantity of primes found among the positive terms would be more or less equal to the quantity of primes found among the negative terms, which represents statistical fairness. It is evident that through a non-trivial zero the order or pattern of the distribution of the primes could be observed.

Next, we look at the error term in the following J function for estimating the quantity of primes less than a given number:-

$$J(n) = Li(n) - \sum_{p} Li(n^{p}) - \log 2 + \int_{n}^{\infty} dt/(t(t^{2} - 1) \log t) \tag{2.2}$$

where the first term $Li(n)$ is generally referred to as the "principal term" and the second term $\sum_{p} Li(n^{p})$ had been called the "periodic terms" by Riemann, Li being the logarithmic integral

$\sum_{p} Li(n^{p})$, the secondary term of the function, the error term, represents the sum taken over all the non-trivial zeros of the Riemann zeta function ζ. n here is a real number raised to the power of p, which is in this instance a complex number of the form $1/2 + bi$, for some real number b, $n^{1/2}$ being $\sqrt{n}$. If the Riemann hypothesis is true, for a given number n, when computing the values of n^{p} for a number of different zeta zeros p, the numbers we obtain are scattered round the circumference of a circle of radius $\sqrt{n}$ in the complex plane, centered on zero, and are either in the positive half or negative half of the complex plane.

To evaluate $\sum_{p} Li(n^{p})$ each zeta zero has to be paired with its mirror image, i.e., complex conjugate, in the south half of the argument plane. These pairs have to be taken in ascending order of the positive imaginary parts as follows:-

zeta zero: $1/2 + 14.134725i$ & its complex conjugate: $1/2 - 14.134725i$
zeta zero: $1/2 + 21.022040i$ & its complex conjugate: $1/2 - 21.022040i$
zeta zero: $1/2 + 25.010858i$ & its complex conjugate: $1/2 - 25.010858i$

.
.
.

(Note: The complex conjugates are all also zeros.)

If, e.g., we let $n = 100$, then the error term for $n = 100$ would be $\sum_p Li(100^p)$. To calculate this error term, we have to first raise 100 to the power of a long list of zeta zeros in ascending order of the positive imaginary parts (the first 3 zeta zeros are shown above), the longer the list of zeta zeros the better, e.g., 100,000 zeta zeros, in order to achieve the highest possible accuracy in the error term. Then we take the logarithmic integrals of the above powers (100,000 pairs of zeta zeros & their complex conjugates) and add them up, which is as follows:-

$$
\begin{aligned}
&100^{1/2 + 14.134725i} + 100^{1/2 - 14.134725i} \\
&+ 100^{1/2 + 21.022040i} + 100^{1/2 - 21.022040i} \\
&+ 100^{1/2 + 25.010858i} + 100^{1/2 - 25.010858i}
\end{aligned}
$$

.
.
.

The imaginary parts of the zeta zeros would cancel out the imaginary parts of their complex conjugates, leaving behind their respective real parts. For example, for the first zeta zero $1/2 + 14.134725i$, its imaginary part $+ 14.134725i$ would cancel out the imaginary part $- 14.134725i$ in its complex conjugate $1/2 - 14.134725i$, leaving behind only the real parts $100^{1/2}$ for each of them. That is, for $100^{1/2 + 14.134725i} + 100^{1/2 - 14.134725i}$, we only have to add together the logarithmic integral of $100^{1/2}$(from $100^{1/2 + 14.134725i}$) and the logarithmic integral of $100^{1/2}$(from $100^{1/2 - 14.134725i}$) to get the first term. The same is to be carried out for the next 99,999 powers in ascending order of the positive imaginary parts, giving altogether a total of 200,000 logarithmic integrals (of both the zeta zeros & their complex conjugates) to be added together to give the 100,000 terms. These terms have either positive or negative values, an equal or almost equal number of positive and negative values, which depend on whether they are in the positive or negative half of the complex plane, as is described above. The positive values and the negative values of these 100,000 terms are added together and should cancel out each other, slowly converging. The difference between the positive values and the negative values of these 100,000 terms constitutes the error term. (Note: The Riemann hypothesis posits that the difference between the true quantity of primes $p(n)$ and the estimated quantity of primes $q(n)$ would be not much larger than $\sqrt{n}$ - not much larger than $\sqrt{100}$ ($\sqrt{100}$ is also expressed as $100^{1/2}$) in the above case. Like the case of tossing a coin whereby the statistical probability is that in the long run the number of heads would practically equal the number of tails, there should be equal or almost equal quantities of positive terms and negative terms, i.e., 50,000 or thereabout positive terms and 50,000 or thereabout negative terms, which would be statistically fair, the discrepancy if any being the error.)

All this is evidently a laborious process, though the ingenuity of the ideas behind the Riemann hypothesis should be acknowledged.

It may be compared to the sieve of Eratosthenes, which could count the exact quantity of primes less than a given number without any error or error term at all, and could thus be regarded as a better, more straight-forward and perhaps less laborious and faster algorithm for finding the quantity of primes less than a given number as compared to Riemann's.

It is evident that the Riemann zeta function ζ has the property of prime sieving encoded within it (compare: sieve of Eratosthenes), the properties of the prime counting function $\pi(n)$ being somehow encoded in the properties of the Riemann zeta function ζ. (Refer to author's paper *Non-Trivial Zeros of Riemann Zeta Function and Riemann Hypothesis* published in Bulletin of Pure and Applied Sciences Section - E - Mathematics & Statistics Vol. 41E, No.1, January-June 2022, P.88-99 for more details.) The close link between the Riemann zeta function ζ and the prime numbers is shown below:-

The Riemann zeta function $\zeta(s)$ below is the sum over all natural numbers n:

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \ldots$$

This function could also be written in the following manner (using Euler's product formula) showing its connection with the prime numbers:

$$\zeta(s) = \prod_{p \text{ prime}} p^s/p^s - 1 = 2^s/2^s - 1 \text{ x } 3^s/3^s - 1 \text{ x } 5^s/5^s - 1 \text{ x } 7^s/7^s - 1 \text{ x } \ldots \qquad (2.3)$$

where the product is over the consecutive prime numbers p, providing the first indication that the Riemann zeta function $\zeta(s)$ is closely linked to the prime numbers; it could be seen above that the Riemann zeta function $\zeta(s)$ has the property of prime sieving encoded within it (compare: sieve of Eratosthenes)

3 Conclusions

It is clear that the non-trivial zeros of the Riemann zeta function ζ are an important and effective tool which could be used to somehow estimate with accuracy the quantity of primes less than a given number, as is explained in the paper; at the same time the mystery surrounding these non-trivial zeros should have been dispelled by the paper. Compared to the sieve of Eratosthenes, Riemann's *J* function for estimating the quantity of primes less than a given number appears a more complex method as well as one having some margin of error which is signified by its error term - the difference between the two methods is that while both methods could count or estimate with accuracy the quantity of primes less than a given number, only the sieve of Eratosthenes could produce a full, complete and exact list of the prime numbers smaller than a given number. However, the insights of Riemann in formulating all these mathematical concepts are highly impressive and are indicative of the superb capability of the human mind.

References

[1] H. M. Edwards, 2001, Riemann’s Zeta Function, Dover Publications, Inc.
[2] G. H. Hardy and E. M. Wright, 1979, An Introduction To Theory Of Numbers, Oxford, England: Clarendon Press
[3] A. Ivic, 2003, The Riemann Zeta-Function: Theory And Applications, Dover Publications, Inc.
[4] B. Riemann, 1859, On The Number Of Prime Numbers Less Than A Given Quantity, Berlin Academy of Sciences
[5] B. Wong, Non-Trivial Zeros Of Riemann Zeta Function And Riemann Hypothesis, Bulletin of Pure and Applied Sciences Section - E - Mathematics & Statistics Vol. 41E, No.1, January-June 2022, P.88-99