THE SERIES LIMIT OF $\sum_k \cos(a \log k)/[k \log k]$

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ABSTRACT. The slowly converging series $\sum_{k=2}^{\infty} \cos(a \log k)/[k \log k]$ is evaluated numerically for $a = 1/2, 1, 3/2, \ldots 4$. After some initial terms, the infinite tail of the sum is replaced by the integral of the associated interpolating function, an Exponential Integral, and the "second form" of the Euler-Maclaurin corrections is derived from the analytic equations for higher order derivatives.

1. Aim and Scope

We aim at a precise numerical evaluation of the series

$$C_a \equiv \sum_{k=2}^{\infty} \frac{\cos(a \log k)}{k \log k}, \tag{1}$$

$$S_a \equiv \sum_{k=2}^{\infty} \frac{\sin(a \log k)}{k \log k}, \tag{2}$$

where log denotes the natural logarithm. Without the cosine term in the numerator, the sum is divergent (a result of the integral test). Direct numerical summation is a futile strategy to estimate the series limit; Figures 1 and 2 illustrate the poor convergence of the series for upper limits up to $k \le 2.1 \times 10^6$. We employ a Euler-Maclaurin formalism [5] to compute many digits of C_a.

2. Numerical strategy

2.1. Romberg Integration.

Some initial terms up to $k = N$ are summed directly. The smoothly interpolating integral is easily evaluated with the substitution $y = \log x$ as an Exponential Integral [1, 5.2.27]

$$I_{a,N} \equiv \int_{x=N+1/2}^{\infty} \frac{\cos(a \log x)}{x \log x} dx = -\operatorname{Ci}(a \log(N+1/2)). \tag{3}$$

The half-infinite interval is cut into abscissa sections of unit width centered at integer k,

$$I_{a,N} = \sum_{k=N+1}^{\infty} A_{a,k}, \tag{4}$$

such that [2, Thrm. 1]

$$A_{a,k} = \frac{\cos(a \log k)}{k \log k} + \sum_{s=1}^{\infty} \frac{1}{4^s (2s+1)!} \frac{d^{2s}}{dx^{2s}} \frac{\cos(a \log x)}{x \log x}_{|x=k} \tag{5}$$

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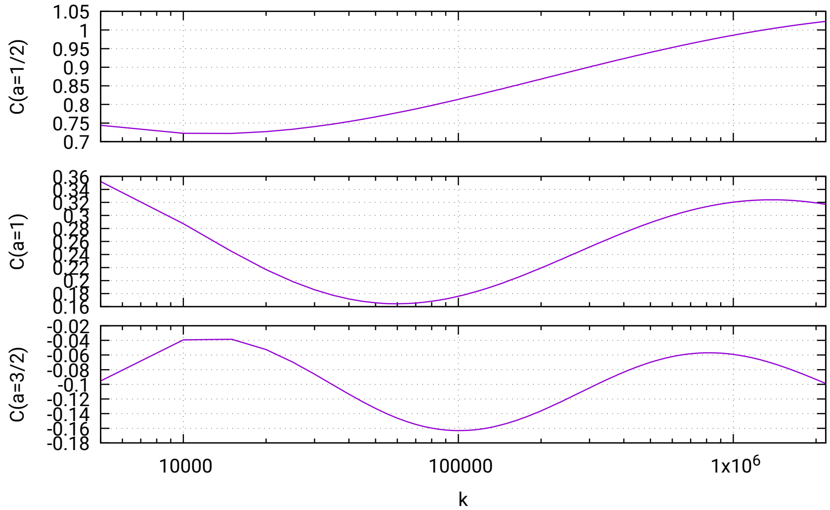

FIGURE 1. The partial sums of C_a for three values of a as a function of the upper limit on k on a logarithmic k-scale.

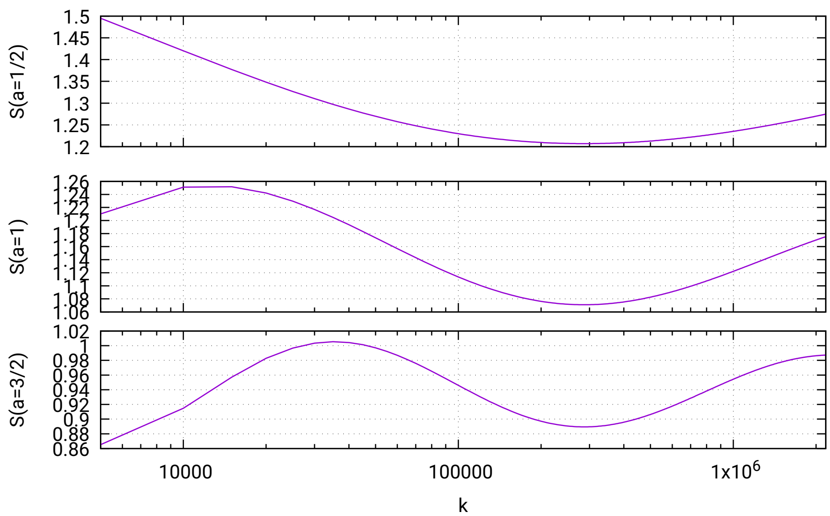

FIGURE 2. The partial sums of S_a for three values of a as a function of the upper limit on k on a logarithmic k-scale.

to yield

Algorithm 1. *(Romberg)*

$$C_a = \sum_{k=2}^{N} \frac{\cos(a\log k)}{k\log k} + I_{a,N} - \sum_{s=1}^{\infty} \frac{1}{4^s(2s+1)!} \sum_{k=N+1}^{\infty} \frac{d^{2s}}{dx^{2s}} \frac{\cos(a\log x)}{x\log x}_{|k}. \tag{6}$$

2.2. **Euler-Maclaurin.** Recursive use of this technique for the sums originating from the curvature approximations generates an Euler-Maclaurin formula [4, 5]

Algorithm 2. *(Centered Euler-Maclaurin)*

$$C_a = \sum_{k=2}^{N} \frac{\cos(a\log k)}{k\log k} + I_{a,N} + \sum_{s=1}^{\infty} \frac{1}{2^{2s-1}}\beta(s)\frac{d^{2s-1}}{dx^{2s-1}}\frac{c(0,1)}{x} \tag{7}$$

where

$$\beta(s) = (2^{2s-1}-1)\frac{B_{2s}}{(2s)!} \tag{8}$$

in terms of signed Bernoulli numbers B [1, Ch. 23]. With this approach, derivatives of odd order are evaluated at $x = N + \frac{1}{2}$. The derivatives share a common format, which suggests the notational shortcut

Definition 1. *(Atoms of Derivatives)*

$$s(i,j) \equiv \frac{\sin(a\log x)a^i}{\log^j x}; \quad c(i,j) \equiv \frac{\cos(a\log x)a^i}{\log^j x} \tag{9}$$

The dominant (smallest) orders are

$$\frac{d}{dx}\frac{c(0,1)}{x} = \frac{1}{x^2}[-s(1,1) - c(0,1) - c(0,2)]; \tag{10}$$

$$\begin{aligned}\frac{d^3}{dx^3}\frac{c(0,1)}{x} = \frac{1}{x^4}[&-11s(1,1) - 12s(1,2) - 6s(1,3) + s(3,1)\\ &- 6c(0,1) - 11c(0,2) - 12c(0,3) - 6c(0,4) + 6c(2,1) + 3c(2,2)];\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{d^5}{dx^5}\frac{c(0,1)}{x} = \frac{1}{x^6}[&-274s(1,1) - 450s(1,2) - 510s(1,3) - 360s(1,4) - 120s(1,5)\\ &+ 85s(3,1) + 60s(3,2) + 20s(3,3) - s(5,1)\\ &- 120c(0,1) - 274c(0,2) - 450c(0,3) - 510c(0,4) - 360c(0,5) - 120c(0,6)\\ &+ 225c(2,1) + 255c(2,2) + 180c(2,3) + 60c(2,4) - 15c(4,1) - 5c(4,2)].\end{aligned} \tag{12}$$

Numerical examples are shown in Table 1 while restricting the s-sum in (7) to upper limits $\hat{s}$.

Remark 1. *Integer powers of cosines in the numerator can be reduced to finite sums of our format (1) [3, 1.320.7]:*

$$\cos^{2n+1}(a\log k) = \frac{1}{2^{2n}}\sum_{l=0}^{n}\binom{2n+1}{l}\cos[(2n-2l+1)a\log k]. \tag{13}$$

Cases with even powers $\cos^{2n}(a\log k)$, however, require the diverging $\sum_k 1/(k\log k)$ implied by a constant term. This is one way of proving that their associated series do not *converge.*

Table 1. Convergence Algorithm 2 for C_a accumulating terms $s \le \hat{s}$, last digits rounded.

a	$\hat{s}$	$N=20$	$N=40$	$N=80$
1/2	0	0.9197364825071931	0.91971836267638194256	0.919717017657257338874051
1/2	3	0.9197174730564872	0.91971747305660365462	0.919717473056603921316555
1/2	4	0.9197174730566045	0.91971747305660392220	0.919717473056603921829781
1/2	5	0.9197174730566039	0.91971747305660392183	0.919717473056603921829589
1/2	6	0.9197174730566039	0.91971747305660392183	0.919717473056603921829589
1	0	0.2539159647888925	0.25394433488581123740	0.253953402983123183854083
1	3	0.2539553648211005	0.25395536482117354871	0.253955364821173476991499
1	4	0.2539553648211740	0.25395536482117347593	0.253955364821173475890836
1	5	0.2539553648211735	0.25395536482117347589	0.253955364821173475891098
1	6	0.2539553648211735	0.25395536482117347589	0.253955364821173475891098
3/2	0	-0.1057202379424350	-0.10566433973251785623	-0.105661588842199800006077
3/2	3	-0.1056639558654949	-0.10566395586555039914	-0.105663955865551072702073
3/2	4	-0.1056639558655510	-0.10566395586555107542	-0.105663955865551074722321
3/2	5	-0.1056639558655511	-0.10566395586555107472	-0.105663955865551074721601
3/2	6	-0.1056639558655511	-0.10566395586555107472	-0.105663955865551074721602
2	0	-0.3287304274390670	-0.32874098686689542975	-0.328756804513726732918771
2	3	-0.3287571396165611	-0.32875713961686295000	-0.328757139616863821065301
2	4	-0.3287571396168649	-0.32875713961686382089	-0.328757139616863819501605
2	5	-0.3287571396168638	-0.32875713961686381950	-0.328757139616863819501613
2	6	-0.3287571396168638	-0.32875713961686381950	-0.328757139616863819501613
5/2	0	-0.4681329062638261	-0.46822990435116153953	-0.468227945718873905180268
5/2	3	-0.4682242390623630	-0.46822423906288517177	-0.468224239062884555844739
5/2	4	-0.4682242390628873	-0.46822423906288455087	-0.468224239062884550461398
5/2	5	-0.4682242390628845	-0.46822423906288455046	-0.468224239062884550463317
5/2	6	-0.4682242390628846	-0.46822423906288455046	-0.468224239062884550463316
3	0	-0.5474404456463450	-0.54745401803332864382	-0.547430531612033597243483
3	3	-0.5474344945208571	-0.54743449452110684129	-0.547434494521103976441456
3	4	-0.5474344945211076	-0.54743449452110397786	-0.547434494521103981143736
3	5	-0.5474344945211039	-0.54743449452110398115	-0.547434494521103981144149
3	6	-0.5474344945211040	-0.54743449452110398114	-0.547434494521103981144148
7/2	0	-0.5788992940283214	-0.57875938865610197386	-0.578776494716326354879199
7/2	3	-0.5787765575941155	-0.57877655759320278387	-0.578776557593202272116999
7/2	4	-0.5787765575932021	-0.57877655759320227941	-0.578776557593202284135375
7/2	5	-0.5787765575932023	-0.57877655759320228414	-0.578776557593202284130085
7/2	6	-0.5787765575932023	-0.57877655759320228413	-0.578776557593202284130087
4	0	-0.5697701203018853	-0.56973166270688474820	-0.569752961500326555268268
4	3	-0.5697478070611435	-0.56974780705908076001	-0.569747807059087757149690
4	4	-0.5697478070590776	-0.56974780705908774338	-0.569747807059087738522870
4	5	-0.5697478070590878	-0.56974780705908773853	-0.569747807059087738523139
4	6	-0.5697478070590877	-0.56974780705908773852	-0.569747807059087738523142

3. The sine Numerator

If the numerator in (1) is generalized introducing a phase shift b as $\cos(b + a\log k) = \cos b\cos(a\log k) - \sin b\sin(a\log k)$ [1, 4.3.17], a complementary companion emerges: the series (2). Equation 3 is replaced by [1, 5.2.26]

$$\int_{x=N+1/2}^{\infty} \frac{\sin(a\log x)}{x\log x}dx = -\operatorname{si}(a\log x) \tag{14}$$

and the derivatives starting at (7) are replaced by

$$\frac{d}{dx}\frac{s(0,1)}{x} = \frac{1}{x^2}[-s(0,1) - s(0,2) + c(1,1)]; \tag{15}$$

$$\begin{aligned}\frac{d^3}{dx^3}\frac{s(0,1)}{x} = \frac{1}{x^4}[&-6s(0,1) - 11s(0,2) - 12s(0,3) - 6s(0,4) + 6s(2,1) + 3s(2,2)\\ &+ 11c(1,1) + 12c(1,2) + 6c(1,3) - c(3,1)];\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{d^5}{dx^5}\frac{s(0,1)}{x} = \frac{1}{x^6}[&-120s(0,1) - 274s(0,2) - 450s(0,3) - 510s(0,4) - 360s(0,5) - 120s(0,6)\\ &+ 225s(2,1) + 255s(2,2) + 180s(2,3) + 60s(2,4) - 15s(4,1) - 5s(4,2)\\ &+ 274c(1,1) + 450c(1,2) + 510c(1,3) + 360c(1,4) + 120c(1,5)\\ &- 85c(3,1) - 60c(3,2) - 20c(3,3) + c(5,1)].\end{aligned} \tag{17}$$

These are essentially sign flips and swaps $c(.,.) \leftrightarrow s(.,.)$. Numerical evaluation yields Table 2.

Remark 2. *Integer powers of sines in the numerator can be reduced to finite sums of our format* (2) [3, 1.320.3]*:*

$$\sin^{2n+1}(a\log k) = \frac{1}{2^{2n}}\sum_{l=0}^{n}(-)^{n+l}\binom{2n+1}{l}\sin[(2n-2l+1)a\log k]. \tag{18}$$

Remark 3. *Outlook: The generalized sums* $\sum_{k=2}^{\infty}\frac{\cos(a\log k)}{k(\log k)^q}$ *with* $q > 1$ *may approached in the same manner because the interpolating integral is also of the Exponential Integral format; a recurrence obtained by partial integration lowers the exponent in the denominator* [3, 3.351.4]*:*

$$\int_{x=N+1/2}^{\infty}\frac{\cos(a\log x)}{x\log^q x}dx = \frac{1}{q-1}\left[-\frac{\cos(a\log x)}{\log^{q-1}x} + a\int_{x=N+1/2}^{\infty}\frac{\sin(a\log x)}{x\log^{q-1}x}dx\right] \tag{19}$$

such that closed-form expressions of the generalized $I_{a,N}$ *are available.*

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TABLE 2. Convergence of Algorithm 2 for S_a accumulating terms $s \le \hat{s}$, last digits rounded.

a	$\hat{s}$	$N = 20$	$N = 40$	$N = 80$
1/2	0	1.3596492469983137	1.35961597266189652209	1.359608536114376847685329
1/2	3	1.3596066477454514	1.35960664774552058741	1.359606647745520871996315
1/2	4	1.3596066477455212	1.35960664774552087339	1.359606647745520873064292
1/2	5	1.3596066477455209	1.35960664774552087306	1.359606647745520873063978
1/2	6	1.3596066477455209	1.35960664774552087306	1.359606647745520873063979
1	0	1.1496843087793079	1.14964766298547568932	1.149645237914789118809333
1	3	1.1496464767116954	1.14964647671184881650	1.149646476711849319540887
1	4	1.1496464767118500	1.14964647671184932132	1.149646476711849320679884
1	5	1.1496464767118493	1.14964647671184932068	1.149646476711849320679468
1	6	1.1496464767118493	1.14964647671184932068	1.149646476711849320679468
3/2	0	0.9421844852132328	0.94220509443202287813	0.942217007698393468180485
3/2	3	0.9422185774680283	0.94221857746825929440	0.942218577468259596673977
3/2	4	0.9422185774682608	0.94221857746825959608	0.942218577468259595458040
3/2	5	0.9422185774682596	0.94221857746825959546	0.942218577468259595458204
3/2	6	0.9422185774682596	0.94221857746825959546	0.942218577468259595458204
2	0	0.7386996205696191	0.73877565626495505671	0.738777218633926767386172
2	3	0.7387737963896372	0.73877379638982435903	0.738773796389823626869313
2	4	0.7387737963898251	0.73877379638982362311	0.738773796389823623515648
2	5	0.7387737963898236	0.73877379638982362352	0.738773796389823623516824
2	6	0.7387737963898236	0.73877379638982362352	0.738773796389823623516824
5/2	0	0.5410056063817872	0.54100668826579758312	0.540986588610980686939069
5/2	3	0.5409882960785833	0.54098829607841026970	0.540988296078408576855175
5/2	4	0.5409882960784091	0.54098829607840857704	0.540988296078408579345573
5/2	5	0.5409882960784086	0.54098829607840857935	0.540988296078408579345775
5/2	6	0.5409882960784086	0.54098829607840857935	0.540988296078408579345775
3	0	0.3509500500488555	0.35083157489646823548	0.350839855224607962656655
3	3	0.3508424733437655	0.35084247334293786310	0.350842473342938102124874
3	4	0.3508424733429353	0.35084247334293810844	0.350842473342938110434696
3	5	0.3508424733429381	0.35084247334293811044	0.350842473342938110431524
3	6	0.3508424733429381	0.35084247334293811043	0.350842473342938110431526
7/2	0	0.1707055950209379	0.17067937798496536224	0.170703742304541987589945
7/2	3	0.1706983078230975	0.17069830782208806516	0.170698307822092612009934
7/2	4	0.1706983078220852	0.17069830782209260674	0.170698307822092602555546
7/2	5	0.1706983078220927	0.17069830782209260255	0.170698307822092602555119
7/2	6	0.1706983078220926	0.17069830782209260256	0.170698307822092602555120
4	0	0.0032315763474963	0.00339208230827944991	0.003364903514924475734437
4	3	0.0033682234166344	0.00336822341716551395	0.003368223417167289852031
4	4	0.0033682234171609	0.00336822341716728348	0.003368223417167274340241
4	5	0.0033682234171674	0.00336822341716727433	0.003368223417167274348992
4	6	0.0033682234171673	0.00336822341716727435	0.003368223417167274348989

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