Considerations on the 3n+1 problem

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Abstract
This paper presents some considerations on the 3n+1 problem. In particular on the next odd elements in the sequence lower than the starting number.

3n + 1 problem (or conjecture)

In the $3\cdot n+1$ problem[1] it is possible to define the function:

$$f(x)=\begin{cases} 3\cdot x+1 & if\ x\equiv 1(mod\,2) \\ \frac{x}{2} & if\ x\equiv 0(mod\,2) \end{cases}$$

The sequence obtained using the function is as follows:

$$n_i=\begin{cases} n & for\ i=0 \\ f(n_{i-1}) & for\ i>0 \end{cases}$$

the sequence can be rewritten considering only the odd terms, then starting from an odd number n if x is an odd integer we have $f(x)=\frac{3\cdot x+1}{2^a}$ (Syracuse function[1]) with $f(x)$ odd and $a\geq 1$ therefore:

$$n_1=\frac{3\cdot n+1}{2^{a_1}}$$

$$n_2=\frac{3\cdot n_1+1}{2^{a_2}}=\frac{3\cdot\frac{3\cdot n+1}{2^{a_1}}+1}{2^{a_2}}$$

$\dots$

if $k>1$

$$n_k=\frac{3\cdot n_{k-1}+1}{2^{a_k}}=\frac{3^k\cdot n+3^{k-1}+\sum_{j=0}^{k-2}3^j\cdot 2^{\sum_{i=1}^{k-1-j}a_i}}{2^{\sum_{i=1}^{k}a_i}}$$

using this formula for n_k to find the next odd number lower than n it is possible to obtain a pattern given a certain classes of numbers modulo 2^c example:

$$43\rightarrow 130\rightarrow 65\rightarrow 196\rightarrow 98\rightarrow 49\rightarrow 148\rightarrow 74\rightarrow 37\rightarrow\dots$$

the next odd number in the sequence lower than 43 is $37=\frac{27\cdot 43+23}{32}$

the next odd number in the sequence less than $107=43+64$ is $91=\frac{27\cdot 107+23}{32}$

then next odd number in the sequence less than $n\equiv 43(mod\,64)$ is $\frac{27\cdot n+23}{32}$

As it is possible to analyze also in OEIS A177789[2] the study based on the residue classes modulo 2^d with d obtained from the sequence OEIS A020914[3] and if we consider the corresponding 3^k result from the sequence OEIS A020914 we can obtain the following results:

for the numbers $n\equiv 1(mod\ 4)$ we have $d=2$ and $k=1$ then

$$n\rightarrow\frac{3\cdot n+1}{2^{2\cdot x+2}}\quad \text{if}\quad n\equiv\frac{2^{2\cdot x+2}-1}{3}(mod\ 2^{2\cdot x+3})$$

$$n\rightarrow\frac{3\cdot n+1}{2^{2\cdot x+3}}\quad \text{if}\quad n\equiv\frac{2^{2\cdot x+3}-1}{3}(mod\ 2^{2\cdot x+4})$$

for the numbers $n\equiv 3(mod\ 16)$ we have $d=4$ and $k=2$ then

$$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+4}}\quad \text{if}\quad n\equiv\frac{11\cdot 2^{6\cdot x+4}-5}{9}(mod\ 2^{6\cdot x+5})$$

$$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+5}}\quad \text{if}\quad n\equiv\frac{2^{6\cdot x+5}-5}{9}(mod\ 2^{6\cdot x+6})$$

$$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+6}}\quad \text{if}\quad n\equiv\frac{5\cdot 2^{6\cdot x+6}-5}{9}(mod\ 2^{6\cdot x+7})$$

$$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+7}}\quad \text{if}\quad n\equiv\frac{7\cdot 2^{6\cdot x+7}-5}{9}(mod\ 2^{6\cdot x+8})$$

$$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+8}}\quad \text{if}\quad n\equiv\frac{17\cdot 2^{6\cdot x+8}-5}{9}(mod\ 2^{6\cdot x+9})$$

$$n\rightarrow\frac{9\cdot n+5}{2^{6\cdot x+9}}\quad \text{if}\quad n\equiv\frac{13\cdot 2^{6\cdot x+9}-5}{9}(mod\ 2^{6\cdot x+10})$$

for the numbers $n\equiv 23(mod\ 32)$ we have $d=5$ and $k=3$ then

$$n\rightarrow\frac{27\cdot n+19}{2^{18\cdot x+5}}\quad \text{if}\quad n\equiv\frac{47\cdot 2^{18\cdot x+5}-19}{27}(mod\ 2^{18\cdot x+6})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+6}} \quad \text{if} \quad n \equiv \frac{37 \cdot 2^{18 \cdot x+6}-19}{27} (mod\ 2^{18 \cdot x+7})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+7}} \quad \text{if} \quad n \equiv \frac{5 \cdot 2^{18 \cdot x+7}-19}{27} (mod\ 2^{18 \cdot x+8})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+8}} \quad \text{if} \quad n \equiv \frac{43 \cdot 2^{18 \cdot x+8}-19}{27} (mod\ 2^{18 \cdot x+9})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+9}} \quad \text{if} \quad n \equiv \frac{35 \cdot 2^{18 \cdot x+9}-19}{27} (mod\ 2^{18 \cdot x+10})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+10}} \quad \text{if} \quad n \equiv \frac{31 \cdot 2^{18 \cdot x+10}-19}{27} (mod\ 2^{18 \cdot x+11})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+11}} \quad \text{if} \quad n \equiv \frac{29 \cdot 2^{18 \cdot x+11}-19}{27} (mod\ 2^{18 \cdot x+12})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+12}} \quad \text{if} \quad n \equiv \frac{2^{18 \cdot x+12}-19}{27} (mod\ 2^{18 \cdot x+13})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+13}} \quad \text{if} \quad n \equiv \frac{41 \cdot 2^{18 \cdot x+13}-19}{27} (mod\ 2^{18 \cdot x+14})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+14}} \quad \text{if} \quad n \equiv \frac{7 \cdot 2^{18 \cdot x+14}-19}{27} (mod\ 2^{18 \cdot x+15})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+15}} \quad \text{if} \quad n \equiv \frac{17 \cdot 2^{18 \cdot x+15}-19}{27} (mod\ 2^{18 \cdot x+16})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+16}} \quad \text{if} \quad n \equiv \frac{49 \cdot 2^{18 \cdot x+16}-19}{27} (mod\ 2^{18 \cdot x+17})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+17}} \quad \text{if} \quad n \equiv \frac{11 \cdot 2^{18 \cdot x+17}-19}{27} (mod\ 2^{18 \cdot x+18})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+18}} \quad \text{if} \quad n \equiv \frac{19 \cdot 2^{18 \cdot x+18}-19}{27} (mod\ 2^{18 \cdot x+19})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+19}} \quad \text{if} \quad n \equiv \frac{23 \cdot 2^{18 \cdot x+19}-19}{27} (mod\ 2^{18 \cdot x+20})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+20}} \quad \text{if} \quad f\, n \equiv \frac{25 \cdot 2^{18 \cdot x+20}-19}{27} (mod\ 2^{18 \cdot x+21})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+21}} \quad \text{if} \quad n \equiv \frac{53 \cdot 2^{18 \cdot x+21}-19}{27} (mod\ 2^{18 \cdot x+22})$$

$$n \rightarrow \frac{27 \cdot n+19}{2^{18 \cdot x+22}} \quad \text{if} \quad n \equiv \frac{13 \cdot 2^{18 \cdot x+22}-19}{27} (mod\ 2^{18 \cdot x+23})$$

for the numbers $n \equiv 11 (mod\ 32)$ we have $d=5$ and $k=3$ then

$$n \rightarrow \frac{27 \cdot n+23}{2^{18 \cdot x+5}} \quad \text{if} \quad n \equiv \frac{37 \cdot 2^{18 \cdot x+5}-23}{27} (mod\ 2^{18 \cdot x+6})$$

$$n \rightarrow \frac{27 \cdot n+23}{2^{18 \cdot x+6}} \quad \text{if} \quad n \equiv \frac{5 \cdot 2^{18 \cdot x+6}-23}{27} (mod\ 2^{18 \cdot x+7})$$

…

for the numbers $n \equiv 15 (mod\ 128)$ we have $d=7$ and $k=4$ then

$$n \rightarrow \frac{81 \cdot n+65}{2^{54 \cdot x+7}} \quad \text{if} \quad n \equiv \frac{91 \cdot 2^{54 \cdot x+7}-65}{81} (mod\ 2^{54 \cdot x+8})$$

$$n \rightarrow \frac{81 \cdot n+65}{2^{54 \cdot x+8}} \quad \text{if} \quad n \equiv \frac{5 \cdot 2^{54 \cdot x+8}-65}{81} (mod\ 2^{54 \cdot x+9})$$

…

We can conjecture that for any value of $k>1$ integer we can find $n_k<n$

$$n_k = \frac{3^k \cdot n+3^{k-1}+\sum_{j=0}^{k-2} 3^j \cdot 2^{\sum_{i=1}^{k-1-j} a_i}}{2^{\sum_{i=1}^{k} a_i}} = \frac{3^k \cdot n+s}{2^e}$$

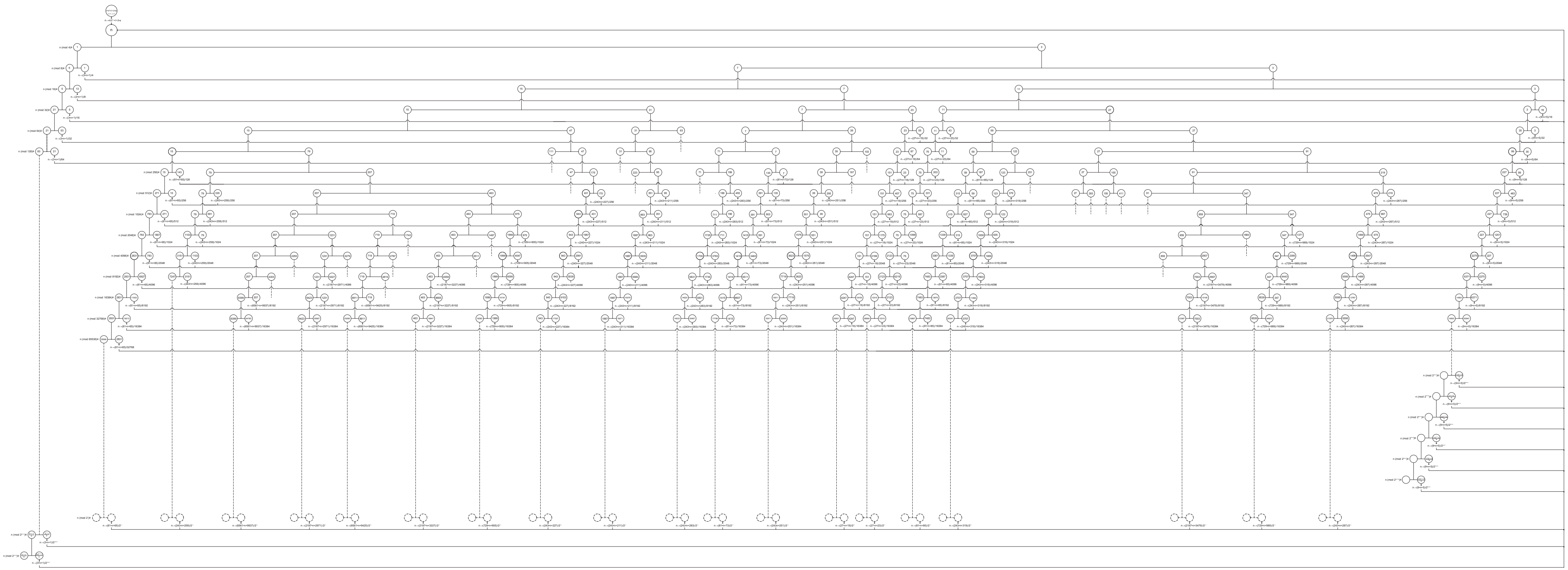
with s odd integer not divisible by 3 and $3^k-2^k \leqslant s<m$ and $m=2 \cdot 3^{k-1}$

then

$$n \rightarrow \frac{3^k \cdot n+s}{2^{m \cdot b+d+r}} \quad \text{if} \quad n \equiv \frac{t \cdot 2^{m \cdot b+d+r}-s}{3^k} (mod\ 2^{m \cdot b+d+r+1})$$

with $b \geq 0$ integer, t odd integer not divisible by 3 and $t<m$, $0 \leqslant r<m$ integer and d number of digits in the base-2 representation of 3^k .

This can be summarized in a graph (shown on the next page) where for each row of the graph the different residue classes modulo 2^c are shown.



References

[1] https://en.wikipedia.org/wiki/Sieve_of_Eratosthenes

[2] https://oeis.org/A177789

[3] https://oeis.org/A020914