Location and Radius of a Triangle's Incircle Via Geometric Algebra

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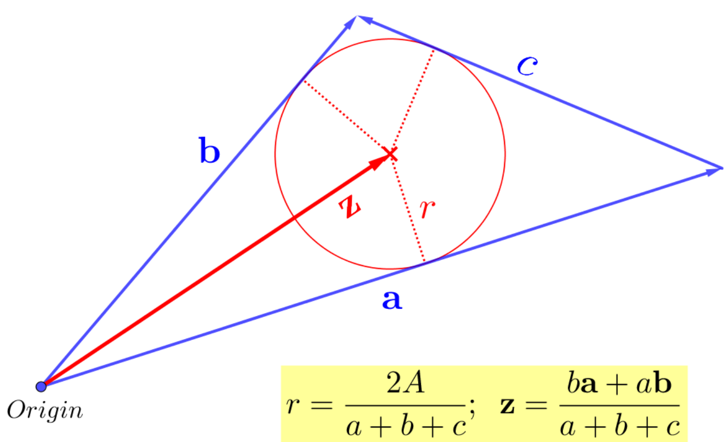
Abstract

We show how to use the GA concept of the "rejection" of vectors, and also the related outer product, to derive equations for the location and radius of a triangle's incircle.



$$r = \frac{2A}{a+b+c}; \quad \mathbf{z} = \frac{b\mathbf{a} + a\mathbf{b}}{a+b+c}$$

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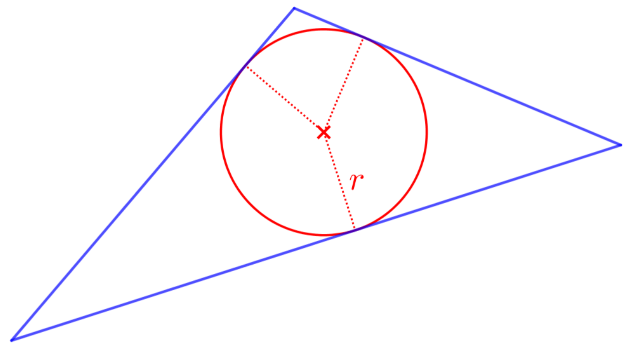


Figure 1: The Our task is to use GA to derive formulas for the radius and center of a triangle's incenter.

1 Introduction

Our task is to derive formulas for the radius and center of a triangle's incenter (Fig. 1). We will express those characteristics in terms of the triangle's area and the lengths of the triangle's sides.

If need be, the radius and center can be expressed in terms of the lengths alone by using Heron's theorem for the area. See Eq. (6.7).

2 Some of the Ideas that We Will Find Useful

1. A line that is tangent to a circle at a point P is perpendicular to the radius that is drawn to that point (Fig. 2).

2. The "rejection" of a vector $\mathbf{v}$ with respect to a vector $\mathbf{u}$ (denoted $\mathsf{Rej}_{\mathbf{u}}(\mathbf{v})$—Fig. 2). For our purposes, it is important that

$$\|\mathsf{Rej}_{\mathbf{u}}(\mathbf{v})\| = \frac{\|\mathbf{v}\wedge\mathbf{u}\|}{u} . \tag{2.1}$$

3. For any two vectors $\mathbf{u}$ and $\mathbf{v}$, the magnitude of the outer product $\mathbf{u}\wedge\mathbf{v}$ is equal to twice the area of the triangle that is formed by those vectors (Fig. 4.):

$$\|\mathbf{u}\wedge\mathbf{v}\| = 2A . \tag{2.2}$$

3 Examination and Analysis of the Problem

The key features of the diagram are that (1) the center of the incircle lies the intersection of the bisectors of the three angles; and (2) the center is equidistant from the three sides (Fig. 5).

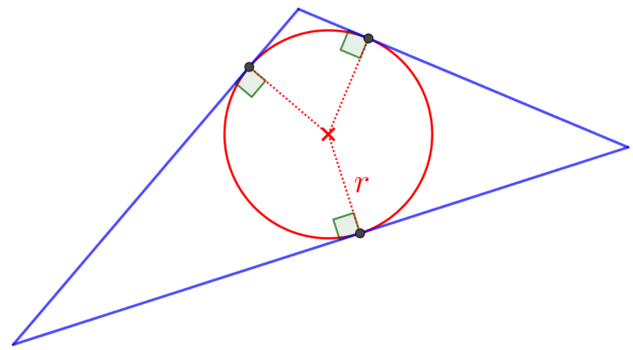


Figure 2: A line that is tangent to a circle at any point is perpendicular to the radius that is drawn to that point.

4 Our Strategy

An examination of Fig. 5 suggests two reasonable ideas:

- Find the center of the incircle by calculating the point of intersection of two of the bisectors, then find r by finding the distance from that intersection to any one of the sides; and
- Locate the point, along any of the three bisectors, that is equidistant from the three sides (Fig. 6).

We will follow the second strategy.

5 Formulation of the Problem in GA Terms

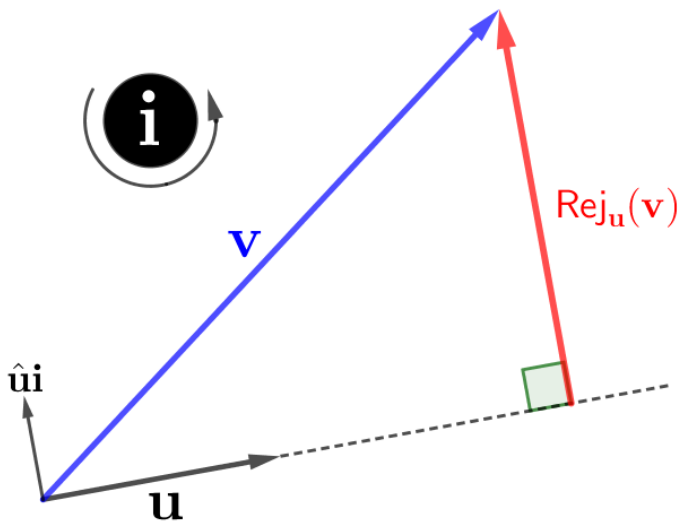
The formulation shown in Fig 7 will enable us to carry out our strategy. The center must lie along the direction of $\hat{\mathbf{a}} + \hat{\mathbf{b}}$ in order to be equidistant from the sides a and b.

6 Solution

The geometric interpretation of Eq. (6.1) is that the area of the parallelogram in Fig. 4, divided by the length of the base, is equal to the height of the parallelogram.

We will express r in two ways, both of which involve rejections that we will write in terms of γ. As a preliminary, we note an important inference from Eqs. 2.1 and (2.2):

$$\|\mathsf{Rej}_{\mathbf{u}}\left(\mathbf{v}\right)\| = \frac{2A}{u} \ . \tag{6.1}$$



$$
\begin{aligned}
\mathsf{Rej}_{\mathbf{u}}(\mathbf{v}) &= \{\mathbf{v}\wedge\mathbf{u}\}\mathbf{u}^{-1} \\
&= \{\mathbf{v}\wedge\mathbf{u}\}\left[\frac{\mathbf{u}}{u^2}\right] \\
&= \left\{\frac{\mathbf{v}\wedge\mathbf{u}}{u}\right\}\left[\frac{\mathbf{u}}{u}\right] \\
&= \left\{\frac{\mathbf{v}\wedge\mathbf{u}}{u}\right\}\hat{\mathbf{u}} \\
&= +\ \text{or}\ -\left\{\left[\frac{\|\mathbf{v}\wedge\mathbf{u}\|}{u}\right]\mathbf{i}\right\}\hat{\mathbf{u}} \\
&= +\ \text{or}\ -\left[\frac{\|\mathbf{v}\wedge\mathbf{u}\|}{u}\right]\mathbf{i}\hat{\mathbf{u}} \\
&= -\ \text{or}\ +\left[\frac{\|\mathbf{v}\wedge\mathbf{u}\|}{u}\right]\hat{\mathbf{u}}\mathbf{i}
\end{aligned}
$$

$$\therefore\ \|\mathsf{Rej}_{\mathbf{u}}(\mathbf{v})\| = \frac{\|\mathbf{v}\wedge\mathbf{u}\|}{u}$$

Figure 3: The "rejection" of a vector $\mathbf{v}$ with respect to the vector $\mathbf{u}$ (denoted $\mathsf{Rej}_{\mathbf{u}}(\mathbf{v})$. In this diagram, $\mathsf{Rej}_{\mathbf{u}}(\mathbf{v}) = +\left[\frac{\|\mathbf{v}\wedge\mathbf{u}\|}{u}\right]\hat{\mathbf{u}}\mathbf{i}$ because the sense of the rotation from $\mathbf{u}$ to $\mathbf{v}$ is that same as the sense or rotation of $\mathbf{i}$. In this way, we express $\mathsf{Rej}_{\mathbf{u}}(\mathbf{v})$ as a scalar multiple of a 90-degree rotation of $\mathbf{i}$. Note, especially, that $\|\mathsf{Rej}_{\mathbf{u}}(\mathbf{v})\| = \|\mathbf{v}\wedge\mathbf{u}\|/u$

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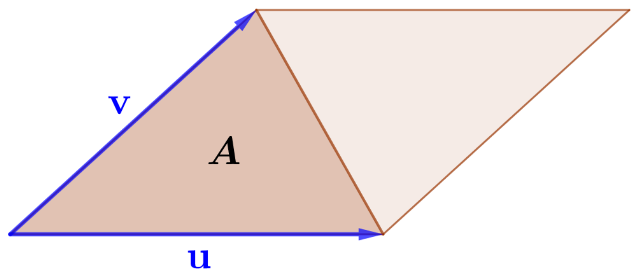


$$\|\mathbf{u} \wedge \mathbf{v}\| = 2A$$

Figure 4: For any two vectors $\mathbf{u}$ and $\mathbf{v}$, the magnitude of the outer product $\mathbf{u} \wedge \mathbf{v}$ is equal to twice the area (A) of the triangle that is formed by those vectors.

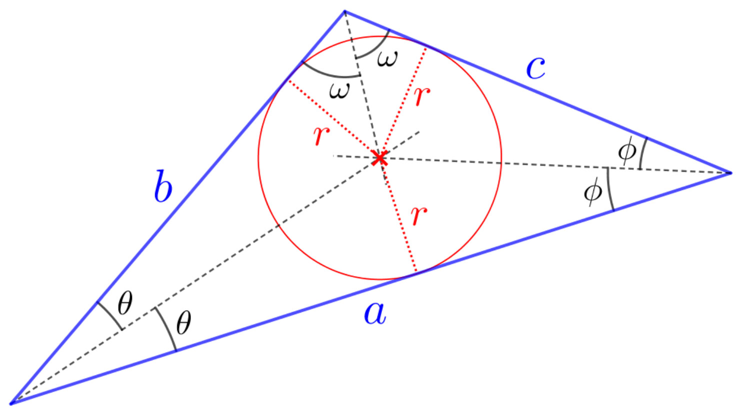


Figure 5: The key features of the diagram are that (1) the center of the incircle lies the intersection of the bisectors of the three angles; and (2) the center is equidistant from the three sides.

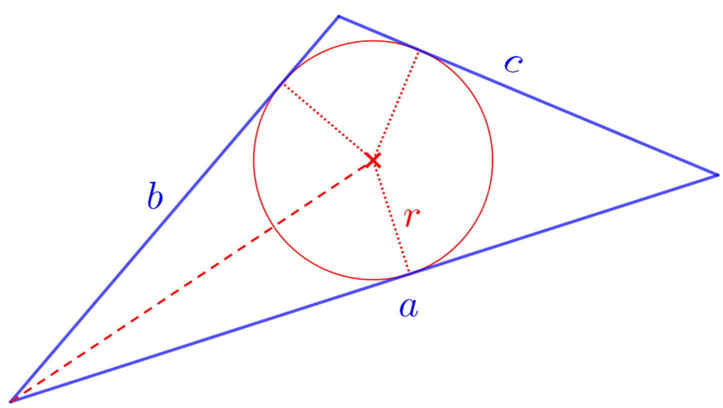
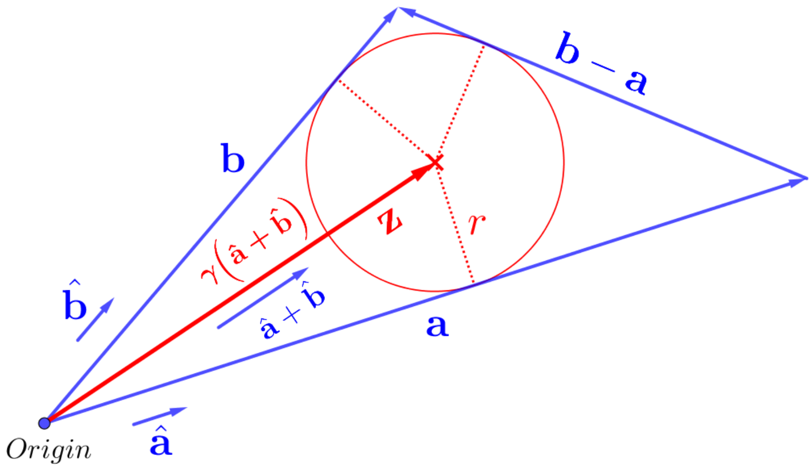


Figure 6: Our strategy will be to locate the point, along one of the three bisectors, that is equidistant from the three sides.



$$\|\mathbf{a}\| = a; \quad \|\mathbf{b}\| = b; \quad \|\mathbf{b}-\mathbf{a}\| = c; \quad \mathbf{z} = \gamma\left(\hat{\mathbf{a}} + \hat{\mathbf{b}}\right)$$

Figure 7: Formulation of the problem for our strategy of locating the point, along any one of the three bisectors, that is equidistant from the three sides. The center must lie along the direction of $\hat{\mathbf{a}} + \hat{\mathbf{b}}$ in order to be equidistant from the sides a and b.

6.1 Preliminary Expressions for r and γ

6.1.1 Our First Way of Expressing r, and the Resulting Equation for γ

As shown in Fig. 8, expressing r in terms of $\mathsf{Rej}_{\mathbf{a}}(\mathbf{z})$ leads to

$$r = \frac{\left\| \left\{ \gamma \left(\hat{\mathbf{a}} + \hat{\mathbf{b}} \right) \wedge \mathbf{a} \right\} \right\|}{a}. \tag{6.2}$$

Now, we'll use that equation to derive an equation for γ in terms of A, r, and the sides of the triangle.

$\hat{\mathbf{a}} \wedge \mathbf{a} = 0.$

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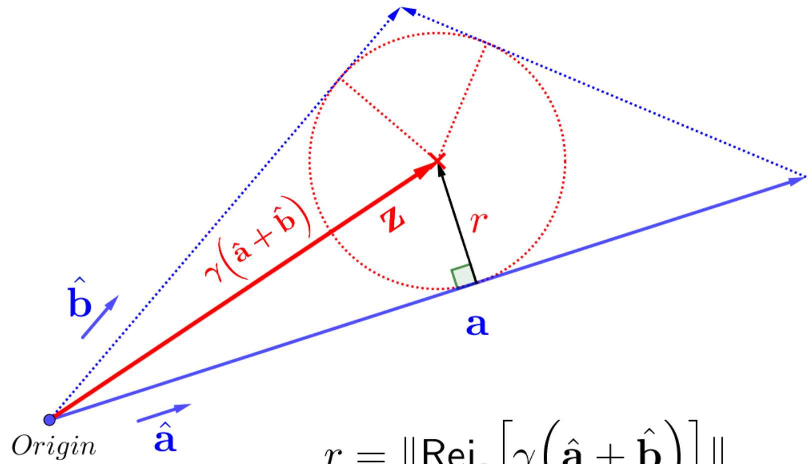
We multiply by the scalar b to convert $\hat{\mathbf{b}} \wedge \mathbf{a}$ into $\mathbf{b} \wedge \mathbf{a}$, so that we may then use Eq. (6.1).

$$\begin{aligned}
r &= \frac{\left\| \left\{ \gamma \left(\hat{\mathbf{a}} + \hat{\mathbf{b}} \right) \wedge \mathbf{a} \right\} \right\|}{a} \\
&= \gamma \left[\frac{\|\hat{\mathbf{b}} \wedge \mathbf{a}\|}{a} \right] \\
&= \gamma \left[\frac{\|b\hat{\mathbf{b}} \wedge \mathbf{a}\|}{ab} \right] \\
&= \gamma \left[\frac{\|\mathbf{b} \wedge \mathbf{a}\|}{ab} \right] \\
&= \gamma \left[\frac{2A}{ab} \right] \text{ (because of Eq. (6.1)).} \\
\therefore \gamma &= \frac{abr}{2A}.
\end{aligned} \tag{6.3}$$



$$r = \|\mathsf{Rej}_{\mathbf{a}}\left[\gamma\left(\hat{\mathbf{a}}+\hat{\mathbf{b}}\right)\right]\|$$

$$= \frac{\|\left\{\gamma\left(\hat{\mathbf{a}}+\hat{\mathbf{b}}\right)\wedge\mathbf{a}\right\}\|}{a}$$

Figure 8: Simplifying the diagram to derive the relationship between r and γ.

6.1.2 Our Second Expression for r

As shown in Fig. (9),

$$r = \|\mathsf{Rej}_{\mathbf{b-a}}\left(\mathbf{a}\right)\| - \|\mathsf{Rej}_{\mathbf{b-a}}\left(\mathbf{z}\right)\|. \tag{6.4}$$

Let's simplify the first term on the right-hand side. Because $\mathbf{b}-\mathbf{a}$ and $\mathbf{a}$ are two sides of the triangle, Eq. (6.1) leads to

$$\|\mathsf{Rej}_{\mathbf{b-a}}\left(\mathbf{a}\right)\| = \frac{2A}{\|\mathbf{b}-\mathbf{a}\|} = \frac{2A}{c}. \tag{6.5}$$

The term $\|\mathsf{Rej}_{\mathbf{b-a}}\left(\mathbf{z}\right)\|$ requires a bit more work because $\mathbf{z}$ is not one of the

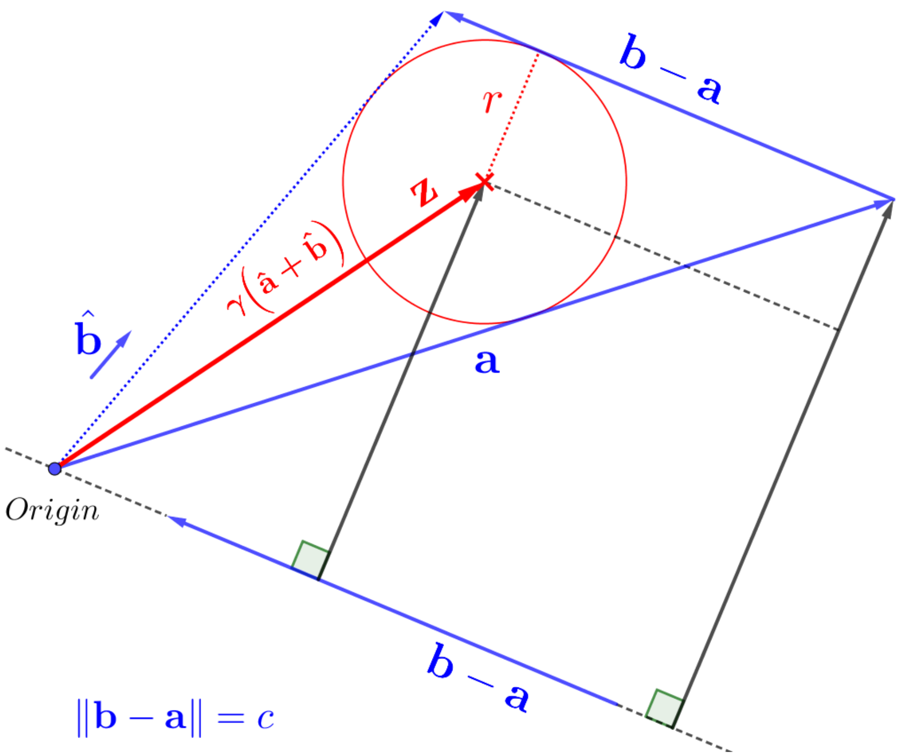


Figure 9: A key idea: $r=\|\mathsf{Rej}_{\mathbf{b}-\mathbf{a}}\left(\mathbf{a}\right)\|-\|\mathsf{Rej}_{\mathbf{b}-\mathbf{a}}\left(\mathbf{z}\right)\|$.

sides of the triangle. Therefore, we proceed as follows, based upon Eq. (2.1):

$$\begin{aligned}
\|\mathsf{Rej}_{\mathbf{b}-\mathbf{a}}(\mathbf{z})\| &= \frac{\mathbf{z}\wedge(\mathbf{b}-\mathbf{a})}{\|\mathbf{b}-\mathbf{a}\|} \\
&= \frac{\left[\gamma\left(\hat{\mathbf{a}}+\hat{\mathbf{b}}\right)\right]\wedge(\mathbf{b}-\mathbf{a})}{c} \\
&= \gamma\left[\frac{\hat{\mathbf{a}}\wedge\mathbf{b}+\mathbf{a}\wedge\hat{\mathbf{b}}}{c}\right] \\
&= \gamma\left[\frac{ab\left(\hat{\mathbf{a}}\wedge\mathbf{b}+\mathbf{a}\wedge\hat{\mathbf{b}}\right)}{abc}\right] \\
&= \gamma\left[\frac{b\,\mathbf{a}\wedge\mathbf{b}+a\,\mathbf{a}\wedge\mathbf{b}}{abc}\right]. \\
&= \gamma\left[\frac{2A\,(b+a)}{abc}\right].
\end{aligned}$$

Combining that result with Ecs. (6.4) and (6.5),

$$r = \frac{2A}{c} - \gamma\left[\frac{2A\,(b+a)}{abc}\right]. \tag{6.6}$$

6.2 The Resulting Final Equations for r and γ

From Eq. (6.3), $\gamma = \frac{abr}{2A}$. By making that substitution in Eq. (6.6), then solving for r, we obtain

$$r = \frac{2A}{a+b+c}. \tag{6.7}$$

If we wish, we can now express A in terms of a, b, and c via Heron's formula:

$$\begin{aligned}
&A = \sqrt{s(s-a)(s-b)(s-d)};\ \text{with} \\
&s = (a+b+c)/2.
\end{aligned}$$

Because $a+b+c=2s$, Eq. (6.7) thus becomes

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Substituting the result from Eq. (6.7) into Eq. (6.6), and solving for γ,

$$\gamma = \frac{ab}{a+b+c}. \tag{6.8}$$

Therefore, the vector from the origin to the circle's center is

$$\begin{aligned}
\mathbf{z} &= \left[\frac{ab}{a+b+c}\right]\left(\hat{\mathbf{a}}+\hat{\mathbf{b}}\right) \\
&= \frac{b\mathbf{a}+a\mathbf{b}}{a+b+c}.
\end{aligned} \tag{6.9}$$

The results are illustrated in Fig. 10.

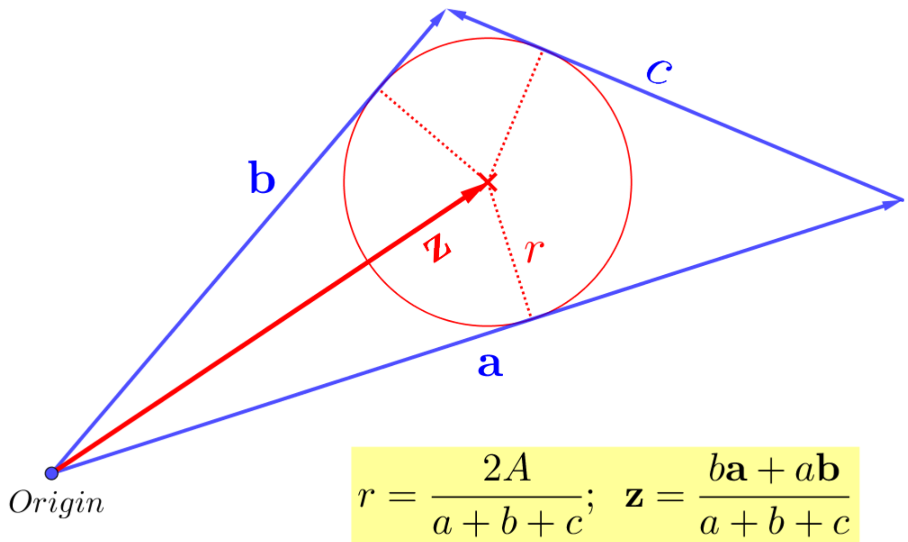


Figure 10: The results.

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).