Model $\lambda(\varphi^{2n})_4, n \geq 2$ Quantum Field Theory: A Nonstandard Approach Based on Nonstandard Pointwise-Defined Quantum Fields

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Abstract. A new non-Archimedean approach to interacted quantum fields is presented. In proposed approach, a field operator $\varphi(x,t)$ no longer a standard tempered operator-valued distribution, but a non-classical operator-valued function. We prove using this novel approach that the quantum field theory with Hamiltonian $P(\varphi)_4$ exists and that the corresponding C^*- algebra of bounded observables satisfies all the Haag-Kastler axioms except Lorentz covariance. We prove that the $\lambda(\varphi^{2n})_4, n \geq 2$ quantum field theory models are Lorentz covariant.

1. Introduction

Extending the real numbers $\mathbb{R}$ to include infinite and infinitesimal quantities originally enabled D. Laugwitz [1] to view the delta distribution $\delta(x)$ as a nonstandard point function. Independently A. Robinson [2] demonstrated that distributions could be viewed as generalized polynomials. Luxemburg [3] and Sloan [4] presented an alternate representative of distributions as internal functions within the context of canonical Robinson's theory of nonstandard analysis. For further information on classical model theoretical nonstandard analysis namely NSA, we refer to [5]-[8].

Abbreviation 1.1 In this paper we adopt the following canonical notations. For a standard set E we often write E_{st}. For a set E_{st} let ${}^{\sigma}E_{\mathrm{st}}$ be a set ${}^{\sigma}E_{\mathrm{st}} = \{{}^*x | x \in E_{\mathrm{st}}\}$. We identify z with ${}^{\sigma}z$ i.e., $z \equiv {}^{\sigma}z$ for all $z \in \mathbb{C}$. Hence, ${}^{\sigma}E_{\mathbf{st}} = E_{\mathrm{st}}$ if $E \subseteq \mathbb{C}$, e.g., ${}^{\sigma}\mathbb{C} = \mathbb{C}$, ${}^{\sigma}\mathbb{R} = \mathbb{R}$, ${}^{\sigma}P = P$, ${}^{\sigma}L_+^{\uparrow} = L_+^{\uparrow}$, etc.

Let ${}^*\mathbb{R}_{\approx}$, ${}^*\mathbb{R}_{\approx+}$, ${}^*\mathbb{R}_{\mathrm{fin}}$, ${}^*\mathbb{R}_{\infty}$, and ${}^*\mathbb{N}_{\infty}$ denote the sets of infinitesimal hyper-real numbers, positive infinitesimal hyper-real numbers, finite hyper-real numbers, infinite hyper-real numbers and infinite hyper natural numbers, respectively.

Note that: ${}^*\mathbb{R}_{\mathrm{fin}} = {}^*\mathbb{R} \backslash {}^*\mathbb{R}_{\infty}$, ${}^*\mathbb{C} = {}^*\mathbb{R} + \mathrm{i}{}^*\mathbb{R}$, ${}^*\mathbb{C}_{\mathrm{fin}} = {}^*\mathbb{R}_{\mathrm{fin}} + \mathrm{i}{}^*\mathbb{R}_{\mathrm{fin}}$.

Definition 1.1 Let $\{X, \|\cdot\|\}$ be a standard Banach space. For $x \in {}^*X$ and $\varepsilon > 0, \varepsilon \approx 0$ we define the open $\approx$-ball about x of radius ε to be the set $B_{\varepsilon}(x) = \{y \in {}^*X | {}^*\|x - y\| < \varepsilon\}$.

Definition 1.2 Let $\{\{X, \|\cdot\|\}$ be a standard Banach space, $Y \subset X$, thus ${}^*Y \subset {}^*X$ and let $x \in {}^*X$.Then x is an $*$- accumulation point of ${}^*Y$ if for any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$ there is a hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$in ${}^*Y$ such that $\{x_n\}_{n=1}^{{}^*\infty} \cap (B_{\varepsilon}(x) \backslash \{x\} \neq \emptyset)$.

Definition 1.3 Let $\{\{X, \|\cdot\|\}$ be a standard Banach space, let${}^*Y \subseteq {}^*X$,${}^*Y$ is $*$ -closed if any $*$-accumulation point of ${}^*Y$ is an element of ${}^*Y$.

Definition 1.4 Let $\{\{X, \|\cdot\|\}$ be a standard Banach space. We shall say that internal hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$in ${}^*X$ is $*$ -converges to $x \in {}^*X$ as $n \to {}^*\infty$ if for any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$ there is $N \in {}^*\mathbb{N}$ such that for any $n > N$: ${}^*\|x - y\| < \varepsilon$.

Definition 1.5 Let $\{\{X, \|\cdot\|_X\}$, $\{\{Y, \|\cdot\|_Y\}$ be a standard Banach spaces. A linear internal operator $A: D(A) \subseteq {}^*X \to {}^*Y$ is $*$ -closed if for every internal hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$ in $D(A)$ $*$ -converging to $x \in {}^*X$ such that $Ax_n \to y \in {}^*Y$ as $n \to {}^*\infty$ one has $x \in D(A)$ and $Ax = y$. Equivalently , A is $*$-closed if its graph is $*$ -closed in the direct sum ${}^*X \oplus {}^*Y$.

Definition 1.6 Let H be a standard external Hilbert space. The graph of the internal linear transformation $T: {}^*H \to {}^*H$ is the set of pairs $\{\langle\varphi, T\varphi\rangle | \varphi \in D(T)\}$. The graph of T, denoted by $\Gamma(\mathrm{T})$, is thus a subset of ${}^*H \times {}^*H$ which is internal Hilbert space with inner product $(\langle\varphi_1, \psi_1\rangle, \langle\varphi_2, \psi_2\rangle) =$

$(\varphi_1, \varphi_2) + (\psi_1, \psi_2)$.The operator T is called a $*$-closed operator if $\Gamma(T)$ is a $*$ -closed subset of Cartesian product ${}^*H \times {}^*H$.

Definition 1.7 Let H be a standard Hilbert space. Let T_1 and T be internal operators on internal Hilbert space ${}^*H$. Note that if $\Gamma(T_1) \supset \Gamma(T)$, then T_1 is said to be an extension of T and we write $T_1 \supset T$. Equivalently, $T_1 \supset T$ if and only if $D(T_1) \supset D(T)$ and $T_1\varphi = T\varphi$ for all $\varphi \in D(T)$.

Definition 1.8 Any internal operator T on ${}^*H$ is $*$-closable if it has a $*$-closed extension. Every $*$-closable internal operator T has a smallest $*$-closed extension, called its $*$-closure, which we denote by $*$-$\bar{T}$.

Definition 1.9 Let H be a standard Hilbert space. Let T be a $*$-densely defined internal linear operator on internal Hilbert space ${}^*H$. Let $D(T^*)$ be the set of $\varphi \in {}^*H$ for which there is a vector $\xi \in {}^*H$ with $(T\psi, \varphi) = (\varphi, \xi)$ for all $\psi \in D(T)$, then for each $\varphi \in D(T^*)$, we define $T^*\varphi = \xi$. T^* is called the $*$-adjoint of T. Note that $S \subset T$ implies $T^* \subset S^*$.

Definition 1.10 Let H is a standard Hilbert space. A $*$-densely defined internal linear operator T on internal Hilbert space ${}^*H$ is called symmetric (or Hermitian) if $T \subset T^*$. Equivalently, T is symmetric if and only if $(T\varphi, \psi) = (\varphi, T\psi)$ for all $\varphi, \psi \in D(T)$.

Definition 1.11 Let H be a standard Hilbert space. A symmetric internal linear operator T on internal Hilbert space ${}^*H$ is called essentially self- $*$-adjoint if its $*$-closure $*$-$\bar{T}$ is self- $*$-adjoint. If T is $*$-closed, a subset $D \subset D(T)$ is called a $*$-core for T if $*$- $\left(\overline{T \upharpoonright D}\right) = T$. If T is essentially self- $*$-adjoint, then it has one and only one self -$*$-adjoint extension.

Let F be the standard Fock space [9],[10] for a massive, neutral scalar field in four-dimensional space-time [10]. The elements of ${}^*F$ are internal sequences of functions on internal momentum space ${}^*\mathbb{R}^3$. Let the standard annihilation and creation operators be normalized by the relation

$$\left[a(k), a^\dagger(k')\right] = \delta^3(k - k'). \tag{1.1}$$

so that the free-field Hamiltonian with finite momentum cut-off $\sigma \in {}^\sigma\mathbb{R}$ is

$$H_{0,\sigma} = \int_{|k|\leq\sigma} a^\dagger(k')a(k)\mu(k)d^3k\,, \mu(k) = \sqrt{k_1^2 + k_2^2 + k_3^2}\,. \tag{1.2}$$

From (1.1) by transfer one obtains

$$\left[{}^*a(k), {}^*a^\dagger(k')\right] = {}^*\delta^3(k - k'), \tag{1.3}$$

so that internal free-field Hamiltonian with hyperfinite cut-off $\varkappa \in {}^*\mathbb{R}_{+;\infty}$ is

$${}^*H_{0,\varkappa} = {}^*\!\!\int_{|k|\leq\varkappa} {}^*a^\dagger(k')\left({}^*a(k)\right)\left({}^*\mu(k)\right)d^3k. \tag{1.4}$$

The $t = 0$ internal field ${}^*\varphi_\varkappa(x)$ with hyperfinite momentum cut-off $\varkappa \in {}^*\mathbb{R}_{+;\infty}$ is

$${}^*\varphi_\varkappa(x) = \frac{1}{(2\pi)^{3/2}}\, {}^*\!\!\int_{|k|\leq\varkappa} {}^*e^{-i(k,x)}\left[{}^*a^\dagger(k\,) + {}^*a(-k)\right]\frac{d^3k}{{}^*\left(\sqrt{2\mu(k)}\right)}. \tag{1.5}$$

The spatially cut-off internal interaction Hamiltonian with hyperfinite momentum cut-off $\varkappa \in {}^*\mathbb{R}_{+;\infty}$ is

$${}^*H_{I,\varkappa}(g) = \sum_{j=0}^{4} \binom{4}{j}\, {}^*\!\!\int_{|k_1|\leq\varkappa} \cdots\, {}^*\!\!\int_{|k_{j+1}|\leq\varkappa} \cdots\, {}^*\!\!\int_{|k_4|\leq\varkappa} {}^*a^\dagger(k_1\,) \cdots {}^*a^\dagger\left(k_j\,\right){}^*a\left(-k_{j+1}\right) \times$$

$$\times\ {}^*a(-k_4)\left({}^*\hat{g}\left(\sum_{i=1}^{4} k_i\right)\right)\prod_{i=1}^{4} {}^*\mu(k)^{1/2} d^3k_i. \tag{1.6}$$

We also need internal number operator with hyperfinite momentum cut-off $\varkappa \in {}^*\mathbb{R}_{+;\infty}$

$${}^*N_\varkappa = {}^*\!\int_{|k|\leq\varkappa} {}^*a^\dagger(k\ )\, {}^*a(k) d^3k \tag{1.7}$$

and the domain

$$D_{0,\varkappa} = \bigcap_{n\in{}^*\mathbb{N}} D\left({}^*H_{0,\varkappa}^n\right). \tag{1.8}$$

Remark 1.1 Note that the domain $D_{0,\varkappa}$ is a nonstandard *external* set so there is no standard set $\mathcal{D}$ such that $D_{0,\varkappa} = {}^*\mathcal{D}$.

Proposition 1.1 Let W_σ be a standard operator $W_\sigma : F \to F$ of the form

$$W_\sigma = \int_{|k_1|\leq\sigma} \cdots \int_{|k_m|\leq\sigma} w(k_1, \ldots, k_m)\ a^\dagger(k_{\ 1}) \cdots\ a(-k_m) \prod_{i=1}^{m} d^3k_i \tag{1.9}$$

and let N_σ be a standard operator $N_\sigma : F \to F$ of the form

$$N_\sigma = \int_{|k|\leq\varkappa} a^\dagger(k\ )\ (k) d^3k. \tag{1.10}$$

Assume that for all σ such that $0 < \sigma < \infty$ the inequality holds

$$\int \cdots \int \chi_\sigma(k_1, \ldots, k_m) w^2(k_1, \ldots, k_m) \prod_{i=1}^{m} d^3k_i < \infty,$$

where $\chi_\sigma(k_1, \ldots, k_m) = 1$ if $|k_i| \leq \sigma$ for all $1 \leq i \leq m$, and $\chi_\kappa(k_1, \ldots, k_m) = 0$ otherwise. Then for all σ such that $0 < \sigma < \infty$ and for all j such that $|j| \leq m$ the inequality holds

$$\left\|(N_\sigma + I)^{-\frac{j}{2}} W_\sigma (N_\sigma + I)^{\frac{(m-j)}{2}}\right\| \leq$$

$$\leq \left(\int \cdots \int \chi_\sigma(k_1, \ldots, k_m) w^2(k_1, \ldots, k_m) \prod_{i=1}^{m} d^3k_i\right)^{\frac{1}{2}}. \tag{1.11}$$

Proposition 1.2 Let ${}^*W_\varkappa$ be internal operator ${}^*W_\varkappa : {}^*F \to {}^*F$ of the form

$${}^*W_\varkappa = {}^*\!\int_{|k_1|\leq\varkappa} \cdots {}^*\!\int_{|k_{\boldsymbol{m}}|\leq\varkappa} {}^*w(k_1, \ldots, k_{\boldsymbol{m}})\, {}^*a^\dagger(k_{\ 1}) \cdots {}^*a(-k_{\boldsymbol{m}}) \prod_{i=1}^{\boldsymbol{m}} d^3k_i. \tag{1.12}$$

Then for all $\varkappa$ such that $\varkappa \in {}^*\mathbb{R}_{+}$,and for all j such that $|j| \leq m, m \in {}^*\mathbb{N}_\infty$ the inequality holds

$$\left\|(N_\varkappa + I)^{-\frac{j}{2}} W_\varkappa (N_\varkappa + I)^{\frac{(\boldsymbol{m}-j)}{2}}\right\| \leq$$

$$\leq \left({}^*\!\int \cdots {}^*\!\int {}^*\chi_\varkappa(k_1, \ldots, k_{\boldsymbol{m}})\, {}^*w^2(k_1, \ldots, k_m) \prod_{i=1}^{\boldsymbol{m}} d^3k_i\right)^{\frac{1}{2}}. \tag{1.13}$$

Proof It follows directly from (1.11) by transfer.

Remark 1.2 It follows from (2.11) that:

(1) ${}^*H_{I,\varkappa}(g)$ is well defined on the domain $D_{0,\varkappa}$,

(2) there is a $*$-closure $*\text{-}\overline{{}^*H_{I,\varkappa}(g)}$ with domain $D\left(*\ \text{-}\overline{{}^*H_{I,\varkappa}(g)}\right) \supset D_{0,\varkappa}$,

(3) external set $D_{0,\varkappa}$ is a $*$-core for ${}^*H_{I,\varkappa}(g)$ i.e., $*\text{-}\left(\overline{{}^*H_{I,\varkappa}(g) \upharpoonright D_{0,\varkappa}}\right) = {}^*H_{I,\varkappa}(g)$

Remark 1.3 The operator $*\text{-}\overline{{}^*H_{I,\varkappa}(g)}$ is external mapping $*\text{-}\overline{{}^*H_{I,\varkappa}(g)}: {}^*F \to {}^*F$ i.e., there is no standard operator $T: F \to F$ with domain $D(T)$ such that:

(1) ${}^*D(T) = D\left(*\text{-}\overline{{}^*H_{I,\varkappa}(g)}\right)$ and (2) ${}^*T \upharpoonright {}^*D(T) = *\text{-}\overline{{}^*H_{I,\varkappa}(g)} \upharpoonright D\left(*\text{-}\overline{{}^*H_{I,\varkappa}(g)}\right)$.

Thus we cannot derive the desired properties of the operator $*\text{-}\overline{{}^*H_{I,\varkappa}(g)}$ by using Robinson transfer principle [2]-[7].

As that has been explained in [8] classical model theoretical nonstandard analysis NSA does not power enough to resolve the stated in [8] problems in constructive quantum field theory related to physical dimension $d = 4$,

In order to avoid any difficultness mentioned above, in this paper as in [8] we deal by using a non-conservative extension of NSA developed in [11].

Remind that Robinson nonstandard analysis (NSA) many developed using set theoretical objects called super-structures [5]-[7]. A superstructure V(S) over a set S is defined in the following way: $\mathrm{V}_0(\mathrm{S}) = \mathrm{S}$, $\mathrm{V}_{n+1}(\mathrm{S}) = \mathrm{V}_n(\mathrm{S}) \cup \mathrm{P}\left(\mathrm{V}_n(\mathrm{S})\right)$, $\mathrm{V}(\mathrm{S}) = \bigcup_{n\in\mathbb{N}} \mathrm{V}_{n+1}(\mathrm{S})$. Making $\mathrm{S} = \mathbb{R}$ will suffice for virtually any construction necessary in analysis. Bounded formulas are formulas where all quantifiers occur in the form: $\forall x\ (x \in y \to \cdots)$, $\exists x\ (x \in y \to \cdots)$. A nonstandard embedding is a mapping $*: \mathrm{V}(X) \to \mathrm{V}(Y)$ from a superstructure $\mathrm{V}(X)$ called the standard universe, into another superstructure $\mathrm{V}(Y)$ called nonstandard universe, satisfying the following postulates:
1. $Y = {}^*X$
2. ***Transfer Principle*** For every bounded formula $\Phi(x_1, \dots, x_n)$ and elements $a_1, \dots, a_n \in V(X)$ the property $\Phi(a_1, \dots, a_n)$ is true for $a_1, \dots, a_n$ in the standard universe if and only if it is true for ${}^*a_1, \dots, {}^*a_n$ in the nonstandard universe $V(X) \models \Phi(x_1, \dots, x_n) \leftrightarrow V(Y) \models \Phi({}^*a_1, \dots, {}^*a_n)$.
3. ***Non-triviality*** For every infinite set A in the standard universe, the set $\{{}^*a | a \in A\}$ is a proper subset of ${}^*A$.
Definition 1.12 A set x is internal if and only if x is an element of ${}^*A$ for some $A \in \mathrm{V}(\mathbb{R})$. Let X be a set and $A = \{A_i\}_{i\in I}$ a family of subsets of X .Then the collection A has the infinite intersection property, if any infinite sub collection $J \subset I$ has non-empty intersection. Nonstandard universe is σ-saturated if whenever $\{A_i\}_{i\in I}$ is a collection of internal sets with the infinite intersection property and the cardinality of I is less than or equal to σ.
Remark 1.4 For each standard universe $U = V(X)$ there exists canonical language L_U and for each nonstandard universe $W = V(Y)$ there exists corresponding canonical nonstandard language ${}^*L = L_W$ [5],[7]

4.***The restricted rules of conclusion*** If Let A and B well formed, closed formulas so that $A, B \in {}^*L$. If $W \models A$, then $\neg A \nvdash_{RMP} B$. Thus, if a statement A holds in nonstandard universe, we cannot obtain from formula $\neg A$ any formula B whatsoever.
Definition 1.13 [8] A set $S \subset {}^*\mathbb{N}$ is a hyper inductive if the following statement holds in $V(Y)$:

$$\bigwedge_{\alpha\in{}^*\mathbb{N}}(\alpha \in S \to \alpha^+ \in S).$$

Here $\alpha^+ = \alpha + 1$.Obviously a set ${}^*\mathbb{N}$ is a hyper inductive.

5. ***Axiom of hyper infinite induction***

$$\forall S(S \subset {}^*\mathbb{N})\left\{\forall\beta(\beta \subset {}^*\mathbb{N})\left[\bigwedge_{1\le\alpha<\beta}(\alpha \in S \to \alpha^+ \in S)\right] \to S = {}^*\mathbb{N}\right\}.$$

Example 1.1 Remind the proof of the following statement: structure $(\mathbb{N},<,=)$ is a well-ordered set.
Proof Let X be a nonempty subset of $\mathbb{N}$. Suppose X does not have a <-least element. Then consider the set $\mathbb{N}\backslash X$. Case1. $\mathbb{N}\backslash X=\emptyset$. Then $X=\mathbb{N}$ and so 0 is a $<$ -least element but this is a contradiction. Case2. $\mathbb{N}\backslash X\neq\emptyset$. Then $1\in\mathbb{N}\backslash X$ otherwise 1 is a $<$ -least element but this is a contradiction. Assume now that there exists some $n\in\mathbb{N}\backslash X$ such that $n\neq 1$, but since we have supposed that X does not have a $<$ -least element, thus $n+1\notin X$. Thus we see that for all n the statement $n\in\mathbb{N}\backslash X$ implies that $n+1\in\mathbb{N}\backslash X$. We can conclude by axiom of induction that $n\in\mathbb{N}\backslash X$ for all $n\in\mathbb{N}$. Thus $\mathbb{N}\backslash X=\mathbb{N}$ implies $X=\emptyset$. This is a contradiction to X being a non-empty subset of $\mathbb{N}$. Remind that structure $({}^*\mathbb{N},<,=)$ is not a well-ordered set [5]-[7]. We set now $X_1={}^*\mathbb{N}\backslash\mathbb{N}$ and thus ${}^*\mathbb{N}\backslash X_1=\mathbb{N}$. In contrast with a set X mentioned above the assumption $n\in{}^*\mathbb{N}\backslash X_1$ implies that $n+1\in{}^*\mathbb{N}\backslash X_1$ if and only if n is finite, since for any infinite $n\in{}^*\mathbb{N}\backslash\mathbb{N}$ the assumption $n\in{}^*\mathbb{N}\backslash X_1$ contradicts with a true statement $V(Y)\vDash n\notin{}^*\mathbb{N}\backslash X_1=\mathbb{N}$ and therefore in accordance with postulate 4 we cannot obtain from $n\in{}^*\mathbb{N}\backslash X_1$ any closed formula B whatsoever.
For further information on non-classical nonstandard analysis namely$NSA^{\#}$, we refer to [8]-[13].
Abbreviation1.2 In this paper we adopt the following notations [8]. For a standard set E we often write $E_{\rm st}$, let ${}^\sigma E_{\rm st}=\{{}^*x|x\in E_{\rm st}\}$.We identify z with ${}^\sigma z$ i.e., $z\equiv{}^\sigma z$ for all $z\in\mathbb{C}$. Hence, ${}^\sigma E_{\rm st}=E_{\rm st}$ if $E\subseteq\mathbb{C}$, e.g., ${}^\sigma\mathbb{C}=\mathbb{C}$, ${}^\sigma\mathbb{R}=\mathbb{R}$, etc. Let ${}^*\mathbb{R}_c^{\#}$, ${}^*\mathbb{R}_{c,\approx}^{\#}$, ${}^*\mathbb{R}_{c,\approx+}^{\#}$, ${}^*\mathbb{R}_{c,\rm fin}^{\#}$, ${}^*\mathbb{R}_{c,\infty}^{\#}$, ${}^*\mathbb{N}_\infty$ de-note the sets of Cauchy hyper-real numbers, Cauchy infinitesimal hyper-real numbers, Cauchy positive infinitesimal hyperreal numbers, Cauchy finite hyper-real numbers, Cauchy infinite hyper-real numbers and infinite hypernatural numbers, respectively. Note that ${}^*\mathbb{R}_{c,\rm fin}^{\#}={}^*\mathbb{R}_c^{\#}\backslash{}^*\mathbb{R}_{c,\infty}^{\#}$.
Definition 1.13 Let H be external hyper infinite dimensional vector space over the complex field ${}^*\mathbb{C}_c^{\#}={}^*\mathbb{R}_c^{\#}+i{}^*\mathbb{R}_c^{\#}$. An inner product on H is a${}^*\mathbb{C}_c^{\#}$-valued function, $\langle\cdot,\cdot\rangle\colon H\times H\to{}^*\mathbb{C}_c^{\#}$, such that (1) $\langle ax+by,z\rangle=\langle ax,z\rangle+\langle by,z\rangle$, (2) $\overline{\langle x,y\rangle}=\langle y,x\rangle$, (3) $\|x\|^2\equiv\langle x,x\rangle\geq 0$ with equality $\langle x,x\rangle=0$ if and only if $x=0$.
Theorem 1.1 (Generalized Schwarz Inequality) Let $\{H,\langle\cdot,\cdot\rangle\}$be an inner product space, then for all $x,y\in H$: $|\langle x,y\rangle|\leq\|x\|\|y\|$ and equality holds if and only if x and y are linearly dependent.
Theorem 1.2 Let $\{H,\langle\cdot,\cdot\rangle\}$be an inner product space, and $\|x\|_{\#}=\sqrt{\langle x,x\rangle}$. Then $\|\cdot\|_{\#}$ is a ${}^*\mathbb{R}_c^{\#}$ -valued #-norm on a space H. Moreover $\langle x,x\rangle$ is #-continuous on Cartesian product $H\times H$, where H is viewed as the #-normed space $\{H,\|\cdot\|_{\#}\}$.
Definition 1.14 A non-Archimedean Hilbert space H is a #-complete inner product space.
Two elements x and y of non-Archimedean Hilbert space H are called orthogonal if $\langle x,y\rangle=0$.
Definition 1.15 The graph of the linear transformation $T\colon H\to H$ is the set of pairs $\{\langle\phi,T\phi\rangle|(\phi\in D(T))\}$. The graph of the operator T, denoted by $\Gamma(\mathrm{T})$, is thus a subset of $H\times H$ which is a non-Archimedean Hilbert space with the following inner product $(\langle\phi_1,\psi_1\rangle,\langle\phi_2,\psi_2\rangle)$. Operator T is called a #-closed operator if $\Gamma(\mathrm{T})$ is a #-closed subset of $H\times H$.
Definition 1.16 Let T_1 and T be operators on H. If $\Gamma(T_1)\supset\Gamma(\mathrm{T})$, then T_1 is said to be an extension of T and we write $T_1\supset T$. Equivalently: $T_1\supset T$ if and only if $D(T_1)\supset D(T)$ and $T_1\phi=T\phi$ for all $\phi\in D(T)$.
Definition 1.17 An operator T is #-closable if it has a #-closed extension. Every #-closable operator has a smallest #-closed extension, called its #-closure, which we denote by #-T.
Theorem 1.3 If T is #-closable, then $\Gamma(\#\text{-}\overline{T})=\#\text{-}\overline{\Gamma(T)}$.
Definition 1.18 Let $D(T^*)$ be the set of $\varphi\in H$ for which there is an $\xi\in H$ with $(T\psi,\varphi)=(\psi,\xi)$ for all $\psi\in D(T)$. For each $\varphi\in D(T^*)$, we define $T^*\varphi=\xi$.The operator T^* is called the #-adjoint of T. Note that $\varphi\in D(T^*)$ if and only if $|(T\psi,\varphi)|\leq C\|\psi\|_{\#}$ for all $\psi\in D(T)$. Note that $S\subset T$ implies $T^*\subset S$.
Remark 1.5 Note that for ξ to be uniquely determined by the condition $(T\psi,\varphi)=(\psi,\xi)$ one need

the fact that $D(T)$ is #-dense in H. If the domain $D(T^{*})$ is #-dense in H, then we can define $T^{**}=(T^{*})^{*}$.

Theorem 1.4 Let T be a #-densely defined operator on a non-Archimedean Hilbert space H. Then: (a) T^{*} is #-closed. (b) The operator T is #-closabie if and only if $D(T^{*})$ is -dense in which case $T=T^{**}$. (c) If T is #-closable, then $(\#\text{-}\bar{T})^{*}=T^{*}$.

Definition 1.19 Let T be a #-closed operator on a non-Archimedean Hilbert space H. A complex number $\lambda\in{}^{*}\mathbb{C}_{c}^{\#}$ is in the resolvent set $\rho(T)$, if $\lambda I-T$ is a bijection of $D(T)$ onto H with a finitely or hyper finitely bounded inverse. If complex number $\lambda\in\rho(T)$, $R_{\lambda}=(\lambda I-T)^{-1}$ is called the resolvent of T at λ.

Definition 1.20 A #-densely defined operator T on a non-Archimedean Hilbert space is called symmetric or Hermitian if $T\subset T^{*}$, that is, $D(T)\subset D(T^{*})$ and $T\varphi=T^{*}\varphi$ for all $\varphi\in D(T)$ and equivalently, T is symmetric if and only if $(T\varphi,\psi)=(\varphi,T\psi)$ for all $\varphi,\psi\in D(T)$.

Definition 1.21 A #-densely defined operator T is called self-#-adjoint if $T=T^{*}$, that is, if and only if T is symmetric and $D(T)=D(T^{*})$.

Remark 1.6 A symmetric operator T is always #-closable, since $D(T)$ #-dense in H. If T is symmetric, T^{*} is a #-closed extension of T so the smallest #-closed extension T^{**} of T must be contained in T^{*}. Thus for symmetric operators, we have $T\subset T^{**}\subset T^{*}$, for #-closed symmetric operators we have $T=T^{**}\subset T^{*}$ and, for self-#-adjoint operators we have $T=T^{**}=T^{*}$. Thus a #-closed symmetric operator T is self-#-adjoint if and only if T^{*} is symmetric.

Definition 1.22 A symmetric operator T is called essentially self-#-adjoint if its #-closure #-$\bar{T}$ is self-#-adjoint. If T is #-closed, a subset $D\subset D(T)$ is called a core for T if #-$\overline{T\restriction D}=T$.

Remark 1.7 If T is essentially self-#-adjoint, then it has one and only one self-#-adjoint extension.

Theorem 1.5 [8] (see [8], sect.15.1) If $g\in S_{\text{fin}}^{\#}\left({}^{*}\mathbb{R}_{c}^{\#\mathbf{3}}\right)$ is real, then

$$H_{I,\varkappa}(g)=Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}}:\varphi_{\varkappa}^{\#4}(x):g(x)\,d^{\#3}x \tag{1.14}$$

is essentially self #-adjoint on the domain $D_{0,\varkappa}^{\#}=\bigcap_{n=0}^{{}^{*}\infty}D\left(H_{0,\varkappa}^{n}\right)$.

Here $\varphi_{\varkappa}^{\#}(x)$ is a nonstandard pointwise-defined operator valued function $\varphi_{\varkappa}^{\#}:{}^{*}\mathbb{R}_{c}^{\#3}\to L\left(\mathcal{F}^{\#}\right)$

$$\varphi_{\varkappa}^{\#}(x)=\frac{1}{(2\pi)^{3/2}}\,Ext\text{-}\int_{|k|\leq\varkappa}(Ext\text{-}\exp[-i(k,x)])\left[a^{\dagger}(k)+a(-k)\right]\frac{d^{\#3}k}{\sqrt{2\mu(k)}}, \tag{1.15}$$

where $\varkappa\in{}^{*}\mathbb{R}_{c+,\infty}^{\#}$.

The main purpose of the present paper is to extend the result of [8] to $\lambda(\varphi^{2n})_{4},n>2$. Our notation and definitions are the same as in [8].

We remind that for every function $f\in C_{0}^{{}^{*}\infty}\left({}^{*}\mathbb{R}_{c,\text{fin}}^{\#4},{}^{*}\mathbb{R}_{c,\text{fin}}^{\#}\right),$), the averaged free quantum field

$$\varphi_{\varkappa}^{\#}(f)=\frac{1}{(2\pi)^{3/2}}\,Ext\text{-}\int_{|k|\leq\varkappa}(Ext\text{-}\exp[\,t\mu(k)-i(k,x)])\left[a^{\dagger}(k)+a(-k)\right]f(x)\frac{d^{\#3}k}{\sqrt{2\mu(k)}}d^{\#4}x, \tag{1.16}$$

is a self-#-adjoint operator on a non-Archimedean Fock space $\mathcal{F}^{\#}$ [8].

A non -Archimedean $C_{\#}^{*}$-algebra of local observables $\mathfrak{A}^{\#}$ is defined as the #-norm #-closure [8]

$$\mathfrak{A}^{\#}=\#\text{-}\overline{\bigcup_{O}\mathfrak{A}^{\#}(O)}\,, \tag{1.17}$$

where the union takes place over bounded regions O of space-time, and $\mathfrak{A}^{\#}(O)$ is the von Neumann #-algebra generated by [8]:

$$\left\{Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)+i\pi_{\varkappa}^{\#}(f)\right)|f\in C_0^{^*\infty}\left({}^*\mathbb{R}_{c,\mathrm{fin}}^{\#4},{}^*\mathbb{R}_{c,\mathrm{fin}}^{\#}\right)\right\}.$$

A non –Archimedean near standard $C_{\#}^{*\approx}$-algebra of physical local observables $\mathfrak{A}_{\approx}^{\#}(O)$ is defined as

$$\mathfrak{A}_{\approx}^{\#}(O)=\left\{Q\in\mathfrak{A}^{\#}(O)|\|Q\|_{\#}\in{}^*\mathbb{R}_{c+,\mathrm{fin}}^{\#}\right\}.$$

Let ${}^{\sigma}G$ be the restricted Poincare group of transformations of 4-dimensional Minkowski space-time $\mathcal{M}_4$. Poincare transformations $\left\{a,\Lambda_{\beta_i}^{(i)}\right\}\in{}^{\sigma}G$ generated by a Lorentz boosts along the x^i-direction $i=1,2,3$ and space-time translation $x\to x+a, a=(\alpha^1,\alpha^2,\alpha^3,\tau)$ are

$$\left\{a,\Lambda_{\beta_1}^{(1)}\right\}(x,t)=$$

$$=(\alpha^1+x^1\cosh\beta_1+t\sinh\beta_1,\tau\ +x^1\sinh\beta_1+t\cosh\beta_1,\alpha^2+x^2,\alpha^3+x^3), \qquad (1.18)$$

$$\left\{a,\Lambda_{\beta_2}^{(2)}\right\}(x,t)=$$

$$=(\alpha^1+x^1,\alpha^2+x^2\cosh\beta_2+t\sinh\beta_2,\alpha^3+x^3,\tau\ +x^2\sinh\beta_2+t\cosh\beta_2), \qquad (1.19)$$

$$\left\{a,\Lambda_{\beta_3}^{(3)}\right\}(x,t)=$$

$$=(\alpha^1+x^1,\alpha^2+x^2,\alpha^3+x^3\cosh\beta_3+t\sinh\beta_3,\tau+x^1\sinh\beta_3+t\cosh\beta_3). \qquad (1.20)$$

Theorem 1.6 For every $\left\{a,\Lambda_{\beta_i}^{(i)}\right\}\in{}^{\sigma}G, i=1,2,3$ and for every bounded set $O\subset{}^*\mathbb{R}_{c,\mathrm{fin}}^{\#3}$ there exists a unitary operators $U_O^{(i)}, i=1,2,3$ such that, for all $f\in C_0^{^*\infty}\left({}^*\mathbb{R}_{c,\mathrm{fin}}^{\#4},{}^*\mathbb{R}_{c,\mathrm{fin}}^{\#}\right)$

$$U_O^{(i)}\left[Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(f)\right)\right]\left(U_O^{(i)}\right)^*\approx Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}\left(f_{\left\{a,\Lambda_{\beta_i}^{(i)}\right\}}\right)\right), i=1,2,3, \qquad (1.21)$$

where $f_{\left\{a,\Lambda_{\beta_i}^{(i)}\right\}}(x,t)=f\left(\left\{a,\Lambda_{\beta_i}^{(i)}\right\}(x,t)\right)$. This mappings extends to a representation $\sigma_{\left\{a,\Lambda_{\beta_i}^{(i)}\right\}}$ of $*$-automorphisms of $\mathfrak{A}^{\#}$such that

$$\sigma_{\left\{a,\Lambda_{\beta_i}^{(i)}\right\}}\left(\mathfrak{A}_{\approx}^{\#}(O)\right)\approx\mathfrak{A}_{\approx}^{\#}\left(\left\{a,\Lambda_{\beta_i}^{(i)}\right\}O\right), i=1,2,3. \qquad (1.22)$$

The formal expressions for the Hamiltonian and Lorentz transformation generators are given by [8]

$$H_{\varkappa}=H_{0,\varkappa}+H_{I,\varkappa}=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\left(T_{0,\varkappa}(x)+T_{I,\varkappa}(x)\right)d^{\#3}x, \qquad (1.23)$$

$$M_{\varkappa}^{0k}=M_{0,\varkappa}+M_{I,\varkappa}=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}x^k\left(T_{0,\varkappa}(x)+T_{I,\varkappa}(x)\right)d^{\#3}x,=1,2,3, \qquad (1.24)$$

where

$$T_{0,\varkappa}(x)=\frac{1}{2}\left[:\pi_{\varkappa}^{\#2}(x):+m^2:\varphi_{\varkappa}^{\#2}(x):+:\left(\partial_{x_1}^{\#}\varphi_{\varkappa}^{\#}(x)\right)^2:+:\left(\partial_{x_2}^{\#}\varphi_{\varkappa}^{\#}(x)\right)^2:+:\left(\partial_{x_3}^{\#}\varphi_{\varkappa}^{\#}(x)\right)^2:\right] \quad (1.25)$$

is the free energy density with hyperfinite cut-off $\varkappa\in{}^*\mathbb{R}_{c+,\infty}^{\#}$, and where the interaction energy density $T_{I,\varkappa}(x)$ reads

$$T_{I,\varkappa}(x) =: \varphi_{\varkappa}^{\#2n}(x):. \tag{1.26}$$

Formally one verifies the commutation relations

$$[iH_{\varkappa}, M_{\varkappa}^{0k}] = P_{\varkappa}^{k}, k = 1,2,3 \tag{1.27}$$

and

$$[iH_{\varkappa}, P_{\varkappa}^{k}] = 0, k = 1,2,3, \tag{1.28}$$

where $P_{\varkappa}^{k}, k = 1,2,3$ are the momentum operators $P_{\varkappa}^{k} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} P_{\varkappa}^{k}(x)\, d^{\#3}x$ with densities defined by

$$P_{\varkappa}^{k}(x) = \frac{1}{2}\left[: \pi_{\varkappa}^{\#}(x)\partial_{x_k}^{\#}\varphi_{\varkappa}^{\#}(x): + : \partial_{x_k}^{\#}\varphi_{\varkappa}^{\#}(x)\pi_{\varkappa}^{\#}(x): \right]. \tag{1.29}$$

We wish to prove that $Ext\text{-}\exp(i\beta)M_{\varkappa}^{0k}$ implements Lorentz rotations on suitable domain

$$\left[Ext\text{-}\exp\left(i\beta M_{\varkappa}^{0k}\right)\right]\varphi_{\varkappa}^{\#}(x,t)\left[Ext\text{-}\exp\left(-i\beta M_{\varkappa}^{0k}\right)\right] = \varphi_{\varkappa}^{\#}\left(\Lambda_{\beta}^{(k)}(x,t)\right), k = 1,2,3, \tag{1.30}$$

where

$$\varphi_{\varkappa}^{\#}(x,t) = [Ext\text{-}\exp(itH_{\varkappa})]\varphi_{\varkappa}^{\#}(x)[Ext\text{-}\exp(-itH_{\varkappa})], \tag{1.31}$$

and $\Lambda_{\beta}^{(k)}(x,t) = \left\{0, \Lambda_{\beta}^{(k)}\right\}(x,t)$.

In differential form (1.30) becomes

$$[iM_{\varkappa}^{0k}, \varphi_{\varkappa}^{\#}(x,t)] \approx t\partial_{x_k}^{\#}\varphi_{\varkappa}^{\#}(x,t) + x_k\partial_t^{\#}\varphi_{\varkappa}^{\#}(x,t), k = 1,2,3. \tag{1.32}$$

We define now

$$M_{\varkappa}^{0k}(t) = [Ext\text{-}\exp(-itH_{\varkappa})]M_{\varkappa}^{0k}[Ext\text{-}\exp(itH_{\varkappa})], k = 1,2,3, \tag{1.33}$$

and using the commutation relations (1.27) and (1.28) we obtain

$$M_{\varkappa}^{0k}(t) \equiv Ext\text{-}\frac{\sum_{r=0}^{*\infty}\left(\mathrm{ad}(-itH_{\varkappa})\right)^{r} M_{\varkappa}^{0k}}{r!^{\#}} = M_{\varkappa}^{0k} - tP_{\varkappa}^{k}, \tag{1.34}$$

since second order and higher terms in t vanish identically. Thus we get

$$[iM_{\varkappa}^{0k}, \varphi_{\varkappa}^{\#}(x,t)] = [Ext\text{-}\exp(itH_{\varkappa})][iM_{\varkappa}^{0k}(t), \varphi_{\varkappa}^{\#}(x,0)][Ext\text{-}\exp(-itH_{\varkappa})] =$$

$$= [Ext\text{-}\exp(itH_{\varkappa})][iM_{\varkappa}^{0k} - itP_{\varkappa}^{k}, \varphi_{\varkappa}^{\#}(x,0)][Ext\text{-}\exp(-itH_{\varkappa})], k = 1,2,3. \tag{1.35}$$

Since $\varphi_{\varkappa}^{\#}(x,0)$ commutes with $M_{I,\varkappa}$ by a standard computation we get

$$[iM_{\varkappa}^{0k}, \varphi_{\varkappa}^{\#}(x,0)] = \left[iM_{0,\varkappa}^{0k}, \varphi_{\varkappa}^{\#}(x,0)\right] = x_k\pi_{\varkappa}^{\#}(x,0), k = 1,2,3. \tag{1.36}$$

Also we get

$$[iP_{\varkappa}^{k}, \varphi_{\varkappa}^{\#}(x,0)] = -\partial_{x_k}^{\#}\varphi_{\varkappa}^{\#}(x,0), k = 1,2,3. \tag{1.37}$$

Substituting (1.36) and (1.37) into (1. 35), we obtain the desired commutation relation (1. 32).

The three main steps to convert the above argument into a rigorous proof are (a) to introduce a spatial cut-off into the Lorentz boost generators in such a way that we obtain a self-#-adjoint operators $M^{0k}_{\varkappa,g}$, $k = 1,2,3$; (b) to show that for suitable bounded regions $O \subset {}^*\mathbb{R}^{\#3}_{c,\text{fin}}$, (1.34) holds in the sense that for every $f \in C_0^{*\infty}\left({}^*\mathbb{R}^{\#3}_{c,\text{fin}}, {}^*\mathbb{R}^{\#}_{c,\text{fin}}\right)$,

$$\left[iM^{0k}_{\varkappa,g}(t), \varphi^{\#}_{\varkappa}(f)\right] \approx \left[iM^{0k}_{\varkappa,g} - iP^{k}_{\varkappa,g}, \varphi^{\#}_{\varkappa}(f)\right], \tag{1.38}$$

where $P^{k}_{\varkappa,g}$, $k = 1,2,3$ are the locally correct momentum operators. Note that (1. 38) states that $M^{0k}_{\varkappa,g}$ are the locally correct Lorentz boost generators for the region O corresponding to the exact cancellation of higher order terms in (1.34) is the fact that second and higher order terms in $M^{0k}_{\varkappa,g}(t)$ are localized $\approx$ -outside region O and hence $\approx$ -commutes with $\varphi^{\#}_{\varkappa}(f)$. From (1. 38) one obtains the relations

$$\left[iM^{0k}_{\varkappa,g}(t), \varphi^{\#}_{\varkappa}(f)\right] \approx -\varphi^{\#}_{\varkappa}\left(t\frac{\partial^{\#} f}{\partial^{\#} x_k} + x_k\frac{\partial^{\#} f}{\partial^{\#} t}\right), k = 1,2,3, \tag{1.39}$$

and its direct consequence

$$\left[Ext\text{-}\exp\left(i\beta M^{0k}_{\varkappa,g}\right)\right]\varphi^{\#}_{\varkappa}(x,t)\left[Ext\text{-}\exp\left(-i\beta M^{0k}_{\varkappa,g}\right)\right] \approx \varphi^{\#}_{\varkappa}\left(\Lambda^{(k)}_{\beta}(x,t)\right), k = 1,2,3. \tag{1.40}$$

Definition 1.23 If $I^3 = [a,b]^3 = [a,b] \times [a,b] \times [a,b]$ is a cube in ${}^*\mathbb{R}^{\#3}_{c,\text{fin}}$, where $[a,b]$ is an #-closed interval in ${}^*\mathbb{R}^{\#}_{c,\text{fin}}$. A causal shadow of I^3 is defined to be the diamond

$$O_{I^3} = \{(x_1, x_2, x_3, t) | a + |t| < x_k < b - |t|; k = 1,2,3\}. \tag{1.41}$$

Remark 1.8 Note that because we can always translate in the positive $x_k, k = 1,2,3$ directions, it is sufficient to prove Theorem 1.6 for sets O such that both O and $\Lambda^{(k)}_{\beta}O, k = 1,2,3$ are contained in O_{I^3} for some #-closed interval $I \subset {}^*\mathbb{R}^{\#}_{c,\text{fin}+}$. The advantage of working over ${}^*\mathbb{R}^{\#\mathbf{3}}_{\boldsymbol{c},\mathbf{fin}+}$ is that the locally correct Lorentz boost generators $M^{0k}_{\varkappa,g}, k = 1,2,3$ are bounded be1ow.

2. Properties of the Lorentz boost generators $M^{0k}_{\varkappa,g}, k = 1,2,3$

In this section we consider the basic properties of $H_{\varkappa,g}$ and $M^{0k}_{\varkappa,g}, k = 1,2,3$ in particular, the first order estimates they satisfy. Note that $H_{\varkappa,g}$ and $M^{0k}_{\varkappa,g}, k = 1,2,3$ are well defined operators on a non-Archimedean Fock space $\mathcal{F}^{\#}$. We take the definition of $\mathcal{F}^{\#}$ and the definition of the pointwise-defined time-zero field operators on $\mathcal{F}^{\#}$ as in [8] (see [8, Section 9]). The spatially cut-off Hamiltonian is defined as self-#-adjoint operator on a non-Archimedean Fock space $\mathcal{F}^{\#}$ [8].

Let $g = \{g_0, g_1\}$, where $g_0 = \left\{g^{(k)}_0\right\}, k = 1,2,3$, $g^{(k)}_0, g_1 \in C_0^{*\infty}\left({}^*\mathbb{R}^{\#3}_{c,\text{fin}+}, {}^*\mathbb{R}^{\#}_{\text{c,fin}}\right)$ and $g^{(k)}_0, g_1 \geq 0, k = 1,2,3$. The spatially cut-off Hamiltonian reads

$$H_{\varkappa,g} = H_{\varkappa}(g) = H_{0,\varkappa} + T_{I,\varkappa}(g_1), \tag{2.1}$$

where $T_{I,\varkappa}(f) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_{c}} f(x) T_{I,\varkappa}(x) d^{\#3}x$ and

$$T_{I,\varkappa}(x) =: \varphi^{\#2n}_{\varkappa}(x): \tag{2.2}$$

is the interaction energy density. The operator $H_{\varkappa}(g)$ has been studied in [8] and is known to be a self-#-adjoint semibounded operator on $\mathcal{F}^{\#}$. For the region O_{I^3}, defined above in section 1 we set now

$$M^{0k}_{\varkappa,g} = \alpha H_{0,\varkappa} + T_{0,\varkappa}\left(x_k g^{(k)}_0\right) + T_{I,\varkappa}(x_k g_1) \quad (2.3)$$

with $\alpha > 0$, and

$$T_{0,\varkappa}(f) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} f(x) T_{0,\varkappa}(x) d^{\#3}x.$$

We assume now that

$$\alpha + x_k g^{(k)}_0(x) = x_k g_1(x) = x_k, k = 1,2,3 \text{ on } I^3 = [a,b]^3 \subset {}^*\mathbb{R}^{\#3}_{c,\text{fin}+} \quad (2.4)$$

and two additional technical conditions on the $g = \{g_0, g_1\}$

$$x_k g^{(k)}_0(x) = h^2_k(x) \geq 0, h_k \in C^{*\infty}_0\left({}^*\mathbb{R}^{\#3}_{c,\text{fin}+}, {}^*\mathbb{R}^{\#}_{c,\text{fin}}\right), k = 1,2,3 \quad (2.5)$$

and

$$x_k g_1(x) = \left[\alpha + x_k g^{(k)}_0(x)\right] g_1(x). \quad (2.6)$$

We rewrite now the operator $T_{0,\varkappa}(f)$ as

$$T_{0,\varkappa}(f) = T^{(1)}_{0,\varkappa}(f) + T^{(2)}_{0,\varkappa}(f) = Ext\text{-}\int_{|k_1|\leq\varkappa} Ext\text{-}\int_{|k_2|\leq\varkappa} t^{(1)}(k_1,k_2) a^*(k_1) a(k_2)\, d^{\#3}k_1\, d^{\#3}k_2 \quad (2.7)$$

$$+Ext\text{-}\int_{|k_1|\leq\varkappa} Ext\text{-}\int_{|k_2|\leq\varkappa} t^{(2)}(k_1,k_2)[a^*(k_1)a^*(-k_2) + a(-k_1)a(k_2)]\, d^{\#3}k_1\, d^{\#3}k_2 =$$

$$Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} \Theta(k_1,\varkappa)\Theta(k_2,\varkappa) t^{(1)}(k_1,k_2) a^*(k_1) a(k_2)\, d^{\#3}k_1\, d^{\#3}k_2$$

$$+Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} \Theta(k_1,\varkappa)\Theta(k_2,\varkappa) t^{(2)}(k_1,k_2)[a^*(k_1)a^*(-k_2) + a(-k_1)a(k_2)]\, d^{\#3}k_1\, d^{\#3}k_2,$$

$$t^{(1)}(k_1,k_2) = \text{const}\cdot\Theta(k_1,\varkappa)\Theta(k_2,\varkappa)\left[Ext\text{-}\hat{f}(k_1-k_2)\right] \times [\mu(k_1) + \mu(k_2) + \langle k_1,k_2\rangle + m^2] \times$$

$$\times [\mu(k_1)\mu(k_2)]^{-1/2}, \quad (2.8)$$

$$t^{(2)}(k_1,k_2) = \text{const}\cdot\Theta(k_1,\varkappa)\Theta(k_2,\varkappa)\left[Ext\text{-}\hat{f}(k_1-k_2)\right][-\mu(k_1) + \mu(k_2) + \langle k_1,k_2\rangle + m^2] \times$$

$$\times [\mu(k_1)\mu(k_2)]^{-1/2}, \quad (2.9)$$

where

$$\Theta(k,\varkappa) = \begin{cases} 1 \text{ if } |k| \leq \varkappa, \\ 0 \text{ if } |k| > \varkappa. \end{cases} \quad (2.10)$$

Note that $t^{(1)}, t^{(2)} \in L^{\#}_2\left({}^*\mathbb{R}^{\#6}_c\right)$.

It follows that $T^{(i)}_{0,\varkappa}(f)(N_\varkappa + I)^{-1}, i = 1,2$ are bounded,

$$\left\| T^{(i)}_{0,\varkappa}(f)(N_\varkappa + I)^{-1} \right\|_{\#} \leq \text{const}\cdot \left\| t^{(i)} \right\|_{L^{\#}_2}.$$

Let $P^k_\varkappa(f)$

$$P^k_\varkappa(f) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} f(x) P^k_\varkappa(x) d^{\#3}x, \quad (2.11)$$

Where $P_{\varkappa}^{k}(x)$ is given by (1.29) and $f \in C_0^{^*\infty}\left({}^*\mathbb{R}_{c,\text{fin}}^{\#3}, {}^*\mathbb{R}_{c,\text{fin}}^{\#}\right)$.

Here $N_{\varkappa}$ is the number operator with hyperfinite cut-off $\varkappa$ and we have used the $N_{\varkappa}$-estimate [8]: Let W be a Wick monomial

$$W_{\varkappa} = Ext\text{-}\int_{|k_1|\leq\varkappa} d^{\#3}k_1 \ldots Ext\text{-}\int_{|k_r|\leq\varkappa} d^{\#3}k_r w(k_1, \ldots, k_r) a^{\dagger}(k_1) \cdots a(k_r) \tag{2.12}$$

with a kernel $w \in L_2^{\#}\left({}^*\mathbb{R}_{\boldsymbol{c}}^{\#3r}\right)$, then

$$\left\|(N_{\varkappa} + I)^{-a/2} W (N_{\varkappa} + I)^{-b/2}\right\|_{\#} \leq \text{const} \cdot \|w\|_{L_2^{\#}}, \tag{2.13}$$

where $a + b \geq r$. A similar decomposition holds for $P_{\varkappa}^{k}(f), k = 1,2,3$. The result reads:

Proposition 2.1[8] Let $A = T_{0,\varkappa}^{(i)}(f), i = 1,2$ or $P_{\varkappa}^{k}(f)$ with $f \in C_0^{^*\infty}\left({}^*\mathbb{R}_{c,\text{fin}}^{\#3}, {}^*\mathbb{R}_{c,\text{fin}}^{\#}\right)$. Then,

$$\left\|\left(H_{0,\varkappa} + I\right)^{-i/2} A \left(H_{0,\varkappa} + I\right)^{-j/2}\right\|_{\#} < {}^*\infty. \tag{2.14}$$

That is convenient to approximate the operators $M_{\varkappa,g}^{0k}, k = 1,3,3$ by the operators $M_{\varkappa,\kappa,g}^{0k}, k = 1,3,3$ with an additional momentum cut-off

$$M_{\varkappa,\kappa,g}^{0k} = \alpha H_{0,\varkappa,\kappa} + T_{0,\varkappa,\kappa}\left(x_k g_0^{(k)}\right) + T_{I,\varkappa,\kappa}(x_k g_1),$$

where $T_{0,\varkappa,\kappa}$ and $T_{I,\varkappa,\kappa}$ are defined by cutting off all the momentum integrals at $|k| > \kappa$. That is, $T_{0,\varkappa}$ and $T_{I,\varkappa}$, are expressed as a sum of Wick monomials (2.12) each of which is replaced in the definition of $T_{0,\varkappa,\kappa}$ and $T_{I,\varkappa,\kappa}$ by

$$W_{\varkappa,\kappa} = Ext\text{-}\int_{|k_1|\leq\varkappa} d^{\#3}k_1 \ldots Ext\text{-}\int_{|k_r|\leq\varkappa} d^{\#3}k_r \chi_{\kappa}(k_1, \ldots, k_r) w(k_1, \ldots, k_r) a^{\dagger}(k_1) \cdots a(k_r).$$

Here $\chi_{\kappa}(k_1, \ldots, k_r) = 1$ if $|k_i| \leq \kappa \leq \varkappa$ for all $1 \leq i \leq r$, and $\chi_{\kappa}(k_1, \ldots, k_r) = 0$ otherwise. We abbreviate also

$$M_{0,\varkappa,\kappa,g}^{0k} = \alpha H_{0,\varkappa,\kappa} + T_{0,\varkappa,\kappa}\left(x_k g_0^{(k)}\right), k = 1,2,3.$$

Note that as a rule, estimates that hold for $M_{\varkappa,g}^{0k}$ also hold for $M_{\varkappa,\kappa,g}^{0k}$, uniformly in κ. For example, for all $\kappa \in {}^*\mathbb{R}_{c+,\infty}^{\#}, \kappa \leq \varkappa$:

$$\left\|\left(H_{0,\varkappa,\kappa} + I\right)^{-l_1/2} T_{0,\varkappa,\kappa}^{(i)}(f) \left(H_{0,\varkappa,\kappa} + I\right)^{-l_2/2}\right\|_{\#} \leq \text{const.}, i = 1,2 \tag{2.15}$$

and

$$\left\|\left(N_{\varkappa,\kappa} + I\right)^{-l_1/2} T_{0,\varkappa,\kappa}^{(i)}(f) \left(N_{\varkappa,\kappa} + I\right)^{-l_2/2}\right\|_{\#} \leq \text{const.}, i = 1,2 \tag{2.16}$$

for $l_1 + l_2 \geq 2$, where the constants are independent of κ. As a domain of admissible vectors in $\mathcal{F}^{\#}$

$$\mathcal{D}_{\text{fin}}^{\#} = \left\{\psi | \psi = (\psi_0, \psi_1, \ldots) \in \mathcal{F}^{\#}, \psi_n \in C_0^{^*\infty}\left({}^*\mathbb{R}_{c,\text{fin}+}^{\#3n}, {}^*\mathbb{R}_{c,\text{fin}}^{\#}\right), \psi_n \equiv 0 \text{ for large } n \in {}^*\mathbb{N}\right\}. \tag{2.17}$$

Remark 2.1 The operators $M_{\varkappa,g}^{0k}, k = 1,2,3$ as constructed above, enjoys the property of being semibounded.

Theorem 2.2 Let $g = \{g_0, g_1\}$ satisfy the condition (2.4). Then there are constants a and b such that for all $\kappa < \varkappa$

$$H_{0,\varkappa} \leq a\left(M_{\varkappa,\kappa,g}^{0k} + b\right), k = 1,2,3 \tag{2.18}$$

on the domain $\mathcal{D}_{\text{fin}}^{\#} \times \mathcal{D}_{\text{fin}}^{\#}$.
Proof For $\varepsilon > 0$, there is a constant d such that [8]

$$0 \leq H_{0,\varkappa} + T_{I,\varkappa,\kappa}\left(x_k g_1(x)\right) + d, k = 1,2,3 \tag{2.19}$$

on the domain $\mathcal{D}_{\text{fin}}^{\#} \times \mathcal{D}_{\text{fin}}^{\#}$. For $\varepsilon > 0$, there is a constant c such that [8]

$$0 \leq H_{0,\varkappa} + T_{0,\varkappa,\kappa}\left(x_k g_0^{(k)}(x)\right) + c, k = 1,2,3 \tag{2.20}$$

on the domain $\mathcal{D}_{\text{fin}}^{\#} \times \mathcal{D}_{\text{fin}}^{\#}$. The inequalities (2.18) follows from adding (2.19) and (2.20).
Proposition 2.3 There are positive constants a, b, c such that

$$M_{\varkappa}^{0k} \leq a(H_{\varkappa} + b) \leq c\left(M_{\varkappa}^{0k} + b\right), k = 1,2,3 \tag{2.21}$$

on the domain $\mathcal{D}_{\text{fin}}^{\#} \times \mathcal{D}_{\text{fin}}^{\#}$.
Proof Note that for $k = 1,2,3$

$$a(H_{\varkappa} + b) - M_{\varkappa}^{0k} = (a - \alpha)H_{0,\varkappa} - T_{0,\varkappa}\left(x_k g_0^{(k)}(x)\right) + T_{I,\varkappa}\left((a - x_k)g_1(x)\right) + ab.$$

By choosing constant a larger than $\max_k[\sup\{x_k | g_1(x) \neq 0\}]$, we have $(a - x_k)g_1(x) > 0$ and therefore as in (2.19)

$$H_{0,\varkappa} + T_{I,\varkappa}\left((a - x_k)g_1(x)\right) \geq 0.$$

Moreover, by (2.14) we can choose a so that

$$(a - \alpha - 1)\left(H_{0,\varkappa} + I\right) - T_{0,\varkappa}\left(x_k g_0^{(k)}(x)\right) \geq 0.$$

The second part of (2.21) follows by a similar consideration,

3. Quadratic estimates

In this section we prove the self-#-adjointness of the operators $M_{\varkappa,\kappa}^{0k}, k = 1,2$, by interpreting the operator $T_{0,\varkappa,\kappa}$ as generalized Kato perturbation [8]. Thus we need proving quadratic inequalities

$$\left(H_{0,\varkappa} + I\right)^2 \leq a_\kappa\left(H_{0,\varkappa} + \lambda T_{0,\varkappa,\kappa}\left(f_{0,k}\right) + T_{I,\varkappa,\kappa}(f_1) + b\right)^2, \tag{3.1}$$

where a_κ and b are constants with a_κ depending on κ. Here λ is finite constant and $f_{0,k} = \alpha^{-1} x_k g_0^{(k)}(x)$ where $g_0^{(k)}(x)$ satisfies conditions (2.5).

Theorem 3.1 The operators $M_{0,\varkappa,\kappa}^{ik}, k = 1,2,3$ are essentially self-#-adjoint on $D^{\#}$. There are constants a and b independent of κ, such that for $\kappa < \varkappa$ and $k = 1,2,3$

$$\left(H_{0,\varkappa} + I\right)^2 \leq a\left(M_{0,\varkappa,\kappa}^{0k} + b\right). \tag{3.2}$$

Remark 3.1 For φ_4^{2n} we use the “pull through formula” (3.5). Let $T_\varkappa = \#\text{-}\overline{(H_{0,\varkappa} + V_\varkappa)}$ and $R(z) = (T_\varkappa - z)^{-1}$. Then

$$a(\boldsymbol{k})R(z) = R\big(z - \mu(\boldsymbol{k})\big)a(\boldsymbol{k}) - R\big(z - \mu(\boldsymbol{k})\big)[a(\boldsymbol{k}), V]R(z). \tag{3.3}$$

We shall always be concerned with operators T that are essentially self-#-adjoint on domain $\mathcal{D}_{\text{fin}}^{\#}$ defined in (2.17), and whose perturbation V is a finite sum of Wick monomials with #-smooth kernels. It follows that $a(\boldsymbol{k})$ is defined on the #-dense domain

$$\mathcal{D}_{\text{fin}}^{\#\prime} = (T_\varkappa - z)\mathcal{D}_{\text{fin}}^{\#} \tag{3.4}$$

and that (3.3) holds on this domain.

Lemma 3.2 Suppose that $T_\varkappa = \#\text{-}\overline{(H_{0,\varkappa} + V_\varkappa)}$ satisfies the above conditions. Let $\psi \in \mathcal{D}_{\text{fin}}^{\#\prime}$, where $(z - c)$ is in the resolvent set of $T_\varkappa$ for all $c \geq 0$. Then for $r \in \mathbb{N}$ a positive integer

$$a_{(1,r)}R\psi = Ext\text{-}\textstyle\sum_{\text{part.}}(-1)^j R_{J_1}V_\varkappa^{I_1}R_{J_2} \cdots R_{J_j}V_\varkappa^{I_j}R_{J_{j+1}}a_{I_{j+1}}\psi, \tag{3.5}$$

where $I = \{i_1, \ldots, i_s\}$ be a set of distinct ordered positive integers, $(1, r) = \{1, 2, \ldots, r\}$, $a_I = Ext\text{-}\prod_{l=1}^{s} a\left(k_{i_l}\right)$ for $s > 0$, $a_I = \mathrm{I}$ for $s = 0$. The sum in (3.5) takes place over all partitions of $\{1, 2, \ldots, r\}$ into disjoint subsets $I_1, \ldots, I_{j+1}$ (including permutations among the subsets) for $j = 0, 1, \ldots, r$. The elements of each I_i are taken in natural order. Let $R_{J_l} = R(\zeta), R(z) = (T_\varkappa - z)^{-1}$, where $\zeta = z - Ext\text{-}\sum_{i \in J_l}\mu(k_i)$ and $J_l = I_l \cup I_{j+1} \cup \ldots \cup I_{j+1}$. Let $V^I = \left[a\left(k_{i_1}\right), \ldots, \left[a\left(k_{i_s}\right), V\right] \ldots\right]$ for $s > 0$ and $V^I = 0$ for $s = 0$. Note that the sum (3.5) includes terms where J_{j+1} is empty but not $I_1, \ldots, I_j$; this convention adjusts the sign $(-1)^j$ correctly. The $j = 0$ term is simply $R_1 a_{(1,r)}\psi$.

Proof In order to apply (3.5) to the proof of (3.1) we must be able to estimate the commutators

$$X_{\varkappa,\kappa}^{(i)}(k) = \left[a(k), T_{0,\varkappa,\kappa}(f)\right] \tag{3.6}$$

$i = 1, 2$, for sufficiently large k, where $f \in {}^*C_0^{\infty}\left({}^*\mathbb{R}_{c,\text{fin}+}^{\#3}, {}^*\mathbb{R}_{c,\text{fin}}^{\#}\right)$.

Lemma 3.3

$$\left\| X_\varkappa^{(2)}(k)(N_\varkappa + I)^{-1/2} \right\|_\# = O([\mu(k)]^{-1}). \tag{3.7}$$

Proof $X_\varkappa^{(2)}(k))$ is certainly #-densely defined, say on domain D; it is sufficient to prove (3.7) on D and then $X_\varkappa^{(2)}(k)(N_\varkappa + I)^{-1/2}$ extends to a bounded operator on all vectors of $\mathcal{F}^\#$. Now we set

$$X_\varkappa^{(2)}(k) = Ext\text{-}\int_{|k| \leq \varkappa} w(k,p)^* a(-p) d^{\#3}p,$$

where by (2.9) the kernel $w(k, p)$ can be estimated by

$$|w(k,p)| = |h(k - p)|[\mu(k)]^{-1/2}[\mu(p)]^{-1/2}$$

where $h \in S_{\text{fin}}^{\#}\left({}^*\mathbb{R}_c^{\#3}\right)$ is rapidly decreasing. According to (2.13), by a simple calculation one obtains

$$\left\| X_\varkappa^{(2)}(k)(N_\varkappa + I)^{-1/2} \right\|_\# \leq \text{const.} \times \|w(k,\cdot)\|_{\#2} = O([\mu(k)]^{-1}).$$

Lemma 3.4 For arbitrary $\psi \in \mathcal{F}^\#$ and $c > 0$

$$A = Ext\text{-}\int_{|k|\leq\varkappa} d^{\#3}k \left\| \left(H_{0,\varkappa} + c + \mu(k)\right)^{-\frac{1}{2}} X_{\varkappa}^{(1)}(k)\left(H_{0,\varkappa} + c\right)^{-\frac{1}{2}}\psi \right\|_{\#}^{2} \leq \text{const.}\times \|\psi\|_{\#}^{2}. \qquad (3.8)$$

Proof Let $\mathcal{F}_n^{\#}, n \in {}^*\mathbb{N}$ be the n-particle Fock space. Now $X_{\varkappa}^{(1)}(k)$ is defined on D for all k and since $X_{\varkappa}^{(1)}(k)$ maps $\mathcal{F}_r^{\#}$ into $\mathcal{F}_{r-1}^{\#}$, it is sufficient to prove that (3.8) holds for $\psi \in D \cap \mathcal{F}_n^{\#}$ with the constant independent of n. We remark that by the methods of the previous lemma it is easy to show that the integrand in (3.8) is uniformly bounded in k, but different methods are necessary to prove it integrability. Now we define

$$X_{\varkappa}^{(1)}(k) = Ext\text{-}\int_{|k|\leq\varkappa} t^{(1)}(k,p)a(\,p)d^{\#3}p,$$

where $t^{(1)}(k,p)$ is given by (2.9); therefore we obtain

$$A_{\varkappa} \leq Ext\text{-}\int_{|k|\leq\varkappa} d^{\#3}k\, Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1 \cdots Ext\text{-}\int_{|p_{n-1}|\leq\varkappa} d^{\#3}p_{n-1} \times$$

$$\times \left[\left(Ext\text{-}\sum_{i=1}^{n-1}\mu(p_i) + \mu(k) + c\right)^{-1/2} n^{1/2} \times\right.$$

$$\left.\times\, Ext\text{-}\int_{|p|\leq\varkappa} d^{\#3}p \left|t^{(1)}(k,p)\right|\left(Ext\text{-}\sum_{i=1}^{n-1}\mu(p_i) + \mu(k) + c\right)^{-1/2}|\psi(p_1,\dots,p_{n-1},p)|\right]^2, \quad (3.9)$$

where $a(p)$ has destroyed a particle by

$$(a(p)\psi)(p_1,\dots,p_{n-1},p) = n^{1/2}\psi(p_1,\dots,p_{n-1},p). \qquad (3.10)$$

By the definition (2.9) we obtain

$$\left|t^{(1)}(k,p)\right|(Ext\text{-}\sum_{i=1}^{n}\mu(p_i) + \mu(k) + c)^{-1/2} \leq \text{const.}\times [\mu(k)]^{1/2}\left(\left|Ext\text{-}\hat{f}(k-p)\right|\right).$$

Replacing now k by p_n in (3.9) we get

$$A_{\varkappa} \leq a \times n \times Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1 \cdots Ext\text{-}\int_{|p_{n-1}|\leq\varkappa} d^{\#3}p_n \times$$

$$\left[[\mu(p_n)]^{1/2}(Ext\text{-}\sum_{i=1}^{n}\mu(p_i) + c)^{-1/2} Ext\text{-}\int_{|p|\leq\varkappa} d^{\#3}p\,\left(\left|Ext\text{-}\hat{f}(p_n-p)\right|\right)|\psi(p_1,\dots,p_{n-1},p)|\right]^2 =$$

$$= a \times Ext\text{-}\sum_{j=1}^{n} Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1 \cdots Ext\text{-}\int_{|p_n|\leq\varkappa} d^{\#3}p_n \times$$

$$\times \left[Ext\text{-}\int_{|p|\leq\varkappa} d^{\#3}p E_j(p_1,\dots,p_n)\left(\left|Ext\text{-}\hat{f}(p_j-p)\right|\right)\left|\psi\left(p_1,\dots,p_{j-1},p,p_{j+1},\dots,p_n\right)\right|\right]^2, \qquad (3.11)$$

where a is a constant and

$$E_j(p_1,\dots,p_n) = \left[\mu(p_j)/(Ext\text{-}\sum_{i=1}^{n}\mu(p_i) + c)\right]^{1/2}$$

We shall write this symbolically as $E_j(p_j)$, suppressing the other variables. In obtaining (3.11) we have interchanged p_j and p_n, and exploited the symmetry of ψ. In (3.1 I) we wish to replace $E_j(p_j)$ by $E_j(p)$ to get

$$A'_{\varkappa} = a \times Ext\text{-}\sum_{j=1}^{n} Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1 \cdots Ext\text{-}\int_{|p_n|\leq\varkappa} d^{\#3}p_n \times$$

$$\times\left[Ext\text{-}\int_{|p|\le\varkappa} d^{\#3}pE_j(p)\left(\left|Ext\text{-}\hat{f}(p_j-p)\right|\right)\left|\psi(p_1,\dots,p_{j-1},p,p_{j+1},\dots,p_n)\right|\right]^2$$

For then the integral over p is a convolution between

$$\phi_j(p)=E_j(p)\left|\psi(p_1,\dots,p_{j-1},p,p_{j+1},\dots,p_n)\right|$$

and $h(p)=\left|Ext\text{-}\hat{f}(p)\right|$, and the integral over p_j is the square of the $L_2^{\#}$ #-norm of this convolution. Now we get

$$Ext\text{-}\int_{|p_j|\le\varkappa} d^{\#3}p_j\left[Ext\text{-}\int_{|p|\le\varkappa} h(p_j-p)\phi_j(p)d^{\#3}p\right]^2=\left\|\left(Ext\text{-}\hat{h}\right)\times\left(Ext\text{-}\widehat{\phi_j}\right)\right\|_{\#2}^2\le$$

$$\le\left\|Ext\text{-}\hat{h}\right\|_{^*\infty}^2\times\left\|\phi_j\right\|_{\#2}^2$$

and

$$\left\|Ext\text{-}\hat{h}\right\|_{^*\infty}^2=Ext\text{-}\int_{|p|\le\varkappa}\left(Ext\text{-}\hat{f}(p)\right)d^{\#3}p<\infty.$$

Therefore,

$$A'_\varkappa\le\text{const.}\times Ext\text{-}\sum_{j=1}^n\left\|E_j(p_j)\psi(p_1,\dots,p_n)\right\|_{\#2}^2=\text{const.}\times\left\|\left(Ext\text{-}\sum_{j=1}^n E_j^2\right)^{1/2}\psi\right\|_{\#2}^2\le$$

$$\le\text{const.}\times\|\psi\|_{\#2}^2.$$

In order to justify the replacement of $E_j(p_j)$ by $E_j(p)$, we set

$$E_j(p_j)=E_j(p)+\left(E_j(p_j)-E_j(p)\right)$$

and therefore we obtain

$$\left[Ext\text{-}\int_{|p|\le\varkappa} d^{\#3}pE_j(p_j)\left|\left(Ext\text{-}\hat{f}\right)\psi\right|\right]^2=\left[Ext\text{-}\int_{|p|\le\varkappa} d^{\#3}pE_j(p)\left|\left(Ext\text{-}\hat{f}\right)\psi\right|\right]^2+$$

$$+\left[Ext\text{-}\int_{|p|\le\varkappa} d^{\#3}p\left(E_j(p_j)-E_j(p)\right)\left|\left(Ext\text{-}\hat{f}\right)\psi\right|\right]^2+2\left[Ext\text{-}\int_{|p|\le\varkappa} d^{\#3}pE_j(p)\left|\left(Ext\text{-}\hat{f}\right)\psi\right|\right]\times$$

$$\times\left[Ext\text{-}\int_{|p|\le\varkappa} d^{\#3}p\left(E_j(p_j)-E_j(p)\right)\left|\left(Ext\text{-}\hat{f}\right)\psi\right|\right]. \tag{3.12}$$

Applying the operation $a\times Ext\text{-}\sum_{j=1}^n Ext\text{-}\int_{|p_1|\le\varkappa} d^{\#3}p_1\cdots Ext\text{-}\int_{|p_n|\le\varkappa} d^{\#3}p_n$ to (3.12), we obviously get $A_\varkappa$ on the left and $A'_\varkappa$ from the first term on the right. To estimate the second term, we note that

$$\left|E_j(p_j)-E_j(p)\right|\le\left|E_j(p_j)^2-E_j(p)^2\right|^{\frac{1}{2}}=$$

$$\left|\left(Ext\text{-}\sum_{i\ne j}\mu(p_i)+c\right)\left(\mu(p_j)-\mu(p)\right)\right|^{1/2}\left|\left(Ext\text{-}\sum\mu(p_i)+c\right)\left(Ext\text{-}\sum_{i\ne j}\mu(p_i)+\mu(p)+c\right)\right|^{-1/2}$$

$$\le\text{const.}\times n^{-\frac{1}{2}}\left|\mu(p_j)-\mu(p)\right|^{\frac{1}{2}}\le\text{const.}\times n^{-\frac{1}{2}}\left|\left\|p_j\right\|_{\#}-\|p\|_{\#}\right|^{\frac{1}{2}}\le\text{const.}\times n^{-\frac{1}{2}}\left\|p_j-p\right\|_{\#}^{1/2},$$

where $\|\cdot\|_{\#}$is Euclidian #- norm in ${}^{*}\mathbb{R}_{c}^{\#3}$.Therefore the integral of the second term in (3.12) can be estimated by

$$\text{const.}\times n^{-1}\times Ext\text{-}\sum_j Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1 \cdots Ext\text{-}\int_{|p_n|\leq\varkappa} d^{\#3}p_n \times$$

$$\left[Ext\text{-}\int_{|p\,|\leq\varkappa} d^{\#3}p\|p_j - p\|_{\#}^{1/2}\left|\left(Ext\text{-}\hat{f}(p_j - p)\right)\psi(p_1,\ldots,p_{j-1},p,p_{j+1},\ldots,p_n)\right|\right].$$

But, as before, this is the square of the $L_2^{\#}$- #-norm of the convolution of the function ψ with a rapidly decreasing function and so it can be estimated by

$$\text{const.}\times n\times Ext\text{-}\sum_j\|\psi\|_{\#}^2 \leq \text{const.}\times\|\psi\|_{\#}^2,$$

where the constant is independent of $n\in{}^{*}\mathbb{N}$. The third term resulting from (3.12) can then be estimated by the generalized Schwarz inequality applied to $Ext\text{-}\sum_{j=1}^{n} Ext\text{-}\int_{|p_1|\leq\varkappa} d^{\#3}p_1\cdots$ $Ext\text{-}\int_{|p_n|\leq\varkappa} d^{\#3}p_n$. Hence $A_{\varkappa}$ is bounded as claimed. The single commutators (3.6) are all that we need estimate. For let $I=\{i_1,\ldots,i_s\}$; then $\left(T_{0,\varkappa}^{(1)}(f)\right)^{I}=0$ if $\left(T_{0,\varkappa}^{(2)}(f)\right)^{I}$ and $\left(T_{0,\varkappa}^{(2)}(f)\right)^{I}=0$ when $s>2$. When $s=2$, $\left(T_{0,\varkappa}^{(2)}(f)\right)^{I}$reduces to the constant $2\Theta(k\,,\varkappa)t^{(2)}(k_1-k_2)$; thus for all s, $T_{0,\varkappa}^{(2)}(f)$ satisfies

$$\left\|\left(T_{0,\varkappa}^{(2)}(f)\right)^{I}(N_{\varkappa}+I)^{-1/2}\right\|_{\#}\leq\text{const.}\times Ext\text{-}\prod_{i\in I}[\mu(k_i)]^{-1/2} \tag{3.13}$$

by virtue of (3.7) and (2.11).

Remark 3.2 We now go to prove (3.1) by using the formula (3.5). For convenience, we work now with operators

$$T_{\varkappa,\kappa}^{0k}(\lambda)=\#\text{-}\overline{\left[\left(H_{0,\varkappa}+\lambda T_{0,\varkappa,\kappa}(f_{0,k})+T_{I,\varkappa,\kappa}(f_1)\right)\upharpoonright D\right]} \tag{3.14}$$

which are $M_{\varkappa,\kappa}^{0k}$ up to constants. To apply the pull-through formula (3.5) it is necessary to know that the operators $T_{\varkappa,\kappa}^{0k}, k=1,2,3$ are self-#-adjoint. For the moment we assume this, postponing the proof until Theorem 3.8. We remark though that in the case $\lambda=0$, $T_{\varkappa,\kappa}^{0k}$ reduces to $H_{\varkappa,\kappa}(f_1)$ which is known to be self-#-adjoint. The next lemma gives an estimate on commutators such that

$$X_{\varkappa,\kappa}^{(3)}(k)=\left[a(k),T_{I,\varkappa,\kappa}(f_1)\right] \tag{3.15}$$

which is finite or hyperfinite polynomial of degree $(2n-1)$ in the field $\varphi_{\varkappa}^{\#}(x)$. Since $T_{\varkappa,\kappa}^{0k}$ remains semibounded (Theorem 2.2) when perturbed by a polynomial in the field of degree less than $2n$, we have the following estimate in terms of the resolvent $R_{\varkappa,\kappa}(z)=\left(T_{\varkappa,\kappa}-z\right)^{-1}$:

Lemma 3.5 Let $r\in{}^{*}\mathbb{N}$ be a positive integer. There is a $z_0<0$ independent of κ and r such that, for $z_1\leq z_0, z_2\leq z_0$

$$\left\|R_{\varkappa,\kappa}^{1/2}(z_2)T_{\varkappa,\kappa}^{(1,r)}R_{\varkappa,\kappa}^{1/2}(z_1)\right\|_{\#}\leq\text{const.}\times\prod_{i=1}^{r}[\mu(k)]^{-\frac{1}{2}}, \tag{3.16}$$

where the constant is independent of z_1, z_2. Here, in the notation of Lemma 3.2,

$$T_{I,\varkappa,\kappa}^{(1,r)0k} = \left[a(k_1), \left[\cdots \left[a(k_r), T_{I,\varkappa,\kappa}(f_1)\right] \cdots\right]\right].$$

Theorem 3.6 Assume that the operators $T_{\varkappa,\kappa}^{0k}$ are given by (3.14) is self-#-adjoint, where $k \leq \varkappa$. Then there are positive constants $b, c(k)$, and $d(k)$ all independent of λ such that

$$\left(H_{0,\varkappa} + I\right)^2 \leq \left(c(k) + \lambda^2 d(k)\right)\left(T_{\varkappa,\kappa}^{0k} + b\right)^2. \tag{3.17}$$

Proof Obviously it is sufficient to prove that

$$\| \left(H_{0,\varkappa,\kappa} + I\right) R_{\varkappa,\kappa}(-b)\psi\|_{\#}^2 \leq \left(c(k) + \lambda^2 d(k)\right)\|\psi\|_{\#}^2 \tag{3.18}$$

for ψ in the dense set $D_{1,k} = \left(T_{\varkappa,\kappa}^{0k} + b\right)D$ as in (3.4). This choice of ψ ensures that $R_{\varkappa,\kappa}(-b)\psi \in D_{1,k}$ is in the domain of all the operators we wish to apply to it. Here b is chosen so large that

$$\left\| \left(H_{0,\varkappa,\kappa} + I\right)^{1/2} R_{\varkappa,\kappa}(-b)^{1/2}\right\|_{\#}^2 \leq \text{const.}, \tag{3.19}$$

(see 2.18) and so that (3.16) holds with $r = 1$,

$$\left\|R_{\varkappa,\kappa}^{1/2}(z_2) X_{\varkappa,\kappa}^{(3)}(k) R_{\varkappa,\kappa}^{1/2}(z_1)\right\|_{\#} \leq \text{const.} \times \Theta(k\,,\kappa)[\mu(k)]^{-\frac{1}{2}} \tag{3.20}$$

for $z_i < -b$. Now we get

$$\| \left(H_{0,\varkappa,\kappa} + I\right) R_{\varkappa,\kappa}(-b)\psi\|_{\#}^2 =$$

$$Ext\text{-}\int_{|k|\leq\kappa} \left\|\left(H_{0,\varkappa,\kappa} + I + \mu(k)\right)^{1/2} a(k) R_{\varkappa,\kappa}(-b)\psi\right\|_{\#}^2 \mu(k) d^{\#3}k. \tag{3.21}$$

But by the pull-through formula (3.3) we get

$$a(k)R_{\varkappa,\kappa}(-b)\psi = R_{\varkappa,\kappa}\left(-b-\mu(k)\right)a(k)\psi - R_{\varkappa,\kappa}\left(-b-\mu(k)\right) \times$$

$$\times \left[\lambda X_{\varkappa,\kappa}^{(1)}(k) + \lambda X_{\varkappa,\kappa}^{(2)}(k) + X_{\varkappa,\kappa}^{(3)}(k)\right] R_{\varkappa,\kappa}(-b)\psi,$$

where $X_{\varkappa,\kappa}^{(i)}(k), i = 1, 2$, are defined by (3.6) with a momentum cut-off κ. Substituting this into (3.21), we obtain by generalized Schwarz' inequality,

$$\left\|\left(H_{0,\varkappa,\kappa} + I\right) R_{\varkappa,\kappa}(-b)\psi\right\|_{\#}^2 \leq$$

$$\leq 4Ext\text{-}\int_{|k|\leq\kappa} d^{\#3}k\mu(k)\left\{\|Aa(k)\psi\|_{\#}^2 + \left\|AX_{\varkappa,\kappa}^{(3)}(k)\, R_{\varkappa,\kappa}(-b)\psi\right\|_{\#}^2 + \right.$$

$$\left. +\lambda^2 \sum_{i=1}^2 \left\|AX_{\varkappa,\kappa}^{(i)}(k)\, R_{\varkappa,\kappa}(-b)\psi\right\|_{\#}^2\right\}, \tag{3.22}$$

where $A = \left(H_{0,\varkappa,\kappa} + I + \mu(k)\right)^{1/2} R_{\varkappa,\kappa}\left(-b-\mu(k)\right)$. But by (3.19) we obtain

$$\|A\psi\|_{\#} \leq \text{const.} \times \left\|R_{\varkappa,\kappa}^{1/2}\left(-b-\mu(k)\right)\psi\right\|_{\#} \leq \text{const.} \times \left\|\left(H_{0,\varkappa,\kappa} + \mu(k)\right)^{-\frac{1}{2}}\psi\right\|_{\#}.$$

Therefore from (3.22) we get

$$\left\|\left(H_{0,\varkappa,\kappa}+I\right) R_{\varkappa,\kappa}(-b)\psi\right\|_{\#} \leq$$

$$\text{const.}\times\ Ext\text{-}\int_{|k|\leq\kappa} d^{\#3}k\mu(k)\left\{\left\|\left(H_{0,\varkappa,\kappa}+\mu(k)\right)^{-\frac{1}{2}} a(k)\psi\right\|_{\#}^{2}+\left\|AX_{\varkappa,\kappa}^{(3)}(k)\, R_{\varkappa,\kappa}(-b)\psi\right\|_{\#}^{2}+\right.$$

$$\left.+\lambda^{2}\sum_{i=1}^{2}\left\|\left(H_{0,\varkappa,\kappa}+\mu(k)\right)^{-\frac{1}{2}} X_{\varkappa,\kappa}^{(i)}(k)\, R_{\varkappa,\kappa}(-b)\psi\right\|_{\#}^{2}\right\}.$$

The integral of the first term on the right can be written as

$$Ext\text{-}\int_{|k|\leq\kappa}\mu(k)\left\|a(k)H_{0,\varkappa,\kappa}^{-1/2}\psi\right\|_{\#}^{2} d^{\#3}k=\left\|H_{0,\varkappa,\kappa}^{1/2}H_{0,\varkappa,\kappa}^{-1/2}\psi\right\|_{\#}^{2}\leq\|\psi\|_{\#}^{2},$$

where $H_{0,\varkappa,\kappa}^{-1/2}$ is taken equal to zero on the Fock vacuum. The terms in the integrand involving the $X_{\varkappa,\kappa}^{(i)}(k)$, $i = 1,2,3$, are all bounded by const.× $\Theta(k,\kappa)$ by virtue of (3.20) and (2.13). Hence the integral is hyperfinite and the bound (3.18) holds. We remark that because of the momentum cut-off it was not necessary to use the full force of Lemmas 3.3-3.5, but only the estimates

$$\left\|R_{\varkappa,\kappa}^{1/2}X_{\varkappa,\kappa}^{(i)}(k)R_{\varkappa,\kappa}^{1/2}\right\|_{\#}\leq\text{const.}\times\ \Theta(k,\kappa). \tag{3.23}$$

Remark 3.3 We now prove the self-#-adjointness of $M_{g,\varkappa,\kappa}^{0k}, k=1,2,3$ by treating $T_{0,\varkappa,\kappa}^{0k}$ as a Kato perturbation. Generalized Kato's criterion is [8]:

Proposition 3.7 Let T is a self-#-adjoint operator and let D be a #-core for T. Suppose that A is symmetric and that there are positive constants a and b with $a<1$ such that

$$\|A\psi\|_{\#}\leq a\|(T+b)\psi\|_{\#}$$

for all $\psi\in D(T)$. Then $T+A$ is self-#-adjoint on $D(T)$ and essentially self-#-adjoint on D.

Theorem 3.8 For $\kappa\leq\varkappa$ and g satiating (2.4), $M_{g,\varkappa,\kappa}^{0k}, k=1,2,3$ are essentially self-#-adjoint on D.

Proof We show that $T_{\varkappa,\kappa}^{0k}$ given by (3.14) is self-#-adjoint where $f_{0,k}=\left\{x_k g_0^{(k)}/\alpha\right\}, f_1=x_k g_1/\alpha,$ $k=1,2,3$ and $\lambda=1$; this is equivalent to the statement of the theorem. We use Theorem 3.6 to prove Theorem 3.8 in spite of the fact that the conclusion of the second theorem appears as a hypothesis of the first. By Lemma 2.1 we know that there is a constant c_1 such that

$$\left\|T_{\varkappa,\kappa}^{0k}(f)\psi\right\|_{\#}\leq c_1\left\|\left(H_{0,\varkappa}+I\right)\psi\right\|_{\#} \tag{3.24}$$

for all $\psi\in D\left(H_{0,\varkappa}\right)$. We choose J to be a sufficiently large integer such that $c_1\left(c(k)+d(k)\right)^{1/2}<J$,where $c(k)$ and $d(k)$ are the constants in (3.17). Let us consider the sequence of values $\lambda=j/J, j=0,\ldots,J$. Let $P_{j,k}$ be the statement that $T_{\varkappa,\kappa}^{0k}(j/J)$ is self-#-adjoint and $Q_{j,k}$ the statement that $J^{-1}T_{0,\varkappa,\kappa}^{0k}\left(f_{0,k}\right)$ is a Kato perturbation of $T_{\varkappa,\kappa}^{0k}(j/J)$, i.e., $\left\|J^{-1}T_{0,\varkappa,\kappa}^{0k}\left(f_{0,k}\right)\psi\right\|_{\#}\leq a\left\|\left(T_{\varkappa,\kappa}^{0k}(j/J)+b\right)\psi\right\|_{\#}$ for constants a and b with a < 1. As we have already observed, $P_{0,k}$ holds since $T_{\varkappa,\kappa}^{0k}(0)$ reduces to the Hamiltonian $H_{f_1,\varkappa,\kappa}$. Note that $P_{j,k}$ implies $Q_{j,k}, k=1,2,3$ since, for $\psi\in D\left(T_{\varkappa,\kappa}^{0k}(j/J)\right)$,

$$\left\|J^{-1}T_{0,\varkappa,\kappa}^{0k}\left(f_{0,k}\right)\psi\right\|_{\#}\leq c_1J^{-1}\left\|\left(H_{0,\varkappa}+I\right)\psi\right\|_{\#}\leq c_1J^{-1}c_1\left(c(k)+d(k)\right)^{1/2}\left\|\left(T_{\varkappa,\kappa}^{0k}(j/J)+b\right)\psi\right\|_{\#}$$

by the inequality (3.24) and (3.17). However, by Proposition 3.7, the statement $Q_{j,k}$implies $P_{j+1,k}, k = 1,2,3$.

4. Higher order estimates

In this section we derive higher order estimates of the following form

$$H_{0,\varkappa}^{j} \leq a_{\kappa}\left(M_{\varkappa,\kappa}^{0k}+b\right) \leq c_{\kappa}\left(H_{0,\varkappa}+I\right)^{nj} \tag{4.1}$$

and

$$H_{0,\varkappa}^{2}+N_{\varkappa}^{2n} \leq a\left(M_{\varkappa,\kappa}^{0k}+b\right)^{2n}, \tag{4.2}$$

where a_{κ} and c_{κ} are constants depending on κ. The estimates (4.1) are used to prove that the powers $\left(M_{0,\varkappa,\kappa}^{0k}\right)^{j}$ are essentially self-#-adjoint on $\mathcal{D}_{\text{fin}}^{\#}$ and do not survive in the #-limit: $\kappa \to_{\#} \varkappa$; on the other hand, the estimate (4.2) does transfer to the #-limit $\kappa = \varkappa$ and, in fact, enables us to prove that this #-limit exists. For real $\tau \in {}^{*}\mathbb{R}_{c}^{\#}$ we define the generalized number operator with hyperfinite momentum cut-off $\varkappa \in {}^{*}\mathbb{R}_{c,\infty}^{\#}$

$$N_{\varkappa,\tau} = \ Ext\text{-}\int_{|k|\leq\varkappa} a^{\dagger}(k)[\mu(k)]^{\tau}\ a(k)d^{\#3}k. \tag{4.3}$$

Note that $N_{\varkappa,0} = N_{\varkappa}$ and $N_{\varkappa,1} = H_{0,\varkappa}$.

Lemma 4.1 (1) If $\tau \leq \nu$, then

$$N_{\varkappa,\tau} \leq \text{const.}\cdot N_{\varkappa,\tau}. \tag{4.4}$$

(2) If $\tau > 0, r > 0$, then

$$N_{\varkappa}^{r(1+\tau)} \leq H_{0,\varkappa}^{\tau r} N_{\varkappa,\tau}^{r}. \tag{4.5}$$

(3) Let $\tau \in {}^{*}\mathbb{R}_{c}^{\#}$ and $r \in {}^{*}\mathbb{N}$ a positive integer, then for any vector $\psi \in D\left(N_{\varkappa,\tau}^{r/2}\right)$,

$$\left\|N_{\varkappa,\tau}^{\frac{r}{2}}\psi\right\|_{\#} =$$

$$Ext\text{-}\sum_{j=1}^{r}\left[Ext\text{-}\int d^{\#3}k_{1}\cdots d^{\#3}k_{j}p_{rj}\left(\mu_{1}^{\tau},\dots,\mu_{j}^{\tau}\right)\left(Ext\text{-}\prod_{j=1}^{r}\Theta\left(k_{j},\varkappa\right)\right)\left\|a_{(1,j)}\psi\right\|_{\#}^{2}\right], \tag{4.6}$$

where $\Theta(k,\varkappa)$ is defined by (2.10), $a_{(1,j)}$ is defined in Lemma 3.2, and p_{rj} is a homogeneous polynomial of degree $r \in {}^{*}\mathbb{N}$ with positive coeficients that satisfies, for $x_{i} > 0$,

$$\left(Ext\text{-}\prod_{l=1}^{j}x_{l}\right)\left(Ext\text{-}\sum_{l=1}^{j}x_{l}\right)^{r-j} \leq p_{rj}\left(x_{1},\dots,x_{j}\right) \leq \text{const.}\cdot\left(Ext\text{-}\prod_{l=1}^{j}x_{l}\right)\left(Ext\text{-}\sum_{l=1}^{j}x_{l}\right)^{r-j}. \tag{4.7}$$

In this section we set

$$M_{\varkappa,\kappa}^{0k} = \#\text{-}\overline{\left[\left(H_{0,\varkappa}+V_{\varkappa,\kappa}\right)\restriction D\right]},$$

where $V_{\varkappa,\kappa}^{(k)} = T_{0,\varkappa,\kappa}\left(f_{0,k}\right)+T_{I,\varkappa,\kappa}\left(f_{1}\right), k = 1,2,3$. Let $R_{k}(-b) = \left(M_{\varkappa,\kappa}^{0k}+b\right)^{-1}$.

Lemma 4.2 Let $r \in {}^{*}\mathbb{N}$ be a positive integer. Then there are constants a_{κ} and b where a_{κ} depends on $\kappa < \varkappa$, such that

$$\left\|\left(H_{0,\varkappa,\kappa}+I\right)^{r/2}\psi\right\|_{\#}\leq a_{\kappa}\left\|\left(M^{0k}_{\varkappa,\kappa}+b\right)^{\frac{r}{2}}\psi\right\|_{\#}, k=1,2,3 \quad (4.8)$$

for all $\psi\in D\left(\left(M^{0k}_{\varkappa,\kappa}+b\right)^{\frac{r}{2}}\right)$.

Proof (4.8) is proved by hyper infinite induction on $r\in{}^{*}\mathbb{N}$: the cases $r=1,2$ are already known by Theorem 2.2 and 3.6. Let $\psi\in D_{1,k}=D\left(M^{0k}_{\varkappa,\kappa}+b\right), k=1,2,3$, where $b=-z_0$ is chosen sufficiently large that (3.16) and (3.19) hold. By (4.6),

$$A_{r+1,,\varkappa,\kappa}=\left\|\left(H_{0,\varkappa,\kappa}+I\right)^{(r+1)/2}R_k(-b)\psi\right\|^2_{\#}=$$

$$Ext\text{-}\sum_{j=1}^{r}Ext\text{-}\int_{|k_1|\leq\kappa}d^{\#3}k_1\cdots Ext\text{-}\int_{|k_n|\leq\kappa}d^{\#3}k_j p_{rj}\left(\mu_1,\dots,\mu_j\right)\times$$

$$\times\left\|\left(H_{0,\varkappa,\kappa}+Ext\text{-}\sum_{i=1}^{j}\mu(k_i)\ +I\right)^{1/2}a_{(1,j)}R_k\psi\right\|^2_{\#}, \quad (4.9)$$

where me have converted all but one $\left(H_{0,\varkappa,\kappa}+I\right)^{1/2}$ into an integral of products of annihilation operators. We apply the pull through formula (3.5) to pull the $a_{(1,j)}$ through the R_k, and we dominate the factor $\left(H_{0,\varkappa,\kappa}+Ext\text{-}\sum_{i=1}^{j}\mu(k_i)\ +I\right)^{1/2}$ by

$$R^{1/2}_{k,J_1}=R_k\left(-b-Ext\text{-}\sum\mu(k_i)\right)$$

by using (3.19). This gives

$$A_{r+1,,\varkappa,\kappa}\leq Ext\text{-}\sum_{j=1}^{r}Ext\text{-}\int_{|k_1|\leq\kappa}d^{\#3}k_1\cdots Ext\text{-}\int_{|k_n|\leq\kappa}d^{\#3}k_j p_{rj}\left(\mu_1,\dots,\mu_j\right)\times$$

$$\times\left(Ext\text{-}\sum_{\text{part.of }(1,j)}\left\|R^{1/2}_{J_1}V^{I_1}_{\varkappa}R^{1/2}_{J_2}\cdots R^{1/2}_{J_i}V^{I_i}_{\varkappa}R^{1/2}_{J_{i+1}}a_{I_{i+1}}\psi\right\|^2_{\#}\right). \quad (4.10)$$

Let us consider a typical factor $R_{J_l}V^{I_l}_{\varkappa}R_{J_{l+1}}$, regarded as a function of the variables $k_{i_1},\dots,k_{i_t}$, where $i_\nu\in I_l, \nu=1,\dots,t$. Because of the momentum cut-off, the estimates (3.16) and (3.23) hold:

$$\left\|R^{1/2}_{J_l}V^{I_l}_{\varkappa}R^{1/2}_{J_{l+1}}\right\|_{\#}\leq\text{const.}\times\chi_{\varkappa}\left(k_{i_1},\dots,k_{i_t}\right), \chi_{\varkappa}\left(k_{i_1},\dots,k_{i_t}\right)=Ext\text{-}\prod_{m=1}^{t}\Theta\left(k_{i_m},\varkappa\right).$$

Note that when $t\geq 2$, $\left(T_{0,\varkappa,\kappa}\left(f_{0,k}\right)\right)^{I_l}$ is a multiple of the identity. Therefore, from (4.10) and (3.19),

$$A_{r+1,,\varkappa,\kappa}\leq\text{const.}\times$$

$$\times Ext\text{-}\sum_{j=1}^{r}Ext\text{-}\int_{|k_1|\leq\kappa}d^{\#3}k_1\cdots Ext\text{-}\int_{|k_n|\leq\kappa}d^{\#3}k_j\, Ext\text{-}\sum_{(1,j)=Z_1\cup Z_2}p_{rj}\,\chi_{\varkappa}(Z_1)\times$$

$$\times\left\|\left(H_{0,\varkappa,\kappa}+Ext\text{-}\sum_{i\in Z_2}\mu(k_i)\ +I\right)^{-1/2}a_{Z_2}\psi\right\|^2_{\#}, \quad (4.11)$$

where we have set

$$Z_1=\bigcup_{p=1}^{i}I_p=\left\{i_1,\dots,i_{j-s}\right\}, Z_2=I_{i+1}=\{j_1,\dots,j_s\}, \chi_{\varkappa}(Z_1)=\chi_{\varkappa,\kappa}\left(k_{i_1},\dots,k_{i_{j-s}}\right).$$

By the binomial expansion and (4.7) we get

$$p_{rj}(\mu_1,\dots,\mu_j) \le \text{const.}\times [\mu(\kappa)]^{j-s} \times Ext\text{-}\prod_{i\in Z_2}\mu(k_i)\,[(j-s)\mu(\kappa)+]^{r-j}$$

Here the const. depends on $\kappa \le \varkappa$ and

$$p_{s+t,s}(Z_2) \le \text{const.}\times p_{s+t,s}\left(\mu(k_{j_1}),\dots,\mu(k_{j_s})\right).$$

By (4.7), since $\mu(k) > m > 0$,

$$p_{ts}(\mu_1,\dots,\mu_j) \le \text{const.}\times p_{t's}(\mu_1,\dots,\mu_s).$$

if $t < t'$. In the above sum over $t, s + t < r$; therefore,

$$p_{rj}(\mu_1,\dots,\mu_j) \le p_{rs}(Z_2)$$

Integrating out the variables in Z_1, in (4.11), we obtain

$$A_{r+1,,\varkappa,\kappa} \le Ext\text{-}\sum_{j=1}^{r} Ext\text{-}\sum_{Z_{2\subset(1,j)}} Ext\text{-}\int Ext\text{-}\prod_{i\in Z_2}\Theta(k_i,\kappa)d^{\#3}k_i\, p_{rs}(Z_2)\times$$

$$\times \left\|a_{Z_2}\left(H_{0,\varkappa,\kappa}+I\right)^{-1/2}\psi\right\|_{\#}^{2} \le \text{const.}\times \left\|\left(H_{0,\varkappa,\kappa}\right)^{r/2}\left(H_{0,\varkappa,\kappa}+I\right)^{-1/2}\psi\right\|_{\#}^{2}$$

by virtue of (4.6) with $\tau = 1$. Setting $\psi = \left(M_{\varkappa,\kappa}^{0k}+b\right)\phi, k = 1,2,3$, where ϕ is an arbitrary element of the domain D, we obtain

$$\left\|\left(H_{0,\varkappa,\kappa}+I\right)^{(r+1)/2}\phi\right\|_{\#} \le \text{const.}\times \left\|\left(H_{0,\varkappa,\kappa}+I\right)^{(r-1)/2}\left(M_{\varkappa,\kappa}^{0k}+b\right)\phi\right\|_{\#} \qquad (4.12)$$

By the inductive assumption we have

$$\left\|\left(H_{0,\varkappa,\kappa}+I\right)^{(r-1)/2}\left(M_{\varkappa,\kappa}^{0k}+b\right)\phi\right\|_{\#} \le \text{const.}\times \left\|\left(M_{\varkappa,\kappa}^{0k}+b\right)^{\frac{(r+1)}{2}}\phi\right\|_{\#}, \qquad (4.13)$$

which appears to prove the lemma. However, we do not yet know that D is a #-core for $\left(M_{\varkappa,\kappa}^{0k}+b\right)^{\frac{(r+1)}{2}}$ and so we must argue more carefully. Define now the operators

$$B_k(\lambda)=\left(H_{0,\varkappa,\kappa}+I\right)^{(r-1)/2}\left(H_{0,\varkappa,\kappa}+\lambda T_{0,\varkappa,\kappa}^{0k}+T_{I,\varkappa,\kappa}^{0k}+b\right),$$

$k = 1,2,3$ on the domain D. It is sufficient to prove that D is a #-core for $B_k(1)$. For then (4.12) extends from D to $D(B_k(1))$; by induction (4.13) holds on $D\left(\left(M_{\varkappa,\kappa}^{0k}+b\right)^{\frac{(r+1)}{2}}\right) \subset D(B_k(1))$, and the proof of the lemma is complete. As in the proof of Theorem 3.8, we consider a sequence of values $\lambda_j = j/J, j = 0,1,\dots,J$, and regard the operator

$$\mathrm{C}_{\varkappa,\kappa} = J^{-1}\left(H_{0,\varkappa,\kappa}+I\right)^{(r-1)/2}T_{0,\varkappa,\kappa}$$

as a perturbation of $B_k(\lambda_j)$. By (4.12)

$$\left\|\left(H_{0,\varkappa,\kappa}+I\right)^{(r+1)/2}\phi\right\|_{\#} \le c\left\|B_k(\lambda_j)\phi\right\|_{\#}$$

for any $\phi \in D$, where the constants b and c are seen to be independent of $\lambda_j \in [0,1]$. But, as in the next lemma,

$$\left\|\left(H_{0,\varkappa,\kappa}+I\right)^{(r-1)/2} T_{0,\varkappa,\kappa}\left(H_{0,\varkappa,\kappa}+I\right)^{-(r+1)/2}\right\|_{\#} \leq c_\kappa < {}^*\infty.$$

Hence, by choosing hyperinteger $J \in {}^*\mathbb{N}_\infty, J > cc_\kappa$, we have for $\phi \in D$,

$$\left\|\mathrm{C}_{\varkappa,\kappa}\phi\right\|_{\#} \leq J^{-1}c_\kappa \left\|\left(H_{0,\varkappa,\kappa}+I\right)^{\frac{(r+1)}{2}}\phi\right\|_{\#} \leq \left\|B_k(\lambda_j)\phi\right\|_{\#}, \tag{4.14}$$

where $a = J^{-1}cc_\kappa < 1$. That is, C is a Kato perturbation of $B_k(\lambda_j)$. Note that domain D is a #-core for $B_k(0)$. This follows from the facts that (4.8) holds when $\lambda = 0,$ i.e. when $M^{0k}_{\varkappa,\kappa}$ is replaced by $H^{0k}_{\varkappa,\kappa} = H_{0,\varkappa,\kappa} + T^{0k}_{I,\varkappa,\kappa}$, and that powers $\left(H^{0k}_{\varkappa,\kappa}\right)^r$ are essentially self -#-adjoint on D. From 4.14 we see that D is also a #-core for $B_k(0) + C = B_k(\lambda_1)$ and that $D(B_k(\lambda_1)) = D(B_k(0))$ Continuing in this way we reach the conclusion that D is a #-core for $B_k(1)$. To complete the estimate (4.1), we dominate powers of $M^{0k}_{\varkappa,\kappa}$, $k = 1,2,3$ by powers of $H_{0,\varkappa,\kappa}$.

Lemma 4.3 Let $j \in {}^*\mathbb{N}$ be a positive hyperinteger. Then there are positive constants b and c_κ, where c_κ depends on κ such that

$$\left\|\left(M^{0k}_{\varkappa,\kappa}\right)^j\psi\right\|_{\#} \leq c_\kappa \left\|\left(H_{0,\varkappa,\kappa}+b\right)^{nj}\psi\right\|_{\#}, k = 1,2,3. \tag{4.15}$$

Here $2n$ is the order of the interaction.

Proof Here $2n$ is the order of the interaction. Since $\left(H_{0,\varkappa,\kappa}+b\right)^{nj}$ is essentially self-#-adjoint on D it is sufficient to prove (4.15) for $\psi \in D$. Now because of the momentum cutoff, $M^{0k}_{0,\varkappa,\kappa}$ has the form $M^{0k}_{0,\varkappa,\kappa} = H_{0,\varkappa,\kappa} + \sum W_i$, where W_i is a Wick monomial (2.12) whose kernel has #-compact support. Each such monomial W_i maps domain D into a set of vectors which have a finite number of particles and which are of #-compact support and $C^{{}^*\infty}\left({}^*\mathbb{R}^{\#}_{\boldsymbol{c}}\right)$ #-almost everywhere in the momentum variables. It follows that $\left(M^{0k}_{0,\varkappa,\kappa}\right)^j$ can be expanded on D into a sum of welldefined products of the form $A = Ext\text{-}\prod_{m=0}^{r} H^{i_{2m+1}}_{0,\varkappa,\kappa} W^{i_{2m+2}}$ where $Ext\text{-}\sum_{l=1}^{s} i_l = j$, and W represents a typical Wick monomial in $M^{0k}_{0,\varkappa,\kappa}$. Each such product can be dominated by $\left(H_{0,\varkappa,\kappa}+b\right)^{nj}$provided that b is chosen sufficiently large, say $b > 2nj\mu(\kappa)$. It suffices to show that

$$W\left(H_{0,\varkappa,\kappa}+a\right)^{-i} = \left(H_{0,\varkappa,\kappa}+a-2n\mu(\kappa)\right)^{-i+n} B, \tag{4.16}$$

where B is a bounded operator. For then it is clear by hyper infinite induction that $A\left(H_{0,\varkappa,\kappa}+b\right)^{-nj}$ is bounded. Take W of the form (2.12) with $r < 2n$. Then

$$W\left(H_{0,\varkappa,\kappa}+a\right)^{-i} =$$

$$= \left(H_{0,\varkappa,\kappa}+a-2n\mu(\kappa)\right)^{-i} Ext\text{-}\int_{|k_1|\leq\kappa} d^{\#3}k_1 \cdots Ext\text{-}\int_{|k_n|\leq\kappa} d^{\#3}k_r v(k_1,\ldots,k_r) \times$$

$$\times a^*(k_1)\cdots a(k_r),$$

where

$$v(k_1,\dots,k_r)=\left(H_{0,\varkappa,\kappa}+a-2n\mu(\kappa)\right)^{i}\left(H_{0,\varkappa,\kappa}+a\pm\mu(k_1)\pm\cdots\pm\mu(k_r)\right)^{-i},$$

where the $\pm$ is chosen according to whether the corresponding $a^{\#}(k)$ is an a or $a^{*}(k)$. Since

$$-2n\mu(\kappa)\leq\pm\mu(k_1)\pm\cdots\pm\mu(k_r)$$

the operator #-norm

$$\|v(k_1,\dots,k_r)\|_{\#}\leq|w(k_1,\dots,k_r)|.$$

By an extension of the basic estimate (2.13) to cover the case of operator-valued kernels, it follows that

$$B=\left(H_{0,\varkappa,\kappa}+a-2n\mu(\kappa)\right)^{-n}Ext\text{-}\int_{|k_1|\leq\kappa}d^{\#3}k_1\cdots Ext\text{-}\int_{|k_n|\leq\kappa}d^{\#3}k_r v(k_1,\dots,k_r)\times$$

$$\times a^{*}(k_1)\cdots a(k_r).$$

is a bounded operator. This completes the proof of the lemma. Note that by the generalized spectral theorem [8], the κ dependence of b can be incorporated into constant c_{κ}.

Theorem 4.4 Let $j\in{}^{*}\mathbb{N}$ be a positive integer. Then the operators $\left(M^{0k}_{0,\varkappa,\kappa}\right)^{j}, k=1,2,3$ are essentially self-#-adjoint on D.

Proof Let $C_k=D_k=D\left(\left(M^{0k}_{\varkappa,\kappa}+B\right)^{nj}\right), k=1,2,3,$ where b is a large positive number. By the previous two lemmas we have that

$$D_k\subset C_k\subset D\left(\left(H_{0,\varkappa,\kappa}+B\right)^{nj}\right)\subset D\left(\left(M^{0k}_{\varkappa,\kappa}\right)^{j}\right)\tag{4.17}$$

Since D is a #-core for $\left(H_{0,\varkappa,\kappa}+B\right)^{nj}$, it follows from (4.15) that

$$D\left(\#\text{-}\overline{\left(M^{0k}_{\varkappa,\kappa}\right)^{j}\restriction D}\right)\supset D\left(\left(H_{0,\varkappa,\kappa}+B\right)^{nj}\right).$$

Therefore, by (4.17),

$$\#\text{-}\overline{\left(M^{0k}_{\varkappa,\kappa}\right)^{j}\restriction D}\supset\#\text{-}\overline{\left(M^{0k}_{\varkappa,\kappa}\right)^{j}\restriction C_k}$$

since C_k is a core for $\left(M^{0k}_{\varkappa,\kappa}\right)^{j}, k=1,2,3.$

Theorem 4.5 Let $\tau>0$ and $r\in{}^{*}\mathbb{N}$ be a positive integer. Then there are constants a and b independent of κ such that

$$\left\|H^{1/2}_{0,\varkappa,\kappa}N^{(r-1)/2}_{\varkappa,,\kappa,-\tau}\psi\right\|_{\#}\leq a\left\|\left(M^{0k}_{\varkappa,\kappa}+b\right)^{\frac{r}{2}}\psi\right\|_{\#}\tag{4.18}$$

for all $\psi\in D\left(\left(M^{0k}_{\varkappa,\kappa}+b\right)^{\frac{r}{2}}\right)$.

Proof The proof is by hyper infinite induction on r, the case $r=1$ being (3.19). By the previous theorem it is sufficient to prove (4.18) for $\psi\in D$. We set now $\phi=\left(M^{0k}_{\varkappa,\kappa}+b\right)\psi\in D_1$, where $b=-z_0$ is chosen sufficiently large that (3.16) and (3.19) hold. By a now familiar procedure we expand

$$A_{r+1,\varkappa,\kappa} = \left\| H_{0,\varkappa,\kappa}^{1/2} N_{\varkappa,\kappa,-\tau}^{r/2} \psi \right\|_{\#}^{2}$$

by (4.6) and apply the pull-through formula. The result is similar to (4.10)

$$A_{r+1,\varkappa,\kappa} \leq \text{const.}\times\ Ext\text{-}\sum_{j=1}^{r} Ext\text{-}\int_{|k_1|\leq\kappa} d^{\#3}k_1 \cdots Ext\text{-}\int_{|k_n|\leq\kappa} d^{\#3}k_j p_{rj}\left(\mu_1^{-\tau},\ldots,\mu_j^{-\tau}\right) \times$$

$$\times \left(Ext\text{-}\sum_{\text{part.of }(1,j)} \left\| R_{J_1}^{1/2} V_{\varkappa}^{I_1} R_{J_2}^{1/2} \cdots R_{J_i}^{1/2} V_{\varkappa}^{I_i} R_{J_{i+1}} a_{I_{i+1}} \phi \right\|_{\#}^{2} \right). \qquad (4.19)$$

By (4.7) one obtains

$$p_{rj}\left(\mu_1^{-\tau},\ldots,\mu_j^{-\tau}\right) \leq \text{const.}\times \left(\mu_1 \times \ldots \times \mu_j\right)^{-\tau}.$$

We insert this inequality into (4.19) and estimate the integral over the “variables” of I_1. Say $I_1 = \{i_1,\ldots,i_t\}$. We must estimate

$$Ext\text{-}\int_{|k_1|\leq\kappa} d^{\#3}k_1 \cdots Ext\text{-}\int_{|i_t|\leq\kappa} d^{\#3}k_{i_t} \left(\mu\left(k_{i_1}\right)\times\ldots\times\mu\left(k_{i_t}\right)\right)^{-\tau} \times \left\| R_{J_1}^{\frac{1}{2}} V_{\varkappa}^{I_1} R_{J_2}^{\frac{1}{2}} \phi_1 \right\|_{\#}^{2}, \quad (4.20)$$

where

$$\phi_1 = R_{J_1}^{1/2} V_{\varkappa}^{I_1} R_{J_2}^{1/2} \cdots R_{J_i}^{1/2} V_{\varkappa}^{I_i} R_{J_{i+1}} a_{I_{i+1}} \phi$$

does not depend on the variables of I_1, for which we recall that $J_l = I_l \cup I_{l+1} \cup \ldots \cup I_{i+1}$. Now

$$V_{\varkappa,\kappa} = T_{0,\varkappa,\kappa}^{(1)} + T_{0,\varkappa,\kappa}^{(2)} + T_{I,\varkappa,\kappa}$$

and by the triangle inequality it is sufficient to estimate each of these three contributions to (4.20) separately. By (3.16) the contribution of $T_{I,\varkappa,\kappa}$ can be dominated by

$$\text{const.}\times\ Ext\text{-}\int_{|k_1|\leq\kappa} d^{\#3}k_1 \cdots Ext\text{-}\int_{|k_{i_t}|\leq\kappa} d^{\#3}k_{i_t} \left(\mu\left(k_{i_1}\right)\times\ldots\times\mu\left(k_{i_t}\right)\right)^{-1-\tau} \times \|\phi_1\|_{\#}^{2} \leq$$

$$\leq \text{const.}\times \|\phi_1\|_{\#}^{2}, \qquad (4.21)$$

where the constant is independent of κ. As for the $T_{T_{0,\varkappa,\kappa}}^{(i)}$ terms, when $t > 1$ we have

$$\left(T_{0,\varkappa,\kappa}^{(1)}\right)^{I_1} = 0,$$

and by (3.13) and (3.19), we have

$$\left\| R^{1/2} \left(T_{0,\varkappa,\kappa}^{(2)}\right)^{I_1} R^{1/2} \right\|_{\#} \leq \text{const.}\times\ Ext\text{-}\prod_{i\in I_1}\left[\mu\left(k_i\right)\right]^{-1/2}$$

for all t. Thus the contribution of $T_{0,\varkappa,\kappa}^{(2)}$ to the integral (4.20) is bounded as in (4.21). It remains to estimate

$$\left(T_{0,\varkappa,\kappa}^{(1)}\right)^{I_1} = X^{(1)}\left(k_{i_1}\right)$$

when $t = 1$. By (3.19) and (3.8), we have

$$Ext\text{-}\int_{|k_{i_1}|\le\kappa} d^{\#3}k_{i_1}\ \left\|R_{J_1}^{1/2}X^{(1)}\left(k_{i_1}\right)R_{J_2}^{1/2}\phi_1\right\|_{\#}^2 \le \|\phi_1\|_{\#}^2$$

Hence we have integrated out the variables of I_1

$$A_{r+1,\varkappa,\kappa} \le \text{const.}\times Ext\text{-}\sum_{j=1}^{r} Ext\text{-}\sum_{\text{part.of}\,(1,j)} Ext\text{-}\int_{|k_l|\le\kappa} Ext\text{-}\prod_{l\in I_2\cup\ldots\cup I_i} d^{\#3}k_l[\mu(k_l)]^{-\tau}\times$$

$$\times\left\|R_{J_1}^{1/2}V_{\varkappa}^{I_1}\cdots R_{J_{i+1}}a_{I_{i+1}}\phi\right\|_{\#}^2.$$

In this way we integrate over the variables of $I_2\cup\ldots\cup I_i$ to obtain

$$A_{r+1,\varkappa,\kappa} \le \text{const.}\times Ext\text{-}\sum_{j=1}^{r} Ext\text{-}\sum_{I\subset(1,j)} Ext\text{-}\int_{|k_l|\le\kappa} Ext\text{-}\prod_{l\in I} d^{\#3}k_l[\mu(k_l)]^{-\tau}\left\|R_I^{\frac{1}{2}}a_I\phi\right\|_{\#}^2.$$

By a change of variables we can rewrite the sum over j and I as a sum over subsets $\{1,2,\ldots,s\}$ of $(1,2,\ldots,r)$. Using the estimates (3.19) and (4.7), we get

$$A_{r+1,\varkappa,\kappa} \le \text{const.}\times Ext\text{-}\sum_{s=0}^{r}\int_{|k_1|\le\kappa} d^{\#3}k_1\cdots Ext\text{-}\int_{|k_s|\le\kappa} d^{\#3}k_s p_{ss}(\mu_1^{-\tau},\ldots,\mu_s^{-\tau})\times$$

$$\times\left\|\left(H_{0,\varkappa,\kappa}+Ext\text{-}\sum_{i=1}^{s}\mu_i^{-\tau}+I\right)^{-1/2}a_{(1,s)}\phi\right\|_{\#}^2,$$

where the $s=0$ term is simply $\left\|\left(H_{0,\varkappa,\kappa}+I\right)^{-1/2}\phi\right\|_{\#}^2$. It follows from the expansion (4.6) that

$$A_{r+1,\varkappa,\kappa} \le \text{const.}\times\left\|\left(N_{\varkappa,\kappa,-\tau}^{r}+I\right)^{1/2}\left(H_{0,\varkappa,\kappa}+I\right)^{-1/2}\phi\right\|_{\#}^2 \le \text{const.}\times\left\|(N_{-\tau}^{r}+I)^{1/2}\,\phi\right\|_{\#}^2$$

by (4.4). Since $\phi\in D_1\subset D\left(\left(M_{0,\varkappa,\kappa}^{0k}+b\right)^{\frac{(r-1)}{2}}\right)$, we obtain by the inductive hypothesis,

$$A_{r+1,\varkappa,\kappa} = \left\|H_{0,\varkappa,\kappa}^{1/2}N_{\varkappa,\kappa,-\tau}^{r/2}\psi\right\|_{\#}^2 \le \text{const.}\times\left\|\left(M_{0,\varkappa,\kappa}^{0k}+b\right)^{\frac{(r-1)}{2}}\phi\right\|_{\#} =$$

$$= \text{const.}\times\left\|\left(M_{0,\varkappa,\kappa}^{0k}+b\right)^{\frac{r}{2}}\psi\right\|_{\#},$$

where the constant is independent of κ.

Corollary 4.6 Let $\delta>0$ and r be a positive integer. Then there are constants a and b independent of κ such that

$$\left\|H_{0,\varkappa,\kappa}^{(1-\delta)/2}N_{\varkappa,\kappa,-\tau}^{(r+\delta)/2}\psi\right\|_{\#} \le a\left\|\left(M_{\varkappa,\kappa}^{0k}+b\right)^{\frac{(r+1)}{2}}\psi\right\|_{\#} \qquad (4.22)$$

for all $\psi\in D\left(\left(M_{\varkappa,\kappa}^{0k}+b\right)^{\frac{(r+1)}{2}}\right), k=1,2,3.$

Proof The Corollary follows immediately from the Theorem by means of (4.5).

Remark 4.1 The estimates (3.19) and (4.22) do not permit us to dominate the operator $H_{0,\varkappa,\kappa}$ itself by the operators $M^{0k}_{\varkappa,\kappa} + b, k = 1,2,$. However we can dominate $H_{0,\varkappa,\kappa}$ as in (4.2) if we abandon the requirement that the powers of $H_{0,\varkappa,\kappa}$ and $M^{0k}_{\varkappa,\kappa}$ agree. The inequality

$$H^2_{0,\kappa} \leq a\left(M^{0k}_{\varkappa,\kappa} + b\right)^j \quad (4.23)$$

we prove with $j = 2n$.

Corollary 4.7 There are constants a and b independent of κ such that for all $\psi \in D\left(\left(M^{0k}_{0,\varkappa,\kappa}\right)^n\right)$

$$\left\|H_{0,\varkappa,\kappa}\psi\right\|_{\#} \leq a\left\|\left(M^{0k}_{\varkappa,\kappa} + b\right)^n\psi\right\|_{\#}. \quad (4.24)$$

Proof By Theorem 4.4 it is sufficient to prove (4.24) for $\psi \in D$. Since $D \subset D\left(M^{0k}_{0,\varkappa,\kappa}\right) \cap D\left(T_{I,\kappa}\right)$ obviously we have

$$\left(M^{0k}_{0,\varkappa,\kappa} + b\right)\psi = \left(M^{0k}_{\varkappa,\kappa} + b\right)\psi - T_{I,\kappa}\psi. \quad (4.25)$$

Since $T_{I,\kappa}(f_1)$ is a sum of Wick monomials with $L^{\#}_2$-kernels and maximum order $2n$ [8], it follows from the basic estimate (2.13) that

$$\left\|T_{I,\varkappa,\kappa}\left(N_{0,\varkappa,\kappa} + I\right)^{-n}\right\|_{\#} \leq \text{const.}, \quad (4.26)$$

where the constant is independent of κ. Therefore from the identity (4.25) we obtain

$$\left\|\left(M^{0k}_{0,\varkappa,\kappa} + b\right)\psi\right\|_{\#} \leq \left\|\left(M^{0k}_{\varkappa,\kappa} + b\right)\psi\right\|_{\#} + \left\|T_{I,\varkappa,\kappa}\left(N_{0,\varkappa,\kappa} + I\right)^{-n}\right\|_{\#} \times \left\|\left(N_{0,\varkappa,\kappa} + I\right)^{-n}\psi\right\|_{\#}$$

by (4.22). But by Theorem 3.1 we obtain

$$\left\|H_{0,\varkappa,\kappa}\psi\right\|_{\#} \leq \text{const.}\times \left\|\left(M^{0k}_{\varkappa,\kappa} + b\right)\psi\right\|_{\#}$$

and therefore the estimate (4.24) is proved.

5. Essential self-#-adjointness of the #-limit $M^{0k}_{\kappa,g}$ **as** $\kappa \to_{\#} \varkappa$

In the previous two sections we established a number of properties of the hyperfinite ultraviolet cut-off Lorentz boost generators $M^{0k}_{\varkappa,\kappa}, k = 1,2,3$ by methods that depended on $\kappa < \varkappa$ being hyperfinite. Now we take the #-limit $\kappa \to_{\#} \varkappa$ and find that many of the properties of $M^{0k}_{\varkappa,\kappa}$ transfer to the #-limiting operators $M^{0k}_{\varkappa}, k = 1,2,3$. As the next lemma states, $M^{0k}_{\varkappa,\kappa}, k = 1,2,3$ #-converges to $M^{0k}_{\varkappa}, k = 1,2,3$ on the #-dense domain

$$D_n = D\left(H_{0,\varkappa}\right) \cap D\left(N^n_{\varkappa}\right), n \in {}^*\mathbb{N}. \quad (5.1)$$

Note that #-convergence in this sense is not strong enough to control the #-limiting operator and in Theorem 5.3 we prove that the resolvents $R^{(k)}_{\varkappa,\kappa}(z) = \left(M^{0k}_{\varkappa,\kappa} - z\right)^{-1}, k = 1,2,3$ #-converge in #-norm. From this it follows that the operators $M^{0k}_{\varkappa}, k = 1,2,3$ are essentially self-#-adjoint on D.

Lemma 5.1 Let $\psi \in D_n$, then $M^{0k}_{\varkappa,\kappa}\psi \to_{\#} M^{0k}_{0,\varkappa}\psi, k = 1,2,3$ as $\kappa \to_{\#} \varkappa$.

Proof We write now $M^{0k}_{\varkappa,\kappa,g} = H_{0,\varkappa,\kappa} + T_{0,\varkappa,\kappa}\left(x_k g^{(k)}_0\right) + T_{I,\varkappa,\kappa}(x_k g_1), k = 1,2,3$ of the form

$$M^{0k}_{\varkappa,\kappa} = H_{0,\varkappa,\kappa} + T_{0,\varkappa,\kappa}\left(f_{0,k}\right) + T_{I,\varkappa,\kappa}(f_1), k = 1,2,3.$$

By the estimates (2.15), (2.16), and (4.26), $T_{0,\varkappa,\kappa}(f_{0,k})$ and $T_{I,\varkappa,\kappa}$ are defined on domain D_n, for $\kappa \leq \varkappa$. In fact, precisely these estimates prove #-convergence. For consider the difference

$$A_{\varkappa,\kappa} = T_{I,\varkappa}(f_1) - T_{I,\varkappa,\kappa}(f_1).$$

$A_{\varkappa,\kappa}$ can be written as a sum of Wick monomials whose kernels are the tails of $L_2^{\#}$ kernels. Therefore, by (2.13), $\|A_{\varkappa,\kappa}(N_\varkappa + I)^{-n}\|_{\#}$ bounded by the $L_2^{\#}$-#-norms of these tails which go to zero as $\kappa \to_{\#} \varkappa$. Since a similar argument can be made for $T^{(2)}_{0,\varkappa,\kappa}(f)$ it follows that on D_n

$$T^{(2)}_{0,\varkappa,\kappa} + T_{I,\varkappa,\kappa} \to_{\#} T^{(2)}_{0,\varkappa} + T_{I,\varkappa}. \tag{5.2}$$

The strong #-convergence of the differences

$$B^{(k)}_{\varkappa,\kappa} = T_{0,\varkappa}(f_{0,k}) - T_{0,\varkappa,\kappa}(f_{0,k}), k = 1,2,3$$

to zero on $D(H_{0,\varkappa})$ does not follow from a corresponding statement of #-norm #-convergence, since

$$\left\| B^{(k)}_{\varkappa,\kappa}(H_{0,\varkappa} + I)^{-1} \right\|_{\#} \nrightarrow_{\#} 0 \tag{5.3}$$

as $\kappa \to_{\#} \varkappa$. However, by (2.15) $\left\| B^{(k)}_{\varkappa,\kappa}(H_{0,\varkappa} + I)^{-1} \right\|_{\#}$ is uniformly bounded in κ. It is thus sufficient to show that $B^{(k)}_{\varkappa,\kappa}\psi_r \to_{\#} 0$ for $r \in {}^*\mathbb{N}$ particle vector $\psi_r = \psi(p_1, \dots, p_r) \in D$. By (2.8) one obtains

$$\left(B^{(k)}_{\varkappa,\kappa}\, \psi_r\right)(p_1, \dots, p_1) = Ext\text{-}\textstyle\sum_{j=1}^{r} Ext\text{-}\int d^{\#3}k w_{\varkappa,\kappa}(k, p_j)\psi(p_1, \dots, p_{j-1}, k, p_{j+1}, \dots, p_r), \tag{5.4}$$

where

$$w_{\varkappa,\kappa}(k, p) = t^{(1)}(k, p)\big(\Theta(k\,, \varkappa)\Theta(p\,, \varkappa) - \Theta(k\,, \kappa)\Theta(p\,, \kappa)\big), \tag{5.5}$$

where $\Theta(k\,, \kappa)$ is defined by (2.10) with $\varkappa = \kappa$. Therefore,

$$\left|B^{(k)}_{\varkappa,\kappa}\psi\right| \leq 2Ext\text{-}\textstyle\sum_{j=1}^{r} Ext\text{-}\int_{|k|>\kappa} d^{\#3}kt^{(1)}(k, p_j)\psi(p_1, \dots, p_{j-1}, k, p_{j+1}, \dots, p_r), \tag{5.6}$$

where by (2.15) the right side is an $L_2^{\#}$ function in variables $(p_1, \dots, p_r)$ whose #-norm is bounded by const. $\left\|(H_{0,\varkappa} + I)^{-1}\, \psi_r\right\|_{\#}$. Moreover, as $\kappa \to_{\#} \varkappa$, $\left(B^{(k)}_{\varkappa,\kappa}\, \psi_r\right)(p_1, \dots, p_r) \to_{\#} 0$ pointwise so that by the dominated #-convergence theorem $\left\|B^{(k)}_{\varkappa,\kappa}\, \psi_r\right\|_{\#} \to_{\#} 0$. For the proof of resolvent #- convergence we require a #-norm #-convergent statement for $T^{(1)}_{0,\varkappa,\kappa}(f_{0,k})$. The failure in (5.3) is to be expected, for, roughly speaking; we can regard $T^{(1)}_{0,\varkappa,\kappa}(f_{0,k})$ as $H_{0,\varkappa,\kappa}$ and obviously $C_{\varkappa,\kappa} = (H_{0,\varkappa} - H_{0,\varkappa,\kappa})(H_{0,\varkappa} + I)^{-1}$ does not #-converge to zero in #-norm. However, this argument indicates that $\left\|B^{(k)}_{\varkappa,\kappa}(H_{0,\varkappa} + I)^{-\tau}\right\|_{\#} \to_{\#} 0$ for $\tau > 1$.

Lemma 5.2 Let $i, j \in {}^*\mathbb{N}$ be nonnegative integers, and $f \in C_0^{{}^*\infty}({}^*\mathbb{R}^{\#3n}_{c,\text{fin}+}, {}^*\mathbb{R}^{\#}_{c,\text{fin}})$.
(1) For $i + j > 2$,

$$\left\|(H_{0,\varkappa} + I)^{-i/2}\left(T^{(1)}_{0,\varkappa}(f) - T^{(1)}_{0,\varkappa,\kappa}(f)\right)(H_{0,\varkappa} + I)^{-j/2}\right\|_{\#} \to_{\#} 0 \;\text{ as } \kappa \to_{\#} \varkappa \tag{5.7}$$

(2) For $i + j \geq 2$,

$$\left\| (H_{0,\varkappa} + I)^{-i/2} \left(T_{0,\varkappa}^{(2)}(f) - T_{0,\varkappa,\kappa}^{(2)}(f) \right) (H_{0,\varkappa} + I)^{-j/2} \right\|_{\#} \to_{\#} 0 \text{ as } \kappa \to_{\#} \varkappa \tag{5.8}$$

(3) For $i + j \geq 2n$,

$$\left\| (H_{0,\varkappa} + I)^{-i/2} \left(T_{I,\varkappa}(f) - T_{I,\varkappa,\kappa}(f) \right) (H_{0,\varkappa} + I)^{-j/2} \right\|_{\#} \to_{\#} 0 \text{ as } \kappa \to_{\#} \varkappa \tag{5.9}$$

Proof Equation (5.7) is a consequence of estimates developed in [8] for Wick monomials with one creating and one annihilating leg. These estimates involve $L_1^{\#}$ - $L_{^*\infty}^{\#\approx}$ #-norms on the kernels such that

$$\|w\|_{\#1,\tau} =\approx \text{-essup}_k \left([\mu(k)]^{-\tau} \left(Ext\text{-}\int_{|p|\leq\varkappa} |w(k,p)|\, d^{\#3}p \right) \right). \tag{5.10}$$

Given a #-measurable function $f\colon {}^*\mathbb{R}_c^{\#3} \to {}^*\mathbb{R}_c^{\#}$ [16], the $\approx$ -essential supremum of f is the smallest number α such that the set $\{x \in {}^*\mathbb{R}_c^{\#3} | f(x) > \alpha\}$ has infinite small Lebesgue #-measure, i.e., $\mu^{\#}(\{x|f(x) > \alpha\}) \approx 0$. The essential supremum of a function f is denoted $\approx$ -essup$_x(f)$. The essential supremum of the absolute value of a function $|f|$ is denoted $\|f\|_{^*\infty}^{\#\approx}$ and this serves as the #-norm for $L_{^*\infty}^{\#\approx}$ -infty-space.

As an example of (5.7), we consider the case $i = 1$ and $j = 2$. As in (5.4),

$$B_{\varkappa,\kappa} = T_{0,\varkappa}^{(1)}(f) - T_{0,\kappa}^{(1)}(f) = Ext\text{-}\int w_{\varkappa,\kappa}(k,p) a^*(k)\, a(p) d^{\#3}k d^{\#3}p.$$

We see that for r particle vector $\psi_r = \psi(p_1, \dots, p_r)$ the inequality holds

$$\left| B_{\varkappa,\kappa} (H_{0,\varkappa} + I)^{-\frac{1}{2}} \psi(p_1, \dots, p_r) \right| \leq$$

$$Ext\text{-}\left\{ \sum_{j=1}^{r} Ext\text{-}\int d^{\#3}k \frac{|w_{\varkappa,\kappa}(k,p_j)|}{[\mu(p_j)]^{1/2}} |\psi(p_1, \dots, p_{j-1}, k, p_{j+1}, \dots, p_r)| \right\}.$$

Therefore $\left\| B_{\varkappa,\kappa} (H_{0,\varkappa} + I)^{-1/2} \psi_r \right\|_{\#}$ is bounded by the #-norm of

$$A_{\varkappa,\kappa} |\psi_r| = Ext\text{-}\int |w_{\varkappa,\kappa}(k,p)| [\mu(p)]^{-1/2} \Theta(p,\varkappa)\, a^*(k) a(p) d^{\#3}k d^{\#3}p |\psi_r|$$

and

$$\left\| (H_{0,\varkappa} + I)^{-\frac{1}{2}} B_{\varkappa,\kappa} (H_{0,\varkappa} + I)^{-1} \right\|_{\#} \leq \left\| (H_{0,\varkappa} + I)^{-\frac{1}{2}} A_{\varkappa,\kappa} (H_{0,\varkappa} + I)^{-\frac{1}{2}} \right\|_{\#} \leq$$

$$\leq \left\| w_{\varkappa,\kappa}(k,p) [\mu(p)]^{-1/2} \right\|_{\#1,1}.$$

see **[8]**. According to the definition (5.10) by (5.5) and (2.9) we obtain

$$\left\| w_{\varkappa,\kappa}(k,p) [\mu(p)]^{-1/2} \right\|_{\#1,1} = \sup_k \left\{ [\mu(p)]^{-1} Ext\text{-}\int \left| w_{\varkappa,\kappa}(k,p) [\mu(p)]^{-1/2} \right| d^{\#3}p \right\} \leq \text{const.}\times$$

$$\times \left(\approx \text{-esssup}_k \left\{ [\mu(k)]^{-\frac{1}{2}} Ext\text{-}\int \left| Ext\text{-}\hat{f}(k-p) \right| \left(\Theta(k,\varkappa)\Theta(p,\varkappa) - \Theta(k,\kappa)\Theta(p,\kappa) \right) d^{\#3}p \right\} \right)$$

$$= \delta(\varkappa, \kappa) \to_{\#} 0 \text{ as } \kappa \to_{\#} \varkappa. \tag{5.11}$$

Theorem 5.3 There is a semibounded self-#-adjoint operator $\boldsymbol{T_{\varkappa}}$ such that for $\boldsymbol{z}$ sufficiently negative

$$\left\|\left(\left(M_{\varkappa,\kappa}^{0k} - z\right)^{-1}\right) - (T_{\varkappa} - z)^{-1}\right\|_{\#} \to_{\#} 0 \quad \text{as } \kappa \to_{\#} \varkappa. \tag{5.12}$$

Proof We first establish the #-norm #-convergence of the $2n$-th powers $[R_{\kappa}(-b)]^{2n}$ of the resolvents for all b sufficiently large. Then the #-norm #-convergence of $R_{\kappa}(-b)$ follows by taking $2n$-th roots and applying the generalized Stone-Weierstrass Theorem [8]. Let $\kappa \leq \varkappa$ be two values of the hyperfinite ultraviolet cut-off. We use the following formula

$$R_{\kappa}^{2n} - R_{\varkappa}^{2n} = Ext\text{-}\sum_{i=1}^{2n} R_{\kappa}^{2n+1-i}\left(M_{\varkappa}^{0k} - M_{\kappa}^{0k}\right) R_{\varkappa}^{i}. \tag{5.13}$$

The differences $M_{\varkappa}^{0k} - M_{\kappa}^{0k}, k = 1,2,3$ contain of three terms

$$B^{(1)} = T_{0,\varkappa}^{(1)} - T_{0,\kappa}^{(1)}, B^{(2)} = T_{0,\varkappa}^{(2)} - T_{0,\kappa}^{(2)}, B^{(3)} = T_{I,\varkappa} - T_{I,\kappa}.$$

By (4.22) we get

$$\left\|R_{\kappa}^{2n+1-i} B^{(j)} R_{\varkappa}^{i}\right\|_{\#} \leq \text{const} \times \left\|(N_{\varkappa} + I)^{-2n-1+i} B^{(j)} (N_{\varkappa} + I)^{-i}\right\|_{\#}$$

where the constant is independent of κ. Therefore by (5.8) and (5.9) when $j = 2$ or 3,

$$\left\|R_{\kappa}^{2n+1-i} B^{(j)} R_{\varkappa}^{i}\right\|_{\#} \to_{\#} 0 \text{ as } \kappa \to_{\#} \varkappa.$$

As for $B^{(1)}$, at least one of i or $2n + 1 - i$ is greater than n. Therefore by (4.24) and (3.19),

$$\left\|R_{\kappa}^{2n+1-i} B^{(1)} R_{\varkappa}^{i}\right\|_{\#} \leq \text{const} \times \left\{\left\|\left(H_{0,\varkappa} + I\right)^{-\frac{1}{2}} B^{(1)} \left(H_{0,\varkappa} + I\right)^{-i}\right\|_{\#} + \right.$$

$$\left. + \left\|\left(H_{0,\varkappa} + I\right)^{-\frac{1}{2}} B^{(1)} \left(H_{0,\varkappa} + I\right)^{-i}\right\|_{\#}\right\} \to_{\#} 0 \text{ as } \kappa \to_{\#} \varkappa.$$

by (5.7). This obviously establishes the #-convergence of R_{κ}^{2n}. Let $R_{\varkappa}(z) = \#\text{-}\lim_{\kappa \to_{\#} \varkappa} R_{\kappa}(z)$. As a #-limit of resolvents, $R_{\kappa}(z)$ is itself the resolvent of an operator if and only if the null space $N(R_{\varkappa}(z)) = 0$ for some z [8]. But notice that this is a direct consequence of Lemma 5.1: Suppose that $\psi \in N(R_{\varkappa}(-b))$ where b is sufficiently large so that $R_{\kappa}(-b)$ #-converges. Take vector θ arbitrary in D_n . Then

$$\langle\theta, \psi\rangle_{\#} = \langle\left(M_{\kappa}^{0k} + b\right)\theta, R_{\kappa}(-b)\psi\rangle_{\#} \to_{\#} \langle\left(M_{\varkappa}^{0k} + b\right)\theta, R_{\varkappa}(-b)\psi\rangle_{\#} = 0,$$

so that $\psi = 0$. Therefore, $R_{\varkappa}(-b)$ is invertible, and $T = [R_{\varkappa}(-b)\,]^{-1} - b$ as a #-densely defined, #-closed, symmetric operator with the sufficiently negative real axis in its resolvent set, is actually self-#-adjoint and bounded below.

Theorem 5.4 $M_{\varkappa}^{0k}, k = 1,2,3$ are essentially self-#-adjoint on D.

Proof From the strong #-convergence of $M_{\varkappa,\kappa}^{0k}$ to $M_{\varkappa}^{0k}$ on D_n it follows by a simple argument that

$$M_{\varkappa}^{0k} \upharpoonright D_n \subset T_{\varkappa}. \tag{5.14}$$

Note that by the independence of κ- cutoff, the estimate (4.2) transfers to $T_{\varkappa}$, i.e.,

$$H_{0,\varkappa}^2 + N_\varkappa^{2n} \leq a(T_\varkappa + b)^{2n} \tag{5.15}$$

and therefore $C = D(T_\varkappa^{2n}) \subset D_n$, and from (5.14) one obtains $T_\varkappa \upharpoonright C \subset M_\varkappa^{0k} \upharpoonright D_n$. Now the domain C is a #-core for $T_\varkappa$, hence

$$T_\varkappa = \#\text{-}\overline{T_\varkappa \upharpoonright C} \subset \#\text{-}\overline{M_\varkappa^{0k} \upharpoonright D_n}$$

a symmetric extension of a self-#-adjoint operator and therefore we conclude that

$$T_\varkappa = \#\text{-}\overline{M_\varkappa^{0k} \upharpoonright D_n}.$$

Essential self-#-adjointness of $M_\varkappa^{0k}, k = 1,2,3$ on the domain D follows from self-#-adjointness on the domain D_n by a standard argument.

Corollary 5.5 For suitable constants a, b, c and $k = 1,2,3$

$$H_\varkappa \leq a\left(M_\varkappa^{0k} + b\right), \tag{5.16}$$

$$H_\varkappa^2 \leq c\left(H_{0,\varkappa}^2 + N_\varkappa^{2n} + I\right) \leq a\left(M_\varkappa^{0k} + b\right)^{2n}. \tag{5.17}$$

The same inequalities hold with the roles of $H_\varkappa$ and $M_\varkappa^{0k}$ interchanged so that

$$D\left((H_\varkappa + b)^{1/2}\right) = D\left(\left(M_\varkappa^{0k} + b\right)^{1/2}\right), \tag{5.18}$$

$$D(H_\varkappa^n) \subset D\left(M_\varkappa^{0k}\right), \tag{5.19}$$

$$D\left(\left(M_\varkappa^{0k}\right)^n\right) \subset D(H_\varkappa). \tag{5.20}$$

Proof Since D is a #-core for $M_\varkappa^{0k}, k = 1,2,3$, it is a #-core for $\left(M_\varkappa^{0k} + b\right)^{1/2}$ and (5.16) follows from closing (2.2). (5.17) is just a restatement of (5.15). Since $H_\varkappa$ is a special case of $M_\varkappa^{0k}$ obtained by setting, $g_0^{(k)}(x) = 0$, it is clear that the higher order estimates (5.15) hold for $T_\varkappa = H_\varkappa$; hence the roles of $H_\varkappa$ and $M_\varkappa^{0k}, k = 1,2,3$ can be interchanged in (5.16) and (5.17).

6. Lorentz covariance

According to the discussion in Section 1 this amounts to showing that if $I^3 = [a,b]^3 \subset {}^*\mathbb{R}^{\#3}_{c,,\text{fin}+}$ and if f is a $C_0^{*\infty}\left({}^*\mathbb{R}^{\#4}_{c,\text{fin}+}, {}^*\mathbb{R}^{\#}_{\text{c,fin}}\right)$ function with $\operatorname{supp}(f) \cup \operatorname{supp}\left(f_{\Lambda_\beta}\right) \subset O_{I^3}$, then on suitable near standard domain

$$\left[Ext\text{-}\exp\left(iM_\varkappa^{0k}\beta\right)\right]\varphi_\varkappa(f)\left[Ext\text{-}\exp\left(-iM_\varkappa^{0k}\beta\right)\right] \approx \varphi_\varkappa\left(f_{\Lambda_\beta}\right). \tag{6.1}$$

Notice that (6.1) is operator equality**,** since for ${}^*\mathbb{R}^{\#}_{\text{c,fin}}$ valued function f, $\varphi_\varkappa(f)$ is a self-#-adjoint operator whose domain includes $D\left(\left(M_\varkappa^{ik} + b\right)^{1/2}\right)$. In addition, we prove on the domain $D\left(\left(M_\varkappa^{ik} + b\right)^{1/2}\right) \times D\left(\left(M_\varkappa^{ik} + b\right)^{1/2}\right)$ that

$$\left[Ext\text{-}\exp\left(iM_\varkappa^{ik}\beta\right)\right]\varphi_\varkappa(x,t)\left[Ext\text{-}\exp\left(-iM_\varkappa^{ik}\beta\right)\right] = \varphi_\varkappa\left(\Lambda_\beta(x,t)\right). \tag{6.2}$$

Here the vectors (x,t) and $\Lambda_\beta(x,t)$ are in O_{I^3}, and the forms in (6.2) are #-continuous in x and t by the first-order estimate (5.16) and results of [8] sect.6.

Notice that the main part in the proof of (6.1) is to verify the commutation relation (1.15) for $f \in C_0^{*\infty}(O_{I^3}, {}^*\mathbb{R}^{\#}_{c,\text{fin}})$ and g a cut-off function for the region O_{I^3} . For convenience, we assume that a function f with support contained in the region O_ε defined by

$$O_\varepsilon = \{(x_1, x_2, x_3, t) | a + \varepsilon + |t| < x_k < b - \varepsilon - |t|, k = 1,2,3; |t| < \varepsilon\}, \quad (6.3)$$

and where $\varepsilon > 0$ is some small enough number. This represents no loss of generality since any f in $C_0^{*\infty}(O_{I^3}, {}^*\mathbb{R}^{\#}_{c,\text{fin}})$ can be presented as a sum of such f. It follows from this assumption that if $|s| < \varepsilon$, then external integral

$$\left[Ext\text{-}\exp\left(iH_\varkappa(t+s)\right)\right]\left\{Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}} \varphi_\varkappa(x) f(x,t) d^{\#3}x\right\}\left[Ext\text{-}\exp\left(-iH_\varkappa(t+s)\right)\right] \quad (6.4)$$

is related to a non-Archimedean von Neumann algebra $\mathfrak{R}(I^3)$ generated by the set

$$\left\{Ext\text{-}\exp\left(i\varphi_\varkappa(h_1)\right) + Ext\text{-}\exp\left(i\pi_\varkappa(h_2)\right)\} | h_i \in C_0^{*\infty}\left({}^*\mathbb{R}^{\#3}_{c,\text{fin}+}, {}^*\mathbb{R}^{\#}_{c,\text{fin}}\right), \text{supp}(h_i) \subset I^3, i = 1,2\right\}.$$

The main parts of the proof are as follows:

Part1. For $\psi \in D\left(H_\varkappa^{n+3}\right)$ we define

$$F_{ik}(t) = \langle \psi, \left[iM_\varkappa^{ik}(t), \varphi_\varkappa(f)\right]\psi\rangle_\# \quad (6.5)$$

where $M_\varkappa^{ik}(t) = [Ext\text{-}\exp(-itH_\varkappa)]M_\varkappa^{ik}[Ext\text{-}\exp(itH_\varkappa)]$. Note that $F_{ik}(t)$ is well-defined and three times #-continuously #-differentiable by (5.19) and [8, Section 6]:

$$\left\|(H_\varkappa + b)^{j/2}\varphi_\varkappa(f)(H_\varkappa + b)^{-(j+1)/2}\right\|_\# <. \quad (6.6)$$

for $j = 0, 1, 2, \ldots$. Obviously one obtains,

$$\frac{d^\# F_{ik}(t)}{d^\# t} = \langle \psi, \left[\left[H_\varkappa, M_\varkappa^{ik}(t)\right], \varphi_\varkappa(f)\right]\psi\rangle_\#, \quad (6.7)$$

$$\frac{d^{\#2} F_{ik}(t)}{d^\# t^2} = -i\,\langle \psi, \left[\left[H_\varkappa, \left[H_\varkappa, M_\varkappa^{ik}(t)\right]\right], \varphi_\varkappa(f)\right]\psi\rangle_\#. \quad (6.8)$$

Part2. The commutators in (6.7)-(6.8) can be evaluated. On $D_n^\# \times D_n^\#$ one obtains, in the sense of bilinear forms,

$$\left[iH_\varkappa, M_\varkappa^{ik}\right] = P_\varkappa^k + Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}} 2n{:}\,\varphi_\varkappa^{2n-1}(x)\pi_\varkappa(x){:}\, g_1(x)\left(x_k - \alpha - x_k g_0^{(k)}(x)\right) d^{\#3}x \quad (6.9)$$

where $P_\varkappa^k, k = 1,2,3$ is a locally correct momentum operators

$$P_\varkappa^k \equiv P_\varkappa^k\left(\frac{d^\#}{d^\# x_k}\left(x_k g_0^{(k)}(x)\right)\right). \quad (6.10)$$

By (2.6) the integral in (6.9) vanishes, and in analogy to (1.27),

$$[iH_\varkappa, M_\varkappa^{0k}] = P_\varkappa^k \quad (6.11)$$

on the domain $D\left(H_{\varkappa}^{n}\right) \times D\left(H_{\varkappa}^{n}\right) \subset D_{n}^{\#} \times D_{n}^{\#}$. Since the operators $P_{\varkappa}^{k}$ and $M_{\varkappa}^{ik}$ are defined on $D\left(H_{\varkappa}^{n}\right)$, extends to an operator equality on $D\left(H_{\varkappa}^{n+1}\right)$. Therefore, we obtain on the domain $D\left(H_{\varkappa}^{n+2}\right) \times D\left(H_{\varkappa}^{n+2}\right)$ that

$$\left[iH_{\varkappa}, [iH_{\varkappa}, M_{\varkappa}^{0k}]\right] = [iH_{\varkappa}, P_{\varkappa}^{k}] = S^{k}, \tag{6.12}$$

where

$$S^{k} = T_{0,\varkappa}\left(\frac{d^{\#2}}{d^{\#}x_{k}^{2}}\left(x_{k} g_{0}^{(k)}(x)\right)\right) -$$

$$-m^{2} Ext\text{-}\int_{^{*}\mathbb{R}_{c}^{\#3}} :\varphi_{\varkappa}^{2}(x): \frac{d^{\#2}}{d^{\#}x_{k}^{2}}\left(x_{k} g_{0}^{(k)}(x)\right) d^{\#3}x - T_{I,\varkappa}\left(\frac{d^{\#}(g_{1})}{d^{\#}x_{k}}\right). \tag{6.13}$$

Part3. Since $S^{k}, k = 1,2,3$ are local operators whose kernels vanishes on I^{3} we expect that $S^{k}, k = 1,2,3$ commutes with $\Re(I^{3})$. The exact statement is $[S^{k}, \Re(I^{3})] = 0, k = 1,2,3$ on domain $D_{n}^{\#} \times D_{n}^{\#}$. Note that $D_{n}^{\#} \subset D\left(S^{k}\right), k = 1,2,3$. It follows from (6.4) and (6.6) on domain $D_{n}^{\#} \times D_{n}^{\#}$ that

$$\left[S^{k}, [Ext\text{-}\exp(isH_{\varkappa})\varphi_{\varkappa}(f) Ext\text{-}\exp(-isH_{\varkappa})]\right] = 0 \tag{6.14}$$

for $|s| < \varepsilon$ and $\operatorname{supp}(f) \subset O_{\varepsilon}$

Part4. The rigorous counterpart of the formal expansion (1.34) is to write $F_{ik}(t)$ in terms of its generalized Taylor series **[**8,Theorem 2.27**].** For some s, $|s| \leq |t|$

$$F_{ik}(t) = F_{ik}(0) + tF_{ik}^{\#\prime}(0) + \tfrac{t^{2}}{2} F_{ik}^{\#\prime}(s). \tag{6.15}$$

For $|t| \leq \varepsilon$ (6.15) on domain $D\left(H_{\varkappa}^{n+3}\right) \times D\left(H_{\varkappa}^{n+3}\right)$ reads

$$[iM_{\varkappa}^{0k}(t), \varphi_{\varkappa}(f)] = \left[iM_{\varkappa}^{ik}, \varphi_{\varkappa}(f)\right] - i[iP_{\varkappa}^{k}, \varphi_{\varkappa}(f)]. \tag{6.16}$$

Part5. The commutators on the right of (6.16) can be evaluated by passing to the sharp time fields,

$$\varphi_{\varkappa}(f_{s}, t) = Ext\text{-}\int_{^{*}\mathbb{R}_{c}^{\#3}} f(x,s)\, \varphi_{\varkappa}(x,t) d^{\#3}x.$$

where the subscript s indicates the time dependence of a function f. The result for $|t| \leq \varepsilon$ reads

$$[iM_{\varkappa}^{0k}(t), \varphi_{\varkappa}(f_{t}, 0)] = \pi_{\varkappa}(x_{k} f_{t}, 0) - t\varphi_{\varkappa}\left(\frac{\partial^{\#} f_{t}}{\partial^{\#} x_{k}}, 0\right)$$

on domain $D\left(H_{\varkappa}^{n+3}\right) \times D\left(H_{\varkappa}^{n+3}\right)$. That is, for $|t| \leq \varepsilon$ we get

$$[iM_{\varkappa}^{0k}(t), \varphi_{\varkappa}(f_{t}, 0)] = \pi_{\varkappa}(x_{k} f_{t}, t) - \varphi_{\varkappa}\left(t\frac{\partial^{\#} f_{t}}{\partial^{\#} x_{k}}, t\right). \tag{6.17}$$

Since $\operatorname{supp}(f) \subset O_{\varepsilon}$, we can integrate (6.17) with respect to t and thus on domain $D\left(H_{\varkappa}^{n+3}\right) \times D\left(H_{\varkappa}^{n+3}\right)$ we obtain

$$[iM_{\varkappa}^{0k}(t), \varphi_{\varkappa}(f_{t}, 0)] = \pi_{\varkappa}(x_{k} f, t) - \varphi_{\varkappa}\left(t\frac{\partial^{\#} f}{\partial^{\#} x_{k}}, t\right) = -\varphi_{\varkappa}\left(x_{k}\frac{\partial^{\#} f}{\partial^{\#} t} + t\frac{\partial^{\#} f_{t}}{\partial^{\#} x_{k}}\right). \tag{6.18}$$

Part6. In order to deduce (6.1) from (6.18) we must show that the equality (6.18) holds on a domain of the form $D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)\times D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)$. Note that if $\psi\in D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)$, then $Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\psi\in D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)$ and

$$\mathcal{G}_{k}(x,t,\beta)=\langle Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\psi,\varphi_{\varkappa}(x,t)Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\rangle_{\#}$$

is a #-continuous function of x and t [8, Section 6] with a distribution #-derivative in β,

$$\langle Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\psi,\left\{x_{k}\frac{\partial^{\#}\varphi_{\varkappa}(x,t)}{\partial^{\#}t}+t\frac{\partial^{\#}\varphi_{\varkappa}(x,t)}{\partial^{\#}x_{k}}\right\}Ext\text{-}\exp\left(-iM_{\varkappa}^{0k}\beta\right)\rangle_{\#}$$

by the equality (6.18).Thus $\mathcal{G}_{k}(x,t,\beta)$ satisfies the distribution differential equation in partial #-derivatives

$$\frac{\partial^{\#}\mathcal{G}_{k}(x,t,\beta)}{\partial^{\#}\beta}=x_{k}\frac{\partial^{\#}\mathcal{G}_{k}(x,t,\beta)}{\partial^{\#}t}+t\frac{\partial^{\#}\mathcal{G}_{k}(x,t,\beta)}{\partial^{\#}x_{k}}. \tag{6.19}$$

The distribution differential equation (6.19) has a unique solution with initial condition $\mathcal{G}_{k}(x,t,0)$:

$$\mathcal{G}_{k}(x,t,0)=\langle\psi,\varphi_{\varkappa}(x,t)\psi\rangle_{\#}.$$

This proves (6.2) on $D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)\times D\left(\left(M_{\varkappa}^{0k}\right)^{j}\right)$ and, by extension, on the domain

$$D\left(\left(M_{\varkappa}^{0k}+b\right)^{1/2}\right)\times D\left(\left(M_{\varkappa}^{0k}+b\right)^{1/2}\right).$$

Obviously the operator statement (6.1) is follows immediately. It remains only to prove the following.

Lemma 6.1 Let $I^{3}\subset{}^{*}\mathbb{R}_{c,\text{fin}+}^{\#3}$, g satisfy (2.4)-(2.6), $\varepsilon>0$, and $f\in C_{0}^{*\infty}\left(O_{\varepsilon},{}^{*}\mathbb{R}_{c,\text{fin}}^{\#}\right)$. Then, in the sense of bilinear forms

$$\left[iM_{\varkappa}^{0k}(t),\varphi_{\varkappa}(f)\right]=-\varphi_{\varkappa}\left(x_{k}\frac{\partial^{\#}f}{\partial^{\#}t}+t\frac{\partial^{\#}f}{\partial^{\#}x_{k}}\right) \tag{6.20}$$

on $D\left(H_{\varkappa}\right)\times D\left(H_{\varkappa}\right)$ or on $D\left(M_{\varkappa}^{0k}\right)\times D\left(M_{\varkappa}^{0k}\right)$.

Proof As we know that (6.20) holds on $D\left(H_{\varkappa}^{n+3}\right)\times D\left(H_{\varkappa}^{n+3}\right)$. Let $\psi\in D\left(H_{\varkappa}\right)$; since $D\left(H_{\varkappa}^{n+3}\right)$ is a #-core for $H_{\varkappa}$, there exists a hyper infinite sequence $\psi_{l}, l\in{}^{*}\mathbb{N}$ in $D\left(H_{\varkappa}^{n+3}\right)$ such that $\psi_{l}\rightarrow_{\#}\psi$ and $H_{\varkappa}\psi_{l}\rightarrow_{\#}H_{\varkappa}\psi$ as $l\rightarrow{}^{*}\infty$. By the first order estimate, we have for some constants a and b

$$\left\|\left(M_{\varkappa}^{0k}+a\right)^{1/2}\left(H_{\varkappa}+b\right)^{-1/2}\right\|_{\#}<{}^{*}\infty. \tag{6.21}$$

and by (6.6) we get

$$\left\|\varphi_{\varkappa}(u_{k})\left(H_{\varkappa}+b\right)^{-1/2}\right\|_{\#}<{}^{*}\infty. \tag{6.22}$$

where $u_{k}=x_{k}\frac{\partial^{\#}f}{\partial^{\#}t}+t\frac{\partial^{\#}f}{\partial^{\#}x_{k}}$ is in $C_{0}^{*\infty}\left({}^{*}\mathbb{R}_{c,\text{fin}+}^{\#4},{}^{*}\mathbb{R}_{c,\text{fin}}^{\#}\right)$. Therefore,

$$\left(M_{\varkappa}^{0k}+a\right)^{1/2}\psi_{l}\rightarrow_{\#}\left(M_{\varkappa}^{0k}+a\right)^{1/2}\psi \tag{6.23}$$

and

$$\varphi_\varkappa(u_k)\psi_l \to_\# \varphi_\varkappa(u_k)\psi \tag{6.24}$$

Moreover, by (6.6) we obtain

$$\left\|(H_\varkappa + b)^{1/2}\varphi_\varkappa(f)(H_\varkappa + b)^{-1/2}\right\|_\# < {}^*\infty. \tag{6.25}$$

From (6.21) and (6.25) one obtains $D(H_\varkappa) \subset D\left((H_\varkappa + b)^{1/2}\varphi_\varkappa(f)\right)$ and that

$$\left(M_\varkappa^{0k} + a\right)^{1/2}\varphi_\varkappa(f)\psi_l \to_\# \left(M_\varkappa^{0k} + a\right)^{1/2}\varphi_\varkappa(f)\psi. \tag{6.26}$$

Note that

$$\langle\psi_l, [iM_\varkappa^{0k}(t), \varphi_\varkappa(f)]\psi_l\rangle_\# = i\,\langle\left(M_\varkappa^{0k} + a\right)^{1/2}\psi_l, \left(M_\varkappa^{0k} + a\right)^{1/2}\varphi_\varkappa(f)\psi_l\rangle_\# -$$

$$-i\,\langle\left(M_\varkappa^{0k} + a\right)^{1/2}\varphi_\varkappa(f)\psi_l, \left(M_\varkappa^{0k} + a\right)^{1/2}\psi_l\rangle_\#.$$

And therefore from (6.23) (6.24), and (6.26) we conclude that (6.20) extends by #-continuity to domain $D(H_\varkappa) \times D(H_\varkappa)$. By (5.20), (6.20) is then exactly valid when restricted to $D\left(\left(M_\varkappa^{0k}\right)^n\right) \times D\left(\left(M_\varkappa^{0k}\right)^n\right)$. Finally, the extension to domain $D\left(M_\varkappa^{0k}\right) \times D\left(M_\varkappa^{0k}\right)$ follows directly as above from the inequality

$$\left\|\varphi_\varkappa(f)\left(M_\varkappa^{0k} + b\right)^{-1/2}\right\|_\# < {}^*\infty.$$

§ 7. The spectral theorem related to bounded in ${}^*\mathbb{R}_c^\#$ operators.

In this section, we will discuss the generalized spectral theorem in its many aspects. This structure theorem is a concrete description of all self-#-adjoint operators. There are several apparently distinct formulations of the spectral theorem. In some sense they are all equivalent. The form we prefer in this section, says that every bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operator is a multiplication operator. This means that given a bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operator A on a non-Archimedean Hilbert space $H^\#$, we can always find a #-measure $\mu^\#$ on a #-measure space M and a unitary operator $U\colon H^\# \to L_2^\#(M, d^\#\mu^\#)$ so that $(UAU^{-1}f)(x) = F(x)f(x)$ for some bounded ${}^*\mathbb{R}_c^\#$-valued #-measurable function F on M. In practice, M will be a union of copies of ${}^*\mathbb{R}_c^\#$ and F will be x so the core of the proof of the theorem will be the construction of certain #-measures. Our main goal in this section will be to make sense out of $f(A)$, for f a #-continuous function. We will consider also the #-measure defined by the functional: $f \mapsto \langle\psi, f(A)\psi\rangle_\#$ for fixed $\psi \in H^\#$.

Definition 7.1. The operator #-norm of a linear operator $A\colon H^\# \to H^\#$ is the largest value by which A stretches an element of $H^\#$,

$$\|A\|_{\#\mathrm{op}} = \|A\|_{\mathcal{L}(H^\#)} = \sup\{\|Ax\|_\# | x \in H^\#, \|x\|_\# = 1\}.$$

An operator A is called bounded in ${}^*\mathbb{R}_c^\#$ if $\|A\|_{\#\mathrm{op}} < {}^*\infty$, otherwise operator A is called unbounded in ${}^*\mathbb{R}_c^\#$. We often write bounded operator instead bounded in ${}^*\mathbb{R}_c^\#$ and unbounded operator correspondingly.

Definition7.2. A linear operator $A\colon H^\# \to H^\#$ is called finitely bounded if $\|A\|_{\mathcal{L}(H^\#)} = \|A\|_{\#\mathrm{op}} \in {}^*\mathbb{R}_{c,\mathrm{fin}}^\#$ i.e., if $\|A\|_{\#\mathrm{op}}$ is a near standard number.

Definition7.3. Let $\mathrm{C}^\#(U)$ be the linear space of ${}^*\mathbb{C}_c^\#$- valued #-continuous functions of #-compact support $U \subset {}^*\mathbb{R}_c^\#$ endowed with the essential sup #-norm $\|f\|_{^*\infty} = \operatorname{ess\,sup}_{x\in U}\{f(x)\}$. An function f in $C^\#(U)$ is called finitely bounded if $\|f\|_{^*\infty} \in {}^*\mathbb{R}_{c,\text{fin}}^\#$ i.e., if $\|f\|_{^*\infty}$ is a near standard number.

Definition7.4. We define now $\mathrm{C}_{\text{fin}}^\#(U) \subsetneq C^\#(U)$ by

$$\mathrm{C}_{\text{fin}}^\#(U) = \left\{f \,|\, [f \in \mathrm{C}^\#(U)] \wedge \left[\|f\|_{^*\infty} \in {}^*\mathbb{R}_{c,\text{fin}}^\#\right]\right\}.$$

An function f is called finitely bounded if $f \in \mathrm{C}_{\text{fin}}^\#(U)$ i.e. if $\|f\|_{^*\infty} \in {}^*\mathbb{R}_{c,\text{fin}}^\#$. Note that $\mathrm{C}_{\text{fin}}^\#(U)$ is a linear space over field ${}^*\mathbb{R}_{c,\text{fin}}^\#$.

Theorem7.1. (#-continuous functional calculus) Let A be a bounded in${}^*\mathbb{R}_c^\#$ self-#-adjoint operator on a non-Archimedean Hilbert space $H^\#$. Then there is a unique map $\phi: C^\#(\sigma(A)) \to \mathcal{L}(H^\#)$ with the following properties:

(a) ϕ is an algebraic $*$-homomorphism, that is,
$\phi\,(fg) = \phi\,(f)\phi\,(g), \phi\,(\lambda f) = \lambda\,\phi\,(f), \phi\,(1) = I, \phi\,(f) = \phi\,(f)^*$.

(b) ϕ is #-continuous, that is, $\|\phi\,(f)\|_{\mathcal{L}(H^\#)} \leq C\|f\|_{^*\infty}$.

(c) Let f be the function $f(x) = x$; then $\phi\,(f) = A$.

Moreover, ϕ have the additional properties:

(d) If A$\psi = \lambda\psi$, then $\phi\,(f)\psi = f(\lambda)\psi$.

(e) $\sigma[\phi\,(f)] = \{f(\lambda)|\lambda \in \sigma(A)\}$ [Spectral mapping theorem].

(f) If $f \geq 0$, then $\phi\,(f) \geq 0$.

(g) $\|\phi\,(f)\|_{\mathcal{L}(H^\#)} = \|f\|_{^*\infty}$.

Remark 7.1. The proof which we give below is quite simple, (a) and (c) uniquely determine ϕ(P) for any hyperfinite polynomial $P(x)$. By the generalized Weierstrass theorem 7.3, the set of hyperfinite polynomials is #-dense in $\mathrm{C}^\#(\sigma(A))$ so the main part of the proof is showing that $\|P(A)\|_{\#op} = \sup_{\lambda\in\sigma(A)}|P(\lambda)|$. The existence and uniqueness of φ then follow from the generalized B.L.T. theorem 7.4. To prove the crucial equality, we first prove a special case of (e) which holds for arbitrary bounded in ${}^*\mathbb{R}_c^\#$ operators.

Lemma7.1. Let $P(x) = Ext\text{-}\sum_{n=1=0}^{N} c_n x^n, N \in {}^*\mathbb{N}$. $P(A) = Ext\text{-}\sum_{n=1=0}^{N} c_n A^n$. Then

$$\sigma(P(A)) = \{P(\lambda)|\lambda \in \sigma(A)\}.$$

Proof Let $\lambda \in \sigma(A)$. Since $x = \lambda$ is a root of $P(x) - P(\lambda)$, we have $P(x) - P(\lambda) = (x - \lambda)Q(x)$, so $P(A) - P(\lambda) = (A - \lambda)Q(A)$. Since $(A - \lambda)$ has no inverse neither does $P(A) - P(\lambda)$ that is, $P(\lambda) \in \sigma(P(A))$. Conversely, let $\mu \in \sigma(P(A))$ and let $\lambda_1, \ldots, \lambda_n$ be the roots of $P(x) - \mu$, that is, $P(x) - \mu = a(Ext\text{-}\prod_{i=1}^{n}(x - \lambda_i)$. If $\lambda_1, \ldots, \lambda_n \notin \sigma(A)$, then $(P(A) - \mu)^{-1} = a^{-1}[Ext\text{-}\prod_{i=1}^{n}(x - \lambda_i)^{-1}]$ so we conclude that some $\lambda_i \in \sigma(A)$ that is, $\mu = P(\lambda)$ for some $\lambda \in \sigma(A)$.

Definition 7.5. Let $r(A) = \sup_{\lambda\in\sigma(A)} |\lambda|$.Then $r(A)$ is called the spectral radius of A.

Theorem 7.2. Let X be a non-Archimedean Banach space, $A \in \mathcal{L}(X)$. Then #-$\lim_{n\to{}^*\infty} \sqrt[n]{\|A^n\|_{\#op}}$ exists and is equal to $r(A)$. If X is a non-Archimedean Hilbert space and A is self-#-adjoint, then

$r(A) = \|A\|_{\mathcal{L}(X)}$

Lemma 7.2 Let A be a bounded self-#-adjoint operator. Then

$$\|P(A)\|_{\#op} = \sup_{\lambda\in\sigma(A)}|P(\lambda)|.$$

Proof By theorem 6.2 we obtain

$$\|P(A)\|_{\#op}^2 = \|P(A)^*P(A)\|_{\#op} = \|(\bar{P}P)(A)\|_{\#op} = \sup_{\lambda\in\sigma((\bar{P}P)(A))}|\lambda|.$$

By Lemma 7.1 we obtain

$$\sup_{\lambda\in\sigma((\bar{P}P)(A))}|\lambda| = \sup_{\lambda\in\sigma(A)}|\bar{P}P(\lambda)| = \left(\sup_{\lambda\in\sigma(A)}|P(\lambda)|\right)^2.$$

Notation7.1. We often write $\phi_A\,(f)$ or $f(A)$ for $\phi\,(f)$ in order to emphasize the dependence on operator A.

Definition7.6. (Hyperfinite Bernstein Polynomials) For each $n \in {}^*\mathbb{N}$, the n-th hyperfinite Bernstein Polynomial $B_n^\#(x,f)$ of a function $f \in \mathrm{C}^\#\left([a,b], {}^*\mathbb{R}_c^\#\right)$ is defined as

$$B_n^\#(x,f) = Ext\text{-}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

Theorem7.3. (Generalized Weierstrass approximation theorem) Let $f \in \mathrm{C}^\#\left([a,b], {}^*\mathbb{R}_c^\#\right)$, $[a,b] \subset {}^*\mathbb{R}_c^\#$. Then there is a hyper infinite sequence of polynomials $p_b(x), n \in {}^*\mathbb{N}$ that #-converges uniformly to $f(x)$ on $[a,b]$.

Proof Consider first $f \in \mathrm{C}^\#\left([0,1], {}^*\mathbb{R}_c^\#\right)$. Once the theorem is proved for this case, the general theorem will follow by a change of variables. Since $[0,1]$ is #-compact, the #-continuity of f implies uniform #-continuity. So, given $\varepsilon > 0$, there exists δ > 0 such that: $\forall x,y(\,x,y \in [0,1])[|x-y| \leq \delta \Longrightarrow |f(x)-f(y)| \leq \varepsilon/2]$. Now, let $M = \|f\|_{^*\infty}$. Note that M exists since f is a #-continuous function on a #-compact set. Now, fix $\xi \in [0,1]$. Then, if $|x-\xi| \leq \delta$, then the inequality holds $|f(x) - f(\xi)| \leq \varepsilon/2$ by #-continuity. Alternatively, if $|x-\xi| \geq \delta$, then

$$|f(x) - f(\xi)| \leq 2M \leq 2M\left(\frac{x-\xi}{\delta}\right)^2 + \varepsilon/2.$$

From the above two inequalities, we obtain that

$$\forall x\,(x \in [0,1])\left[|f(x) - f(\xi)| \leq 2M\left(\frac{x-\xi}{\delta}\right)^2 + \varepsilon/2\right].$$

The hyperfinite Bernstein Polynomials can be used to approximate $f(x)$ on $[0,1]$. First, note that

$$B_n^\#(x, f - f(\xi)) = B_n^\#(x,f) - f(\xi)B_n^\#(x,1)$$

and for all $n \in {}^*\mathbb{N}$

$$B_n^\#(x,1) = Ext\text{-}\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = (x + (1-x))^n = 1,$$

where the generalized Binomial Theorem was used in the second equality. Thus,

$$\left|B_n^\#(x, f - f(\xi))\right| \leq B_n^\#\left(x, 2M\left(\frac{x-\xi}{\delta}\right)^2 + \frac{\varepsilon}{2}\right) = \frac{2M}{\delta^2}B_n^\#(x,(x-\xi)^2) + \varepsilon/2,$$

where in the second step the fact that $0 \leq B_n^{\#}(x,f)$ for $0 \leq f$ and $B_n^{\#}(x,g) \leq B_n^{\#}(x,f)$ if $g \leq f$ were used. Both can be proven directly from the definition of $B_n^{\#}(x,f)$. It can also be shown that

$$B_n^{\#}(x,(x-\xi)^2) = x^2 + n^{-1}(x-x^2) - 2\xi x + \xi^2.$$

So

$$\left|B_n^{\#}(x,f-f(\xi))\right| \leq \frac{\varepsilon}{2} + \frac{2M(x-\xi^2)}{\delta^2} + \frac{2M(x-x^2)}{n\delta^2}.$$

In particular,

$$\left|B_n^{\#}(\xi,f-f(\xi))\right| \leq \frac{\varepsilon}{2} + \frac{2M(\xi-\xi^2)}{n\delta^2}.$$

A simple calculation shows that on $[0,1]$, the maximum of $z-z^2$ is $1/4$. Thus,

$$\left|B_n^{\#}(\xi,f-f(\xi))\right| \leq \frac{\varepsilon}{2} + \frac{2M}{n\delta^2}.$$

So, take $N \geq \frac{M}{2\delta^2\varepsilon}$, for $n \geq N$ we get

$$\left\|B_n^{\#}(\xi,f-f(\xi))\right\|_{^*\infty}.$$

This proves the theorem for #-continuous functions on $[0,1]$. Now we let $g \in \mathrm{C}^{\#}([a,b])$. Consider the function $\varphi:[0,1] \to [a,b]$ defined by $\varphi: x \mapsto (b-a)x-a, \varphi$ is clearly a homeomorphism. Thus, the composite function $f = g \circ \varphi$ is a #-continuous on $[0,1]$. By application of the theorem for functions on $[0,1]$, the case for an arbitrary interval $[a,b]$ follows.

Theorem 7.4. (Generalized B.L.T. theorem) Suppose that Z is a non-Archimedean normed space, Y is a non-Archimedean Banach space, and $S \subset Z$ is a #-dense linear subspace of Z. If $T: S \to Y$ is a bounded in${}^*\mathbb{R}_c^{\#}$ linear transformation (i.e. there exists $C < {}^*\infty$ such that $\|Tz\|_{\#} \leq C\,\|z\|_{\#}$ for all $z \in S$), then T has a unique extension to an element of $\mathcal{L}(Z,Y)$.

Definition 7.7. (Unital Sub-Algebra, Separating Points). Let K be a #-compact metric space. Consider the non-Archimedean Banach algebra $\mathrm{C}^{\#}(K,{}^*\mathbb{R}_c^{\#}) = \{f : K \to {}^*\mathbb{R}_c^{\#} \mid f\ is\ \#\text{-continuous}\}$ equipped with the sup-norm, $\|f\|_{^*\infty}$. Then, (1) $A \subset C^{\#}(K,{}^*\mathbb{R}_c^{\#})$ is a unital sub-algebra if $1 \in A$ and if $f,g \in A, \alpha,\beta \in {}^*\mathbb{R}_c^{\#}$ implies that $\alpha f + \beta g \in A$ and $fg \in A$. (2) $A \subset \mathrm{C}^{\#}(K,{}^*\mathbb{R}_c^{\#})$ separates points of K if for all $s,t \in K$ with $s \neq t$, there exists $f \in A$ such that $f(s) \neq f(t)$.

Proof of the Theorem 7.1. Let $\phi(P) = P(A)$. Then $\|\phi(P)\|_{\mathcal{L}(H^{\#})} = \|P\|_{C^{\#}(\sigma(A))}$ so ϕ has a unique linear extension to the #-closure of the polynomials in $\mathrm{C}^{\#}(\sigma(A)$. Since the polynomials are an algebra containing I, containing complex conjugates, and separating points, this #-closure is all of $\mathrm{C}^{\#}(\sigma(A)$. Properties (a), (b), (c), (g) are obvious and if $\tilde{\phi}$ obeys (a), (b), (c) it agrees with ϕ on polynomials and thus by #-continuity on $C^{\#}(\sigma(A)$. In order to prove (d), note that $\phi(P)\psi = P(\lambda)\psi$ and apply #-continuity. To prove (f), notice that if $f \geq 0$, then $f = g^2$ with g is ${}^*\mathbb{R}_c^{\#}$-valued and $g \in \mathrm{C}^{\#}(\sigma(A)$. Thus $\phi(f) = \phi(g)^2$ with $\phi(g)$ self-#-adjoint, so $\phi(f) \geq 0$.

Remark 7.2 Notice that in addition the following statements hold:

(1) $\phi(f) \geq 0$ if and only if $f \geq 0$.

(2) Since $fg = gf$ for all f,g, $\{f(A) | f \in \mathrm{C}^{\#}(\sigma(A))\}$ forms an abelian algebra closed under adjoints.

(3) Since $\|\phi(f)\|_{\mathcal{L}(H^{\#})} = \|f\|_{^*\infty}$ and $\mathrm{C}^{\#}(\sigma(A))$ is #-complete, $\{f(A) | f \in \mathrm{C}^{\#}(\sigma(A))\}$ is #-norm-#-closed. It is thus a non-Archimedean an abelian C^* algebra over field ${}^*\mathbb{C}_c^{\#}$ of operators.

(4) Ran(ϕ) is actually the non-Archimedean C^*-algebra generated by A that is, the smallest $\mathbf{C}^*$-algebra

over field ${}^{*}\mathbb{C}_{c}^{\#}$ containing A.
(5) Notice that $\mathrm{C}^{\#}(\sigma(A))$ and the non-Archimedean C^{*}-algebra generated by A are #-isometrically isomorphic.
(6) The statement (b) actually follows from (a) and Proposition 7.1. Thus (a) and (c) alone determine ϕ uniquely.
Proposition 7.1 Suppose that $\phi\colon \mathrm{C}^{\#}(X) \to \mathcal{L}(H^{\#})$ is an algebraic $*$-homomorphism, X a #-compact metric space. Then: (a) if $f \geq 0$, then ϕ (f) ≥0, (b) $\|\phi\,(f)\|_{\mathcal{L}(\,H^{\#})} \leq \|f\|_{{}^{*}\infty}$.

8. The spectral #-measures.

We are now going to introduce the #-measures corresponding to a bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ self-#-adjoint operators. Let A be bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ self-#-adjoint operator. Let $\psi \in H^{\#}$. Then

$$f \longmapsto \langle \psi, f(A)\psi \rangle_{\#}$$

is a positive ${}^{*}\mathbb{R}_{c}^{\#}$-valued linear functional on $\mathrm{C}^{\#}(\sigma(A))$. Thus, by the generalized Riesz-Markov theorem, see Theorem 8.2, there is a unique #-measure $\mu_{\psi}^{\#}$ on the #-compact set $\sigma(A)$ with the property

$$\langle \psi, f(A)\psi \rangle_{\#} = Ext\text{-}\int_{\sigma(A)} f(\lambda) d^{\#}\, \mu_{\psi}^{\#}$$

Definition 8.1.The #-measure $\mu_{\psi}^{\#}$ is called the spectral #-measure associated with the vector $\psi \in H^{\#}$. The first and simplest application of the $\mu_{\psi}^{\#}$ is to allow us to extend the #-continuous functional calculus to $B^{\#}({}^{*}\mathbb{R}_{c}^{\#})$, the bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ #-Borel functions on ${}^{*}\mathbb{R}_{c}^{\#}$. Let $g \in B^{\#}({}^{*}\mathbb{R}_{c}^{\#})$. It is natural way to define $g(A)$ so that $\langle \psi, g(A)\psi \rangle_{\#} = Ext\text{-}\int_{\sigma(A)} g(\lambda) d^{\#}\, \mu_{\psi}^{\#}$. The polarization identity lets us recover $\langle \psi, g(A)\psi \rangle_{\#}$ from the functional $\langle \psi, g(A)\psi \rangle_{\#}$ and then the Generalized Riesz lemma lets us construct $g(A)$.
Theorem 8.2. (Generalized Riesz-Markov theorem) Let X be a locally #-compact non-Archimedean metric space endowed with ${}^{*}\mathbb{R}_{c}^{\#}$-valued metric. Let $\mathrm{C}_{c}^{\#}(X)$ be the space of #-continuous #-compactly supported ${}^{*}\mathbb{C}_{c}^{\#}$-valued functions on X. For any positive linear functional Φ on $\mathrm{C}_{c}^{\#}(X)$, there is a unique #-measure $\mu_{\psi}^{\#}$ on X such that
$\forall f \in \mathrm{C}_{c}^{\#}(X)\colon \Phi(f) = Ext\text{-}\int_{X} f(x) d^{\#}\, \mu^{\#}(x)$.
Theorem 8.3. (Generalized Riesz lemma) Let Y be a #-closed proper vector subspace of a #-normed space $(X, \|\cdot\|_{\#})$ and let $\alpha \in {}^{*}\mathbb{R}_{c}^{\#}$ be any real number satisfying $0 < \alpha < 1$.Then there exists a vector $u \in X$ of unit #-norm $\|u\|_{\#} = 1$ such that $\|u - y\|_{\#} \geq \alpha$ for all $y \in Y$.
Theorem 8.4. (spectral theorem-functional calculus form) Let A be a bounded in ${}^{*}\mathbb{R}_{c}^{\#}$ self-#-adjoint operator on non-Archimedean Hilbert space $H^{\#}$. There is a unique map $\hat{\phi}\colon B^{\#}({}^{*}\mathbb{R}_{c}^{\#}) \to \mathcal{L}(H^{\#})$ so that: (a) $\hat{\phi}$ is an algebraic $*$ -homomorphism.
(b) $\hat{\phi}$ is #-norm #-continuous: $\|\hat{\phi}(f)\|_{\mathcal{L}(\,H^{\#})} \leq \|f\|_{{}^{*}\infty}$.
(c) Let f be the function $f(x) = x$; then $\hat{\phi}(f) = A$.
(d) Suppose $f_{n}(x) \to_{\#} f(x)$ for each x as $n \to_{\#} {}^{*}\infty$ and hyper infinite sequence $\|f_{n}\|_{{}^{*}\infty}$ is bounded in ${}^{*}\mathbb{R}_{c}^{\#}$.Then $\hat{\phi}(f_{n}) \to_{\#} \hat{\phi}(f)$, as $n \to_{\#} {}^{*}\infty$ strongly.
Moreover $\hat{\phi}$ has the properties:
(e) If $\mathrm{A}\psi = \lambda\psi$, then $\hat{\phi}(f) = f(\lambda)\psi$.
(f) If $f \geq 0$, then $\hat{\phi}(f) \geq 0$.
(g) If $BA = AB$ then $\hat{\phi}(f)B = B\hat{\phi}(f)$.

9. The spectral projections

Definition 9.1. Let A be a bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operator and Ω a #-Borel set of ${}^*\mathbb{R}_c^\#$. $P_\Omega = \chi_\Omega(A)$ is called a spectral projection of A.

As the definition suggests, P_Ω is an orthogonal projection since $\chi_\Omega = \chi_\Omega^2 = 1$pointwise. The properties of the family of projections $\{P_\Omega | \Omega$ an arbitrary #-Borel set$\}$ is given by the following elementary translation of the functional calculus.

Proposition 9.1. The family $\{P_\Omega\}$ of spectral projections of a bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operator, A, has the following properties:

(a) Each P_Ω is an orthogonal projection.

(b) $P_\emptyset = 0$; $P_{(-a,a)} = I$ for some $a \in {}^*\mathbb{R}_{c+}^\#$.

(c) If $\Omega = Ext\text{-}\bigcup_{n=1}^{^*\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ for all $n \neq m$ then

$$P_\Omega = s\text{-}\#\text{-}\lim_{N \to {}^*\infty}\left(Ext\text{-}\sum_{n=1}^{N} P_{\Omega_n}\right)$$

(d) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

Definition 9.2. A family of projections obeying (a)-(c) is called a projection-valued #-measure (p.v. #-m.).

Remark 9.1. Note that (d) follows from (a) and (c) by abstract considerations. As one might guess, one can integrate with respect to a p.v.#-m. If P_Ω is a p.v. #-m., then $\langle \varphi, P_\Omega \varphi\rangle_\#$ is an ordinary #-measure for any φ. We will use the symbol $d^\#\langle \varphi, P_\lambda \varphi\rangle_\#$ to mean integration with respect to this #-measure. By generalized Riesz lemma methods, there is a unique operator B with $\langle \varphi, B\varphi\rangle_\# = \int_{{}^*\mathbb{R}_c^\#} f(\lambda) d^\#\langle \varphi, P_\lambda \varphi\rangle_\#$.

Theorem 9.1. If P_Ω is a p.v. #-m. and f a bounded in ${}^*\mathbb{R}_c^\#$ #-Borel function on $\mathrm{supp}(P_\Omega)$, then there is a unique operator B which we denote $\int_{{}^*\mathbb{R}_c^\#} f(\lambda) d^\#\langle \varphi, P_\lambda \varphi\rangle_\#$ so that

$$\langle \varphi, B\varphi\rangle_\# = \int_{{}^*\mathbb{R}_c^\#} f(\lambda) d^\#\langle \varphi, P_\lambda \varphi\rangle_\#.$$

10. The spectral theorem related to unbounded in ${}^*\mathbb{R}_c^\#$ self - #- adjoint operators.

In this section we will show how the spectral theorem for bounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operators which we developed in section 9 can be extended to unbounded in ${}^*\mathbb{R}_c^\#$ self-#-adjoint operators.

Proposition 10.1. Let $\langle M, \mu^\#\rangle$ be a #-measure space with $\mu^\#$ a hyperfinite #-measure. Suppose that f is a #-measurable, ${}^*\mathbb{R}_c^\#$ -valued function on M which is finite or hyperfinite $\mu^\#$-a.e.. Then the operator $T_f : \phi \to f\phi$ on $L_2^\#(M, d^\#\mu^\#)$ with domain $D(T_f) = \{\varphi | f\varphi \in L_2^\#(M, d^\#\mu^\#)\}$ is self-#-adjoint and $\sigma(T_f)$ is the essential range of T_f.

Proposition10.2. Let f and T_f satisfy the conditions in Proposition 6.4.1. Suppose in addition that $f \in L_p^\#(M, d^\#\mu^\#)$ for $2 < p < {}^*\infty$. Let D be any #-dense set in $L_q^\#(M, d^\#\mu^\#)$, where $q^{-1} + p^{-1} = 1/2$. Then D is a #-core for T_f.

Theorem 10.1.(Spectral theorem-multiplication operator form) Let A be a self-#-adjoint operator on a ${}^*\infty$- dimensional a non-Archimedean Hilbert space $H^\#$ with domain $D(A)$. Then there is a #-measure space $\langle M, \mu^\#\rangle$ with $\mu^\#$ a hyperfinite #-measure, a unitary operator U: $H^\# \to L_2^\#(M, d^\#\mu^\#)$ and a ${}^*\mathbb{R}_c^\#$ -valued function f on M which is finite or hyperfinite $\mu^\#$-a.e. so that

(a) $\psi \in D(A)$ if and only if $f(\cdot)(U\psi)(\cdot) \in L_2^\#(M, d^\#\mu^\#)$.

(b) If $\varphi \in U[D(A)]$, then $(UAU^{-1}\varphi)(m) = f(m)\phi(m)$.

Remark 10.1. There is a natural way to define functions of a self-#-adjoint operator by using the Theorem 10.1. Given a bounded #-Borel function h on ${}^*\mathbb{R}_c^\#$ we define

$$h(A) = UT_{h(f)}U^{-1} \quad (10.1)$$

where $T_{h(f)}$ is the operator on $L_2^{\#}(M, d^{\#}\mu^{\#})$ which acts by multiplication by the function $h(f(m))$. Using this definition the following theorem follows easily from Theorem 6.4.1.

Theorem 10.2. (Spectral theorem -functional calculus form) Let A be a self-#-adjoint operator on $H^{\#}$. Then there is a unique map $\hat{\phi}$ from the bounded #-Borel functions on ${}^{*}\mathbb{R}_c^{\#}$ into $\mathcal{L}(H^{\#})$, so that

(a) $\hat{\phi}$ is an algebraic $*$-homomorphism.

(b) $\hat{\phi}$ is #-norm #-continuous, that is, $\|\hat{\phi}(h)\|_{\mathcal{L}(H^{\#})} \leq \|h\|_{^*\infty}$.

(c) Let $h_n(x), n \in {}^{*}\mathbb{N}$ be a hyper infinite sequence of bounded in ${}^{*}\mathbb{R}_c^{\#}$ #-Borel functions with #-$\lim_{n\to{}^*\infty} h_n(x) = x$, for each x and $|h_n(x)| \leq |x|$ for all x and $n \in {}^{*}\mathbb{N}$. Then, for any $\psi \in D(A)$,

$$\#\text{-}\lim_{n\to{}^*\infty} \left(\hat{\phi}(h_n)\psi\right) = A\psi.$$

(d) If $h_n(x) \to_{\#} h(x)$ pointwise and if the hyper infinite sequence $\|h_n(x)\|_{^*\infty}, n \in {}^{*}\mathbb{N}$ is bounded in ${}^{*}\mathbb{R}_c^{\#}$, then $\hat{\phi}(h_n) \to_{\#} \hat{\phi}(h)$ strongly.

In addition:

(e) If A$\psi = \lambda\psi$ then $\hat{\phi}(h) = h(\lambda)\psi$.

(f) If $h \geq 0$, then $\hat{\phi}(h) \geq 0$.

The spectral theorem in its projection-valued #-measure form follows directly from the functional calculus. Let P_Ω be the operator $\chi_\Omega(A)$ where χ_Ω is the characteristic function of the #-measurable set $\Omega \subset {}^{*}\mathbb{R}_c^{\#}$. The family of operators $\{P_\Omega\}$ has the following properties:

Proposition 10.3. The family $\{P_\Omega\}$ of spectral projections of abounded in ${}^{*}\mathbb{R}_c^{\#}$ self-#-adjoint operator, A, has the following properties:

(a) Each P_Ω is an orthogonal projection.

(b) $P_\emptyset = 0$; $P_{(-{}^*\infty,{}^*\infty)} = I$.

(c) If $\Omega = Ext\text{-}\bigcup_{n=1}^{^*\infty}\Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ for all $n \neq m$ then

$$P_\Omega = s\text{-}\#\text{-}\lim_{N\to{}^*\infty}\left(Ext\text{-}\sum_{n=1}^{N} P_{\Omega_n}\right)$$

(d) $P_{\Omega_1}P_{\Omega_2} = P_{\Omega_1\cap\Omega_2}$.

Definition 10.1. A family of projections obeying (a)-(c) is called a projection-valued #-measure (p.v. #-m.).

Remark 10.2. This is a generalization of the notion of bounded projection -valued #-measure introduced in Section 9. In that we only require $P_{(-{}^*\infty,{}^*\infty)} = I$ rather than $P_{(-a,a)} = I$ for some $a \in {}^{*}\mathbb{R}_{c+}^{\#}$. For vector $\varphi \in H^{\#}$, $\langle\varphi, P_\Omega\varphi\rangle_{\#}$ is a well-defined Borel #-measure on ${}^{*}\mathbb{R}_c^{\#}$ which we denote by $\langle\varphi, P_\lambda\varphi\rangle_{\#}$ as in § 4.3. The complex ${}^{*}\mathbb{C}_c^{\#}$-valued #-measure $d^{\#}\langle\varphi, P_\lambda\psi\rangle_{\#}$ is defined by polarization. Thus, given a bounded in ${}^{*}\mathbb{R}_c^{\#}$ #-Borel function g we can define $g(A)$ by

$$\langle\varphi, g(A)\,\varphi\rangle_{\#} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#}} g(\lambda)d^{\#}\langle\varphi, P_\lambda\varphi\rangle_{\#}. \quad (10.2)$$

It is not difficult to show that this map $g \mapsto g(A)$ has the properties (a)-(d) of Theorem 10.1, so $g(A)$ as defined by (10.2) coincides with the definition of $g(A)$ given by Theorem 10.1. Now, suppose g is an unbounded ${}^{*}\mathbb{C}_c^{\#}$-valued #-Borel function and let

$$D_g = \left\{\varphi | Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda) d^\# \langle \varphi, P_\lambda \varphi \rangle_\# < {}^*\infty \right\}. \quad (10.3)$$

Then, D_g is #-dense in $H^\#$ and an operator $g(A)$ is defined on D_g by

$$\langle \varphi, g(A)\, \varphi \rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda) d^\# \langle \varphi, P_\lambda \varphi \rangle_\#. \quad (10.4)$$

As in Section 9, we write symbolically

$$g(A) = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda) d^\# P_\lambda. \quad (10.5)$$

In particular, for $\varphi, \psi \in D(A)$,

$$\langle \varphi, g(A)\, \psi \rangle_\# = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} g(\lambda) d^\# \langle \varphi, P_\lambda \psi \rangle_\#. \quad (10.6)$$

if g is ${}^*\mathbb{R}_c^\#$ -valued, then $g(A)$ is self-#-adjoint on D_g. We summarize:

Theorem 10.3. (Spectral theorem-projection valued #-measure form).There is a one-to-one correspondence between self-#-adjoint operators A and projection-valued #-measures $\{P_\Omega\}$ on $H^\#$ the correspondence being given by

$$A = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} \lambda d^\# P_\lambda. \quad (10.7)$$

We use now the functional calculus developed above in order to define Ext-$\exp(itA)$.

Theorem 10.4 Let A be a self-#-adjoint operator and define $U(t) = Ext$-$\exp(itA)$. Then

(a) For each $t \in {}^*\mathbb{R}_c^\#$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s, t \in {}^*\mathbb{R}_c^\#$.

(b) If $\phi \in H^\#$ and $t \to_\# t_0$, then $U(t)\phi \to_\# U(t_0)\phi$.

(c) For any $\psi \in D(A)$: $((U(t)\psi - \psi)/t) \to_\# iA\psi$ as $t \to_\# 0$.

(d) If #-$\lim_{t \to_\# 0}((U(t)\psi - \psi)/t)$ exists, then $\psi \in D(A)$.

Proof (a) follows immediately from the functional calculus and the corresponding statements for the ${}^*\mathbb{C}_c^\#$- valued function Ext-$\exp(it\lambda)$. To prove (b) observe that

$$\|Ext\text{-}\exp(itA)\psi - \psi\|_\#^2 = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} |Ext\text{-}\exp(it\lambda) - 1|^2 d^\# g(\lambda) d^\# \langle P_\lambda \varphi, \psi \rangle_\#.$$

Since $|Ext\text{-}\exp(it\lambda) - 1|^2$ is dominated by the #-integrable function $g(\lambda) = 2$ and since for each $\lambda \in {}^*\mathbb{R}_c^\#$ $|Ext\text{-}\exp(it\lambda) - 1|^2 \to_\# 0$ as $t \to_\# 0$ we conclude that $(U(t)\psi - \psi) \to_\# 0$ as $t \to_\# 0$, by the generalized Lebesgue dominated-#-convergence theorem. Thus $t \mapsto U(t)$ is strongly #-continuous at $t = 0$, which by the group property proves $t \mapsto U(t)$ is strongly #-continuous everywhere. The proof of (c), again uses the dominated #-convergence theorem and the estimate $|Ext\text{-}\exp(itx) - 1|^2 \leq |x|$. To prove (d), we define

$$D(B) = \left\{\psi |\ \#\text{-}\lim_{t \to_\# 0} \left(\frac{U(t)\psi - \psi}{t}\right) \text{ exists}\right\}$$

and let $iB\psi = \#\text{-}\lim_{t \to_\# 0} \left(\frac{U(t)\psi - \psi}{t}\right)$. A simple computation shows that B is symmetric. By (c), B $\supset A$, so $B = A$.

Definition 10.2. An operator-valued function $U(t)$ satisfying (a) and (b) is called a strongly #-continuous one-parameter unitary group.

Definition 10.3 If $U(t)$ is a strongly #-continuous one-parameter unitary group, then the self-#-adjoint operator A with $U(t) = Ext$-$\exp(itA)$ is called the #-infinitesimal generator of $U(t)$.

Theorem 10.5. Let $U(t)$ be a strongly #-continuous one-parameter unitary group on a non-Archimedean Hilbert space $H^{\#}$. Then, there is a self-#-adjoint operator A on $H^{\#}$ so that $U(t) = Ext\text{-}\exp(itA)$.

Theorem 10.6. Let $U(t)$ be a one-parameter group of unitary operators on a hyper infinite dimensional non-Archimedean Hilbert space $H^{\#}$. Suppose that for all $\phi, \psi \in H^{\#}$, $\langle U(t)\psi, \phi\rangle_{\#}$ is #-measurable.Then $U(t)$ is strongly #-continuous.

Theorem 10.7. Suppose that $U(t)$ is a strongly #-continuous one-parameter unitary group. Let D be a #-dense domain which is invariant under $U(t)$ and on which $U(t)$ is strongly #-differentiable. Then i^{-1} times the strong #-derivative of $U(t)$ is essentially self-#-adjoint on D and its #-closure is the #-infinitesimal generator of $U(t)$.

Theorem 10.8. Let A be a self-adjoint operator on $H^{\#}$ and D be a #-dense linear set contained in $D(A)$. If for all $t, Ext\text{-}\exp(itA): D \to D$ then D is a #-core for A.

Remark 6.4.3. Finally, we have the following generalization of Theorem 10.5. If $g(\lambda)$ is a ${}^{*}\mathbb{R}_c^{\#}$-valued #-BoreI function on ${}^{*}\mathbb{R}_c^{\#}$, then $g(A) = Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#}} g(\lambda) d^{\#} P_{\lambda}$. defined on D_g (10.3) is self-#-adjoint. If g is bounded, $g(A)$ coincides with $\hat{\phi}(g)$ in Theorem 10.2.

Theorem 10.9. Let $U(\boldsymbol{t}) = U(t_1, \ldots, t_n)$ be a strongly #-continuous map of ${}^{*}\mathbb{R}_c^{\#n}$ into the unitary operators on a hyper infinite dimensional Hilbert space $H^{\#}$ satisfying $U(\boldsymbol{t}+\boldsymbol{s}) = U(\boldsymbol{t})U(\boldsymbol{s})$ Let D be the set of hyperfinite linear combinations of vectors of the form

$$\varphi_f = Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#n}} f(\boldsymbol{t}) U(\boldsymbol{t}) d^{\#n} t, \tag{10.8}$$

where $\phi \in H^{\#}, f \in C_0^{\#^{*}\infty}\left({}^{*}\mathbb{R}_c^{\#n}\right)$. Then D is a domain of essential self-#-adjointness for each of the generators A_j of the one-parameter subgroups $U(0,0,\ldots,t_j,\ldots,0)$, each $A_j : D \to D$ and the A_j commute, $j = 1, \ldots, n$. Furthermore, there is a projection-valued #-measure P_{Ω} on ${}^{*}\mathbb{R}_c^{\#n}$ so that

$$\langle \varphi, U(\boldsymbol{t})\psi\rangle_{\#} = Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#n}} [Ext\text{-}\exp(i\langle \boldsymbol{t}, \boldsymbol{\lambda}\rangle)]\, d^{\#}\langle \varphi, P_{\boldsymbol{\lambda}}\psi\rangle_{\#} \tag{10.9}$$

for all $\phi, \psi \in H^{\#}$.

Remark 10.4.Suppose that A and B are two unbounded self-#-adjoint operators on a non-Archimedean Hilbert space $H^{\#}$.We would like to find a reasonable meaning for the statement: "A and B commute." This cannot be done in the straightforward way since the operator $C = AB - BA$ may not make sense on any vector $\psi \in H^{\#}$ for example one might have $(\mathrm{Ran}(A)) \cap D(B) = \emptyset$ in which case BA does not have a meaning. This suggests that we find an equivalent formulation of commutativity for bounded self-#-adjoint operators.The spectral theorem for bounded self-#-adjoint operators A and B shows that in that case $AB - BA = 0$ if and only if all their projections, P_{Ω}^{A} and P_{Ω}^{B}, commute. We take this as our definition in the unbounded case.

Definition 10.3 Two (possibly unbounded in ${}^{*}\mathbb{R}_{\mathbf{c}}^{\#}$ self-#-adjoint operators A and B are said to commute if and only if all the projections in their associated projection-valued #-measures commute.

Remark 10.5.The spectral theorem shows that if A and B commute, then all the bounded in ${}^{*}\mathbb{R}_{\mathbf{c}}^{\#}$ #-Borel functions of A and B also commutes. In particular, the resolvents $R_{\lambda}(A)$ and $R_{\mu}(B)$ commute and the unitary groups $Ext\text{-}\exp(itA)$ and $Ext\text{-}\exp(itA)$ commute. The converse statement is also true and this shows that the above definition of "commute" is reasonable.

Theorem 10.10 Let A and B be self-#-adjoint operators on a non-Archimedean Hilbert space $H^{\#}$.Then the following three statements are equivalent:

(a) $P_{(a,b)}^{A}$ and $P_{(c,d)}^{B}$ commute.

(b) If $\mathrm{Im}\lambda$ and $\mathrm{Im}\mu$ are nonzero, then $R_{\lambda}(A)R_{\mu}(B) - R_{\mu}(B)R_{\lambda}(A) = 0$.

(c) For all $s,t \in {}^*\mathbb{R}^\#_{\boldsymbol{c}}$, $[Ext\text{-}\exp(itA)]\,[Ext\text{-}\exp(itA)] = [Ext\text{-}\exp(itA)][Ext\text{-}\exp(itA)]$.

Proof The fact that (a) implies (b) and (c) follows from the functional calculus. The fact that (b) implies (a) easily follows from the formula which expresses the spectral projections of A and B as strong #-limits of the resolvents together with the fact that $s\text{-}\#\text{-}\lim_{\varepsilon\to_{\#}0} i\varepsilon\, R_{a+i\varepsilon}(A) = P^A_{\{a\}}$. To prove that (c) implies (a), we use some simple facts about the Fourier transform. Let $f \in S^\#\left({}^*\mathbb{R}^\#_{\boldsymbol{c}}\right)$, then, by generalized Fubini's theorem [16],

$$Ext\text{-}\int_{{}^*\mathbb{R}^\#_{\boldsymbol{c}}} f(t)\,\langle [Ext\text{-}\exp(itA)]\varphi,\psi\rangle_\# =$$

$$= Ext\text{-}\int_{{}^*\mathbb{R}^\#_{\boldsymbol{c}}} f(t)\left(Ext\text{-}\int_{{}^*\mathbb{R}^\#_{\boldsymbol{c}}}([Ext\text{-}\exp(-it\lambda)])\,d^\#_\lambda\langle P^A_\lambda\varphi,\psi\rangle_\#\right)d^\# t =$$

$$= \sqrt{2\pi_\#}\,Ext\text{-}\int_{{}^*\mathbb{R}^\#_{\boldsymbol{c}}} \hat{f}(\lambda)\,d^\#_\lambda\langle P^A_\lambda\varphi,\psi\rangle_\# = \sqrt{2\pi_\#}\langle\varphi,\hat{f}(A)\psi\rangle_\#.$$

Thus, using (c) and generalized Fubini's theorem again,

$$\langle\varphi,\hat{f}(A)\hat{g}(B)\psi\rangle_\# =$$

$$= Ext\text{-}\int_{{}^*\mathbb{R}^\#_c} Ext\text{-}\int_{{}^*\mathbb{R}^\#_c} f(t)\,g(s)\,\langle\varphi,[Ext\text{-}\exp(-itA)][Ext\text{-}\exp(isB)]\psi\rangle_\# =$$

$$= \langle\varphi,\hat{g}(B)\hat{f}(A)\psi\rangle_\#$$

so, for all $f,g \in S^\#\left({}^*\mathbb{R}^\#_{\boldsymbol{c}}\right), \hat{f}(A)\hat{g}(B) - \hat{g}(B)\hat{f}(A) = 0$. Since the Fourier transform maps $S^\#\left({}^*\mathbb{R}^\#_{\boldsymbol{c}}\right)$ onto $S^\#\left({}^*\mathbb{R}^\#_{\boldsymbol{c}}\right)$ we conclude that $f(A)g(B) - g(B)f(A) = 0$ for all $f,g \in S^\#\left({}^*\mathbb{R}^\#_{\boldsymbol{c}}\right)$. But, the characteristic function, $\chi_{(a,b)}$ can be expressed as the pointwise #-limit of a hyper infinite sequence $f_n, n \in {}^*\mathbb{N}$ of uniformly bounded functions such that $f_n \in S^\#\left({}^*\mathbb{R}^\#_{\boldsymbol{c}}\right), n \in {}^*\mathbb{N}$. By the functional calculus we get

$$s\text{-}\#\text{-}\lim_{N\to{}^*\infty} f_n(A) = P^A_{(a,b)}.$$

Similarly, we find uniformly bounded $g_n \in S^\#\left({}^*\mathbb{R}^\#_{\boldsymbol{c}}\right), n \in {}^*\mathbb{N}$ #-converging pointwise to $\chi_{(c,d)}$ and therefore

$$s\text{-}\#\text{-}\lim_{N\to{}^*\infty} g_n(B) = P^B_{(c,d)}.$$

Since the f_n and g_n are uniformly bounded in ${}^*\mathbb{R}^\#_{\boldsymbol{c}}$ and $f_n(A)g_n(B) = g_n(B)f_n(A)$ for each $n \in {}^*\mathbb{N}$, we conclude that $P^A_{(a,b)}$ and $P^B_{(c,d)}$ commute which proves (a).

References

[1] D. Laugwitz, Eineeinfuhrung der δ-funktionen, S.-B. Bayerische Akad. Wiss., 4 (1958), 41-59.
[2].A. Robinson, Nonstandard Analysis, North-Holland, (1974).
[3] W. Luxemburg, Nonstandard Analysis. Lectures on A. Robinson's Theory of Infinitesimal and

Infinitely Large Numbers Mimeographed Notes, California Institute of Technology, (1962).
[4] A. Sloan, A Note on exponentials of distributions, Pacific journal of mathematics Vol. 79, No. 1, 197.
[5] S. Albeverio, R. Høegh-Krohn, J. E. Fenstad, T. Lindstrøm, Nonstandard Methods in Stochastic Analysis and Mathematical Physics, ISBN-13: 978-0486468990, ISBN-10: 0486468992
[6] K. D. Stroyan, W. A. J. Luxemburg, Introduction to the theory of infinitesimals, New York: Academic Press 1976. ISBN: 0126741506
[7] M. Davis, Applied Nonstandard Analysis (Dover Books on Mathematics) (2005) ISBN-13: 978-0486442297, ISBN-10: 0486442292
[8] J.Foukzon, Model $P(\varphi)_4$ Quantum Field Theory. A Nonstandard Approach Based on Nonstandard Pointwise-Defined Quantum Fields (July 14, 2022). Available at SSRN: https://ssrn.com/abstract=4163159 or http://dx.doi.org/10.2139/ssrn.4163159
[9] M. Reed, B. Simon, Functional Analysis: Volume 1 (Methods of Modern Mathematical Physics,Vol.1) December 28th, 1980 ISBN: 9780125850506 ISBN-10: 0125850506
[10] M. Reed, B. Simon, Fourier Analysis, Self-Adjointness (Methods of Modern Mathematical Physics, Vol. 2) 1975 ISBN-13: 978-0125850025 ISBN-10: 0125850026
[11] J. Foukzon, (2022). Internal Set Theory IST# Based on Hyper Infinitary Logic with Restricted Modus Ponens Rule: Nonconservative Extension of the Model Theoretical NSA. Journal of Advances in Mathematics and Computer Science, 37(7), 16-43. https://doi.org/10.9734/jamcs/2022/v37i730463
[12] J. Foukzon, Set Theory INC# Based on Intuitionistic Logic with Restricted Modus Ponens Rule (Part. I). Journal of Advances in Mathematics and Computer Science, 36(2), 73-88, (2021). https://doi.org/10.9734/jamcs/2021/v36i230339
[13] J. Foukzon, Set Theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ Based on Hyper Infinitary Intuitionistic Logic with Restricted Modus Ponens Rule (Part. II) Hyper Inductive Definitions. Journal of Advances in Mathematics and Computer Science, 36(4), 90-112, (2021). https://doi.org/10.9734/jamcs/2021/v36i430359
[14] J. Foukzon, Set Theory $\mathrm{NC}^{\#}_{\infty}$ Based on Bivalent Infinitary Logic with Restricted Modus Ponens Rule. Basic Real Analysis on External Non-Archimedean Field ${}^{*}\mathbb{R}^{\#}_{c}$. Basic Complex Analysis on External Field ${}^{*}\mathbb{C}^{\#}_{c} = {}^{*}\mathbb{R}^{\#}_{c} + i\,{}^{*}\mathbb{R}^{\#}_{c}$. (December 20, 2021) Available at SSRN: https://ssrn.com/abstract=3989960
[15] J. Foukzon, The Solution of the Invariant Subspace Problem. Part I. Complex Hilbert Space (February 20, 2022). Available at SSRN: https://ssrn.com/abstract=4039068 or http://dx.doi.org/10.2139/ssrn.4039068
[16] Foukzon, Jaykov, Basic non-Archimedean functional analysis over a non-Archimedean field ${}^{*}\mathbb{R}^{\#}_{c}$ (August 10, 2022). Available at SSRN: https://ssrn.com/abstract=4186963 or http://dx.doi.org/10.2139/ssrn.4186963