A simple Markov chain for the Collatz problem

Wiroj Homsup and Nathawut Homsup

May 30, 2023

Abstract

We show that the iteration of the Collatz function is represented by a simple three states Markov chain. This simple model is implemented to show the probabilistic convergence of the algorithm to the equilibrium point set {1,2}.

Introduction

Define the iterating function introduced by R. Terras[1] :

$$a_{n+1} = (3^b a_n + b)/2 \qquad (1)$$

where b = 1 when a_n is odd and b = 0 when a_n is even. The Collatz conjecture asserts that by starting with any positive integer a_0, there exists a natural number k such that a_k =1.

1. The Markov Chain and the transition probability

The Eq. (1) can be represented by a Markov chain with three states.

Let partition positive natural numbers N in three sets (states):

A : {3,5,7,9,……………}

B : {4,6,8,10,…………..}

C : {1,2}

Then, consider the following Markov chain:

$$X_{i+1} = PX_i \qquad (2)$$

Where X_i is a vector with three components each of which represents the probability of a number to belong to one of the above defined sets, i.e. $X_i\ (1,1) =$ Prob.{ a number is in A}, $X_i\ (2,1) =$ Prob.{a number is in B} and $X_i\ (3,1) =$ Prob.{a number is in C}. P is a 3x3 real transition matrix whose element, i.e. p_{ij} is the probability of transition from state j to state i. The Markov chain of the Collatz problem is shown in Figure 1.

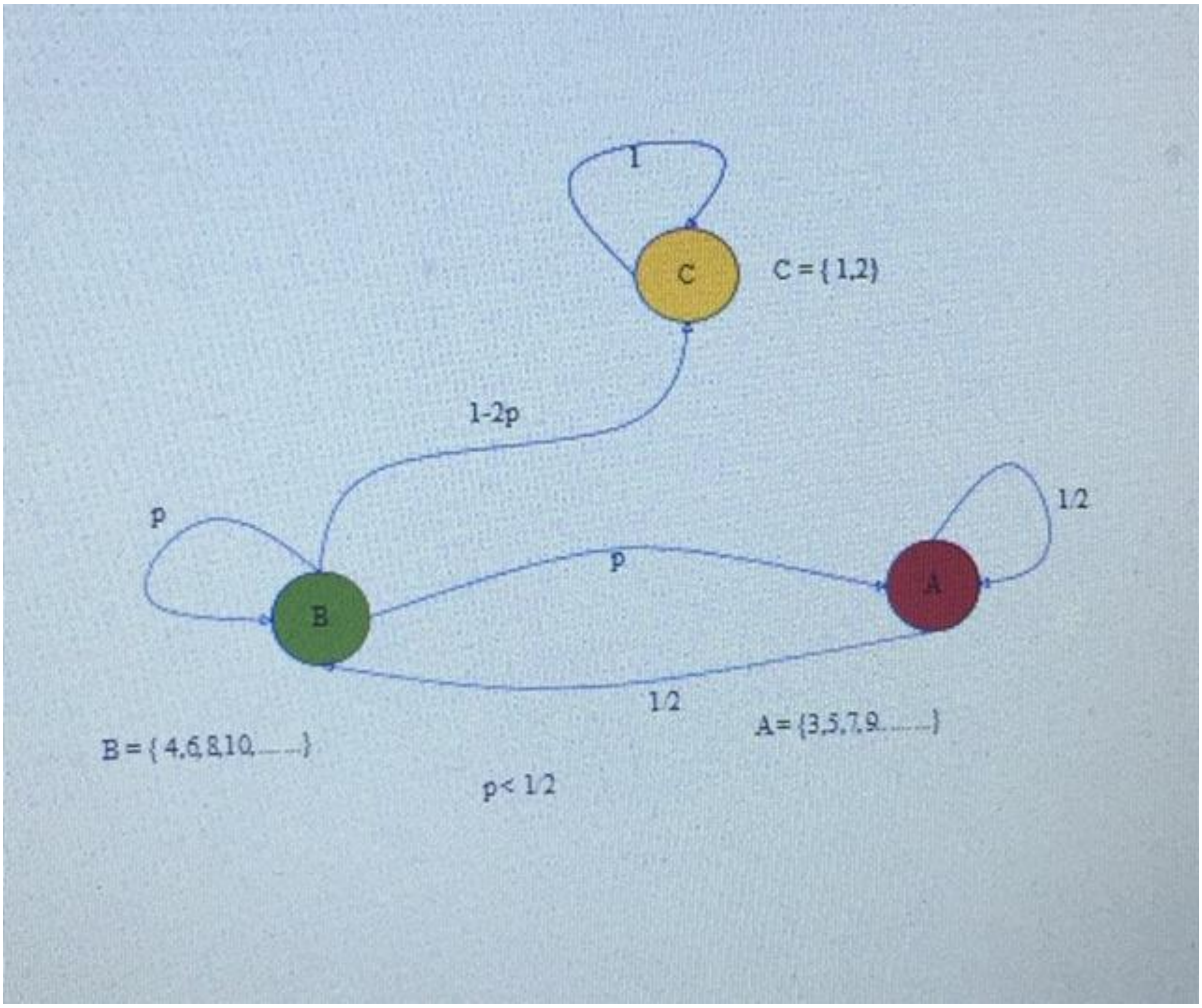

Figure 1. Markov chain of the Collatz problem.

$$P = \begin{bmatrix} 1/2 & p & 0 \\ 1/2 & p & 0 \\ 0 & 1-2p & 1 \end{bmatrix}; \quad p < ½$$

2. **The probabilistic convergence of the dynamic system**

The limiting of X_i can be defined as

$$X_\infty = \lim_{n\to\infty} P^n X_0 \qquad (3)$$

Let

$$P = SVS^{-1} \qquad (4)$$

where S and V are a 3x3 eigen vector and eigen value matrix , respectively.

$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p+1/2 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 2/c & \frac{-1}{d} \\ 0 & \frac{-1}{cp} & \frac{-1}{d} \\ 1 & \frac{(1-2p)}{cp} & \frac{(2-4p)}{d(1-2p)} \end{bmatrix}$$

where

$$c^2 = \frac{(8p^2-4p+2)}{p^2}$$

$$d^2 = \frac{(6p^2-6p+1.5)}{(0.5-p)^2}$$

and S^{-1} is shown to represent in a matrix form as

$$S^{-1} = \begin{bmatrix} 1 \quad 1 \quad 1 \\ [M]_{2x3} \end{bmatrix}$$

where $[M]_{2x3}$ is a real 2x3 matrix.

As n → ∞,

$$\lim_{n\to\infty} P^n = S \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Then a convergence of eq. (3) follows as

$$X_\infty = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} X_0 \qquad (5)$$

The matrix structure which yields the above condition, show that the Markov chain has an absorbing state which is a state C. Once entered in state C it remains in state C, i.e. a number will alternate between 1 and 2.

3. **Conclusions**

In the paper, it has been shown that using a simple structured Markov chain, eq. (1) representing the Collatz iteration, converges to 1 with probability 1.

References

[1] R . Terras, (1976). " A stopping time problem on the positive integers". Acta Arithmetica, 30(3), 241-252.