Proof of the Collatz Conjecture

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Abstract

Take any positive integer N. If it is odd, multiply it by three and add one. If it is even, divide it by two. Repeatedly do the same operations to the results, forming a sequence. It is found that, whatever the initial starting number we choose, the sequence will eventually descend and reach number 1, where it enters an eternal closed loop of 1- 4 - 2 - 1. This has been numerically confirmed to numbers up to 2^{60}. This is known as the Collatz conjecture. So far no proof has ever been found that this holds for every positive integer. This problem has been stated by some as perhaps the simplest math problem to state, yet perhaps the most difficult to solve. This paper completely solves this problem by using new insights.

Introduction

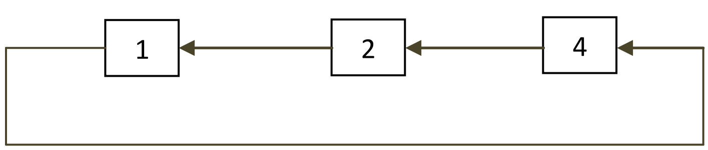
The Collatz function is defined as:

$$C(N) = \begin{cases} 3N + 1, & \text{if n is odd} \\ \frac{N}{2}, & \text{if n is even} \end{cases}$$

The Collatz conjecture :

Take any positive number *N*. If *N* is odd, multiply it by three and add one. If *N* is even, divide it by two. Repeatedly do this to form a sequence. The Collatz conjecture says that this sequence always ends in 1.

All sequences finally end inthe closed loop:



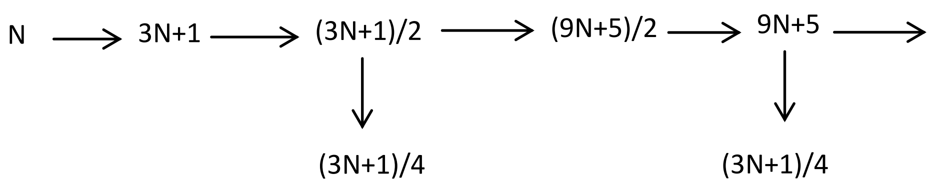
To prove the Collatz conjecture, one has to show that:

1. The sequence will not diverge to infinity.

2. The sequence will not enter some closed loop other than the 1- 4- 2- 1 loop, i.e. no closed loops other than the 1-4-2-1 loop exist.

Intuitive explanation on why the Collatz sequence cannot diverge to infinity

Let the starting number *N* be odd..



We can see that ***no successive Collatz operations can be odd operations, whereas multiple successive even operations can occur before an odd operation occurs***. This is because any even number can have 2^m as a factor. Therefore, after every odd Collatz operation there are one or more even operations. This shows that the Collatz number going to infinity is impossible and in fact, in the long run the sequence necessarily converges and descends.

As the starting number is made larger and larger, the probability that many more even operations occur before an odd operation increases (see Appendix). This leads to the conclusion that the probability that the next number being greater than the current number during any Collatz operation necessarily approaches zero as the starting number approaches infinity.

$$\lim_{N\to\infty}(Probability\ that\ C(N+1) > C(N)) = 0$$

where N is the starting number or any number in the sequence.

In other words, the Collatz sequence always wanders around before it eventually (and frequently) lands on a number Q that has 2^m as a factor, where *m* is a positive integer. I refer to such numbers as ‘wells’. Once the sequence lands on these wells, it descends all the way down by a factor of 2^m, more than losing everything it gained by the possible previous odd operations.

$$P = 2^m * Q$$

Let us call the number P a ‘perfect/pure even number’ if it is a power of 2 :

$$P = 2^m$$

On the other hand, an ‘imperfect’ even number is of the form:

$$P = 2^m * Q$$

where Q does not have 2 or any powers of 2 as a factor. If the Collatz sequence lands on an ‘imperfect’ even number, as opposed to perfect even number, the sequence will descend significantly but the descent will stop somewhere.

One can still ask if there is a greater than zero probability that the sequence will diverge to infinity. This is completely disproved by the fact that there is always a greater than zero probability that the sequence will land on one of the *perfect* / *pure* even numbers. One can think of the column (shown below) formed by these numbers as ‘eternal well’ because once the sequence lands on such number, it will descend all the way down to one, where it enters the 1-4-2-1 loop.

1- 2 – 4 – 8 – 16 – 32 – 64 – 128 – 256 – 512 – 1024 – 2048 - . . . .

Now let us define one Collatz cycle. For any Collatz operation on an odd number, the next number is necessarily an even number. Therefore, after every Collatz odd operation, there is an even operation. Let us call this the Collatz cycle.

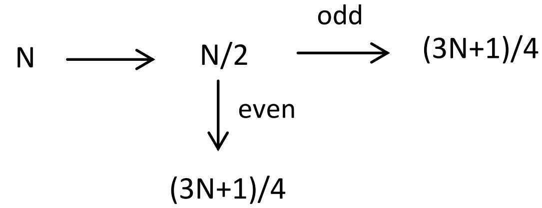
$$N \longrightarrow 3N+1 \longrightarrow (3N+1)/2$$

We can see that an odd number N becomes N’ =(3N+1)/2 after one Collatz cycle.

$$N' = \frac{3N+1}{2} = 1.5N + 0.5 \approx 1.5\,N \quad \text{for} \quad 3N \gg 1$$

Therefore, an odd number is approximately increased by a factor of 1.5 after one Collatz cycle.

Now consider an even number.



One Collatz cycle on an even number *N* has two cases depending on whether the Collatz operation on *N* results in an even integer or an odd integer.

1. If the Collatz operation on *N* results in an even integer, the number at the end of one cycle will be *N/*4.

2. If the Collatz operation on *N* results in an odd integer, the number at the end of one cycle will be 3(*N*/2)+1.

In the first case, the number *N'* after one Collatz operation on *N* will be:

$$N' = \frac{N}{4}$$

In the second case, the number *N'* after one Collatz operation on *N* will be:

$$N' = 3\left(\frac{N}{2}\right) + 1 = \frac{3N}{2} + 1 \approx 1.5\,N \;\; , for\ \frac{3N}{2} \gg 1$$

We defined one Collatz operation to help us compare the probabilities of the sequence increasing or decreasing. We can see that the maximum possible increase of *N* due to any odd operations is approximately:

$$N' - N \approx 1.5N - N = 0.5N$$

which is a fifty percent increase.

The maximum possible decrease of the sequence due to even operations is:

$$N' - N = \frac{N}{4} - N = -0.75N$$

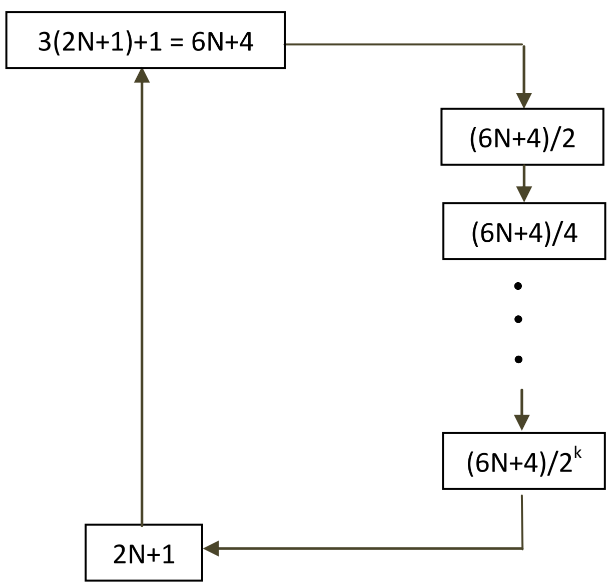
A Collatz cycle on an odd number *N* always increases it to *3N* +1, however a Collatz cycle on an even number *N* may decrease it to *N*/4 or increase it to *(3N/2)+1*. One might think that this will result in slightly higher probability of an ascending Collatz sequence than probability of descending Collatz sequence. However, all this is nullified by the argument we made above: whatever gains (ascents) the Collatz sequence has gained due to multiple odd operations is soon more than lost because the sequence inevitably lands on ‘wells’ of varying ‘depth’.

These arguments seal the impossibility of the Collatz sequence diverging to infinity.

Non- existenceof closedloop other than 1-4-2-1 loop

Next we present a new approach to disprove non-existence of any other closed loop other than the 1-4-2-1 loop as follows.

We start with an odd integer, *2N + 1*, where *N* is an integer greater than or equal to zero. Since it is odd, we multiply it by 3 and add 1. To form a loop, successive Collatz operations (divide by 2) on the result must give the original integer, i.e.



$$3\,(2N+1)+1=(2N+1)\,2^k\ ,\quad k \text{ is a positive integer}$$

$$\Longrightarrow\ 6N+4=(2N+1)\,2^k$$

$$\Longrightarrow \frac{6N+4}{2N+1}\ =\ 2^k$$

$$\Longrightarrow\ \frac{2(\,3N+2\,)}{2N+1}\ =\ 2^k$$

$$\Longrightarrow\ \frac{(\,3N+2\,)}{2N+1}\ =\ 2^{k-1}$$

$$\Rightarrow \frac{(2N + 1\,) + (N + 1)}{2N + 1} = 2^{k-1}$$

$$\Rightarrow 1 + \frac{N + 1}{2N + 1} = 2^{k-1}$$

Since the right hand side is always a positive integer that is a power of two, the left hand side must also be an integer (not a fraction) and a power of two for both sides to be equal. This is possible only if $N = 0$.

$$\Rightarrow N = 0$$

Since $N = 0$, our starting odd number *2N+1* will be:

$$\Rightarrow 2N + 1 = 2 * 0 + 1 = 1$$

Thus we have proved that no other closed loop exists other than the 4-2-1-4 loop.

A rigorous proof of non-existence of closed loop other than the 1-4-2-1 loop

The above proof of non-existence of closed loop other than 1-4-2-1is only meant to show the approach to be used and obviously is far from complete and therefore not rigorous. Next we present a new approach towards a rigorous proof that no other closed loop can exist.

We consider two cases: the Collatz sequence starting number is:

1) Odd
2) Even

<u>Odd starting number</u>

Re-writing the Collatz function:

$$\mathrm{C(N)} = \begin{cases} \mathrm{3N + 1\,,} & \text{if n is odd} \\ \frac{\mathrm{N}}{2}, & \text{if n is even} \end{cases}$$

as

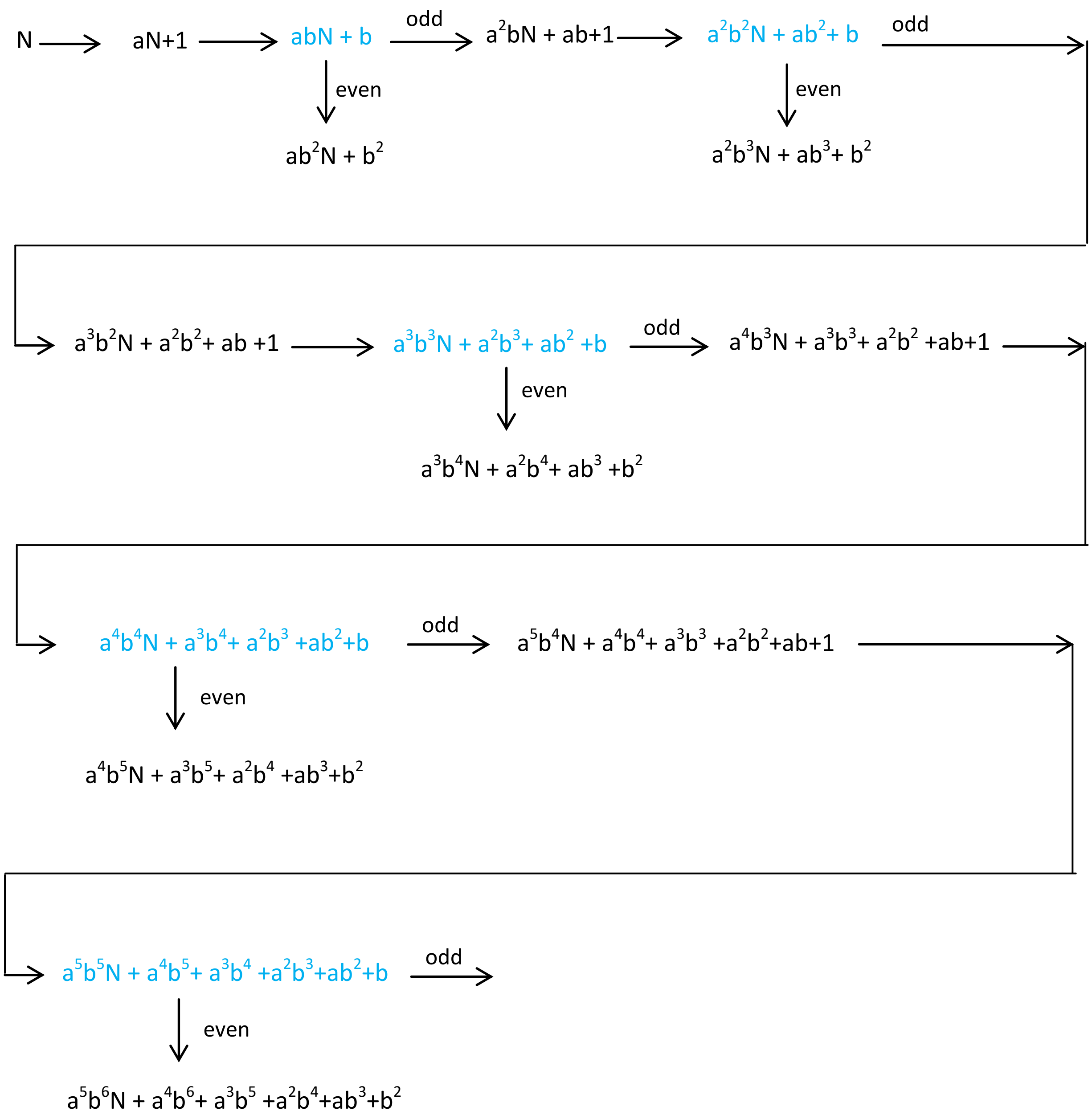
$$\mathrm{C(N)} = \begin{cases} \mathrm{aN + 1\,,} & \text{if n is odd} \\ \mathrm{bN,} & if\ n\ is\ even \end{cases}$$

where

$$a = 3 \quad and \quad b = \frac{1}{2}$$

Case 1: Starting number N is odd

We assume that the initial Collatz number is an odd number that successivelyresults in an odd number after multiplying it by three and adding one, and then dividing the result by two. Note that multiplying an odd number by three and adding one always gives an even number.



From the above, the last Collatz number is:

$$a^5b^5N + a^4b^5 + a^3b^4 + a^2b^3 + ab^2 + b$$

We can generalize this as follows:

$$C(N,x) = a^x b^x N + a^{x-1}b^x + a^{x-2}b^{x-1} + a^{x-3}b^{x-2} + ... + a^3b^4 + a^2b^3 + ab^2 + b$$

We can see that the part:

$$(a^{x-1}b^x + a^{x-2}b^{x-1} + a^{x-3}b^{x-2} + ... + a^3b^4 + a^2b^3 + ab^2 + b)$$

is a geometric sequence with ratio $r = ab$ and first term b .

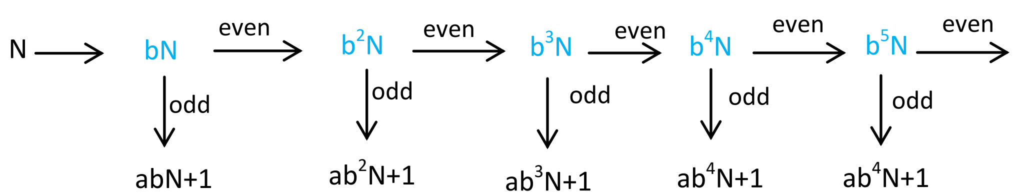
$$(a^{x-1}b^x + a^{x-2}b^{x-1} + a^{x-3}b^{x-2} + ... + a^3b^4 + a^2b^3 + ab^2 + b) = b\frac{1-(ab)^x}{1-ab}$$

Therefore, the x^{th} number will be:

$$C(N,x) = a^x b^x N + b\frac{1-(ab)^x}{1-ab} \quad \dots\dots\dots\dots (1)$$

<u>Case 2: Starting number *N* is even</u>

In this case we assume that the starting number is an even number that successively results in an even number after dividing the result by two.



The y^{th} number will be:

$$C(N,y) = b^y N \quad \dots\dots\dots\dots\dots (2)$$

We can reach any number along the Collatz sequence by combining equations (1) and (2).

To check if other loops exist, we shall consider two cases and develop separate general equations for each:

1. Collatz sequence starting with odd number and ending in odd number
2. Collatz sequence starting with even number and ending in even number

Collatz sequence starting with odd number and ending in odd number

Any number in the sequence can be expressed as:

$$C\,(N) =$$

$$(((\,(\,a^{x1}b^{x1}\,N\;+\;b\frac{1-(ab\,)^{x1}}{1-ab}\,)\,b^{y1}\,a^{x2}b^{x2}\;+\;b\frac{1-(ab\,)^{x2}}{1-ab})b^{y2}\,a^{x3}b^{x3}$$

$$+\,b\frac{1-(ab\,)^{x3}}{1-ab}\;)\;b^{y3}\,a^{x4}b^{x4}\;+\;\;b\frac{1-(ab\,)^{x4}}{1-ab}\;)b^{y4}\,a^{x5}b^{x5}$$

$$+\;\;b\frac{1-(ab\,)^{x5}}{1-ab}\qquad \dots\dots\dots\dots(3)$$

This can be generalized as:

$$C(N) =$$

$$(((\dots\;(\,a^{x1}b^{x1}\,N\;+\;b\frac{1-(ab\,)^{x1}}{1-ab}\,)\,b^{y1}\,a^{x2}b^{x2}\;+\;b\frac{1-(ab\,)^{x2}}{1-ab})b^{y2}\,a^{x3}b^{x3}$$

$$+\,b\frac{1-(ab\,)^{x3}}{1-ab}\;)\;b^{y3}\,a^{x4}b^{x4}\;+\;\;b\frac{1-(ab\,)^{x4}}{1-ab}\;)b^{y4}\,a^{x5}b^{x5}\;+\;.\;.\;.$$

$$.\;\;.\;\;.+\,b\frac{1-(ab\,)^{xn-1}}{1-ab})b^{yn-1}\,a^{xn}b^{xn}\;\;+\;\;b\frac{1-(ab\,)^{xn}}{1-ab}\qquad \dots\dots\dots\quad(4)$$

OR

$$C(N) = (((\dots (a^{x1} b^{x1} N + b \frac{1-(ab)^{x1}}{1-ab}) b^{y1} a^{x2} b^{x2} + b \frac{1-(ab)^{x2}}{1-ab}) b^{y2} a^{x3} b^{x3}$$

$$+ b \frac{1-(ab)^{x3}}{1-ab}) b^{y3} a^{x4} b^{x4} + b \frac{1-(ab)^{x4}}{1-ab}) b^{y4} a^{x5} b^{x5} + \dots$$

$$\dots + b \frac{1-(ab)^{xn-1}}{1-ab}) b^{yn-1} \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (5)$$

Rearranging equation (3)

$$C(N) = (ab)^{x1+x2+x3+x4+x5} b^{y1+y2+y3+y4} N +$$

$$b \frac{1-(ab)^{x1}}{1-ab} (ab)^{x2+x3+x4+x5} b^{y1+y2+y3+y4} +$$

$$b \frac{1-(ab)^{x2}}{1-ab} (ab)^{x3+x4+x5} b^{y2+y3+y4} + b \frac{1-(ab)^{x3}}{1-ab} (ab)^{x4+x5} b^{y3+y4} +$$

$$b \frac{1-(ab)^{x4}}{1-ab} (ab)^{x5} b^{y4} + b \frac{1-(ab)^{x5}}{1-ab} \quad \dots\dots\dots\dots\dots \quad (6)$$

This can be generalized as:

$$C(N) = (ab)^{x1+x2+\dots+xn} b^{y1+y2+\dots+yn-1} N +$$

$$b \frac{1-(ab)^{x1}}{1-ab} (ab)^{x2+x3+\dots+xn} b^{y1+y2+\dots+yn-1} +$$

$$b \frac{1-(ab)^{x2}}{1-ab} (ab)^{x3+x4+\dots+xn} b^{y2+y3+\dots+yn-1} +$$

$$b \frac{1-(ab)^{x3}}{1-ab} (ab)^{x4+x5+\dots+xn} b^{y3+y4+\dots+yn-1} + \dots +$$

$$b \frac{1-(ab)^{(xn-1)}}{1-ab} (ab)^{xn} b^{yn-1} + b \frac{1-(ab)^{xn}}{1-ab} \quad \dots\dots\dots \quad (7)$$

Collatz sequence starting with even number and ending in even number

Any number in the sequence can be expressed as:

$$((((a^{x1}b^{x1}\,Nb^{y1} + b\frac{1-(ab)^{x1}}{1-ab})b^{y2}a^{x2}b^{x2} + b\frac{1-(ab)^{x2}}{1-ab})b^{y3}a^{x3}b^{x3} +$$

$$b\frac{1-(ab)^{x3}}{1-ab})\,b^{y4}a^{x4}b^{x4} + b\frac{1-(ab)^{x4}}{1-ab})\,b^{y5} \quad \dots\dots\dots \quad (8)$$

OR

$$(((a^{x1}b^{x1}\,Nb^{y1} + b\frac{1-(ab)^{x1}}{1-ab})b^{y2}a^{x2}b^{x2} + b\frac{1-(ab)^{x2}}{1-ab})b^{y3}a^{x3}b^{x3} +$$

$$b\frac{1-(ab)^{x3}}{1-ab})\,b^{y4}a^{x4}b^{x4} + b\frac{1-(ab)^{x4}}{1-ab} \quad \dots\dots\dots \quad (9)$$

Equation (8) can be generalized as:

$$C(N) =$$

$$((..((a^{x1}b^{x1}\,Nb^{y1} + b\frac{1-(ab)^{x1}}{1-ab})b^{y2}a^{x2}b^{x2} + b\frac{1-(ab)^{x2}}{1-ab})b^{y3}a^{x3}b^{x3} +$$

$$b\frac{1-(ab)^{x3}}{1-ab})\,b^{y4}a^{x4}b^{x4} + ...)b^{yn-1}a^{xn-1}b^{xn-1} + b\frac{1-(ab)^{xn-1}}{1-ab})\,b^{yn} \quad \dots\dots\dots. \quad (10)$$

Equation (9) can be generalized as:

$$C(N) =$$

$$((..((a^{x1}b^{x1}\,Nb^{y1} + b\frac{1-(ab)^{x1}}{1-ab})b^{y2}a^{x2}b^{x2} + b\frac{1-(ab)^{x2}}{1-ab})b^{y3}a^{x3}b^{x3} +$$

$$b\frac{1-(ab)^{x3}}{1-ab})\,b^{y4}a^{x4}b^{x4} + ...)b^{yn}a^{xn}b^{xn} + b\frac{1-(ab)^{xn}}{1-ab} \quad \dots\dots\dots. \quad (11)$$

Rearranging equation (8):

$$C\ (N) = \ (ab)^{x1+x2+x3+x4}\ b^{y1+y2+y3+y4+y5}\ \ N\ +$$

$$b\frac{1-(ab\ )^{x1}}{1-ab}(ab)^{x2+x3+x4}b^{y2+y3+y4+y5}\ +$$

$$b\frac{1-(ab\ )^{x2}}{1-ab}(ab)^{x3+x4}b^{y3+y4+y5}\ +\ \ b\frac{1-(ab\ )^{x3}}{1-ab}(ab)^{x4}b^{y4+y5}\ +$$

$$b\frac{1-(ab\ )^{x4}}{1-ab}b^{y5}\qquad \dots\dots\dots\dots\dots\ (12)$$

This can be generalized as:

$$C\ (N) = \ (ab)^{x1+x2+...+xn-1}\ b^{y1+y2+..+yn}\ \ N\ +$$

$$b\frac{1-(ab\ )^{x1}}{1-ab}(ab)^{x2+x3+..+xn-1}b^{y2+y3+..+yn}\ +$$

$$b\frac{1-(ab\ )^{x2}}{1-ab}(ab)^{x3+x4+..+xn-1}b^{y3+y4+..+yn}\ +\ .\ .\ .$$

$$b\frac{1-(ab\ )^{xn-1}}{1-ab}b^{yn}\qquad \dots\dots\dots\dots..\ (13)$$

To illustrate what *x1*, *x2*, *x3*, ... , *y1*, *y2*, *y3*, . . . are let us see what these are for a particular starting number.

Consider an odd starting number 715. The Collatz sequence will be:

715, 2146, 1073, 3220, 1610, 805, 2416, 1208, 604,302,151,454,227,682,341, 1024,512,256,128,

64,32,16,8,4,2,1,4,2,1, 4,2,1,4,2,. . .

We can see that:

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13	x14			
	2	**0**	**1**	**0**	**3**	**0**	**1**	**0**	**1**	**0**	**1**						
		y1	y2	y3	y4	y5	y6	y7	y8	y9	y10	y11	y12	y13			
		1		**3**	**0**	**9**	**0**	**1**	**0**	**1**	**0**						

Consider another odd number 1433.

1433,4300,2150,1075,3226,1613,4840,2420,1210,605,1816,908,454,227,682,341,1024,512,256,

128,64,32,16,8,4,2,1,4,2,1, . . .

	x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13	x14			
	1	**0**	**2**	**0**	**1**	**0**	**2**	**0**	**1**								
		y1	y2	y3	y4	y5	y6	y7	y8	y9	y10	y11	y12	y13			
	0	**1**	**0**	**2**	**0**	**2**	**0**	**9**		**1**							

Consider an even starting number 1738.

1738,869,2608,1304,652,326,163,490,245,736,368,184,92,46,23,70,35,106,53,160,80,40,20,10,

5,16,8,4,2,1,4,2,1, . . .

	y1	y2	y3	y4	y5	y6	y7	y8	y9	y10	y11	y12	y13	y14			
	1	**0**	**3**	**0**	**4**	**0**	**4**	**0**	**3**	**0**							
		x1	x2	x3	x4	x5	x6	x7	x8	x9	x10	x11	x12	x13			
		1	**0**	**2**	**0**	**3**	**0**	**1**	**0**	**1**							

Test of closed loop

A closed loop is formed if :

$$C(N) = N$$

where C(N) is as expressed in equation (4), (5), (10), or (11), or their rearranged forms and checking the above equality for each.

After substituting a = 3 and b = ½ in the above equations, a valid value of *N* cannot be obtained for any values of *x*'s and *y*'s except for those corresponding to the 1-4-2-1 loop, if there are no loops other than the 1-4-2-1 loop.

Re-writing equation (7) below:

$$C(N) = (ab)^{x1+x2+...+xn} b^{y1+y2+...+yn-1} N +$$

$$b\frac{1-(ab)^{x1}}{1-ab}(ab)^{x2+x3+...+xn} b^{y1+y2+...+yn-1} +$$

$$b\frac{1-(ab)^{x2}}{1-ab}(ab)^{x3+x4+....+xn} b^{y2+y3+...+yn-1} +$$

$$b\frac{1-(ab)^{x3}}{1-ab}(ab)^{x4+x5+...+xn} b^{y3+y4+...+yn-1} + ... +$$

$$b\frac{1-(ab)^{(xn-1)}}{1-ab}(ab)^{xn} b^{yn-1} + b\frac{1-(ab)^{xn}}{1-ab}$$

Let us test it for $x_1 = y_1 = 1$,which is the case of the 1-4-2-1 loop:

$$C(N) = (ab)^1 b^1 N + b\frac{1-(ab)^1}{1-ab} b^1$$

$$C(N) = N$$

$$(ab)^1 b^1\, N \; + \; b\frac{1-(ab\,)^1}{1-ab}b^1 \; = \; N$$

$$(\,\frac{3}{2})^1(\frac{1}{2})^1\, N \; + \; \frac{1}{2}\frac{1-\left(\frac{3}{2}\right)^1}{1-\frac{3}{2}}(\,\frac{1}{2})^1 \; = \; N$$

$$\frac{3}{4}\, N \; + \; \frac{1}{4} \; = \; N$$

$$\frac{1}{4} \; = \; N\left(1-\frac{3}{4}\right) \quad \Longrightarrow \quad N = 1$$

Thus we have reduced the Collatz conjecture to the problem of whether C(N) = N in the above equations, for some positive integer N. We need to show that C(N) = N only for the 1-4-2-1 loop, i.e. if no other loops exist (which is the more likely case).

$$C(N) = N$$

Using C(N) from equation (7):

$$C\,(N\,) = \; (ab)^{x1+x2+\,.\,.\,.+xn}\, b^{y1+y2+\,.\,.\,.+yn-1} \;\; N \;\; +$$

$$b\frac{1-(ab\,)^{x1}}{1-ab}(ab)^{x2+x3+\,.\,.\,.+xn} b^{y1+y2+\;.\,.\,.+yn-1} \;\; +$$

$$b\frac{1-(ab\,)^{x2}}{1-ab}(ab)^{x3+x4+\,.\,.\,.\,.+xn} b^{y2+y3+.\,.\,.+yn-1} \; +$$

$$b\frac{1-(ab\,)^{x3}}{1-ab}(ab)^{x4+x5+\,.\,.\,.+xn} b^{y3+y4+\,.\,.\,.+yn-1} \;\; + \;\; . \;\; . \;\; . \; +$$

$$b\frac{1-(ab\,)^{(xn-1)}}{1-ab}(ab)^{xn} b^{yn-1} \;\; + \;\;\; b\frac{1-(ab\,)^{xn}}{1-ab}$$

But

$$\frac{b}{1-ab} = \frac{1/2}{1-(\frac{3}{2})} = -1$$

Therefore,

$$C(N) = (ab)^{x1+x2+...+xn} b^{y1+y2+...+yn-1} N +$$

$$-(1-(ab)^{x1})(ab)^{x2+x3+...+xn} b^{y1+y2+...+yn-1} +$$

$$-(1-(ab)^{x2})(ab)^{x3+x4+....+xn} b^{y2+y3+...+yn-1} +$$

$$-(1-(ab)^{x3})(ab)^{x4+x5+...+xn} b^{y3+y4+...+yn-1} + ... +$$

$$-(1-(ab)^{xn-1})(ab)^{xn} b^{yn-1} - (1-(ab)^{xn})$$

$$\Longrightarrow C(N) = (ab)^{x1+x2+...+xn} b^{y1+y2+...+yn-1} N +$$

$$+((ab)^{x1}-1)(ab)^{x2+x3+...+xn} b^{y1+y2+...+yn-1} +$$

$$+((ab)^{x2}-1)(ab)^{x3+x4+....+xn} b^{y2+y3+...+yn-1} +$$

$$+((ab)^{x3}-1)(ab)^{x4+x5+...+xn} b^{y3+y4+...+yn-1} + ... +$$

$$+((ab)^{xn-1}-1)(ab)^{xn} b^{yn-1} + ((ab)^{xn}-1) \quad \dots\dots\dots\dots\dots \quad (14)$$

$$\Longrightarrow \quad C\,(N\,) = \left(\frac{3}{2}\right)^{x1+x2+\,.\,.\,.+xn} \left(\frac{1}{2}\right)^{y1+y2+\,.\,.\,.+yn-1} \; N \; +$$

$$+ \left(\left(\frac{3}{2}\right)^{x1} - 1 \right) \left(\frac{3}{2}\right)^{x2+x3+\,.\,.\,.+xn} \left(\frac{1}{2}\right)^{y1+y2+\,.\,.\,.+yn-1} \; +$$

$$+ \left(\left(\frac{3}{2}\right)^{x2} - 1 \right) \left(\frac{3}{2}\right)^{x3+x4+\,.\,.\,.\,.+xn} \left(\frac{1}{2}\right)^{y2+y3+.\,.\,.+yn-1} \; +$$

$$+ \left(\left(\frac{3}{2}\right)^{x3} - 1 \right) \left(\frac{3}{2}\right)^{x4+x5+\,.\,.\,.+xn} \left(\frac{1}{2}\right)^{y3+y4+\,.\,.\,.+yn-1} \; + \; . \; . \; . \; +$$

$$+ \left(\left(\frac{3}{2}\right)^{xn-1} - 1 \right) \left(\frac{3}{2}\right)^{xn} \left(\frac{1}{2}\right)^{yn-1} \; + \; \left(\left(\frac{3}{2}\right)^{xn} - 1 \right)$$

For closed loop to occur,

$$C(N) = \; N$$

$$\Longrightarrow \quad N \; =$$

$$\frac{\left(\left(\frac{3}{2}\right)^{x1} - 1 \right) \left(\frac{3}{2}\right)^{x2+x3+\,.\,.\,.+xn} \left(\frac{1}{2}\right)^{y1+y2+\,.\,.\,.+yn-1} + \left(\left(\frac{3}{2}\right)^{x2} - 1 \right) \left(\frac{3}{2}\right)^{x3+x4+\,.\,.\,.\,.+xn} \left(\frac{1}{2}\right)^{y2+y3+.\,.\,.+yn-1} + \,.\,.\,.}{1 \; - \; \left(\frac{3}{2}\right)^{x1+x2+\,.\,.\,.+xn} \left(\frac{1}{2}\right)^{y1+y2+\,.\,.\,.+yn-1}}$$

$$= \frac{\left(\frac{3^{x1} - 2^{x1}}{2^{x1}} \right) \left(\frac{3}{2}\right)^{x2+x3+\,.\,.\,.+xn} \left(\frac{1}{2}\right)^{y1+y2+\,.\,.\,.+yn-1} + \left(\frac{3^{x2} - 2^{x2}}{2^{x2}} \right) \left(\frac{3}{2}\right)^{x3+x4+\,.\,.\,.\,.+xn} \left(\frac{1}{2}\right)^{y2+y3+.\,.\,.+yn-1} + \,.\,.\,.}{\frac{2^{x1+x2+\,.\,.+y1+y2+\,.\,.\,.} - 3^{x1+x2+\,.\,.\,.}}{2^{x1+x2+\,.\,.+y1+y2+\,.\,.\,.}}}$$

$$= \quad 2^{x1+x2+\,.\,.+y1+y2+\,.\,.\,.} \quad *$$

$$\frac{\left(\frac{3^{x1}-2^{x1}}{2^{x1}}\right)\left(\frac{3}{2}\right)^{x2+x3+\,.\,.\,.}\left(\frac{1}{2}\right)^{y1+y2+\,.\,.\,.} + \left(\frac{3^{x2}-2^{x2}}{2^{x2}}\right)\left(\frac{3}{2}\right)^{x3+x4+\,.\,.\,.\,.}\left(\frac{1}{2}\right)^{y2+y3+.\,.\,.} + \,.\,.\,.}{2^{x1+x2+\,.\,.+y1+y2+\,.\,.\,.} - 3^{x1+x2+\,.\,.\,.}}$$

$$\Longrightarrow$$

$$N =$$

$$\frac{(3^{x1} - 2^{x1})3^{x2+x3+\,.\,.\,.} + 2^{x1+y1}(3^{x2} - 2^{x2})3^{x3+x4+\,.\,.\,.} + 2^{x1+x2++y1+y2}(3^{x3} - 2^{x3})3^{x4+x5+\,.\,.\,.}\,.\,.\,.}{2^{x1+x2+\,.\,.+y1+y2+\,.\,.\,.} - 3^{x1+x2+\,.\,.\,.}}$$

$$.\,.\,.\,.\,.\,.\,.\,. \qquad (15)$$

Since we have assumed that the starting number N is odd in this case,

$$x1 \neq 0$$

We have also seen that if *x2* is different from zero *y1* must be zero and if *x2* is equal to zero *y1* must be different from zero. This applies to all pairs:

$$(x2, y1)\,, (x3, y2), (x4, y3), \quad . \quad . \quad .$$

Now we determine which element in each pair must be zero and which one must be different from zero.

Consider the first pair (*x2, y1*). Suppose that *y1* = 0 and *x2* ≠ 0. This is impossible since the denominator of equation (15) becomes negative, making N negative. Note that the numerator is always positive.

$$2^{x1+x2} - 3^{x1+x2} < 0$$

Therefore, *y1* must be different from zero and *x2* must be equal to zero. The same holds for all the x_i's in the pairs!!!

That is:

$$x2 = x3 = x4 = \ .\ .\ .\ = \ 0$$

In this case equation (15) becomes:

$$N = \frac{(3^{x1} - 2^{x1})3^0}{2^{x1+y1+y2+\ .\ .\ .} \ - \ 3^{x1}}$$

So far we have proved that

$$x1 \neq 0\ , \qquad y1 \neq 0$$

What about *y2*, *y3*, *y4*, . . . ?
These all must be zero, otherwise N will become negative, which is invalid!

$$y2 = y3 = y4 = \ .\ .\ .\ = \ 0$$

Therefore, equation (15) becomes:

$$N = \frac{3^{x1} - 2^{x1}}{2^{x1+y1} \ - \ 3^{x1}} \qquad \ldots\ldots\ldots\ldots\ldots \ (16)$$

Now we need to solve equation (15). We have proved that x1 and x2 must be different from zero, but we haven't determined their specific values yet.

Suppose that :

$$x1 = 1$$

Then the only possible value of *y1* is:

$$y1 = 1$$

This is because any value of *y1* greater than 1 will return a fractional N which is not valid.

In this case:

$$N = \frac{3^1 - 2^1}{2^2 \ - \ 3^1} = \frac{1}{4 \ - \ 3} = 1$$

So we have proved that number 1 is one of the odd numbers that leads to a closed loop of Collatz sequence!!!

Is $N = 1$ the only solution of equation (16)?

Thus, the problem of Collatz conjecture is reduced to a problem of solving a simple exponential equation!!!

The author has checked some values of $x1$ and $x2$ and found out that this always gave either fractional or negative values of N.

Therefore, if $N = 1$ is the only solution of equation (16), we have proved that number 1 is the only odd number that will lead to a closed loop of the Collatz sequence !!!

It follows that numbers 2 and 4 are the only even numbers leading to a closed loop of the Collatz sequence !!!

Conclusion

Proving the Collatz conjecture requires proving that:

1. The sequence will not diverge to infinity
2. There are no other closed loops other than the 1-4-2-1 loop.

We have presented an intuitive yet conclusive proof of the first. For the second, we have reduced the Collatz conjecture to a problem of solving a simple exponential equation. Thus we have completely proved the Collatz conjecture in this paper.

Thanks to Almighty God Jesus Christ and His Mother Our Lady Saint Virgin Mary

References

Appendix

Sequence	0 if even , 1 if odd	Sequence	0 if even , 1 if odd	Sequence	0 if even , 1 if odd	Sequence	0 if even , 1 if odd
5.64359E+11		76736783426		4.56734E+11		87654768546	
1.69308E+12	0	38368391713	1	1.3702E+12	0	43827384273	1
8.46538E+11	0	1.15105E+11	0	6.85101E+11	0	1.31482E+11	0
4.23269E+11	1	57552587570	0	3.42551E+11	0	65741076410	0
1.26981E+12	0	28776293785	1	1.71275E+11	0	32870538205	1
6.34903E+11	0	86328881356	0	85637673148	0	98611614616	0
3.17452E+11	1	43164440678	0	42818836574	0	49305807308	0
9.52355E+11	0	21582220339	1	21409418287	1	24652903654	0
4.76178E+11	0	64746661018	0	64228254862	0	12326451827	1
2.38089E+11	0	32373330509	1	32114127431	1	36979355482	0
1.19044E+11	0	97119991528	0	96342382294	0	18489677741	1
59522190127	1	48559995764	0	48171191147	1	55469033224	0
1.78567E+11	0	24279997882	0	1.44514E+11	0	27734516612	0
89283285191	1	12139998941	1	72256786721	1	13867258306	0
2.6785E+11	0	36419996824	0	2.1677E+11	0	6933629153	1
1.33925E+11	1	18209998412	0	1.08385E+11	0	20800887460	0
4.01775E+11	0	9104999206	0	54192590041	1	10400443730	0
2.00887E+11	1	4552499603	1	1.62578E+11	0	5200221865	1
6.02662E+11	0	13657498810	0	81288885062	0	15600665596	0
3.01331E+11	0	6828749405	1	40644442531	1	7800332798	0
1.50666E+11	1	20486248216	0	1.21933E+11	0	3900166399	1
4.51997E+11	0	10243124108	0	60966663797	1	11700499198	0
2.25998E+11	0	5121562054	0	1.829E+11	0	5850249599	1
1.12999E+11	1	2560781027	1	91449995696	0	17550748798	0
3.38997E+11	0	7682343082	0	45724997848	0	8775374399	1
1.69499E+11	0	3841171541	1	22862498924	0	26326123198	0
84749368366	0	11523514624	0	11431249462	0	13163061599	1
42374684183	1	5761757312	0	5715624731	1	39489184798	0
1.27124E+11	0	2880878656	0	17146874194	0	19744592399	1
63562026275	1	1440439328	0	8573437097	1	59233777198	0
1.90686E+11	0	720219664	0	25720311292	0	29616888599	1
95343039413	1	360109832	0	12860155646	0	88850665798	0
2.86029E+11	0	180054916	0	6430077823	1	44425332899	1
1.43015E+11	0	90027458	0	19290233470	0	1.33276E+11	0
71507279560	0	45013729	1	9645116735	1	66637999349	1
35753639780	0	135041188	0	28935350206	0	1.99914E+11	0
17876819890	0	67520594	0	14467675103	1	99956999024	0

8938409945	1	33760297	1	43403025310	0	49978499512	0
26815229836	0	101280892	0	21701512655	1	24989249756	0
13407614918	0	50640446	0	65104537966	0	12494624878	0
6703807459	1	25320223	1	32552268983	1	6247312439	1
20111422378	0	75960670	0	97656806950	0	18741937318	0
10055711189	1	37980335	1	48828403475	1	9370968659	1
30167133568	0	113941006	0	1.46485E+11	0	28112905978	0
15083566784	0	56970503	1	73242605213	1	14056452989	1
7541783392	0	170911510	0	2.19728E+11	0	42169358968	0
3770891696	0	85455755	1	1.09864E+11	0	21084679484	0
1885445848	0	256367266	0	54931953910	0	10542339742	0
942722924	0	128183633	1	27465976955	1	5271169871	1
471361462	0	384550900	0	82397930866	0	15813509614	0
235680731	1	192275450	0	41198965433	1	7906754807	1
707042194	0	96137725	1	1.23597E+11	0	23720264422	0
353521097	1	288413176	0	61798448150	0	11860132211	1
1060563292	0	144206588	0	30899224075	1	35580396634	0
530281646	0	72103294	0	92697672226	0	17790198317	1
265140823	1	36051647	1	46348836113	1	53370594952	0
795422470	0	108154942	0	1.39047E+11	0	26685297476	0
397711235	1	54077471	1	69523254170	0	13342648738	0
1193133706	0	162232414	0	34761627085	1	6671324369	1
596566853	1	81116207	1	1.04285E+11	0	20013973108	0
1789700560	0	243348622	0	52142440628	0	10006986554	0
894850280	0	121674311	1	26071220314	0	5003493277	1
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223712570	0	182511467	1	39106830472	0	7505239916	0
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335568856	0	273767201	1	9776707618	0	1876309979	1
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166088	0	4877972	0	29033479	1	928667	1
83044	0	2438986	0	87100438	0	2786002	0
41522	0	1219493	1	43550219	1	1393001	1
20761	1	3658480	0	130650658	0	4179004	0
62284	0	1829240	0	65325329	1	2089502	0
31142	0	914620	0	195975988	0	1044751	1
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46714	0	228655	1	48993997	1	1567127	1
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70072	0	342983	1	73490996	0	2350691	1
35036	0	1028950	0	36745498	0	7052074	0
17518	0	514475	1	18372749	1	3526037	1
8759	1	1543426	0	55118248	0	10578112	0
26278	0	771713	1	27559124	0	5289056	0
13139	1	2315140	0	13779562	0	2644528	0
39418	0	1157570	0	6889781	1	1322264	0

19709	1	578785	1	20669344	0	661132	0
59128	0	1736356	0	10334672	0	330566	0
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11087	1	325567	1	1937752	0	371888	0
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24947	1	732527	1	726658	0	23243	1
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37421	1	1098791	1	1089988	0	34865	1
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56132	0	1648187	1	272497	1	52298	0
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42100	0	7416844	0	204373	1	39224	0
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31576	0	5562634	0	153280	0	4903	1
15788	0	2781317	1	76640	0	14710	0
7894	0	8343952	0	38320	0	7355	1
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11842	0	2085988	0	9580	0	11033	1
5921	1	1042994	0	4790	0	33100	0
17764	0	521497	1	2395	1	16550	0
8882	0	1564492	0	7186	0	8275	1
4441	1	782246	0	3593	1	24826	0
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6662	0	1173370	0	5390	0	37240	0
3331	1	586685	1	2695	1	18620	0
9994	0	1760056	0	8086	0	9310	0
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7496	0	220007	1	6065	1	6983	1
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1874	0	330011	1	9098	0	10475	1
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2812	0	495017	1	13648	0	15713	1
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703	1	742526	0	3412	0	23570	0
2110	0	371263	1	1706	0	11785	1

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3166	0	556895	1	2560	0	17678	0
1583	1	1670686	0	1280	0	8839	1
4750	0	835343	1	640	0	26518	0
2375	1	2506030	0	320	0	13259	1
7126	0	1253015	1	160	0	39778	0
3563	1	3759046	0	80	0	19889	1
10690	0	1879523	1	40	0	59668	0
5345	1	5638570	0	20	0	29834	0
16036	0	2819285	1	10	0	14917	1
8018	0	8457856	0	5	1	44752	0
4009	1	4228928	0	16	0	22376	0
12028	0	2114464	0	8	0	11188	0
6014	0	1057232	0	4	0	5594	0
3007	1	528616	0	2	0	2797	1
9022	0	264308	0	1	1	8392	0
4511	1	132154	0	4	0	4196	0
13534	0	66077	1	2	0	2098	0
6767	1	198232	0	1	1	1049	1
20302	0	99116	0	4	0	3148	0
10151	1	49558	0	2	0	1574	0
30454	0	24779	1	1	1	787	1
15227	1	74338	0	4	0	2362	0
45682	0	37169	1	2	0	1181	1
22841	1	111508	0	1	1	3544	0
68524	0	55754	0	4	0	1772	0
34262	0	27877	1	2	0	886	0
17131	1	83632	0	1	1	443	1
51394	0	41816	0	4	0	1330	0
25697	1	20908	0	2	0	665	1
77092	0	10454	0	1	1	1996	0
38546	0	5227	1	4	0	998	0
19273	1	15682	0	2	0	499	1
57820	0	7841	1	1	1	1498	0
28910	0	23524	0	4	0	749	1
14455	1	11762	0	2	0	2248	0
43366	0	5881	1	1	1	1124	0
21683	1	17644	0	4	0	562	0
65050	0	8822	0	2	0	281	1
32525	1	4411	1	1	1	844	0
97576	0	13234	0	4	0	422	0
48788	0	6617	1	2	0	211	1
24394	0	19852	0	1	1	634	0

12197	1	9926	0	4	0	317	1
36592	0	4963	1	2	0	952	0
18296	0	14890	0	1	1	476	0
9148	0	7445	1	4	0	238	0
4574	0	22336	0	2	0	119	1
2287	1	11168	0	1	1	358	0
6862	0	5584	0	4	0	179	1
3431	1	2792	0	2	0	538	0
10294	0	1396	0	1	1	269	1
5147	1	698	0	4	0	808	0
15442	0	349	1	2	0	404	0
7721	1	1048	0	1	1	202	0
23164	0	524	0	4	0	101	1
11582	0	262	0	2	0	304	0
5791	1	131	1	1	1	152	0
17374	0	394	0	4	0	76	0
8687	1	197	1	2	0	38	0
26062	0	592	0	1	1	19	1
13031	1	296	0	4	0	58	0
39094	0	148	0	2	0	29	1
19547	1	74	0	1	1	88	0
58642	0	37	1	4	0	44	0
29321	1	112	0	2	0	22	0
87964	0	56	0	1	1	11	1
43982	0	28	0	4	0	34	0
21991	1	14	0	2	0	17	1
65974	0	7	1	1	1	52	0
32987	1	22	0	4	0	26	0
98962	0	11	1	2	0	13	1
49481	1	34	0	1	1	40	0
148444	0	17	1	4	0	20	0
74222	0	52	0	2	0	10	0
37111	1	26	0	1	1	5	1
111334	0	13	1	4	0	16	0
55667	1	40	0	2	0	8	0
167002	0	20	0	1	1	4	0
83501	1	10	0	4	0	2	0
250504	0	5	1	2	0	1	1
125252	0	16	0	1	1	4	0
62626	0	8	0	4	0	2	0
31313	1	4	0	2	0	1	1
93940	0	2	0	1	1	4	0
46970	0	1	1	4	0	2	0

23485	1	4	0	2	0	1	1
70456	0	2	0	1	1	4	0
35228	0	1	1	4	0	2	0
17614	0	4	0	2	0	1	1
8807	1	2	0	1	1	4	0
26422	0	1	1	4	0	2	0
13211	1	4	0	2	0	1	1
39634	0	2	0	1	1	4	0
19817	1	1	1	4	0	2	0
59452	0	4	0	2	0	1	1
29726	0	2	0	1	1	4	0
14863	1	1	1	4	0	2	0
44590	0	4	0	2	0	1	1
22295	1	2	0	1	1	4	0
66886	0	1	1	4	0	2	0
33443	1	4	0	2	0	1	1
100330	0	2	0	1	1	4	0
50165	1	1	1	4	0	2	0
150496	0	4	0	2	0	1	1
75248	0	2	0	1	1	4	0
37624	0	1	1	4	0	2	0
18812	0	4	0	2	0	1	1
9406	0	2	0	1	1	4	0
4703	1	1	1	4	0	2	0
14110	0	4	0	2	0	1	1
7055	1	2	0	1	1	4	0
21166	0	1	1	4	0	2	0
10583	1	4	0	2	0	1	1
31750	0	2	0	1	1	4	0
15875	1	1	1	4	0	2	0
47626	0	4	0	2	0	1	1
23813	1	2	0	1	1	4	0
71440	0	1	1	4	0	2	0
35720	0	4	0	2	0	1	1
17860	0	2	0	1	1	4	0
8930	0	1	1	4	0	2	0
4465	1	4	0	2	0	1	1
13396	0	2	0	1	1	4	0
6698	0	1	1	4	0	2	0
3349	1	4	0	2	0	1	1
10048	0	2	0	1	1	4	0
5024	0	1	1	4	0	2	0
2512	0	4	0	2	0	1	1

1256	0	2	0	1	1	4	0
628	0	1	1	4	0	2	0
314	0	4	0	2	0	1	1
157	1	2	0	1	1	4	0
472	0	1	1	4	0	2	0
236	0	4	0	2	0	1	1
118	0	2	0	1	1	4	0
59	1	1	1	4	0	2	0
178	0	4	0	2	0	1	1
89	1	2	0	1	1	4	0
268	0	1	1	4	0	2	0
134	0	4	0	2	0	1	1
67	1	2	0	1	1	4	0
202	0	1	1	4	0	2	0
101	1	4	0	2	0	1	1
304	0	2	0	1	1	4	0
152	0	1	1	4	0	2	0
76	0	4	0	2	0	1	1
38	0	2	0	1	1	4	0
19	1	1	1	4	0	2	0
58	0	4	0	2	0	1	1
29	1	2	0	1	1	4	0
88	0	1	1	4	0	2	0
44	0	4	0	2	0	1	1
22	0	2	0	1	1	4	0
11	1	1	1	4	0	2	0
34	0	4	0	2	0	1	1
17	1	2	0	1	1	4	0
52	0	1	1	4	0	2	0
26	0	4	0	2	0	1	1
13	1	2	0	1	1	4	0
40	0	1	1	4	0	2	0
20	0	4	0	2	0	1	1
10	0	2	0	1	1	4	0
5	1	1	1	4	0	2	0
16	0	4	0	2	0	1	1
8	0	2	0	1	1	4	0
4	0	1	1	4	0	2	0
2	0	4	0	2	0	1	1
1	1	2	0	1	1	4	0
4	0	1	1	4	0	2	0