Quantum geometry of a Planck-scale black hole

Christian Lembke

Lüneburg, Germany

ORCID: 0009-0007-4658-9585

Published online: May 22, 2023

Abstract

The size of the Planck area can be determined indirectly via the measured value of the gravitational constant. Its size is thus subject to the measurement accuracy of the gravitational constant. In this work, the Planck area is derived from the quantum geometry of theoretical black holes in the Planck scale range and will be calculated exactly as a physical constant.

The exact value of the Planck area makes it possible to calculate the gravitational constant with the two known natural constants, reduced Planck constant and the speed of light.

From the quantum geometric description of black holes, the radius to the surface of a black hole can be calculated.

Keywords

Bekenstein-Hawking entropy, gravitational constant, Planck area, Planck length, quantum geometry, quantum gravity, black hole, Schwarzschild radius

1. Introduction

The general theory of relativity describes gravity as a property of curved four-dimensional spacetime. Spacetime is described mathematically by Einstein's field equations. Many scientific experiments and cosmological investigations have proven the correctness of the general theory of relativity [1]. In some areas of cosmology, such as in the case of massive bodies, the black holes, the field equations behind the event horizon result in a singularity and thus have no statement about spacetime. Another theory is needed to describe these areas.

To calculate gravity, knowledge of the magnitude of the gravitational constant is required. For over 200 years [2], about 300 experiments [3] have been carried out to measure the gravitational constant. In the process, the methods for measuring the gravitational constant have been further developed and made more precise. Nevertheless, the accuracy of the measurements is low compared to the values of most other natural constants. The current value of the gravitational constant was determined by the Committee on Data for Science and Technology CODATA, 2018 [4].

$$G_{CODATA} = 6{,}67430(15)\ \cdot 10^{-11}\,\frac{m^3}{kg\ s^2} \tag{1}$$

The Planck units, which depend on the gravitational constant, could also only be determined with the accuracy of the measured gravitational constant. Alternative approaches to determine the Planck length by measurement were first described by Espen Gaarder Haug [5], [6].

Max Planck (1858-1947) introduced the Planck units named after him [7]. The Planck units show the physical laws on minimal space and time quantities (Planck scale) in dependence of the constants, reduced Planck constant ħ, gravitational constant G and speed of light c. The essential quantities for spacetime are according to CODATA [4]:

$$l_P = \sqrt{\frac{\hbar\, G}{c^3}} \qquad l_P \approx 1.616\,255(18) \cdot 10^{-35}\,\text{m, Planck length} \tag{2}$$

$$t_P = \sqrt{\frac{\hbar\, G}{c^5}} \qquad t_P \approx 5.391\,247(60) \cdot 10^{-44}\,\text{s, Planck time} \tag{3}$$

$$m_P = \sqrt{\frac{\hbar\, c}{G}} \qquad m_P \approx 2.176\,434(24) \cdot 10^{-8}\,\text{kg, Planck mass} \tag{4}$$

From the square of the Planck length the Planck area is calculated A_P with

$$A_P = {l_P}^2 = \frac{\hbar\, G}{c^3} \tag{5}$$

In the following, we attempt to derive the Planck units from the quantum geometry of a theoretically smallest and simplest black hole. On the one hand, the calculation of this black hole is carried out on a relativistic basis with the help of the Schwarzschild metric and on the other hand thermodynamically via the state variable of entropy. With the exact knowledge of the Planck units, it is possible to calculate the gravitational constant according to equation (5).

2. Schwarzschild radius in the Planck scale range

The radius, named after Karl Schwarzschild (1873-1916), is a solution of the Einstein field equation for a spherical, symmetrical, and stationary black hole and describes the distance from the centre of a black hole, from which no information can penetrate to the outside. This area is also called event horizon because events beyond this boundary are no longer visible to an outside observer and no information is transmitted. The Schwarzschild radius r_s of the mass M is calculated according to [1]

$$r_s = \frac{2\, G\, M}{c^2} \tag{6}$$

The Schwarzschild radius $r_{s(m_P)}$ of a black hole with the theoretically smallest mass, the Planck mass m_P is given by

$$r_{s(m_P)} = \frac{2\, G\, m_P}{c^2} \tag{7}$$

By squaring equation (4) and substituting into the squared Eq. (7) it follows

$${r_{s(m_P)}}^2 = \frac{4\, \hbar\, G}{c^3} \tag{8}$$

and after substituting eq. (5) into Eq. (8) follows

$${r_{s(m_P)}}^2 = 4\, A_P = 4\, {l_P}^2 \tag{9}$$

The square root of Eq. (9) gives the Schwarzschild radius

$$r_{s(m_P)} = 2\, l_P \tag{10}$$

3. Entropy of black holes in the Planck scale range

Black holes are described with the solutions of the Einstein field equations. The Schwarzschild metric is one of these descriptions. However, a black hole can also be described thermodynamically via the state variable of entropy [8]. There are various descriptive interpretations for the entropy of a system. Entropy can be understood as a measure of the disorder of a system [9]. In statistical physics, entropy is a measure of the different microstates of a system [10]. A system with a large number of microstates is thus more disorderly and has a higher entropy than an ordered system with few microstates.

The calculation of the microstates Ω is carried out according to the calculation rules of combinatorics for the permutations of n elements [11].

$$\Omega = n! \tag{11}$$

The thermodynamic consideration of a theoretically smallest black hole leads to a system with minimal entropy. This means that the number of equally probable microstates Ω is minimal. The smallest number of different states is two. For two different states the number of equally probable microstates is calculated with

$$\Omega = 2! \tag{12}$$

An example of a three-dimensional geometric body that has only two different microstates is a tetrahedron. Of all the space-enclosing polygons, the tetrahedron is the simplest body [11]. It consists of four equilateral triangles and can be depicted in two different two-dimensional developments (Figure 1). This development is also called a net.

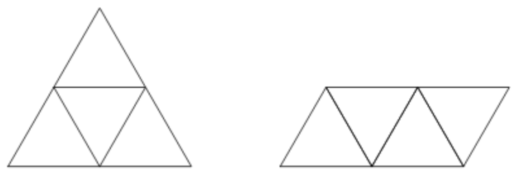

Figure 1

The number n of nets into which a body can be decomposed is equivalent to the number of different microstates. For the tetrahedron with two nets is $\Omega = 2!$

Ludwig Bolzmann (1844-1906) described the relationship between entropy and the number of possible microstates as early as 1877 [12] and Max Planck elaborated on this approach a few years later [9].

$$S = ln\, \Omega \ \cdot k_B \tag{13}$$

For a black hole with two possible microstates, the following applies

$$S_{bh(2!)} = ln\ 2! \ \cdot k_B \tag{14}$$

A black hole according to the Schwarzschild metric is defined only by its mass. With an increase in mass, the surface of the event horizon increases proportionally [1]. Jacob D. Bekenstein (1947-2015) published his theory "Black Holes and Entropy" [8] in 1973, in which he established the connection between the entropy of a black hole and its surface. Based on the work of Stephen Hawking (1942-2018) on black hole entropy [13], the theory is referred to as the Bekenstein-Hawking entropy.

According to the Bekenstein-Hawking entropy, the entropy S_{bh} of a black hole depends only on the surface A_{bh} of a black hole [10].

$$S_{bh} = \frac{k_B\, c^3 A_{bh}}{4\, \hbar\, G} \tag{15}$$

All other quantities, such as the Boltzmann constant k_B, the reduced Planck constant ħ, the gravitational constant G and the speed of light c are natural constants.

For the entropy of a theoretically smallest black hole, thus applies

$$S_{bh(2!)} = \frac{k_B\, c^3 A_{bh(2!)}}{4\, \hbar\, G} \tag{16}$$

By equating Eq. (14) and Eq. (16), it follows that,

$$ln\ 2!\ \cdot k_B = \frac{k_B\, c^3 A_{bh(2!)}}{4\, \hbar\, G} \tag{17}$$

and for a spherical surface

$$ln\ 2 = \frac{c^3\, \pi\, {r_{bh(2!)}}^2}{\hbar\, G} \tag{18}$$

After rearranging Eq. (18), we get for the square radius ${r_{bh(2!)}}^2$ of a black hole

$${r_{bh(2!)}}^2 = \frac{\hbar\, G \ln 2}{c^3\, \pi} \tag{19}$$

and after inserting Eq. (5) it follows

$${r_{bh(2!)}}^2 = {l_P}^2 \frac{\ln 2}{\pi} \tag{20}$$

4. The Planck area and gravitational constant in quantum gravity

If one compares the square of the area ${r_{s(m_P)}}^2$ of a theoretically smallest black hole that was relativistically calculated on the basis of the Schwarzschild metric with mass m_P, with the square of the area of a black hole ${r_{bh(2!)}}^2$ of a black hole that thermodynamically has the entropy $S_{bh(2!)} = ln\ 2! \cdot k_B$ i.e. has the smallest number of different microstates, it becomes clear that the surface areas are not equal. The Bekenstein-Hawking entropy assumes that the entropy of a black hole calculated according to equation (15) is equal to the surface of the event horizon [10]. However, in the microscopic description of a black hole, surface differences become apparent which are considered below.

The difference ΔA of the areas ${r_{s(m_P)}}^2$ and ${r_{bh(2!)}}^2$ results from

$$\Delta A = {r_{s(m_P)}}^2 - {r_{bh(2!)}}^2 \tag{21}$$

and with Eq. (9) and Eq. (20) follows

$$\Delta A = 4\, {l_P}^2 - {l_P}^2 \frac{\ln 2}{\pi} \tag{22}$$

with the numerical value for l_P according to Eq. (2) results in ΔA according to CODATA

$\Delta A = 9{,}872075\ \ldots \cdot 10^{-70} m^2$

The above-mentioned sequence of numbers is very similar to the result from π^2. Therefore, the heuristic assumption is made that

$$\Delta A = \pi^2 \cdot 10^{-70} m^2 \tag{23}$$

corresponds. With this presumption the connection follows

$$\pi^2 \cdot 10^{-70} m^2 = 4\, {l_P}^2 - {l_P}^2 \frac{\ln 2}{\pi} \tag{24}$$

and after transformation one obtains

$$l_P{}^2 = A_P = \frac{\pi^2 \cdot 10^{-70}}{4 - \frac{\ln 2}{\pi}} \tag{25}$$

Knowing the Planck area according to equation (25), the Planck length can be calculated from the root of the area. The Planck length is calculated with

$$l_P = \frac{\pi \cdot 10^{-35}}{\sqrt{4 - \frac{\ln 2}{\pi}}} \tag{26}$$

$$l_P = 1{,}615996771 \ldots \cdot 10^{-35}\ \mathrm{m} \tag{27}$$

The gravitational constant can be calculated from the Planck area and the known natural constants, reduced Planck constant ħ and the speed of light c, without a measurement method, only on a numerical basis. After transforming equation (5), one obtains for the gravitational constant

$$G = \frac{c^3 l_P{}^2}{\hbar} \tag{28}$$

After inserting Eq. (25) it follows

$$G = \frac{c^3\ \pi^2 \cdot 10^{-70}}{\hbar\left(4 - \frac{ln\ 2}{\pi}\right)} \tag{29}$$

$$G = 6{,}672167267 \ldots \cdot 10^{-11} \frac{m^3}{kg\ s^2} \tag{30}$$

The calculated magnitude of the gravitational constant is about 320 ppm lower than the official value according to CODATA, 2018 [4]. Current measurements by means of atomic interferometry yield a value of $G = 6{,}67191(77)(65) \cdot 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$ [3]. The difference between the measured value and the calculated gravitational constant is about 40 ppm. Future measurements of the gravitational constant should the calculated value and thus also the quantum geometry of black holes described above.

5. The surface of a black hole

By observing a simple and theoretically smallest black hole, geometric relationships between the surface of a black hole and its event horizon become clear. From the connection of equation (24) it follows, that the surface areas of $r_{bh(2!)}{}^2$ and $\pi^2 \cdot 10^{-70}$ according to the Pythagorean theorem, are equal to the area of the Schwarzschild radius squared $r_{s(m_P)}{}^2$.

$$r_{s(m_P)}{}^2 = r_{bh(2!)}{}^2 + \pi^2 \cdot 10^{-70} \tag{31}$$

The angle between $\pi \cdot 10^{-35}$ and $r_{s(m_P)}$ is α, therefore

$$sin\,\alpha = \frac{r_{bh(2!)}}{r_{s(m_P)}} \tag{32}$$

$$sin\,\alpha = \frac{l_P\sqrt{\frac{\ln 2}{\pi}}}{2\ l_P} \tag{33}$$

$$sin\, \alpha = \sqrt{\frac{\ln 2}{4\, \pi}} \tag{34}$$

From this it follows from Eq. (32) and Eq. (34)

$$r_{bh(2!)} = r_{s(m_P)} \cdot \sqrt{\frac{\ln 2}{4\, \pi}} \tag{35}$$

Assuming that the angle α is a constant quantity, Eq. (6) gives for the radius r_{bh} to the two-dimensional surface of a black hole

$$r_{bh} = \frac{2\, G\, M}{c^2} \sqrt{\frac{\ln 2}{4\, \pi}} \tag{36}$$

This geometric relationship is assumed for all black holes.

6. Summary and Discussion

In this work, the Planck area and Planck length were derived from the entropy of a theoretically smallest black hole. With knowledge of the Planck area, the gravitational constant can be calculated exactly. Thus, the accuracy of the gravitational constant is only dependent on the constants, reduced Planck constant ħ and the speed of light c. From the quantum geometric description of a theoretically smallest black hole, the radius to the two-dimensional surface of a black hole can be calculated as a further result.

The possibility of deriving the gravitational constant from the geometry of black holes points to a quantum gravity that is discrete at Planck level.

According to the Bekenstein-Hawking entropy, the space structure of black holes is two-dimensional and spherical. In the general theory of relativity, the space is three-dimensional and connected with time to a four-dimensional space-time. It seems that black holes can be understood as two-dimensional spheres whose surfaces increase by the input of mass and energy in discrete steps. Between the event horizon according to the Schwarzschild metric and the two-dimensional spherical surface of a black hole, transformation and symmetry enhancement from a three-dimensional to a two-dimensional space probably occurs. This transformation process points to a quantum structure of the space.

The input of mass and energy into a black hole increases its surface area, and thus the entropy of the black hole. This entropy causes an entropic force which can be understood as a measure of gravity [14]. According to this description, a black hole is a gravitational sphere, or Gravisphere for short, with a defined radius that is hidden from outside observers behind its event horizon.

Acknowledgments

I thank my wife Stephanie and my daughters Anna and Emma for their support, their confidence in my work, and their patience over the years. I thank Christiane and Thomas for their moral support and for their proofreading of the many preliminary versions. Last but not least, I would like to thank Patrick, who helped me with proofreading and formatting.

References

[1] Kiefer, C., Gravitation: Himmelsmechanik, Allgemeine Relativitätstheorie, Einstein-Gleichungen, Raumzeit, Orig.-Ausg. Fischer-Taschenbücher Fischer kompakt 15357. Frankfurt am Main: Fischer-Taschenbuch-Verl., 2003.

[2] Cavendish, H., "Experiments to Determine the Density of the Earth. By Henry Cavendish, Esq. F. R. S. and A. S," Philosophical Transactions of the Royal Society of London, vol. 88, pp. 469–526, 1798. http://www.jstor.org/stable/106988

[3] Rosi, G., Sorrentino, F., Cacciapuoti, L., Prevedelli, M., and Tino, G. M., "Precision measurement of the Newtonian gravitational constant using cold atoms," Nature, vol. 510, no. 7506, pp. 518–521, 2014. https://arxiv.org/pdf/1412.7954

[4] Committee on Data for Science and Technology CODATA

[5] Haug, E. G., "Can the Planck Length Be Found Independent of Big G?" APR, vol. 9, no. 6, p. 58, 2017. https://doi.org/10.5539/apr.v9n6p58

[6] Haug, E., Finding the Planck Length Independent of Newton's Gravitational Constant and the Planck Constant: The Compton Clock Model of Matter, 2018. http://dx.doi.org/10.20944/preprints201809.0396.v1

[7] Max Planck, „Über irreversible Strahlungsvorgänge. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin." 1899 - Erster Halbband (Verl. d. Kgl. Akad. d. Wiss., Berlin 1899). Seite 440–480.

[8] Bekenstein, J. D., "Black Holes and Entropy," Phys. Rev. D, vol. 7, no. 8, pp. 2333–2346, 1973. https://link.aps.org/doi/10.1103/PhysRevD.7.2333

[9] Max Planck, „Über das Gesetz der Energieverteilung im Normalspektrum," 1900

[10] Bekenstein, J., "Bekenstein-Hawking entropy," Scholarpedia, http://dx.doi.org/10.4249/scholarpedia.7375

[11] Netz, H., Formeln der Mathematik, 6., verb. Aufl. München, Wien: Hanser, 1986

[12] Ludwig Bolzmann, „Über die Beziehungen zwischen dem zweiten Hauptsatze der mechanischen Wärmetheorie und der Wahrscheinlichkeitsrechnung respektive den Sätzen über das Wämegleichgewicht," Vienna 1877.

[13] Hawking: Gravitational radiation from colliding black holes, https://doi.org/10.1103/PhysRevLett.26.1344

[14] Erik Verlinde, "On the origin of gravity and the laws of Newton," https://doi.org/10.1007/JHEP04(2011)029