PROOFS OF ABC CONJECTURE

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Abstract

Several crucial properties of ABC conjecture are presented and proven. Therefore, the ABC conjecture is proven.

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The abc conjecture says the following. For every positive real number ϵ, and triplet (a, b, c) of pairwise coprime positive integers, with $a+b = c$, holds $k < K(\epsilon) < \infty$, with $k = c/r^{1+\epsilon}$, where $r = \text{rad}(a\,b\,c)$. The conjecture is regarded as unproven [1].

1. Ternary Goldbach Conjecture implies abc conjecture

The ternary Goldbach Conjecture was proven in Ref. [3]. Why? Even if the paper is not published in a journal, the consensus of experts says that the article is accurate. So, any number c (odd or even) can be presented as a sum of four primes $a + q + p + r = c$. Hereby, even primes are allowed.

Let me arrange the prime numbers $a \geq q \geq p \geq r$. Then $c \leq 4\,a$, and

$$k = \frac{c}{r^{1+\epsilon}} \leq \frac{4\,a}{a^{1+\epsilon}\,(\text{rad}((q+p+r)\,c))^{1+\epsilon}} < \frac{4\,a}{(4a)^{1+\epsilon}} < 1\,. \tag{1}$$

Because $\text{rad}((q+p+r)\,c) = \text{rad}(q+p+r)\,\text{rad}(c) > 4$. The $q+p+r \neq 1$, because prime $r \geq 2$. So, for any value of c, there is a triplet $(a, b, c = a+b)$ with $k < 1$. Hereby $k = 0$ as $c \to \infty$. Why? Because $a \to \infty$ implies $c \to \infty$, and $\epsilon \neq 0$.

Notably, the a cannot be a prime factor of c. Why? Because the abc conjecture is formulated for co-primes. But does it mean that my idea is not applicable in some cases? In such cases would be $c = a\,n$, $c \leq 4\,a$, so, $n \leq 4$. Therefore, $n = 2$, $n = 3$, or $n = 4$. But then I can

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write $1+u=a\,n$, where a, $n=2$ or $n=3$ are primes with

$$k=\frac{a\,n}{(a\,n)^{1+\epsilon}\,(\mathrm{rad}(u))^{1+\epsilon}}<1. \tag{2}$$

Case $n=4$ means

$$k=\frac{4\,a}{(2\,a)^{1+\epsilon}\,(\mathrm{rad}(u))^{1+\epsilon}}<\frac{4\,a}{(4\,a)^{1+\epsilon}}<1. \tag{3}$$

So, there are no counter-examples to the conclusion: "for any c, there is a triplet with $k<1$."

The problem with the above proof is that a and b are special numbers, not a general integers. Namely, the a is a prime, and the $b=q+p+r<3\,a$ represents an arbitrary odd number (due to validity of ternary Goldbach Conjecture). In the following, I am dealing with this issue.

If $a+b=c$ implies finitness of $k<\infty$, then $a+b+0=c$, where $0=x-x$, implies finitness of k as well. This means, e.g., $a^*+b^*=c$, where $a^*=a-x$, $b^*=b+x$, or $a+b^*=c^*$, where $b^*=b+b$, $c^*=c+b$. Why? Because if abc conjecture is true, it cannot become untrue by replacing $a+b\to a+b+0$. The $b^*=b+b$ is even, and $a^*=a-x$ can become any integer, not only a prime.

2. The signature of abc conjecture

The abc conjecture implies that in the limit $c\to\infty$, one has $r=\infty$. Otherwise, for every single $\epsilon>0$ one has $K(\epsilon)=\infty$. For arbitrary $m>0$ one has

$$c/r^{1+m}=U\,W\,, \tag{4}$$

where

$$U=c/r^{1+\epsilon}\,,\quad W=r^{\epsilon}/r^{m}\,, \tag{5}$$

and $\epsilon>0$ is arbitrary. For $\epsilon>m$, in the limit $r\to\infty$ the abc conjecture implies $U=0$, as $W=\infty$; because the abc conjecture implies finiteness of $c/r^{1+m}<\infty$ as well. One concludes that in the limit $r\to\infty$, the abc conjecture implies $k=c/r^{1+\epsilon}=0$. If, for some triplet, the $U\neq 0$ happens in the limit $r\to\infty$, the abc conjecture is wrong because then $c/r^{1+m}=\infty$. Therefore, the limit exists. Accordingly, in this limit there is an infinite number of triplets (a,b,c) with k arbitrarily close to zero. In other words, the abc conjecture implies that for an arbitrary constant $\delta>0$ there is an infinite number of triplets (a,b,c) satisfying $c/r^{1+\epsilon}<\delta$, $c<\delta\,r^{1+\epsilon}$.

2.1. **Realization of the signature.** Because a, b, c have no common factors, one has $r = \mathrm{rad}(a\,b)\,\mathrm{rad}(c)$.

Accordingly, $c < \delta\,(\mathrm{rad}(ab))^{1+\epsilon}\,(\mathrm{rad}(c))^{1+\epsilon}$. Here and in the following, δ is a fixed parameter. Let us study such numbers c which are prime numbers, namely $c = 2, 3, 5, \ldots, \infty$. Then $c = \mathrm{rad}(c)$. Therefore, $1 < \delta\,(\mathrm{rad}(ab))^{1+\epsilon}\,(\mathrm{rad}(c))^{\epsilon}$. By increasing c, $\mathrm{rad}(c)$ tends to infinity, $(\mathrm{rad}(ab))^{1+\epsilon} \geq 1$, and there is an infinite amount of different primes. Therefore, the infinite amount of triplets satisfies $1 < \delta\,(\mathrm{rad}(ab))^{1+\epsilon}\,(\mathrm{rad}(c))^{\epsilon}$. This holds for any combination of a and b for a given $c = a + b$.

In the following, c is again an arbitrary integer. Because there are several ways to put $c = a + b$, k can take several values for a given c. The maximum value $S(c) = \max k(c)$ saturates at zero. This means the limit $k(c) \leq S(c) = 0$, $c \to \infty$.

3. No transitions between $k = 0$ and $k = \infty$

The first part of the paper has shown that there are infinitely many triplets at $k < 1$. Therefore, if the abc conjecture fails, the k starts endless bouncing (while the increase of c) between near zero and large values ($k \gg 1$). There are an infinite number of forth (in values of k) and back trans-passings; each one leaves behind a trace of the triplets. Hence, an infinite number of triplets would be expected within a gap $k_1 < k < k_2$, where $k_1 \neq 0$. An alternative formulation of the abc conjecture is that for $k \geq 1$, there is a finite number of triplets [2]. Hence, the number of triplets within $1 < k < k_2$ has to be finite. Otherwise, even if $k < K(\epsilon)$ the conjecture fails because there is an infinite amount of triplets with $k \geq 1$. But if $k < K(\epsilon)$, the conjecture cannot fail. We came to a disagreement. Hence, the number of triplets within $1 < k < k_2$ is finite.

4. The boundary of limit

Let us define

$$Z = \frac{r(c+Y)}{r(c)}\,\frac{r(c)}{r(c-1)} = \frac{r(c-1+1+Y)}{r(c-1)}\,. \tag{6}$$

Such an integer Y exists within $2 - c \leq Y < \infty$ so that

$$Z > G \tag{7}$$

together with

$$\frac{r(c+Y)}{r(c)} < M \tag{8}$$

because non-vanishing G can be arbitrarily small, and the finite M can be arbitrarily large. The $Y = Y(c)$.

Eqs. (6), (7), (8) imply

$$\frac{r(c)}{r(c-1)} > \frac{G}{M}\,, \tag{9}$$

which implies

$$\frac{r(c+1)}{r(c)} > L \neq 0\,. \tag{10}$$

The ratio reads

$$\frac{c}{c+1}\left(\frac{r(c+1)}{r(c)}\right)^{1+\epsilon} = \frac{k(c)}{k(c+1)} = \beta\,. \tag{11}$$

Let us assume for a moment that the abc conjecture fails. Because there are infinitely many triplets at $k = 0$ while increasing c, k starts to jump abruptly from nearly zero to unlimitedly large values. Then if abc conjecture fails, β changes repeatedly from zero to infinity and from infinity to zero in the limit $c \to \infty$. Therefore, $r(c+1)/r(c)$ changes repeatedly from zero to infinity and from infinity to zero during the growth of c. But this comes into a disagreement with Eq. (10).

5. Conclusion

Several crucial properties of abc conjecture are presented and proven. Therefore, the abc conjecture is proven.

References

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