Disproof of the Riemann Hypothesis and the Non-Trivial Zeros of the Zeta Function

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Abstract

This paper disproves the Riemann hypothesis by disproving the non-trivial zeros of the zeta function.

Introduction

The Riemann zeta function is defined for complex *s* by the series:

$$\zeta\,(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

where

$$s = \sigma + it \quad , \quad i = \sqrt{-1} \quad , \quad \sigma \text{ and } t \text{ are real}$$

for $\Re(s) > 1$.

The series converges for $\Re(s) > 1$, and diverges for $\Re(s) \leq 1$. By analytical continuation, it is possible to redefine the zeta function for all complex numbers, except *s* = 1.

Euler had already proved that :

$$\zeta\,(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \frac{1}{1 - \frac{1}{P^s}} \qquad \ldots\ldots\ldots\ldots\ldots\ (2)$$

for real values of *s*, showing a connection between the zeta function and distribution of prime numbers.

The zeta function can be extended to $\Re(s) > 0$ by the Dirichlet *eta* function (the alternating zeta function) :

$$\left(1 - \frac{2}{2^s}\right) \zeta\,(s) = \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \ .\ .\ . \quad \ldots\ldots(3)$$

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This relation gives an equation for calculating $\zeta(s)$ in the region $0 < \text{Re}(s) < 1$, [1][2]

$$\zeta(s) = \frac{1}{\left(1 - \frac{2}{2^s}\right)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \quad \dots\dots\dots\dots\dots \quad (4)$$

The Dirichlet *eta* function η(s) converges for $\Re(s) > 0$, except for values of *s* for which:

$$(1 - 2^{1-s}) = 0$$

i.e.

$$s = 1 + i\,\frac{2\pi n}{\ln 2}$$

where *n* is any integer different from zero.

The zeta function can be extended over all complex values of $s \neq 1$, by use of the functional equation :

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s) \quad \dots\dots\dots\dots\dots\dots \quad (5)$$

where Γ (s) is the gamma function.

$$\Gamma(s) = \int_0^{\infty} e^{-t}\, t^{s-1} dt \quad , \quad \Re(s) > 0$$

The Riemann Hypothesis

The Riemann Hypothesis concerns the zeros of the zeta function for $0 < \Re(s) < 1$:

$$\zeta(s) = 0 \quad \dots\dots\dots\dots \quad (6)$$

As can be seen from (5),

$$\zeta(s) = 0 \quad for \quad s = -2, -4, -6, -8, \ . \ . \ .$$

because of the sine term. These are known as the *trivial zeros* of the zeta function, which are negative even integers.

The Riemann Hypothesis is a conjecture that, in addition to the trivial zeros, ζ(s) also has zeros in the critical strip $0 < \Re(s) < 1$, known as *non-trivial zeros*, and that *all* non-trivial zeros have real part equal to ½ .

This is a 160 year old unsolved problem in mathematics. There have been numerous attempts by mathematicians to prove or disprove this hypothesis. All attempts at analytical proof (or disproof) of this hypothesis have failed so far. Computer numerical computations of trillions of zeros, with precision of up to one thousand decimal places, searching for a zero with a real part different from ½ , have so far failed to find a counter example to disproof the hypothesis. The Riemann hypothesis is one of the Clay Mathematics Institute Millennium Prize Problems.

In this paper, the existence of the non-trivial zeros of the Riemann zeta function, and thereby the Riemann hypothesis, is disproved.

Statement of the Riemann hypothesis

The Riemann hypothesis can be stated as [3][4] [5][6][7][8]:

the Dirichlet eta function has infinite number of (non-trivial) zeros in the critical strip $0 < \Re(s) < 1$ *and all the (non-trivial) zeros have real part* $\Re(s) = 1/2$.

Disproof of the non-trivial zeros of the Riemann zeta Function

Re-writing the Dirichlet eta series:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \ . \ . \ .$$

$$\eta(s) = 0 \implies 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \ . \ . \ . = 0$$

$$\implies 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ . = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} \ . \ . \ . \qquad \dots\dots(6)$$

$$\implies 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ . = \frac{1}{2^s} \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \ . \ . \ . \right)$$

$$\implies 2^s = \frac{1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \ . \ . \ .}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .}$$

$$\Rightarrow 2^s = \frac{\left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .\right) + \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} \ . \ . \ .\right)}{\left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .\right)}$$

$$\Rightarrow 2^s = \frac{\left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .\right)}{\left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .\right)} + \frac{\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} \ . \ . \ .\right)}{\left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .\right)}$$

$$\Rightarrow 2^s = 1 + \frac{\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} \ . \ . \ .\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .}$$

$$\Rightarrow 2^s = 1 + \frac{\frac{1}{2^s}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \ . \ . \ .\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .}$$

$$\Rightarrow 2^s = 1 + \frac{1}{2^s} \frac{\left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .\right) + \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} \ . \ . \ .\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .}$$

$$\Rightarrow 2^s = 1 + \frac{1}{2^s}\left(1 + \frac{\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} \ . \ . \ .\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .}\right)$$

$$\Rightarrow 2^s = 1 + \frac{1}{2^s} + \frac{1}{2^s} \frac{\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} \ . \ . \ .\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ . \ . \ .}$$

$$\Rightarrow 2^s = 1 + \frac{1}{2^s} + \frac{1}{2^s} \frac{\frac{1}{2^s}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \ldots\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ldots}$$

$$\Rightarrow 2^s = 1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 \frac{\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \ldots\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ldots}$$

$$\Rightarrow 2^s = 1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 \frac{\left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \ldots\right) + \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \ldots\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ldots}$$

$$\Rightarrow 2^s = 1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 \left(1 + \frac{\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \ldots\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ldots}\right)$$

$$\Rightarrow 2^s = 1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \left(\frac{1}{2^s}\right)^2 \frac{\frac{1}{2^s}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \ldots\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ldots})$$

$$\Rightarrow 2^s = 1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \left(\frac{1}{2^s}\right)^3 \frac{\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} \ldots\right)}{1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \ldots})$$

We can see that the right hand side is a geometric progression. Therefore,

$$\Rightarrow 2^s = \frac{\left(1 - \left(\frac{1}{2^s}\right)^{n+1}\right)}{1 - \frac{1}{2^s}} \quad \ldots\ldots\ldots\ldots.\ (7)$$

Using:

$$2^s = e^{s\,ln2}$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{2^s}\right)^{n+1}$$

$$= \lim_{n\to\infty} \left(\frac{1}{e^{s\ln 2}}\right)^{n+1}$$

$$= \lim_{n\to\infty} \left(\frac{1}{e^{(\sigma+it)\ln 2}}\right)^{n+1} = \lim_{n\to\infty} \left(\frac{1}{e^{(\sigma\ln 2}}\right)^{n+1} \left(\frac{1}{e^{(it\ln 2}}\right)^{n+1} = 0$$

Therefore, from equation (7):

$$\Rightarrow 2^s = \frac{(1-0)}{1-\frac{1}{2^s}}$$

$$\Rightarrow 2^s - 1 = 1$$

$$\Rightarrow 2^s = 2$$

$$\Rightarrow s = 1$$

Since s = 1 is not in the domain of the zeta function, the Riemann zeta function has no zeros for $\Re(s) > 0$.

This is a simple and flawless disproof of the Riemann hypothesis. However, one might still wonder about all the numerical computations that have confirmed trillions of non-trivial zeros. The Riemann hypothesis is equivalent to the statement that real part (*x*) and imaginary part (*y*) of the Riemann zeta function,

$$\zeta\,(s) = x\ + i\,y$$

both become zero for some,

$$s = \sigma + it = \tfrac{1}{2} + i\,t$$

known as the non-trivial zeros and that all non-trivial zeros have real part equal to ½. Considering the historical importance of the Riemann hypothesis, the author suggests numerical computations to test and disprove at least one of the non-trivial zeros (for example 0.5 + i 14.134725…) by numerical computations of sufficiently high precision, say 10,000 decimal places. In this case the numerical computation would test the following:

$$\nexists(s)\ (\ |x| = |y| = 0\ )$$

near the selected non-trivial zero.

Conclusion

The Riemann hypothesis (RH) is widely considered as one of the greatest unsolved problem in mathematics. The proof (or disproof) of RH has eluded mathematicians for nearly one and a half century. The existence of the non-trivial zeros has been (almost) universally accepted/ assumed. This led mathematicians to see the problem of proving (or disproving) the RH as a problem of searching and not finding (or finding) non-trivial zeros with real part different from 1/2, analytically or numerically. This paper has clearly shown that the assumption of the existence of non-trivial zeros has been wrong. Considering the importance of the RH, this author suggests that at least one of the non-trivial zeros be tested and disproved by numerical computations of sufficiently high precision. Considering the simplicity of the solution presented in this paper, one can only say that the disparity between the perceived difficulty of the Riemann hypothesis and the simplicity of its solution is profound.

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