A Category is a Partial Algebra

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Abstract A category consists of arrows and objects. We may define a language $\mathfrak{L} := \{\mathrm{dom}, \mathrm{cod}, \circ\}$. Then a category is a partial algebra of the language $\mathfrak{L}$. Hence a functor is a homomorphism of partial algebras. And a natural transformation of functors is a natural transformation of homomorphisms. And we may define a limit of a homomorphism like a limit of functor. Then a limit of a homomorphism forms a homomorphism of partial algebras.

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1. Introduction

We may define a language[1] $\mathfrak{L} := \{\mathrm{dom}, \mathrm{cod}, \circ\}$ where 'dom' and 'cod' are partial unary operations, '$\circ$' is a partial binary operation. Then we have that every category[2] is a partial algebra[5] of the language $\mathfrak{L}$, see proposition 3.1 and corollary 3.1.1 for more details.

So a functor of categories is a homomorphism of the partial algebras, see proposition 3.2. Suppose that $\boldsymbol{A}$ is a partial algebra of the language $\mathfrak{L}$. Let $\boldsymbol{A}'$, $\boldsymbol{A}''$ be partial subalgebras of $\boldsymbol{A}$, and let $\varepsilon\colon \boldsymbol{A}' \to \boldsymbol{A}''$ be a natural homomorphism. Then we may define a natural transformation(cf. [2,3]) along ε, see definitions 3.1 and 3.2 for the details. And we may define a natural transformation of homomorphisms, see notation 3.2, definition 3.3, and proposition 3.3 for more details.

Suppose that $\boldsymbol{A}, \boldsymbol{B}$ are partial algebras of the language $\mathfrak{L}$. Then the set $\mathrm{Hom}\,(\boldsymbol{A}, \boldsymbol{B})$ together with the set of the natural transformations constitutes a partial algebra of the language $\mathfrak{L}$, see proposition 3.4.

Suppose that $\boldsymbol{I}, \boldsymbol{A}$ are partial algebras of the language $\mathfrak{L}$. Let $\varphi\colon \boldsymbol{I} \to \boldsymbol{A}$ be a homomorphism. Then we have that a limit(cf. [2, 3]) of φ is an object $\varprojlim \varphi$ of $\boldsymbol{A}$ together with a natural transformation $\tau\colon \Delta(\varprojlim \varphi) \dot{\to} \varphi$ such that $\upsilon\colon \Delta(x) \dot{\to} \varphi$ factors uniquely through τ for every object $x \in \boldsymbol{A}$, see notation 3.3 and definition 3.5 for more details. And we have that $\varprojlim$ is a homomorphism from $\boldsymbol{B}^{\boldsymbol{A}}$ to $\boldsymbol{B}$, see proposition 3.5 for the details.

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2. Preliminaries

2.1. Partial Algebra.

Definition 2.1 ([1,5])**.** An ordered pair $\langle L, \sigma\rangle$ is said to be a **language** provided that

- L is a nonempty set,
- $\sigma\colon L \to \mathbb{Z}$ is a mapping.

A language $\langle L, \sigma\rangle$ is denoted by $\mathfrak{L}$. If $f \in \mathfrak{L}$ and $\sigma(f) \geq 0$ then f is called an **operation symbol**, and $\sigma(f)$ is called the **arity** of f. If $r \in \mathfrak{L}$ and $\sigma(r) < 0$, then r is called a **relation symbol**, and $-\sigma(r)$ is called the **arity** of r. A language is said to be **algebraic** if it has no relation symbols.

Definition 2.2 ([1])**.** Let X be a nonempty class and n a nonnegative integer. Then an n-ary **partial operation** on X is a mapping from a subclass of X^n to X. If the domain of the mapping is X^n, then it is called an n-ary **operation**. And an n-ary **relation** is a subclass of X^n where $n > 0$. An operation(relation) is said to be **unary**, **binary** or **ternary** if the arity of the operation(relation) is 1, 2 or 3, respectively. And an operation is called **nullary** if the arity is 0.

Definition 2.3 ([1])**.** An ordered pair $\boldsymbol{A} := \langle A, \mathfrak{L}\rangle$ is said to be a **structure** of a language $\mathfrak{L}$ if A is a nonempty class and there exists a mapping which assigns to every n-ary operation symbol $f \in \mathfrak{L}$ an n-ary operation f^A on $\boldsymbol{A}$ and assigns to every n-ary relation symbol $r \in \mathfrak{L}$ an n-ary relation r^A on $\boldsymbol{A}$. If all operation on $\boldsymbol{A}$ are partial operations, then $\boldsymbol{A}$ is called a **partial structure**. A (partial)structure $\boldsymbol{A}$ is said to be a (**partial**)**algebra** if the language $\mathfrak{L}$ is algebraic.

Definition 2.4 ([1,5])**.** Let $\boldsymbol{A}, \boldsymbol{B}$ be structures of a language $\mathfrak{L}$. A mapping $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ is said to be a **homomorphism** provided that

$$\varphi(f^A(a_1, \ldots, a_n)) = f^B(\varphi(a_1), \ldots, \varphi(a_n)) \text{ for every } n\text{-ary operation } f;$$
$$r^A(a_1, \ldots, a_n) \Longrightarrow r^B(\varphi(a_1), \ldots, \varphi(a_n)) \text{ for every } n\text{-ary relation } r.$$

If φ is a homomorphism, then $\varphi(\boldsymbol{A})$ is a substructure of $\boldsymbol{B}$. We denote the class of all homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$ by $\operatorname{Hom}(\boldsymbol{A}, \boldsymbol{B})$.

2.2. Category.

Definition 2.5 ([2])**.** A **graph** consists of objects, arrows(morphisms) and two unary operations, as follows:

Domain: For every arrow f, $\operatorname{dom}(f)$ is an object;
Codomain: For every arrow f, $\operatorname{cod}(f)$ is an object.

Definition 2.6 ([2])**.** A graph $\mathcal{C}$ is called a **category** if it satisfies the following properties:

Identity: For every object a, there exists an arrow $\operatorname{id}_a\colon a \to a$;
Unit law: If $f\colon a \to b$, then $f \circ \operatorname{id}_a = \operatorname{id}_b \circ f = f$;
Composition: If f, g are arrows with $\operatorname{dom}(g) = \operatorname{cod}(f)$, then the composition $g \circ f\colon \operatorname{dom}(f) \to \operatorname{cod}(g)$ is an arrow;
Associative: If $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

Let a, b be objects in $\mathcal{C}$. The class of all arrows from a to b is denoted $\mathcal{C}(a, b)$.

Definition 2.7 ([2,3])**.** Let $\mathcal{C}$, $\mathcal{D}$ be categories. A **functor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of a mapping of objects and a mapping of arrows satisfying the following properties:

- $F(g \circ f) = F(g) \circ F(f)$ if $a \xrightarrow{f} b \xrightarrow{g} c \in \mathcal{C}$;
- $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ for every object $a \in \mathcal{C}$.

Definition 2.8 ([3])**.** Let $\mathcal{C}$, $\mathcal{D}$ be categories, and $F, T \colon \mathcal{C} \to \mathcal{D}$ functors. A **natural transformation** is a class $(\tau_a \colon F(a) \to T(a))_{a \in \mathcal{C}}$ of arrows in $\mathcal{D}$ such that the following diagram is commutative for every arrow $f \colon a \to b$ in $\mathcal{C}$.

$$\begin{array}{ccc} F(a) & \xrightarrow{\tau_a} & T(a) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle T(f)} \\ F(b) & \xrightarrow{\tau_b} & T(b) \end{array}$$

3. A Category is a Partial Algebra

***Notation* 3.1.** Suppose that $\mathfrak{L} := \{\mathrm{dom}, \mathrm{cod}, \circ\}$ is a language where 'dom' and 'cod' are partial unary operations, '$\circ$' is a partial binary operation.

Let $\mathcal{G}$ be a graph, $E(\mathcal{G})$ the class of arrows in $\mathcal{G}$, and $V(\mathcal{G})$ the class of objects in $\mathcal{G}$. Suppose that $\boldsymbol{A}$ is the union $V(\mathcal{G}) \cup E(\mathcal{G})$.

Proposition 3.1. *The class $\langle \boldsymbol{A}, \mathrm{dom}, \mathrm{cod}, \circ \rangle$ together with two partial unary operations (dom, cod) and a partial binary operation($\circ$) is a partial algebra of the language $\mathfrak{L}$, where 'dom', 'cod' are defined in definition 2.5, and '$\circ$' is defined in definition 2.6.*

Proof. By definition 2.5, we have that 'dom', 'cod' are the mappings from $E(\mathcal{G})$ to $V(\mathcal{G})$. Hence 'dom', 'cod' are partial unary operations on $\boldsymbol{A} = E(\mathcal{G}) \cup V(\mathcal{G})$. And if $f, g \in E(\mathcal{G})$ with $\mathrm{dom}(g) = \mathrm{cod}(f)$, then $g \circ f \in E(\mathcal{G})$. It follows that '$\circ$' is a mapping from a subset of $E(\mathcal{G}) \times E(\mathcal{G})$ to $E(\mathcal{G})$. Hence we have that '$\circ$' is a partial binary operation on $\boldsymbol{A}$. By definitions 2.2 and 2.3, $\boldsymbol{A}$ is a partial algebra. □

Corollary 3.1.1. *Suppose that $\mathcal{C}$ is a category. Let $V(\mathcal{C})$, $E(\mathcal{C})$ be the class of objects and arrows of $\mathcal{C}$, respectively. And let $\boldsymbol{A} = V(\mathcal{C}) \cup E(\mathcal{C})$. Then $\langle \boldsymbol{A}, \mathrm{dom}, \mathrm{cod}, \circ \rangle$ is a partial algebra.*

Proof. By definition 2.6, the category $\mathcal{C}$ is a graph. Then it follows from proposition 3.1 that $\boldsymbol{A}$ is a partial algebra. □

Proposition 3.2. *Let $\mathcal{CAT}$ be the category of all categories, and $\mathcal{PA}$ the category of all partial algebras of the language $\mathfrak{L}$. Then there exists a functor $PA \colon \mathcal{CAT} \to \mathcal{PA}$ defined as follows:*

- *For every object $\mathcal{C} \in \mathcal{CAT}$, $PA(\mathcal{C})$ is the partial algebra defined as in corollary 3.1.1.*
- *For every morphism $F \colon \mathcal{C} \to \mathcal{D}$ in $\mathcal{CAT}$, $PA(F)$ is the homomorphism $PA(F) \colon PA(\mathcal{C}) \to PA(\mathcal{D})$ given by*

$$\begin{aligned} f &\mapsto F(f) \quad \text{if } f \colon a \to b \text{ is a morphism in } \mathcal{C}; \\ a &\mapsto F(a) \quad \text{if } a \in \mathcal{C} \text{ is an object.} \end{aligned}$$

Proof. Let φ denote the mapping $\mathrm{PA}(F)$. And let $f \colon a \to b$, $g \colon b \to c$ be morphisms in $\mathcal{C}$. Since $F(f) \colon F(a) \to F(b)$, we have that $\varphi(f) \colon \varphi(a) \to \varphi(b)$. It follows that $\varphi(\mathrm{dom}(f)) = \mathrm{dom}(\varphi(f))$ and $\varphi(\mathrm{cod}(f)) = \mathrm{cod}(\varphi(f))$. And we have that $F(g \circ f) = F(g) \circ F(f)$ implies $\varphi(g \circ f) = \varphi(g) \circ \varphi(f)$. Hence φ is a homomorphism. Therefore, it is clear that PA is a functor. □

We have seen that every category is a partial algebra of the language $\mathfrak{L}$ and every functor is a homomorphism of the partial algebras. Now, we may define a natural transformation.

Definition 3.1. Let $\boldsymbol{A}$ be a partial algebra of the language $\mathfrak{L}$. Suppose that $\boldsymbol{A}'$ and $\boldsymbol{A}''$ are partial subalgebras of $\boldsymbol{A}$. And let $\varepsilon\colon \boldsymbol{A}' \to \boldsymbol{A}''$ be a homomorphism of the partial subalgebras. If there exists an arrow $a' \to \varepsilon(a')$ in $\boldsymbol{A}$ for all object $a' \in \boldsymbol{A}'$, then we say that ε is **natural**.

Definition 3.2. Let $\boldsymbol{A}$ be a partial algebra of the language $\mathfrak{L}$. Suppose that $\boldsymbol{A}'$ and $\boldsymbol{A}''$ are partial subalgebras of $\boldsymbol{A}$. And let $\varepsilon\colon \boldsymbol{A}' \to \boldsymbol{A}''$ be a natural homomorphism of the partial subalgebras. Then a **natural transformation** $\tau\colon \boldsymbol{A}' \dot{\to} \boldsymbol{A}''$ **along** ε is a subset $\tau \subseteq E(\boldsymbol{A})$ such that there exists an arrow $\tau_{a'} \in \tau$ which makes the following diagram commute for every arrow $f' \in \boldsymbol{A}'$.

$$\begin{array}{ccc} a' & \xrightarrow{\tau_{a'}} & \varepsilon(a') \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \varepsilon(f')} \\ b' & \xrightarrow[\tau_{b'}]{} & \varepsilon(b') \end{array}$$

***Notation* 3.2.** If $\boldsymbol{A}, \boldsymbol{B}$ are partial algebras of the language $\mathfrak{L}$, then the direct product $\boldsymbol{A} \times \boldsymbol{B}$ is a partial algebra of the language $\mathfrak{L}$. And let $\varphi\colon \boldsymbol{A} \to \boldsymbol{B}$ be a natural homomorphism. It is clear that the class

$$\vec{\varphi} := \{\langle x, \varphi(x)\rangle \mid x \in \boldsymbol{A}\} \tag{3.1}$$

is a partial subalgebra of $\boldsymbol{A} \times \boldsymbol{B}$. If $\varphi, \psi\colon \boldsymbol{A} \to \boldsymbol{B}$ are homomorphisms, then $\vec{\varphi}$ and $\vec{\psi}$ are partial subalgebras of $\boldsymbol{A} \times \boldsymbol{B}$. Let $\vec{\varepsilon}\colon \vec{\varphi} \to \vec{\psi}$ be a homomorphism given by

$$\begin{array}{rcll} \langle a, \varphi(a)\rangle & \mapsto & \langle a', \psi(a')\rangle & \text{for every object } a \in \boldsymbol{A}; \\ \langle f, \varphi(f)\rangle & \mapsto & \langle f', \psi(f')\rangle & \text{for every arrow } f \in \boldsymbol{A}, \end{array}$$

such that the following diagram is commutative where $\tau_{\square}$ is an arrow in $\boldsymbol{A}$ and $\tau_{\varphi(\square)}$ is an arrow in $\boldsymbol{B}$.

$$\begin{array}{ccc} \langle a, \varphi(a)\rangle & \xrightarrow{\langle \tau_a, \tau_{\varphi(a)}\rangle} & \langle a', \psi(a')\rangle \\ {\scriptstyle \langle f, \varphi(f)\rangle}\downarrow & & \downarrow{\scriptstyle \langle f', \psi(f')\rangle} \\ \langle b, \varphi(b)\rangle & \xrightarrow{\langle \tau_b, \tau_{\varphi(b)}\rangle} & \langle b', \psi(b')\rangle \end{array}$$

If $f = f'$ and $\tau_{\square}$ is an identity arrow, then the homomorphism $\vec{\varepsilon}$ is called a **canonical** homomorphism.

Now we may define the natural transformation of homomorphisms.

Definition 3.3 (cf. [2, 3])**.** Let $\boldsymbol{A}, \boldsymbol{B}$ be two partial algebras of the language $\mathfrak{L}$. Suppose that φ and ψ are homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$. Then a **natural transformation** $\tau\colon \varphi \dot{\to} \psi$ is a natural transformation of subalgebras along a canonical homomorphism $\vec{\tau}\colon \vec{\varphi} \to \vec{\psi}$. Let $Nat(\varphi, \psi)$ denote the set of all natural transformations from φ to ψ.

Proposition 3.3. *Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F, T\colon \mathcal{C} \to \mathcal{D}$ two functors. If $\tau\colon F \dot{\to} T$ is a natural transformation of functors, then $PA(\tau)$ is a natural transformation $PA(\tau)\colon PA(F) \dot{\to} PA(T)$ of homomorphisms where PA is a functor defined in proposition 3.2.*

Proof. By definition 2.8, we have that $\tau_b \circ F(f) = T(f) \circ \tau_a$ for all arrow $f\colon a \to b$. And it is obvious that τ forms a natural transformation $\hat{\tau}\colon \mathrm{PA}(\mathcal{C}) \times \mathrm{PA}(F(\mathcal{C})) \dot{\to} \mathrm{PA}(\mathcal{C}) \times \mathrm{PA}(T(\mathcal{C}))$ along a canonical homomorphism given by $\langle f, \mathrm{PA}(F(f))\rangle \mapsto \langle f, \mathrm{PA}(T(f))\rangle$. It implies that $\mathrm{PA}(\tau)$ is a natural transformation by definitions 3.2 and 3.3 and notation 3.2. □

Proposition 3.4 (cf. [2, 3])**.** *If $\boldsymbol{A}, \boldsymbol{B}$ are partial algebras of the language $\mathfrak{L}$ then* Hom $(\boldsymbol{A}, \boldsymbol{B})$ *together with the set of natural transformations constitutes a partial algebra of the language $\mathfrak{L}$. Let $\boldsymbol{B}^{\boldsymbol{A}}$ denote it.*

Proof. It is obvious. □

***Notation* 3.3** (cf. [2, 3])**.** Let $\boldsymbol{J}, \boldsymbol{A}$ be partial algebras of the language $\mathfrak{L}$. A homomorphism $\Delta\colon \boldsymbol{A} \to \boldsymbol{A}^{\boldsymbol{J}}$ is called a **diagonal homomorphism** if Δ assigns to every object $a \in \boldsymbol{A}$ a homomorphism $\Delta(a)\colon \boldsymbol{J} \to \boldsymbol{A}$ that is defined as follows:

$$\begin{array}{lll} i & \mapsto a & \text{for all object } i \in \boldsymbol{J}; \\ f & \mapsto id_a & \text{for all arrow } f \in \boldsymbol{J}. \end{array}$$

Let $a, b \in \boldsymbol{A}$. It is obvious that the codomain of $\Delta(a)$ is the partial subalgebra $\{a, id_a\}$. And if $(a \xrightarrow{f} b) \in \boldsymbol{A}$, then there exists a natural transformation $\tau\colon \Delta(a) \dot{\to} \Delta(b)$, and we have that $\Delta(f) = \tau$.

Definition 3.4 (cf. [2, 3])**.** Let $\boldsymbol{I} := \bullet \to \bullet \leftarrow \bullet$, $\boldsymbol{A}$ be two partial algebras of the language $\mathfrak{L}$. Suppose that $\varphi\colon \boldsymbol{I} \to \boldsymbol{A}$ is a homomorphism. Then the **pullback** of φ is an object $p \in \boldsymbol{A}$ together with a natural transformation $\pi\colon \Delta(p) \dot{\to} \varphi$ such that every natural transformation $\tau\colon \Delta(x) \dot{\to} \varphi$ factors uniquely through π for every object $x \in \boldsymbol{A}$.

Definition 3.5 (cf. [2, 3])**.** Suppose that $\boldsymbol{I}, \boldsymbol{A}$ are partial algebras of the language $\mathfrak{L}$. Let $\varphi\colon \boldsymbol{I} \to \boldsymbol{A}$ be a homomorphism. Then the **limit** of φ, denoted $\varprojlim \varphi$, is an object in $\boldsymbol{A}$ together with a natural transformation $\pi\colon \Delta(\varprojlim \varphi) \dot{\to} \varphi$ such that $\upsilon\colon \Delta(x) \dot{\to} \varphi$ factors uniquely through π for every object $x \in \boldsymbol{A}$ and every $\upsilon \in Nat(\Delta(x), \varphi)$.

We have seen that $\boldsymbol{B}^{\boldsymbol{A}}$ is a partial algebra of the language $\mathfrak{L}$ in proposition 3.4. Hence we have the following proposition.

Proposition 3.5 (cf. [2, 3])**.** *Let $\boldsymbol{A}, \boldsymbol{B}$ be partial algebras of the language $\mathfrak{L}$. Suppose that every object of $\boldsymbol{B}^{\boldsymbol{A}}$ has a limit. Then we have that the mapping $\varprojlim\colon \boldsymbol{B}^{\boldsymbol{A}} \to \boldsymbol{B}$ given by $\varphi \mapsto \varprojlim \varphi$ is a homomorphism.*

Proof. Let $\varphi, \psi \in \boldsymbol{B}^{\boldsymbol{A}}$ be two objects. If $\tau\colon \varphi \dot{\to} \psi$ is an arrow in $\boldsymbol{B}^{\boldsymbol{A}}$, then τ uniquely determines an arrow $\varprojlim \varphi \to \varprojlim \psi$ in $\boldsymbol{B}$. Therefore, it is clear that $\varprojlim$ is a homomorphism. □

Remark 3.1. Suppose that $\boldsymbol{A}, \boldsymbol{B}$ are partial algebras of the language $\mathfrak{L}$. Let φ, ψ be homomorphisms from $\boldsymbol{A}$ to $\boldsymbol{B}$. Then $\vec{\varphi} := \{\langle x, \varphi(x)\rangle \mid x \in \boldsymbol{A}\}$ and $\vec{\psi} := \{\langle x, \psi(x)\rangle \mid x \in \boldsymbol{A}\}$ are subalgebras of $\boldsymbol{A} \times \boldsymbol{B}$. Suppose that $\varepsilon\colon \vec{\varphi} \to \vec{\psi}$ is a natural homomorphism that is not a canonical homomorphism. Then a natural transformation $\zeta\colon \vec{\varphi} \to \vec{\psi}$ along ε makes the following diagram commute for every arrow $(f\colon a \to b) \in \boldsymbol{A}$.

$$
\begin{array}{ccc}
\langle a,\varphi(a)\rangle & \xrightarrow{\zeta_{\langle a,\varphi(a)\rangle}} & \varepsilon\big(\langle a,\varphi(a)\rangle\big) \\
{\scriptstyle \langle f,\varphi(f)\rangle}\downarrow & & \downarrow{\scriptstyle \varepsilon\left(\langle f,\varphi(f)\rangle\right)} \\
\langle b,\varphi(b)\rangle & \xrightarrow[\zeta_{\langle b,\varphi b\rangle}]{} & \varepsilon\big(\langle b,\varphi(b)\rangle\big)
\end{array}
\tag{3.2}
$$

And the arrow $\zeta_{\langle a,\varphi(a)\rangle}$ is an ordered pair $\langle \zeta_a, \zeta_{\varphi(a)}\rangle$ where ζ_a is an arrow in $\boldsymbol{A}$ and $\zeta_{\varphi(a)}$ is an arrow in $\boldsymbol{B}$. Hence the diagram (3.2) consists of two commutative diagrams:

$$
\begin{array}{ccc}
a & \xrightarrow{\zeta_a} & a' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
b & \xrightarrow[\zeta_b]{} & b'
\end{array}
\qquad\qquad
\begin{array}{ccc}
\varphi(a) & \xrightarrow{\zeta_{\varphi(a)}} & \psi(a') \\
{\scriptstyle \varphi(f)}\downarrow & & \downarrow{\scriptstyle \psi(f')} \\
\varphi(b) & \xrightarrow[\zeta_{\varphi(b)}]{} & \psi(b')
\end{array}
$$

And we have that $\varepsilon(\langle f,\varphi(f)\rangle) = \langle f', \psi(f')\rangle$.

4. Examples

Example 4.1 (Natural numbers). Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of all natural numbers. Then $\langle \mathbb{N}, \mathfrak{L}\rangle$ is a partial algebra consists of

$$
\begin{array}{rl}
\text{Object:} & \text{prime and } 1; \\
\text{Arrow:} & \text{otherwise,}
\end{array}
$$

together with three partial operations defined as follows:

$$
\begin{array}{rl}
n \circ m = qpr & \text{if } m = qp \text{ and } n = pr \text{ where } p \text{ is a prime;} \\
\begin{array}{r} \mathrm{dom}(q) = p_1 \\ \mathrm{cod}(q) = p_n \end{array} & \text{if } q = p_1 p_2 \dots p_n \text{ with } p_i \le p_{i+1} \text{ where } p_i \text{ is a prime.}
\end{array}
$$

There is *not* identity arrows in $\langle \mathbb{N}, \mathfrak{L}\rangle$. Suppose that $\mathbb{P}$ is the set of all prime numbers. Let $\varphi\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a homomorphism of the partial algebra of the language $\mathfrak{L}$. Then we have that φ is generated by a mapping of $\mathbb{P}$.

Example 4.2 (Chain Complex). Let $\mathfrak{L}' = \mathfrak{L} \cup \{\mathbf{0}\}$ where $\mathbf{0}$ is a nullary operation symbol. Hence we have that $\mathfrak{L}' = \langle \mathrm{dom}, \mathrm{cod}, \circ, \mathbf{0}\rangle$ is a language. Let C be a partial ordered set. We may construct a partial algebra $\boldsymbol{C}$ of the language $\mathfrak{L}'$ over the set C as follows:

$$
\begin{array}{rcl}
\text{Object:} & c & c \in C \\
\text{Arrow:} & \begin{array}{c} a \longrightarrow b \\ \mathbf{0} \end{array} & \begin{array}{l} \text{if } a < b \text{ and } a \text{ is adjacent to } b \text{ for } a, b \in C; \\ \text{otherwise;} \end{array} \\
\circ\colon & g \circ f = \mathbf{0} & \text{for all } (a \xrightarrow{f} b \xrightarrow{g} c) \in C.
\end{array}
$$

Let J be an ordered set. So $\boldsymbol{J} = \langle J, \mathfrak{L}'\rangle$ is a partial algebra of the language $\mathfrak{L}'$. Suppose that $\varphi\colon \boldsymbol{J} \to \boldsymbol{C}$ is a monomorphism[3, 5]. Then the image of φ is a partial subalgebra which is an ordered set. If $\varphi(\boldsymbol{J})$ is a set of abelian groups, then $\varphi(\boldsymbol{J})$ is a chain complex, cf. [4]. Let $\psi\colon \boldsymbol{J} \longrightarrow \boldsymbol{C}$ be a monomorphism and let $\tau\colon \varphi \dot{\to} \psi$ be a natural transformation. Then we have that τ is a chain map[4].

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