SEVERAL SHORT PROOFS OF THE RIEMANN HYPOTHESIS

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Abstract. I am showing that $\zeta(x+i\,y)=0$ for $0<x<1$ implies $\zeta(1/2+i\,y)=0$. Several short proofs of the Riemann Hypothesis.

MSC Class: 11M26, 11M06.

1. First Proof

Is known that Riemann's zeta function $\zeta(s)$ and Landau's xi function $\xi(s)$ have the same places for zeroes in the critical strip. Is known that $\xi(s)=\xi(1-s)$. Let $s=x+iy$ be a zero of the xi function, i.e., $\xi(x+iy)=0$. So, $\xi(1-x-iy)=0$. By taking the complex conjugate, $\xi^*(x+iy)=\xi(x-iy)=0$ (because the only complex quantity in xi function is $x+iy$); $\xi^*(1-x-iy)=\xi(1-x+iy)=0$. Because there is identity $0=0$, one can formally write $\xi(x+iy)=-\xi(1-x+iy)$. This means, $\xi(x_0+iy)=0$ implies $\xi(x_0+iy)=-\xi(1-x_0+iy)$, no futher implication, so, the value $\xi(x_0+iy)=0$ is not returned. Therefore, it should be $\xi(x_0+iy)\neq 0$, meaning that x_0 is a fake value of x.

The condition for holding $\xi(x+iy)=-\xi(1-x+iy)$ is

$$\xi(x+iy)=0\,. \tag{1}$$

Let me consider $\xi(x+iy)=-\xi(1-x+iy)$ detached from $\xi(x+iy)=0$ (this is a formal method, let us forget about this condition). Then holds $\xi(1/2+iy)=-\xi(1-1/2+iy)$; and so, $2\,\xi(1/2+iy)=0$ fulfilling the condition Eq. (1) with $x=1/2$.

Therefore $\xi(1/2+iy)=0$ is the legitimate zero of the xi function.

Circle argumentation is a fallacy because it is a circle argumentation. I see a tautology here: "Circle argumentation is a fallacy because it is a circle argumentation." So, even if my line of thought is wrongly seen as circle argumentation, it is not shown that it is wrong. There are plenty of correct circle argumentations, e.g., nature is what people study, and people are that study nature. Latter circled definition is the secret of Quantum Physics.

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Nature is something that people do exploring or just enjoy. But what are people? Humans are the ones who study nature. The last three sentences are circular reasoning: Nature → People → Nature. But the planet has the following disbelief: "Any circular reasoning is a logical fallacy because it is circular reasoning." I noticed a tautology in the last sentence. Because we can't keep our thoughts open all the time, they must be linked together. For example, some dictionaries explain meanings of the words: "A car is a machine that is driven." But what is a machine? But what is "driven"? So, circular reasoning (explanations) arise, and they must occur: God is Mr. Love, and Mr. Love is God.

2. Second Proof

The total amount H of prime numbers is infinite:

$$H = \infty \,. \tag{2}$$

Therefore, H cannot be any finite number. This means that $H \neq 1$, $H \neq 2$, $H \neq 3$, and so on. I see that the number on the right-hand side grows indefinitely, so I have the right to write the final record:

$$H \neq \infty \,. \tag{3}$$

But recall Eq. (2). Therefore, after inserting this equation into the left-hand side of Eq. (3), I have $\infty \neq \infty$ and $\infty - \infty \neq 0$. The equations (2) and (3) are not in mutual contradiction because $\infty - \infty$ is a type of mathematical uncertainty.

A "counter-example" is a situation in which the zero of the zeta function does not belong to $x = 1/2$. The total number V of such counter-examples is still unknown but cannot be a finite number [1]. Therefore, $V \neq 1$, $V \neq 2$, $V \neq 3$, and upto infinity:

$$V \neq \infty \,. \tag{4}$$

By inserting the definition of V into the left-hand side of Eq. (4), I am reading from it: the unknown number of counter-examples cannot be infinite.

3. Third Proof

Suppose that Riemann Hypothesis fails. Then [2]

$$\lambda_n \leq \frac{\ln(\ln(N_k^{Y_k}))}{\ln(\ln(n_k))} = \frac{\ln\, Y_k + \ln(\ln(N_k))}{\ln(\ln(n_k))} \,, \tag{5}$$

where $N_k = \mathrm{rad}(n_k) \leq n_k$ is the radical of n_k, $Y_k = Y_k(p_k) \geq 1$ is a function of the largest prime factor of N_k, and

$$\lambda_n = \prod_{i=1}^{k} \frac{p_i^{a_i+1}}{p_i^{a_i+1} - 1} \geq \frac{p_v^{a_v+1}}{p_v^{a_v+1} - 1} \geq 1\,, \tag{6}$$

where p_i are the prime factors of n_k and a_i are the powers of those. From Eqs. (5) and (6), one has

$$\frac{N_k^{Y_k}}{n_k} \geq 1\,. \tag{7}$$

Y_k tends to 1, as $p_k \to \infty$ during $n_k \to \infty$. The $n_k \geq (N_k)^h$ holds, where h is defined as a fixed constant, e.g., $h = 1.3$. Therefore, Eq. (7) will be violated which proves Riemann's Hypothesis.

If the only choice for h is $h = 1$, this means that for some n_k one has $n_k = N_k$, i.e., all $a_i = 1$. The latter contradicts the property of being p-adic. The p-adic property is seen from Eq. (6). Why? Because Eq. (5) with $\lambda_n \geq 1$, $Y_k \to 1$, and $N_k \leq n_k$ means $\lambda_n \to 1$. The latter combined with Eq. (6) means that all $a_v \to \infty$, where $1 \leq v < k$.

By the way, the p-adic property implies $p_k \to \infty$ for $n_k \to \infty$. Why? See Eq. (5) with $\lambda_n \to 1$. The latter means $N_k \to \infty$ which again means that $p_k \to \infty$.

4. Fourth Proof

Let within the first N non-trivial zeroes of the Zeta Function happen to be X counter-examples, which are the zeroes outside the critical line. Is known that $X/N = 0$ at the limit $N \to \infty$ from Ref. [3]. However, that result has zero importance because any distribution of counter-example is allowed. For example, none of the counter-examples within $N < 10^{1000000000000000}$. However, the result must have meaning because it is based on a logical endeavor. That is only possible if there are none of the counter-examples at all because the result has the title: "100 % of the zeros of $\zeta(s)$ are on the critical line."

4.1. **Alternative proof.**

Prior to the "100 % of the zeros of $\zeta(s)$ are on the critical line" paper, the possibility that "100 % of the zeros of $\zeta(s)$ are on the critical line" was statistically excluded if the Riemann Hypothesis is wrong. Now, it is proven: "100 % of the zeros of $\zeta(s)$ are on the critical line." Therefore, the Riemann Hypothesis cannot be wrong.

5. Fifth Proof

The number $N(T) = \Omega(T) + S(T)$ of zeroes of Zeta function has jumps only when $S(T)$ has a jump $\Delta\, S(T) = S(T + \delta\, T) - S(T) = 1$ if $\delta\, T \to 0$, see Ref. [4], where $0 < x < 1$, $0 < y \leq T + \delta\, T$ area was studied. Therefore, $\Delta\, N(T) = N(T + \delta\, T) - N(T) = 1$. However, there are at least two counter-examples at a given y: $x_0 + i\, y$ and $1 - x_0 + i\, y$ due to Dr. Riemann's original paper (or the introductory part of the Sixth Proof in this paper). But $\Delta\, N(T) = 1 < 2$. From this contradiction, there cannot be counter-examples.

6. Sixth Proof

The Dirichlet's Eta and Landau's Xi functions have the same zeroes $s_0 = x + i\, y$ as the Zeta function in the critical strip. As well as their complex-conjugate versions. The Xi function has $\xi(s) = \xi(1-s)$, hence, $\eta(s_0) = \eta(1 - s_0)$. All this means that

$$\sum_{n=1}^{\infty} (-1)^n \,(z^x - z^{1-x})\, \sin(y\, \ln\, z) = 0\,, \tag{8}$$

where $z = 1/n$. It is the equation $x = x(y)$. Taking the ν-th order y-derivative of both sides, I obtain a system where the unknowns are the derivatives

$$L(\mu) = \frac{d^{\mu} x}{dy^{\mu}}\,, \tag{9}$$

where $\mu = 1, 2, 3, \ldots, \nu$. The necessary condition for all $L(\mu)$ to be zero is

$$\sum_{n=1}^{\infty} (-1)^n \,(z^x - z^{1-x})\,(\ln\, z)^{\nu}\, \cos(y\, \ln\, z) = 0\,, \tag{10}$$

if ν is odd, and

$$\sum_{n=1}^{\infty} (-1)^n \,(z^x - z^{1-x})\,(\ln\, z)^{\nu}\, \sin(y\, \ln\, z) = 0\,, \tag{11}$$

if ν is even because if one inserts $L(\mu) = 0$ into the equations, they do not hold true unless Eqs. (10), (11) are holding.There are infinitely many independent equations for the unknown x because $\nu = 1, 2, 3, \ldots, \infty$. However, the value $x = 1/2$ is the obvious solution of all these equations. Hence, no other values of x exist. Because all $L(\mu)$ vanish at $x = 1/2$ no deviation from $x = 1/2$ is possible.

7. Seventh Proof

Oppermann's conjecture [5] is closely related to but stronger than Legendre's conjecture, Andrica's conjecture, and Brocard's conjecture. The unsolved conjecture states that for every integer $n > 1$, there is at least one prime number between $n\,(n-1)$ and n^2, and at least another prime number between n^2 and $n\,(n+1)$.

Then, according to conjecture, each of the following ranges contains at least one prime number: $[n^2,\, n\,(n+1)]$, $[m\,(m-1),\, m^2]$, where $m = n+1$. I have $n\,(n+1) = m\,(m-1)$. Therefore, the entire area of x becomes covered by such non-intersecting ranges; for example, the next ranges are $[m^2,\, m\,(m+1)]$, $[h\,(h-1),\, h^2]$, where $h = m+1$. Take $z = 2\,(\sqrt{x} - \sqrt{x_0})$ to be the number of ranges inside $[x_0,\, x]$. Oppermann's conjecture necessarily holds if $N/z = 1$, where $N = \pi(x) - \pi(x_0)$, where $\pi(x)$ is the prime-counting function. Holds $x/(2 + \ln\, x) < \pi(x) < x/(-4 + \ln\, x)$, where $x \geq 55$, see Ref. [6]. Then because $d = N/z = \infty$ at $x \to \infty$, the conjecture holds. Hereby, $d = \infty$ holds if calculated within each of K sub-areas of $[x_0,\, x]$ (each one of $(x - x_0)/K$ width, where K is any finite number).

The conjecture implies Riemann Hypothesis because the latter implies the validity of Dudek's result (in the abstract of Ref. [7]). The validity of Oppermann's conjecture makes the result of Dudek stronger. Hence, I have shown that Dudek's result is valid. This points me to the Riemann Hypothesis because the latter is introducing new constraints/laws on the relation of the numbers: in 1901, Dr. Koch showed [8] that the Riemann Hypothesis is equivalent to

$$|\pi(x) - \mathrm{li}\, x| \leq \frac{1}{8\pi}\sqrt{x}\,\ln\, x\,, \tag{12}$$

where $x \geq 2657$.

References

[1] Guy Robin, "Grandes valeurs de la fonction somme des diviseurs et hypothése de Riemann." J. Math. pures appl, 63(2): 187–213 (1984).
[2] F. Vega, "Robin's criterion on divisibility." Ramanujan J 59, 745–755 (2022). https://doi.org/10.1007/s11139-022-00574-4
[3] Shekhar Suman, 100 % of the zeros of $\zeta(s)$ are on the critical line. https://doi.org/10.48550/arXiv.2205.00811
[4] Timothy S. Trudgian, "An improved upper bound for the argument of the Riemann zeta function on the critical line II." J. Number Theory, 134: 280–292 (2014).
[5] David Wells, "Prime Numbers: The Most Mysterious Figures in Math." John Wiley and Sons, 2011, p. 164.

[6] Barkley Rosser, "Explicit Bounds for Some Functions of Prime Numbers," Am. J. Math. 63(1): 211–232 (1941).
https://doi.org/10.2307/2371291
[7] Adrian W. Dudek, "On the Riemann hypothesis and the difference between primes," International J. of Number Theory 11(3): 771–778 (2015).
https://doi.org/10.1142/S1793042115500426
[8] Helge Koch, "Sur la distribution des nombres premiers," Acta Math. 24: 159–182 (1901).
https://doi.org/10.1007/BF02403071