

# 各种自旋粒子方程及其量子化的新表述

## —常数不变张量分析与应用

施水荣 著

Email: [hytvssr@hotmail.com](mailto:hytvssr@hotmail.com)

广东省惠州市惠阳区淡水开城大道中73号广播电视大楼(516211)

2023年11月27日

# 序言

本书是对相对论、粒子物理和量子场论的充实、完善和进一步发展，总体上采用严格、解析、优美的描述方法，尽量使整个文章具有一种数学和物理的美感。本书前十八章是基础部分，提出了多个十分有用的数学工具，特别是我独立发展和创建了用于物理研究的常数不变张量分析方法，并按我自己的方式重新表述了经典物理，大部分内容属于经典场论和量子力学领域；十八章后是高级进阶部分，大部分内容属于量子场论领域，特别是给出了新的量子化程式，此新程式不依赖哈密顿量直接给出量子化规则，本书按此程式完成了任意时空中任意自旋线性粒子的量子化，极大丰富和扩展了量子场论的内容。

在此特别说明一点，新的量子化程式本质上属于基于方程的量子化方法，特别适合无法写出哈密顿量的系统进行量子化，是对传统基于作用量量子化方法的补充。对于高自旋粒子一般难于直接写出其哈密顿或拉式量，故而无法直接采用传统基于作用量的量子化方法，但新的量子化程式就可以避开这个困难，可以直接对高自旋粒子进行量子化，并成功建立了相应的高自旋量子场论。为了更紧凑流畅地表述物理，使文章逻辑上更清晰简单，我重新编排了各个章节内容，将大量复杂繁琐的推广章节剥离文章主线，放到文章后段，这样既保持了内容的完整性，也兼顾了逻辑流畅性，使读者不会被复杂问题阻断和产生厌烦，相对容易并愉快地读下去。这样的安排更符合阅读的规律，从简单到复杂，逐渐深入，一步一步推广，深刻完美展示完整的物理和数学内容。

具体来说，在第一至第三章中我独自发展创立了常数不变张量分析法<sup>[1-3]</sup>，发现了某些美妙的数学性质，提出了很多重要有用的常数不变张量，为物理研究提供了一个十分有用的数学工具。第四章在高低维时空和无限维表示两个维度上推广了常数不变张量。在第五章中指明了常数不变张量与表象变换之间的本质联系。在第六至九章中我运用前面章节建立的常数张量分析法等数学工具，重新表述了电磁场、Yang-Mills场、引力场和引力微子场的方程<sup>[4-14]</sup>，提出了多种等价的表述形式<sup>[1,2]</sup>，并严格解析地证明了各种表述形式之间的等价性，特别是分析得到了引力场毕安奇恒等式<sup>[11-14]</sup>的旋量形式。

第十章是本书的重中之重，也是我最开始写作此书的初衷。在此章中我独立创新地提出了一种全新的粒子方程表述形式：自旋方程，此方程直接用自旋和自旋张量矩阵构造，并注意到自旋张量同时也是场量相应表示的变换矩阵，所以此方程物理意义十分明确，可以根据粒子场量变换规律简单直接地写出相应的粒子方程，它正确描述了中微子<sup>[5]</sup>、电磁场<sup>[7,8]</sup>、Yang-Mills场<sup>[6]</sup>和电子<sup>[4]</sup>等经典方程，并发现其无质量表示完全等价于全对称的Penrose旋量方程<sup>[1,2]</sup>，当然它比Penrose旋量方程更广泛，可以描述更多物理方程。我继续利用自旋表述这个思想，进一步得到了正确描述Einstein引力场和引力微子的低一阶导数自旋方程。在以上这些自旋表述形式中可以很自然地引入一个标量场，从而推广得到了一个更有意思的方程：开关型自旋方程，在标量场为零时，自由粒子可以存在，在标量场不为零时，自由粒子不存在。此标量场的作用就象开关一样，控制粒子的产生与湮灭，这就为粒子的产生与湮灭提供了一种新的物理机制，同时也能回答宇宙暴涨期<sup>[15]</sup>为何只由标量场就可以完全描述。并且此方程本身对标量场有内在固有的限制，从而使标量场自动量子化，每一种量子化值对应截然不同的物理方程，一种对应经典粒子方程，一种对应类似扭量的方程，一种对应常数平凡解。这为五种超弦<sup>[16]</sup>的统一表述提供了一种新的思路和启示。

在第十一至十二章中我对Penrose旋量方程<sup>[1,2]</sup>、Penrose扭量方程<sup>[3]</sup>和Bargmann-Wigner方程<sup>[17]</sup>进行了全面深入的分析。并在平坦时空中严格证明了Bargmann-Wigner方程在半整数自旋情形<sup>[18,19]</sup>等价于Rarita-Schwinger方程<sup>[20,21]</sup>，在整数自旋情形<sup>[19]</sup>等价于Klein-Gordon方程<sup>[21,22]</sup>，揭示了Bargmann-Wigner方程的深刻物理内涵。通过对比研究，发现Bargmann-Wigner方程更适合描写有质量粒子，而Penrose旋量方程或自旋方程更适合描写无质量粒子。

第十三章是对前面章节内容的进一步充实和深化，我换成从表象变换的角度去研究同一个物理问题，为后续的各种自旋粒子洛伦兹变换多项式表示的证明提供了一个数学基础，并同时用表象变换技术提出了全新的粒子耦合理论。在第十四章中我对洛伦兹变换<sup>[23-25]</sup>作了细致深入的分析，特别是得到了各种常见自旋粒子洛伦兹变换的多项式表示，为以后各种自旋粒子物理的研究提供了又一个十分有用的数学工具。在第十五至十八章中建立了螺旋度、自旋代数、特殊拟微分算子和矩阵连乘迹的数学分析，为下一步任意自旋无质量粒子的成功量子化提

供了多个十分有用的数学工具。

在第十九章中，充分运用四维傅里叶变换技巧，汇聚讨论了非相对论性粒子的二次量子化细节。由于大多量子场论的书均没有详细论述Majorana粒子和中微子的量子化<sup>[26]</sup>，我也一直没找到相应内容，为了弥补这个缺憾，决定自己动手推导演算，在第二十章中我应用洛伦兹推动变换先给出了Dirac粒子的量子化<sup>[26, 27]</sup>，然后在此基础上采用类似的技巧又给出了Majorana粒子和中微子的详细量子化细节。2019.8.2 重大灵感产生，在第二十一至二十四章中我应用以上章节创立的数学工具和常数不变张量分析，按新的协变量子化程式成功完成了各种无质量自旋粒子的量子化，特别对标量场和电磁场分别单独一章进行了详细讨论，从而对比经典结果印证了新的量子化程式的合理性和正确性，然后在此基础上对各种无质量自旋粒子按相同、统一的程式成功进行了量子化。

在第二十五章至第二十七章中，基于Bargmann-Wigner方程并对照无质量粒子量子化的成功经验，对各种有质量自旋粒子按新的程式统一进行了量子化，给出了几种等价场或势的协变对易规则。在第二十八章中，我对各种对称和反对称平面波解采用类似的数学技巧进行了统一的处理。在第二十九章中我重新梳理分析了前面章节各种方程的自旋基，理清了自旋基分解的逻辑推演关系和不同表象下的自旋基分解关系。并进一步论证了自旋基就是一般自旋算符的共同本征态，且自旋基分解系数就是CG系数。在第三十章中我推广发展了全对称指标的多项式定理，为上一步按新的程式统一量子化有质量粒子提供了数学支持，同时基于历史上Behrends和Fronsdal构造出来的公式，并结合自己新的结论，得到了一个十分有意义的投影算子猜想。在第三十一至第三十三章中，我详细论述了有质量矢量粒子、有质量引力微子和有质量引力子的具体量子化方案和结果。在第三十四至第三十七章中，我试着按新的程式去统一量子化所有高、低维时空中的粒子，进行了很多有意义的推广和探索，取得了一系列成果，得到了与四维时空中完全类似的结论。特别是自然而然地涌现出了反对称张量场，这是在我意料之外的一个美妙结论。

在第三十八至第四十一章中我具体研究了二阶、多重、Dirac型和反对称的完美常数不变张量，进一步发展和丰富了常数不变张量分析法的内容。在第四十二章中，类比无质量情形，我利用Dirac型完美常数不变张量建立了有质量粒子的高自旋场量子化方案。以上第三十八至第四十二章正是我从主线剥离出来的内容，放在此处，感兴趣的读者可以对照前面对应的章节一块阅读，不感兴趣的可以略过。在第四十三章中，我大胆猜测提出一种新型的相互作用：粒子内部相互作用，是否正确有待实践验证。在第四十四章中我讨论了CG系数、自旋耦合和量子纠缠之间的本质联系，并基于表象变换技术对自旋纠缠态提供了一种美妙的数学描述。

在第四十五、四十六章中，我讨论了玻色弦、二维旋量、二维矢量和任意子的相关内容，也对二维超对称、超弦作了很初步的探讨，做这些是为了将新的量子化程式应用于超对称和超弦理论方面作好初步准备，但目前还没找到合适的切入点。在此特别说明一点，本书实际上是一个开放性的课题，有一些问题并没有得到解决，比如新的量子化程式对非线性的Yang-Mills场和Einstein引力场量子化无能为力，因为此时不能只作为自由场处理，必须要考虑自相互作用和相应的费曼规则，这也是下一步要努力解决和开拓的方向之一。

本书的数学和物理都具有较强的原创性，一些数学和物理概念、方法、内容也有一定新颖性，均是我自己本人独立地一步步演算严格建立起来的。比较正规的研究起始时间远的可以追溯到十几年前<sup>[28]</sup>，最早始于2004年5月，那时只是初步建立了目前这个理论体系的基础部分，持续了好几年，但极大地影响了我正常的生活，故一度想把它彻底忘掉，所以中间又断了几年。虽然如此，但是我对理论物理的兴趣依然没有减退，后来于八年前又开启了新一轮的研究，即从2015年3月起连续写作<sup>[29-33]</sup>至今，一直没停。它耗费了我大量的时间与精力，并利用业余时间长期写作而成，在此特别感谢家人多年的理解与支持！由于本书很多课题涉及了全新的领域，具有挑战性和开放性，充满了好多猜想、试探、验证和证明，有些数学和物理问题只是大部分得到了解决，但并没有彻底完全解决。再加上本人水平有限，不能面面俱到，书中难免有错误之处，欢迎大家指正！

备注：2022年5月后，没有再像之前七年一样灵感想法不断，并满怀激情。虽然还有几个重要问题没有得到彻底解决，但是研究的激情和强烈的好奇心似乎已离我而去了。也许是思想疲劳了，也许是厌倦了，也许是想办法枯竭了。总之，一下子突然感觉没意思了。故在此之后我没再连续写作，只有零星的进展、研究和写作；2022年9月初又产生了少些新的灵感，理清和证明了之前的一些猜想，之后只有零星缓慢的进展。直到2023年7月8日那天之后，灵感又开始不断出现了，我开始进一步具体研究清楚了高阶、多重和Dirac型的完美常数不变张量；在2023年8月15日又产生了一个新的重要灵感，我一下就求出了近三年未解决的二元数列的通项

公式；在2023年9月5日-9月7日这三天灵感突发，我终于把Dirac矩阵连乘积那一章节的猜想都进行了严格的证明；在2023年9月15日又严密化了一个多项式展开的重要定义并初步证明了一个相关猜想；在2023年10月22日-24日我又提出了B-G方程的高旋量场量子化等价方案；在2023年10月25日-26日我轻松获得并证明了反对称排序规律，由此顺带一举解决了两年前的对称排序猜想，严格证明了它。并基于新的对称排序和之前章节全对称完美常数不变张量内容，类比发展了全反对称完美常数不变张量，使常数不变张量理论得到了进一步扩展和完善。总之，这段时间灵感不断迸发，整体有所进展。

作者：施水荣

2015年3月-2023年11月于淡水



# 目录

第一章 常数不变张量分析	1
1 类Penrose抽象指标 <sup>[1,2]</sup>	1
1.1 一般表象的厄密自旋矩阵 $\sigma(s)$	1
1.2 厄密自旋矩阵的一个具体表象 <sup>[34]</sup>	1
1.3 自旋矩阵复杂性质的梳理	1
1.4 正交标架下洛伦兹变换参数 $\vartheta^{ab}$ 和自旋张量 $S_{ab}(s, \varsigma)$ (本书采用此标架)	3
1.5 其它标架下洛伦兹变换参数 $\vartheta^{ab}$ 和自旋张量 $S_{ab}(s, \varsigma)$ 的对比	3
1.6 s-自旋旋量指标	4
1.7 $\frac{1}{2}$ -自旋旋量指标	5
1.8 复矢量指标	5
1.9 矢量指标	5
2 常用矩阵	6
2.1 泡利矩阵	6
2.2 光子矩阵	6
2.3 旋转生成元矩阵	6
2.4 $SO(4)$ 群生成元矩阵	6
3 常数不变张量的发现与证明	7
3.1 各种常见度规常数不变张量	7
3.2 基本定理一及其相关常数不变张量	8
3.3 基本定理二及其相关常数不变张量	9
3.4 基本定理三及其相关常数不变张量	11
3.5 基本定理四及其相关常数不变张量	12
3.6 基本定理五及其相关常数不变张量	13
3.7 基本定理六及其相关常数不变张量	13
3.8 新常数不变张量的各种获得方法	14
3.9 对六个基本定理的评述	14
4 几个直观基本常数不变张量的性质	14
4.1 基本常数不变张量 $\varepsilon^{A_\varsigma B_\varsigma}, \varepsilon_{A_\varsigma B_\varsigma}, \varepsilon^{A'_\varsigma B'_\varsigma}, \varepsilon_{A'_\varsigma B'_\varsigma}$ 的性质 <sup>[1,2]</sup>	14
4.2 基本常数不变张量 $(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma}, (\sigma, i\varsigma)_{A_\varsigma A'_\varsigma}^a$ 的性质 <sup>[1,2]</sup>	14
4.3 基本常数不变张量 $\sigma_{\alpha'_\varsigma}^{A'_\varsigma B'_\varsigma}, \sigma^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}$ 的性质	15
4.4 基本常数不变张量 $\sigma_{+ab}^\alpha, \sigma_{-ab}^{\alpha'}, \sigma_{\varsigma ab}^{\alpha_\varsigma}, \sigma_{-\varsigma ab}^{\alpha'_\varsigma}$ 的性质	16
4.5 基本常数不变张量 $\sigma_{\alpha'_\varsigma}^{k'_\varsigma l'_\varsigma}(s), \sigma^{\alpha_\varsigma}_{k_\varsigma l_\varsigma}(s)$ 的性质	17
5 几个直观复合常数不变张量的性质	19
5.1 扩展常数不变张量 $(\sigma, i\kappa)_{\alpha'_\varsigma}^{A'_\varsigma B'_\varsigma}, (\sigma, i\kappa)^{\alpha_\varsigma}_{A_\varsigma B_\varsigma}$ 的性质	19
5.2 扩展常数不变张量 $(\sigma_\varsigma, i\kappa)_{ab}^{\alpha_\varsigma}, (\sigma_{-\varsigma}, i\kappa)_{ab}^{\alpha'_\varsigma}$ 的性质	19
5.3 自旋常数不变张量 $S_{ab}^{k'_\varsigma l'_\varsigma}(s, -\varsigma), S_{abk_\varsigma l_\varsigma}(s, \varsigma)$ 的性质	20
5.4 自旋常数不变张量 $S_{ab}^{A'_\varsigma B'_\varsigma}, S_{abA_\varsigma B_\varsigma}$ 的性质	21
5.5 几个基本常数不变张量之间的关系	21
5.6 矢量自旋张量 $S_{abcd}$ 和反对称张量 $\varepsilon_{abcd}$ 的性质	22

5.7	自旋张量 $S_{ab}(\zeta)$ 的性质	22
5.8	Dirac 自旋张量 $S_{ab}(e, \zeta)$ 的性质 <sup>[19]</sup>	22
5.9	常数不变张量 $\varepsilon_{abcd}, \gamma_a(\zeta)$ 之间的关系 <sup>[19]</sup>	23
5.10	迹 $tr[\gamma_a(\zeta)\gamma_b(\zeta)\cdots]$ 的性质	23
5.11	常数不变张量 $\varepsilon_{abcd}, (\sigma, i\zeta)_a$ 之间的关系	23
5.12	常数不变张量 $\varepsilon_{abcd}, \delta_{ab}$ 之间的关系	24
5.13	常数不变张量 $\varepsilon_{\alpha\zeta\beta\zeta\gamma\zeta}, \sigma_{\alpha\zeta}$ 之间的关系	25
5.14	常数不变张量 $\varepsilon_{\alpha\zeta\beta\zeta\gamma\zeta}, \varepsilon_{abcd}$ 之间的关系	25
5.15	常数不变张量 $\varepsilon_{A\zeta B\zeta}, \varepsilon_{\alpha\zeta\beta\zeta\gamma\zeta}$ 之间的关系	25
6	几个非直观复合常数不变张量的性质	25
6.1	复合常数不变张量 $\sigma_{\alpha\zeta}^{k'l'}(s), \sigma_{k\zeta l\zeta}^{\alpha\zeta}(s), \sigma_{k'l'\zeta}^{\alpha\zeta}(s), \sigma_{\alpha\zeta}^{k\zeta l\zeta}(s)$ 的性质	25
6.2	复合常数不变张量 $\sigma_{\alpha\zeta}^{A' B'\zeta}, \sigma_{A\zeta B\zeta}^{\alpha\zeta}, \sigma_{A'\zeta B'\zeta}^{\alpha\zeta}, \sigma_{\alpha\zeta}^{A\zeta B\zeta}$ 的性质	26
6.3	常数不变张量 $\sigma_{\alpha\zeta}^{ab}, \sigma_{ab}^{\alpha\zeta}$ 的性质	27
6.4	自旋常数不变张量 $S_{ab}^{k\zeta l\zeta}(s, \zeta), S_{ab}^{k'l'\zeta}(s, -\zeta), S_{ab}^{k\zeta l\zeta}(s, \zeta), S_{ab}^{k'l'\zeta}(s, -\zeta)$ 的性质	29
6.5	几个基本常数不变张量之间的重要联系	30
6.6	自旋常数不变张量 $S_{ab}^{A\zeta B\zeta}, S_{ab}^{A'\zeta B'\zeta}, S_{ab}^{A\zeta B\zeta}, S_{ab}^{A'\zeta B'\zeta}$ 的性质	30
6.7	不变常数自旋张量之间的重要关系	30
6.8	复合常数不变张量 $\Sigma_{k\zeta l\zeta}^{k's' l's'}(s, s')$ 的性质	35
7	常数不变张量的一般理论	35
7.1	常数不变张量的一般定义	35
7.2	常数不变张量的协变导数为零	35
<b>第二章 完美常数不变张量</b>		<b>36</b>
1	对称指标排序	36
1.1	二阶对称指标排序	36
1.2	四阶对称指标排序	37
1.3	四阶对称指标正向排序规律的归纳小结	39
1.4	$w+1$ 阶对称指标排序	40
1.5	$w+1$ 阶反对称指标正向排序规律的严格证明(从0排起)	41
2	完美常数不变张量 $\Gamma_{A\zeta B\zeta C\zeta}^{k\zeta}(s; w), \Gamma_{k\zeta}^{A\zeta B\zeta C\zeta}(s; w)$	41
2.1	常数不变张量 $\Gamma_{A\zeta B\zeta C\zeta}^{k\zeta}(s), \Gamma_{k\zeta}^{A\zeta B\zeta C\zeta}(s)$ 的引入	41
2.2	常数不变张量 $\Gamma_{A\zeta B\zeta C\zeta}^{k\zeta}(s; w), \Gamma_{k\zeta}^{A\zeta B\zeta C\zeta}(s; w)$ 的引入	42
2.3	定义的解读	42
2.4	常数矩阵 $\Gamma(s; w), \bar{\Gamma}(s; w)$ 的引入	42
2.5	常数不变张量 $\Gamma_{A\zeta B\zeta C\zeta}^{k\zeta}(s; w), \Gamma_{k\zeta}^{A\zeta B\zeta C\zeta}(s; w)$ 的基本性质	43
2.6	度规常数不变张量 $\varepsilon_{k\zeta l\zeta}(s; w)$ 的引入及其性质(存在 $\varepsilon_{A\zeta B\zeta}$ 为前提条件)	43
2.7	常数不变张量 $\Omega_{A\zeta B\zeta C\zeta}^{A' B' C' \zeta}(s; w), \Omega(s; w)$ 的引入及其基本性质	44
2.8	常数不变张量 $\Omega_{ab A\zeta B\zeta C\zeta}^{A' B' C' \zeta}(s; w), \Omega_{ab}(s, \zeta; w)$ 的引入及其基本性质	45
2.9	自旋常数不变张量 $\sigma_{k\zeta}^{\alpha\zeta l\zeta}(s; w), \sigma(s; w)$ 的引入	45
2.10	自旋常数不变张量 $S_{ab k\zeta}^{l\zeta}(s; w), S_{ab}(s, \zeta; w)$ 的引入	45
2.11	常数矩阵 $\Omega(s; w)$ 的两个重要引理	45
2.12	常数矩阵 $\Omega_{ab}(s; w)$ 的两个重要引理	46
2.13	关于常数矩阵 $\bar{\Gamma}(s; w), \Omega(s; w), \sigma(s; w), \Gamma(s; w)$ 的置换性质及其推论	46

2.14	关于常数矩阵 $\bar{\Gamma}(s; w), \Omega_{ab}(s, \varsigma; w), S_{ab}(s, \varsigma; w), \Gamma(s; w)$ 的置换性质及其推论	47
2.15	关于常数矩阵 $\Omega(s; w), \sigma(s; w)$ 的重要推论及洛伦兹群表示	47
2.16	关于常数矩阵 $\Omega_{ab}(s, \varsigma; w), S_{ab}(s, \varsigma; w)$ 的重要推论及洛伦兹群表示	48
2.17	常数矩阵 $\Omega_{ab}(s, \varsigma; w), S_{ab}(s, \varsigma; w)$ 与 $\Omega(s; w), \sigma(s; w)$ 之间关系的讨论	49
2.18	推论：常数矩阵 $\Gamma(s; w), \bar{\Gamma}(s; w)$ 的几个恒等式	49
2.19	常数矩阵 $\Gamma(s; w), \bar{\Gamma}(s; w)$ 的对称置换性质	49
2.20	矩阵 $\Gamma(s; w), \bar{\Gamma}(s; w)$ 的常数不变张量性质	50
2.21	矩阵 $\Gamma(s), \bar{\Gamma}(s)$ 的常数不变张量性质	51
2.22	常数矩阵 $I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w), I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)$ 的置换性质	51
2.23	推论：常数矩阵 $\Gamma(s - \frac{1}{2}; w), \bar{\Gamma}(s - \frac{1}{2}; w)$ 的几个恒等式	52
2.24	矩阵 $I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w), I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)$ 的常数不变张量性质	53
2.25	矩阵 $I \otimes \bar{\Gamma}(s - \frac{1}{2}), I \otimes \Gamma(s - \frac{1}{2})$ 的常数不变张量性质	53
2.26	两个定理的另外一种证明方法(强调思路启发而不求全)	53
3	全对称高自旋张量(关于 $w > 1$ 情形的讨论和澄清)	54
3.1	全对称旋量例子	54
3.2	矩阵 $\Gamma(s; 2\tau), \bar{\Gamma}(s; 2\tau)$ 无穷种常数不变张量的性质	54
3.3	全对称复合旋量例子	54
3.4	全对称空间旋量例子	55
3.5	自洽性质验证	55
3.6	具体例子	55
3.7	高维时空中全对称旋量例子	55
4	完美常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$	56
4.1	常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的引入	56
4.2	常数矩阵 $N_{A_\zeta}(s; w), N^{A_\zeta}(s; w); \bar{N}_{A_\zeta}(s; w), \bar{N}^{A_\zeta}(s; w); N(s; w), \bar{N}(s; w)$ 的引入	56
4.3	常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的基本性质	57
4.4	常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的正交性质	57
4.5	常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的升降指标(存在 $\varepsilon_{A_\zeta B_\zeta}$ 为前提条件)	59
4.6	常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的自旋变换	60
4.7	常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的置换性质	61
4.8	常数不变张量 $N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; w)$ 的引入及其性质	62
4.9	推论1：常数矩阵 $N(s; w), \bar{N}(s; w)$ 的几个恒等式	63
4.10	推论2：常数矩阵 $N(s; w), \bar{N}(s; w)$ 的另外几个恒等式	63
4.11	矩阵 $N(s; w), \bar{N}(s; w)$ 的常数不变张量性质	64
4.12	矩阵 $N(s), \bar{N}(s)$ 的常数不变张量性质	64
4.13	矩阵 $(\sigma \otimes I_{C_{2s-1}^{2s-1}}, -i\zeta)_a$ 的常数不变张量性质	64
4.14	矩阵 $(\sigma \otimes I_{2s}, -i\zeta)_a$ 的常数不变张量性质	65
5	完美常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ (存在 $\varepsilon_{A_\zeta B_\zeta}$ 为前提)	65
5.1	完美常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的引入	65
5.2	常数矩阵 $X(s; w), \bar{X}(s; w)$ 的引入	65
5.3	常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的升降指标	65
5.4	常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的正交性	66
5.5	矩阵 $N(s), \bar{N}(s), X(s), \bar{X}(s)$ 的联合正交性质	66
5.6	常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的自旋变换	67

5.7	常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的置换性质	68
5.8	推论：关于常数矩阵 $X(s; w), \bar{X}(s; w)$ 的重要性质	69
5.9	矩阵 $X(s; w), \bar{X}(s; w)$ 的常数不变张量性质	69
5.10	矩阵 $X(s), \bar{X}(s)$ 的常数不变张量性质	69
5.11	常数矩阵 $\Omega(s; w), \sigma(s-1; w)$ 的置换性质	70
5.12	常数矩阵 $\Omega(s-l; w), [\vec{v} \cdot \Omega(s-l; w)]^n, e^{\vec{v} \cdot \Omega(s-l; w)}$ 的同构性表示	70
<b>第三章 重要的复合常数不变张量</b>		<b>71</b>
1	本章需满足的代数同构前提条件	71
2	复合常数不变张量 $X^{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}}(s, n, l; w)$	71
2.1	复合常数不变张量 $X^{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}}(s, n, l; w)$ 的引入	71
2.2	复合常数不变张量 $M^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w)$ 和 $N^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w)$	72
2.3	复合常数不变张量 $\Gamma^{\alpha_{1\zeta} \cdots \alpha_{n\zeta} k_\zeta l_\zeta}(s; w)$ 的引入	76
2.4	$M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w), N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w), \Gamma^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w)$ 的展开式	78
2.5	应用迭代法对前几项展开系数进行具体计算	80
2.6	应用迭代法求解一般展开系数	80
3	$s$ 解析延拓后数列的纯数学递推关系和初始条件	81
3.1	解析延拓后几个数列的重新定义	81
3.2	解析延拓后等价的归一化定义	82
4	解析延拓后几个数列通项公式的求解	82
4.1	$m(s, n; i)$ 通项公式的正向推导	82
4.2	$m(s, n; i)$ 通项公式的反向推导	85
4.3	$m(s, n; n-2l)$ 的通项公式	87
4.4	$n(s, n; n-2l)$ 的通项公式	88
4.5	$\bar{n}(s, n-2k; n-2l)$ 的定义	89
4.6	$c(s, n; n-2l)$ 通项公式的寻找(未获成功)	89
4.7	展开系数的线性代数解法	94
5	常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w), Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w)$	100
5.1	常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w), Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w)$ 的引入	100
5.2	常数矩阵 $Z_a^{A'_\zeta}(s, \zeta; w), Z_{A'_\zeta}^a(s, \zeta; w)$ 的引入	100
5.3	常数不变张量矩阵 $Z_a(s, \zeta; w), \bar{Z}_a(s, \zeta; w)$ 的引入	100
5.4	矩阵 $Z_a(s, \zeta; w), \bar{Z}_a(s, \zeta; w)$ 的常数不变张量性质	100
5.5	常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w), Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w)$ 的性质	100
5.6	猜想(不具一般性)	101
5.7	常数不变张量矩阵 $Z_a(s, \zeta; w), \bar{Z}_a(s, \zeta; w)$ 的性质(不具一般性)	101
6	常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta), Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta)$	102
6.1	常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta), Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta)$ 的引入	102
6.2	常数矩阵 $Z_a^{A'_\zeta}(s, \zeta), Z_{A'_\zeta}^a(s, \zeta)$ 的引入	102
6.3	常数不变张量矩阵 $Z_a(s, \zeta), \bar{Z}_a(s, \zeta)$ 的引入	102
6.4	矩阵 $Z_a(s, \zeta), \bar{Z}_a(s, \zeta)$ 的常数不变张量性质	102
6.5	常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta), Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta)$ 的性质I	102
6.6	常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta), Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta)$ 的性质II	103

6.7 常数不变张量矩阵 $Z_a(s, \varsigma), \bar{Z}_a(s, \varsigma)$ 的性质	103
7 几个重要的复合常数不变张量	104
7.1 复合常数不变张量 $\Gamma_{\alpha\varsigma\beta\varsigma\dots}^{k\varsigma}(n), \Gamma_{k\varsigma}^{\alpha\varsigma\beta\varsigma\dots}(n)$	104
7.2 复合常数不变张量 $\Gamma_{abcd\dots}^{k\varsigma}(n), \Gamma_{k\varsigma}^{abcd\dots}(n)$	105
7.3 复合常数不变张量 $N_{l\varsigma l\varsigma a}^{k\varsigma k\varsigma'}(s), N_{k\varsigma' k\varsigma}^{l\varsigma l\varsigma a}(s)$ 的引入	106
7.4 复合常数不变张量 $\Gamma_{abcd\dots}^{a_1 b_1 c_1 d_1 \dots a_2 b_2 c_2 d_2 \dots}(n)$ 的引入	106
7.5 复合常数不变张量 $\Gamma_{\alpha\varsigma\beta\varsigma\dots}^{A\varsigma B\varsigma C\varsigma D\varsigma\dots}(n), \Gamma_{A\varsigma B\varsigma C\varsigma D\varsigma\dots}^{\alpha\varsigma\beta\varsigma\dots}(n)$ 的引入	106
8 复合常数不变张量 $\Gamma_{abc\dots}^{k\varsigma k\varsigma'}(s), \Gamma_{k\varsigma k\varsigma'}^{abc\dots}(s)$	106
8.1 复合常数不变张量 $\Gamma_{abc\dots}^{k\varsigma k\varsigma'}(s), \Gamma_{k\varsigma k\varsigma'}^{abc\dots}(s)$ 的引入	106
8.2 复合常数不变张量 $\Gamma_{abc\dots}^{k\varsigma k\varsigma'}(s, w), \Gamma_{k\varsigma k\varsigma'}^{abc\dots}(s, w)$ 的引入	107
8.3 复合常数不变张量 $\Gamma_{abcd\dots}^{\alpha\varsigma\alpha'\beta\varsigma\beta'\dots}(n), \Gamma_{\alpha\varsigma\alpha'\beta\varsigma\beta'\dots}^{abcd\dots}(n)$ 的引入	107
8.4 复合常数不变张量 $\Gamma_{abc\dots}^{k\varsigma k\varsigma'}(s), \Gamma_{k\varsigma k\varsigma'}^{abc\dots}(s)$ 前几项的具体展开	108
8.5 $\Gamma_{+}^{abc\dots}(s)$ 和 $\Gamma_{-}^{abc\dots}(s)$ 的定义	109
8.6 算符 $\Gamma_{\pm}^{abc\dots}(s) p_a p_b p_c \dots$ 和 $\Gamma_{\pm}^{abc\dots}(s) \partial_a \partial_b \partial_c \dots$ 的基本性质	109
8.7 算符 $\hat{p}_a$ 和 $\hat{\partial}_a$ 的定义	110
8.8 算符 $\Gamma_{abc\dots}^{abc\dots}(s) \hat{p}_a \hat{p}_b \hat{p}_c \dots$ 和 $\Gamma_{abc\dots}^{abc\dots}(s) \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots$ 的基本性质	110
8.9 算符 $\Gamma_{k\varsigma k\varsigma'}^{ij\dots\pi\dots\pi}(s) \hat{p}_i \hat{p}_j \dots$ 和 $\Gamma_{k\varsigma k\varsigma'}^{ij\dots\pi\dots\pi}(s) \hat{\partial}_i \hat{\partial}_j \dots$ 的展开式	110
8.10 算符 $\Gamma_{abc\dots}^{abc\dots}(s) \hat{p}_a \hat{p}_b \hat{p}_c \dots$ 和 $\Gamma_{abc\dots}^{abc\dots}(s) \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots$ 的展开式	111
8.11 算符 $\Gamma_{abc\dots}^{abc\dots}(s) \hat{p}_a \hat{p}_b \hat{p}_c \dots$ 和 $\Gamma_{abc\dots}^{abc\dots}(s) \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots$ 展开系数的线性代数解法	111
8.12 特殊复合常数不变张量之间的关系	112
9 最基本的常数不变张量	113
第四章 高维常数不变张量初步	114
1 N+1维时空的洛伦兹群表示	114
1.1 N+1维时空Dirac矩阵的递推表象	114
1.2 N+1维时空Dirac矩阵、最小旋量和实表象形式	115
1.3 N+1维时空Dirac矩阵简洁构造	116
1.4 N+1=2n维时空通过Dirac矩阵构造费米算子 <sup>[35]</sup>	116
1.5 N+1维时空Dirac矩阵的递推关系	116
1.6 N+1维时空的自旋张量	116
1.7 N+1维时空的洛伦兹群表示	117
1.8 N+1维时空的度规张量和电荷共轭矩阵	117
1.9 N+1维时空中的常数不变张量(终于成功推广了)	118
1.10 n=N+1维时空中的常数不变张量性质	118
1.11 n=N+1维时空中的Penrose变换	118
2 外部时空对称变换 <sup>[26, 27]</sup>	118
2.1 彭加莱变换	118
2.2 洛伦兹变换	119
2.3 平移变换	119
2.4 彭加莱群生成元对易关系	119
2.5 生成元的意义	119

3 无穷维常数不变张量	119
3.1 量子力学意义上的无穷维不变张量 [26,27]	119
3.2 量子场论意义上的无穷维不变张量 [26,27]	120
3.3 量子场论意义上的一般不变张量 [26,27]	121
3.4 量子场论算子代数	121
3.5 任意算子的本征态和本征值如何求	121
<b>第五章 常数不变张量与表象变换</b>	<b>122</b>
1 一般 $s$ -自旋场各种旋量的定义	122
1.1 Penrose抽象符号规则 [1,2]	122
1.2 场旋量 $\psi(s, \varsigma; w), \tilde{\psi}(s, \varsigma; w), \hat{\psi}(s, \varsigma; w)$ 的引入	122
1.3 源旋量 $\tilde{J}(s, \varsigma; w), \hat{J}(s, \varsigma; w)$ 的引入	122
1.4 旋量 $\psi_{k_\varsigma}(s; w), \psi^{k_\varsigma}(s; w)$ 和 $\psi_{A_\varsigma l_\varsigma}(s; w), \psi^{A_\varsigma l_\varsigma}(s; w)$ 的引入	122
1.5 旋量 $J^{A'_\varsigma l_\varsigma}(s; w), J_{A'_\varsigma l_\varsigma}(s; w)$ 的引入及其 $\tilde{J}(s, \varsigma; w), \hat{J}(s, \varsigma; w)$ 之间的关系	123
1.6 旋量 $J^{A'_\varsigma B_\varsigma C_\varsigma D_\varsigma \dots}$ 的引入及其 $\hat{J}(n), \hat{J}(n, \varsigma)$ 之间的关系	123
1.7 $\psi_{\alpha_\varsigma \beta_\varsigma \dots}(n), \psi^{\alpha_\varsigma \beta_\varsigma \dots}(n), \Psi(n, \varsigma), \hat{\Psi}(n, \varsigma)$ 的定义	124
1.8 旋量 $J^{A'_\varsigma B_\varsigma C_\varsigma D_\varsigma \dots}$ 的引入及其 $\hat{J}(n), \hat{J}(n, \varsigma)$ 之间的关系	125
2 全对称性条件分析	125
2.1 场量全对称性条件分析	125
2.2 源对称性条件分析	126
3 电磁场的全对称旋量与表象变换	128
3.1 电磁场的全对称旋量 $\psi_{A_\varsigma B_\varsigma}$	128
3.2 $\psi(1, \varsigma), \hat{\psi}(1, \varsigma)$ 之间的关系	128
3.3 $\tilde{\psi}(1, \varsigma), \hat{\psi}(1, \varsigma)$ 之间的关系	128
3.4 $\psi(1, \varsigma), \tilde{\psi}(1, \varsigma)$ 之间的关系	128
3.5 表象变换矩阵是常数不变张量	128
3.6 电磁场量的旋量关系与表象变换	129
3.7 电磁场源的旋量关系与表象变换	129
4 引力场的全对称旋量与表象变换	129
4.1 引力场的全对称条件	129
4.2 引力场的全对称旋量	130
4.3 $\psi(2, \varsigma), \hat{\psi}(2, \varsigma)$ 之间的关系	130
4.4 $\tilde{\psi}(2, \varsigma), \hat{\psi}(2, \varsigma)$ 之间的关系	130
4.5 $\psi(2, \varsigma), \tilde{\psi}(2, \varsigma)$ 之间的关系	131
4.6 引力场量的旋量关系与表象变换	131
4.7 引力场源的旋量关系与表象变换	131
4.8 表象变换间关系	132
4.9 纯虚相似变换	132
4.10 纯虚表象变换	132
5 $s$ -自旋粒子各种场量的关系	133
5.1 场旋量 $\hat{\psi}(s, \varsigma; w), \tilde{\psi}(s, \varsigma; w), \psi(s, \varsigma; w)$ 之间的恒等表象变换关系	133
5.2 源旋量 $\hat{J}(s, \varsigma; w), \tilde{J}(s, \varsigma; w)$ 之间的恒等表象变换关系	133
5.3 $s$ -自旋粒子各种场量之间的变换关系	133

5.4 同步表象变换	134
<b>6 常用矩阵汇总</b>	<b>134</b>
6.1 自旋矩阵的一个非厄密表象	134
6.2 $\sigma$ 轮换表象变换矩阵	135
6.3 $\sigma(s)$ 轮换表象变换矩阵	135
6.4 电磁纯虚表象变换矩阵和交换矩阵	135
6.5 引力纯虚表象变换	136
6.6 引力纯虚相似变换矩阵	136
6.7 全对称旋量条件矩阵	136
<b>第六章 电磁场方程的重新表述</b>	<b>137</b>
1 应用常数张量定义电磁场 <sup>[7]</sup> 的各种旋量	137
1.1 电磁场强的经典描述	137
1.2 电磁场强的复矢量描述	137
1.3 电磁场强的基本性质	137
1.4 电磁场强的 $\frac{1}{2}$ -旋量描述 <sup>[1,2]</sup>	138
1.5 电磁场强的1-旋量描述	138
1.6 电磁场源的 $\frac{1}{2}$ -旋量描述 <sup>[1,2]</sup>	138
2 电磁场方程的多种等价表述形式	138
2.1 电磁场规范理论的标准描述	138
2.2 电磁场方程的经典形式	139
2.3 电磁场方程的复矢量表述形式	139
2.4 电磁场方程的 $\frac{1}{2}$ -旋量表述形式	139
2.5 猜测	140
2.6 电磁场方程 <sup>[7]</sup> 的自旋张量表述形式	140
2.7 电磁场方程的经典分离形式	142
<b>第七章 Yang-Mills场方程的重新表述</b>	<b>143</b>
1 应用常数张量定义Yang-Mills场 <sup>[7]</sup> 的各种旋量	143
1.1 Yang-Mills场强的经典描述	143
1.2 Yang-Mills场强的复矢量描述	143
1.3 Yang-Mills场强的 $\frac{1}{2}$ -旋量描述 <sup>[1,2]</sup>	143
1.4 Yang-Mills场强的1-旋量描述	144
1.5 Yang-Mills场源的 $\frac{1}{2}$ -旋量描述 <sup>[1,2]</sup>	144
2 Yang-Mills场方程的多种等价表述形式	144
2.1 Yang-Mills理论的标准描述	144
2.2 Yang-Mills理论的分量描述	145
2.3 Yang-Mills方程的标架描述	145
2.4 Yang-Mills场方程的经典形式	145
2.5 Yang-Mills场方程的复矢量表述形式	145
2.6 Yang-Mills场方程的 $\frac{1}{2}$ -旋量表述形式	146
2.7 猜测	146
2.8 Yang-Mills场方程 <sup>[7]</sup> 的自旋张量表述形式	146
2.9 Yang-Mills场方程的经典分离形式	148

<b>第八章 引力场方程的重新表述</b> .....	149
1 引力场 <sup>[11-14]</sup> 物理量的各种描述 .....	149
1.1 引力场 <sup>[11-14]</sup> 物理量的经典描述 .....	149
1.2 引力场物理量的Yang-Mills描述 <sup>[6]</sup> .....	149
1.3 引力场曲率张量的Ashtekar规范表述 <sup>[36]</sup> .....	150
1.4 引力场物理量的复张量描述 .....	152
1.5 引力场物理量的 $\frac{1}{2}$ -旋量描述 <sup>[1,2]</sup> .....	154
2 引力场毕安奇恒等式的各种表述形式 .....	157
2.1 引力场方程的经典表述形式 .....	157
2.2 引力场毕安奇恒等式的Yang-Mills形式 .....	158
2.3 毕安奇恒等式的复矢量表述形式 .....	162
2.4 毕安奇恒等式的 $\frac{1}{2}$ -旋量表述形式 <sup>[1,2]</sup> .....	162
2.5 毕安奇恒等式的全 $\frac{1}{2}$ -旋量表述形式 .....	163
2.6 猜测 .....	163
2.7 毕安奇恒等式 <sup>[7]</sup> 的自旋张量表述形式 .....	163
2.8 毕安奇恒等式的外尔表述形式 .....	165
2.9 毕安奇恒等式的经典分离形式 .....	167
2.10 毕安奇恒等式的特殊类电磁场表述形式 .....	168
3 引力场的物理Yang-Mills规范方程 .....	168
3.1 Einstein方程 <sup>[11]</sup> 和引力场Yang-Mills规范方程 .....	168
3.2 引力场Yang-Mills规范方程的旋量表述形式 .....	169
3.3 自我评述 .....	169
4 广义相对论Einstein方程 <sup>[11-14]</sup> 的等价矩阵形式 .....	169
4.1 准备 .....	169
4.2 Einstein方程的等价矩阵形式 .....	169
4.3 Einstein方程等价的低一阶导数自旋张量新形式 .....	170
<b>第九章 引力微子场方程的重新表述</b> .....	171
1 应用常数张量定义引力微子场 <sup>[7]</sup> 的各种旋量 .....	171
1.1 引力微子理论的场强描述 .....	171
1.2 引力微子场强的经典描述 .....	171
1.3 引力微子场强的复矢量描述 .....	171
1.4 引力微子场强的 $\frac{1}{2}$ -旋量描述 <sup>[1,2]</sup> .....	172
1.5 引力微子场强的1-旋量描述 .....	172
1.6 引力微子场源的 $\frac{1}{2}$ -旋量描述 <sup>[1,2]</sup> .....	172
1.7 引力微子场的对称性条件证明 .....	173
2 平坦时空中Penrose型引力微子场方程的多种等价表述形式 .....	173
2.1 引力微子方程的标架描述 .....	173
2.2 引力微子场方程的经典形式 .....	173
2.3 引力微子场方程的复矢量表述形式 .....	173
2.4 引力微子场方程的 $\frac{1}{2}$ -旋量表述形式 <sup>[1,2]</sup> .....	174
2.5 引力微子场方程的全 $\frac{1}{2}$ -旋量表述形式 .....	174
2.6 全对称方程(广义协变推广) .....	174



2.7 猜测	175
2.8 引力微子场方程 <sup>[7]</sup> 的自旋张量表述形式	175
3 Rarita-Schwinger方程的分析 <sup>[20]</sup>	177
3.1 准备	177
3.2 有质量Rarita-Schwinger方程的等价形式	177
3.3 无质量Rarita-Schwinger方程的等价形式	178
3.4 Weyl型R-S方程的等价形式	178
3.5 Weyl型R-S方程等价的低一阶导数自旋张量新形式	178
4 方程的对比	178
4.1 Weyl型引力微子方程和Penrose型引力微子方程的对比	178
4.2 引力场Einstein方程和引力场规范方程的对比	178
<b>第十章 各种粒子的自旋方程</b>	<b>179</b>
1 自旋矢量 $W_a$ 的描述	179
1.1 自旋矢量 $W_a, W_a(s, \varsigma)$ 定义	179
1.2 自旋矢量 $W_a(s, \varsigma)$ 的性质	179
1.3 自旋矢量 $W_a(s, \varsigma)$ 在特殊坐标系中的性质	180
1.4 自旋矢量 $W_a(s, \varsigma)$ 和 $p_a, S_{ab}(s, \varsigma)$ 的对易关系	180
1.5 有质量粒子彭加勒群的卡西米算子 <sup>[8]</sup>	180
1.6 无质量粒子彭加勒群的卡西米算子 <sup>[8]</sup>	180
1.7 自旋张量 $S_{ab}(s, \varsigma)$ 的统一描述	181
2 自旋方程的构造	181
2.1 由自旋量直接构造的全新粒子方程	181
2.2 新粒子方程的性质	181
2.3 自旋方程的定义	182
2.4 自旋方程的一种等价表述	182
2.5 开关型自旋方程的定义	182
2.6 开关型自旋方程的一种等价表述	182
3 各种粒子自旋方程	183
3.1 中微子 <sup>[5]</sup> 自旋方程	183
3.2 任意 $N+1$ 维时空中的电子 <sup>[4]</sup> 自旋方程	183
3.3 Yang-Mills场 <sup>[6]</sup> 自旋方程	183
3.4 s-自旋粒子的自旋方程：全对称性Penrose方程 <sup>[1,2]</sup>	183
3.5 s-自旋方程的各种等价形式	185
3.6 偶数维时空中s-自旋粒子的自旋方程：偶数维时空全对称性Penrose方程 <sup>[1,2]</sup> 错??	185
3.7 广义自旋方程	186
4 开关型自旋方程	186
4.1 开关型无源中微子自旋方程	186
4.2 开关型无源电磁场自旋方程	187
4.3 任意 $N+1$ 维时空中的矢量场自旋方程及其无源开关型自旋方程	188
4.4 任意 $N+1$ 维时空中标量场的来源	188
4.5 任意 $N+1$ 维时空中的无源开关型电子自旋方程	189
4.6 s-自旋粒子的无源开关型自旋方程	189

4.7 s-自旋粒子开关型自旋方程的各种等价形式	190
<b>5 低一阶导数的自旋方程新形式</b>	<b>190</b>
5.1 低一阶导数的s自旋方程	190
5.2 低一阶导数的 $s = \frac{1}{2}$ 自旋方程：引力微子方程	190
5.3 低一阶导数的 $s = 1$ 自旋方程：Einstein方程	190
5.4 低一阶导数的s-自旋方程的各种等价形式	191
5.5 低一阶导数的开关型自旋方程新形式	191
5.6 低一阶导数的s-自旋开关型方程的各种等价形式	191
5.7 两种自旋方程的对比	191
5.8 两种开关型自旋方程的对比	191
5.9 猜测：一个新的物理方程	191
<b>第十一章 Penrose方程和扭量方程</b>	<b>192</b>
1 任意自旋全对称Penrose方程 <sup>[1,2]</sup> 的重新表述	192
1.1 任意自旋全对称Penrose方程的整旋量等价形式	192
1.2 任意自旋全对称Penrose方程的矩阵等价形式	193
1.3 任意自旋全对称Penrose方程的自旋方程形式	193
2 扭量方程的重新表述	193
2.1 Penrose扭量方程 <sup>[2,3]</sup>	193
2.2 类扭量方程的等价形式	193
2.3 类扭量方程的解	194
2.4 开关型自旋方程与类扭量方程的关系	194
<b>第十二章 Bargmann-Wigner方程的分析</b>	<b>195</b>
1 Bargmann-Wigner方程	195
1.1 Bargmann-Wigner方程 <sup>[17]</sup>	195
2 二阶矩阵完备展开	195
2.1 二阶矩阵完备泡利基展开	195
2.2 二阶矩阵对称和反对称基展开	195
3 四阶矩阵完备展开	196
3.1 四阶矩阵双泡利基展开	196
3.2 电荷共轭矩阵C <sup>[4,9]</sup>	196
3.3 特殊表象下狄拉克矩阵 <sup>[4,9]</sup>	197
3.4 四阶矩阵狄拉克基 <sup>[4,9]</sup> 展开	197
3.5 四阶矩阵对称和反对称基展开	198
3.6 对称四阶矩阵展开	198
4 自旋-1的Bargmann-Wigner方程 <sup>[17]</sup>	198
4.1 有质量自旋-1的Bargmann-Wigner方程的分析	198
4.2 有质量自旋-1的Bargmann-Wigner方程	200
4.3 无质量自旋-1的Bargmann-Wigner方程的分析	200
4.4 无质量自旋-1的Bargmann-Wigner方程	200

5 自旋 $-\frac{3}{2}, 2$ 的Bargmann-Wigner方程 <sup>[17]</sup> .....	201
5.1 有质量自旋 $-\frac{3}{2}$ 的Bargmann-Wigner方程的分析.....	201
5.2 弯曲时空中有质量自旋 $-\frac{3}{2}$ 的Bargmann-Wigner方程.....	202
5.3 平坦时空中有质量自旋 $-\frac{3}{2}$ 的Bargmann-Wigner方程的源项要求?.....	203
5.4 平坦时空中有质量自旋 $-\frac{3}{2}$ 的Bargmann-Wigner方程 <sup>[18]</sup> .....	203
5.5 平坦时空中有质量自旋 $-2$ 的Bargmann-Wigner方程.....	204
6 平坦时空中任意自旋粒子的Bargmann-Wigner方程.....	204
6.1 平坦时空中有质量自旋 $s = n$ 的Bargmann-Wigner方程 <sup>[17,19,22]</sup> .....	204
6.2 平坦时空中有质量自旋 $s = n + \frac{1}{2}$ 的Bargmann-Wigner方程 <sup>[17,19,20]</sup> .....	205
6.3 用数学归纳法严格证明以上两个定理.....	205
6.4 对有质量自旋的Bargmann-Wigner方程的评述.....	207
6.5 Bargmann-Wigner自旋方程形式.....	207
6.6 Bargmann-Wigner自旋方程形式.....	207
7 反对称性的Dirac方程 <sup>[4]</sup> .....	208
7.1 有质量反对称性Dirac方程的分析.....	208
7.2 无质量反对称性Dirac方程的分析.....	209
7.3 有质量膺标量场方程.....	210
7.4 无质量膺标量场方程.....	210
7.5 有质量膺矢量场方程.....	210
7.6 无质量膺矢量场方程.....	211
<b>第十三章 高级表象变换技术</b> .....	<b>212</b>
1 高级表象变换技巧.....	212
1.1 表象变换与常数不变张量.....	212
1.2 表象变换 $\tilde{S}(s)$ 和常数矩阵 $\Sigma(s)$ 的引入及其性质.....	212
1.3 表象变换矩阵 $\hat{S}(s)$ 的引入.....	214
1.4 关于表象变换矩阵 $\hat{S}(s)$ 的一个重要定理及其证明.....	215
1.5 常数矩阵 $\pi(s, s')$ 的表象变换.....	217
1.6 常数矩阵 $\pi(s, s')$ 的表象变换一般形式.....	217
1.7 表象变换矩阵 $S(s)$ 的引入和性质.....	218
1.8 引力子的表象变换.....	219
1.9 常数矩阵 $\pi(s, s')$ 的深入分析.....	219
1.10 多重态自旋方程.....	220
1.11 自旋 $s = 2$ 的表象变换性质一.....	220
1.12 自旋 $s = 2$ 的表象变换性质二.....	221
1.13 更一般的表象变换性质(猜测).....	221
2 高级表象变换的物理应用.....	221
2.1 一般的新型耦合理论.....	221
2.2 具体的新型耦合理论.....	222
<b>第十四章 洛伦兹变换的深入分析</b> .....	<b>223</b>
1 $3+1$ 维时空的洛伦兹群表示 <sup>[8,12]</sup> .....	223
1.1 彭加勒群表示 <sup>[8]</sup> .....	223
1.2 从自旋张量中提取矢量 $\vec{X}, \vec{Y}, \vec{a}, \vec{b}$ .....	223

1.3	洛伦兹群表示关系命题的正向证明	223
1.4	洛伦兹群表示关系命题的反向证明	224
1.5	洛伦兹群表示关系命题的综合结论	224
2	单粒子的相对论洛伦兹推动变换	225
2.1	坐标的洛伦兹变换 <sup>[23-25]</sup>	225
2.2	速度合成公式	225
2.3	有质量粒子四动量的洛伦兹推动变换	225
2.4	不同速度参考系间的速度洛伦兹变换猜想	225
2.5	无质量粒子四动量的洛伦兹推动变换	226
2.6	单粒子外力的洛伦兹推动变换	226
2.7	单粒子外力的广义相对论变换关系假设	226
3	多粒子质点系的相对论洛伦兹变换	226
3.1	多粒子质点系的洛伦兹推动变换	226
3.2	不同速度参考系质点系间的洛伦兹推动变换	227
3.3	动质点系推动变换到静质点系	227
3.4	静质点系推动变换到动质点系	227
3.5	单方向多光子系的洛伦兹推动变换	227
3.6	质点系的普适静止质量公式	228
3.7	相互作用质点系势能的洛伦兹推动变换假设	228
3.8	多粒子相互作用质点系的洛伦兹推动变换	228
3.9	不同质心速度相互作用质点系间的洛伦兹推动变换	229
3.10	相互作用质点系的普适静止质量公式	229
3.11	氢原子的普适静止质量公式(质心系)???	229
4	各种旋量的洛伦兹推动变换	229
4.1	反对称张量和电磁旋量的洛伦兹变换规律 <sup>[23-25]</sup>	229
4.2	角动量变换规律	230
4.3	从光子旋量变换规律猜测得到旋量的变换规律	230
4.4	从旋量变换规律推得光子旋量的变换规律	231
4.5	从旋量变换规律推得矢量的变换规律	231
4.6	洛伦兹推动变换小结	231
5	各种自旋粒子洛伦兹变换的多项式表示	231
5.1	数学准备	232
5.2	多项式展开的重要定理(终于严密化了2023.09.15)	232
5.3	归一化约束条件下的多项式展开	234
5.4	自然数拆分	238
5.5	平方零约束条件下的多项式展开	239
5.6	更具体的直接求法	240
5.7	中微子旋量的洛伦兹变换	240
5.8	光子旋量洛伦兹变换的多项式表示	241
5.9	引力微子旋量洛伦兹变换的多项式表示	242
5.10	引力子旋量洛伦兹变换的多项式表示	243
5.11	s旋量洛伦兹变换的多项式统一表示	243
5.12	s旋量的洛伦兹推动变换的多项式表示???	244

5.13	矢量洛伦兹变换的多项式表示	245
5.14	矢量洛伦兹变换多项式表示的一般展开	245
5.15	任意运动电荷的电磁场 <sup>[23]</sup>	246
5.16	自旋矢量的变换规律 <sup>[23]</sup>	247
5.17	无质量粒子的角动量变换规律	247
5.18	质点系的角动量变换规律	248
5.19	魏格纳小群 <sup>[37]</sup>	248
<b>第十五章 螺旋度的数学分析</b>		<b>249</b>
1	单位矢量的空间旋转变换	249
1.1	自旋-1旋量的空间旋转变换	249
1.2	单位矢量 $\hat{p}$ 的空间旋转变换	249
1.3	Wigner SO(2)小群	249
1.4	单位矢量 $\hat{p}$ 洛伦兹推动变换 <sup>[25]</sup> 与空间旋转变换的等价性	249
2	螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的分析	250
2.1	螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的具体求法 <sup>[38]</sup>	250
2.2	螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的正交性与完备性	250
2.3	螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的升降算符	250
2.4	螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的基本性质	251
2.5	螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的复杂性质	251
2.6	螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的导数性质	252
2.7	螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数导数性质的继续探索一?	255
2.8	螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数导数性质的继续探索二?	255
3	螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数的分析	255
3.1	螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数的具体求法I	255
3.2	螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数的具体求法II	256
3.3	螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数正交性与完备性的验证	256
3.4	螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数的正交性与完备性	256
3.5	螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数导数的性质	257
3.6	螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数导数性质的小结	258
4	螺旋度 $\gamma \cdot \hat{p}$ 本征函数的分析	258
4.1	螺旋度 $\gamma \cdot \hat{p}$ 本征函数	258
4.2	螺旋度 $\gamma \cdot \hat{p}$ 本征函数的基本性质	259
4.3	螺旋度 $\gamma \cdot \hat{p}$ 本征函数的正交性与完备性	259
4.4	螺旋度 $\gamma \cdot \hat{p}$ 本征函数的复杂性质	259
4.5	螺旋度 $\gamma \cdot \hat{p}$ 本征函数的导数性质一	260
4.6	螺旋度 $\gamma \cdot \hat{p}$ 本征函数的导数性质二?	261
5	螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数的分析	261
5.1	自旋-2洛伦兹变换 $e^{i\omega \cdot \sigma(2)}$	261
5.2	螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数的具体求法I	261
5.3	螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数	262
5.4	螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数的具体求法II	263
5.5	螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数的正交性与完备性	263

5.6 螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数导数性质的小结	263
6 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数的分析	264
6.1 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数的定义	264
6.2 螺旋度 $\sigma(s) \cdot \hat{p}$ 的z-方向本征函数	264
6.3 螺旋度 $\sigma(s) \cdot \hat{p}$ 的一般本征函数	264
6.4 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数的正交性与完备性	264
6.5 自旋洛伦兹变换性质的合理性猜测(还需严格化)	265
6.6 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数的性质	265
6.7 自旋矢量算符本征态 $\lambda(\hat{p}, -s_C)$	266
6.8 自旋矢量算符本征态 $\lambda(\hat{p}, -s_C)$ 的性质	266
6.9 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征态 $\lambda(\hat{p}, h; s)$ 分解为 $\frac{1}{2}$ -自旋本征态	267
6.10 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数的性质	268
6.11 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数的通用解法一	269
6.12 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数的通用解法二	269
6.13 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数的通用解法三	271
6.14 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数性质的小结	272
6.15 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数的通用解法四	272
6.16 特例:1-自旋本征态 $\lambda(\hat{p}, h; 1)$ 分解为 $\frac{1}{2}$ -自旋本征态	273
6.17 螺旋度本征函数的算符化-新数学工具	274
7 螺旋度 $\sigma(s) \cdot \hat{p}$ 的解析延拓(还需严格化)	275
7.1 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}, \hat{p} \in C$ 本征函数的分析	275
7.2 螺旋度 $\sigma(s) \cdot \hat{p}, \hat{p} \in C$ 本征函数的分析	275
7.3 螺旋度 $\sigma(s) \cdot \hat{p}, \hat{p} \in C$ 杂分析	275
<b>第十六章 自旋代数的数学分析</b>	<b>277</b>
1 自旋-单位矢量算符的基本代数性质	277
1.1 $\sigma(s), \hat{p}$ 的基本运算规则	277
1.2 自旋-单位矢量算符叉乘代数的分类	278
2 基本展开式	279
2.1 $\sigma(s)[ \hat{p} ^n]$ 和 $i^n[\hat{p} \times  \hat{p} ^n \sigma(s)]$ 的两种展开式	279
2.2 $[\sigma(s) \cdot \hat{p}]^n \sigma(s)$ 和 $\sigma(s)[\sigma(s) \cdot \hat{p}]^n$ 的递推公式	280
2.3 $i^{-n}[\sigma(s) \times  \hat{p} ^n]$ 和 $i^{-n}[\hat{p} \times \sigma(s)]^n$ 的通项公式	281
3 两类基本自旋复合算符的通项公式	285
3.1 $\sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s)$ 的通项公式	285
3.2 $i^{-1} \sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\}$ 的通项公式	288
4 复杂自旋复合算符的通项公式	292
4.1 $i^{-n}[\sigma(s) \times  \hat{p} ^n \sigma(s)]$ 和 $i^{-n} \sigma(s)[\sigma(s) \cdot \hat{p}]^n$ 的通项公式	292
4.2 $i^{-n}[\sigma(s) \times  \hat{p} ^l \sigma(s)]$ 和 $i^{-n} \sigma(s)[\sigma(s) \cdot \hat{p}]^l$ 的通项公式	293
5 复杂自旋复合算符通项公式的另一种独立求法	296
5.1 $i^{-1} \hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\}\}$ 通项公式的另一种求法	296
5.2 $i^{-n} \hat{p} \cdot [ \sigma(s) \times  \hat{p} ^l \sigma(s)]$ 和 $i^{-n} \hat{p} \cdot [\sigma(s)[\sigma(s) \cdot \hat{p}]^l]$ 通项公式的另一种求法	299

6	各种通项公式汇总小结	302
6.1	基本通项公式小结	302
6.2	基本叉乘型通项公式小结	302
6.3	基本叉乘型扩展通项公式小结	302
6.4	基本标积型通项公式小结	302
6.5	基本标积型扩展通项公式小结	302
6.6	更一般各种通项公式的探讨	303
7	特殊的通项公式	303
7.1	一种特殊的通项公式	303
7.2	该特殊通项公式的推论	303
8	$[\sigma(s) \cdot \hat{p}]^{2s+1}$ 的低阶展开	304
8.1	$[\sigma(s) \cdot \hat{p}]^{2s+1}$ 低阶展开的猜想	304
8.2	$[\sigma(s) \cdot \hat{p}]^{2s+1}$ 的低阶展开式系数方程的严格证明	304
8.3	$[\sigma(s) \cdot \hat{p}]^{2s+1}$ 的低阶展开式求解	305
8.4	$C_{\{a_1, a_2, \dots, a_n\}}^k$ 的性质	306
8.5	$[\sigma(s) \cdot \hat{p}]^{2s+1}$ 的低阶展开式的特例验证	306
8.6	$[\sigma(s) \cdot \hat{p}]^{2s+1}$ 的低阶展开式系数方程的更严格证明	307
8.7	$[\sigma(s) \cdot \hat{p}]^{2s+1+m}$ 的低阶展开式系数	309
9	$e^{\vec{v} \cdot \sigma(s)}$ 的多项式展开	310
9.1	$e^{\vec{v} \cdot \sigma(s)}$ 多项式展开系数的求解(原则上解决了)	310
9.2	多项式展开系数求解的关键I	310
9.3	多项式展开系数求解的关键II	311
10	$[\sigma(s) \cdot \hat{p}]^{2s+1}$ 低阶展开系数的同构性	314
10.1	一个重要的递推关系	314
11	展开系数的同构性	315
11.1	$z^{2s+1}$ 展开系数的同构性	315
11.2	$z^{2s+2}$ 展开系数的同构性	316
11.3	$[\sigma(s) \cdot \hat{p}]^{2s+1}$ 展开系数的同构性	317
11.4	$[\sigma(s) \cdot \hat{p}]^{2s+2}$ 展开系数的同构性	317
12	$e^{\vec{v} \cdot \sigma(s)}$ 低阶展开系数的同构性	317
12.1	一个重要定理及其推论	317
12.2	一个重要猜想及其推论	318
12.3	猜想推论： $e^{\vec{v} \cdot \sigma(n-\frac{1}{2})}$ 多项式展开系数之间的关系	318
12.4	猜想推论： $e^{\vec{v} \cdot \sigma(n)}$ 多项式展开系数之间的关系	318
12.5	$e^{\vec{v} \cdot \sigma(s)}$ 泰勒展开系数的相等性	318
12.6	推理过程的梳理	318
<b>第十七章 自旋代数的推广</b>		<b>320</b>
1	自旋-单位矢量算符的基本代数性质	320
1.1	$\sigma(s; w)$ 的基本前提性质	320
1.2	$\sigma(s; w), p$ 的基本运算规则	320

2 基本展开式	321
2.1 $\sigma(s; w)[ \times \hat{p} ^n$ 和 $i^n[\hat{p} \times ]^n\sigma(s; w)$ 的两种展开式	321
2.2 $[\sigma(s; w) \cdot \hat{p}]^n\sigma(s; w)$ 和 $\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^n$ 的递推公式	322
2.3 $i^{-n}[\sigma(s; w) \times ]^n\hat{p}$ 和 $i^{-n}\hat{p}[ \times \sigma(s; w)]^n$ 的通项公式	323
3 两类基本自旋复合算符的通项公式	325
3.1 $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n\sigma_\alpha(s; w)$ 的通项公式	325
3.2 $i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n\sigma(s; w)\}$ 的通项公式	327
4 复杂自旋复合算符的通项公式	329
4.1 $i^{-n}[\sigma(s; w) \times ]^n[\sigma(s; w) \cdot \hat{p}]^n\sigma(s; w)$ 和 $i^{-n}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}][ \times \sigma(s; w)]^n$ 的通项公式	329
4.2 $i^{-n}[\sigma(s; w) \times ]^n[\sigma(s; w) \cdot \hat{p}]^l\sigma(s; w)$ 和 $i^{-n}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l[ \times \sigma(s; w)]^n$ 的通项公式	330
5 复杂自旋复合算符通项公式的另一种独立求法	334
5.1 $i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n\sigma(s; w)\}\}$ 通项公式的另一种求法	334
5.2 $i^{-n}\hat{p} \cdot [ \sigma(s; w) \times ]^n[\sigma(s; w) \cdot \hat{p}]^l\sigma(s; w)$ 和 $i^{-n}\hat{p} \cdot  \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l[ \times \sigma(s; w)]^n$ 通项公式的另一种求法	335
6 各种通项公式汇总小结	337
6.1 基本通项公式小结	337
6.2 基本叉乘型通项公式小结	338
6.3 基本叉乘型扩展通项公式小结	338
6.4 基本标积型通项公式小结	338
6.5 基本标积型扩展通项公式小结	338
6.6 更一般各种通项公式的探讨	338
7 特殊的通项公式	339
7.1 一种特殊的通项公式	339
<b>第十八章 特殊拟微分算子和矩阵连乘积</b>	<b>340</b>
1 新数学工具的建立	340
1.1 特殊拟微分算子的引入	340
1.2 有质量特殊拟微分算子的基本性质	340
1.3 无质量特殊拟微分算子的基本性质	341
2 四维时空中矩阵连乘迹	342
2.1 自旋矩阵连乘迹 $tr[\sigma_{\alpha_1}(s) \cdots \sigma_{\alpha_n}(s)]$ 的性质	342
2.2 泡利矩阵连乘迹 $tr[\sigma_{\alpha_1} \cdots \sigma_{\alpha_n}]$ 的性质	342
2.3 Dirac矩阵连乘迹 $tr[\gamma_{a_1} \cdots \gamma_{a_n}]$ 的一般性质	342
2.4 Dirac矩阵连乘迹 $tr[\gamma_a(\zeta)\gamma_b(\zeta) \cdots]$ 的具体性质	342
3 $N+1=n$ 维时空中的Dirac矩阵连乘积	343
3.1 $N+1=n$ 维时空中Dirac矩阵连乘积的一个重要定理及其推论	343
3.2 $N+1=n$ 维时空中Dirac矩阵连乘积的重要推论	344
3.3 $N+1=n$ 维时空中Dirac矩阵连乘积的重要猜想	345
3.4 $N+1=n$ 维时空中Dirac矩阵连乘积猜想的优化表述	345
3.5 $N+1=n$ 维时空中前几阶Dirac矩阵连乘积猜想的验证	346
3.6 $N+1=n$ 维时空中Dirac矩阵连乘积常数项的具体计算	348



4	$N+1=n$ 维时空Dirac矩阵连乘积正交性和完备性的证明	349
4.1	$N+1=n$ 维时空Dirac矩阵连乘积正交性的证明	349
4.2	$N+1=n$ 维时空Dirac矩阵连乘积正交性汇总	350
4.3	$N+1=n$ 维时空Dirac矩阵连乘积完备性的证明	350
4.4	$N+1=n$ 维时空中Dirac矩阵连乘迹性质	351
4.5	$N+1=n$ 维时空中Dirac矩阵连乘迹有关定理的严格证明	351
5	$N+1$ 维时空中 $\delta$ 函数积和式的性质	352
5.1	$N+1$ 维时空中 $\delta$ 函数积和式的指标单调轮换求和规律	352
5.2	$w+1$ 阶全对称张量的展开与缩并	354
5.3	$N+1=n$ 维时空中反对称张量的展开与缩并	354
6	$N+1=n$ 维时空中的Q积	355
6.1	$N+1=n$ 维时空中Q积的具体计算	355
<b>第十九章 非相对论性粒子的量子化</b>		<b>357</b>
1	傅里叶分析技巧和平面波解展开 <sup>[39]</sup>	357
1.1	波函数的傅里叶展开	357
1.2	波函数的傅里叶展开与洛仑兹协变性	357
1.3	特殊函数的洛仑兹协变性	358
2	粒子守恒协变量	358
2.1	电流源守恒方程和守恒协变量	358
3	非相对论性粒子	359
3.1	复标量场方程的平面波解	359
3.2	复标量场方程的两个非相对论性分支	360
3.3	正能解薛定谔方程的作用量及其泊松括号	360
3.4	正能解薛定谔方程的平面波解	360
3.5	负能解薛定谔方程的作用量及其泊松括号	361
3.6	负能解薛定谔方程的平面波解	361
4	正、负能解薛定谔方程的二次量子化	362
4.1	正、负能解薛定谔方程的二次量子化讨论	362
4.2	粒子的量子描述	362
5	玻色子的量子描述	362
5.1	玻色子的基本对易关系	362
5.2	玻色子的粒子数算符性质	363
5.3	玻色子的能量动量算符性质	363
5.4	玻色子解的一般构造	364
5.5	玻色子量子态的构造一	364
5.6	玻色子量子态的构造二	364
5.7	玻色子坐标空间和动量空间的对应	364
5.8	玻色子量子态的存在性	365
6	费米子的量子描述	365
6.1	费米子的基本对易关系	365
6.2	费米子的粒子数算符性质	365

6.3 费米子量子态的存在性	366
6.4 占有数表象	366
<b>第二十章 Majorana粒子和中微子的量子化</b>	<b>367</b>
1 洛伦兹推动变换应用—Dirac方程平面波的求解 <sup>[26,27]</sup>	367
1.1 一般表象下Dirac旋量的洛伦兹推动变换	367
1.2 一般表象下Dirac方程的静止解和运动解	367
1.3 特殊表象下Dirac方程的洛伦兹推动变换和平面波解	367
1.4 一般表象下Dirac方程的平面波解	369
1.5 一般表象下Dirac方程的自旋基及其性质	369
1.6 一般表象下Dirac方程的等时量子化	370
1.7 一般表象下Dirac方程的协变反对易规则	370
1.8 一般表象下Dirac方程的守恒荷	371
2 实表象下Majorana方程的平面波解和量子化	372
2.1 严格求解Majorana方程的平面波解 <sup>[26]</sup>	372
2.2 实表象下Majorana方程自旋基的性质	374
2.3 实表象下Majorana方程的平面波解	374
2.4 实表象下Majorana方程的守恒荷	374
2.5 实表象下Majorana方程的量子化	375
3 任意表象下Majorana方程的平面波解和量子化	375
3.1 任意表象下Majorana方程自旋基的性质	375
3.2 任意表象下Majorana方程的平面波解	376
3.3 任意表象下Majorana方程的量子化	376
4 Majorana方程和有质量中微子方程的等价关系	376
4.1 Majorana方程和有质量中微子方程等价的正则反对易关系	376
4.2 Dirac表象下Majorana作用量和中微子作用量	377
5 有质量中微子方程的平面波解和直接量子化 <sup>[40]</sup>	378
5.1 有质量中微子方程自旋基的性质	378
5.2 从Dirac表象下Majorana方程变换得到有质量中微子方程平面波解	379
5.3 有质量中微子方程平面波解系数和量子化条件的直接验证	379
5.4 有质量中微子方程反对易规则小结	382
5.5 有质量中微子方程的三种等价描述小结	382
6 无质量中微子方程的平面波解和量子化初步(后面章节会详细展开)	382
6.1 无质量中微子方程的平面波解	382
6.2 无质量中微子方程的量子化	383
6.3 中微子Weyl方程出发回到Dirac表象	383
7 $s$ -自旋方程的平面波解和另类量子化方案	384
7.1 $s$ -自旋方程的平面波解	384
7.2 $s$ -自旋方程的能量动量算符	384
7.3 $s$ -自旋方程的量子洛伦兹不变性	384
7.4 $s$ -自旋方程与量子洛伦兹不变性的自洽性	384

<b>第二十一章 标量场协变量子化方案</b> .....	<b>385</b>
1 标量场的经典正则量子化方案 <sup>[26, 27, 39, 40]</sup> .....	385
1.1 实标量场的经典描述 .....	385
1.2 实标量场的量子描述 .....	386
1.3 复标量粒子的量子理论 .....	388
2 标量场的协变量子化方案 <sup>[26, 27, 39, 40]</sup> .....	389
2.1 标量场的守恒量 .....	389
2.2 标量场方程及其平面波解 .....	390
2.3 标量场数学上一般的协变对易规则 .....	390
2.4 标量场物理的协变对易规则 .....	391
2.5 标量场的等时对易规则 .....	391
2.6 标量场对易规则小结 .....	391
2.7 单复标量场方程及其平面波解 .....	392
2.8 单个复标量场对易规则 .....	392
2.9 无质量标量场的因果函数 .....	392
2.10 标量场的对易函数、因果函数和费曼传播子 .....	393
3 标量场各种算符的提取 .....	393
3.1 标量场能量动量算符的提取 .....	393
3.2 标量场自旋算符和粒子数算符的提取 .....	394
3.3 标量场空间角动量算符的提取 .....	395
3.4 对易和反对易公式 .....	396
3.5 标量场的彭加莱代数 .....	396
3.6 标量场的彭加莱代数严格证明 .....	397
<b>第二十二章 电磁场协变量子化新方案</b> .....	<b>402</b>
1 电磁场方程的规范势分析 <sup>[23, 25]</sup> .....	402
1.1 有质量电磁场方程的规范势描述 .....	402
1.2 电磁场方程的一般规范势描述 .....	402
1.3 电磁场方程的辐射规范势描述( $\theta = 0$ ) .....	403
1.4 一般电磁场场强协变对易关系分析 .....	406
1.5 电磁场方程的洛伦兹 $\lambda$ -规范势描述( $\theta = \frac{\partial_t \phi}{-\nabla^2}$ )有内在矛盾 .....	407
1.6 洛伦兹规范势和辐射规范势两种描述的等价转换 .....	409
2 辐射规范下的电磁场方程 .....	411
2.1 无源电磁场方程的辐射规范势描述 .....	411
2.2 辐射规范势的洛伦兹变换性质 .....	412
2.3 电磁场方程势解的分析与探讨 .....	412
2.4 辐射规范下电磁场方程 $z$ -方向的势与场平面波解 .....	412
2.5 辐射规范下电磁场方程的势与场平面波通解 .....	412
2.6 辐射规范势解的详细分析 .....	413
3 辐射规范下电磁势的对易规则 <sup>[26, 27, 39, 40]</sup> .....	413
3.1 辐射规范下电磁势的对易-反对易关系 .....	413
3.2 辐射规范下复场强的对易-反对易关系 .....	415
3.3 辐射规范下电磁势和场的同型对易-反对易关系 .....	415

3.4	辐射规范下电磁势和场的异型对易-反对易关系	416
3.5	辐射规范下势和复场强正则对易-反对易关系汇总分析	418
3.6	辐射规范下物理的势和复场强对易关系汇总	419
3.7	辐射规范下经典反对易关系(零模态)	420
4	从传统辐射规范量子化方案导出新方案 <sup>[26,27,39,40]</sup>	420
4.1	从传统辐射规范势等时对易关系得到复场强的等时对易关系	420
4.2	在辐射规范下势对易关系与复场强协变对易关系的等价性	421
4.3	从传统量子化方案得到复场强 $\Psi(\vec{r}, t)$ 表达的能量动量算符	422
4.4	从传统量子化方案得到复场强 $\Psi(\vec{r}, t)$ 算符的运动方程	422
4.5	复场强 $\Psi(\vec{r}, t)$ 算符演化方程和约束方程	422
4.6	复场强 $\Psi(\vec{r}, t)$ 算符的量子标积方程(是否最基本?)	423
4.7	复场强能量动量算符的Fock表示	423
4.8	复场强 $\Psi(\vec{r}, t)$ 算符的Fock对易关系	424
4.9	正规化能量动量算符的Fock表示	425
5	量子电动力学之场方案	425
5.1	电磁相互作用之场表述方案	425
5.2	电磁相互作用之场表述的S矩阵	426
<b>第二十三章 无质量粒子的协变量子化方案</b>		427
1	全新的协变量子化程式	427
1.1	新量子化程式	427
2	无质量复标量场协变量子化方案 <sup>[26,27,39,40]</sup>	427
2.1	复标量场方程及其平面波解	427
2.2	复标量场协变常数不变张量 $\Gamma(0)$ 的性质	427
2.3	复标量场数学上一般的协变对易规则	427
2.4	复标量场物理的协变对易规则	428
2.5	复标量场的对易函数、因果函数和费曼传播子	428
2.6	复标量场能量动量算符的提取	430
2.7	复标量场的彭加莱对称性	430
3	中微子场协变量子化方案	431
3.1	中微子自旋算符方程及其平面波解	431
3.2	中微子自旋算符方程平面波解的洛伦兹变换	431
3.3	中微子场协变常数不变张量的性质	432
3.4	中微子场数学上一般的协变对易规则	433
3.5	中微子场物理的协变反对易规则	434
3.6	中微子场的等时反对易规则	434
3.7	中微子场反对易规则小结	435
3.8	中微子场的对易函数、因果函数和费曼传播子	435
3.9	中微子场能量动量算符的提取	436
3.10	中微子场轻子数算符的提取	437
3.11	中微子场粒子数算符的提取	437
3.12	中微子场角动量算符的提取	437
3.13	中微子场角动量算符小结	441

3.14	中微子场正规化能量动量算符	441
3.15	中微子场的量子方程	441
3.16	数学引理	441
3.17	中微子场的彭加莱对称性	442
4	光子旋量场协变量子化方案	444
4.1	光子旋量自旋算符方程及其平面波解	444
4.2	光子旋量场协变常数不变张量的性质	445
4.3	光子旋量场数学上一般的协变对易规则	445
4.4	光子旋量场的协变对易规则	446
4.5	光子旋量场的等时对易规则	446
4.6	光子旋量场的对易规则小结	447
4.7	电磁场多种旋量形式的等价对易规则	447
4.8	光子旋量场的对易函数、因果函数和费曼传播子	448
4.9	光子旋量场的量子方程	449
4.10	光子旋量场的彭加莱对称性	450
4.11	光子自旋的彭加莱对称性	451
4.12	光子角动量的彭加莱对称性	452
5	电磁场复场强协变量子化新方案	454
5.1	电磁场方程各种等价形式 <sup>[23, 25]</sup>	454
5.2	电磁场复场强自旋方程及其平面波解	454
5.3	电磁场常数不变张量 $\sigma_{\alpha\zeta}^{ab}$ 的性质	454
5.4	电磁场数学上一般的协变对易规则	455
5.5	电磁场物理的协变对易规则	456
5.6	电磁场的等时对易规则	457
5.7	电磁场对易规则小结	458
5.8	电磁场的对易函数、因果函数和费曼传播子(好像跟波戈留波夫的差一个负号)	458
5.9	电磁场能量动量算符的提取	459
5.10	电磁场类电荷算符的提取	460
5.11	电磁场粒子数算符的提取	460
5.12	电磁场能量动量正规化算符	460
5.13	电磁场角动量算符的提取	461
5.14	电磁场角动量算符的小结	463
5.15	对易和反对易公式	465
6	自由无耦合Yang-Mills场协变量子化新方案	465
6.1	自由无耦合Yang-Mills场方程各种等价形式 <sup>[23, 25]</sup>	465
6.2	自由无耦合Yang-Mills场复场强自旋方程及其平面波解	465
6.3	自由无耦合Yang-Mills场对易规则	465
6.4	自由无耦合Yang-Mills场的对易函数、因果函数和费曼传播子	466
6.5	自由无耦合Yang-Mills场在辐射规范下势对易关系与复场强协变对易关系的等价性	466
6.6	在 $\lambda$ -规范下导出自由无耦合Yang-Mills场的协变对易规则	467
7	引力微子场协变量子化方案	467
7.1	引力微子自旋算符方程及其平面波解	467
7.2	引力微子场协变常数不变张量的性质	468

7.3	引力微子场数学上一般的协变对易规则	468
7.4	引力微子场物理的协变反对易规则	469
7.5	引力微子场的等时反对易规则	469
7.6	引力微子场的对易函数、因果函数和费曼传播子	470
7.7	引力微子的量子方程	471
7.8	引力微子的彭加莱对称性	472
7.9	引力微子场的彭加莱对称性	472
8	自由无耦合引力子场协变量子化方案	474
8.1	引力子自旋算符方程及其平面波解	474
8.2	引力子场协变常数不变张量的性质	475
8.3	引力子场数学上一般的协变对易规则	476
8.4	引力子场物理的协变对易规则	476
8.5	引力子场的等时对易规则	477
8.6	引力子场对易规则小结	478
8.7	引力子场的对易函数、因果函数和费曼传播子	478
8.8	引力子场的量子方程	479
8.9	对易和反对易公式	479
8.10	引力子场的第二种量子方程	480
8.11	引力子场的彭加莱对称性	481
<b>第二十四章 s-自旋方程的协变量子化方案</b>		<b>484</b>
1	自旋波函数的傅里叶变换性质(无需满足自旋方程)	484
1.1	坐标-动量空间的一阶对应性质	484
1.2	坐标-动量空间的二阶对应性质	484
1.3	坐标空间和动量空间的高阶对应性质	485
2	坐标空间中的自旋方程	485
2.1	s-自旋方程及其平面波解	485
2.2	动量空间中自旋方程的平面波解	485
2.3	几个重要引理	486
2.4	s-自旋方程的多种等价形式(证明略)	486
2.5	约束方程的多种等价形式	487
2.6	约束方程的重要推论	487
2.7	自旋方程的重要推论	489
3	动量空间中的自旋方程	490
3.1	动量空间中s-自旋方程的多种等价形式	490
3.2	动量空间中s-自旋方程的平面波解	491
3.3	动量空间中平面波解的性质	491
4	自旋和角动量算符的各种性质(满足自旋方程)	492
4.1	自旋波函数的一般性质	492
4.2	角动量算符的性质一	493
4.3	角动量算符的性质二	493
4.4	角动量算符的性质???	494
4.5	角动量算符的性质三	495

4.6	自旋算符的性质一	495
4.7	自旋算符的性质二	495
4.8	自旋算符的性质三	496
5	常数不变张量 $\Gamma_{k_c k'_c}^{abc\cdots}(s)$ 的数学分析	496
5.1	s-自旋场协变常数不变张量 $\Gamma_{k_c k'_c}^{abc\cdots}(s)$ 的性质	496
5.2	重要关系	497
5.3	s-自旋场协变常数不变张量 $\Gamma_{k_c k'_c}^{abc\cdots}(s)$ 的重要定理	498
5.4	算符 $\hat{p}_a, \hat{\partial}_a$ 和 $\Gamma_{+}^{abc\cdots}(s), \Gamma_{-}^{abc\cdots}(s)$ 定义的重新汇总	499
5.5	算符 $\Gamma_{\pm}^{abc\cdots}(s)\partial_a\partial_b\partial_c\cdots$ 和 $\Gamma_{\pm}^{abc\cdots}(s)\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\cdots$ 在质壳条件下的性质	499
5.6	算符 $\Gamma^{abc\cdots}(s)\partial_a\partial_b\partial_c\cdots\Delta(x-x') _{t=t'}$ 的性质	499
5.7	算符 $\Gamma^{abc\cdots}(s)\partial_a\partial_b\partial_c\cdots \partial_{\pi}\Delta(x-x') _{t=t'}$ 的性质	500
5.8	几个重要定理	500
6	s-自旋场的对易规则	501
6.1	对易和反对易公式	501
6.2	s-自旋场数学上一般的协变对易规则	501
6.3	s-自旋场物理的协变对易规则	502
6.4	s-自旋场的等时对易规则	503
6.5	s-自旋场的对易函数、因果函数和费曼传播子	504
6.6	s-自旋场能量动量算符的提取	507
6.7	s-自旋场方程的各种物理算符	509
6.8	s-自旋场能量动量算符的小结	510
6.9	s-自旋场合理的哈密顿型能量动量算符	510
6.10	导出的能量动量算符和角动量算符	511
6.11	s-自旋场量子方程的提取	511
6.12	对易和反对易公式	512
6.13	猜测：s-自旋场的能量动量算符和角动量算符的可能错误表述	512
6.14	s-自旋场的量子方程	512
6.15	s-自旋场的彭加莱对易代数	512
7	s-自旋粒子的彭加莱对称性	513
7.1	玻色子的彭加莱对称性	513
7.2	费米子的彭加莱对称性	515
7.3	费米子自旋的彭加莱对称性	517
8	Penrose全对称方程的协变量子化	519
8.1	Penrose全对称方程 <sup>[1,2]</sup> 及其平面波解	519
8.2	Penrose全对称方程的自旋基及其性质	519
8.3	Penrose全对称方程的各种物理算符	519
8.4	Penrose全对称方程的协变对易规则	521
8.5	光子Penrose全对称方程的协变量子化	522
8.6	光子Penrose一般方程的协变对易规则	523
8.7	Penrose全对称方程平面波的直接解法	523
8.8	从Penrose全对称方程平面波解重新证明协变对易关系	524
8.9	Penrose全对称方程的等时量子化规则	525
8.10	Penrose全对称方程的对易函数、因果函数和费曼传播子	526

<b>第二十五章 有质量复粒子的场协变量子化方案</b> .....	<b>528</b>
<b>1 基础准备部分</b> .....	<b>528</b>
1.1 Dirac自旋基的引入 .....	528
1.2 四维矢量自旋基的引入 .....	530
1.3 Dirac自旋基的数学分析 .....	533
1.4 Dirac自旋基与四维矢量基之间关系的分析 .....	535
1.5 二阶Bargmann-Wigner自旋基与四维矢量基之间的关系 .....	538
1.6 Dirac方程的协变反对易规则 .....	541
<b>2 重要的组合学恒等式</b> .....	<b>541</b>
2.1 基本恒等式 .....	541
2.2 一组之前章节的组合学恒等式 .....	541
2.3 一个组合学恒等式及其证明 .....	542
2.4 一个重要组合学恒等式及其证明 .....	542
2.5 以上重要组合学恒等式的推论 .....	542
<b>3 Bargmann-Wigner方程<sup>[17]</sup>的自旋基和平面波解</b> .....	<b>543</b>
3.1 Dirac方程自旋基的广义二项式定理及其推论 .....	543
3.2 Bargmann-Wigner方程平面波解的合理推测(本章节后面将会给出严格的证明) .....	544
3.3 实表象下Bargmann-Wigner方程的平面波解 .....	544
3.4 Bargmann-Wigner方程的自旋基 .....	545
3.5 Bargmann-Wigner方程的两种自旋基之间的关系 .....	545
3.6 Bargmann-Wigner方程自旋基的正交性质(可以直接看出来) .....	546
<b>4 Bargmann-Wigner方程自旋基的递推关系</b> .....	<b>546</b>
4.1 Bargmann-Wigner方程自旋基递推关系(枚举试探法) .....	546
4.2 Bargmann-Wigner方程U-自旋基的分解(组合学方法) .....	548
4.3 Bargmann-Wigner方程V-自旋基的分解(组合学方法) .....	548
4.4 Bargmann-Wigner方程自旋基的合成 .....	549
4.5 Bargmann-Wigner方程自旋基合成的推论 .....	549
<b>5 Bargmann-Wigner方程的准投影算子</b> .....	<b>550</b>
5.1 Bargmann-Wigner方程准投影算子的定义和性质 .....	550
5.2 Bargmann-Wigner方程真正的两类投影算子 .....	550
5.3 Bargmann-Wigner方程准投影算子的递推关系(严格证明) .....	551
5.4 Bargmann-Wigner方程准投影算子之间的关系 .....	552
<b>6 Bargmann-Wigner方程的对易规则</b> .....	<b>552</b>
6.1 Bargmann-Wigner方程的协变对易规则 .....	552
6.2 Bargmann-Wigner方程协变对易规则的反向推理 .....	554
6.3 Bargmann-Wigner方程协变对易规则的小结 .....	557
6.4 Bargmann-Wigner方程的对易函数、因果函数和费曼传播子 .....	557
6.5 可分离表象下Bargmann-Wigner方程的协变量子化规则的重要推论 .....	559
6.6 Bargmann-Wigner方程对易规则两种描述的等价性证明 .....	560
6.7 有质量玻色子对易规则小结 .....	561
6.8 有质量费米子反对易规则小结 .....	561



7 Bargmann-Wigner方程 <sup>[17]</sup> 各种量子算符的提取	561
7.1 Bargmann-Wigner方程的等时对易规则	561
7.2 Bargmann-Wigner方程能量算符的提取	562
7.3 Bargmann-Wigner方程动量算符的提取	563
7.4 Bargmann-Wigner方程能量动量算符的小结	565
7.5 Bargmann-Wigner方程的各种物理算符	565
7.6 Bargmann-Wigner方程的量子方程	567
7.7 Bargmann-Wigner方程玻色子能量动量算符	568
7.8 Bargmann-Wigner方程的玻色子量子方程	568
<b>第二十六章 有质量复粒子的势协变量子化方案</b>	<b>570</b>
1 Klein-Gordon方程的对易规则	570
1.1 有质量自旋-n的Bargmann-Wigner方程等价于Klein-Gordon方程 <sup>[17,19,22]</sup>	570
1.2 Bargmann-Wigner方程玻色自旋基分解为自旋-1基	571
1.3 Klein-Gordon方程自旋基的分解	573
1.4 数学归纳法严格证明Klein-Gordon方程自旋基的完整分解	574
1.5 Klein-Gordon方程准投影算子的递推关系	576
1.6 推导到自旋-n粒子Klein-Gordon方程的平面波解	577
1.7 自旋-n粒子Klein-Gordon方程自旋基的正确猜想(最初的想法仍保留)	578
1.8 自旋-n粒子Klein-Gordon方程的平面波解	580
2 K-G方程自旋基和准投影算子的几个例子	580
2.1 自旋-1粒子Klein-Gordon方程的准投影算子	580
2.2 自旋-1粒子C-K自旋基和1-自旋基的关系	580
2.3 自旋-2粒子Klein-Gordon方程的准投影算子	580
2.4 另一种证法	581
2.5 自旋-3粒子Klein-Gordon方程的CG系数和自旋基	584
2.6 自旋-n粒子Klein-Gordon方程的准投影算子	584
3 Rarita-Schwinger方程的反对易规则	585
3.1 有质量自旋- $n + \frac{1}{2}$ 的Bargmann-Wigner方程等价于Rarita-Schwinger方程 <sup>[17,19,20]</sup>	585
3.2 自旋 $s = n + \frac{1}{2}$ 粒子Rarita-Schwinger方程的自旋基	586
3.3 自旋 $s = n + \frac{1}{2}$ 粒子Rarita-Schwinger方程的平面波解	587
3.4 自旋- $\frac{n}{2}$ 粒子Rarita-Schwinger方程的自旋基	588
4 R-S方程自旋基和准投影算子的几个例子	589
4.1 自旋-1粒子R-S自旋基和Dirac自旋基的关系	589
4.2 自旋-2粒子R-S自旋基和Dirac自旋基的关系	590
4.3 自旋- $\frac{3}{2}$ 粒子Rarita-Schwinger方程的准投影算子	590
4.4 自旋- $n + \frac{1}{2}$ 粒子Rarita-Schwinger方程的准投影算子猜想证明	591
4.5 自旋- $n + \frac{1}{2}$ 粒子Rarita-Schwinger方程自旋基的合成	594
5 各种势准投影算子的缩减模式	595
5.1 各种势准投影算子的正式定义	595
5.2 各种势准投影算子之间的关系-最简缩减模式	595
5.3 各种势准投影算子最简缩减模式的重新整理	597
5.4 各种势准投影算子之间的关系-物理缩减模式	597

5.5 各种势准投影算子物理缩减模式的重新整理	598
5.6 各种势准投影算子的通用性质	598
6 势对易规则的直接解法(等价变换法)	599
6.1 引理	599
6.2 势 $A_{abc}$ 对易规则的变换求法	599
6.3 势 $A_{abc \cdot \tau_c}$ 反对易规则的变换求法	600
6.4 Bargmann-Wigner全对称方程的等时量子化规则	600
7 有质量粒子协变对易规则的总结与梳理	601
7.1 有质量玻色子协变对易规则梳理	601
7.2 有质量玻色子对易规则与准投影算子之间的关系	602
7.3 有质量费米子协变反对易规则梳理	602
7.4 有质量费米子对易规则与准投影算子之间的关系	603
<b>第二十七章 有质量实粒子的协变量子化方案</b>	<b>604</b>
1 Majorana方程	604
1.1 实表象和Dirac分离表象下的Majorana方程 <sup>[4]</sup>	604
1.2 实表象和Dirac分离表象下的Majorana条件	604
2 Majorana B-W方程	604
2.1 实表象和Dirac分离表象下的Majorana B-W方程 <sup>[17]</sup>	604
2.2 实表象和Dirac分离表象下的Majorana条件	604
3 分离表象下Majorana B-W方程的平面波解	605
3.1 引理	605
3.2 分离表象下Majorana B-W方程 <sup>[17]</sup> 的平面波解(证明还需补上)	605
3.3 分离表象下Majorana B-W方程的协变对易规则	606
3.4 分离表象下Majorana B-W方程协变对易规则的反向推理	608
3.5 分离表象下Majorana B-W方程协变对易规则的小结	612
3.6 分离表象下Majorana B-W方程的协变量子化规则的重要推论	612
4 分离表象下Majorana B-W方程对易规则的两种等价表述	613
4.1 分离表象下Majorana B-W方程的等价协变对易规则	613
4.2 分离表象下有质量Majorana玻色子对易规则小结	613
4.3 分离表象下有质量Majorana费米子反对易规则小结	614
4.4 分离表象下推导到自旋-n粒子Klein-Gordon方程的平面波解	614
5 分离表象下Majorana B-W方程各种量子算符	614
5.1 分离表象下Majorana B-W方程各种算符的提取	614
6 Klein-Gordon方程的对易规则	614
6.1 有质量自旋-n的Majorana B-W方程等价于Klein-Gordon方程 <sup>[17, 19, 22]</sup>	614
6.2 自旋-n粒子Klein-Gordon方程的平面波解	615
7 Rarita-Schwinger方程的对易规则	615
7.1 有质量自旋 $-n + \frac{1}{2}$ 的Majorana B-W方程等价于Rarita-Schwinger方程 <sup>[17, 19, 20]</sup>	615
7.2 分离表象下自旋 $s = n + \frac{1}{2}$ 粒子Rarita-Schwinger方程的平面波解	616
7.3 分离表象下Majorana B-W方程的等时量子化规则	616

8 分离表象下有质量Majorana粒子协变对易规则的总结与梳理	616
8.1 定义	616
8.2 分离表象下有质量Majorana玻色子协变对易规则梳理(取 $\theta = 0$ )	617
8.3 分离表象下有质量Majorana费米子协变反对易规则梳理	618
8.4 实表象和Dirac分离表象的Majorana方程	619
9 实表象下Bargmann-Wigner方程	620
9.1 实表象B-W自旋基	620
9.2 实表象下B-W自旋基之间关系	620
9.3 实表象下B-W自旋基和矢量基之间的关系	620
9.4 实表象下矢量基	621
9.5 实表象下Majorana B-W方程 <sup>[17]</sup> 平面波解	621
9.6 实表象下Majorana B-W方程各种算符的提取	622
9.7 实表象下Klein-Gordon方程和Rarita-Schwinger方程的平面波解	622
9.8 实表象下Majorana B-W方程协变对易规则	623
10 实表象下有质量Majorana粒子协变对易规则的总结与梳理	623
10.1 实表象下有质量Majorana玻色子协变对易规则梳理(取 $\theta = 0$ )	623
10.2 实表象下有质量Majorana费米子协变反对易规则梳理	624
<b>第二十八章 各种对称和反对称方程的平面波解</b>	<b>627</b>
1 Bargmann-Wigner方程的平面波解	627
1.1 两个推论	627
1.2 两个关于U-自旋基的引理	627
1.3 两个关于V-自旋基的引理	628
1.4 用数学归纳法严格求解Bargmann-Wigner方程的平面波解	630
2 N+1维时空中Bargmann-Wigner方程的平面波解	632
2.1 N+1维时空中Bargmann-Wigner方程U-自旋基的性质	632
2.2 N+1维时空中Bargmann-Wigner方程V-自旋基的性质	638
2.3 用数学归纳法严格求解N+1维时空中Bargmann-Wigner方程的平面波解	644
3 K-G方程平面波解的直观解法	646
3.1 直观解法的数学基础	646
3.2 K-G方程平面波解的直观解法?	649
4 四维时空中反对称张量场方程的平面波解	651
4.1 自旋-1粒子Klein-Gordon方程的平面波解	651
4.2 n=N+1维时空中自旋-1粒子Klein-Gordon方程的平面波解	651
4.3 n=N+1维时空中自旋-2粒子Klein-Gordon方程的平面波解	652
<b>第二十九章 自旋基和CG系数的数学分析</b>	<b>653</b>
1 Bargmann-Wigner方程自旋基的重新梳理分析	653
1.1 Dirac自旋基是自旋、螺旋度和电荷三个算符的共同本征态	653
1.2 Dirac方程准投影算子 $\gamma^a = (\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x)$	653
1.3 特殊表象 $\gamma^a = (\sigma \otimes \sigma_y, I \otimes \sigma_x; I \otimes \sigma_z)$ 下相应自旋基	653
1.4 自旋基组合性质 $(\gamma_a, \gamma_5) := [(\sigma \otimes \sigma_y, I \otimes \sigma_z), -I \otimes \sigma_x]$	654
1.5 定义-自旋基分解: $s = (s - \frac{1}{2}) \oplus \frac{1}{2}$	654

1.6 推论- $U_{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)$ 是自旋本征态	655
1.7 特殊表象 $\gamma^a = (\sigma \otimes \sigma_y, I \otimes \sigma_z; -I \otimes \sigma_x)$ 下Dirac自旋基的升降算符	655
1.8 一般表象Dirac自旋基的升降算符	657
1.9 Bargmann-Wigner方程自旋基的升降算符	658
1.10 推论- $U_{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)$ 的正交性	660
1.11 推论-自旋基分解: $1 = \frac{1}{2} \oplus \frac{1}{2}$	660
1.12 推论-自旋基分解: $0 = \frac{1}{2} \ominus \frac{1}{2}$	660
1.13 推论-自旋基分解: $s = (s - 1) \oplus 1$	661
1.14 推论-自旋基分解: $s + s' = s \oplus s'$	662
1.15 推论-自旋基反向合成	663
1.16 推论-自旋基分解: $s_1 + s_2 + s_3 = s_1 \oplus s_2 \oplus s_3$	663
1.17 推论-自旋基分解: $s_1 + s_2 \dots + s_n = s_1 \oplus s_2 \dots \oplus s_n$	664
1.18 一个重要的数学推论	664
1.19 推论-自旋基分解: $s = \frac{1}{2} \oplus \frac{1}{2} \dots \oplus \frac{1}{2}$	664
1.20 推论- $U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)$ 的全对称性	664
1.21 有质量 $s$ -自旋粒子的准投影算子	665
1.22 有质量1-自旋粒子的准投影算子	666
1.23 Bargmann-Wigner方程平面波解的算子表述	666
2 Klein-Gordon方程自旋基的重新梳理分析	666
2.1 静止系的有质量矢量自旋基	666
2.2 $z$ 轴方向的有质量矢量自旋基	667
2.3 $e^{i\vec{\omega} \cdot R} R e^{-i\vec{\omega} \cdot R}$ 的性质	668
2.4 $e^{i\vec{\omega} \cdot R} L e^{-i\vec{\omega} \cdot R}$ 的性质	669
2.5 $\varepsilon(\vec{p}, h)$ 的升降算符	669
2.6 定义-自旋基分解: $n = (n - 1) \oplus 1$	670
2.7 推论- $\varepsilon_{a \dots bc}(\vec{p}, h)$ 是自旋本征态	670
2.8 Klein-Gordon方程自旋基的升降算符	671
2.9 推论- $\varepsilon_{ab \dots c}(\vec{p}, h)$ 的正交性	673
2.10 推论- $p^a \varepsilon_{ab \dots c}(\vec{p}, h)$ 的零性	673
2.11 推论- $\varepsilon_{ab \dots c}(\vec{p}, h)$ 的无迹性	674
2.12 推论-自旋基分解: $2 = 1 \oplus 1$	674
2.13 推论-自旋基分解: $n = (n - 2) \oplus 2$	675
2.14 推论-自旋基分解: $n + n' = n \oplus n'$	676
2.15 推论-自旋基反向合成	677
2.16 推论-自旋基分解: $n_1 + n_2 \dots + n_n = n_1 \oplus n_2 \dots \oplus n_n$	677
2.17 一个重要的数学推论	677
2.18 推论-自旋基分解: $s = 1 \oplus 1 \dots \oplus 1$	677
2.19 推论- $\varepsilon_{ab \dots c}(\vec{p}, h)$ 的全对称性	678
2.20 $\varepsilon_{ab \dots c}(\vec{p}, h)$ 性质的小结	678
2.21 Klein-Gordon方程平面波解的算子表述	678
3 Rarita-Schwinger方程自旋基的重新梳理分析	678
3.1 定义-自旋基分解: $n + \frac{1}{2} = n \oplus \frac{1}{2}$	678
3.2 推论- $\varepsilon_{ab \dots c \tau_\zeta}(\vec{p}, h)$ 是自旋本征态	679
3.3 Rarita-Schwinger方程自旋基的升降算符	679

3.4	推论- $\varepsilon_{ab\cdots c\tau_\zeta}(\vec{p}, h)$ 的正交性	680
3.5	推论- $p^a \varepsilon_{ab\cdots c\tau_\zeta}(\vec{p}, h)$ 的零性	680
3.6	推论- $\varepsilon_{ab\cdots c\tau_\zeta}(\vec{p}, h)$ 的无迹性	680
3.7	推论- $\varepsilon_{ab\cdots c\tau_\zeta}(\vec{p}, h)$ 的全对称性	681
3.8	推论-自旋基分解: $n + n' + \frac{1}{2} = n \oplus n' + \frac{1}{2}$	681
3.9	推论-自旋基反向合成	681
3.10	推论-自旋基分解: $n_1 + n_2 \cdots + n_n + \frac{1}{2} = n_1 \oplus n_2 \cdots \oplus n_n \oplus \frac{1}{2}$	681
3.11	Rarita-Schwinger方程平面波解的算子表述	681
4	Penrose方程自旋基的重新梳理分析	682
4.1	中微子自旋基是自旋、螺旋度的共同本征态	682
4.2	定义-自旋基分解: $s = (s - \frac{1}{2}) \oplus \frac{1}{2}$	682
4.3	推论- $\lambda_{A_\zeta \cdots B_\zeta C_\zeta}(\hat{p}, h)$ 是自旋本征态	682
4.4	Penrose方程自旋基的升降算符	682
4.5	推论- $\lambda_{A_\zeta \cdots B_\zeta C_\zeta}(\hat{p}, h)$ 的正交性	684
4.6	推论-自旋基分解: $1 = \frac{1}{2} \oplus \frac{1}{2}$	685
4.7	推论-自旋基分解: $0 = \frac{1}{2} \ominus \frac{1}{2}$	685
4.8	推论-自旋基分解: $s = (s - 1) \oplus 1$	685
4.9	推论-自旋基分解: $s + s' = s \oplus s'$	686
4.10	推论-自旋基反向合成	687
4.11	推论-自旋基分解: $s_1 + s_2 + s_3 = s_1 \oplus s_2 \oplus s_3$	687
4.12	推论-自旋基分解: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$	688
4.13	一个重要的数学推论	688
4.14	推论-自旋基分解: $s = \frac{1}{2} \oplus \frac{1}{2} \cdots \oplus \frac{1}{2}$	688
4.15	推论- $\lambda_{A_\zeta \cdots B_\zeta C_\zeta}(\hat{p}, h)$ 的全对称性	688
5	自旋方程自旋基的重新梳理分析	689
5.1	定义-自旋基分解: $s = (s - \frac{1}{2}) \oplus \frac{1}{2}$	689
5.2	推论- $\lambda(\hat{p}, h; s)$ 是自旋本征态	689
5.3	推论-自旋本征态 $\lambda(\hat{p}, h; s)$ 的升降算符	689
5.4	推论- $\lambda_{A_\zeta \cdots B_\zeta C_\zeta}(\hat{p}, h)$ 的正交性	690
5.5	推论-自旋基分解: $1 = \frac{1}{2} \oplus \frac{1}{2}$	690
5.6	推论-自旋基分解: $s + s' = s \oplus s'$	690
5.7	推论-自旋基分解: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$	690
5.8	引入一个新的常数不变张量	690
6	关于自旋基的么正变换	690
6.1	常数不变张量:动量变换等价于么正变换	690
6.2	物理自旋基分解: $1 = \frac{1}{2} \oplus \frac{1}{2}$	691
6.3	么正变换 $\rightarrow$ B-W方程自旋基分解的z-方向表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$	692
6.4	么正变换 $\rightarrow$ B-W方程自旋基分解的静止表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$	692
6.5	么正变换 $\rightarrow$ K-G方程自旋基分解的z-方向表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$	693
6.6	么正变换 $\rightarrow$ K-G方程自旋基分解的静止表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$	693
6.7	么正变换 $\rightarrow$ Penrose方程自旋基分解的z-方向表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$	693
6.8	么正变换 $\rightarrow$ 自旋方程之自旋基分解的z-方向表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$	693

7 么正变换→自旋基分解的真实物理表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$ .....	694
7.1 有质量任意自旋粒子情形 .....	694
7.2 有质量整数自旋粒子情形 .....	695
7.3 无质量任意自旋粒子情形 .....	696
7.4 自我评述 .....	696
8 自旋耦合和CG系数的一般理论 <sup>[34, 41-43]</sup> .....	697
8.1 单自旋本征态 .....	697
8.2 多自旋耦合本征态 .....	697
8.3 自旋本征态展开 .....	697
8.4 自旋本征态的矩阵变换 .....	697
8.5 自旋本征态的正交性 .....	698
8.6 自旋本征态的完备性 .....	698
8.7 两个角动量耦合CG系数的一般Racah公式 .....	698
8.8 两个粒子合成一个粒子CG系数的Racah公式 .....	698
8.9 两个粒子合成一个粒子CG系数的具体表述 .....	699
9 多个粒子合成一个粒子的CG系数公式 <sup>[34, 41-43]</sup> .....	700
9.1 多个光子合成一个粒子的CG系数公式 .....	700
9.2 特殊情形下多个角动量耦合的CG系数公式 .....	701
9.3 多个粒子合成一个粒子的CG系数公式 .....	702
10 两个角动量耦合CG系数的一些特例 .....	702
10.1 特例: $\frac{1}{2} \oplus \frac{1}{2} = (0, 0)$ .....	702
10.2 特例: $1 \oplus 1 = (0, 0)$ .....	703
10.3 特例: $1 \oplus 1 = (1, 0)$ .....	703
10.4 特例: $1 \oplus 1 = (1, 1)$ .....	704
10.5 特例: $1 \oplus 1 = (1, -1)$ .....	704
10.6 特例: $1 \oplus 1 = (1, -1)$ .....	704
11 不变张量算符 .....	704
11.1 不变张量算符 .....	704
12 球谐函数 .....	705
12.1 数学准备 .....	705
12.2 球谐函数 .....	705
12.3 一般算子方法 <sup>[34]</sup> .....	706
<b>第三十章 B-F公式和投影算子猜想</b> .....	<b>707</b>
<b>1 多项式定理及其全对称指标推广</b> .....	<b>707</b>
1.1 零阶全对称指标二项式展开 .....	707
1.2 零阶全对称指标多项式展开 .....	707
1.3 一阶全对称指标二项式展开 .....	707
1.4 一阶全对称指标多项式展开 .....	707
1.5 二阶全对称指标二项式展开 .....	707
1.6 二阶全对称指标多项式展开 .....	708
1.7 多阶全对称指标多项式展开 .....	708
1.8 全对称指标多项式性质 .....	708

2	多项式定理及其反对称指标推广	708
2.1	一阶全对称指标二项式展开	708
2.2	一阶全对称指标多项式展开	709
2.3	反对称指标二项式展开	709
2.4	反对称指标多项式展开(错的?)	709
3	自旋-n粒子Klein-Gordon方程 <sup>[22]</sup> 投影算子	709
3.1	自旋-n粒子Klein-Gordon方程投影算子的经典表述	709
3.2	自旋-n粒子Klein-Gordon方程投影算子的定义表述	710
3.3	自旋-n粒子Klein-Gordon方程投影算子的重要猜想	710
4	从物理到数学抽象：投影算子猜想	711
4.1	从自旋-n粒子Klein-Gordon方程投影算子引出的数学猜想	711
4.2	投影算子猜想得到的新组合学恒等式	711
4.3	以上组合学恒等式的推论	711
4.4	投影算子猜想的 $C(2i, 2j)$ 项情形	713
4.5	投影算子猜想的 $C(2i+1, 2j+1)$ 项情形	714
4.6	与投影算子猜想等价的组合学恒等式	715
4.7	投影算子组合学恒等式的等价分析	715
4.8	投影算子组合学恒等式的严格证明???	715
4.9	投影算子组合学恒等式特例的试探	716
5	投影算子猜想具体情形表述和验证	716
5.1	取值范围分析	716
5.2	$i=0, j=0$ 情形	717
5.3	$i=1, j=0$ 和 $i=0, j=1$ 情形	718
5.4	$i=1, j=1$ 情形	719
6	Klein-Gordon方程投影算子递推关系的分析	720
6.1	自旋-1粒子Klein-Gordon方程投影算子的基本性质	720
6.2	自旋-2粒子Klein-Gordon方程投影算子的基本性质	720
6.3	自旋-3粒子Klein-Gordon方程投影算子的基本性质	720
6.4	自旋-n粒子Klein-Gordon方程投影算子的基本性质	721
6.5	自旋-2粒子Klein-Gordon方程投影算子的展开(唯一)	721
6.6	自旋-3粒子Klein-Gordon方程投影算子的展开(不唯一, 故无大的物理意义)	721
6.7	自旋-4粒子Klein-Gordon方程投影算子的展开(不唯一且复杂, 暂时放放)	722
6.8	自旋-n粒子Klein-Gordon方程投影算子的展开(不唯一且复杂)	723
7	平移准投影算子	723
7.1	自旋-1基的完备性	723
7.2	一般自旋基的完备性猜想	723
7.3	质量猜想(高维时空中的运动)	723
7.4	数学准备(以前章节的结论)	723
7.5	二阶平移准投影算子	724
<b>第三十一章 有质量矢量粒子的协变量子化方案</b>		<b>726</b>
1	有质量矢量粒子两种等价描述的相互转换	726
1.1	自旋-1粒子的B-W全对称方程和K-G矢量方程两种等价描述 <sup>[17, 19, 22]</sup>	726

1.2	自旋-1粒子Bargmann-Wigner方程 <sup>[17]</sup> 的平面波解	726
1.3	推导到自旋-1粒子Klein-Gordon方程 <sup>[26,39,40]</sup> 的平面波解	727
1.4	再回到自旋-1粒子Bargmann-Wigner方程 <sup>[17]</sup> 的平面波解	727
2	有质量矢量粒子方程的第三种等价描述	728
2.1	有质量矢量粒子自旋方程等价描述	728
2.2	有质量矢量粒子场 $F_{ab}$ 的平面波解和投影算子	728
2.3	有质量矢量粒子场 $\Psi_{\alpha_\zeta}$ 的平面波解和投影算子	728
2.4	有质量矢量粒子方程的第三种等价描述小结	729
2.5	$m \rightarrow 0$ 形式推得光子情形	729
3	有质量矢量场对易规则两种表述等价性的直观证明方法	730
3.1	数学准备	730
3.2	有质量矢量场对易规则两种表述等价性的第一种直观证明方法	731
3.3	有质量矢量场对易规则两种表述等价性的第二种直观证明方法	732
4	自旋-1粒子Bargmann-Wigner方程各种量子算符的提取	733
4.1	自旋-1粒子Bargmann-Wigner方程 <sup>[17]</sup> 的等时对易规则	733
4.2	自旋-1粒子Bargmann-Wigner方程能量算符的提取	733
5	有质量Majorana矢量场(取 $\theta = 0$ )	734
5.1	有质量Majorana矢量场与有质量矢量场的对比	734
5.2	不依赖方程导出自旋-1粒子各种物理量之间等价对易规则	734
5.3	$\psi_{\lambda_\zeta\mu_\zeta}$ 与 $(A_a, F_{ab})$ 的等价对易规则	734
5.4	从 $A_a$ 导出 $F_{ab}$ 的对易关系	736
5.5	$\psi_{\alpha_\zeta}$ 与 $F_{ab}$ 的等价对易关系	737
5.6	$\psi_{\alpha_\zeta}$ 与 $\psi_{A_\zeta B_\zeta}$ 的等价对易关系	740
5.7	$\psi_{k_\zeta}$ 与 $\psi_{A_\zeta B_\zeta}$ 的等价对易关系	741
5.8	$\psi_{\alpha_\zeta}$ 与 $\psi_{k_\zeta}$ 的等价对易关系	742
5.9	有质量复矢量场 $\psi_{\lambda_\zeta\mu_\zeta}$ 和复势 $A_a$ 的等价对易关系	743
5.10	自旋-1 Majorana粒子从实势出发导出各种物理量对易规则	743
5.11	有质量Majorana矢量场对易关系	744
<b>第三十二章 有质量引力微子的协变量子化方案</b>		<b>746</b>
1	有质量引力微子两种等价描述的相互转换	746
1.1	自旋- $\frac{3}{2}$ 粒子的B-W全对称方程和R-S矢量方程两种等价描述 <sup>[17,19,20]</sup>	746
1.2	自旋- $\frac{3}{2}$ 粒子Bargmann-Wigner方程 <sup>[17]</sup> 的平面波解	746
1.3	推导到自旋- $\frac{3}{2}$ 粒子R-S方程的平面波解	747
1.4	再回到自旋- $\frac{3}{2}$ 粒子Bargmann-Wigner方程 <sup>[17]</sup> 的平面波解	749
1.5	自旋- $\frac{3}{2}$ 粒子场 $F_{ab\tau_\zeta}, \psi_{\alpha_\zeta\tau_\zeta}$ 的反对易关系	749
1.6	有质量引力微子场能量动量算符的提取	749
<b>第三十三章 有质量引力子的协变量子化方案</b>		<b>751</b>
1	有质量引力子两种等价描述的相互转换	751
1.1	自旋-2粒子的B-W全对称方程和K-G矢量方程两种等价描述 <sup>[17,19,22]</sup>	751
1.2	自旋-2粒子Bargmann-Wigner方程 <sup>[17]</sup> 的平面波解	751
1.3	推导到自旋-2粒子Klein-Gordon方程 <sup>[17]</sup> 的平面波解	752
1.4	再回到自旋-2粒子Bargmann-Wigner方程 <sup>[17]</sup> 的平面波解	752



2 有质量引力子方程的第三种等价描述	753
2.1 有质量引力子自旋方程等价描述	753
2.2 有质量引力子场 $F_{ab}$ 的平面波解和投影算子	753
2.3 有质量引力子场 $\Psi_{\alpha\kappa}$ 的平面波解和投影算子	753
3 有质量引力子方程的第四种等价描述	754
3.1 有质量引力子各种物理量定义	754
3.2 有质量粒子一般对易规则两种等价描述	754
3.3 线性引力场 $\psi_{\alpha\kappa\beta\kappa}$ 的对易规则	755
3.4 线性引力场 $C_{abcd}$ 的对易规则	755
4 有质量引力子势 $A_{ab}$ 的协变对易规则复杂的直接计算	756
4.1 数学准备	756
4.2 反过来验证场对易关系	758
4.3 有质量引力子场能量动量算符的提取	759
5 四维时空中有质量自旋 $s = 2$ 的势对易关系求解新方法(试探还有错)	760
5.1 四维时空中有质量自旋 $s = 2$ 的Bargmann-Wigner方程势对易关系	760
5.2 逐项求解一	762
5.3 逐项求解二	764
5.4 逐项求解三	764
<b>第三十四章 高维时空有质量粒子的协变量子化</b>	<b>766</b>
1 $N+1$ 维时空中的洛伦兹推动变换	766
1.1 $N+1$ 维时空中矢量的洛伦兹推动变换	766
1.2 $N+1$ 维时空中中微子旋量的洛伦兹推动变换	766
1.3 $N+1$ 维时空中电子旋量的洛伦兹推动变换	766
1.4 $N+1$ 维时空中光子旋量的洛伦兹推动变换的多项式表示	766
1.5 $N+1$ 维时空中引力微子旋量的洛伦兹推动变换的多项式表示	766
1.6 $N+1$ 维时空中引力子旋量的洛伦兹推动变换的多项式表示	766
2 $N+1$ 维时空中电子的协变量子化	767
2.1 $N+1$ 维时空中的电子方程 <sup>[4]</sup>	767
2.2 $N+1$ 维时空中的电子静止解和运动解	767
2.3 $N+1$ 维时空中电子平面波解的性质	767
2.4 $N+1$ 维时空中的电子自旋基	768
2.5 $N+1$ 维时空中Dirac自旋基是自旋、螺旋度和电荷三个算符的共同本征态	768
2.6 $N+1$ 维时空中的电子自旋空间	768
2.7 $N+1$ 维时空中的电子自旋基性质(在一般表象下也成立)	768
2.8 $N+1$ 维时空中电子的平面波解	768
2.9 $N+1$ 维时空中电子的协变量子化规则	769
2.10 $N+1$ 维时空中Dirac方程的守恒荷	769
3 $N+1$ 维时空中Bargmann-Wigner方程的协变量子化	770
3.1 $N+1$ 维时空中的电子正常自旋基性质(在一般表象下也成立)	770
3.2 $N+1$ 维时空中Dirac方程自旋基的广义多项式定理	770
3.3 $N+1$ 维时空中Bargmann-Wigner方程的自旋基	771
3.4 $N+1$ 维时空中Bargmann-Wigner方程自旋基的正交性质(可以直接看出来)	771

3.5	$N+1$ 维时空中Bargmann-Wigner方程U-自旋基的分解	771
3.6	$N+1$ 维时空中Bargmann-Wigner方程V-自旋基的分解	772
3.7	$N+1$ 维时空中Bargmann-Wigner方程的准投影算子	772
3.8	$N+1$ 维时空中Bargmann-Wigner方程 <sup>[17]</sup> 平面波解猜想(后面章节会给出证明)	773
3.9	$N+1$ 维时空中Bargmann-Wigner方程的协变量子化规则	773
3.10	$N+1$ 维时空中Bargmann-Wigner方程协变对易规则的反向推理	775
3.11	$N+1$ 维时空中Bargmann-Wigner方程协变对易规则的小结	777
3.12	$N+1$ 维时空中Bargmann-Wigner方程的各种物理算符	778
3.13	$N+1$ 维时空中的Bargmann-Wigner全对称方程势对易规则猜想(很可能错?)	780
<b>第三十五章 高维时空无质量粒子的协变量子化</b>		<b>781</b>
1	$N+1$ 维时空中的螺旋度本征函数	781
1.1	$N+1=2n$ 偶数维时空中分离表象的电子方程 <sup>[4]</sup>	781
1.2	$N+1=2n$ 偶数维时空中的中微子方程 <sup>[5]</sup>	781
1.3	$N+1=2n$ 偶数维时空中运动方向的螺旋度本征函数	781
1.4	$N+1=2n-1$ 奇数维时空中运动方向的螺旋度本征函数	781
1.5	$N+1$ 维时空中螺旋度 $\Gamma \cdot \hat{p}$ 的本征函数	781
1.6	$N+1$ 维时空中螺旋度 $\Gamma \cdot \hat{p}$ 本征函数的正交性与完备性	782
1.7	$N+1$ 维时空中高自旋螺旋度的本征函数	782
2	$N+1$ 维时空中Penrose全对称方程自旋基及其基本性质	782
2.1	$N+1$ 维时空中Penrose全对称方程自旋基的广义多项式定理	782
2.2	$N+1$ 维时空中Penrose全对称方程的自旋基	783
2.3	$N+1$ 维时空中Penrose全对称方程自旋基的正交性质	783
2.4	$N+1$ 维时空中Penrose全对称方程自旋基的分解	783
2.5	$N+1$ 维时空中Penrose全对称方程的投影算子	784
3	$N+1$ 维时空中Penrose全对称方程的协变量子化	784
3.1	$N+1$ 维时空中Penrose全对称方程 <sup>[1,2]</sup> 的平面波解及其对易规则猜想	784
3.2	$N+1$ 维时空中Penrose全对称方程自旋基之间的递推关系	785
3.3	用数学归纳法严格求解 $N+1$ 维时空中Penrose全对称方程的平面波解	791
3.4	$N+1$ 维时空中Penrose全对称方程的协变对易规则	792
3.5	$N+1$ 维时空中Penrose全对称方程的各种物理算符	793
3.6	Bargmann-Wigner方程的作用量	795
<b>第三十六章 低维时空粒子的协变量子化</b>		<b>796</b>
1	三维时空中有质量粒子的协变量子化	796
1.1	三维时空中有质量粒子的Bargmann-Wigner方程	796
1.2	三维时空中的Bargmann-Wigner方程	796
1.3	三维时空中有质量粒子势方程的具体表述	797
1.4	三维时空中有质量玻色子势方程的一般表述	799
1.5	三维时空中费米子势方程的一般表述	803
1.6	三维时空中s-自旋方程	805
1.7	三维时空中的Bargmann-Wigner方程 <sup>[17]</sup> 等价于Penrose方程 <sup>[1,2]</sup>	806
1.8	四维时空中z轴受限的Majorana方程等价于三维时空中的Penrose方程 <sup>[1,2]</sup>	806

2	三维时空中广义Bargmann-Wigner方程	807
2.1	三维时空中广义的Bargmann-Wigner方程	807
2.2	三维时空中的广义Majorana B-W方程	808
2.3	三维时空中实表象的广义Bargmann-Wigner方程	808
2.4	三维时空中实表象的广义Majorana B-W方程	809
3	三维时空中直观表象下的Bargmann-Wigner方程	809
3.1	三维时空中直观表象下的Dirac方程	809
3.2	三维时空中直观表象下的类Klein-Gordon方程自旋基 $\varepsilon_a(\vec{p}), \tilde{\varepsilon}_a(\vec{p})$ 及其性质	809
3.3	三维时空中直观表象下s-自旋方程	809
4	三维时空中无质量粒子的Bargmann-Wigner方程(m只是参数)	810
4.1	三维时空中无质量粒子的Penrose方程	810
4.2	三维时空中无质量粒子势方程的具体表述	811
4.3	三维时空中无质量粒子势方程的一般表述	813
4.4	三维时空中无质量s-自旋方程	814
5	二维时空中有质量粒子的协变量子化	815
5.1	二维时空中的Dirac方程	815
5.2	二维时空中的Bargmann-Wigner方程	816
5.3	二维时空中的势方程	818
5.4	二维时空中自旋 $-\frac{3}{2}$ 的Bargmann-Wigner方程 <sup>[17]</sup>	821
5.5	二维时空中的自旋方程	821
5.6	二维时空中有质量粒子协变对易规则	822
6	二维时空中无质量粒子的Bargmann-Wigner方程	823
6.1	二维时空中的无质量Dirac方程	823
6.2	二维时空中无质量Bargmann-Wigner方程 <sup>[17]</sup> 的平面波解	823
6.3	二维时空中无质量Bargmann-Wigner方程的势描述 <sup>[17,19]</sup>	823
6.4	二维时空中无质量s-自旋方程	824
6.5	二维时空中无质量Bargmann-Wigner方程的协变对易规则	824
7	二维时空中无质量粒子的Penrose方程	824
7.1	二维时空中分离表象的Dirac方程 <sup>[4,5]</sup>	824
7.2	二维时空中分离表象的无质量Dirac方程的螺旋度本征函数	824
7.3	二维中的矢量和旋量 <sup>[44]</sup>	825
7.4	二维时空中Penrose方程 <sup>[1,2]</sup> 的平面波解	825
7.5	二维时空中的因果函数	826
7.6	二维时空中s-自旋方程的对易规则	826
7.7	二维时空中Penrose方程 <sup>[1,2]</sup> 杂烩	827
7.8	二维时空中 $\frac{1}{3}$ -自旋方程的对易规则?	828
<b>第三十七章 N+1维时空B-W方程的势分析</b>		<b>829</b>
1	N+1维时空Dirac矩阵	829
1.1	N+1维时空Dirac矩阵的常规表象	829

2	$N+1=n$ 维时空中二阶矩阵的反对称张量场展开 <sup>[45]</sup> . . . . .	831
2.1	$N+1=n$ 偶数维时空中一般矩阵的反对称张量场展开 . . . . .	831
2.2	$N+1=n$ 奇数维时空中一般矩阵的反对称张量场展开 . . . . .	832
2.3	$N+1=n$ 偶数维时空中对称矩阵的展开 . . . . .	832
2.4	$N+1=n$ 偶数维时空中反对称矩阵的展开 . . . . .	833
2.5	$N+1=n$ 奇数维时空中对称矩阵的展开 . . . . .	833
2.6	$N+1=n$ 奇数维时空中反对称矩阵的展开 . . . . .	833
3	$N+1$ 维时空中基本反对称张量场的共同性质 . . . . .	833
3.1	$N+1$ 维时空中的无质量反对称张量场 . . . . .	833
3.2	$N+1$ 维时空中的有质量反对称张量场 . . . . .	833
3.3	$N+1$ 维时空中反对称张量场对偶基之间的关系 . . . . .	833
3.4	$N+1$ 维时空中反对称张量场的等价对偶表示 . . . . .	834
3.5	$N+1=n$ 偶维时空中B-W方程导出基本的反对称张量场 . . . . .	835
3.6	$N+1=n$ 偶维时空中基本反对称张量场的性质 . . . . .	836
3.7	$N+1=n$ 奇数维时空中基本反对称张量场的性质 . . . . .	836
4	$N+1$ 维时空中基本反对称张量场的协变对易规则 . . . . .	837
4.1	从对称和反对称B-W方程导出基本反对称张量场的对易规则 . . . . .	837
4.2	$N+1=n$ 维时空中基本反对称张量场的通用对易规则猜想 . . . . .	838
5	全耦合反对称张量场集 . . . . .	838
5.1	$N+1=n$ 偶维时空中的Bargmann-Wigner一般矢量场方程 . . . . .	838
5.2	全耦合反对称张量场的对易规则 . . . . .	839
5.3	$N+1=n$ 偶维时空中反对称张量场集基和B-W集基之间的关系 . . . . .	839
5.4	$N+1$ 维时空中无质量反对称张量场的对易规则猜想 . . . . .	840
6	偶数维时空中的对称与反对称B-W矢量场方程 . . . . .	840
6.1	二维时空中的对称Bargmann-Wigner矢量场方程 . . . . .	840
6.2	二维时空中的反对称Bargmann-Wigner矢量场方程 . . . . .	841
6.3	四维时空中的对称Bargmann-Wigner矢量场方程 . . . . .	841
6.4	四维时空中的反对称Bargmann-Wigner矢量场方程 . . . . .	842
6.5	六维时空中的对称Bargmann-Wigner矢量场方程 . . . . .	843
6.6	六维时空中的反对称Bargmann-Wigner矢量场方程 . . . . .	844
6.7	八维时空中的对称Bargmann-Wigner矢量场方程 . . . . .	844
6.8	八维时空中的反对称Bargmann-Wigner矢量场方程 . . . . .	845
6.9	十维时空中的对称Bargmann-Wigner矢量场方程 . . . . .	846
6.10	十维时空中的反对称Bargmann-Wigner矢量场方程 . . . . .	847
6.11	十维时空中Bargmann-Wigner矢量场方程的对易规则 . . . . .	848
7	奇数维时空中的对称和反对称B-W矢量场方程 . . . . .	849
7.1	三维时空中的对称Bargmann-Wigner矢量场方程 . . . . .	849
7.2	三维时空中的反对称Bargmann-Wigner矢量场方程 . . . . .	849
7.3	五维时空中的对称Bargmann-Wigner矢量场方程 . . . . .	850
7.4	五维时空中的反对称Bargmann-Wigner矢量场方程 . . . . .	850
7.5	七维时空中的对称Bargmann-Wigner矢量场方程 . . . . .	850
7.6	七维时空中的反对称Bargmann-Wigner矢量场方程 . . . . .	851
7.7	九维时空中的对称Bargmann-Wigner矢量场方程 . . . . .	851

7.8	九维时空中的反对称Bargmann-Wigner矢量场方程	852
7.9	十一维时空中的对称Bargmann-Wigner矢量场方程	852
7.10	十一维时空中的反对称Bargmann-Wigner矢量场方程	853
8	$N+1=n$ 偶维时空中反对称B-W矢量场方程	853
8.1	关于反对称关系的引理	853
8.2	$N+1=n$ 偶维时空中反对称B-W矢量场方程的平面波解	853
9	$N+1$ 维时空中反对称张量场的直接求解	854
9.1	$N+1$ 维时空中的电子方程 <sup>[4]</sup>	854
9.2	$N+1$ 维时空中的电子自旋基	854
9.3	$N+1$ 维时空中电子的平面波解	854
9.4	$N+1$ 维时空中B-W矢量场的自旋基	855
9.5	$N+1$ 维时空中反对称张量场的自旋基及其性质	855
9.6	$N+1$ 维时空中Dirac矩阵的正交性?	856
10	四维时空中特殊的反对称张量场	856
10.1	相关性质汇总	856
10.2	四维时空中特殊的反对称张量场平面波解	856
10.3	四维时空中特殊的反对称张量场的B-W准投影算子	858
10.4	四维时空中特殊的反对称张量场的势对易规则	858
11	相关性质的具体计算	859
11.1	四维时空中特殊的反对称张量场具体计算一	859
11.2	四维时空中特殊的反对称张量场具体计算二	860
11.3	四维时空中特殊的反对称张量场具体计算三	861
11.4	四维时空中特殊的反对称张量场具体计算四	861
12	$N+1$ 维时空中的基本反对称张量场高阶推广的猜想	862
12.1	$N+1$ 维时空中有质量自旋 $s=n$ 的Bargmann-Wigner方程的猜想	862
12.2	$N+1$ 维时空中有质量自旋 $s=n+\frac{1}{2}$ 的Bargmann-Wigner方程的猜想	863
13	$N+1$ 维时空中的电磁场方程分析	863
13.1	$N+1$ 维时空中的Bargmann-Wigner矢量场方程	863
<b>第三十八章 二阶完美常数不变张量</b>		<b>865</b>
1	二阶完美常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)$	865
1.1	常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)$	865
1.2	常数矩阵 $\Gamma(s; 2j), \bar{\Gamma}(s; 2j)$	865
1.3	常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)$ 的基本性质	865
1.4	度规常数不变张量 $\varepsilon_{k_\zeta l_\zeta}(s; 2j)$ 的引入及其性质	866
1.5	自旋常数不变张量 $\sigma^{\alpha_\zeta}_{k_\zeta l_\zeta}(s; 2j), S_{ab}(s, \zeta; 2j)$	866
1.6	常数不变张量 $\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j), \Omega(s; 2j)$ 的引入及其性质	867
1.7	常数不变张量 $\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j), \Omega_{ab}(s, \zeta; 2j)$ 的引入及其基本性质	867
1.8	自旋常数不变张量 $\sigma^{\alpha_\zeta}_{k_\zeta l_\zeta}(s; 2j), \sigma(s; 2j)$ 的引入	867
1.9	自旋常数不变张量 $S_{ab k_\zeta l_\zeta}(s; 2j), S_{ab}(s, \zeta; 2j)$ 的引入	867
1.10	常数矩阵 $\Omega_{ab}(s, \zeta; 2j), S_{ab}(s, \zeta; 2j)$ 与 $\Omega_{\alpha_\zeta}(s; 2j), \sigma_{\alpha_\zeta}(s; 2j)$ 之间关系	867
1.11	常数矩阵 $\Omega(s; 2j)$ 的两个重要引理	867
1.12	常数矩阵 $\Omega_{ab}(s, \zeta; 2j)$ 的两个重要引理	868

1.13	关于常数矩阵 $\bar{\Gamma}(s; 3), \Omega(s; 3), \sigma(s; 3), \Gamma(s; 3)$ 的置换性质及其推论	868
1.14	关于常数矩阵 $\bar{\Gamma}(s; 2j), \Omega_{ab}(s, \varsigma; 2j), S_{ab}(s, \varsigma; 2j), \Gamma(s; 2j)$ 的置换性质及其推论	868
1.15	常数矩阵 $\Omega(s; 2j), \sigma(s; 2j)$ 的洛伦兹群表示	868
1.16	常数矩阵 $\Omega_{ab}(s, \varsigma; 2j), S_{ab}(s, \varsigma; 2j)$ 的洛伦兹群表示	868
1.17	推论：常数矩阵 $\Gamma(s; 2j), \bar{\Gamma}(s; 2j)$ 的几个恒等式	869
1.18	矩阵 $\Gamma(s; 2j), \bar{\Gamma}(s; 2j)$ 的常数不变张量性质	869
1.19	常数矩阵 $I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j), I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)$ 的置换性质	870
1.20	推论：常数矩阵 $\Gamma(s - \frac{1}{2}; 2j), \bar{\Gamma}(s - \frac{1}{2}; 2j)$ 的几个恒等式	870
1.21	常数矩阵 $\Gamma(s; 2j), \bar{\Gamma}(s; 2j)$ 的对称置换性质	870
1.22	矩阵 $I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j), I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)$ 的常数不变张量性质	871
2	二阶完美常数不变张量 $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$	871
2.1	完美常数不变张量 $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$	871
2.2	常数矩阵 $N_{A_\varsigma}(s; 2j), N^{A_\varsigma}(s; 2j); \bar{N}_{A_\varsigma}(s; 2j), \bar{N}^{A_\varsigma}(s; 2j); N(s; 2j), \bar{N}(s; 2j)$	871
2.3	常数不变张量 $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ 的基本性质	871
2.4	常数不变张量 $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ 的正交性质	872
2.5	常数不变张量 $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ 的升降指标	872
2.6	常数不变张量 $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ 的自旋矩阵变换	872
2.7	常数不变张量 $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ 的置换性质	873
2.8	常数不变张量 $N_{A_{\varsigma 1} \dots A_{\varsigma n} l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_{\varsigma 1} \dots A_{\varsigma n} l_\varsigma}(s; 2j)$ 的性质	873
2.9	推论1：常数矩阵 $N(s; 2j), \bar{N}(s; 2j)$ 的几个恒等式	874
2.10	推论2：常数矩阵 $N(s; 2j), \bar{N}(s; 2j)$ 的另外几个恒等式	874
2.11	矩阵 $N(s; 2j), \bar{N}(s; 2j)$ 的常数不变张量性质	875
3	二阶完美常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$	875
3.1	完美常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$	875
3.2	常数矩阵 $X(s; 2j), \bar{X}(s; 2j)$	875
3.3	常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$ 的升降指标	875
3.4	常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$ 的正交性	875
3.5	常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$ 的自旋变换	875
3.6	常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$ 的置换性质	876
3.7	推论：关于常数矩阵 $X(s; 2j), \bar{X}(s; 2j)$ 的重要性质	876
3.8	矩阵 $X(s; 2j), \bar{X}(s; 2j)$ 的常数不变张量性质	877
3.9	常数矩阵 $\Omega(s; 2j), \sigma(s - 1; 2j)$ 的置换性质	877
3.10	常数矩阵 $\Omega(s - l; 2j), [\vec{\partial} \cdot \Omega(s - l; 2j)]^n, e^{\vec{\partial} \cdot \Omega(s - l; 2j)}$ 的同构性表示	877
4	几个例子	878
4.1	例子一	878
4.2	具体例子	878
5	高阶完美常数不变张量的一般推广	878
5.1	无穷扩展和推广(“二阶”名字的由来)	878
5.2	高阶完美常数不变张量的表示	879
<b>第三十九章 Dirac型完美常数不变张量</b>		<b>880</b>
1	Dirac型完美常数不变张量 $\Gamma_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \dots}^{k_\varsigma}(s; 3), \Gamma_{k_\varsigma}^{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \dots}(s; 3)$	880
1.1	常数不变张量 $\Gamma_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \dots}^{k_\varsigma}(s; 3), \Gamma_{k_\varsigma}^{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \dots}(s; 3)$	880

1.2	常数矩阵 $\Gamma(s; 3), \bar{\Gamma}(s; 3)$ . . . . .	880
1.3	常数不变张量 $\Gamma_{\lambda_s \mu_s \eta_s \dots}^{k_s} (s; 3), \Gamma_{k_s}^{\lambda_s \mu_s \eta_s \dots} (s; 3)$ 的基本性质 . . . . .	880
1.4	度规常数不变张量 $\varepsilon_{k_s l_s} (s; 3)$ 的引入及其性质 . . . . .	881
1.5	常数不变张量 $\Omega_{\lambda_s \mu_s \eta_s \dots}^{\lambda_s' \mu_s' \eta_s' \dots} (s; 3), \Omega^a (s; 3)$ 的引入及其基本性质 . . . . .	881
1.6	常数不变张量 $\Omega_{ab A_s B_s C_s \dots}^{A_s' B_s' C_s' \dots} (s; 3), \Omega_{ab} (s; 3)$ 的引入及其基本性质 . . . . .	882
1.7	自旋常数不变张量 $\gamma_{k_s}^a l_s (s; 3), \gamma^a (s; 3); S_{ab k_s} l_s (s; 3), S_{ab} (s; 3)$ 的引入 . . . . .	882
1.8	常数矩阵 $\Omega_{ab} (s; 3), S_{ab} (s; 3)$ 与 $\Omega_a (s; 3), \gamma_a (s; 3)$ 之间关系 . . . . .	882
1.9	常数矩阵 $\Omega^a (s; 3), \Omega_{ab} (s; 3)$ 的两个重要引理 . . . . .	882
1.10	关于常数矩阵 $\bar{\Gamma}(s; 3), \Omega^a (s; 3), \gamma^a (s; 3), \Gamma(s; 3)$ 的置换性质 . . . . .	882
1.11	关于常数矩阵 $\bar{\Gamma}(s; 3), \Omega_{ab} (s; 3), S_{ab} (s; 3), \Gamma(s; 3)$ 的置换性质 . . . . .	882
1.12	关于常数不变张量 $\gamma_{k_s}^a l_s (s; 3), S_{ab k_s} l_s (s; 3)$ 的重要推论 . . . . .	883
1.13	常数矩阵 $\Omega_{ab} (s; 3), S_{ab} (s; 3)$ 的洛伦兹群表示 . . . . .	883
1.14	推论：常数矩阵 $\Gamma(s; 3), \bar{\Gamma}(s; 3)$ 的几个恒等式 . . . . .	883
1.15	常数矩阵 $\Gamma(s; 3), \bar{\Gamma}(s; 3)$ 的对称置换性质 . . . . .	883
1.16	矩阵 $\Gamma(s; 3), \bar{\Gamma}(s; 3)$ 的常数不变张量性质 . . . . .	883
1.17	常数矩阵 $I_4 \otimes \Gamma(s - \frac{1}{2}; 3), I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)$ 的置换性质 . . . . .	883
1.18	推论：常数矩阵 $\Gamma(s - \frac{1}{2}; 3), \bar{\Gamma}(s - \frac{1}{2}; 3)$ 的几个恒等式 . . . . .	884
1.19	矩阵 $I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3), I_4 \otimes \Gamma(s - \frac{1}{2}; 3)$ 的常数不变张量性质 . . . . .	884
2	Dirac型完美常数不变张量 $N_{\lambda_s l_s}^{k_s} (s; 3), N_{k_s}^{\lambda_s l_s} (s; 3)$ . . . . .	884
2.1	完美常数不变张量 $N_{\lambda_s l_s}^{k_s} (s; 3), N_{k_s}^{\lambda_s l_s} (s; 3)$ . . . . .	884
2.2	常数矩阵 $N_{\lambda_s} (s; 3), N^{\lambda_s} (s; 3); \bar{N}_{\lambda_s} (s; 3), \bar{N}^{\lambda_s} (s; 3); N(s; 3), \bar{N}(s; 3)$ . . . . .	885
2.3	常数不变张量 $N_{\lambda_s l_s}^{k_s} (s; 3), N_{k_s}^{\lambda_s l_s} (s; 3)$ 的基本性质 . . . . .	885
2.4	常数不变张量 $N_{\lambda_s l_s}^{k_s} (s; 3), N_{k_s}^{\lambda_s l_s} (s; 3)$ 的正交性质 . . . . .	885
2.5	常数不变张量 $N_{\lambda_s l_s}^{k_s} (s; 3), N_{k_s}^{\lambda_s l_s} (s; 3)$ 的升降指标 . . . . .	886
2.6	常数不变张量 $N_{\lambda_s l_s}^{k_s} (s; 3), N_{k_s}^{\lambda_s l_s} (s; 3)$ 的自旋矩阵变换 . . . . .	886
2.7	常数不变张量 $N_{\lambda_s l_s}^{k_s} (s; 3), N_{k_s}^{\lambda_s l_s} (s; 3)$ 的置换性质 . . . . .	886
2.8	常数不变张量 $N_{A_s 1 \dots A_s n l_s}^{k_s} (s; 3), N_{k_s}^{A_s 1 \dots A_s n l_s} (s; 3)$ 的性质 . . . . .	887
2.9	推论1：常数矩阵 $N(s; 3), \bar{N}(s; 3)$ 的几个恒等式 . . . . .	887
2.10	推论2：常数矩阵 $N(s; 3), \bar{N}(s; 3)$ 的另外几个恒等式 . . . . .	888
2.11	矩阵 $N(s; 3), \bar{N}(s; 3)$ 的常数不变张量性质 . . . . .	888
3	Dirac型完美常数不变张量 $X_{m_s}^{\lambda_s l_s} (s; 3), X_{\lambda_s l_s}^{m_s} (s; 3)$ . . . . .	888
3.1	完美常数不变张量 $X_{m_s}^{\lambda_s l_s} (s; 3), X_{\lambda_s l_s}^{m_s} (s; 3)$ . . . . .	888
3.2	常数矩阵 $X(s; 3), \bar{X}(s; 3)$ . . . . .	888
3.3	常数不变张量 $X_{m_s}^{\lambda_s l_s} (s; 3), X_{\lambda_s l_s}^{m_s} (s; 3)$ 的升降指标 . . . . .	888
3.4	常数不变张量 $X_{m_s}^{\lambda_s l_s} (s; 3), X_{\lambda_s l_s}^{m_s} (s; 3)$ 的正交性 . . . . .	889
3.5	常数不变张量 $X_{m_s}^{\lambda_s l_s} (s; 3), X_{\lambda_s l_s}^{m_s} (s; 3)$ 的自旋变换 . . . . .	889
3.6	常数不变张量 $X_{m_s}^{\lambda_s l_s} (s; 3), X_{\lambda_s l_s}^{m_s} (s; 3)$ 的置换性质 . . . . .	889
3.7	推论：关于常数矩阵 $X(s; 3), \bar{X}(s; 3)$ 的重要性质 . . . . .	890
3.8	矩阵 $X(s; 3), \bar{X}(s; 3)$ 的常数不变张量性质 . . . . .	890
3.9	常数矩阵 $\Omega^a (s; 3), \gamma^a (s - 1; 3)$ 的置换性质 . . . . .	890
3.10	常数矩阵 $\Omega^a (s - l; 3), [\vec{\partial} \cdot \Omega^a (s - l; 3)]^n, e^{\vec{\partial} \cdot \Omega^a (s - l; 3)}$ 的同构性表示 . . . . .	890
4	高阶Dirac型完美常数不变张量 . . . . .	891
4.1	无穷扩展和推广 . . . . .	891

5 Dirac型常数不变张量的具体计算	891
5.1 Dirac型常数不变张量 $\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s; 3), \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}(s; 3)$ 的引入	891
5.2 Dirac型常数不变张量 $\Gamma_{\lambda_\zeta \mu_\zeta}^{k_\zeta}(1; 3), \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta}(1; 3)$ 的性质	891
5.3 Dirac型常数不变张量 $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(1; 3), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(1; 3)$ 的性质	892
5.4 Dirac型常数不变张量 $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(1; 3), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(1; 3)$ 的性质I	892
5.5 Dirac型常数不变张量 $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(1; 3), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(1; 3)$ 的性质II	893
<b>第四十章 多重完美常数不变张量</b>	<b>897</b>
1 多重完美常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1)$	897
1.1 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1)$	897
1.2 常数矩阵 $\Gamma(s; 2j-1), \bar{\Gamma}(s; 2j-1)$	897
1.3 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1)$ 的基本性质	897
1.4 度规常数不变张量 $\varepsilon_{k_\zeta l_\zeta}(s; 2j-1)$ 的引入及其性质	898
1.5 自旋常数不变张量 $\sigma^{\alpha_\zeta k_\zeta l_\zeta}(s; 2j-1), S_{ab}(s, \zeta; 2j-1)$	898
1.6 常数不变张量 $\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j-1), \Omega(s; 2j-1)$ 的引入及其基本性质	899
1.7 常数不变张量 $\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j-1), \Omega_{ab}(s, \zeta; 2j-1)$ 的引入及其基本性质	899
1.8 自旋常数不变张量 $\sigma^{\alpha_\zeta k_\zeta l_\zeta}(s; 2j-1), \sigma(s; 2j-1)$ 的引入	899
1.9 自旋常数不变张量 $S_{ab k_\zeta l_\zeta}(s; 2j-1), S_{ab}(s, \zeta; 2j-1)$ 的引入	899
1.10 常数矩阵 $\Omega_{ab}(s, \zeta; 2j-1), S_{ab}(s, \zeta; 2j-1)$ 与 $\Omega_{\alpha_\zeta}(s; 2j-1), \sigma_{\alpha_\zeta}(s; 2j-1)$ 之间关系	899
1.11 常数矩阵 $\Omega(s; 2j-1)$ 的两个重要引理	900
1.12 常数矩阵 $\Omega_{ab}(s, \zeta; 2j-1)$ 的两个重要引理	900
1.13 关于常数矩阵 $\bar{\Gamma}(s; 3), \Omega(s; 3), \sigma(s; 3), \Gamma(s; 3)$ 的置换性质及其推论	900
1.14 关于常数矩阵 $\bar{\Gamma}(s; 2j-1), \Omega_{ab}(s, \zeta; 2j-1), S_{ab}(s, \zeta; 2j-1), \Gamma(s; 2j-1)$ 的置换性质及其推论	900
1.15 常数矩阵 $\Omega(s; 2j-1), \sigma(s; 2j-1)$ 的洛伦兹群表示	900
1.16 常数矩阵 $\Omega_{ab}(s, \zeta; 2j-1), S_{ab}(s, \zeta; 2j-1)$ 的洛伦兹群表示	901
1.17 推论：常数矩阵 $\Gamma(s; 2j-1), \bar{\Gamma}(s; 2j-1)$ 的几个恒等式	901
1.18 常数矩阵 $\Gamma(s; 2j-1), \bar{\Gamma}(s; 2j-1)$ 的对称置换性质	901
1.19 矩阵 $\Gamma(s; 2j-1), \bar{\Gamma}(s; 2j-1)$ 的常数不变张量性质	901
1.20 常数矩阵 $I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1), I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)$ 的置换性质	901
1.21 推论：常数矩阵 $\Gamma(s - \frac{1}{2}; 2j-1), \bar{\Gamma}(s - \frac{1}{2}; 2j-1)$ 的几个恒等式	902
1.22 矩阵 $I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1), I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)$ 的常数不变张量性质	902
2 多重完美常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$	903
2.1 完美常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$	903
2.2 常数矩阵 $N_{A_\zeta}(s; 2j-1), N^{A_\zeta}(s; 2j-1); \bar{N}_{A_\zeta}(s; 2j-1), \bar{N}^{A_\zeta}(s; 2j-1); N(s; 2j-1), \bar{N}(s; 2j-1)$	903
2.3 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$ 的基本性质	903
2.4 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$ 的正交性质	903
2.5 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$ 的升降指标	904
2.6 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$ 的自旋矩阵变换	904
2.7 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$ 的置换性质	904
2.8 常数不变张量 $N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; 2j-1)$ 的性质	905
2.9 推论1：常数矩阵 $N(s; 2j-1), \bar{N}(s; 2j-1)$ 的几个恒等式	905
2.10 推论2：常数矩阵 $N(s; 2j-1), \bar{N}(s; 2j-1)$ 的另外几个恒等式	906
2.11 矩阵 $N(s; 2j-1), \bar{N}(s; 2j-1)$ 的常数不变张量性质	906



3 多重完美常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$ .....	906
3.1 完美常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$ .....	906
3.2 常数矩阵 $X(s; 2j-1), \bar{X}(s; 2j-1)$ .....	907
3.3 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$ 的升降指标 .....	907
3.4 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$ 的正交性 .....	907
3.5 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$ 的自旋变换 .....	907
3.6 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$ 的置换性质 .....	907
3.7 推论：关于常数矩阵 $X(s; 2j-1), \bar{X}(s; 2j-1)$ 的重要性质 .....	908
3.8 矩阵 $X(s; 2j-1), \bar{X}(s; 2j-1)$ 的常数不变张量性质 .....	908
3.9 常数矩阵 $\Omega(s; 2j-1), \sigma(s-1; 2j-1)$ 的置换性质 .....	909
3.10 常数矩阵 $\Omega(s-l; 2j-1), [\vec{d} \cdot \Omega(s-l; 2j-1)]^n, e^{\vec{d} \cdot \Omega(s-l; 2j-1)}$ 的同构性表示 .....	909
<b>第四十一章 反对称完美常数不变张量</b> .....	<b>910</b>
1 反对称指标排序 .....	910
1.1 $w+1$ 阶反对称指标正向排序规律的严格证明(从0排起) .....	910
2 完美常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)$ .....	911
2.1 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)$ 的引入 .....	911
2.2 定义的解读 .....	911
2.3 常数矩阵 $\Gamma(s; w), \bar{\Gamma}(s; w)$ 的引入 .....	911
2.4 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)$ 的基本性质 .....	911
2.5 度规常数不变张量 $\varepsilon_{k_\zeta l_\zeta}(s; w)$ 的引入及其性质(存在 $\varepsilon_{A_\zeta B_\zeta}$ 为前提条件) .....	912
2.6 常数不变张量 $\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; w), \Omega(s; w)$ 的引入及其基本性质 .....	913
2.7 常数不变张量 $\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; w), \Omega_{ab}(s; \varsigma; w)$ 的引入及其基本性质 .....	913
2.8 自旋常数不变张量 $\sigma^{\alpha_\zeta k_\zeta l_\zeta}(s; w), \sigma(s; w)$ 的引入 .....	913
2.9 自旋常数不变张量 $S_{ab k_\zeta l_\zeta}(s; w), S_{ab}(s; \varsigma; w)$ 的引入 .....	913
2.10 常数矩阵 $\Omega(s; w)$ 的两个重要引理 .....	914
2.11 常数矩阵 $\Omega_{ab}(s; w)$ 的两个重要引理 .....	914
2.12 关于常数矩阵 $\bar{\Gamma}(s; w), \Omega(s; w), \sigma(s; w), \Gamma(s; w)$ 的置换性质及其推论 .....	914
2.13 关于常数矩阵 $\bar{\Gamma}(s; w), \Omega_{ab}(s; \varsigma; w), S_{ab}(s; \varsigma; w), \Gamma(s; w)$ 的置换性质及其推论 .....	915
2.14 关于常数矩阵 $\Omega(s; w), \sigma(s; w)$ 的重要推论及洛伦兹群表示 .....	915
2.15 关于常数矩阵 $\Omega_{ab}(s; \varsigma; w), S_{ab}(s; \varsigma; w)$ 的重要推论及洛伦兹群表示 .....	916
2.16 常数矩阵 $\Omega_{ab}(s; \varsigma; w), S_{ab}(s; \varsigma; w)$ 与 $\Omega(s; w), \sigma(s; w)$ 之间关系的讨论 .....	917
2.17 推论：常数矩阵 $\Gamma(s; w), \bar{\Gamma}(s; w)$ 的几个恒等式 .....	917
2.18 矩阵 $\Gamma(s; w), \bar{\Gamma}(s; w)$ 的常数不变张量性质 .....	918
2.19 常数矩阵 $I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w), I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)$ 的置换性质 .....	919
2.20 推论：常数矩阵 $\Gamma(s - \frac{1}{2}; w), \bar{\Gamma}(s - \frac{1}{2}; w)$ 的几个恒等式 .....	919
2.21 矩阵 $I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w), I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)$ 的常数不变张量性质 .....	920
2.22 自洽性质验证 .....	920
3 完美常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ .....	920
3.1 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的引入 .....	920
3.2 常数矩阵 $N_{A_\zeta}(s; w), N^{A_\zeta}(s; w); \bar{N}_{A_\zeta}(s; w), \bar{N}^{A_\zeta}(s; w); N(s; w), \bar{N}(s; w)$ 的引入 .....	920
3.3 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的基本性质 .....	921
3.4 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的正交性质 .....	921
3.5 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的升降指标(存在 $\varepsilon_{A_\zeta B_\zeta}$ 为前提条件) .....	923

3.6	常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的自旋矩阵变换	924
3.7	常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的置换性质	926
3.8	常数不变张量 $N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w), N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; w)$ 的引入及其性质	927
3.9	推论1: 常数矩阵 $N(s; w), \bar{N}(s; w)$ 的几个恒等式	927
3.10	推论2: 常数矩阵 $N(s; w), \bar{N}(s; w)$ 的另外几个恒等式	928
3.11	矩阵 $N(s; w), \bar{N}(s; w)$ 的常数不变张量性质	928
3.12	两个定理的另外一种证明	929
4	完美常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ (存在 $\varepsilon_{A_\zeta B_\zeta}$ 为前提)	929
4.1	完美常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的引入	929
4.2	常数矩阵 $X(s; w), \bar{X}(s; w)$ 的引入	929
4.3	常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的升降指标	929
4.4	常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的正交性	930
4.5	常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的自旋变换	931
4.6	常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的置换性质	931
4.7	推论: 关于常数矩阵 $X(s; w), \bar{X}(s; w)$ 的重要性质	933
4.8	矩阵 $X(s; w), \bar{X}(s; w)$ 的常数不变张量性质	933
4.9	常数矩阵 $\Omega(s; w), \sigma(s-1; w)$ 的置换性质	933
4.10	常数矩阵 $\Omega(s-l; w), [\vec{v} \cdot \Omega(s-l; w)]^n, e^{\vec{v} \cdot \Omega(s-l; w)}$ 的同构性表示	933
5	反对称常数不变张量 $\Gamma_{ab}^k(1; 3)$ 的应用-电磁张量的高旋量描述	934
5.1	基础	934
5.2	$SR^x(1; 3)S^+$ 的计算	935
5.3	$SR^y(1; 3)S^+$ 的计算	936
5.4	$SR^z(1; 3)S^+$ 的计算	936
5.5	$SL^x(1; 3)S^+$ 的计算	937
5.6	$SL^y(1; 3)S^+$ 的计算	938
5.7	$SL^z(1; 3)S^+$ 的计算	939
5.8	$SR(1; 3)S^+$ 的计算结果	940
5.9	反对称常数不变张量 $\Gamma_{abc}^k(\frac{3}{2}; 3)$ 及与 $\varepsilon^k_{abc}$ 的关系	940
5.10	反对称常数不变张量 $\Gamma_{abcd}^k(2; 3) = \varepsilon_{abcd}$	940
6	反对称常数不变对偶张量	941
6.1	反对称常数不变对偶张量 $*\Gamma_{ab\dots}^k(s; 3)$ 的引入	941
6.2	常数不变张量 $\Gamma_{\alpha_\zeta \beta_\zeta \dots}^k(n; 3)$ 的引入	941
<b>第四十二章 有质量粒子的高旋量场量子化方案</b>		942
1	Bargmann-Wigner 方程 <sup>[17]</sup> 自旋基和平面波解的高旋量场表述	942
1.1	有质量粒子旋量与无质量粒子旋量的对比	942
1.2	Bargmann-Wigner 方程高旋量场自旋基的定义及推论	942
1.3	Bargmann-Wigner 方程平面波解的高旋量场表述	942
1.4	Bargmann-Wigner 高旋量场的自旋方程	943
1.5	Bargmann-Wigner 高旋量场的类Dirac方程	943
1.6	Bargmann-Wigner 方程高旋量场自旋基的矩阵表述	944
1.7	Bargmann-Wigner 方程平面波解的高自旋矩阵表述	944
1.8	Bargmann-Wigner 方程的两种自旋基之间的关系	944
1.9	实表象下Bargmann-Wigner方程的两种自旋基之间的关系	945

1.10 Bargmann-Wigner方程自旋基的正交性质	945
1.11 Dirac方程自旋基的广义二项式定理及其推论	945
1.12 Bargmann-Wigner方程准投影算子的定义和性质	946
2 Bargmann-Wigner方程对易规则的高旋量场表述	946
2.1 Bargmann-Wigner方程的协变对易规则	946
2.2 Bargmann-Wigner方程玻色子对易规则的两种等价描述	946
2.3 Bargmann-Wigner方程的对易函数、因果函数和费曼传播子	947
2.4 Bargmann-Wigner方程的等时对易规则	948
2.5 Bargmann-Wigner方程的有关引理	948
2.6 Bargmann-Wigner方程能量动量算符的提取	949
2.7 Bargmann-Wigner方程的各种物理算符	949
2.8 Bargmann-Wigner方程的量子方程	949
<b>第四十三章 基本粒子内部分量作用</b>	<b>950</b>
1 基本粒子内部分量假设	950
2 粒子复合理论	950
2.1 两个中微子合成一个光子的物理机制	950
2.2 $2s$ 个中微子合成一个自旋- $s$ 粒子的物理机制	950
3 常数张量与新相互作用	951
3.1 新的相互作用	951
3.2 引力子合成相互作用	952
3.3 引力微子合成相互作用	952
3.4 光子合成相互作用	952
3.5 新的类电磁相互作用	952
4 粒子内部分量作用	952
4.1 光子对内部分量作用	952
<b>第四十四章 自旋纠缠态分析初步</b>	<b>953</b>
1 电子方程的单模平面波解	953
1.1 双电子合成: $\frac{1}{2} \oplus \frac{1}{2} = (1, 1)$	953
1.2 双电子合成: $\frac{1}{2} \oplus \frac{1}{2} = (1, -1)$	953
1.3 双电子合成: $\frac{1}{2} \oplus \frac{1}{2} = (1, 0)$	953
1.4 双电子合成: $\frac{1}{2} \oplus \frac{1}{2} = (0, 0)$	954
2 光子方程的单模平面波解	954
2.1 双光子合成: $1 \oplus 1 = (2, 2)$	954
2.2 双光子合成: $1 \oplus 1 = (2, -2)$	954
2.3 双光子合成: $1 \oplus 1 = (2, 0)$	955
2.4 双光子合成: $1 \oplus 1 = (0, 0)$	955
3 矢量场方程的单模平面波解	955
3.1 双光子合成: $1 \oplus 1 = (2, 2)$	955
3.2 双光子合成: $1 \oplus 1 = (2, -2)$	955
3.3 双光子合成: $1 \oplus 1 \rightarrow (2, 0)$	956
3.4 双光子合成: $1 \oplus 1 \rightarrow (1, 0)$	956

4	粒子自旋方程的单模平面波解	957
4.1	s-自旋方程及其单模平面波解	957
4.2	单模自旋方程的多种等价形式	957
4.3	自旋方程的沿z轴运动的单模平面波解	957
4.4	1-自旋方程的沿z轴运动的单模平面波解	958
4.5	单模平面波解的自旋矢量	959
4.6	正z轴单模平面波解的完整量子态	959
4.7	正z轴单模光子的完整量子态	959
4.8	s-自旋方程的双模同向同步平面波解	959
4.9	s-自旋方程的双模异向同步正平面波解	959
4.10	s-自旋方程的双模异向同步负平面波解	959
4.11	s-自旋方程的双模异向同步纠缠平面波解猜测	960
5	几个粒子的自旋耦合例子	960
5.1	两个电子自旋的对角耦合表象	960
5.2	两个电子自旋的分离非耦合表象	960
5.3	两个电子自旋的分离耦合表象	960
5.4	两个有质量光子的自旋耦合表象	961
5.5	两个无质量光子的自旋耦合	963
5.6	三个电子的自旋耦合	964
6	经典物理+概率论解释光子纠缠的可能性(只是试探性的)	965
6.1	经典电磁学对自旋极化的描述	965
6.2	经典电磁学对线性极化的描述	965
7	多粒子自旋体系	966
7.1	两电子紧耦合体系的总自旋	966
7.2	两电子松耦合体系的总自旋	966
7.3	多电子紧耦合体系的总自旋	966
7.4	多电子松耦合体系的总自旋	966
8	多粒子纠缠方程的猜想	966
8.1	四阶矩阵的物理意义一	966
8.2	四阶矩阵的物理意义二	966
8.3	有质量双光子纠缠方程	967
8.4	双电子纠缠方程	967
8.5	二阶矩阵的物理意义	967
8.6	双中微子纠缠方程	967
8.7	三阶矩阵的物理意义	968
8.8	三阶矩阵的有质量双纠缠光子方程	968
8.9	四阶矩阵的双引力微子纠缠方程	969
8.10	$2s+1$ 阶矩阵的双s-自旋粒子纠缠方程	969
8.11	$(2s_1+1) \times (2s_2+1)$ 矩阵的 $s_1 \otimes s_2$ 纠缠方程猜想	969
<b>第四十五章 相对论性玻色弦平面波解</b>		<b>970</b>
1	数学准备	970
1.1	$\delta(x)$ 函数的性质	970

1.2 代数方程的 $\delta(x)$ 函数解	970
2 不同边界条件玻色弦方程的平面波解 <sup>[44, 46]</sup>	970
2.1 玻色闭弦方程波函数展开(自由闭弦)	970
2.2 玻色N开弦方程波函数展开(对称闭弦)	971
2.3 玻色D开弦方程波函数展开(反对称闭弦)	972
2.4 玻色混合条件开弦方程波函数展开	972
2.5 p-膜	973
2.6 一般边界条件的解	973
2.7 玻色开弦方程的严格简单解法	974
3 玻色弦方程的平面波解完备性疑惑	974
3.1 对比研究—玻色闭弦方程波函数第二种展开方式	974
3.2 对比研究—玻色闭弦方程波函数第三种展开方式	975
3.3 性质	975
3.4 玻色闭弦方程的推广	976
<b>第四十六章 简单超对称理论初步</b>	<b>977</b>
1 二维中的电磁场	977
1.1 二维中的电磁场	977
1.2 二维中的矢量旋量超对称 <sup>[47]</sup>	978
1.3 二维中的Majorana-Weyl任意子方程及其相应作用量	978
1.4 二维中的Majorana-Weyl任意子作用量?	978
1.5 二维中Dirac方程的经典构造	978
2 左超对称弦 <sup>[44, 46, 47]</sup>	979
2.1 左超对称弦作用量及运动方程	979
2.2 左超对称弦作用量的局域玻色变换	979
2.3 左超对称弦作用量的全局玻色变换	980
2.4 左超对称弦作用量的全局变换的封闭性	980
3 右超对称弦 <sup>[44, 46, 47]</sup>	980
3.1 右超对称弦作用量及运动方程	980
3.2 右超对称弦作用量的局域玻色变换	981
3.3 右超对称弦作用量的全局玻色变换	981
3.4 右超对称弦作用量的全局变换的封闭性	981
4 左右超对称弦 <sup>[44, 46, 47]</sup>	981
4.1 质壳左右超对称弦作用量及运动方程	981
4.2 非质壳左右超对称弦作用量及运动方程	981
4.3 左右超对称弦的超对称代数	982
4.4 左右超对称弦的超空间表示	983
5 局域超对称弦作用量 <sup>[44, 46, 47]</sup>	983
5.1 世界面的超对称作用量	983
5.2 时空中的超对称作用量	983
5.3 超对称理论的自旋表述	983
参考文献	984

# 第一章 常数不变张量分析

自我评述：本章我发展创立了常数不变张量分析方法，我是由已知的一些常数不变张量启发而发展起来的。本章发现了一大批新的基本常数不变张量，并且与物理联系紧密，具有天然的协变性与不变性，使用起来十分方便有用，是物理研究的一个新数学工具，其实我发展这个数学工具的初衷就是要用于物理研究。

## 1 类Penrose抽象指标 [1,2]

符号约定： $\sim$ 表示洛伦兹变换， $\prec$ 表示矩阵展开为分量， $\succ$ 表示分量收缩为矩阵， $\varsigma = \pm 1$ 。

自我评述：本节是对Penrose抽象指标的发展与推广，将 $\frac{1}{2}$ -自旋指标推广到了一般自旋指标，并引入了双重表示指标，双重表示指标对应无质量粒子的两种表示。这样做的好处是“一式两份”、一个表达两种表示同时呈现、一次处理同时得到两个结果，这样的抽象指标更美、更完备、更有力。

### 1.1 一般表象的厄密自旋矩阵 $\sigma(s)$

$$\sigma(s) \times \sigma(s) = i\sigma(s), \sigma^2(s) = s(s+1), \sigma^+(s) = \sigma(s) \quad (1.1)$$

### 1.2 厄密自旋矩阵的一个具体表象 [34]

$$\sigma(s) = \left( \frac{1}{2} \begin{bmatrix} 0 & A_1 & 0 & 0 & 0 \\ A_1 & 0 & A_2 & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & A_{2s} \\ 0 & 0 & 0 & A_{2s} & 0 \end{bmatrix}, \frac{i}{2} \begin{bmatrix} 0 & -A_1 & 0 & 0 & 0 \\ A_1 & 0 & -A_2 & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -A_{2s} \\ 0 & 0 & 0 & A_{2s} & 0 \end{bmatrix}, \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix} \right) \quad (1.2a)$$

$$A_n = \sqrt{n \cdot \sqrt{2s+1-n}}, n = 1, 2, \dots, 2s; \sigma(s) \prec \sigma_{\alpha_\varsigma k_\varsigma}^{l_\varsigma}(s) \simeq \sigma_{\alpha'_\varsigma k'_\varsigma}^{l'_\varsigma}(s) \quad (1.2b)$$

$$\sigma(s) \times \sigma(s) = i\sigma(s), \sigma^2(s) = s(s+1), \sigma^+(s) = \sigma(s), s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots \quad (1.2c)$$

与此自旋矩阵相对应的度规张量如下：

$$\varepsilon(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & (-1)^0 \\ 0 & 0 & 0 & (-1)^1 & 0 \\ 0 & 0 & (-1)^2 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \\ (-1)^{2s} & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 \\ (-1)^{2s} & 0 & 0 & 0 & 0 \end{bmatrix}, \sigma^*(s) = (-1)^{2s+1} \varepsilon(s) \sigma(s) \varepsilon(s) \quad (1.3)$$

$$\varepsilon(s) \prec \varepsilon_{k_\varsigma l_\varsigma}(s) \simeq \varepsilon^{k'_\varsigma l'_\varsigma}(s) \simeq \varepsilon_{k'_\varsigma l'_\varsigma}(s), \varepsilon^2(s) = (-1)^{2s} \quad (1.4)$$

自我评述：自旋矩阵选取本质上是有无穷种选法，可以是厄密的，也可以不是厄密的。本书主要采用的是以上特殊表示的厄密自旋矩阵，这样做的理由是量子力学中观测量要求是厄密的，另一个理由是在这种表示自旋矩阵下存在下一章的完美常数不变张量，若采用其它表示的自旋矩阵，得不到这样完美的常数不变张量。事实上一开始我采用的是一个全整数的自旋矩阵，表面上似乎更漂亮，但没有以上两个优点，最终我还是放弃了它而采用了现在的这种厄密表示。

### 1.3 自旋矩阵复杂性质的梳理

$$\sigma_x^2(s) = \frac{1}{4} \begin{bmatrix} A_1^2 & 0 & A_1 A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2 + A_1^2 & 0 & A_2 A_3 & 0 & 0 & 0 \\ A_1 A_2 & 0 & A_3^2 + A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2 A_3 & 0 & \cdots & 0 & A_{2s-2} A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2 + A_{2s-2}^2 & 0 & A_{2s-1} A_{2s} \\ 0 & 0 & 0 & A_{2s-2} A_{2s-1} & 0 & A_{2s}^2 + A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1} A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \quad (1.5)$$

$$\sigma_y^2(s) = \frac{1}{4} \begin{bmatrix} A_1^2 & 0 & -A_1 A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2 + A_1^2 & 0 & -A_2 A_3 & 0 & 0 & 0 \\ -A_1 A_2 & 0 & A_3^2 + A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & -A_2 A_3 & 0 & \cdots & 0 & -A_{2s-1} A_{2s} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2 + A_{2s-2}^2 & 0 & -A_{2s-1} A_{2s} \\ 0 & 0 & 0 & -A_{2s-2} A_{2s-1} & 0 & A_{2s}^2 + A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & -A_{2s-1} A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \quad (1.6)$$

$$\sigma_z^2(s) = \begin{bmatrix} s^2 & 0 & 0 & 0 & 0 \\ 0 & (s-1)^2 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1)^2 & 0 \\ 0 & 0 & 0 & 0 & -s^2 \end{bmatrix} \quad (1.7)$$

$$\sigma_z(s)\sigma_x(s) = \frac{1}{2} \begin{bmatrix} 0 & sA_1 & 0 & 0 & 0 \\ (s-1)A_1 & 0 & (s-1)A_2 & 0 & 0 \\ 0 & (s-2)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(s-1)A_{2s} \\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix} \quad \sigma_x(s)\sigma_z(s) = \frac{1}{2} \begin{bmatrix} 0 & (s-1)A_1 & 0 & 0 & 0 \\ sA_1 & 0 & (s-2)A_2 & 0 & 0 \\ 0 & (s-1)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -sA_{2s} \\ 0 & 0 & 0 & -(s-1)A_{2s} & 0 \end{bmatrix} \quad (1.8)$$

$$\sigma_z(s)\sigma_y(s) = \frac{i}{2} \begin{bmatrix} 0 & -sA_1 & 0 & 0 & 0 \\ (s-1)A_1 & 0 & -(s-1)A_2 & 0 & 0 \\ 0 & (s-2)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & (s-1)A_{2s} \\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix} \quad \sigma_y(s)\sigma_z(s) = \frac{i}{2} \begin{bmatrix} 0 & -(s-1)A_1 & 0 & 0 & 0 \\ sA_1 & 0 & -(s-2)A_2 & 0 & 0 \\ 0 & (s-1)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & sA_{2s} \\ 0 & 0 & 0 & -(s-1)A_{2s} & 0 \end{bmatrix} \quad (1.9)$$

$$\sigma_x(s)\sigma_y(s) = \frac{i}{4} \begin{bmatrix} A_1^2 & 0 & -A_1A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2 - A_1^2 & 0 & -A_2A_3 & 0 & 0 & 0 \\ A_1A_2 & 0 & A_3^2 - A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_3 & 0 & \cdots & 0 & -A_{2s-2}A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2 - A_{2s-2}^2 & 0 & -A_{2s-1}A_{2s} \\ 0 & 0 & 0 & A_{2s-2}A_{2s-1} & 0 & A_{2s}^2 - A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & -A_{2s}^2 \end{bmatrix} \quad (1.10)$$

$$\sigma_y(s)\sigma_x(s) = \frac{i}{4} \begin{bmatrix} -A_1^2 & 0 & -A_1A_2 & 0 & 0 & 0 & 0 \\ 0 & A_1^2 - A_2^2 & 0 & -A_2A_3 & 0 & 0 & 0 \\ A_1A_2 & 0 & A_3^2 - A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_3 & 0 & \cdots & 0 & -A_{2s-2}A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-2}^2 - A_{2s-1}^2 & 0 & -A_{2s-1}A_{2s} \\ 0 & 0 & 0 & A_{2s-2}A_{2s-1} & 0 & A_{2s-1}^2 - A_{2s}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \quad (1.11)$$

自旋矩阵的反对易关系:

$$\{\sigma_y(s), \sigma_z(s)\} = \frac{i}{2} \begin{bmatrix} 0 & -(2s-1)A_1 & 0 & 0 & 0 \\ (2s-1)A_1 & 0 & -(2s-3)A_2 & 0 & 0 \\ 0 & (2s-3)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & (2s-1)A_{2s} \\ 0 & 0 & 0 & -(2s-1)A_{2s} & 0 \end{bmatrix} \quad (1.12)$$

$$\{\sigma_z(s), \sigma_x(s)\} = \frac{1}{2} \begin{bmatrix} 0 & (2s-1)A_1 & 0 & 0 & 0 \\ (2s-1)A_1 & 0 & (2s-3)A_2 & 0 & 0 \\ 0 & (2s-3)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -(2s-1)A_{2s} \\ 0 & 0 & 0 & -(2s-1)A_{2s} & 0 \end{bmatrix} \quad (1.13)$$

$$\{\sigma_x(s), \sigma_y(s)\} = \frac{i}{2} \begin{bmatrix} 0 & 0 & -A_1A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_2A_3 & 0 & 0 & 0 \\ A_1A_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_3 & 0 & \cdots & 0 & -A_{2s-1}A_{2s} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & -A_{2s}A_{2s+1} \\ 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{2s}A_{2s+1} & 0 & 0 \end{bmatrix} \quad (1.14)$$

$$\{\sigma_x(s), \sigma_x(s)\} = \frac{1}{2} \begin{bmatrix} 0 & 0 & A_1A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_2A_3 & 0 & 0 & 0 \\ A_1A_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_3 & 0 & \cdots & 0 & A_{2s-1}A_{2s} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & A_{2s}A_{2s+1} \\ 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{2s}A_{2s+1} & 0 & 0 \end{bmatrix} + s(s+1) - \sigma_z^2(s) \quad (1.15)$$

$$\{\sigma_y(s), \sigma_y(s)\} = -\frac{1}{2} \begin{bmatrix} 0 & 0 & A_1A_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_2A_3 & 0 & 0 & 0 \\ A_1A_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A_2A_3 & 0 & \cdots & 0 & A_{2s-1}A_{2s} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & A_{2s}A_{2s+1} \\ 0 & 0 & 0 & A_{2s-1}A_{2s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{2s}A_{2s+1} & 0 & 0 \end{bmatrix} + s(s+1) - \sigma_z^2(s) \quad (1.16)$$

$$\{\sigma_z(s), \sigma_z(s)\} = 2 \begin{bmatrix} s^2 & 0 & 0 & 0 & 0 \\ 0 & (s-1)^2 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (s-1)^2 & 0 \\ 0 & 0 & 0 & 0 & s^2 \end{bmatrix} \quad (1.17)$$

$$A_1(s) := \frac{1}{2} \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{2s} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_2(s) := \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{2s} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{2s+1} \end{bmatrix} \quad (1.18)$$

$$A_2^2(s) + A_1^2(s) = s(s+1) - \sigma_z^2(s), A_2^2(s) - A_1^2(s) = \sigma_z(s) \quad (1.19)$$

#### 1.4 正交标架下洛伦兹变换参数 $\vartheta^{ab}$ 和自旋张量 $S_{ab}(s, \varsigma)$ (本书采用此标架)

$\epsilon \in R$ 表示 $O'$ (粒子)相对 $O$ 的速度,  $\omega \in R$ 表示 $O'$ (粒子)相对 $O$ 的旋转角度。

自我评述: 这样规定好, 物理解释才不会混淆, 尤其时间久了, 就很容易搞不清参数与真实物理的对应了, 一般是差个符号。万一忘了, 可以回到此处, 正本清源。

$$g_{ab} \simeq g^{ab} \succ \text{diag}(1, 1, 1, 1), x^a \simeq x_a = (x, y, z, it), \vec{\vartheta} \equiv i\omega + \epsilon \quad (1.20a)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ it' \end{bmatrix} (= x'^a) = e^{\vartheta^a{}_b} \begin{bmatrix} x \\ y \\ z \\ it \end{bmatrix} (= x^b), \vartheta^a{}_b \succ \begin{bmatrix} 0 & \omega_z & -\omega_y & i\epsilon_x \\ -\omega_z & 0 & \omega_x & i\epsilon_y \\ \omega_y & -\omega_x & 0 & i\epsilon_z \\ -i\epsilon_x & -i\epsilon_y & -i\epsilon_z & 0 \end{bmatrix} \prec \vartheta_a{}^b \simeq \vartheta^{ab} \simeq \vartheta_{ab} \succ i\omega \cdot R + \epsilon \cdot L \quad (1.20b)$$

$$\vartheta_{ij} = \varepsilon_{ijk}\omega^k, \omega_k = \frac{1}{2}\varepsilon_{kij}\vartheta^{ij} \quad (1.20c)$$

$$x'^a = (g^a{}_b + \vartheta^a{}_b)x^b, x'^a = (g^{ab} + \vartheta^{ab})x_b, x'_a = (g_a{}^b + \vartheta_a{}^b)x_b, x'_a = (g_{ab} + \vartheta_{ab})x^b \quad (1.20d)$$

$$\begin{cases} \delta x_a = \vartheta_a{}^b x_b = \vartheta_{ab} x^b, \delta x^a = \vartheta^a{}_b x^b = \vartheta^{ab} x_b \\ \vec{S}_{abcd} = -i(g_{ac}g_{bd} - g_{ad}g_{bc}) \succ \begin{bmatrix} 0 & R_z(s) & -R_y(s) & -L_x(s) \\ -R_z(s) & 0 & R_x(s) & -L_y(s) \\ R_y(s) & -R_x(s) & 0 & -L_z(s) \\ L_x(s) & L_y(s) & L_z(s) & 0 \end{bmatrix} \end{cases} \quad (1.20e)$$

$$\begin{cases} \delta\varphi(s) = \frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma)\varphi(s) = \frac{i}{2}\vartheta_{ab}S^{ab}(s, \varsigma)\varphi(s) \\ S^{ab}(s, \varsigma) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -\varsigma\sigma_z(s) \\ \varsigma\sigma_x(s) & \varsigma\sigma_y(s) & \varsigma\sigma_z(s) & 0 \end{bmatrix} \prec S_{ab}(s, \varsigma) = -i[\sigma(s), \frac{i\varsigma}{2}][a[\sigma(s), -\frac{i\varsigma}{2}]b] \end{cases} \quad (1.20f)$$

$$\begin{cases} L_{ab} = x_a p_b - x_b p_a \succ \begin{bmatrix} 0 & xp_y - yp_x & -(z p_x - x p_z) & ix E - itp_x \\ -(xp_y - yp_x) & 0 & yp_z - zp_y & iy E - itp_y \\ zp_x - xp_z & -(yp_z - zp_y) & 0 & iz E - itp_z \\ -(ix E - itp_x) & -(iy E - itp_y) & -(iz E - itp_z) & 0 \end{bmatrix} \\ M_{ab} = L_{ab} + S_{ab}(s, \varsigma) = -i(x_a \partial_b - x_b \partial_a) + S_{ab}(s, \varsigma) \end{cases} \quad (1.20g)$$

自我评述: 标架选取本质上也有无穷种选法, 常用的也有好几种。本书采用的是本节的正交标架, 这样的好处是在这种标架下常数不变张量更简单、更统一、更有规律。当然通过等价变换也可以变换到其它标架表示。

#### 1.5 其它标架下洛伦兹变换参数 $\vartheta^{ab}$ 和自旋张量 $S_{ab}(s, \varsigma)$ 的对比

##### 1.5.1 伪标架

$$g_{ab} = g^{ab} = \text{diag}(1, 1, 1, -1), x^a = (x, y, z, t), x_a = (x, y, z, -t) \quad (1.21a)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} (= x'^a) = e^{\vartheta^a{}_b} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} (= x^b), \vartheta^a{}_b = \begin{bmatrix} 0 & \omega_z & -\omega_y & -\epsilon_x \\ -\omega_z & 0 & \omega_x & -\epsilon_y \\ \omega_y & -\omega_x & 0 & -\epsilon_z \\ -\epsilon_x & -\epsilon_y & -\epsilon_z & 0 \end{bmatrix}, x'_a = (g_a{}^b + \vartheta_a{}^b)x_b \quad (1.21b)$$

$$\vartheta_{ab} = g_{ac}\vartheta^c{}_b \succ \begin{bmatrix} 0 & \omega_z & -\omega_y & -\epsilon_x \\ -\omega_z & 0 & \omega_x & -\epsilon_y \\ \omega_y & -\omega_x & 0 & -\epsilon_z \\ \epsilon_x & \epsilon_y & \epsilon_z & 0 \end{bmatrix}, \vartheta^{ab} = \vartheta^a{}_c g^{cb} \succ \begin{bmatrix} 0 & \omega_z & -\omega_y & \epsilon_x \\ -\omega_z & 0 & \omega_x & \epsilon_y \\ \omega_y & -\omega_x & 0 & \epsilon_z \\ -\epsilon_x & -\epsilon_y & -\epsilon_z & 0 \end{bmatrix}, \vartheta_a{}^b = g_{ac}\vartheta^c{}_d g^{db} \succ \begin{bmatrix} 0 & \omega_z & -\omega_y & \epsilon_x \\ -\omega_z & 0 & \omega_x & \epsilon_y \\ \omega_y & -\omega_x & 0 & \epsilon_z \\ \epsilon_x & \epsilon_y & \epsilon_z & 0 \end{bmatrix} \quad (1.21c)$$

$$x'^a = (g^a{}_b + \vartheta^a{}_b)x^b, x'^a = (g^{ab} + \vartheta^{ab})x_b, x'_a = (g_a{}^b + \vartheta_a{}^b)x_b, x'_a = (g_{ab} + \vartheta_{ab})x^b \quad (1.21d)$$

$$\delta x^a = \vartheta_a{}^b x_b = \vartheta_{ab} x^b, \delta x_a = \vartheta^a{}_b x^b = \vartheta^{ab} x_b, \delta\psi(s) = \frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma)\psi(s) = \frac{i}{2}\vartheta_{ab}S^{ab}(s, \varsigma)\psi(s) \quad (1.21e)$$



$$\vec{S}_{abcd} \succ \begin{bmatrix} 0 & R_z(s) & -R_y(s) & L_x(s) \\ -R_z(s) & 0 & R_x(s) & L_y(s) \\ R_y(s) & -R_x(s) & 0 & L_z(s) \\ -L_x(s) & -L_y(s) & -L_z(s) & 0 \end{bmatrix} = -i(g_{ac}g_{bd} - g_{ad}g_{bc}), \delta x_a = \vartheta^{cd} \vec{S}_{abcd} x^b, \delta x^a = \vartheta_{cd} S^{abcd} x_b \quad (1.21f)$$

$$S^{ab}(s, \varsigma) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & i\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & i\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & i\varsigma\sigma_z(s) \\ -i\varsigma\sigma_x(s) & -i\varsigma\sigma_y(s) & -i\varsigma\sigma_z(s) & 0 \end{bmatrix}, S_{ab}(s, \varsigma) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -i\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -i\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -i\varsigma\sigma_z(s) \\ i\varsigma\sigma_x(s) & i\varsigma\sigma_y(s) & i\varsigma\sigma_z(s) & 0 \end{bmatrix}, \varsigma = \pm 1 \quad (1.21g)$$

$$L_{ab} = x_a p_b - x_b p_a \succ \begin{bmatrix} 0 & xp_y - yp_x & -(zp_x - xp_z) & -(xE - tp_x) \\ -(xp_y - yp_x) & 0 & yp_z - zp_y & -(yE - tp_y) \\ zp_x - xp_z & -(yp_z - zp_y) & 0 & -(zE - tp_z) \\ xE - tp_x & yE - tp_y & zE - tp_z & 0 \end{bmatrix}, M_{ab} = L_{ab} + S_{ab}(s, \varsigma) \quad (1.21h)$$

### 1.5.2 负伪标架

$$g_{ab} = g^{ab} = -diag(1, 1, 1, -1), x^a = (x, y, z, t), x_a = -(x, y, z, -t) \quad (1.22a)$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} (= x'^a) = e^{\vartheta^a_b} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} (= x^b), \vartheta^a_b = \begin{bmatrix} 0 & \omega_z & -\omega_y & -\epsilon_x \\ -\omega_z & 0 & \omega_x & -\epsilon_y \\ \omega_y & -\omega_x & 0 & -\epsilon_z \\ -\epsilon_x & -\epsilon_y & -\epsilon_z & 0 \end{bmatrix}, x'_a = (g_a^b + \vartheta_a^b) x_b \quad (1.22b)$$

$$\vartheta_{ab} = g_{ac} \vartheta^c_b \succ - \begin{bmatrix} 0 & \omega_z & -\omega_y & -\epsilon_x \\ -\omega_z & 0 & \omega_x & -\epsilon_y \\ \omega_y & -\omega_x & 0 & -\epsilon_z \\ \epsilon_x & \epsilon_y & \epsilon_z & 0 \end{bmatrix}, \vartheta^{ab} = \vartheta^a_c g^{cb} \succ - \begin{bmatrix} 0 & \omega_z & -\omega_y & \epsilon_x \\ -\omega_z & 0 & \omega_x & \epsilon_y \\ \omega_y & -\omega_x & 0 & \epsilon_z \\ -\epsilon_x & -\epsilon_y & -\epsilon_z & 0 \end{bmatrix}, \vartheta_a^b = g_{ac} \vartheta^c_a g^{db} \succ \begin{bmatrix} 0 & \omega_z & -\omega_y & \epsilon_x \\ -\omega_z & 0 & \omega_x & \epsilon_y \\ \omega_y & -\omega_x & 0 & \epsilon_z \\ \epsilon_x & \epsilon_y & \epsilon_z & 0 \end{bmatrix} \quad (1.22c)$$

$$x'^a = (g_a^b + \vartheta_a^b) x^b, x'^a = (g^{ab} + \vartheta^{ab}) x_b, x'_a = (g_a^b + \vartheta_a^b) x_b, x'_a = (g_{ab} + \vartheta_{ab}) x^b \quad (1.22d)$$

$$\delta x^a = \vartheta_a^b x_b = \vartheta_{ab} x^b, \delta x_a = \vartheta^a_b x^b = \vartheta^{ab} x_b, \delta \psi(s) = \frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma) \psi(s) = \frac{i}{2} \vartheta_{ab} S^{ab}(s, \varsigma) \psi(s) \quad (1.22e)$$

$$\vec{S}_{abcd} \succ \begin{bmatrix} 0 & R_z(s) & -R_y(s) & L_x(s) \\ -R_z(s) & 0 & R_x(s) & L_y(s) \\ R_y(s) & -R_x(s) & 0 & L_z(s) \\ -L_x(s) & -L_y(s) & -L_z(s) & 0 \end{bmatrix} = -i(g_{ac}g_{bd} - g_{ad}g_{bc}), \delta x_a = \vartheta^{cd} \vec{S}_{abcd} x^b, \delta x^a = \vartheta_{cd} S^{abcd} x_b \quad (1.22f)$$

$$S^{ab}(s, \varsigma) \succ - \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & i\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & i\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & i\varsigma\sigma_z(s) \\ -i\varsigma\sigma_x(s) & -i\varsigma\sigma_y(s) & -i\varsigma\sigma_z(s) & 0 \end{bmatrix}, S_{ab}(s, \varsigma) \succ - \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -i\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -i\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -i\varsigma\sigma_z(s) \\ i\varsigma\sigma_x(s) & i\varsigma\sigma_y(s) & i\varsigma\sigma_z(s) & 0 \end{bmatrix}, \varsigma = \pm 1 \quad (1.22g)$$

$$L_{ab} = x_a p_b - x_b p_a \succ - \begin{bmatrix} 0 & xp_y - yp_x & -(zp_x - xp_z) & -(xE - tp_x) \\ -(xp_y - yp_x) & 0 & yp_z - zp_y & -(yE - tp_y) \\ zp_x - xp_z & -(yp_z - zp_y) & 0 & -(zE - tp_z) \\ xE - tp_x & yE - tp_y & zE - tp_z & 0 \end{bmatrix}, M_{ab} = L_{ab} + S_{ab}(s, \varsigma) \quad (1.22h)$$

### 1.6 s-自旋旋量指标

s-自旋旋量指标定义:用阿拉伯小写字母 $\{i, j, k, l, m, n, p, q, r, s\}$ 特别表示一般自旋情形。

$$k_\varsigma \sim e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma)} = e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s)} \Leftrightarrow k_+ := k \sim e^{(i\omega + \epsilon) \cdot \sigma(s)} \quad k_- := k' \sim e^{(i\omega - \epsilon) \cdot \sigma(s)} \quad (1.23)$$

$$k_\varsigma \sim e^{-\frac{i}{2} \vartheta^{ab} S_{ab}^T(s, \varsigma)} = e^{-(i\omega + \varsigma\epsilon) \cdot \sigma^T(s)} \Leftrightarrow k_+ := k \sim e^{-(i\omega + \epsilon) \cdot \sigma^T(s)} \quad k_- := k' \sim e^{-(i\omega - \epsilon) \cdot \sigma^T(s)} \quad (1.24)$$

$$k'_\varsigma \sim e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, -\varsigma)} = e^{(i\omega - \varsigma\epsilon) \cdot \sigma(s)} \Leftrightarrow k'_+ := k' \sim e^{(i\omega - \epsilon) \cdot \sigma(s)} \quad k'_- := k \sim e^{(i\omega + \epsilon) \cdot \sigma(s)} \quad (1.25)$$

$$k'_\varsigma \sim e^{-\frac{i}{2} \vartheta^{ab} S_{ab}^T(s, -\varsigma)} = e^{-(i\omega - \varsigma\epsilon) \cdot \sigma^T(s)} \Leftrightarrow k'_+ := k' \sim e^{-(i\omega - \epsilon) \cdot \sigma^T(s)} \quad k'_- := k \sim e^{-(i\omega + \epsilon) \cdot \sigma^T(s)} \quad (1.26)$$

指标关系: 指标恒等, 旋量也恒等, 是定义。

$$\begin{cases} k_\varsigma \equiv k'_{-\varsigma} \\ k_{-\varsigma} \equiv k'_\varsigma \end{cases} \quad \begin{cases} k_\varsigma \equiv k'_{-\varsigma} \\ k_{-\varsigma} \equiv k'_\varsigma \end{cases} \quad \begin{cases} k_+ \equiv k'_{-} \equiv k \\ k_- \equiv k'_+ \equiv k' \end{cases} \quad \begin{cases} k_+ \equiv k'_{-} \equiv k \\ k_- \equiv k'_+ \equiv k' \end{cases} \quad (1.27)$$

共轭指标: 指标相等, 旋量可能等, 可能不等。而且在特殊的厄密表象下才成立。

$$\begin{cases} (k_\zeta)^* = k'_\zeta & \begin{cases} (k'_\zeta)^* = k_\zeta \\ (k_\zeta)^* = k'_\zeta \end{cases} & \begin{cases} (k)^* = k' \\ (k')^* = k \end{cases} & \begin{cases} (k')^* = k \\ (k)^* = k' \end{cases} \end{cases} \quad (1.28)$$

s-自旋旋量指标对应的度规张量和自洽的升降规则如下:

$$\begin{cases} \varepsilon_{k_\zeta l_\zeta}(s) \simeq \varepsilon^{k'_\zeta l'_\zeta}(s) \simeq \varepsilon_{k'_\zeta l'_\zeta}(s) \simeq \varepsilon^{k_\zeta l_\zeta}(s) \succ \varepsilon(s) \\ \psi_{k_\zeta} = (-\zeta)^{2s} \varepsilon_{k_\zeta l_\zeta}(s) \psi^{l_\zeta}, \psi^{k_\zeta} = \zeta^{2s} \varepsilon^{k_\zeta l_\zeta}(s) \psi_{l_\zeta} \\ \psi_{k'_\zeta} = (-\zeta)^{2s} \varepsilon_{k'_\zeta l'_\zeta}(s) \psi^{l'_\zeta}, \psi^{k'_\zeta} = \zeta^{2s} \varepsilon^{k'_\zeta l'_\zeta}(s) \psi_{l'_\zeta} \end{cases} \quad (1.29)$$

自洽的升降规则本质如下, 与Penrose的规则一致, 且Penrose指标与我的指标一致。

$$\begin{cases} \varepsilon_{k'l'}(s) = [\varepsilon_{kl}(s)]^* \simeq \varepsilon_{kl}(s), \varepsilon^{k'l'}(s) = [\varepsilon^{kl}(s)]^* \simeq \varepsilon^{kl}(s) \\ \psi_k = (-1)^{2s} \varepsilon_{kl}(s) \psi^l, \psi^k = \varepsilon^{kl}(s) \psi_l \\ \psi_{k'} = (-1)^{2s} \varepsilon_{k'l'}(s) \psi^{l'}, \psi^{k'} = \varepsilon^{k'l'}(s) \psi_{l'} \end{cases} \quad (1.30)$$

自我评述: 为什么要这样规定新抽象指标的升降规则, 主要考虑是两点, 一是尽量与Penrose抽象指标规则一致; 二是新抽象指标内在自洽性要求的, 也是后面大量常数不变张量实际计算实践的总结与提炼。

### 1.7 $\frac{1}{2}$ -自旋旋量指标

$\frac{1}{2}$ -自旋旋量指标是s-自旋旋量指标的特例, 为了与Penrose抽象指标一致, 仍用阿拉伯大写字母 $\{A, B, C, \dots\}$ 特别表示 $\frac{1}{2}$ -自旋情形, 定义与规则如上节所示(取 $s = \frac{1}{2}$ )。

### 1.8 复矢量指标

光子自旋矩阵: 用希腊字母 $\{\alpha, \beta, \gamma, \dots\}$ 特别表示复矢量指标。

$$\gamma = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \quad (1.31)$$

形式上取 $\sigma(1) = \gamma$ , 得复矢量指标如下:

$$\alpha_\zeta \sim e^{(i\omega + \varepsilon) \cdot \gamma} \quad \Leftrightarrow \alpha_+ := \alpha \sim e^{(i\omega + \varepsilon) \cdot \gamma} \quad \alpha_- := \alpha' \sim e^{(i\omega - \varepsilon) \cdot \gamma} \quad (1.32)$$

$$\alpha'_\zeta \sim e^{(i\omega - \varepsilon) \cdot \gamma} \quad \Leftrightarrow \alpha'_+ := \alpha' \sim e^{(i\omega - \varepsilon) \cdot \gamma} \quad \alpha'_- := \alpha \sim e^{(i\omega + \varepsilon) \cdot \gamma} \quad (1.33)$$

指标关系:

$$\alpha_\zeta \equiv \alpha'_{-\zeta}, \alpha_{-\zeta} \equiv \alpha'_\zeta; \alpha_+ \equiv \alpha'_- \equiv \alpha, \alpha_- \equiv \alpha'_+ \equiv \alpha \quad (1.34)$$

共轭关系:

$$(\alpha_\zeta)^* \equiv \alpha'_\zeta, (\alpha'_\zeta)^* \equiv \alpha_\zeta; (\alpha)^* \equiv \alpha', (\alpha')^* \equiv \alpha \quad (1.35)$$

复矢量指标对应的度规张量及升降规则:

$$\begin{cases} g_{\alpha_\zeta \beta_\zeta} = \delta_{\alpha_\zeta \beta_\zeta} \succ I, g^{\alpha_\zeta \beta_\zeta} = \delta^{\alpha_\zeta \beta_\zeta} \succ I \\ g_{\alpha'_\zeta \beta'_\zeta} = \delta_{\alpha'_\zeta \beta'_\zeta} \succ I, g^{\alpha'_\zeta \beta'_\zeta} = \delta^{\alpha'_\zeta \beta'_\zeta} \succ I \end{cases}, \begin{cases} \psi_{\alpha_\zeta} = g_{\alpha_\zeta \beta_\zeta} \psi^{\beta_\zeta}, \psi^{\alpha_\zeta} = g^{\alpha_\zeta \beta_\zeta} \psi_{\beta_\zeta} \\ \psi_{\alpha'_\zeta} = g_{\alpha'_\zeta \beta'_\zeta} \psi^{\beta'_\zeta}, \psi^{\alpha'_\zeta} = g^{\alpha'_\zeta \beta'_\zeta} \psi_{\beta'_\zeta} \end{cases} \quad (1.36)$$

此时度规张量是单位矩阵, 可以不用区分逆变和协变张量, 即上标与下标可以随意同步调换。

自我评述: 复矢量指标来源于对电磁场、Yang-Mills场和引力场的描述, 所以也特别适合描述它们。

### 1.9 矢量指标

空间旋转矩阵R和洛伦兹推动矩阵L:

$$R = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}, L = \left\{ \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \right\} \quad (1.37)$$

矢量指标如下:用阿拉伯小写字母 $\{a, b, c, d, e, f, g, h, u, v, w\}$ 特别表示矢量指标.

$$a_\zeta \sim e^{(i\omega \cdot R + \zeta \epsilon \cdot L)} \quad \Leftrightarrow a_+ := a \sim e^\vartheta = e^{(i\omega \cdot R + \epsilon \cdot L)} \quad a_- := a' \sim e^{\vartheta^*} = e^{(i\omega \cdot R - \epsilon \cdot L)} \quad (1.38)$$

$$a'_\zeta \sim e^{(i\omega \cdot R - \zeta \epsilon \cdot L)} \quad \Leftrightarrow a'_+ := a' \sim e^{\vartheta^*} = e^{(i\omega \cdot R - \epsilon \cdot L)} \quad a'_- := a \sim e^\vartheta = e^{(i\omega \cdot R + \epsilon \cdot L)} \quad (1.39)$$

指标关系:

$$a_\zeta \equiv a'_{-\zeta}, a_{-\zeta} \equiv a'_\zeta; a_+ \equiv a'_- \equiv a, a_- \equiv a'_+ \equiv a' \quad (1.40)$$

共轭关系:

$$(a_\zeta)^* \equiv a'_\zeta, (a'_\zeta)^* \equiv a_\zeta; (a)^* \equiv a', (a')^* \equiv a \quad (1.41)$$

矢量指标对应的度规张量及升降规则:

$$\begin{cases} g_{a_\zeta b_\zeta} = \delta_{a_\zeta b_\zeta} \succ I, g^{a_\zeta b_\zeta} = \delta^{a_\zeta b_\zeta} \succ I \\ g_{a'_\zeta b'_\zeta} = \delta_{a'_\zeta b'_\zeta} \succ I, g^{a'_\zeta b'_\zeta} = \delta^{a'_\zeta b'_\zeta} \succ I \end{cases}, \begin{cases} \psi_{a_\zeta} = g_{a_\zeta b_\zeta} \psi^{b_\zeta}, \psi^{a_\zeta} = g^{a_\zeta b_\zeta} \psi_{b_\zeta} \\ \psi_{a'_\zeta} = g_{a'_\zeta b'_\zeta} \psi^{b'_\zeta}, \psi^{a'_\zeta} = g^{a'_\zeta b'_\zeta} \psi_{b'_\zeta} \end{cases} \quad (1.42)$$

此时度规张量是单位矩阵, 可以不用区分逆变和协变张量, 即上标与下标可以随意同步调换.

## 2 常用矩阵

### 2.1 泡利矩阵

$$\sigma = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}, tr(\sigma_{\alpha_\zeta}) = 0, tr(\sigma_{\alpha_\zeta} \sigma_{\beta_\zeta}) = 2\delta_{\alpha_\zeta \beta_\zeta} \quad (1.43)$$

$$[\sigma_{\alpha_\zeta}, \sigma_{\beta_\zeta}] = 2i\varepsilon_{\alpha_\zeta \beta_\zeta} \gamma_\zeta \sigma_{\gamma_\zeta}, \{\sigma_{\alpha_\zeta}, \sigma_{\beta_\zeta}\} = 2\delta_{\alpha_\zeta \beta_\zeta}, \sigma^2(\frac{1}{2}) = \frac{1}{2}(\frac{1}{2} + 1) \quad (1.44)$$

### 2.2 光子矩阵

$$\gamma = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, tr(\gamma_{\alpha_\zeta}) = 0, tr(\gamma_{\alpha_\zeta} \gamma_{\beta_\zeta}) = 2\delta_{\alpha_\zeta \beta_\zeta} \quad (1.45)$$

$$[\gamma_{\alpha_\zeta}, \gamma_{\beta_\zeta}] = i\varepsilon_{\alpha_\zeta \beta_\zeta} \gamma_\zeta \gamma_{\gamma_\zeta}, \gamma^2 = 1(1+1), \gamma_{\alpha_\zeta} \prec \gamma_{\alpha_\zeta \beta_\zeta} \gamma_\zeta \equiv -i\varepsilon_{\alpha_\zeta \beta_\zeta} \gamma_\zeta \quad (1.46)$$

### 2.3 旋转生成元矩阵

空间旋转生成元矩阵:

$$R = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}, tr(R_{\alpha_\zeta}) = 0, tr(R_{\alpha_\zeta} R_{\beta_\zeta}) = 2\delta_{\alpha_\zeta \beta_\zeta} \quad (1.47a)$$

洛伦兹推动生成元矩阵:

$$L = \left\{ \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \right\}, tr(L_{\alpha_\zeta}) = 0, tr(L_{\alpha_\zeta} L_{\beta_\zeta}) = 2\delta_{\alpha_\zeta \beta_\zeta} \quad (1.47b)$$

$$[R_{\alpha_\zeta}, R_{\beta_\zeta}] = i\varepsilon_{\alpha_\zeta \beta_\zeta} \gamma_\zeta R_{\gamma_\zeta}, [L_{\alpha_\zeta}, L_{\beta_\zeta}] = i\varepsilon_{\alpha_\zeta \beta_\zeta} \gamma_\zeta R_{\gamma_\zeta}, [R_{\alpha_\zeta}, L_{\beta_\zeta}] = [L_{\alpha_\zeta}, R_{\beta_\zeta}] = i\varepsilon_{\alpha_\zeta \beta_\zeta} \gamma_\zeta L_{\gamma_\zeta} \quad (1.47c)$$

$$R^2 = diag(2, 2, 2, 1), L^2 = diag(0, 0, 0, 3) \quad (1.47d)$$

### 2.4 $SO(4)$ 群生成元矩阵

$SO(4)$ 群生成元矩阵的正分支:

$$\sigma_+ = R + L = \left\{ \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \right\} \quad (1.48a)$$

$$\sigma_+ = \{-\sigma_y \otimes \sigma_x, -I \otimes \sigma_y, \sigma_y \otimes \sigma_z\}, tr(\sigma_{+\alpha_\zeta}) = 0, tr(\sigma_{+\alpha_\zeta} \sigma_{+\beta_\zeta}) = 4\delta_{\alpha_\zeta \beta_\zeta} \quad (1.48b)$$

$SO(4)$ 群生成元矩阵的负分支:

$$\sigma_- = R - L = \left\{ \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \right\} \quad (1.49a)$$

$$\sigma_- = \{\sigma_x \otimes \sigma_y, -\sigma_z \otimes \sigma_y, \sigma_y \otimes I\}, tr(\sigma_{-\alpha_\zeta}) = 0, tr(\sigma_{-\alpha_\zeta} \sigma_{-\beta_\zeta}) = 4\delta_{\alpha_\zeta \beta_\zeta} \quad (1.49b)$$

$SO(4)$ 群生成元矩阵两个分支间的关系。

$$[\sigma_{+\alpha_\zeta}, \sigma_{+\beta_\zeta}] = i\varepsilon_{\alpha_\zeta\beta_\zeta} \gamma_\zeta \sigma_{+\gamma_\zeta}, \{\sigma_{+\alpha_\zeta}, \sigma_{+\beta_\zeta}\} = 2\delta_{\alpha_\zeta\beta_\zeta}, \sigma_+^2 = \frac{1}{2}(\frac{1}{2} + 1) \quad (1.50a)$$

$$[\sigma_{-\alpha_\zeta}, \sigma_{-\beta_\zeta}] = i\varepsilon_{\alpha_\zeta\beta_\zeta} \gamma_\zeta \sigma_{-\gamma_\zeta}, \{\sigma_{-\alpha_\zeta}, \sigma_{-\beta_\zeta}\} = 2\delta_{\alpha_\zeta\beta_\zeta}, \sigma_-^2 = \frac{1}{2}(\frac{1}{2} + 1) \quad (1.50b)$$

$$[\sigma_{+\alpha_\zeta}, \sigma_{-\beta_\zeta}] = [\sigma_{-\alpha_\zeta}, \sigma_{+\beta_\zeta}] = 0 \quad (1.50c)$$

$SO(4)$ 群生成元矩阵的统一表示

$$\sigma_\zeta = \left\{ \begin{bmatrix} 0 & 0 & 0 & i\zeta \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i\zeta & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i\zeta \\ -i & 0 & 0 & 0 \\ 0 & -i\zeta & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i\zeta \\ 0 & 0 & -i\zeta & 0 \end{bmatrix} \right\} \quad (1.51a)$$

$$[\sigma_{\kappa\alpha_\zeta}, \sigma_{\tau\beta_\zeta}] = i\delta_{\kappa\tau}\varepsilon_{\alpha_\zeta\beta_\zeta} \gamma_\zeta \sigma_{\kappa\gamma_\zeta}, \{\sigma_{\kappa\alpha_\zeta}, \sigma_{\tau\beta_\zeta}\} = 2\delta_{\kappa\tau}\delta_{\alpha_\zeta\beta_\zeta}, \sigma_\zeta^2 = \frac{1}{2}(\frac{1}{2} + 1) \quad (1.51b)$$

$$\text{推论2.4.1. } \sigma_{\zeta\beta_\zeta}^{ab} = \sigma_\zeta^a | \beta_\zeta^b = \left\{ \begin{bmatrix} 0 & 0 & 0 & i\zeta \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i\zeta \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i\zeta \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -i\zeta & 0 & 0 & 0 \\ 0 & -i\zeta & 0 & 0 \\ 0 & 0 & -i\zeta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

$$\text{推论2.4.2. } \sigma_{-\zeta\beta'_\zeta}^{ab} = \sigma_{-\zeta}^a | \beta'_\zeta^b = \left\{ \begin{bmatrix} 0 & 0 & 0 & -i\zeta \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i\zeta \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\zeta \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} i\zeta & 0 & 0 & 0 \\ 0 & i\zeta & 0 & 0 \\ 0 & 0 & i\zeta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

自我评述：这对常数不变张量本质上就是 $SO(4)$ 两个生成元矩阵；特霍夫特也用过，叫特霍夫特 $\eta$ 矩阵；这对常数不变张量也出现于圈量子引力中Ashtekar作用量的构造，不过他们没叫常数不变张量。在这里它出现的地方更多，无处不在，应用十分广泛，是一个十分有用的基本常数不变张量，与自旋矩阵密切相关。应用它们也可以得到电磁场、Yang-Mills场方程的一种新表述-整旋量表述。

### 3 常数不变张量的发现与证明

#### 3.1 各种常见度规常数不变张量

自我评述：这节的常数不变张量在我发展这个数学理论之前已经存在于数学与物理中了，我正是受它们和Penrose旋量分析的启发，才想发展出一般的常数不变张量理论，并应用到物理中，为物理研究提供一个有用的数学工具。

##### 3.1.1 四矢量度规常数不变张量 $\delta_{a_\zeta b_\zeta}, \delta^{\alpha_\zeta \beta_\zeta}, \delta_{a'_\zeta b'_\zeta}, \delta^{\alpha'_\zeta \beta'_\zeta}$

定理3.1.1.  $I_4 = e^{(i\omega \cdot R + \zeta \epsilon \cdot L)} I_4 e^{(i\omega \cdot R + \zeta \epsilon \cdot L)^T}$ , 即 $\delta_{a_\zeta b_\zeta}, \delta^{\alpha_\zeta \beta_\zeta}$ 是常数不变张量。

推论3.1.1.  $I_4 = e^{(i\omega \cdot R - \zeta \epsilon \cdot L)} I_4 e^{(i\omega \cdot R - \zeta \epsilon \cdot L)^T}$ , 即 $\delta_{a'_\zeta b'_\zeta}, \delta^{\alpha'_\zeta \beta'_\zeta}$ 是常数不变张量。

##### 3.1.2 复矢量度规常数不变张量 $\delta_{\alpha_\zeta \beta_\zeta}, \delta^{\alpha_\zeta \beta_\zeta}, \delta_{\alpha'_\zeta \beta'_\zeta}, \delta^{\alpha'_\zeta \beta'_\zeta}$

定理3.1.2.  $I_3 = e^{(i\omega + \zeta \epsilon) \cdot \gamma} I_3 e^{(i\omega + \zeta \epsilon) \cdot \gamma^T}$ , 即 $\delta_{\alpha_\zeta \beta_\zeta}, \delta^{\alpha_\zeta \beta_\zeta}$ 是常数不变张量。

推论3.1.2.  $I_3 = e^{(i\omega - \zeta \epsilon) \cdot \gamma} I_3 e^{(i\omega - \zeta \epsilon) \cdot \gamma^T}$ , 即 $\delta_{\alpha'_\zeta \beta'_\zeta}, \delta^{\alpha'_\zeta \beta'_\zeta}$ 是常数不变张量。

##### 3.1.3 s-旋量度规常数不变张量 $\varepsilon^{k_\zeta l_\zeta}(s), \varepsilon_{k_\zeta l_\zeta}(s), \varepsilon_{k'_\zeta l'_\zeta}(s), \varepsilon^{k'_\zeta l'_\zeta}(s)$

引理3.1.1.  $\sigma^T(s) = (-1)^{2s+1} \varepsilon(s) \sigma(s) \varepsilon(s)$

定理3.1.3.  $\varepsilon(s) = e^{(i\omega + \zeta \epsilon) \cdot \sigma(s)} \varepsilon(s) e^{(i\omega + \zeta \epsilon) \cdot \sigma^T(s)}$ , 即 $\varepsilon^{k_\zeta l_\zeta}(s)$ 是常数不变张量。

推论3.1.3.  $\varepsilon(s) = e^{(i\omega - \zeta \epsilon) \cdot \sigma(s)} \varepsilon(s) e^{(i\omega - \zeta \epsilon) \cdot \sigma^T(s)}$ , 即 $\varepsilon_{k'_\zeta l'_\zeta}(s)$ 是常数不变张量。

推论3.1.4.  $\varepsilon(s) = e^{-(i\omega + \zeta \epsilon) \cdot \sigma^T(s)} \varepsilon(s) e^{-(i\omega + \zeta \epsilon) \cdot \sigma(s)}$ , 即 $\varepsilon_{k_\zeta l_\zeta}(s)$ 是常数不变张量。

推论3.1.5.  $\varepsilon(s) = e^{-(i\omega - \zeta \epsilon) \cdot \sigma^T(s)} \varepsilon(s) e^{-(i\omega - \zeta \epsilon) \cdot \sigma(s)}$ , 即 $\varepsilon^{k'_\zeta l'_\zeta}(s)$ 是常数不变张量。

##### 3.1.4 反对称 $\frac{1}{2}$ -旋量度规张量 $\varepsilon^{A_\zeta B_\zeta}, \varepsilon_{A_\zeta B_\zeta}, \varepsilon_{A'_\zeta B'_\zeta}, \varepsilon^{A'_\zeta B'_\zeta}$

上节取 $s = \frac{1}{2}$ 可得： $\varepsilon^{A_\zeta B_\zeta}, \varepsilon_{A_\zeta B_\zeta}, \varepsilon_{A'_\zeta B'_\zeta}, \varepsilon^{A'_\zeta B'_\zeta}$ 是常数不变张量。

## 3.2 基本定理一及其相关常数不变张量

### 3.2.1 引理

引理3.2.1.  $\vartheta_a^b(\Gamma, i\zeta)_b \equiv (-\omega \times \Gamma - \zeta\epsilon, -i\epsilon \cdot \Gamma)_a, \vartheta_a^b \succ \vartheta \equiv (i\omega \cdot R + \epsilon \cdot L)$

引理3.2.2.  $\frac{1}{2}i\omega \cdot [\Gamma, \Gamma_{\alpha\zeta}] = (\omega \times \Gamma)_{\alpha\zeta}, \forall \omega \rightarrow 0 \Leftrightarrow [\Gamma_{\alpha\zeta}, \Gamma_{\beta\zeta}] = 2i\varepsilon_{\alpha\zeta\beta\zeta} \gamma_\zeta \Gamma_{\gamma\zeta}$

引理3.2.3.  $\frac{1}{2}\epsilon \cdot \{\Gamma, \Gamma_{\alpha\zeta}\} = \epsilon_{\alpha\zeta}, \forall \epsilon \rightarrow 0 \Leftrightarrow \{\Gamma_{\alpha\zeta}, \Gamma_{\beta\zeta}\} = 2\delta_{\alpha\zeta\beta\zeta}$

引理3.2.4.  $\vartheta_{ij} = \varepsilon_{ijk}\omega^k, \vartheta_{i\pi} = i\epsilon_i, \vartheta_{\pi j} = -i\epsilon_j, \omega_k = \frac{1}{2}\varepsilon_{kij}\vartheta^{ij}$

### 3.2.2 基本定理一

在4维时空中存在如下定理：

定理3.2.1.  $(\Gamma, i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a^b e^{(i\omega + \zeta\epsilon) \cdot \frac{1}{2}\Gamma} (\Gamma, i\zeta)_b e^{-(i\omega - \zeta\epsilon) \cdot \frac{1}{2}\Gamma} \Leftrightarrow \begin{cases} [\Gamma_{\alpha\zeta}, \Gamma_{\beta\zeta}] = 2i\varepsilon_{\alpha\zeta\beta\zeta} \gamma_\zeta \Gamma_{\gamma\zeta} \\ \{\Gamma_{\alpha\zeta}, \Gamma_{\beta\zeta}\} = 2\delta_{\alpha\zeta\beta\zeta} \end{cases}$

证明:  $(\Gamma, i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a^b e^{(i\omega + \zeta\epsilon) \cdot \frac{1}{2}\Gamma} (\Gamma, i\zeta)_b e^{-(i\omega - \zeta\epsilon) \cdot \frac{1}{2}\Gamma}, \forall \omega, \forall \epsilon$   
 $\Leftrightarrow (\Gamma, i\zeta)_a = (\delta_a^b + \vartheta_a^b)(1 + (i\omega + \zeta\epsilon) \cdot \frac{1}{2}\Gamma)(\Gamma, i\zeta)_b(1 - (i\omega - \zeta\epsilon) \cdot \frac{1}{2}\Gamma), \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0$   
 $\Leftrightarrow 0 = \vartheta_a^b(\Gamma, i\zeta)_b + \frac{1}{2}i\omega \cdot [\Gamma, (\Gamma, i\zeta)_a] + \frac{1}{2}\epsilon \cdot \{\Gamma, (\Gamma, i\zeta)_a\}, \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0$   
 $\Leftrightarrow 0 = (-\omega \times \Gamma - \zeta\epsilon, -i\epsilon \cdot \Gamma)_a + \frac{1}{2}i\omega \cdot [\Gamma, (\Gamma, i\zeta)_a] + \frac{1}{2}\epsilon \cdot \{\Gamma, (\Gamma, i\zeta)_a\}, \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0$   
 $\Leftrightarrow 0 = (-\omega \times \Gamma - \zeta\epsilon)_{\alpha\zeta} + \frac{1}{2}i\omega \cdot [\Gamma, \Gamma_{\alpha\zeta}] + \frac{1}{2}\epsilon \cdot \{\Gamma, \Gamma_{\alpha\zeta}\}, \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0$   
 $\Leftrightarrow \frac{1}{2}i\omega \cdot [\Gamma, \Gamma_{\alpha\zeta}] = (\omega \times \Gamma)_{\alpha\zeta}, \frac{1}{2}\epsilon \cdot \{\Gamma, \Gamma_{\alpha\zeta}\} = \epsilon_{\alpha\zeta}, \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0$   
 $\Leftrightarrow [\Gamma_{\alpha\zeta}, \Gamma_{\beta\zeta}] = 2i\varepsilon_{\alpha\zeta\beta\zeta} \gamma_\zeta \Gamma_{\gamma\zeta}, \{\Gamma_{\alpha\zeta}, \Gamma_{\beta\zeta}\} = 2\delta_{\alpha\zeta\beta\zeta}$  □

以上定理表明：对易关系 $[\Gamma_{\alpha\zeta}, \Gamma_{\beta\zeta}] = 2i\varepsilon_{\alpha\zeta\beta\zeta} \gamma_\zeta \Gamma_{\gamma\zeta}$  和反对易关系 $\{\Gamma_{\alpha\zeta}, \Gamma_{\beta\zeta}\} = 2\delta_{\alpha\zeta\beta\zeta}$  意味着 $(\Gamma, i\zeta)_a$ 是常数不变张量，反之亦然。

### 3.2.3 常数不变张量 $(\sigma, i\zeta)_a^{A_\zeta A'_\zeta}, (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta}$ [1, 2]

推论3.2.1.  $(\sigma, i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a^b e^{(i\omega + \zeta\epsilon) \cdot \frac{1}{2}\sigma} (\sigma, i\zeta)_b e^{-(i\omega - \zeta\epsilon) \cdot \frac{1}{2}\sigma}$ , 即 $(\sigma, i\zeta)_a^{A_\zeta A'_\zeta}$ 是常数不变张量。

推论3.2.2.  $(\sigma, -i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a^b e^{(i\omega - \zeta\epsilon) \cdot \frac{1}{2}\sigma} (\sigma, -i\zeta)_b e^{-(i\omega + \zeta\epsilon) \cdot \frac{1}{2}\sigma}$ , 即 $(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta}$ 是常数不变张量。

自我评述：这对常数不变张量就是Penrose旋量分析<sup>[1, 2]</sup>的主角，这也是启示我发展一般常数不变张量理论的起因之一，在这里只是按我的方法重新发现而已。

### 3.2.4 基本定理一的推广

在任意N+1维时空中存在如下定理：（终于成功推广了）

定理3.2.2.  $[\Gamma, i\zeta]^a = [e^\vartheta]_a^b e^{\frac{1}{8}\vartheta^{ij}[\Gamma_i, \Gamma_j] + \zeta\epsilon \cdot \frac{1}{2}\Gamma} [\Gamma, i\zeta]^b e^{-\frac{1}{8}\vartheta^{ij}[\Gamma_i, \Gamma_j] + \zeta\epsilon \cdot \frac{1}{2}\Gamma} \Leftrightarrow \{\Gamma_i, \Gamma_j\} = 2\delta_{ij}; S_{ij} := -\frac{i}{4}[\Gamma_i, \Gamma_j]$

证明:  $[\Gamma, i\zeta]^a = [e^\vartheta]_a^b e^{\frac{1}{8}\vartheta^{ij}[\Gamma_i, \Gamma_j] + \zeta\epsilon \cdot \frac{1}{2}\Gamma} [\Gamma, i\zeta]^b e^{-\frac{1}{8}\vartheta^{ij}[\Gamma_i, \Gamma_j] + \zeta\epsilon \cdot \frac{1}{2}\Gamma}$   
 $\Leftrightarrow [\Gamma, i\zeta]^a = [e^\vartheta]_a^b e^{\frac{i}{2}\vartheta^{ij}S_{ij} + \zeta\epsilon \cdot \frac{1}{2}\Gamma} [\Gamma, i\zeta]^b e^{-\frac{i}{2}\vartheta^{ij}S_{ij} + \zeta\epsilon \cdot \frac{1}{2}\Gamma}$   
 $\Leftrightarrow [\Gamma, i\zeta]^a = (\delta_a^b + \vartheta_a^b)$   
 $[1 + \frac{i}{2}\vartheta^{ij}S_{ij} + \zeta\epsilon \cdot \frac{1}{2}\Gamma][\Gamma, i\zeta]^b [1 - \frac{i}{2}\vartheta^{ij}S_{ij} + \zeta\epsilon \cdot \frac{1}{2}\Gamma]$   
 $\Leftrightarrow 0 = \vartheta_a^b [\Gamma, i\zeta]^b$   
 $+ [\frac{i}{2}\vartheta^{ij}S_{ij} + \zeta\epsilon \cdot \frac{1}{2}\Gamma][\Gamma, i\zeta]^a + [\Gamma, i\zeta]^a [-\frac{i}{2}\vartheta^{ij}S_{ij} + \zeta\epsilon \cdot \frac{1}{2}\Gamma]$   
 $\Leftrightarrow 0 = \vartheta_a^b [\Gamma, i\zeta]^b + \frac{i}{2}\vartheta^{ij}[S_{ij}, [\Gamma, i\zeta]^a] + \frac{1}{2}\zeta\epsilon \{[\Gamma, i\zeta]^a, \epsilon \cdot \Gamma\}$   
 $\Leftrightarrow 0 = \vartheta^{ab} [\Gamma, i\zeta]_b + \frac{i}{2}\vartheta^{ij}[S_{ij}, [\Gamma, i\zeta]^a] + \frac{1}{2}\zeta\epsilon \{[\Gamma, i\zeta]^a, \epsilon \cdot \Gamma\}$   
 $\Leftrightarrow \begin{cases} 0 = \vartheta^{kb} [\Gamma, i\zeta]_b + \frac{i}{2}\vartheta^{ij}[S_{ij}, [\Gamma, i\zeta]^k] + \frac{1}{2}\zeta\epsilon \{[\Gamma, i\zeta]^k, \epsilon \cdot \Gamma\} \\ 0 = \vartheta^{\pi b} [\Gamma, i\zeta]_b + \frac{i}{2}\vartheta^{ij}[S_{ij}, [\Gamma, i\zeta]^\pi] + \frac{1}{2}\zeta\epsilon \{i\zeta, \epsilon \cdot \Gamma\} \end{cases}$

$$\begin{aligned}
&\Leftrightarrow \begin{cases} 0 = \vartheta^{kj}\Gamma_j + i\varsigma\vartheta^{k\pi} + \frac{i}{2}\vartheta^{ij}[S_{ij}, \Gamma^k] + \frac{1}{2}\varsigma\epsilon_l\{\Gamma^k, \Gamma^l\} \\ 0 = \vartheta^{\pi j}\Gamma_j + i\epsilon \cdot \Gamma \end{cases} \\
&\Leftrightarrow \begin{cases} 0 = \vartheta^{kj}\Gamma_j + \frac{i}{2}\vartheta^{ij}[S_{ij}, \Gamma^k] + \frac{1}{2}\varsigma\epsilon_l\{\Gamma^k, \Gamma^l\} - \varsigma\epsilon^k \\ 0 = 0 \end{cases} \\
&\Leftrightarrow 0 = \vartheta^{ij}\delta^k{}_i\Gamma_j + \frac{i}{2}\vartheta^{ij}[S_{ij}, \Gamma^k] + \frac{1}{2}\varsigma\epsilon_l\{\Gamma^k, \Gamma^l\} - \varsigma\epsilon^k \\
&\Leftrightarrow \begin{cases} 0 = \frac{1}{2}\varsigma\epsilon_l\{\Gamma^k, \Gamma^l\} - \varsigma\epsilon^k \\ 0 = \vartheta^{ij}\delta^k{}_i\Gamma_j + \frac{i}{2}\vartheta^{ij}[S_{ij}, \Gamma^k] \end{cases} \\
&\Leftrightarrow \begin{cases} 0 = \frac{1}{2}\varsigma\epsilon_l\{\Gamma^k, \Gamma^l\} - \varsigma\epsilon^k \\ 0 = \vartheta^{ij}\{\frac{1}{2}\delta^k{}_i\Gamma_j + \frac{i}{2}[S_{ij}, \Gamma^k]\} \end{cases} \\
&\Leftrightarrow \begin{cases} \{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \\ i[\Gamma_k, S_{ij}] = \delta^k{}_i\Gamma_j \end{cases} \\
&\Leftrightarrow \{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \quad \square
\end{aligned}$$

以上定理表明：反对易关系 $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$ 意味着 $(\Gamma, i\varsigma)_a$ 是常数不变张量，反之亦然，基本定理一只是其特殊情形而已。

### 3.2.5 常数不变张量 $[\Gamma(N), i\varsigma]_{A_\varsigma A'_\varsigma}^a, [\Gamma(N), -i\varsigma]_{A_\varsigma A'_\varsigma}^a$

推论3.2.3.

$$\{\Gamma_i(N), \Gamma_j(N)\} = 2\delta_{ij} \Rightarrow [\Gamma(N), i\varsigma]^a = [e^\vartheta]^a{}_b e^{\frac{1}{8}\vartheta^{ij}[\Gamma_i(N), \Gamma_j(N)] + \varsigma\epsilon \cdot \frac{1}{2}\Gamma(N)} [\Gamma(N), i\varsigma]^b e^{-\frac{1}{8}\vartheta^{ij}[\Gamma_i(N), \Gamma_j(N)] + \varsigma\epsilon \cdot \frac{1}{2}\Gamma(N)}$$

自我评述：所以 $[\Gamma(N), i\varsigma]_{A_\varsigma A'_\varsigma}^a$ 和 $[\Gamma(N), -i\varsigma]_{A_\varsigma A'_\varsigma}^a$ 是常数不变张量，是Penrose旋量在高低维时空的推广。

## 3.3 基本定理二及其相关常数不变张量

### 3.3.1 基本定理二

$$\text{定理3.3.1. } \Gamma_{\alpha_\varsigma} = [e^{(i\omega + \varsigma\epsilon) \cdot \gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega + \varsigma\epsilon) \cdot \Gamma} \Gamma_{\beta_\varsigma} e^{-(i\omega + \varsigma\epsilon) \cdot \Gamma} \Leftrightarrow [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} \Gamma_{\gamma_\varsigma}$$

$$\begin{aligned}
&\text{证明: } \Gamma_{\alpha_\varsigma} = [e^{(i\omega + \varsigma\epsilon) \cdot \gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega + \varsigma\epsilon) \cdot \Gamma} \Gamma_{\beta_\varsigma} e^{-(i\omega + \varsigma\epsilon) \cdot \Gamma}, \forall \omega, \forall \epsilon \\
&\Leftrightarrow \Gamma_{\alpha_\varsigma} = [\delta_{\alpha_\varsigma}{}^{\beta_\varsigma} + (i\omega + \varsigma\epsilon) \cdot \gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}][1 + (i\omega + \varsigma\epsilon) \cdot \Gamma] \Gamma_{\beta_\varsigma} [1 - (i\omega + \varsigma\epsilon) \cdot \Gamma], \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0 \\
&\Leftrightarrow 0 = (i\omega + \varsigma\epsilon) \cdot \{\gamma_{\alpha_\varsigma}{}^{\beta_\varsigma} \Gamma_{\beta_\varsigma} + [\Gamma, \Gamma_{\alpha_\varsigma}]\}, \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0 \\
&\Leftrightarrow \gamma_{\alpha_\varsigma}{}^{\beta_\varsigma} \Gamma_{\beta_\varsigma} + [\Gamma, \Gamma_{\alpha_\varsigma}] = 0 \\
&\Leftrightarrow \gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} \Gamma_{\gamma_\varsigma} + [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = 0 (\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} \equiv i\gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}) \\
&\Leftrightarrow [\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} \Gamma_{\gamma_\varsigma} \quad \square
\end{aligned}$$

以上定理表明：对易关系 $[\Gamma_{\alpha_\varsigma}, \Gamma_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} \Gamma_{\gamma_\varsigma}$ 意味着 $\Gamma_{\alpha_\varsigma}$ 是常数不变张量，反之亦然。

### 3.3.2 广义基本定理二

$$\text{定理3.3.2. } T_{\alpha_\varsigma} = [e^{(i\omega + \varsigma\epsilon) \cdot \gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega + \varsigma\epsilon) \cdot \Gamma} T_{\beta_\varsigma} e^{-(i\omega + \varsigma\epsilon) \cdot \Gamma} \Leftrightarrow [\Gamma_{\alpha_\varsigma}, T_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} T_{\gamma_\varsigma}$$

$$\begin{aligned}
&\text{证明: } T_{\alpha_\varsigma} = [e^{(i\omega + \varsigma\epsilon) \cdot \gamma}]_{\alpha_\varsigma}{}^{\beta_\varsigma} e^{(i\omega + \varsigma\epsilon) \cdot \Gamma} T_{\beta_\varsigma} e^{-(i\omega + \varsigma\epsilon) \cdot \Gamma}, \forall \omega, \forall \epsilon \\
&\Leftrightarrow T_{\alpha_\varsigma} = [\delta_{\alpha_\varsigma}{}^{\beta_\varsigma} + (i\omega + \varsigma\epsilon) \cdot \gamma_{\alpha_\varsigma}{}^{\beta_\varsigma}][1 + (i\omega + \varsigma\epsilon) \cdot \Gamma] T_{\beta_\varsigma} [1 - (i\omega + \varsigma\epsilon) \cdot \Gamma], \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0 \\
&\Leftrightarrow 0 = (i\omega + \varsigma\epsilon) \cdot \{\gamma_{\alpha_\varsigma}{}^{\beta_\varsigma} T_{\beta_\varsigma} + [\Gamma, T_{\alpha_\varsigma}]\}, \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0 \\
&\Leftrightarrow \gamma_{\alpha_\varsigma}{}^{\beta_\varsigma} T_{\beta_\varsigma} + [\Gamma, T_{\alpha_\varsigma}] = 0 \\
&\Leftrightarrow \gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} T_{\gamma_\varsigma} + [\Gamma_{\alpha_\varsigma}, T_{\beta_\varsigma}] = 0 (\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} \equiv i\gamma_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma}) \\
&\Leftrightarrow [\Gamma_{\alpha_\varsigma}, T_{\beta_\varsigma}] = i\varepsilon_{\alpha_\varsigma\beta_\varsigma}{}^{\gamma_\varsigma} T_{\gamma_\varsigma} \quad \square
\end{aligned}$$

### 3.3.3 推广：常数不变张量算符 [42]

定理3.3.3.

$$\begin{cases} \hat{T}(j, m) = \sum_{m'=j}^{-j} \langle j, m | U^+(R) | j, m' \rangle U(R) \hat{T}(j, m') U^+(R) \\ U(R) \hat{T}(j, m) U^+(R) = \sum_{m'=j}^{-j} \hat{T}(j, m') \langle j, m' | U(R) | j, m \rangle \\ U(R) | j, m \rangle = \sum_{m'=j}^{-j} | j, m' \rangle \langle j, m' | U(R) | j, m \rangle \end{cases}$$

### 3.3.4 常数不变张量 $\sigma_{\alpha_\zeta k_\zeta}^{\alpha_\zeta l_\zeta}(s), \sigma_{\alpha'_\zeta k'_\zeta l'_\zeta}(s), \sigma_{\alpha_\zeta A_\zeta B_\zeta}, \sigma_{\alpha'_\zeta A'_\zeta B'_\zeta}$

推论3.3.1.  $\sigma_{\alpha_\zeta}(s) = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega+\zeta\epsilon)\cdot\sigma(s)} \sigma_{\beta_\zeta}(s) e^{-(i\omega+\zeta\epsilon)\cdot\sigma(s)}$ , 即  $\sigma_{\alpha_\zeta k_\zeta}^{\alpha_\zeta l_\zeta}(s)$  是常数不变张量。

推论3.3.2.  $\sigma_{\alpha_\zeta} = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma} \sigma_{\beta_\zeta} e^{-(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma}$ , 即  $\sigma_{\alpha_\zeta A_\zeta B_\zeta}$  是常数不变张量。

推论3.3.3.  $\sigma_{\alpha'_\zeta}(s) = [e^{(i\omega-\zeta\epsilon)\cdot\gamma}]_{\alpha'_\zeta \beta'_\zeta} e^{(i\omega-\zeta\epsilon)\cdot\sigma(s)} \sigma_{\beta'_\zeta}(s) e^{-(i\omega-\zeta\epsilon)\cdot\sigma(s)}$ , 即  $\sigma_{\alpha'_\zeta k'_\zeta l'_\zeta}(s)$  是常数不变张量。

推论3.3.4.  $\sigma_{\alpha'_\zeta} = [e^{(i\omega-\zeta\epsilon)\cdot\gamma}]_{\alpha'_\zeta \beta'_\zeta} e^{(i\omega-\zeta\epsilon)\cdot\frac{1}{2}\sigma} \sigma_{\beta'_\zeta} e^{-(i\omega-\zeta\epsilon)\cdot\frac{1}{2}\sigma}$ , 即  $\sigma_{\alpha'_\zeta A'_\zeta B'_\zeta}$  是常数不变张量。

自我评述：这个定理表明自旋矩阵是常数不变张量，结合上一节的定理，发现泡利矩阵可以组合成两类常数不变张量，这很有意思，但其它自旋矩阵没这个性质，所以泡利矩阵很特别。

推论3.3.5.  $\sigma_{\alpha'_\zeta}(s) = [e^{i\omega\cdot\gamma}]_{\alpha'_\zeta \beta'_\zeta} e^{i\omega\cdot\sigma(s)} \sigma_{\beta'_\zeta}(s) e^{-i\omega\cdot\sigma(s)} [\Rightarrow] \sigma(s) \cdot \hat{p} = e^{i\omega\cdot\sigma(s)} \sigma_z(s) e^{-i\omega\cdot\sigma(s)}$

### 3.3.5 反对称常数不变张量 $\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta}, \varepsilon_{\alpha'_\zeta \beta'_\zeta \gamma'_\zeta}$

推论3.3.6.  $\gamma_{\alpha_\zeta}(s) = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega+\zeta\epsilon)\cdot\gamma} \gamma_{\beta_\zeta}(s) e^{-(i\omega+\zeta\epsilon)\cdot\gamma}$ , 即  $\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} (\equiv i\gamma_{\alpha_\zeta \beta_\zeta \gamma_\zeta})$  是常数不变张量。

推论3.3.7.  $\gamma_{\alpha'_\zeta}(s) = [e^{(i\omega-\zeta\epsilon)\cdot\gamma}]_{\alpha'_\zeta \beta'_\zeta} e^{(i\omega-\zeta\epsilon)\cdot\gamma} \gamma_{\beta'_\zeta}(s) e^{-(i\omega-\zeta\epsilon)\cdot\gamma}$ , 即  $\varepsilon_{\alpha'_\zeta \beta'_\zeta \gamma'_\zeta} (\equiv i\gamma_{\alpha'_\zeta \beta'_\zeta \gamma'_\zeta})$  是常数不变张量。

自我评述：以上表明三维反对称张量是四维洛伦兹常数不变张量。

### 3.3.6 过渡

推论3.3.8.  $\sigma_{+\alpha_\zeta} = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_+} \sigma_{+\beta_\zeta} e^{-(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_+}$

推论3.3.9.  $\sigma_{-\alpha_\zeta} = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_-} \sigma_{-\beta_\zeta} e^{-(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_-}$

### 3.3.7 常数不变张量 $\sigma_{+\alpha_\zeta}^{ab}, \sigma_{-\alpha_\zeta}^{ab}, \sigma_{\zeta\alpha_\zeta}^{ab}, \sigma_{-\zeta\alpha'_\zeta}^{ab}$

推论3.3.10.  $\sigma_{+\alpha_\zeta} = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega\cdot R + \zeta\epsilon\cdot L)} \sigma_{\beta_\zeta} e^{-(i\omega\cdot R + \zeta\epsilon\cdot L)}$ , 即  $\sigma_{\zeta\alpha_\zeta}^{a_\zeta b_\zeta}$  是常数不变张量。

证明:  $\sigma_{+\alpha_\zeta} = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_+} \sigma_{+\beta_\zeta} e^{-(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_+}$

$\Leftrightarrow \sigma_{+\alpha_\zeta} = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_+} [e^{(i\omega-\zeta\epsilon)\cdot\frac{1}{2}\sigma_-} \sigma_{+\beta_\zeta} e^{-(i\omega-\zeta\epsilon)\cdot\frac{1}{2}\sigma_-}] e^{-(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_+}$

$\Leftrightarrow \sigma_{\zeta\alpha_\zeta} = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega\cdot R + \zeta\epsilon\cdot L)} \sigma_{\beta_\zeta} e^{-(i\omega\cdot R + \zeta\epsilon\cdot L)}$ , 即  $\sigma_{\zeta\alpha_\zeta}^{a_\zeta b_\zeta}$  是常数不变张量。 □

推论3.3.11.  $\sigma_{-\zeta\alpha'_\zeta} = [e^{(i\omega-\zeta\epsilon)\cdot\gamma}]_{\alpha'_\zeta \beta'_\zeta} e^{(i\omega\cdot R + \zeta\epsilon\cdot L)} \sigma_{-\beta'_\zeta} e^{-(i\omega\cdot R + \zeta\epsilon\cdot L)}$ , 即  $\sigma_{-\zeta\alpha'_\zeta}^{a_\zeta b_\zeta}$  是常数不变张量。

证明:  $\sigma_{-\alpha_\zeta} = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_-} \sigma_{-\beta_\zeta} e^{-(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_-}$

$\Leftrightarrow \sigma_{-\alpha_\zeta} = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_-} [e^{(i\omega-\zeta\epsilon)\cdot\frac{1}{2}\sigma_+} \sigma_{-\beta_\zeta} e^{-(i\omega-\zeta\epsilon)\cdot\frac{1}{2}\sigma_+}] e^{-(i\omega+\zeta\epsilon)\cdot\frac{1}{2}\sigma_-}$

$\Leftrightarrow \sigma_{-\alpha_\zeta} = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]_{\alpha_\zeta \beta_\zeta} e^{(i\omega\cdot R - \zeta\epsilon\cdot L)} \sigma_{\beta_\zeta} e^{-(i\omega\cdot R - \zeta\epsilon\cdot L)}$ , 即  $\sigma_{-\alpha_\zeta}^{a'_\zeta b'_\zeta}$  是常数不变张量。 □

综合  $\sigma_{+\alpha_\zeta}^{a_\zeta b_\zeta}, \sigma_{-\alpha_\zeta}^{a'_\zeta b'_\zeta}$  两者可知:  $\sigma_{+\alpha_\zeta}^{ab}, \sigma_{-\alpha_\zeta}^{ab}$  是常数不变张量, 进一步可知:  $\sigma_{\zeta\alpha_\zeta}^{ab}, \sigma_{-\zeta\alpha'_\zeta}^{ab}$  是常数不变张量。

自我评述：以上严格证明了SO(4)两个生成元矩阵是常数不变张量。

### 3.3.8 基本定理二推广形式

引理3.3.1.  $[\Gamma_i, \Gamma_j] = -2\gamma^k{}_{ij}\Gamma_k \Rightarrow [\gamma_i, \gamma_j] = -\gamma^k{}_{ij}\gamma_k$

定理3.3.4.  $\Gamma_{\alpha_c} = [e^{\frac{i}{2}\omega^{ij}\vec{S}_{ij} + \varsigma\epsilon^k\gamma_k}]_{\alpha_c}{}^{\beta_c} e^{\frac{1}{8}\omega^{ij}[\Gamma_i, \Gamma_j] + \frac{1}{2}\varsigma\epsilon^k\Gamma_k} \Gamma_{\beta_c} e^{-\frac{1}{8}\omega^{ij}[\Gamma_i, \Gamma_j] + \frac{1}{2}\varsigma\epsilon^k\Gamma_k}$

$\Leftrightarrow [\Gamma_i, \Gamma_j] = -2\gamma^k{}_{ij}\Gamma_k, \frac{1}{4}[[\Gamma_i, \Gamma_j], \Gamma_k] = -i[\vec{S}_{ij}]_k{}^l\Gamma_l$

证明:  $\Gamma_{\alpha_c} = [e^{\frac{i}{2}\omega^{ij}\vec{S}_{ij} + \varsigma\epsilon^k\gamma_k}]_{\alpha_c}{}^{\beta_c} e^{\frac{1}{8}\omega^{ij}[\Gamma_i, \Gamma_j] + \frac{1}{2}\varsigma\epsilon^k\Gamma_k} \Gamma_{\beta_c} e^{-\frac{1}{8}\omega^{ij}[\Gamma_i, \Gamma_j] + \frac{1}{2}\varsigma\epsilon^k\Gamma_k}, \forall\omega, \forall\epsilon$

$\Leftrightarrow \Gamma_{\alpha_c} = [\delta_{\alpha_c}{}^{\beta_c} + (\frac{i}{2}\omega^{ij}\vec{S}_{ij} + \varsigma\epsilon^k\gamma_k)_{\alpha_c}{}^{\beta_c}][1 + \frac{1}{8}\omega^{ij}[\Gamma_i, \Gamma_j] + \frac{1}{2}\varsigma\epsilon^k\Gamma_k] \Gamma_{\beta_c} [1 - \frac{1}{8}\omega^{ij}[\Gamma_i, \Gamma_j] - \frac{1}{2}\varsigma\epsilon^k\Gamma_k], \forall\omega \rightarrow 0, \forall\epsilon \rightarrow 0$

$\Leftrightarrow 0 = (\frac{i}{2}\omega^{ij}\vec{S}_{ij} + \varsigma\epsilon^k\gamma_k)_{\alpha_c}{}^{\beta_c} \Gamma_{\beta_c} + [\frac{1}{8}\omega^{ij}[\Gamma_i, \Gamma_j] + \frac{1}{2}\varsigma\epsilon^k\Gamma_k, \Gamma_{\alpha_c}], \forall\omega \rightarrow 0, \forall\epsilon \rightarrow 0$

$\Leftrightarrow 0 = \epsilon^k\gamma_k\alpha_c{}^{\beta_c}\Gamma_{\beta_c} + [\epsilon^k\Gamma_k, \Gamma_{\alpha_c}], 0 = \frac{i}{2}\omega^{ij}[\vec{S}_{ij}]_{\alpha_c}{}^{\beta_c}\Gamma_{\beta_c} + [\frac{1}{8}\omega^{ij}[\Gamma_i, \Gamma_j], \Gamma_{\alpha_c}]$

$\Leftrightarrow [\Gamma_k, \Gamma_{\alpha_c}] = -2\gamma_k\alpha_c{}^{\beta_c}\Gamma_{\beta_c}, \frac{1}{4}[[\Gamma_i, \Gamma_j], \Gamma_{\alpha_c}] = -i[\vec{S}_{ij}]_{\alpha_c}{}^{\beta_c}\Gamma_{\beta_c}$

$\Leftrightarrow [\Gamma_i, \Gamma_j] = -2\gamma^k{}_{ij}\Gamma_k, \frac{1}{4}[[\Gamma_i, \Gamma_j], \Gamma_k] = -i[\vec{S}_{ij}]_k{}^l\Gamma_l$

$\Leftrightarrow [\Gamma_i, \Gamma_j] = -2\gamma^k{}_{ij}\Gamma_k, \frac{1}{4}[[\Gamma_i, \Gamma_j], \Gamma_k] = \gamma^m{}_{ij}\gamma^l{}_{mk}\Gamma_l = \gamma_{ij}{}^m\gamma_{mk}{}^l\Gamma_l = -i[\vec{S}_{ij}]_k{}^l\Gamma_l$   $\square$

以上 $\gamma^k{}_{ij}$ 是四维类似物,  $\vec{S}_{ij}$ 一般与 $[\Gamma_i, \Gamma_j]$ 不直接相关。似乎无法推广, 只有四维情形才满足。

推论3.3.12.  $\Gamma_{\alpha_c} = [e^{\frac{i}{2}\omega^{ij}\vec{S}_{ij}}]_{\alpha_c}{}^{\beta_c} e^{\frac{1}{8}\omega^{ij}[\Gamma_i, \Gamma_j]} \Gamma_{\beta_c} e^{-\frac{1}{8}\omega^{ij}[\Gamma_i, \Gamma_j]} \Leftrightarrow \frac{1}{4}[[\Gamma_i, \Gamma_j], \Gamma_k] = -i[\vec{S}_{ij}]_k{}^l\Gamma_l$

## 3.4 基本定理三及其相关常数不变张量

### 3.4.1 基本定理三

在任意N+1维时空中存在如下定理:

定理3.4.1.  $\Gamma_{ab} = [e^\vartheta]_a{}^c [e^\vartheta]_b{}^d e^{\frac{i}{2}\vartheta^{ef}\Gamma_{ef}} \Gamma_{cd} e^{-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}}, \vartheta_{ab} = -\vartheta_{ba}$

$\Leftrightarrow i[\Gamma_{ab}, \Gamma_{cd}] = \delta_{ad}\Gamma_{bc} - \delta_{ac}\Gamma_{bd} + \delta_{bc}\Gamma_{ad} - \delta_{bd}\Gamma_{ac}$

证明:  $\Gamma_{ab} = [e^\vartheta]_a{}^c [e^\vartheta]_b{}^d e^{\frac{i}{2}\vartheta^{ef}\Gamma_{ef}} \Gamma_{cd} e^{-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}}, \forall\vartheta^{ef}$

$\Leftrightarrow \Gamma_{ab} = [\delta_a{}^c + \vartheta_a{}^c][\delta_b{}^d + \vartheta_b{}^d](1 + \frac{i}{2}\vartheta^{ef}\Gamma_{ef})\Gamma_{cd}(1 - \frac{i}{2}\vartheta^{ef}\Gamma_{ef}), \forall\vartheta^{ef} \rightarrow 0$

$\Leftrightarrow 0 = \vartheta_a{}^c\Gamma_{cb} - \vartheta_b{}^d\Gamma_{da} - \frac{i}{2}\vartheta^{cd}[\Gamma_{ab}, \Gamma_{cd}], \forall\vartheta^{cd} \rightarrow 0$

$\Leftrightarrow i\vartheta^{cd}[\Gamma_{ab}, \Gamma_{cd}] = 2(\vartheta_a{}^c\Gamma_{cb} - \vartheta_b{}^d\Gamma_{da}), \forall\vartheta^{cd} \rightarrow 0$

$\Leftrightarrow i\vartheta^{cd}[\Gamma_{ab}, \Gamma_{cd}] = \vartheta^{cd}(\delta_{ad}\Gamma_{bc} - \delta_{ac}\Gamma_{bd} + \delta_{bc}\Gamma_{ad} - \delta_{bd}\Gamma_{ac}), \forall\vartheta^{cd} \rightarrow 0$

$\Leftrightarrow i[\Gamma_{ab}, \Gamma_{cd}] = \delta_{ad}\Gamma_{bc} - \delta_{ac}\Gamma_{bd} + \delta_{bc}\Gamma_{ad} - \delta_{bd}\Gamma_{ac}$   $\square$

以上定理表明: 对易关系 $i[\Gamma_{ab}, \Gamma_{cd}] = \delta_{ad}\Gamma_{bc} - \delta_{ac}\Gamma_{bd} + \delta_{bc}\Gamma_{ad} - \delta_{bd}\Gamma_{ac}$  意味着 $\Gamma_{ab}$ 是常数不变张量, 反之亦然。

### 3.4.2 广义基本定理三

在任意N+1维时空中存在如下定理:

定理3.4.2.  $T_{ab} = [e^\vartheta]_a{}^c [e^\vartheta]_b{}^d e^{\frac{i}{2}\vartheta^{ef}\Gamma_{ef}} T_{cd} e^{-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}}, \vartheta_{ab} = -\vartheta_{ba}$

$\Leftrightarrow i[T_{ab}, \Gamma_{cd}] = \delta_{ac}T_{db} - \delta_{ad}T_{cb} + \delta_{bc}T_{ad} - \delta_{bd}T_{ac}$

证明:  $T_{ab} = [e^\vartheta]_a{}^c [e^\vartheta]_b{}^d e^{\frac{i}{2}\vartheta^{ef}\Gamma_{ef}} T_{cd} e^{-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}}, \forall\vartheta^{ef}$

$\Leftrightarrow T_{ab} = [\delta_a{}^c + \vartheta_a{}^c][\delta_b{}^d + \vartheta_b{}^d](1 + \frac{i}{2}\vartheta^{ef}\Gamma_{ef})T_{cd}(1 - \frac{i}{2}\vartheta^{ef}\Gamma_{ef}), \forall\vartheta^{ef} \rightarrow 0$

$\Leftrightarrow 0 = \vartheta_a{}^c T_{cb} + \vartheta_b{}^d T_{ad} - \frac{i}{2}\vartheta^{cd}[T_{ab}, \Gamma_{cd}], \forall\vartheta^{cd} \rightarrow 0$

$\Leftrightarrow i\vartheta^{cd}[T_{ab}, \Gamma_{cd}] = 2(\vartheta_a{}^c T_{cb} + \vartheta_b{}^d T_{ad}), \forall\vartheta^{cd} \rightarrow 0$

$\Leftrightarrow i\vartheta^{cd}[T_{ab}, \Gamma_{cd}] = 2\vartheta^{cd}(-\delta_{ad}T_{cb} + \delta_{bc}T_{ad}), \forall\vartheta^{cd} \rightarrow 0$

$\Leftrightarrow i[T_{ab}, \Gamma_{cd}] = \delta_{ac}T_{db} - \delta_{ad}T_{cb} + \delta_{bc}T_{ad} - \delta_{bd}T_{ac}$   $\square$

### 3.4.3 自旋常数不变张量 ${}^{[8]}S_{abA}{}^B$

推论3.4.1.  $S_{ab} = [e^\vartheta]_a{}^c [e^\vartheta]_b{}^d e^{\frac{i}{2}\vartheta^{ef}S_{ef}} S_{cd} e^{-\frac{i}{2}\vartheta^{ef}S_{ef}}$ , 即 $S_{abA}{}^B$ 是常数不变张量。

自我评述: 以上表明在任意时空中物理的自旋张量都是常数不变张量。



### 3.4.4 自旋常数不变张量 $S_{abk_\zeta}{}^{l_\zeta}(s, \varsigma), S_{ab}{}^{k'_\zeta}{}_{l'_\zeta}(s, -\varsigma)$

推论3.4.2.  $S_{ab}(s, \varsigma) = [e^\vartheta]_a{}^c [e^\vartheta]_b{}^d e^{\frac{i}{2}\vartheta^{ef} S_{ef}(s, \varsigma)} S_{cd}(s, \varsigma) e^{-\frac{i}{2}\vartheta^{ef} S_{ef}(s, \varsigma)}$ , 即  $S_{abk_\zeta}{}^{l_\zeta}(s, \varsigma)$  是常数不变张量。

推论3.4.3.  $S_{ab}(s, -\varsigma) = [e^\vartheta]_a{}^c [e^\vartheta]_b{}^d e^{\frac{i}{2}\vartheta^{ef} S_{ef}(s, -\varsigma)} S_{cd}(s, -\varsigma) e^{-\frac{i}{2}\vartheta^{ef} S_{ef}(s, -\varsigma)}$ , 即  $S_{ab}{}^{k'_\zeta}{}_{l'_\zeta}(s, -\varsigma)$  是常数不变张量。

### 3.4.5 自旋常数不变张量 $S_{abA_\zeta}{}^{B_\zeta}(\frac{1}{2}, \varsigma), S_{ab}{}^{A'_\zeta}{}_{B'_\zeta}(\frac{1}{2}, -\varsigma)$

上节取  $s = \frac{1}{2}$  可得:  $S_{abA_\zeta}{}^{B_\zeta}(\frac{1}{2}, \varsigma), S_{ab}{}^{A'_\zeta}{}_{B'_\zeta}(\frac{1}{2}, -\varsigma)$  是常数不变张量。

## 3.5 基本定理四及其相关常数不变张量

### 3.5.1 基本定理四

引理3.5.1.  $[\Gamma_a, [\Gamma_c, \Gamma_d]] = \frac{1}{2}(\{[\Gamma_a, \Gamma_c], \Gamma_d\} - \{\Gamma_c, [\Gamma_d, \Gamma_a]\})$

在任意N+1维时空中存在如下定理:

定理3.5.1.  $\Gamma_a = [e^\vartheta]_a{}^b e^{\frac{i}{2}\vartheta^{cd} \Gamma_{cd}} \Gamma_b e^{-\frac{i}{2}\vartheta^{cd} \Gamma_{cd}} \Leftrightarrow i[\Gamma_a, \Gamma_{cd}] = \delta_{a[c} \Gamma_{d]}$

证明:  $\Gamma_a = [e^\vartheta]_a{}^b e^{\frac{i}{2}\vartheta^{cd} \Gamma_{cd}} \Gamma_b e^{-\frac{i}{2}\vartheta^{cd} \Gamma_{cd}}, \forall \vartheta^{cd}$

$\Leftrightarrow \Gamma_a = [\delta_a{}^b + \vartheta_a{}^b](1 + \frac{i}{2}\vartheta^{cd} \Gamma_{cd}) \Gamma_b (1 - \frac{i}{2}\vartheta^{cd} \Gamma_{cd}), \forall \vartheta^{cd} \rightarrow 0$

$\Leftrightarrow 0 = \vartheta_a{}^b \Gamma_b - \frac{i}{2}\vartheta^{cd} [\Gamma_a, \Gamma_{cd}], \forall \vartheta^{cd} \rightarrow 0$

$\Leftrightarrow i\vartheta^{cd} [\Gamma_a, \Gamma_{cd}] = 2\vartheta_a{}^b \Gamma_b, \forall \vartheta^{cd} \rightarrow 0$

$\Leftrightarrow i\vartheta^{cd} [\Gamma_a, \Gamma_{cd}] = \vartheta^{cd} (\delta_{ac} \Gamma_d - \delta_{ad} \Gamma_c), \forall \vartheta^{cd} \rightarrow 0$

$\Leftrightarrow i[\Gamma_a, \Gamma_{cd}] = \delta_{a[c} \Gamma_{d]}$  □

在任意N+1维时空中存在如下定理:

定理3.5.2.  $\Gamma_a = [e^\vartheta]_a{}^b e^{\frac{i}{2}\vartheta^{cd} S_{cd}} \Gamma_b e^{-\frac{i}{2}\vartheta^{cd} S_{cd}}, S_{cd} = -\frac{i}{4}[\Gamma_c, \Gamma_d] \Leftrightarrow \frac{1}{4}[[\Gamma_c, \Gamma_d], \Gamma_a] = \Gamma_{[c} \delta_{d]a}$

证明:  $\Gamma_a = [e^\vartheta]_a{}^b e^{\frac{i}{2}\vartheta^{cd} S_{cd}} \Gamma_b e^{-\frac{i}{2}\vartheta^{cd} S_{cd}}$

$\Leftrightarrow \Gamma_a = (1 + \vartheta)_a{}^b (1 + \frac{i}{2}\vartheta^{cd} S_{cd}) \Gamma_b (1 - \frac{i}{2}\vartheta^{cd} S_{cd})$

$\Leftrightarrow 0 = \vartheta_a{}^b \Gamma_b + \frac{i}{2}\vartheta^{cd} [S_{cd}, \Gamma_a]$

$\Leftrightarrow 0 = -\frac{1}{2}\vartheta^{cd} \Gamma_{[c} \delta_{d]a} + \frac{i}{2}\vartheta^{cd} [S_{cd}, \Gamma_a]$

$\Leftrightarrow i[\Gamma_a, S_{cd}] = \delta_{a[c} \Gamma_{d]}$

$\Leftrightarrow \frac{1}{4}[\Gamma_a, [\Gamma_c, \Gamma_d]] = \delta_{a[c} \Gamma_{d]}$  □

推论3.5.1.  $\Gamma_k = [e^\vartheta]_k{}^l e^{\frac{i}{2}\vartheta^{ij} S_{ij}} \Gamma_l e^{-\frac{i}{2}\vartheta^{ij} S_{ij}}, S_{ij} = -\frac{i}{4}[\Gamma_i, \Gamma_j] \Leftrightarrow \frac{1}{4}[[\Gamma_i, \Gamma_j], \Gamma_k] = \Gamma_{[i} \delta_{j]k}$

### 3.5.2 $n = N + 1$ 维时空中的常数不变张量 $\Gamma^a{}_{\lambda_\zeta}{}^{\mu_\zeta}(n)$

引理3.5.2.  $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab} \Rightarrow \frac{1}{4}[\Gamma_a, [\Gamma_c, \Gamma_d]] = \delta_{a[c} \Gamma_{d]}$

在任意N+1维时空中存在如下定理:

定理3.5.3.  $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab} \Rightarrow \Gamma_a = [e^\vartheta]_a{}^b e^{\frac{i}{2}\vartheta^{cd} S_{cd}} \Gamma_b e^{-\frac{i}{2}\vartheta^{cd} S_{cd}}, S_{cd} = -\frac{i}{4}[\Gamma_c, \Gamma_d]$

定理3.5.4.  $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab} \Rightarrow \begin{cases} \Gamma_{N+1} = e^{\frac{i}{2}\vartheta^{*ab} S_{ab}} \Gamma_{N+1} e^{-\frac{i}{2}\vartheta^{*ab} S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] \\ \Gamma_{N+1} = e^{\frac{i}{2}\vartheta^{*ab} S_{ab}} \Gamma_{N+1} e^{-\frac{i}{2}\vartheta^{*ab} S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] \end{cases}$

定理3.5.5.  $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab} \Rightarrow \begin{cases} \Gamma_{N+2} = e^{\frac{i}{2}\vartheta^{*ab} S_{ab}} \Gamma_{N+2} e^{-\frac{i}{2}\vartheta^{*ab} S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b], \Gamma_{N+2} = \Gamma_1 \cdots \Gamma_{N+1} \\ \Gamma_{N+2} = e^{\frac{i}{2}\vartheta^{*ab} S_{ab}} \Gamma_{N+2} e^{-\frac{i}{2}\vartheta^{*ab} S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b], \Gamma_{N+2} = \Gamma_1 \cdots \Gamma_{N+1} \end{cases}$

自我评述: 在任意N+1维时空中Dirac矩阵  $\Gamma_a$  和  $\Gamma_{N+1}, \Gamma_{N+2}$  是常数不变张量。

### 3.5.3 常数不变张量 $^{[4]}\gamma_{a\lambda_\zeta}{}^{\mu_\zeta}(\zeta), \gamma_{5\lambda_\zeta}{}^{\mu_\zeta}(\zeta), \delta_{\lambda_\zeta}{}^{\mu_\zeta}, \gamma_4^{\lambda'_\zeta\lambda_\zeta}$

定义3.5.1.  $\gamma_5(\zeta) \equiv \gamma_x(\zeta)\gamma_y(\zeta)\gamma_z(\zeta)\gamma_\pi(\zeta), S_{ab}(e, \zeta) \equiv -\frac{i}{4}[\gamma_a(\zeta), \gamma_b(\zeta)]$

定义3.5.2.  $\lambda_\zeta \sim e^{\frac{i}{2}\vartheta^{ab}S_{ab}(e, \zeta)}, \mu_\zeta \sim e^{-\frac{i}{2}\vartheta^{ab}S_{ab}^T(e, \zeta)}$

定义3.5.3. 一个特殊表象： $\langle \gamma_a(\zeta), \gamma_5(\zeta) \rangle = [(\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x), \zeta I \otimes \sigma_z]$

推论3.5.2.  $\gamma_a(\zeta) = [e^\vartheta]_a{}^b e^{\frac{i}{2}\vartheta^{cd}S_{cd}(e, \zeta)}\gamma_b(\zeta)e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(e, \zeta)}$ , 即 $\gamma_{a\lambda_\zeta}{}^{\mu_\zeta}(\zeta)$ 是常数不变张量。

推论3.5.3.  $\gamma_5(\zeta) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(e, \zeta)}\gamma_5(\zeta)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(e, \zeta)}$ , 即 $\gamma_{5\lambda_\zeta}{}^{\mu_\zeta}(\zeta)$ 是常数不变张量。

推论3.5.4.  $I_4 = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(e, \zeta)}I_4e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(e, \zeta)}$ , 即 $\delta_{\lambda_\zeta}{}^{\mu_\zeta}$ 是常数不变张量。

推论3.5.5.  $\gamma_4 = e^{\frac{i}{2}\vartheta^{*ab}S_{ab}(e, \zeta)}\gamma_4e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(e, \zeta)}$ , 即 $\gamma_4^{\lambda'_\zeta\lambda_\zeta}$ 是常数不变张量。

自我评述：以上表明四维时空中Dirac矩阵和 $\gamma_4, \gamma_5$ 矩阵都是常数不变张量。

## 3.6 基本定理五及其相关常数不变张量

### 3.6.1 基本定理五

定理3.6.1.  $T_\alpha = [e^{\theta^\gamma}f_\gamma]_\alpha{}^\beta e^{i\theta^\gamma T_\gamma}T_\beta e^{-i\theta^\gamma T_\gamma} \Leftrightarrow [T_\alpha, T_\beta] = if_{\alpha\beta}{}^\gamma T_\gamma, f_\alpha \prec f_{\alpha\beta}{}^\gamma$

证明:  $T_\alpha = [e^{\theta^\gamma}f_\gamma]_\alpha{}^\beta e^{i\theta^\gamma T_\gamma}T_\beta e^{-i\theta^\gamma T_\gamma}, \forall \theta^\gamma$

$\Leftrightarrow T_\alpha = (\delta_\alpha{}^\beta + \theta^\gamma f_{\gamma\alpha}{}^\beta)(1 + i\theta^\gamma T_\gamma)T_\beta(1 - i\theta^\gamma T_\gamma), \forall \theta^\gamma \rightarrow 0$

$\Leftrightarrow 0 = \theta^\gamma (f_{\gamma\alpha}{}^\beta T_\beta + i[T_\gamma, T_\alpha]), \forall \theta^\gamma \rightarrow 0$

$\Leftrightarrow 0 = f_{\gamma\alpha}{}^\beta T_\beta + i[T_\gamma, T_\alpha]$

$\Leftrightarrow [T_\alpha, T_\beta] = if_{\alpha\beta}{}^\gamma T_\gamma$  □

以上定理表明：对易关系 $[T_\alpha, T_\beta] = if_{\alpha\beta}{}^\gamma T_\gamma$ 意味着Yang-Mills基 $T_\alpha$ 是常数不变张量，反之亦然。并且基本定理二是本定理的特殊情形( $f_{\alpha\beta}{}^\gamma = \varepsilon_{\alpha\beta}{}^\gamma, i\theta^\gamma = i\omega + \zeta\epsilon, T_\alpha = \Gamma_\alpha$ )，此定理更为广泛一般，可以描述内部空间和外部空间的协变性，并且对 $T_\alpha$ 的线性无关性没有要求。如果 $T_\alpha$ 满足线性无关性，则由群的结构方程可知群结构常数 $f_{\alpha\beta}{}^\gamma$ 也满足类似Yang-Mills基 $T_\alpha$ 的对易关系，即有如下推论：

推论3.6.1.  $[-if_\alpha, -if_\beta] = if_{\alpha\beta}{}^\gamma(-if_\gamma) \Leftrightarrow -if_\alpha = [e^{\theta^\gamma}f_\gamma]_\alpha{}^\beta e^{i\theta^\gamma(-if_\gamma)}(-if_\beta)e^{-i\theta^\gamma(-if_\gamma)}$

上式左侧就是群的结构方程，所以如果基 $T_\alpha$ 满足线性无关性，则群结构常数 $f_{\alpha\beta}{}^\gamma$ 也是常数不变张量。

自我评述：以上表明Yang-Mills基和群结构常数是内部空间的常数不变张量。

## 3.7 基本定理六及其相关常数不变张量

### 3.7.1 基本定理六

定理3.7.1.  $\Gamma = e^{(i\omega \cdot R + \zeta\epsilon \cdot L)}\Gamma e^{-(i\omega \cdot R - \zeta\epsilon \cdot L)} \Leftrightarrow [R, \Gamma] = 0, \{L, \Gamma\} = 0$

证明:  $\Gamma = e^{(i\omega \cdot R + \zeta\epsilon \cdot L)}\Gamma e^{-(i\omega \cdot R - \zeta\epsilon \cdot L)}, \forall \omega, \forall \epsilon$

$\Leftrightarrow \Gamma = [1 + (i\omega \cdot R + \zeta\epsilon \cdot L)]\Gamma[1 - (i\omega \cdot R - \zeta\epsilon \cdot L)], \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0$

$\Leftrightarrow 0 = (i\omega \cdot R + \zeta\epsilon \cdot L)\Gamma - \Gamma(i\omega \cdot R - \zeta\epsilon \cdot L), \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0$

$\Leftrightarrow 0 = i\omega \cdot [R, \Gamma] + \zeta\epsilon \cdot \{L, \Gamma\}, \forall \omega \rightarrow 0, \forall \epsilon \rightarrow 0$

$\Leftrightarrow [R, \Gamma] = 0, \{L, \Gamma\} = 0$  □

推论3.7.1.  $\eta = e^{(i\omega \cdot R + \zeta\epsilon \cdot L)}\eta e^{-(i\omega \cdot R - \zeta\epsilon \cdot L)}, \eta = \text{diag}(1, 1, 1, -1)$ , 即 $\eta^{a_\zeta b'_\zeta}, \eta^{a' b}, \eta^{ab'}, \eta^{a'b}$ 是常数不变张量。

自我评述：这个常数不变张量可以完成矢量与带撇矢量之间的相互转换，形式上与闵斯科夫度规一样。

### 3.8 新常数不变张量的各种获得方法

方法一： $\epsilon \leftrightarrow -\epsilon$

方法二： $\zeta \leftrightarrow -\zeta$

方法三：取复共轭、转置、相似变换、表象变换等矩阵操作

方法四：取直积、直和、缩并、加减乘除等运算

另外在六个基本定理的各种推论中已运用了以上方法得到了各种常数不变张量，各种推论的证明基本上是显而易见的，并且为了内容的紧凑性，大多略去了其证明过程。

### 3.9 对六个基本定理的评述

六个基本定理的数学证明过程中对变换参数 $\omega, \epsilon, \vartheta^{ab}, \theta^\alpha$ 无特别的限制，可以取任意复数，所以得到的常数不变张量具有很大的数学广泛性。但对于具体物理来讲，内部规范变换的各种参数仍可以取复数。然而对于外部时空变换，由于物理上自洽性要求，变换必须满足洛伦兹群表示<sup>[12]</sup>，所以对变换矩阵和变换参数有限制，且 $\omega, \epsilon$ 只能取实数。特别需要指出的是，基本定理一的推广形式、基本定理三和基本定理四不光在四维时空中成立，而且在任意N+1维时空中也成立，这样为高低维时空的物理研究提供了一个数学分析工具。从六个基本定理的证明还可以得到如下启示：矩阵的对易和反对易关系意味着存在对应的常数不变张量，反之，某种常数不变张量意味着存在相应的对易和反对易关系，从这个思路出发可以去寻找更多有意义的常数不变张量。从上还可以知道矩阵的对易和反对易关系意味着其自身的协变性，即该矩阵的对易和反对易关系本身隐含了其在任何参考系中均成立，这是一个很有趣很奇妙的自举数学性质，令人回味无穷。

## 4 几个直观基本常数不变张量的性质

### 4.1 基本常数不变张量 $\varepsilon^{A_\zeta B_\zeta}, \varepsilon_{A_\zeta B_\zeta}, \varepsilon^{A'_\zeta B'_\zeta}, \varepsilon_{A'_\zeta B'_\zeta}$ 的性质<sup>[1,2]</sup>

#### 4.1.1 重要性质

$$\varepsilon^{A_\zeta B_\zeta} \varepsilon_{C_\zeta D_\zeta} = \delta_{C_\zeta}^{[A_\zeta} \delta_{D_\zeta]}^{B_\zeta] = \delta_{[C_\zeta}^{A_\zeta} \delta_{D_\zeta]}^{B_\zeta} \quad \varepsilon_{A'_\zeta B'_\zeta} \varepsilon^{C'_\zeta D'_\zeta} = \delta_{[A'_\zeta}^{C'_\zeta} \delta_{B'_\zeta]}^{D'_\zeta} = \delta_{A'_\zeta}^{[C'_\zeta} \delta_{B'_\zeta]}^{D'_\zeta} \quad (1.52)$$

$$\text{对比: } S_{abcd} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}, S_{ab}{}^{cd} = \delta_a^{[c} \delta_b^{d]} \quad (1.53)$$

$$\varepsilon_{A_\zeta B_\zeta} \varepsilon_{C_\zeta D_\zeta} = \varepsilon_{A_\zeta C_\zeta} \varepsilon_{B_\zeta D_\zeta} - \varepsilon_{A_\zeta D_\zeta} \varepsilon_{B_\zeta C_\zeta} \quad \varepsilon_{A'_\zeta B'_\zeta} \varepsilon_{C'_\zeta D'_\zeta} = \varepsilon_{A'_\zeta C'_\zeta} \varepsilon_{B'_\zeta D'_\zeta} - \varepsilon_{A'_\zeta D'_\zeta} \varepsilon_{B'_\zeta C'_\zeta} \quad (1.54)$$

$$\varepsilon^{A_\zeta B_\zeta} \varepsilon_{C_\zeta D_\zeta} = \varepsilon^{A_\zeta C_\zeta} \varepsilon_{B_\zeta D_\zeta} - \varepsilon^{A_\zeta D_\zeta} \varepsilon_{B_\zeta C_\zeta} \quad \varepsilon^{A'_\zeta B'_\zeta} \varepsilon_{C'_\zeta D'_\zeta} = \varepsilon^{A'_\zeta C'_\zeta} \varepsilon_{B'_\zeta D'_\zeta} - \varepsilon^{A'_\zeta D'_\zeta} \varepsilon_{B'_\zeta C'_\zeta} \quad (1.55)$$

$$\varepsilon_{A_\zeta B_\zeta} \varepsilon_{C_\zeta D_\zeta} + \varepsilon_{A_\zeta C_\zeta} \varepsilon_{D_\zeta B_\zeta} + \varepsilon_{A_\zeta D_\zeta} \varepsilon_{B_\zeta C_\zeta} = 0 \quad \varepsilon_{A_\zeta [B_\zeta} \varepsilon_{C_\zeta D_\zeta]} = 0 \quad (1.56)$$

$$\varepsilon_{A_\zeta}{}^{B_\zeta} = \delta_{A_\zeta}{}^{B_\zeta} = -\varepsilon^{B_\zeta}{}_{A_\zeta} \quad \varepsilon_{A'_\zeta}{}^{B'_\zeta} = \delta_{A'_\zeta}{}^{B'_\zeta} = -\varepsilon^{B'_\zeta}{}_{A'_\zeta} \quad (1.57)$$

#### 4.1.2 复共轭性

$$[\varepsilon^{A_\zeta B_\zeta}]^* = \varepsilon^{A'_\zeta B'_\zeta} \quad [\varepsilon_{A_\zeta B_\zeta}]^* = \varepsilon_{A'_\zeta B'_\zeta} \quad (1.58)$$

### 4.2 基本常数不变张量 $(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta}, (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a$ 的性质<sup>[1,2]</sup>

#### 4.2.1 转换性

转换性

$$\frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} = [\zeta \varepsilon^{A_\zeta B_\zeta}] [\zeta \varepsilon^{A'_\zeta B'_\zeta}] \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{a B_\zeta B'_\zeta} \quad (1.59)$$

$$\frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a = [-\zeta \varepsilon_{A'_\zeta B'_\zeta}] [-\zeta \varepsilon_{A_\zeta B_\zeta}] \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)^{a B'_\zeta B_\zeta} \quad (1.60)$$

$$\frac{(-i\zeta)}{\sqrt{2}} [\sigma, -i(-\zeta)]_a^{A'_\zeta A_\zeta} \simeq \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \quad (1.61)$$

不冗余版本： $\frac{i}{\sqrt{2}} (\sigma, -i)_a^{A'_\zeta A_\zeta}, \frac{i}{\sqrt{2}} (-\sigma, -i)_{A_\zeta A'_\zeta}^a$

以上表明这两个常数不变张量不是独立的，真正独立的只有一个。

### 4.2.2 正交性

缩减一对矢量指标：(右边是Penrose简记法，用 $\overset{P}{=}$ 表示。)

$$\frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_{aA_\zeta}^{A'_\zeta A'_\zeta} \delta_b^a \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b = \delta_{B_\zeta}^{A'_\zeta} \delta_{B'_\zeta}^{A'_\zeta} \quad \delta_b^a \overset{P}{=} \delta_B^A \delta_{B'}^{A'} \quad (1.62)$$

$$\frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_{aA_\zeta}^{A'_\zeta A'_\zeta} \delta^{ab} \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_{bB'_\zeta}^{B'_\zeta B'_\zeta} = \varepsilon^{AB} \varepsilon^{A'B'} \quad \delta^{ab} \overset{P}{=} \varepsilon^{AB} \varepsilon^{A'B'} \quad (1.63)$$

$$\frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \delta_{ab} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b = \varepsilon_{A_\zeta B_\zeta} \varepsilon_{A'_\zeta B'_\zeta} \quad \delta_{ab} \overset{P}{=} \varepsilon_{AB} \varepsilon_{A'B'} \quad (1.64)$$

各种标架下的iPenrose对应规则

$$\frac{i}{\sqrt{2}}(\sigma^*, -i)_{aAA'}^{AA'} x^a|_{++++} = \frac{i}{\sqrt{2}}(\sigma^*, -1)_{aAA'}^{AA'} x^a|_{++++} = \frac{i}{\sqrt{2}}(-\sigma^*, 1)_{aAA'}^{AA'} x^a|_{----} = x^{AA'} \quad (1.65)$$

$$\frac{i}{\sqrt{2}}(-\sigma, -i)_{AA'}^a x_a|_{++++} = \frac{i}{\sqrt{2}}(-\sigma, 1)_{AA'}^a x_a|_{++++} = \frac{i}{\sqrt{2}}(-\sigma, 1)_{AA'}^a x_a|_{----} = x_{AA'} \quad (1.66)$$

$$\frac{i}{\sqrt{2}}(\sigma^*, -i)_{aAA'}^{AA'} \partial^a|_{++++} = \frac{i}{\sqrt{2}}(\sigma^*, -1)_{aAA'}^{AA'} \partial^a|_{++++} = \frac{i}{\sqrt{2}}(-\sigma^*, 1)_{aAA'}^{AA'} \partial^a|_{----} = \partial_{x_{AA'}} = \nabla^{AA'} \quad (1.67)$$

$$\frac{i}{\sqrt{2}}(-\sigma, -i)_{AA'}^a x_a|_{++++} = \frac{i}{\sqrt{2}}(-\sigma, 1)_{AA'}^a x_a|_{++++} = \frac{i}{\sqrt{2}}(-\sigma, 1)_{AA'}^a x_a|_{----} = \partial_{x_{AA'}} = \nabla_{AA'} \quad (1.68)$$

(++++)标架下规则：矢量上标 $a$ 用 $\frac{i}{\sqrt{2}}(\sigma^*, -i)_{aAA'}^{AA'}$ 变换为 $AA'$ ，矢量下标 $a$ 用 $\frac{i}{\sqrt{2}}(-\sigma, -i)_{AA'}^a$ 变换为 $AA'$ 。

(+++-)标架下规则：矢量上标 $a$ 用 $\frac{i}{\sqrt{2}}(\sigma^*, -1)_{aAA'}^{AA'}$ 变换为 $AA'$ ，矢量下标 $a$ 用 $\frac{i}{\sqrt{2}}(-\sigma, 1)_{AA'}^a$ 变换为 $AA'$ 。

(---+)标架下规则：矢量上标 $a$ 用 $\frac{i}{\sqrt{2}}(-\sigma^*, 1)_{aAA'}^{AA'}$ 变换为 $AA'$ ，矢量下标 $a$ 用 $\frac{i}{\sqrt{2}}(-\sigma, 1)_{AA'}^a$ 变换为 $AA'$ 。

$$\frac{i}{\sqrt{2}}(\sigma^*, -i)_{aAA'}^{AA'} \frac{i}{\sqrt{2}}(-\sigma, -i)_{AA'}^a = \frac{i}{\sqrt{2}}(\sigma^*, -1)_{aAA'}^{AA'} \frac{i}{\sqrt{2}}(-\sigma, 1)_{AA'}^a = -\frac{i}{\sqrt{2}}(-\sigma^*, 1)_{aAA'}^{AA'} \frac{i}{\sqrt{2}}(-\sigma, 1)_{AA'}^a = \delta_B^A \delta_{B'}^{A'} \quad (1.69)$$

缩减一对旋量指标：

$$(\sigma, i\zeta)_{aA_\zeta}^{A'_\zeta B'_\zeta} (\sigma, -i\zeta)_{bA'_\zeta}^{A'_\zeta B'_\zeta} = \delta_{ab} \delta_{A'_\zeta}^{B'_\zeta} + 2iS_{abA'_\zeta}^{B'_\zeta} \quad (1.70)$$

$$(\sigma, -i\zeta)_{aA'_\zeta}^{A'_\zeta B'_\zeta} (\sigma, i\zeta)_{bA_\zeta}^{A'_\zeta B'_\zeta} = \delta_{ab} \delta_{A'_\zeta}^{B'_\zeta} + 2iS_{abA'_\zeta}^{B'_\zeta} \quad (1.71)$$

缩减两对指标：

$$(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, -i\zeta)_{bA'_\zeta}^{A'_\zeta A'_\zeta} = 2\delta_b^a \quad tr[(\sigma, i\zeta)_a (\sigma, -i\zeta)_b] = 2\delta_{ab} \quad (1.72)$$

$$(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, -i\zeta)_{aA'_\zeta}^{A'_\zeta B'_\zeta} = 4\delta_{A'_\zeta}^{B'_\zeta} \quad (\sigma, -i\zeta)_{aA'_\zeta}^{A'_\zeta A'_\zeta} (\sigma, i\zeta)_{A_\zeta B'_\zeta}^a = 4\delta_{A'_\zeta}^{B'_\zeta} \quad (1.73)$$

缩减全部指标：

$$(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, -i\zeta)_{aA'_\zeta}^{A'_\zeta A'_\zeta} = 8 \quad (1.74)$$

### 4.2.3 复共轭性

$$[(\sigma, i\zeta)_{A_\zeta B'_\zeta}^a \partial_a]^* = (\sigma, i\zeta)_{B_\zeta A'_\zeta}^a \partial_a \quad [(\sigma, -i\zeta)_{aA'_\zeta}^{A'_\zeta B'_\zeta} \partial^a]^* = (\sigma, -i\zeta)_{aA'_\zeta}^{B'_\zeta A'_\zeta} \partial^a \quad (1.75)$$

$$[(\sigma, i\zeta)^a \partial_a]^+ = (\sigma, i\zeta)^a \partial_a \quad [(\sigma, -i\zeta)_a \partial^a]^+ = (\sigma, -i\zeta)_a \partial^a \quad (1.76)$$

## 4.3 基本常数不变张量 $\sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta}, \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}$ 的性质

### 4.3.1 正交性

三维自旋张量：

$$S_{\alpha'_\zeta \beta'_\zeta}^{A'_\zeta B'_\zeta} = \frac{i}{2} \gamma_{\alpha'_\zeta \beta'_\zeta}^{\gamma'_\zeta} \sigma_{\gamma'_\zeta}^{A'_\zeta B'_\zeta} = \frac{1}{2} \varepsilon_{\alpha'_\zeta \beta'_\zeta}^{\gamma'_\zeta} \sigma_{\gamma'_\zeta}^{A'_\zeta B'_\zeta} \quad S^{\alpha_\zeta \beta_\zeta}_{A_\zeta B_\zeta} = \frac{i}{2} \gamma^{\alpha_\zeta \beta_\zeta}_{\gamma_\zeta} \sigma^{\gamma_\zeta}_{A_\zeta B_\zeta} = \frac{1}{2} \varepsilon^{\alpha_\zeta \beta_\zeta}_{\gamma_\zeta} \sigma^{\gamma_\zeta}_{A_\zeta B_\zeta} \quad (1.77)$$

缩减一对复矢量指标：

$$\sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} = \delta_{D'_\zeta}^{C'_\zeta} \delta_{B'_\zeta}^{A'_\zeta} - \varepsilon_{A'_\zeta C'_\zeta} \varepsilon_{B'_\zeta D'_\zeta} \quad \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta} \sigma^{\alpha_\zeta}_{C_\zeta D_\zeta} = \delta_{A_\zeta}^{D_\zeta} \delta_{B_\zeta}^{C_\zeta} - \varepsilon_{A_\zeta C_\zeta} \varepsilon_{B_\zeta D_\zeta} \quad (1.78)$$

缩减一对旋量指标：

$$\sigma_{\alpha'_\zeta}^{A'_\zeta} \sigma_{C'_\zeta} \sigma_{\beta'_\zeta}^{B'_\zeta} = \delta_{\alpha'_\zeta \beta'_\zeta} \delta_{A'_\zeta B'_\zeta} + 2i S_{\alpha'_\zeta \beta'_\zeta}^{A'_\zeta B'_\zeta} \quad \sigma^{\alpha_\zeta}_{A_\zeta} \sigma^{C_\zeta} \sigma^{\beta_\zeta}_{B_\zeta} = \delta^{\alpha_\zeta \beta_\zeta} \delta_{A_\zeta B_\zeta} + 2i S^{\alpha_\zeta \beta_\zeta}_{A_\zeta B_\zeta} \quad (1.79)$$

缩减两对指标：

$$\sigma_{\alpha'_\zeta}^{A'_\zeta} \sigma_{B'_\zeta} \sigma_{\beta'_\zeta}^{B'_\zeta} = 2\delta_{\alpha'_\zeta \beta'_\zeta} \quad \sigma^{\alpha_\zeta}_{A_\zeta} \sigma^{B_\zeta} \sigma^{\beta_\zeta}_{B_\zeta} = 2\delta^{\alpha_\zeta \beta_\zeta} \quad (1.80)$$

$$\sigma_{\alpha'_\zeta}^{A'_\zeta} \sigma_{C'_\zeta} \sigma^{\alpha'_\zeta}_{C'_\zeta} \sigma_{B'_\zeta} = 3\delta_{\alpha'_\zeta B'_\zeta} \quad \sigma^{\alpha_\zeta}_{A_\zeta} \sigma^{C_\zeta} \sigma_{\alpha_\zeta C_\zeta} \sigma^{B_\zeta} = 3\delta_{A_\zeta B_\zeta} \quad (1.81)$$

$$\text{tr}[\sigma_{\alpha'_\zeta} \sigma_{\beta'_\zeta}] = 2\delta_{\alpha'_\zeta \beta'_\zeta} \quad \text{tr}[\sigma^{\alpha_\zeta} \sigma^{\beta_\zeta}] = 2\delta^{\alpha_\zeta \beta_\zeta} \quad (1.82)$$

缩减全部指标：

$$\sigma_{\alpha'_\zeta}^{A'_\zeta} \sigma_{B'_\zeta} \sigma^{\alpha'_\zeta}_{B'_\zeta} \sigma_{A'_\zeta} = 6 \quad \sigma^{\alpha_\zeta}_{A_\zeta} \sigma^{B_\zeta} \sigma_{\alpha_\zeta B_\zeta} \sigma^{A_\zeta} = 6 \quad (1.83)$$

### 4.3.2 无迹性

$$\sigma_{\alpha'_\zeta}^{A'_\zeta} A'_\zeta = 0 \quad \text{tr}[\sigma_{\alpha'_\zeta}] = 0 \quad \sigma^{\alpha_\zeta}_{A_\zeta} A_\zeta = 0 \quad \text{tr}[\sigma^{\alpha'_\zeta}] = 0 \quad (1.84)$$

### 4.3.3 复共轭性

$$[\sigma_{\alpha'_\zeta}^{A'_\zeta} B'_\zeta]^* = \sigma_{\alpha_\zeta B_\zeta}^{A_\zeta} \quad [\sigma_{\alpha_\zeta A_\zeta}^{B_\zeta}]^* = \sigma_{\alpha'_\zeta B'_\zeta}^{A'_\zeta} \quad (1.85)$$

## 4.4 基本常数不变张量 $\sigma_{+ab}^\alpha, \sigma_{-ab}^\alpha, \sigma_{\zeta ab}^\alpha, \sigma_{-\zeta ab}^\alpha$ 的性质

### 4.4.1 隐藏的复合性

复合性:(事实上可以将下式作为定义)

$$\sigma_{\zeta a}^{\alpha_\zeta b} = \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \sigma^{\alpha_\zeta}_{A_\zeta} \sigma^{B_\zeta} \delta_{A'_\zeta B'_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \quad (1.86)$$

$$\sigma_{-\zeta \alpha'_\zeta}^a b = \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_a^{A_\zeta A'_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta} \sigma_{B'_\zeta} \delta^{A_\zeta B_\zeta} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_{B_\zeta B'_\zeta}^b \quad (1.87)$$

证明:  $(\sigma, i\zeta)_a^{A_\zeta A'_\zeta} (\sigma, -i\zeta)_b^{A'_\zeta B_\zeta} = \delta_{ab} \delta_{A_\zeta B_\zeta} + 2i S_{ab A_\zeta B_\zeta}$

$$\Rightarrow (\sigma, i\zeta)_a^{A_\zeta A'_\zeta} (\sigma, -i\zeta)_b^{A'_\zeta B_\zeta} \sigma^{\beta_\zeta}_{B_\zeta} \sigma^{A_\zeta} = (\delta_{ab} \delta_{A_\zeta B_\zeta} - \sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta A_\zeta}^{B_\zeta}) \sigma^{\beta_\zeta}_{B_\zeta} \sigma^{A_\zeta}$$

$$\Rightarrow (\sigma, i\zeta)_a^{A_\zeta A'_\zeta} (\sigma, -i\zeta)_b^{A'_\zeta B_\zeta} \sigma^{\beta_\zeta}_{B_\zeta} \sigma^{A_\zeta} = -2\sigma_{\zeta ab}^{\alpha_\zeta} \delta^{\alpha_\zeta \beta_\zeta}$$

$$\Rightarrow 2\sigma_{\zeta ab}^{\alpha_\zeta} = (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \sigma^{\alpha_\zeta}_{A_\zeta} \sigma^{B_\zeta} (\sigma, i\zeta)_{b B_\zeta A'_\zeta}$$

$$\Rightarrow \sigma_{\zeta ab}^{\alpha_\zeta} = \frac{1}{2} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \sigma^{\alpha_\zeta}_{A_\zeta} \sigma^{B_\zeta} \delta_{A'_\zeta B'_\zeta} (\sigma, i\zeta)_{b B_\zeta B'_\zeta}$$

$$\Rightarrow \sigma_{\zeta a}^{\alpha_\zeta b} = \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \sigma^{\alpha_\zeta}_{A_\zeta} \sigma^{B_\zeta} \delta_{A'_\zeta B'_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{b B_\zeta B'_\zeta}^b \quad \square$$

基本常数不变张量间的联系：

$$\frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_a^{A_\zeta A'_\zeta} \sigma_{\zeta a}^{\alpha_\zeta b} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_{b B_\zeta B'_\zeta} = \sigma^{\alpha_\zeta}_{A_\zeta} \sigma^{B_\zeta} \delta_{A'_\zeta B'_\zeta} \quad \sigma_{+a}^{\alpha_\zeta b} \stackrel{P}{=} \sigma^{\alpha_\zeta}_{A_\zeta} \delta_{A'_\zeta B'_\zeta} \quad (1.88)$$

$$\frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \sigma_{-\zeta \alpha'_\zeta}^a b \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{b B_\zeta B'_\zeta} = \sigma_{\alpha'_\zeta}^{A'_\zeta} \sigma_{B'_\zeta} \delta^{A_\zeta B_\zeta} \quad \sigma_{-\alpha'_\zeta}^a b \stackrel{P}{=} \sigma_{\alpha'_\zeta}^{A'_\zeta} \delta^{A_\zeta B_\zeta} \quad (1.89)$$

自我评述：以上表明  $\sigma_{\zeta ab}^{\alpha_\zeta}$  和  $\sigma_{\alpha_\zeta A_\zeta}^{B_\zeta}$  这两个基本常数不变张量不是独立的，是相互关联的。

### 4.4.2 正交性

三维自旋张量和四维自旋张量的关系：

$$S^{\alpha_\zeta \beta_\zeta}_{ab} (\frac{1}{2} \sigma_{\zeta}^{\alpha_\zeta}) = \frac{i}{2} \gamma^{\alpha_\zeta \beta_\zeta} \gamma_\zeta \sigma_{\zeta ab}^{\alpha_\zeta} = \frac{1}{2} i \sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta \beta_\zeta} = \frac{1}{2} S_{ab}^{\alpha_\zeta \beta_\zeta} (\gamma, \zeta) \quad (1.90)$$

缩减一对复矢量指标：

$$\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta cd} = -\delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} + \zeta \varepsilon_{abcd} \quad (1.91)$$

$$S_{abcd} = -\frac{1}{2} (\sigma_{-ab}^{\alpha'_\zeta} \sigma_{-\alpha'_\zeta cd} + \sigma_{+ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta cd}) = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} \quad \varepsilon_{abcd} = -\frac{1}{2} (\sigma_{-ab}^{\alpha'_\zeta} \sigma_{-\alpha'_\zeta cd} - \sigma_{+ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta cd}) \quad (1.92)$$

缩减一对矢量指标：

$$\sigma_{\zeta ac}^{\beta\zeta} \delta^{cd} \sigma_{\zeta db}^{\gamma\zeta} = \delta_{ab} \delta^{\beta\zeta\gamma\zeta} - \sigma_{\zeta ab}^{\alpha\zeta} \gamma_{\alpha\zeta}^{\beta\zeta\gamma\zeta} = \delta^{\beta\zeta\gamma\zeta} \delta_{ab} + \frac{i}{2} S^{\beta\zeta\gamma\zeta}{}_{ab} (\frac{1}{2} \sigma_{\zeta}^{\alpha\zeta}) = \delta_{ab} \delta^{\beta\zeta\gamma\zeta} + i S_{ab}^{\beta\zeta\gamma\zeta} (\gamma, \zeta) \quad (1.93)$$

缩减一对矢量和一对复矢量指标：

$$\sigma_{\zeta ac}^{\alpha\zeta} \sigma_{\alpha\zeta}^{\zeta cb} = 3 \delta_a^b \quad (1.94)$$

缩减两对矢量指标：

$$\sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\zeta\beta\zeta}^{\alpha b} = -4 \delta_{\beta\zeta}^{\alpha\zeta} \quad \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{-\zeta\beta\zeta}^{\alpha b} = 0 \quad \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\kappa}^{\beta\kappa ab} = -4 \delta_{\zeta\kappa} \delta^{\alpha\zeta\beta\kappa} \quad (1.95)$$

$$tr(\sigma_{\zeta}^{\alpha\zeta} \sigma_{\zeta}^{\beta\zeta}) = 4 \delta^{\alpha\zeta\beta\zeta} \quad tr(\sigma_{\zeta}^{\alpha\zeta} \sigma_{-\zeta}^{\beta\zeta}) = 0 \quad tr(\sigma_{\zeta}^{\alpha\zeta} \sigma_{\kappa}^{\beta\kappa}) = 4 \delta_{\zeta\kappa} \delta^{\alpha\zeta\beta\kappa} \quad (1.96)$$

缩减全部指标：

$$\sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\zeta\alpha\zeta}^{\alpha b} = -12 \quad \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{-\zeta\alpha\zeta}^{\alpha b} = 0 \quad (1.97)$$

4.4.3 对偶性

$$\sigma_{+ab}^{\alpha} = - * \sigma_{+ab}^{\alpha} \quad \sigma_{-ab}^{\alpha'} = * \sigma_{-ab}^{\alpha'} \quad \sigma_{\zeta ab}^{\alpha\zeta} = -\zeta * \sigma_{\zeta ab}^{\alpha\zeta} \quad (1.98)$$

4.4.4 复共轭性

$$(\sigma_{\zeta ab}^{\alpha\zeta} \partial^a \hat{\partial}^b)^* = -\sigma_{-\zeta ab}^{\alpha\zeta} \partial^a \hat{\partial}^b \quad (\sigma_{-\zeta ab}^{\alpha\zeta} \partial^a \hat{\partial}^b)^* = -\sigma_{\zeta ab}^{\alpha\zeta} \partial^a \hat{\partial}^b \quad (1.99)$$

4.5 基本常数不变张量  $\sigma_{\alpha\zeta}^{\kappa\zeta} l'_{\zeta}(s), \sigma_{\alpha\zeta}^{\kappa\zeta} l_{\zeta}(s)$  的性质

4.5.1 数学准备

利用公式：  $\sum_{k=1}^{2s} k^2 = \frac{8}{3} s(s + \frac{1}{2})(s + \frac{1}{4})$ ，可得

$$tr[\sigma_x^2(s)] = tr[\sigma_y^2(s)] = \frac{1}{4} \sum_{k=1}^{2s} 2k(2s + 1 - k) = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \quad (1.100)$$

$$tr[\sigma_z^2(s)] = \frac{1}{4} \sum_{k=1}^{2s} (2s - 2k)^2 = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \quad (1.101)$$

$$tr[\sigma_x^2(s)] = tr[\sigma_y^2(s)] = tr[\sigma_z^2(s)] = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \quad (1.102)$$

4.5.2 正交性

从旋量角度看：

$$\sigma_{\alpha\zeta}^{\kappa\zeta} l'_{\zeta}(s) \sigma_{\beta\zeta}^{\zeta l'_{\zeta}}(s) = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \delta_{\alpha\zeta}^{\beta\zeta} \quad \sigma_{\alpha\zeta}^{\kappa\zeta} m'_{\zeta}(s) \sigma_{\alpha\zeta}^{\zeta m'_{\zeta}}(s) = s(s + 1) \delta^{\kappa\zeta} l'_{\zeta} \quad (1.103)$$

$$\sigma_{\alpha\zeta}^{\kappa\zeta} l_{\zeta}(s) \sigma_{\beta\zeta}^{\zeta l_{\zeta}}(s) = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \delta^{\alpha\zeta\beta\zeta} \quad \sigma_{\alpha\zeta}^{\kappa\zeta} m_{\zeta}(s) \sigma_{\alpha\zeta}^{\zeta m_{\zeta}}(s) = s(s + 1) \delta_{\kappa\zeta}^{\zeta} \quad (1.104)$$

$$\sigma_{\alpha\zeta}^{\kappa\zeta} l'_{\zeta}(s) \sigma_{\alpha\zeta}^{\zeta l'_{\zeta}}(s) = 2s(s + \frac{1}{2})(s + 1) \quad \sigma_{\alpha\zeta}^{\kappa\zeta} l_{\zeta}(s) \sigma_{\alpha\zeta}^{\zeta l_{\zeta}}(s) = 2s(s + \frac{1}{2})(s + 1) \quad (1.105)$$

从矩阵角度看：

$$tr[\sigma_{\alpha\zeta}^{\zeta}(s) \sigma_{\beta\zeta}^{\zeta}(s)] = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \delta_{\alpha\zeta}^{\beta\zeta} \quad tr[\sigma^{\alpha\zeta}(s) \sigma^{\beta\zeta}(s)] = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \delta^{\alpha\zeta\beta\zeta} \quad (1.106)$$

4.5.3 正交性

缩减一对复矢量指标：

$$\sigma_{\alpha\zeta}^{\kappa\zeta} l'_{\zeta}(s) \sigma_{\alpha\zeta}^{\zeta m'_{\zeta}}(s) = ? \delta_{\alpha\zeta}^{\kappa\zeta} \delta^{m'_{\zeta}} n'_{\zeta} ? - 2 \varepsilon_{\kappa\zeta}^{\zeta} m'_{\zeta}(s) \varepsilon_{l'_{\zeta}} n'_{\zeta}(s) \quad (1.107)$$

$$\sigma_{\alpha\zeta}^{\kappa\zeta} l_{\zeta}(s) \sigma_{\alpha\zeta}^{\zeta m_{\zeta}}(s) = ? \delta_{\kappa\zeta}^{\zeta} \delta_{m_{\zeta}} n_{\zeta} ? - 2 \varepsilon_{\kappa\zeta}^{\zeta} m_{\zeta}(s) \varepsilon^{l_{\zeta}} n_{\zeta}(s) \quad (1.108)$$

缩减一对旋量指标：

$$\sigma_{\alpha\zeta}^{\kappa\zeta} m'_{\zeta}(s) \sigma_{\beta\zeta}^{\zeta m'_{\zeta}}(s) = ? \delta_{\alpha\zeta}^{\kappa\zeta} \delta^{\zeta} l'_{\zeta} ? + \frac{i}{2} S_{\alpha\zeta}^{\beta\zeta} \delta^{\kappa\zeta} l'_{\zeta}(s) \quad (1.109)$$

$$\sigma_{\alpha\zeta}^{\kappa\zeta} m_{\zeta}(s) \sigma_{\beta\zeta}^{\zeta m_{\zeta}}(s) = ? \delta^{\alpha\zeta\beta\zeta} \delta_{\kappa\zeta}^{\zeta} l_{\zeta} ? + \frac{i}{2} S^{\alpha\zeta\beta\zeta} \delta_{\kappa\zeta}^{\zeta} l_{\zeta}(s) \quad (1.110)$$

缩减两对指标:

$$\sigma_{\alpha'_\zeta}^{k'_\zeta} \nu'_\zeta(s) \sigma_{\beta'_\zeta}^{l'_\zeta} k'_\zeta(s) = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \delta_{\alpha'_\zeta \beta'_\zeta} \quad \sigma_{\alpha'_\zeta}^{k'_\zeta m'_\zeta}(s) \sigma^{\alpha'_\zeta m'_\zeta} \nu'_\zeta(s) = s(s + 1) \delta^{k'_\zeta l'_\zeta} \quad (1.111)$$

$$\sigma^{\alpha_\zeta} k_\zeta(s) \sigma^{\beta_\zeta} l_\zeta(s) = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \delta^{\alpha_\zeta \beta_\zeta} \quad \sigma^{\alpha_\zeta} k_\zeta m_\zeta(s) \sigma_{\alpha'_\zeta m_\zeta} l_\zeta(s) = s(s + 1) \delta_{k_\zeta}^{l_\zeta} \quad (1.112)$$

$$\text{tr}[\sigma_{\alpha'_\zeta}(s) \sigma_{\beta'_\zeta}(s)] = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \delta_{\alpha'_\zeta \beta'_\zeta} \quad \text{tr}[\sigma^{\alpha_\zeta}(s) \sigma^{\beta_\zeta}(s)] = \frac{2}{3} s(s + \frac{1}{2})(s + 1) \delta^{\alpha_\zeta \beta_\zeta} \quad (1.113)$$

缩减全部指标:

$$\sigma_{\alpha'_\zeta}^{k'_\zeta} \nu'_\zeta(s) \sigma^{\alpha'_\zeta} l'_\zeta k'_\zeta(s) = 2s(s + \frac{1}{2})(s + 1) \quad \sigma^{\alpha_\zeta} k_\zeta l_\zeta(s) \sigma_{\alpha_\zeta} k_\zeta(s) = 2s(s + \frac{1}{2})(s + 1) \quad (1.114)$$

#### 4.5.4 无迹性

$$\sigma_{\alpha'_\zeta}^{k'_\zeta} k'_\zeta(s) = 0 \quad \text{tr}[\sigma_{\alpha'_\zeta}(s)] = 0 \quad \sigma^{\alpha_\zeta} k_\zeta k_\zeta(s) = 0 \quad \text{tr}[\sigma^{\alpha_\zeta}(s)] = 0 \quad (1.115)$$

#### 4.5.5 复共轭性

$$[\sigma_{\alpha'_\zeta}^{k'_\zeta} \nu'_\zeta(s)]^* = \sigma_{\alpha_\zeta} k_\zeta(s) \quad [\sigma_{\alpha_\zeta} k_\zeta l_\zeta(s)]^* = \sigma_{\alpha'_\zeta} l'_\zeta k'_\zeta(s) \quad (1.116)$$

#### 4.5.6 复杂性质

$$\sigma_{\alpha_\zeta}(s) \sigma(s) \sigma^{\alpha_\zeta}(s) = [s(s + 1) - 1] \sigma(s) \quad \sigma_{\alpha'_\zeta}(s) \sigma(s) \sigma^{\alpha'_\zeta}(s) = [s(s + 1) - 1] \sigma(s) \quad (1.117)$$

$$[\sigma_{\alpha_\zeta}(s), \sigma_{\beta_\zeta}(s)] \sigma^{\beta_\zeta}(s) = \sigma_{\alpha_\zeta}(s) \quad -i \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \sigma^{\beta_\zeta}(s) \sigma^{\gamma_\zeta}(s) = \sigma_{\alpha_\zeta}(s) \quad (1.118)$$

$$\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \varepsilon^{\gamma_\zeta} \rho_\zeta \sigma_\zeta = \delta_{\alpha_\zeta \rho_\zeta} \delta_{\beta_\zeta \sigma_\zeta} - \delta_{\alpha_\zeta \sigma_\zeta} \delta_{\beta_\zeta \rho_\zeta} \quad \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \varepsilon^{\beta_\zeta \gamma_\zeta} \rho_\zeta = 2 \delta_{\alpha_\zeta \rho_\zeta} \quad (1.119)$$

推论4.5.1.  $\sigma_\alpha(s) \sigma_i(s) \sigma^\alpha(s) = [s(s + 1) - 1] \sigma_i(s)$

推论4.5.2.  $\sigma_\alpha(s) \sigma_i(s) \sigma_j(s) \sigma^\alpha(s) = s(s + 1) \delta_{ij} - \sigma_j(s) \sigma_i(s) + [s(s + 1) - 2] \sigma_i(s) \sigma_j(s)$

证明:  $\sigma_\alpha(s) \sigma_i(s) \sigma_j(s) \sigma^\alpha(s)$

$$\begin{aligned} &= [\sigma_\alpha(s), \sigma_i(s)] \sigma_j(s) \sigma^\alpha(s) + \sigma_i(s) \sigma_\alpha(s) \sigma_j(s) \sigma^\alpha(s) \\ &= [\sigma_\alpha(s), \sigma_i(s)] [\sigma_j(s), \sigma^\alpha(s)] + [\sigma_\alpha(s), \sigma_i(s)] \sigma^\alpha(s) \sigma_j(s) + \sigma_i(s) \sigma_\alpha(s) \sigma_j(s) \sigma^\alpha(s) \\ &= \varepsilon_{j l}^\alpha \varepsilon_{\alpha i k} \sigma^k(s) \sigma^l(s) + [s(s + 1) - 2] \sigma_i(s) \sigma_j(s) \\ &= s(s + 1) \delta_{ij} - \sigma_j(s) \sigma_i(s) + [s(s + 1) - 2] \sigma_i(s) \sigma_j(s) \end{aligned} \quad \square$$

推论4.5.3.  $\sigma_\alpha(s) \sigma_{\{i}(s) \sigma_{j\}}(s) \sigma^\alpha(s) = s(s + 1) \delta_{\{ij\}} + [s(s + 1) - 3] \sigma_{\{i}(s) \sigma_{j\}}(s)$

推论4.5.4.  $[\sigma_\alpha(s), \sigma_{\{i}(s)] \sigma_{j\}}(s) \sigma^\alpha(s) = s(s + 1) \delta_{\{ij\}} - 2 \sigma_{\{i}(s) \sigma_{j\}}(s)$

推论4.5.5.  $\sigma_\alpha(s) \sigma_{\{i}(s) \sigma_j(s) \sigma_{k\}}(s) \sigma^\alpha(s) = [3s(s + 1) - 1] \delta_{\{ij\} k} + [s(s + 1) - 6] \sigma_{\{i}(s) \sigma_j(s) \sigma_{k\}}(s)$

推论4.5.6.  $[\sigma_\alpha(s), \sigma_i(s)] \sigma_j(s) \sigma^\alpha(s) = s(s + 1) \delta_{ij} - \sigma_j(s) \sigma_i(s) - \sigma_i(s) \sigma_j(s)$

证明:  $\sigma_\alpha(s) \sigma_i(s) \sigma_j(s) \sigma_k(s) \sigma^\alpha(s)$

$$\begin{aligned} &= [\sigma_\alpha(s), \sigma_i(s)] \sigma_j(s) \sigma_k(s) \sigma^\alpha(s) + \sigma_i(s) \sigma_\alpha(s) \sigma_j(s) \sigma_k(s) \sigma^\alpha(s) \\ &= [\sigma_\alpha(s), \sigma_i(s)] \sigma_j(s) [\sigma_k(s), \sigma^\alpha(s)] + [\sigma_\alpha(s), \sigma_i(s)] \sigma_j(s) \sigma^\alpha(s) \sigma_k(s) + \sigma_i(s) \sigma_\alpha(s) \sigma_j(s) \sigma_k(s) \sigma^\alpha(s) \\ &= \delta_{ik} \sigma_\alpha(s) \sigma_j(s) \sigma^\alpha(s) - \sigma_k(s) \sigma_j(s) \sigma_i(s) + [\sigma_\alpha(s), \sigma_i(s)] \sigma_j(s) \sigma^\alpha(s) \sigma_k(s) + \sigma_i(s) \sigma_\alpha(s) \sigma_j(s) \sigma_k(s) \sigma^\alpha(s) \\ &= \delta_{ik} [s(s + 1) - 1] \sigma_j(s) - \sigma_k(s) \sigma_j(s) \sigma_i(s) \\ &+ s(s + 1) \delta_{ij} \sigma_k(s) - \sigma_j(s) \sigma_i(s) \sigma_k(s) - \sigma_i(s) \sigma_j(s) \sigma_k(s) \\ &+ s(s + 1) \delta_{jk} \sigma_i(s) - \sigma_i(s) \sigma_k(s) \sigma_j(s) + [s(s + 1) - 2] \sigma_i(s) \sigma_j(s) \sigma_k(s) \\ &= \delta_{ik} \sigma_\alpha(s) \sigma_j(s) \sigma^\alpha(s) - \sigma_k(s) \sigma_j(s) \sigma_i(s) + [\sigma_\alpha(s), \sigma_i(s)] \sigma_j(s) \sigma^\alpha(s) \sigma_k(s) + \sigma_i(s) \sigma_\alpha(s) \sigma_j(s) \sigma_k(s) \sigma^\alpha(s) \\ &= s(s + 1) [\delta_{jk} \sigma_i(s) + \delta_{ij} \sigma_k(s) + \delta_{ik} \sigma_j(s)] - \delta_{ik} \sigma_j(s) - \sigma_k(s) \sigma_j(s) \sigma_i(s) - \sigma_j(s) \sigma_i(s) \sigma_k(s) - \sigma_i(s) \sigma_k(s) \sigma_j(s) + [s(s + 1) - 3] \sigma_i(s) \sigma_j(s) \sigma_k(s) \end{aligned} \quad \square$$

$$\begin{aligned}
& \text{证明: } \sigma_\alpha(s)\sigma_i(s)\sigma_j(s)\sigma_k(s)\sigma_l(s)\sigma^\alpha(s) \\
& = [\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma_k(s)\sigma_l(s)\sigma^\alpha(s) + \|\sigma_i(s)\sigma_\alpha(s)\sigma_j(s)\sigma_k(s)\sigma_l(s)\sigma^\alpha(s) \\
& = [\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma_k(s)[\sigma_l(s), \sigma^\alpha(s)] + \|\|[\sigma_\alpha(s), \sigma_i(s)]\sigma_j(s)\sigma_k(s)\sigma^\alpha(s)\sigma_l(s) + \sigma_i(s)\sigma_\alpha(s)\sigma_j(s)\sigma_k(s)\sigma_l(s)\sigma^\alpha(s) \quad \square
\end{aligned}$$

#### 4.5.7 复杂性质

$$\sigma_{\alpha_\varsigma}(s)\sigma(s)\sigma^{\alpha_\varsigma}(s) = [s(s+1) - 1]\sigma(s) \quad \sigma_{\alpha'_\varsigma}(s)\sigma(s)\sigma^{\alpha'_\varsigma}(s) = [s(s+1) - 1]\sigma(s) \quad (1.120)$$

推论4.5.7.  $\sigma_i(s)\sigma_\alpha(s) = i\varepsilon_{i\alpha\beta}\sigma^\beta(s) + \sigma_\alpha(s)\sigma_i(s)$

## 5 几个直观复合常数不变张量的性质

### 5.1 扩展常数不变张量 $(\sigma, i\kappa)_{\alpha'_\varsigma B'_\varsigma}, (\sigma, i\kappa)^{\alpha_\varsigma B_\varsigma}$ 的性质

#### 5.1.1 转换性

转换性:

$$(\sigma, i\kappa)_{\alpha'_\varsigma B'_\varsigma} = -\varepsilon^{A'_\varsigma D'_\varsigma} \varepsilon_{B'_\varsigma C'_\varsigma} (\sigma, -i\kappa)_{\alpha'_\varsigma C'_\varsigma} \quad (\sigma, i\kappa)^{\alpha_\varsigma B_\varsigma} = -\varepsilon_{A_\varsigma D_\varsigma} \varepsilon^{B_\varsigma C_\varsigma} (\sigma, -i\kappa)^{\alpha_\varsigma C_\varsigma} \quad (1.121)$$

#### 5.1.2 正交性

缩减一对复矢量指标:

$$(\sigma, i\kappa)_{\alpha'_\varsigma B'_\varsigma} (\sigma, -i\kappa)^{\alpha'_\varsigma C'_\varsigma} = 2\delta_{D'_\varsigma}^{A'_\varsigma} \delta_{B'_\varsigma}^{C'_\varsigma} \quad (\sigma, i\kappa)^{\alpha_\varsigma B_\varsigma} (\sigma, -i\kappa)_{\alpha_\varsigma C_\varsigma} = 2\delta_{A_\varsigma}^{D_\varsigma} \delta_{B_\varsigma}^{C_\varsigma} \quad (1.122)$$

$$(\sigma, i\kappa)_{\alpha'_\varsigma B'_\varsigma} (\sigma, i\kappa)^{\alpha'_\varsigma C'_\varsigma} = -2\varepsilon^{A'_\varsigma C'_\varsigma} \varepsilon_{B'_\varsigma D'_\varsigma} \quad (\sigma, i\kappa)^{\alpha_\varsigma B_\varsigma} (\sigma, i\kappa)_{\alpha_\varsigma C_\varsigma} = -2\varepsilon_{A_\varsigma C_\varsigma} \varepsilon^{B_\varsigma D_\varsigma} \quad (1.123)$$

缩减一对旋量指标:

$$(\sigma, i\kappa)_{\alpha'_\varsigma C'_\varsigma} (\sigma, -i\kappa)_{\beta'_\varsigma C'_\varsigma} = \delta_{\alpha'_\varsigma \beta'_\varsigma} \delta_{A'_\varsigma B'_\varsigma} + 2iS_{\alpha'_\varsigma \beta'_\varsigma}^{A'_\varsigma B'_\varsigma}(\kappa) \quad (1.124)$$

$$(\sigma, i\kappa)^{\alpha_\varsigma C_\varsigma} (\sigma, -i\kappa)^{\beta_\varsigma C_\varsigma} = \delta^{\alpha_\varsigma \beta_\varsigma} \delta_{A_\varsigma B_\varsigma} + 2iS^{\alpha_\varsigma \beta_\varsigma}_{A_\varsigma B_\varsigma}(\kappa) \quad (1.125)$$

缩减两对指标:

$$(\sigma, i\kappa)_{\alpha'_\varsigma B'_\varsigma} (\sigma, -i\kappa)_{\beta'_\varsigma A'_\varsigma} = 2\delta_{\alpha'_\varsigma \beta'_\varsigma} \quad (\sigma, i\kappa)^{\alpha_\varsigma B_\varsigma} (\sigma, -i\kappa)^{\beta_\varsigma A_\varsigma} = 2\delta^{\alpha_\varsigma \beta_\varsigma} \quad (1.126)$$

$$(\sigma, i\kappa)_{\alpha'_\varsigma C'_\varsigma} (\sigma, -i\kappa)^{\alpha'_\varsigma C'_\varsigma} = 4\delta_{A'_\varsigma B'_\varsigma} \quad (\sigma, i\kappa)^{\alpha_\varsigma C_\varsigma} (\sigma, -i\kappa)_{\alpha_\varsigma C_\varsigma} = 4\delta_{A_\varsigma B_\varsigma} \quad (1.127)$$

缩减全部指标:

$$(\sigma, i\kappa)_{\alpha'_\varsigma B'_\varsigma} (\sigma, -i\kappa)^{\alpha'_\varsigma B'_\varsigma} = 8 \quad (\sigma, i\kappa)^{\alpha_\varsigma B_\varsigma} (\sigma, -i\kappa)_{\alpha_\varsigma B_\varsigma} = 8 \quad (1.128)$$

#### 5.1.3 复共轭性

$$[(\sigma, i\kappa)_{\alpha'_\varsigma B'_\varsigma}]^* = (\sigma, -i\kappa)_{\alpha_\varsigma B_\varsigma} \quad [(\sigma, i\kappa)_{\alpha_\varsigma B_\varsigma}]^* = (\sigma, -i\kappa)_{\alpha'_\varsigma B'_\varsigma} \quad (1.129)$$

### 5.2 扩展常数不变张量 $(\sigma_\varsigma, i\kappa)_{ab}^{\alpha_\varsigma}, (\sigma_{-\varsigma}, i\kappa)_{ab}^{\alpha'_\varsigma}$ 的性质

#### 5.2.1 正交性

缩减一对复矢量指标:

$$(\sigma_\varsigma, i\kappa)_{ab}^{\alpha_\varsigma} (\sigma_\varsigma, i\kappa)_{\alpha_\varsigma cd} = -\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + \varsigma\varepsilon_{abcd} - \delta_{ab}\delta_{cd} \quad (1.130)$$

$$(\sigma_\varsigma, i\kappa)_{ab}^{\alpha_\varsigma} (\sigma_\varsigma, -i\kappa)_{\alpha_\varsigma cd} = -\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + \varsigma\varepsilon_{abcd} + \delta_{ab}\delta_{cd} \quad (1.131)$$

$$(\sigma_{-\varsigma}, i\kappa)_{ab}^{\alpha'_\varsigma} (\sigma_{-\varsigma}, i\kappa)_{\alpha'_\varsigma cd} = -\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \varsigma\varepsilon_{abcd} - \delta_{ab}\delta_{cd} \quad (1.132)$$

$$(\sigma_{-\varsigma}, i\kappa)_{ab}^{\alpha'_\varsigma} (\sigma_{-\varsigma}, -i\kappa)_{\alpha'_\varsigma cd} = -\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \varsigma\varepsilon_{abcd} + \delta_{ab}\delta_{cd} \quad (1.133)$$

缩减一对矢量指标:

$$(\sigma_\varsigma, i\kappa)_{\alpha_\varsigma}^{ac} (\sigma_\varsigma, -i\kappa)_{\beta_\varsigma cb} = \delta_{\alpha_\varsigma \beta_\varsigma} \delta^a_b + 2iS_{\alpha_\varsigma \beta_\varsigma}^a_b [\frac{1}{2}\sigma_\varsigma, \kappa] \quad (1.134)$$

$$(\sigma_{-\varsigma}, i\kappa)_{\alpha'_\varsigma}^{ac} (\sigma_{-\varsigma}, -i\kappa)_{\beta'_\varsigma cb} = \delta_{\alpha'_\varsigma \beta'_\varsigma} \delta^a_b + 2iS_{\alpha'_\varsigma \beta'_\varsigma}^a_b [\frac{1}{2}\sigma_{-\varsigma}, \kappa] \quad (1.135)$$



缩减两对矢量指标：

$$(\sigma_{\varsigma}, i\kappa)_{ab}^{\alpha\varsigma} (\sigma_{\varsigma}, -i\kappa)_{\beta\varsigma}^{ab} = -4\eta_{\beta\varsigma}^{\alpha\varsigma} \quad (\sigma_{\varsigma}, i\kappa)_{ab}^{\alpha\varsigma} (\sigma_{\varsigma}, i\kappa)_{\beta\varsigma}^{ab} = -4\delta_{\beta\varsigma}^{\alpha\varsigma} \quad (1.136)$$

$$\text{tr}[(\sigma_{\varsigma}, i\kappa)^{\alpha\varsigma} (\sigma_{\varsigma}, -i\kappa)^{\beta\varsigma}] = 4\delta^{\alpha\varsigma\beta\varsigma} \quad \text{tr}[(\sigma_{\varsigma}, i\kappa)^{\alpha\varsigma} (\sigma_{\varsigma}, i\kappa)^{\beta\varsigma}] = 4\eta^{\alpha\varsigma\beta\varsigma} \quad (1.137)$$

$$(\sigma_{-\varsigma}, i\kappa)_{ab}^{\alpha'\varsigma} (\sigma_{-\varsigma}, -i\kappa)_{\beta'\varsigma}^{ab} = -4\eta_{\beta'\varsigma}^{\alpha'\varsigma} \quad (\sigma_{-\varsigma}, i\kappa)_{ab}^{\alpha'\varsigma} (\sigma_{-\varsigma}, i\kappa)_{\beta'\varsigma}^{ab} = -4\delta_{\beta'\varsigma}^{\alpha'\varsigma} \quad (1.138)$$

$$\text{tr}[(\sigma_{-\varsigma}, i\kappa)^{\alpha'\varsigma} (\sigma_{-\varsigma}, -i\kappa)^{\beta'\varsigma}] = 4\delta^{\alpha'\varsigma\beta'\varsigma} \quad \text{tr}[(\sigma_{-\varsigma}, i\kappa)^{\alpha'\varsigma} (\sigma_{-\varsigma}, i\kappa)^{\beta'\varsigma}] = 4\eta^{\alpha'\varsigma\beta'\varsigma} \quad (1.139)$$

缩减全部指标：

$$(\sigma_{\varsigma}, i\kappa)_{ab}^{\alpha\varsigma} (\sigma_{\varsigma}, -i\kappa)_{\alpha\varsigma}^{ab} = -8 \quad (\sigma_{\varsigma}, i\kappa)_{ab}^{\alpha\varsigma} (\sigma_{\varsigma}, i\kappa)_{\alpha\varsigma}^{ab} = -16 \quad (1.140)$$

$$(\sigma_{-\varsigma}, i\kappa)_{ab}^{\alpha'\varsigma} (\sigma_{-\varsigma}, -i\kappa)_{\alpha'\varsigma}^{ab} = -8 \quad (\sigma_{-\varsigma}, i\kappa)_{ab}^{\alpha'\varsigma} (\sigma_{-\varsigma}, i\kappa)_{\alpha'\varsigma}^{ab} = -16 \quad (1.141)$$

### 5.2.2 恒等性

$$(\sigma_{+}, -i)_{ab}^{\alpha} |_{ab} = (\sigma_{-}, -i)_{ab}^{\alpha} \quad (\sigma_{-}, i)^{\alpha'} |_{ab} = (\sigma_{+}, i)_{ab}^{\alpha'} \quad (1.142)$$

$$(\sigma_{\varsigma}, -i\varsigma)_{ab}^{\alpha\varsigma} |_{ab} = (\sigma_{-\varsigma}, -i\varsigma)_{ab}^{\alpha\varsigma} \quad (\sigma_{-\varsigma}, i\varsigma)_{ab}^{\alpha'\varsigma} |_{ab} = (\sigma_{\varsigma}, i\varsigma)_{ab}^{\alpha'\varsigma} \quad (1.143)$$

### 5.2.3 复共轭性

$$[(\sigma_{\varsigma}, i\kappa)_{ab}^{\alpha\varsigma} \partial^a \hat{\partial}^b]^* = -(\sigma_{-\varsigma}, i\kappa)_{ab}^{\alpha'\varsigma} \partial^a \hat{\partial}^b \quad [(\sigma_{-\varsigma}, i\kappa)_{ab}^{\alpha'\varsigma} \partial^a \hat{\partial}^b]^* = -(\sigma_{\varsigma}, i\kappa)_{ab}^{\alpha\varsigma} \partial^a \hat{\partial}^b \quad (1.144)$$

## 5.3 自旋常数不变张量 $S_{ab}^{k' l' \varsigma}(s, -\varsigma)$ , $S_{abk_{\varsigma}}^{l \varsigma}(s, \varsigma)$ 的性质

### 5.3.1 复合性

$$S_{ab}(s, -\varsigma) = -i[\sigma(s), -\frac{i\varsigma}{2}]_a [\sigma(s), \frac{i\varsigma}{2}]_b \quad S_{ab}(s, \varsigma) = -i[\sigma(s), \frac{i\varsigma}{2}]_a [\sigma(s), -\frac{i\varsigma}{2}]_b \quad (1.145)$$

$$S_{ab}^{k' l' \varsigma}(s, -\varsigma) = i\sigma_{-\varsigma'}^{\alpha' \varsigma} \sigma_{\alpha' \varsigma}^{k' l' \varsigma}(s) \quad S_{abk_{\varsigma}}^{l \varsigma}(s, \varsigma) = i\sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma k_{\varsigma}}^{l \varsigma}(s) \quad (1.146)$$

$$\sigma_{\alpha' \varsigma}^{k' l' \varsigma}(s) = \frac{i}{4} \sigma_{-\varsigma \alpha' \varsigma}^{ab} S_{ab}^{k' l' \varsigma}(s, -\varsigma) \quad \sigma_{\alpha\varsigma k_{\varsigma}}^{l \varsigma}(s) = \frac{i}{4} \sigma_{\varsigma \alpha\varsigma}^{ab} S_{abk_{\varsigma}}^{l \varsigma}(s, \varsigma) \quad (1.147)$$

### 5.3.2 正交性

缩减两对矢量指标：

$$\begin{cases} S_{ab}^{k' l' \varsigma}(s, -\varsigma) S^{abm' n' \varsigma}(s, -\varsigma) = 4\sigma_{\alpha' \varsigma}^{k' l' \varsigma}(s) \sigma_{\alpha' \varsigma}^{m' n' \varsigma}(s) \\ S_{abk_{\varsigma}}^{l \varsigma}(s, \varsigma) S^{abm_{\varsigma} n_{\varsigma}}(s, \varsigma) = 4\sigma_{\alpha\varsigma k_{\varsigma}}^{l \varsigma}(s) \sigma_{\alpha\varsigma m_{\varsigma} n_{\varsigma}}(s) \end{cases} \quad (1.148)$$

缩减两对旋量指标：

$$\begin{cases} S_{ab}^{k' l' \varsigma}(s, -\varsigma) S_{cdl' k' \varsigma}(s, -\varsigma) = -\frac{2}{3} s(s + \frac{1}{2})(s + 1) \sigma_{-\varsigma ab}^{\alpha' \varsigma} \sigma_{-\varsigma \alpha' cd} \\ S_{abk_{\varsigma}}^{l \varsigma}(s, \varsigma) S_{cdl_{\varsigma} k_{\varsigma}}(s, \varsigma) = -\frac{2}{3} s(s + \frac{1}{2})(s + 1) \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\varsigma \alpha\varsigma cd} \end{cases} \quad (1.149)$$

缩减一对矢量和一对旋量指标：

$$\begin{cases} S_{ac}^{k' l' m' \varsigma}(s, -\varsigma) S^{cbm' l' \varsigma}(s, -\varsigma) = -s(s + 1) \delta_a^b \delta^{k' l' \varsigma} \\ S_{ack_{\varsigma}}^{m_{\varsigma}}(s, \varsigma) S^{cbm_{\varsigma} l_{\varsigma}}(s, \varsigma) = -s(s + 1) \delta_a^b \delta_{k_{\varsigma}}^{l_{\varsigma}} \end{cases} \quad (1.150)$$

缩减三对指标：

$$\begin{cases} S_{ac}^{k' l' \varsigma}(s, -\varsigma) S^{cbl' k' \varsigma}(s, -\varsigma) = -2s(s + 1) \delta_a^b \\ S_{ack_{\varsigma}}^{l_{\varsigma}}(s, \varsigma) S^{cbl_{\varsigma} k_{\varsigma}}(s, \varsigma) = -2s(s + 1) \delta_a^b \end{cases} \quad (1.151)$$

$$\begin{cases} S_{ab}^{k' l' m' \varsigma}(s, -\varsigma) S^{abm' l' \varsigma}(s, -\varsigma) = 4s(s + 1) \delta^{k' l' \varsigma} \\ S_{abk_{\varsigma}}^{m_{\varsigma}}(s, \varsigma) S^{abm_{\varsigma} l_{\varsigma}}(s, \varsigma) = 4s(s + 1) \delta_{k_{\varsigma}}^{l_{\varsigma}} \end{cases} \quad (1.152)$$

缩减全部指标：

$$\begin{cases} S_{ab}^{k'_\zeta l'_\zeta}(s, -\varsigma) S^{abl'_\zeta k'_\zeta}(s, -\varsigma) = 8s(s + \frac{1}{2})(s + 1) \\ S_{abk_\zeta}^{l_\zeta}(s, \varsigma) S^{abl_\zeta k_\zeta}(s, \varsigma) = 8s(s + \frac{1}{2})(s + 1) \end{cases} \quad (1.153)$$

### 5.3.3 对偶性

$$S_{ab}^{k'_\zeta l'_\zeta}(s, -\varsigma) = \varsigma * S_{ab}^{k'_\zeta l'_\zeta}(s, -\varsigma) \quad S_{abk_\zeta}^{l_\zeta}(s, \varsigma) = -\varsigma * S_{abk_\zeta}^{l_\zeta}(s, \varsigma) \quad (1.154)$$

### 5.3.4 复共轭性

$$[S_{ab}^{k'_\zeta l'_\zeta}(s, -\varsigma) \partial^a \hat{\partial}^b]^* = S_{abl_\zeta}^{k_\zeta}(s, \varsigma) \partial^a \hat{\partial}^b \quad [S_{abk_\zeta}^{l_\zeta}(s, \varsigma) \partial^a \hat{\partial}^b]^* = S_{ab}^{l'_\zeta k'_\zeta}(s, -\varsigma) \partial^a \hat{\partial}^b \quad (1.155)$$

## 5.4 自旋常数不变张量 $S_{ab}^{A'_\zeta B'_\zeta}, S_{abA_\zeta B_\zeta}$ 的性质

### 5.4.1 复合性

$$S_{ab}(\frac{1}{2}, -\varsigma) = -\frac{i}{4}(\sigma, -i\varsigma)_{[a}(\sigma, i\varsigma)_{b]} \quad S_{ab}(\frac{1}{2}, \varsigma) = -\frac{i}{4}(\sigma, i\varsigma)_{[a}(\sigma, -i\varsigma)_{b]} \quad (1.156)$$

$$S_{ab}^{A'_\zeta B'_\zeta} \equiv S_{ab}^{A'_\zeta B'_\zeta}(-\varsigma) \equiv S_{ab}^{A'_\zeta B'_\zeta}(\frac{1}{2}, -\varsigma) \quad S_{abA_\zeta B_\zeta} \equiv S_{abA_\zeta B_\zeta}(\varsigma) \equiv S_{abA_\zeta B_\zeta}(\frac{1}{2}, \varsigma) \quad (1.157)$$

$$S_{ab}^{A'_\zeta B'_\zeta} = \frac{i}{2} \sigma_{-\varsigma ab}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \quad S_{abA_\zeta B_\zeta} = \frac{i}{2} \sigma_{\varsigma ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \quad (1.158)$$

$$\sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} = \frac{i}{2} \sigma_{-\varsigma \alpha'_\zeta}^{ab} S_{ab}^{A'_\zeta B'_\zeta} \quad \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta}(s) = \frac{i}{2} \sigma_{\varsigma \alpha_\zeta}^{ab} S_{abA_\zeta B_\zeta} \quad (1.159)$$

### 5.4.2 正交性

缩减两对矢量指标：

$$S_{ab}^{A'_\zeta B'_\zeta} S^{abC'_\zeta D'_\zeta} = \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} \quad S_{abA_\zeta B_\zeta} S^{abC_\zeta D_\zeta} = \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} \quad (1.160)$$

缩减两对旋量指标：

$$S_{ab}^{A'_\zeta B'_\zeta} S_{cd}^{B'_\zeta A'_\zeta} = -\frac{1}{2} \sigma_{-\varsigma ab}^{\alpha'_\zeta} \sigma_{-\varsigma \alpha'_\zeta cd} \quad S_{abA_\zeta B_\zeta} S_{cdB_\zeta A_\zeta} = -\frac{1}{2} \sigma_{\varsigma ab}^{\alpha_\zeta} \sigma_{\varsigma \alpha_\zeta cd} \quad (1.161)$$

缩减一对矢量和一对旋量指标：

$$S_{ac}^{A'_\zeta C'_\zeta} S^{cbC'_\zeta B'_\zeta} = -\frac{3}{4} \delta_a^b \delta_{A'_\zeta B'_\zeta} \quad S_{acA_\zeta C_\zeta} S^{cbC_\zeta B_\zeta} = -\frac{3}{4} \delta_a^b \delta_{A_\zeta B_\zeta} \quad (1.162)$$

缩减三对指标：

$$S_{ac}^{A'_\zeta B'_\zeta} S^{cbB'_\zeta A'_\zeta} = -\frac{3}{2} \delta_a^b \quad S_{acA_\zeta C_\zeta} S^{cbC_\zeta B_\zeta} = -\frac{3}{2} \delta_a^b \quad (1.163)$$

$$S_{ab}^{A'_\zeta C'_\zeta} S^{abC'_\zeta B'_\zeta} = 3\delta_{A'_\zeta B'_\zeta} \quad S_{abA_\zeta C_\zeta} S^{abC_\zeta B_\zeta} = 3\delta_{A_\zeta B_\zeta} \quad (1.164)$$

缩减全部指标：

$$S_{ab}^{A'_\zeta B'_\zeta} S^{abB'_\zeta A'_\zeta} = 6 \quad S_{abA_\zeta B_\zeta} S^{abB_\zeta A_\zeta} = 6 \quad (1.165)$$

### 5.4.3 对偶性

$$S_{ab}^{A'_\zeta B'_\zeta} = \varsigma * S_{ab}^{A'_\zeta B'_\zeta} \quad S_{abA_\zeta B_\zeta} = -\varsigma * S_{abA_\zeta B_\zeta} \quad (1.166)$$

### 5.4.4 复共轭性

$$[S_{ab}^{A'_\zeta B'_\zeta} \partial^a \hat{\partial}^b]^* = S_{abB_\zeta}^{A_\zeta} \partial^a \hat{\partial}^b \quad [S_{abA_\zeta B_\zeta} \partial^a \hat{\partial}^b]^* = S_{ab}^{B'_\zeta A'_\zeta} \partial^a \hat{\partial}^b \quad (1.167)$$

## 5.5 几个基本常数不变张量之间的关系

$$\begin{cases} S_{ab}^{k'_\zeta l'_\zeta}(s, -\varsigma) = i \sigma_{-\varsigma ab}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{k'_\zeta l'_\zeta}(s) \\ S_{abk_\zeta}^{l_\zeta}(s, \varsigma) = i \sigma_{\varsigma ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta}^{l_\zeta k_\zeta}(s) \end{cases} \quad \begin{cases} (\sigma, -i\varsigma)_a^{A'_\zeta A_\zeta} (\sigma, i\varsigma)_{bA_\zeta B'_\zeta} = \delta_{ab} \delta_{A'_\zeta B'_\zeta} + 2i S_{ab}^{A'_\zeta B'_\zeta} \\ (\sigma, i\varsigma)_{aA_\zeta A'_\zeta} (\sigma, -i\varsigma)_{bA'_\zeta B_\zeta} = \delta_{ab} \delta_{A_\zeta B_\zeta} + 2i S_{abA_\zeta B_\zeta} \end{cases} \quad (1.168)$$

$$\begin{cases} S_{ab}^{A'_\zeta B'_\zeta} = -\frac{i}{4}(\sigma, -i\varsigma)_{[a}^{A'_\zeta A_\zeta} (\sigma, i\varsigma)_{b]A_\zeta B'_\zeta} \\ \delta_{ab} \delta_{A'_\zeta B'_\zeta} = \frac{1}{2}(\sigma, -i\varsigma)_{\{a}^{A'_\zeta A_\zeta} (\sigma, i\varsigma)_{b\}A_\zeta B'_\zeta} \end{cases} \quad \begin{cases} S_{abA_\zeta B_\zeta} = -\frac{i}{4}(\sigma, i\varsigma)_{[aA_\zeta A'_\zeta} (\sigma, -i\varsigma)_{b]A'_\zeta B_\zeta} \\ \delta_{ab} \delta_{A_\zeta B_\zeta} = \frac{1}{2}(\sigma, i\varsigma)_{\{aA_\zeta A'_\zeta} (\sigma, -i\varsigma)_{b\}A'_\zeta B_\zeta} \end{cases} \quad (1.169)$$

## 5.6 矢量自旋张量 $S_{abcd}$ 和反对称张量 $\varepsilon_{abcd}$ 的性质

**定理5.6.1.**  $S_{abcd} = -\frac{1}{2}(\sigma_{-ab}^{\alpha'}\sigma_{-\alpha'cd} + \sigma_{+ab}^{\alpha}\sigma_{+\alpha cd}) = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} = \delta_{a[c}\delta_{d]b} = \delta_{c[a}\delta_{b]d}$ ,  $\vec{S}_{ab} := -iS_{ab|cd}$

**定理5.6.2.**  $\varepsilon_{abcd} = -\frac{1}{2}(\sigma_{-ab}^{\alpha'}\sigma_{-\alpha'cd} - \sigma_{+ab}^{\alpha}\sigma_{+\alpha cd})$

对以上两个定理分情况展开即可证明，这两个定理是以下一些推论的基础与前提。

结合(1.250), (1.274), (1.275)式可得Penrose对应记法：

**推论5.6.1.**  $S_{abcd} \stackrel{P}{=} \varepsilon_{AC} \varepsilon_{BD} \varepsilon_{A'C'} \varepsilon_{B'D'} - \varepsilon_{AD} \varepsilon_{BC} \varepsilon_{A'D'} \varepsilon_{B'C'}$

**推论5.6.2.**  $\varepsilon_{abcd} = -\varepsilon_{acbd} \stackrel{P}{=} \varepsilon_{AC} \varepsilon_{BD} \varepsilon_{A'D'} \varepsilon_{B'C'} - \varepsilon_{AD} \varepsilon_{BC} \varepsilon_{A'C'} \varepsilon_{B'D'}$

$$\begin{cases} S_{(*ab)(*cd)} = S_{abcd} & \begin{cases} S_{(*ab)cd} = S_{ab(*cd)} = \varepsilon_{abcd} \\ \varepsilon_{(*ab)(*cd)} = \varepsilon_{abcd} & \begin{cases} \varepsilon_{(*ab)cd} = \varepsilon_{ab(*cd)} = S_{abcd} \end{cases} \end{cases} \end{cases} \quad (1.170)$$

$$S_{abcd} = S_{cdab}, S_{abcd} = -S_{bacd}, S_{abcd} = S_{abd c}, S_{abcd} = \frac{1}{2}S_{abef}S^{ef}{}_{cd}, \vartheta_{ab} = \frac{1}{2}S_{abcd}\vartheta^{cd} \quad (1.171)$$

$$\begin{cases} \sigma_{-ab}^{\alpha'}\sigma_{-\alpha'cd} = -(S_{abcd} + \varepsilon_{abcd}) = (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \varepsilon_{abcd}) \\ \sigma_{+ab}^{\alpha}\sigma_{+\alpha cd} = -(S_{abcd} - \varepsilon_{abcd}) = (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + \varepsilon_{abcd}) \end{cases} \quad (1.172)$$

$$\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha cd} = -(S_{abcd} - \zeta\varepsilon_{abcd}) = (-\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + \zeta\varepsilon_{abcd}) \quad (1.173)$$

**推论5.6.3.**  $\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{-\zeta\alpha cd} = -\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc} + \zeta\varepsilon_{abc'd'}\eta_c^c'\eta_d^{d'}$

**证明:**  $\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{-\zeta\alpha cd}$

$$= \sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha c'd'}\eta_c^c'\eta_d^{d'}$$

$$= -(S_{abc'd'} - \zeta\varepsilon_{abc'd'})\eta_c^c'\eta_d^{d'}$$

$$= (-\delta_{ac'}\delta_{bd'} + \delta_{ad'}\delta_{bc'} + \zeta\varepsilon_{abc'd'})\eta_c^c'\eta_d^{d'}$$

$$= -\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc} + \zeta\varepsilon_{abc'd'}\eta_c^c'\eta_d^{d'} \quad \square$$

## 5.7 自旋张量 $S_{ab}(\zeta)$ 的性质

$$S_{ab}(\zeta) = \frac{i}{2}\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\alpha\zeta} = -\frac{i}{4}(\sigma, i\zeta)_{[a}(\sigma, -i\zeta)_{b]} \quad \delta_{ab} = \frac{1}{2}(\sigma, i\zeta)_{\{a}(\sigma, -i\zeta)_{b\}} \quad (1.174)$$

$$\begin{cases} i[S_{ab}(\zeta), S_{cd}(\zeta)] = \delta_{a[c}S_{d]b}(\zeta) + S_{a[c}(\zeta)\delta_{d]b} = -\delta_{c[a}S_{b]d}(\zeta) - S_{c[a}(\zeta)\delta_{b]d} \\ \{S_{ab}(\zeta), S_{cd}(\zeta)\} = -\frac{1}{2}\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha cd} = \frac{1}{2}(S_{abcd} - \zeta\varepsilon_{abcd}) \end{cases} \quad (1.175)$$

$$\varepsilon_{abcd} = \zeta 2tr[S_{ab}(-\zeta)S_{cd}(-\zeta) - S_{ab}(\zeta)S_{cd}(\zeta)] \quad S_{ab}(\zeta) = -\zeta * S_{ab}(\zeta) \quad (1.176)$$

$$\begin{cases} 2iS_{ab}(\zeta)(\sigma, i\zeta)_c = (\sigma, i\zeta)_{[a}\delta_{b]c} + \zeta\varepsilon_{abcd}(\sigma, i\zeta)^d \\ 2i(\sigma, -i\zeta)_c S_{ab}(\zeta) = \delta_{c[a}(\sigma, -i\zeta)_{b]} - \zeta\varepsilon_{abcd}(\sigma, -i\zeta)^d \end{cases} \quad (1.177)$$

## 5.8 Dirac自旋张量 $S_{ab}(e, \zeta)$ 的性质 [19]

$$[\gamma_a(\zeta), \gamma_5(\zeta)] = [(\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x), \zeta I \otimes \sigma_z] \quad (1.178)$$

$$S_{ab}(e, \zeta) = -\frac{i}{4}[\gamma_a(\zeta), \gamma_b(\zeta)] = S_{ab}(\zeta) \oplus S_{ab}(-\zeta) \quad \delta_{ab} = \frac{1}{2}\{\gamma_a(\zeta), \gamma_b(\zeta)\} \quad (1.179)$$

$$\begin{cases} i[S_{ab}(e, \zeta), S_{cd}(e, \zeta)] = \delta_{a[c}S_{d]b}(e, \zeta) + S_{a[c}(e, \zeta)\delta_{d]b} = -\delta_{c[a}S_{b]d}(e, \zeta) - S_{c[a}(e, \zeta)\delta_{b]d} \\ \{S_{ab}(e, \zeta), S_{cd}(e, \zeta)\} = -\frac{1}{2}\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha'cd} \oplus \frac{1}{2}\sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta\alpha'cd} = \frac{1}{2}[S_{abcd} - \gamma_5(\zeta)\varepsilon_{abcd}] \end{cases} \quad (1.180)$$

$$[S_{ab}(e, \zeta), \gamma_c(\zeta)] = -i\gamma_{[a}\delta_{b]c} \quad \{S_{ab}(e, \zeta), \gamma_c(\zeta)\} = -i\varepsilon_{abcd}\gamma_5(\zeta)\gamma^d(\zeta) \quad (1.181)$$

$$S_{ab}(e, \zeta) = -\gamma_5(\zeta) * S_{ab}(e, \zeta) \quad (1.182)$$

### 5.9 常数不变张量 $\varepsilon_{abcd}, \gamma_a(\varsigma)$ 之间的关系 [19]

$$\varepsilon_{abcd}\gamma^a(\varsigma)\gamma^b(\varsigma)\gamma^c(\varsigma)\gamma^d(\varsigma) = 24\gamma_5(\varsigma) \quad (1.183)$$

$$\varepsilon_{abcd}\gamma^b(\varsigma)\gamma^c(\varsigma)\gamma^d(\varsigma) = -6\gamma_5(\varsigma)\gamma_a(\varsigma) \quad (1.184)$$

$$\varepsilon_{abcd}\gamma^c(\varsigma)\gamma^d(\varsigma) = -4\gamma_5(\varsigma)iS_{ab}(e, \varsigma) \quad (1.185)$$

$$\varepsilon_{abcd}\gamma^d(\varsigma) = \gamma_5(\varsigma)\{\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma) - [\delta_{ab}\gamma_c(\varsigma) + \gamma_{[a}(\varsigma)\delta_{b]c}]\} \quad (1.186)$$

$$\varepsilon_{abcd} = \gamma_5(\varsigma)\{\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma) \quad (1.187)$$

$$- [\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b} + 2i\delta_{ab}S_{cd}(e, \varsigma) + 2iS_{ab}(e, \varsigma)\delta_{cd} + 2i\delta_{a[c}S_{d]b}(e, \varsigma) - 2iS_{a[c}(e, \varsigma)\delta_{d]b}]\} \quad (1.188)$$

### 5.10 迹 $tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\cdots]$ 的性质

$$tr[\gamma_a(\varsigma)] = 0 \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \quad (1.189)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \quad (1.190)$$

$$tr[\gamma_5(\varsigma)] = 0 \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \quad (1.191)$$

$$tr[S_{ab}(e, \varsigma)] = 0 \quad tr[\gamma_c(\varsigma)S_{ab}(e, \varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)] = 0 \quad (1.192)$$

$$tr[\gamma_5(\varsigma)S_{ab}(e, \varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_c(\varsigma)S_{ab}(e, \varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \quad (1.193)$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)] = 4\delta_{ab} \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 4[\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b}] \quad (1.194)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 4\varepsilon_{abcd} \quad (1.195)$$

$$tr[S_{ab}(e, \varsigma)S_{cd}(e, \varsigma)] = S_{abcd} = \delta_{ac}\delta_{db} - \delta_{ad}\delta_{cb} \quad tr[\gamma_5 S_{ab}(e, \varsigma)S_{cd}(e, \varsigma)] = -\varepsilon_{abcd} \quad (1.196)$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)S_{cd}(e, \varsigma)] = 2iS_{abcd} \quad tr[\gamma_5\gamma_a(\varsigma)\gamma_b(\varsigma)S_{cd}(e, \varsigma)] = -2i\varepsilon_{abcd} \quad (1.197)$$

$$tr[S_{ab}(e, \varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 2iS_{abcd} \quad tr[\gamma_5 S_{ab}(e, \varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = -2i\varepsilon_{abcd} \quad (1.198)$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = 2i\{\delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef}\} \quad (1.199)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = -2i\{\delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \quad (1.200)$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)\gamma_f(\varsigma)] = 4\{(\delta_{ab}\delta_{cd} - S_{abcd})\delta_{ef} - (\delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef})\} \quad (1.201)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)\gamma_f(\varsigma)] = 4\{\varepsilon_{abcd}\delta_{ef} + \delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \quad (1.202)$$

$$tr[\gamma_a(\varsigma)S_{bc}(e, \varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = \delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef} - \delta_{bc}S_{adef} \quad (1.203)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)S_{bc}(e, \varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = \delta_{bc}\varepsilon_{adef} - \{\delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \quad (1.204)$$

$$tr[\gamma_a(\varsigma)S_{bc}(e, \varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = \delta_{ad}S_{bcef} + \delta_{a[b}S_{c]def} + \delta_{d[b}S_{c]aef} \quad (1.205)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)S_{bc}(e, \varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = -\{\delta_{ad}\varepsilon_{bcef} + \delta_{a[b}\varepsilon_{c]def} + \delta_{d[b}\varepsilon_{c]aef}\} \quad (1.206)$$

### 5.11 常数不变张量 $\varepsilon_{abcd}, (\sigma, i\varsigma)_a$ 之间的关系

$$\varepsilon_{abcd}(\sigma, i\varsigma)^a(\sigma, -i\varsigma)^b(\sigma, i\varsigma)^c(\sigma, -i\varsigma)^d = 24\varsigma \quad (1.207)$$

$$\varepsilon_{abcd}(\sigma, i\varsigma)^b(\sigma, -i\varsigma)^c(\sigma, i\varsigma)^d = -6\varsigma(\sigma, i\varsigma)^a \quad (1.208)$$

$$\varepsilon_{abcd}(\sigma, i\varsigma)^c(\sigma, -i\varsigma)^d = -4i\varsigma S_{ab}(\varsigma) \quad (1.209)$$

$$\varepsilon_{abcd}(\sigma, i\varsigma)^d = \varsigma\{(\sigma, i\varsigma)_a(\sigma, -i\varsigma)_b(\sigma, i\varsigma)_c - [\delta_{ab}(\sigma, i\varsigma)_c + (\sigma, i\varsigma)_{[a}\delta_{b]c}]\} \quad (1.210)$$

$$\varepsilon_{abcd} = \varsigma\{(\sigma, i\varsigma)_a(\sigma, -i\varsigma)_b(\sigma, i\varsigma)_c(\sigma, -i\varsigma)_d \quad (1.211)$$

$$- [\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b} + 2i\delta_{ab}S_{cd}(\varsigma) + 2iS_{ab}(\varsigma)\delta_{cd} + 2i\delta_{a[c}S_{d]b}(\varsigma) + 2iS_{a[c}(\varsigma)\delta_{d]b}]\} \quad (1.212)$$

5.12 常数不变张量 $\varepsilon_{abcd}, \delta_{ab}$ 之间的关系

$$\begin{aligned}
\varepsilon_{abcd}\varepsilon_{efgh} &= (\delta_{ae}\delta_{bf}\delta_{cg}\delta_{dh} - \delta_{ah}\delta_{be}\delta_{cf}\delta_{dg} + \delta_{ag}\delta_{bh}\delta_{ce}\delta_{df} - \delta_{af}\delta_{bg}\delta_{ch}\delta_{de}) \\
&\quad - (\delta_{ae}\delta_{bf}\delta_{ch}\delta_{dg} - \delta_{ag}\delta_{be}\delta_{cf}\delta_{dh} + \delta_{ah}\delta_{bg}\delta_{ce}\delta_{df} - \delta_{af}\delta_{bh}\delta_{cg}\delta_{de}) \\
&\quad + (\delta_{ae}\delta_{bg}\delta_{ch}\delta_{df} - \delta_{af}\delta_{be}\delta_{cg}\delta_{dh} + \delta_{ah}\delta_{bf}\delta_{ce}\delta_{dg} - \delta_{ag}\delta_{bh}\delta_{cf}\delta_{de}) \\
&\quad - (\delta_{ae}\delta_{bg}\delta_{cf}\delta_{dh} - \delta_{ah}\delta_{be}\delta_{cg}\delta_{df} + \delta_{af}\delta_{bh}\delta_{ce}\delta_{dg} - \delta_{ag}\delta_{bf}\delta_{ch}\delta_{de}) \\
&\quad + (\delta_{ae}\delta_{bh}\delta_{cf}\delta_{dg} - \delta_{ag}\delta_{be}\delta_{ch}\delta_{df} + \delta_{af}\delta_{bg}\delta_{ce}\delta_{dh} - \delta_{ah}\delta_{bf}\delta_{cg}\delta_{de}) \\
&\quad - (\delta_{ae}\delta_{bh}\delta_{cg}\delta_{df} - \delta_{af}\delta_{be}\delta_{ch}\delta_{dg} + \delta_{ag}\delta_{bf}\delta_{ce}\delta_{dh} - \delta_{ah}\delta_{bg}\delta_{cf}\delta_{de})
\end{aligned} \tag{1.213}$$

$$\begin{aligned}
\varepsilon_{abcd}\varepsilon_{efgh}\eta^{dh} &= (\delta_{ae}\delta_{bf}\delta_{cg}2 - \eta_{ag}\delta_{be}\delta_{cf} + \delta_{ag}\eta_{bf}\delta_{ce} - \delta_{af}\delta_{bg}\eta_{ce}) \\
&\quad - (\delta_{ae}\delta_{bf}\eta_{cg} - \delta_{ag}\delta_{be}\delta_{cf}2 + \eta_{af}\delta_{bg}\delta_{ce} - \delta_{af}\eta_{be}\delta_{cg}) \\
&\quad + (\delta_{ae}\delta_{bg}\eta_{cf} - \delta_{af}\delta_{be}\delta_{cg}2 + \eta_{ag}\delta_{bf}\delta_{ce} - \delta_{ag}\eta_{be}\delta_{cf}) \\
&\quad - (\delta_{ae}\delta_{bg}\delta_{cf}2 - \eta_{af}\delta_{be}\delta_{cg} + \delta_{af}\eta_{bg}\delta_{ce} - \delta_{ag}\delta_{bf}\eta_{ce}) \\
&\quad + (\delta_{ae}\eta_{bg}\delta_{cf} - \delta_{ag}\delta_{be}\eta_{cf} + \delta_{af}\delta_{bg}\delta_{ce}2 - \eta_{ae}\delta_{bf}\delta_{cg}) \\
&\quad - (\delta_{ae}\eta_{bf}\delta_{cg} - \delta_{af}\delta_{be}\eta_{cg} + \delta_{ag}\delta_{bf}\delta_{ce}2 - \eta_{ae}\delta_{bg}\delta_{cf})
\end{aligned} \tag{1.214}$$

$$\begin{aligned}
\varepsilon_{abcd}\varepsilon_{a'b'c'd'}\eta^{dd'} &= (\delta_{aa'}\delta_{bb'}\delta_{cc'}2 - \eta_{ac'}\delta_{ba'}\delta_{cb'} + \delta_{ac'}\eta_{bb'}\delta_{ca'} - \delta_{ab'}\delta_{bc'}\eta_{ca'}) \\
&\quad - (\delta_{aa'}\delta_{bb'}\eta_{cc'} - \delta_{ac'}\delta_{ba'}\delta_{cb'}2 + \eta_{ab'}\delta_{bc'}\delta_{ca'} - \delta_{ab'}\eta_{ba'}\delta_{cc'}) \\
&\quad + (\delta_{aa'}\delta_{bc'}\eta_{cb'} - \delta_{ab'}\delta_{ba'}\delta_{cc'}2 + \eta_{ac'}\delta_{bb'}\delta_{ca'} - \delta_{ac'}\eta_{ba'}\delta_{cb'}) \\
&\quad - (\delta_{aa'}\delta_{bc'}\delta_{cb'}2 - \eta_{ab'}\delta_{ba'}\delta_{cc'} + \delta_{ab'}\eta_{bc'}\delta_{ca'} - \delta_{ac'}\delta_{bb'}\eta_{ca'}) \\
&\quad + (\delta_{aa'}\eta_{bc'}\delta_{cb'} - \delta_{ac'}\delta_{ba'}\eta_{cb'} + \delta_{ab'}\delta_{bc'}\delta_{ca'}2 - \eta_{aa'}\delta_{bb'}\delta_{cc'}) \\
&\quad - (\delta_{aa'}\eta_{bb'}\delta_{cc'} - \delta_{ab'}\delta_{ba'}\eta_{cc'} + \delta_{ac'}\delta_{bb'}\delta_{ca'}2 - \eta_{aa'}\delta_{bc'}\delta_{cb'})
\end{aligned} \tag{1.215}$$

$$\begin{aligned}
\varepsilon_{abcd}\varepsilon_{a'b'c'd'}\eta^{dd'}\partial^c\partial^{+c'} &= (\delta_{aa'}\delta_{bb'}\delta_{cc'}2 - \eta_{ac'}\delta_{ba'}\delta_{cb'} + \delta_{ac'}\eta_{bb'}\delta_{ca'} - \delta_{ab'}\delta_{bc'}\eta_{ca'}) \\
&\quad - (\delta_{aa'}\delta_{bb'}\eta_{cc'} - \delta_{ac'}\delta_{ba'}\delta_{cb'}2 + \eta_{ab'}\delta_{bc'}\delta_{ca'} - \delta_{ab'}\eta_{ba'}\delta_{cc'}) \\
&\quad + (\delta_{aa'}\delta_{bc'}\eta_{cb'} - \delta_{ab'}\delta_{ba'}\delta_{cc'}2 + \eta_{ac'}\delta_{bb'}\delta_{ca'} - \delta_{ac'}\eta_{ba'}\delta_{cb'}) \\
&\quad - (\delta_{aa'}\delta_{bc'}\delta_{cb'}2 - \eta_{ab'}\delta_{ba'}\delta_{cc'} + \delta_{ab'}\eta_{bc'}\delta_{ca'} - \delta_{ac'}\delta_{bb'}\eta_{ca'}) \\
&\quad + (\delta_{aa'}\eta_{bc'}\delta_{cb'} - \delta_{ac'}\delta_{ba'}\eta_{cb'} + \delta_{ab'}\delta_{bc'}\delta_{ca'}2 - \eta_{aa'}\delta_{bb'}\delta_{cc'}) \\
&\quad - (\delta_{aa'}\eta_{bb'}\delta_{cc'} - \delta_{ab'}\delta_{ba'}\eta_{cc'} + \delta_{ac'}\delta_{bb'}\delta_{ca'}2 - \eta_{aa'}\delta_{bc'}\delta_{cb'})
\end{aligned} \tag{1.216}$$

$$\begin{aligned}
\varepsilon_{abcd}\varepsilon_{efgh}\delta^{de} &= (\delta_{ah}\delta_{bf}\delta_{cg} - \delta_{ah}\delta_{bg}\delta_{cf} + \delta_{ag}\delta_{bh}\delta_{cf} - 4\delta_{af}\delta_{bg}\delta_{ch}) \\
&\quad - (\delta_{ag}\delta_{bf}\delta_{ch} - \delta_{ag}\delta_{bh}\delta_{cf} + \delta_{ah}\delta_{bg}\delta_{cf} - 4\delta_{af}\delta_{bh}\delta_{cg}) \\
&\quad + (\delta_{af}\delta_{bg}\delta_{ch} - \delta_{af}\delta_{bh}\delta_{cg} + \delta_{ah}\delta_{bf}\delta_{cg} - 4\delta_{ag}\delta_{bh}\delta_{cf}) \\
&\quad - (\delta_{ah}\delta_{bg}\delta_{cf} - \delta_{ah}\delta_{bf}\delta_{cg} + \delta_{af}\delta_{bh}\delta_{cg} - 4\delta_{ag}\delta_{bf}\delta_{ch}) \\
&\quad + (\delta_{ag}\delta_{bh}\delta_{cf} - \delta_{ag}\delta_{bf}\delta_{ch} + \delta_{af}\delta_{bg}\delta_{ch} - 4\delta_{ah}\delta_{bf}\delta_{cg}) \\
&\quad - (\delta_{af}\delta_{bh}\delta_{cg} - \delta_{af}\delta_{bg}\delta_{ch} + \delta_{ag}\delta_{bf}\delta_{ch} - 4\delta_{ah}\delta_{bg}\delta_{cf})
\end{aligned} \tag{1.217}$$

$$\varepsilon_{abcd}\varepsilon_{efgh}\delta^{de} = -(\delta_{af}\delta_{bg}\delta_{ch} - \delta_{af}\delta_{bh}\delta_{cg} + \delta_{ag}\delta_{bh}\delta_{cf} - \delta_{ag}\delta_{bf}\delta_{ch} + \delta_{ah}\delta_{bf}\delta_{cg} - \delta_{ah}\delta_{bg}\delta_{cf}) \tag{1.218}$$

$$\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} \quad (1.219)$$

$$\varepsilon_{ijk}\varepsilon^{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \varepsilon_{ijk}\varepsilon^{jkl} = 2\delta_{il} \quad (1.220)$$

$$\varepsilon_{A_\zeta B_\zeta} \varepsilon_{C_\zeta D_\zeta} = \delta_{A_\zeta C_\zeta} \delta_{B_\zeta D_\zeta} - \delta_{A_\zeta D_\zeta} \delta_{B_\zeta C_\zeta} \quad (1.221)$$

$$\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \varepsilon^{\gamma_\zeta \rho_\zeta \sigma_\zeta} = \delta_{\alpha_\zeta \rho_\zeta} \delta_{\beta_\zeta \sigma_\zeta} - \delta_{\alpha_\zeta \sigma_\zeta} \delta_{\beta_\zeta \rho_\zeta} \quad \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \varepsilon^{\beta_\zeta \gamma_\zeta \rho_\zeta} = 2\delta_{\alpha_\zeta \rho_\zeta} \quad (1.222)$$

### 5.13 常数不变张量 $\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta}, \sigma_{\alpha_\zeta}$ 之间的关系

$$\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \equiv \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta 4} \quad (1.223)$$

$$\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} = -i(\sigma_{\alpha_\zeta} \sigma_{\beta_\zeta} \sigma_{\gamma_\zeta} - \delta_{\beta_\zeta \gamma_\zeta} \sigma_{\alpha_\zeta} + \delta_{\gamma_\zeta \alpha_\zeta} \sigma_{\beta_\zeta} - \delta_{\alpha_\zeta \beta_\zeta} \sigma_{\gamma_\zeta}) \quad (1.224)$$

$$\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \sigma^{\gamma_\zeta} = -i(\sigma_{\alpha_\zeta} \sigma_{\beta_\zeta} - \delta_{\alpha_\zeta \beta_\zeta}) = -\frac{1}{2}i[\sigma_{\alpha_\zeta}, \sigma_{\beta_\zeta}] \quad (1.225)$$

$$\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \sigma^{\beta_\zeta} \sigma^{\gamma_\zeta} = 2i\sigma_{\alpha_\zeta} \quad \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \sigma^{\alpha_\zeta} \sigma^{\beta_\zeta} \sigma^{\gamma_\zeta} = 6i \quad (1.226)$$

$$2S_{\alpha_\zeta \beta_\zeta} \sigma_{\gamma_\zeta} = -i\sigma_{[\alpha_\zeta} \delta_{\beta_\zeta] \gamma_\zeta} + \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \quad 2\sigma_{\gamma_\zeta} S_{\alpha_\zeta \beta_\zeta} = -i\delta_{\gamma_\zeta [\alpha_\zeta} \sigma_{\beta_\zeta]} + \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \quad (1.227)$$

$$[S_{\alpha_\zeta \beta_\zeta}, \sigma_{\gamma_\zeta}] = -i\sigma_{[\alpha_\zeta} \delta_{\beta_\zeta] \gamma_\zeta} \quad \{S_{\alpha_\zeta \beta_\zeta}, \sigma_{\gamma_\zeta}\} = \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \quad (1.228)$$

### 5.14 常数不变张量 $\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta}, \varepsilon_{abcd}$ 之间的关系

$$\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} \equiv \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta 4} \quad (1.229)$$

$$\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta d} A^d \equiv \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} A^4 \quad (1.230)$$

$$\varepsilon_{\alpha_\zeta \beta_\zeta cd} F^{cd} \equiv \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} (F^{\gamma_\zeta 4} - F^{4\gamma_\zeta}) \quad (1.231)$$

$$\varepsilon_{\alpha_\zeta bcd} H^{bcd} \equiv \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} (H^{\beta_\zeta \gamma_\zeta 4} - H^{\beta_\zeta 4\gamma_\zeta} + H^{4\beta_\zeta \gamma_\zeta}) \quad (1.232)$$

$$\varepsilon_{abcd} R^{abcd} \equiv \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} (R^{\alpha_\zeta \beta_\zeta \gamma_\zeta 4} - R^{\alpha_\zeta \beta_\zeta 4\gamma_\zeta} + R^{\alpha_\zeta 4\beta_\zeta \gamma_\zeta} - R^{4\alpha_\zeta \beta_\zeta \gamma_\zeta}) \quad (1.233)$$

### 5.15 常数不变张量 $\varepsilon_{A_\zeta B_\zeta}, \varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta}$ 之间的关系

$$\varepsilon_{A_\zeta B_\zeta} \equiv \varepsilon_{A_\zeta B_\zeta 3} \quad (1.234)$$

$$\varepsilon_{A_\zeta B_\zeta \gamma_\zeta} A^{\gamma_\zeta} \equiv \varepsilon_{A_\zeta B_\zeta} A^3 \quad (1.235)$$

$$\varepsilon_{A_\zeta \beta_\zeta \gamma_\zeta} F^{\beta_\zeta \gamma_\zeta} \equiv \varepsilon_{A_\zeta B_\zeta} (F^{B_\zeta 3} - F^{3B_\zeta}) \quad (1.236)$$

$$\varepsilon_{\alpha_\zeta \beta_\zeta \gamma_\zeta} H^{\alpha_\zeta \beta_\zeta \gamma_\zeta} \equiv \varepsilon_{A_\zeta B_\zeta} (H^{A_\zeta B_\zeta 3} - H^{A_\zeta 3B_\zeta} + H^{3A_\zeta B_\zeta}) \quad (1.237)$$

## 6 几个非直观复合常数不变张量的性质

### 6.1 复合常数不变张量 $\sigma_{\alpha'_\zeta}^{k'_\zeta l'_\zeta}(s), \sigma_{k_\zeta l_\zeta}^{\alpha_\zeta}(s), \sigma_{k'_\zeta l'_\zeta}^{\alpha'_\zeta}(s), \sigma_{\alpha_\zeta}^{k_\zeta l_\zeta}(s)$ 的性质

#### 6.1.1 定义

$$\begin{cases} \sigma_{\alpha'_\zeta}^{k'_\zeta l'_\zeta}(s) := (\zeta)^{2s} \varepsilon_{l'_\zeta m'_\zeta}(s) \sigma_{\alpha'_\zeta}^{k'_\zeta m'_\zeta}(s) \\ \sigma_{k'_\zeta l'_\zeta}^{\alpha'_\zeta}(s) := (-\zeta)^{2s} \varepsilon_{k'_\zeta m'_\zeta}(s) \sigma_{\alpha'_\zeta}^{m'_\zeta l'_\zeta}(s) \end{cases} \quad \begin{cases} \sigma_{k_\zeta l_\zeta}^{\alpha_\zeta}(s) := (-\zeta)^{2s} \varepsilon_{l_\zeta m_\zeta}(s) \sigma_{\alpha_\zeta}^{k_\zeta m_\zeta}(s) \\ \sigma_{\alpha_\zeta}^{k_\zeta l_\zeta}(s) := (\zeta)^{2s} \varepsilon^{k_\zeta m_\zeta}(s) \sigma_{\alpha_\zeta}^{m_\zeta l_\zeta}(s) \end{cases} \quad (1.238)$$

$$\begin{cases} \sigma_{\alpha'_\zeta}^{k'_\zeta l'_\zeta}(s) := (-\zeta)^{2s} [\sigma_{\alpha'_\zeta}(s) \varepsilon(s)]^{k'_\zeta l'_\zeta} \\ \sigma_{k'_\zeta l'_\zeta}^{\alpha'_\zeta}(s) := (-\zeta)^{2s} [\varepsilon(s) \sigma_{\alpha'_\zeta}(s)]^{k'_\zeta l'_\zeta} \end{cases} \quad \begin{cases} \sigma_{k_\zeta l_\zeta}^{\alpha_\zeta}(s) := (\zeta)^{2s} [\sigma_{\alpha_\zeta}(s) \varepsilon(s)]_{k_\zeta l_\zeta} \\ \sigma_{\alpha_\zeta}^{k_\zeta l_\zeta}(s) := (\zeta)^{2s} [\varepsilon(s) \sigma_{\alpha_\zeta}(s)]^{k_\zeta l_\zeta} \end{cases} \quad (1.239)$$

$$\begin{cases} \sigma_{\alpha'_\zeta}^{k'_\zeta l'_\zeta}(s) \simeq (-1)^{2s} \sigma_{k_\zeta l_\zeta}^{\alpha_\zeta}(s) \\ \sigma_{k'_\zeta l'_\zeta}^{\alpha'_\zeta}(s) \simeq (-1)^{2s} \sigma_{\alpha_\zeta}^{k_\zeta l_\zeta}(s) \end{cases} \quad \begin{cases} \sigma_{\alpha'_\zeta}^{k'_\zeta m'_\zeta}(s) \sigma_{\alpha'_\zeta}^{m'_\zeta l'_\zeta}(s) = (-1)^{2s} \sigma_{\alpha'_\zeta}^{k'_\zeta m'_\zeta}(s) \sigma_{\alpha'_\zeta}^{m'_\zeta l'_\zeta}(s) \\ \sigma_{\alpha_\zeta}^{k_\zeta m_\zeta}(s) \sigma_{\alpha_\zeta}^{m_\zeta l_\zeta}(s) = (-1)^{2s} \sigma_{\alpha_\zeta}^{k_\zeta m_\zeta}(s) \sigma_{\alpha_\zeta}^{m_\zeta l_\zeta}(s) \end{cases} \quad (1.240)$$

### 6.1.2 对称与反对称性

$$\sigma^*(s) = (-1)^{2s+1} \varepsilon(s) \sigma(s) \varepsilon(s) \Rightarrow \quad (1.241)$$

$$\begin{cases} \sigma_{\alpha'_\zeta}^{k'_\zeta l'_\zeta}(s) = (-1)^{2s+1} \sigma_{\alpha'_\zeta}^{l'_\zeta k'_\zeta}(s) \\ \sigma_{k'_\zeta l'_\zeta}^{\alpha'_\zeta}(s) = (-1)^{2s+1} \sigma_{l'_\zeta k'_\zeta}^{\alpha'_\zeta}(s) \end{cases} \quad \begin{cases} \sigma_{k_\zeta l_\zeta}^{\alpha_\zeta}(s) = (-1)^{2s+1} \sigma_{l_\zeta k_\zeta}^{\alpha_\zeta}(s) \\ \sigma_{\alpha_\zeta}^{k_\zeta l_\zeta}(s) = (-1)^{2s+1} \sigma_{\alpha_\zeta}^{l_\zeta k_\zeta}(s) \end{cases} \quad (1.242)$$

### 6.1.3 复共轭性

$$\sigma^*(s) = (-1)^{2s+1} \varepsilon(s) \sigma(s) \varepsilon(s) \Rightarrow \quad (1.243)$$

$$[\sigma_{k'_\zeta l'_\zeta}^{\alpha'_\zeta}(s)]^* = (-1)^{2s+1} \sigma_{k_\zeta l_\zeta}^{\alpha_\zeta}(s) \quad [\sigma_{\alpha'_\zeta}^{k'_\zeta l'_\zeta}(s)]^* = (-1)^{2s+1} \sigma_{\alpha_\zeta}^{k_\zeta l_\zeta}(s) \quad (1.244)$$

## 6.2 复合常数不变张量 $\sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta}, \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta}, \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta}, \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta}$ 的性质

### 6.2.1 定义

$$\begin{cases} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} := \zeta \varepsilon^{B'_\zeta C'_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta C'_\zeta} \\ \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} := -\zeta \varepsilon_{A'_\zeta C'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta B'_\zeta} \end{cases} \quad \begin{cases} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} := -\zeta \varepsilon_{B_\zeta C_\zeta} \sigma_{\alpha_\zeta}^{A_\zeta C_\zeta} \\ \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} := \zeta \varepsilon^{A_\zeta C_\zeta} \sigma_{\alpha_\zeta}^{C_\zeta B_\zeta} \end{cases} \quad (1.245)$$

$$\begin{cases} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} = -\zeta [\sigma_{\alpha'_\zeta} \varepsilon]^{A'_\zeta B'_\zeta} \\ \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} = -\zeta [\varepsilon \sigma_{\alpha'_\zeta}]_{A'_\zeta B'_\zeta} \end{cases} \quad \begin{cases} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} = \zeta [\sigma_{\alpha_\zeta} \varepsilon]_{A_\zeta B_\zeta} \\ \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} = \zeta [\varepsilon \sigma_{\alpha_\zeta}]^{A_\zeta B_\zeta} \end{cases} \quad (1.246)$$

$$\begin{cases} \frac{i_\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} = \frac{i_\zeta}{\sqrt{2}} [\sigma_{\alpha_\zeta} \varepsilon]_{A_\zeta B_\zeta} = \frac{i_\zeta}{\sqrt{2}} [-\sigma_z, i, \sigma_x]_{A_\zeta B_\zeta} \\ \frac{i_\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} = \frac{i_\zeta}{\sqrt{2}} [\varepsilon \sigma_{\alpha_\zeta}]^{A_\zeta B_\zeta} = \frac{i_\zeta}{\sqrt{2}} [\sigma_z, i, -\sigma_x]^{A_\zeta B_\zeta} \end{cases} \quad (1.247)$$

$$\begin{cases} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \simeq -\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \\ \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \simeq -\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \end{cases} \quad \begin{cases} \sigma_{\alpha'_\zeta}^{A'_\zeta C'_\zeta} \sigma_{\beta'_\zeta}^{C'_\zeta B'_\zeta} = -\sigma_{\alpha_\zeta}^{A'_\zeta C'_\zeta} \sigma_{\beta_\zeta}^{B'_\zeta C'_\zeta} \\ \sigma_{A_\zeta C_\zeta}^{\alpha_\zeta} \sigma_{\beta_\zeta}^{C_\zeta B_\zeta} = -\sigma_{\alpha_\zeta}^{A_\zeta C_\zeta} \sigma_{\beta_\zeta}^{B_\zeta C_\zeta} \end{cases} \quad (1.248)$$

### 6.2.2 正交性

缩减一对复矢量指标：

$$\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} = \varepsilon_{A_\zeta D_\zeta} \varepsilon_{C_\zeta B_\zeta} - \varepsilon_{A_\zeta C_\zeta} \varepsilon_{B_\zeta D_\zeta} \quad \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} = \varepsilon_{A'_\zeta D'_\zeta} \varepsilon_{C'_\zeta B'_\zeta} - \varepsilon_{A'_\zeta C'_\zeta} \varepsilon_{B'_\zeta D'_\zeta} \quad (1.249)$$

$$\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} = \varepsilon_{A_\zeta D_\zeta} \varepsilon_{C_\zeta B_\zeta} - \varepsilon_{A_\zeta C_\zeta} \varepsilon_{B_\zeta D_\zeta} \quad \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} = \varepsilon_{A'_\zeta D'_\zeta} \varepsilon_{C'_\zeta B'_\zeta} - \varepsilon_{A'_\zeta C'_\zeta} \varepsilon_{B'_\zeta D'_\zeta} \quad (1.250)$$

$$\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{C_\zeta D_\zeta}^{\alpha_\zeta} = -\delta_{C_\zeta}^{(A_\zeta} \delta_{D_\zeta}^{B_\zeta)} = -\delta_{(C_\zeta}^{A_\zeta} \delta_{D_\zeta}^{B_\zeta)} \quad \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \sigma_{C'_\zeta D'_\zeta}^{\alpha'_\zeta} = -\delta_{C'_\zeta}^{(A'_\zeta} \delta_{D'_\zeta}^{B'_\zeta)} = -\delta_{(C'_\zeta}^{A'_\zeta} \delta_{D'_\zeta}^{B'_\zeta)} \quad (1.251)$$

$$\sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} = -\zeta (\varepsilon_{A'_\zeta D'_\zeta} \delta_{B'_\zeta}^{C'_\zeta} + \delta_{A'_\zeta}^{C'_\zeta} \varepsilon_{B'_\zeta D'_\zeta}) \quad \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} = \zeta (\varepsilon_{A_\zeta D_\zeta} \delta_{C_\zeta}^{B_\zeta} + \delta_{C_\zeta}^{A_\zeta} \varepsilon_{B_\zeta D_\zeta}) \quad (1.252)$$

$$\sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} = \zeta (\delta_{D'_\zeta}^{A'_\zeta} \varepsilon_{B'_\zeta C'_\zeta} + \varepsilon_{A'_\zeta C'_\zeta} \delta_{D'_\zeta}^{B'_\zeta}) \quad \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} = -\zeta (\delta_{A_\zeta}^{D_\zeta} \varepsilon_{B_\zeta C_\zeta} + \varepsilon_{A_\zeta C_\zeta} \delta_{B_\zeta}^{D_\zeta}) \quad (1.253)$$

$$\sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} = \delta_{D'_\zeta}^{A'_\zeta} \delta_{B'_\zeta}^{C'_\zeta} - \varepsilon_{A'_\zeta C'_\zeta} \varepsilon_{B'_\zeta D'_\zeta} \quad \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} = \delta_{A_\zeta}^{D_\zeta} \delta_{C_\zeta}^{B_\zeta} - \varepsilon_{A_\zeta C_\zeta} \varepsilon_{B_\zeta D_\zeta} \quad (1.254)$$

### 6.2.3 对称与反对称性

$$\begin{cases} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} = \sigma_{\alpha_\zeta}^{B_\zeta A_\zeta} \\ \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} = \sigma_{B_\zeta A_\zeta}^{\alpha_\zeta} \end{cases} \quad \begin{cases} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} = \sigma_{B'_\zeta A'_\zeta}^{\alpha'_\zeta} \\ \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} = \sigma_{\alpha'_\zeta}^{B'_\zeta A'_\zeta} \end{cases} \quad (1.255)$$

## 6.2.4 复共轭性

$$\sigma^T = \varepsilon \sigma \varepsilon \quad (1.256)$$

$$[\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta}]^* = \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \quad [\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta}]^* = \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \quad (1.257)$$

6.3 常数不变张量  $\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab}, \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta}$  的性质

## 6.3.1 定义

$$\sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} := \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a A'_\zeta A_\zeta \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_b B'_\zeta B_\zeta \frac{i\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \frac{-i\zeta}{\sqrt{2}} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \quad \sigma_{ab}^{\alpha'_\zeta \alpha} \stackrel{P}{=} \frac{1}{2} \sigma_{A'B'}^{\alpha'} \sigma_{AB}^\alpha \quad (1.258)$$

$$\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} := \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)^a_{A_\zeta A'_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)^b_{B_\zeta B'_\zeta} \frac{-i\zeta}{\sqrt{2}} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \quad \sigma_{\alpha\alpha'}^{ab} \stackrel{P}{=} \frac{1}{2} \sigma_\alpha^{AB} \sigma_{\alpha'}^{A'B'} \quad (1.259)$$

性质:

$$\sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} = \sigma_{ba}^{\alpha_\zeta \alpha'_\zeta} \quad \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} = \sigma_{\alpha'_\zeta \alpha_\zeta}^{ba} \quad \delta^{ab} \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} = 0 \quad \delta_{ab} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} = 0 \quad (1.260)$$

$$\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \simeq \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} \quad (\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab})^* = \sigma_{\alpha'_\zeta \alpha_\zeta}^{ab} \quad (\sigma_{ab}^{\alpha'_\zeta \alpha_\zeta})^* = \sigma_{ab}^{\alpha_\zeta \alpha'_\zeta} \quad R^{ab} = \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \psi^{\alpha_\zeta} \psi^{*\alpha'_\zeta} \quad (1.261)$$

推论6.3.1.  $\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} = -\frac{1}{2} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ac} \delta_{cd} \sigma_{-c d}^{ab}, \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} = -\frac{1}{2} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ac} \delta^{cd} \sigma_{-c d}^{\alpha'_\zeta \alpha_\zeta}, \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \simeq \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta}$

证明:  $\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} = \sigma_{-c d}^{\alpha'_\zeta \alpha_\zeta} \delta^{cb}$ 

$$\begin{aligned} &= \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a A'_\zeta A_\zeta \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \delta_{A'_\zeta B'_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)^c_{B_\zeta B'_\zeta} \cdot \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)^c_{C_\zeta C'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} \delta_{C_\zeta D_\zeta} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)^{D'_\zeta D_\zeta}_b \\ &= \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a A'_\zeta A_\zeta \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \delta_{A'_\zeta B'_\zeta} \varepsilon_{B_\zeta C_\zeta} \varepsilon_{B'_\zeta C'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} \delta_{C_\zeta D_\zeta} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)^{D'_\zeta D_\zeta}_b \\ &= -\frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a A'_\zeta A_\zeta \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} (\zeta \varepsilon_{B_\zeta D_\zeta}) (-\zeta \varepsilon_{A'_\zeta C'_\zeta}) \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)^{D'_\zeta D_\zeta}_b \\ &= -\frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a A'_\zeta A_\zeta \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta D'_\zeta} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)^{D'_\zeta D_\zeta}_b \quad \square \end{aligned}$$

推论6.3.2.  $\sigma_{kl}^{\alpha'_\zeta \alpha_\zeta} = \frac{1}{2} (\delta_k^\alpha \delta_l^{\alpha'} + \delta_k^{\alpha'} \delta_l^\alpha - \delta_{kl} \delta^{\alpha\alpha'})$ ,  $\sigma_{\alpha_\zeta \alpha'_\zeta}^{kl} = \frac{1}{2} (\delta_{\alpha_\zeta}^k \delta_{\alpha'_\zeta}^l + \delta_{\alpha'_\zeta}^k \delta_{\alpha_\zeta}^l - \delta^{kl} \delta_{\alpha_\zeta \alpha'_\zeta})$

证明:  $\sigma_{kl}^{\alpha'_\zeta \alpha_\zeta}$ 

$$\begin{aligned} &= \frac{i\zeta}{\sqrt{2}} (\sigma)_k A'_\zeta A_\zeta \frac{i\zeta}{\sqrt{2}} (\sigma)_l B'_\zeta B_\zeta \frac{i\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \frac{-i\zeta}{\sqrt{2}} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \\ &= -\frac{1}{4} (\sigma)_k A'_\zeta A_\zeta (\sigma)_l B'_\zeta B_\zeta \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \\ &= -\frac{1}{4} (\sigma)_k A'_\zeta A_\zeta (\sigma)_l B'_\zeta B_\zeta \zeta [\sigma^{\alpha_\zeta} \varepsilon]_{A_\zeta B_\zeta} \{-\zeta [\varepsilon \sigma^{\alpha'_\zeta}]_{A'_\zeta B'_\zeta}\} \\ &= \frac{1}{4} (\sigma)_k A'_\zeta A_\zeta [\sigma^{\alpha_\zeta} \varepsilon]_{A_\zeta B_\zeta} (\sigma^T)_l B'_\zeta B'_\zeta [\sigma^T \alpha'_\zeta \varepsilon^T]_{B'_\zeta A'_\zeta} \\ &= \frac{1}{4} \text{tr} \{ \sigma_k \sigma^{\alpha_\zeta} \varepsilon \sigma_l^T \sigma^T \alpha'_\zeta \varepsilon^T \} \\ &= \frac{1}{4} \text{tr} \{ \sigma_k \sigma^{\alpha_\zeta} \varepsilon \sigma_l^T \varepsilon^T \sigma^T \alpha'_\zeta \varepsilon^T \} \\ &= \frac{1}{4} \text{tr} \{ \sigma_k \sigma^{\alpha_\zeta} \sigma_l \sigma^{\alpha'_\zeta} \} \\ &= \frac{1}{4} \text{tr} \{ (\delta_k^{\alpha_\zeta} + i \varepsilon_k^{\alpha_\zeta \beta_\zeta} \sigma_{\beta_\zeta}) (\delta_l^{\alpha'_\zeta} + i \varepsilon_l^{\alpha'_\zeta \beta'_\zeta} \sigma_{\beta'_\zeta}) \} \\ &= \frac{1}{2} (\delta_k^{\alpha_\zeta} \delta_l^{\alpha'_\zeta} - \varepsilon_k^{\alpha_\zeta \beta_\zeta} \varepsilon_l^{\alpha'_\zeta \beta'_\zeta} \delta_{\beta_\zeta \beta'_\zeta}) \\ &= \frac{1}{2} (\delta_k^{\alpha_\zeta} \delta_l^{\alpha'_\zeta} + \delta_k^{\alpha'_\zeta} \delta_l^{\alpha_\zeta} - \delta_{kl} \delta^{\alpha_\zeta \alpha'_\zeta}) \quad \square \end{aligned}$$

推论6.3.3.  $\sigma_{k\pi}^{\alpha_\zeta \alpha'_\zeta} = -\frac{\zeta}{2} \varepsilon_k^{\alpha_\zeta \alpha'_\zeta}, \sigma_{\alpha_\zeta \alpha'_\zeta}^{k\pi} = -\frac{\zeta}{2} \varepsilon^k_{\alpha_\zeta \alpha'_\zeta}$

证明:  $\sigma_{k\pi}^{\alpha_\zeta \alpha'_\zeta}$ 

$$\begin{aligned} &= \frac{i\zeta}{\sqrt{2}} (\sigma)_k A'_\zeta A_\zeta \frac{i\zeta}{\sqrt{2}} (-i\zeta)^{B'_\zeta B_\zeta} \frac{i\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \frac{-i\zeta}{\sqrt{2}} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \\ &= i\zeta \frac{1}{4} (\sigma)_k A'_\zeta A_\zeta \delta^{B'_\zeta B_\zeta} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \\ &= i\zeta \frac{1}{4} (\sigma)_k A'_\zeta A_\zeta \delta^{B'_\zeta B_\zeta} \zeta [\sigma^{\alpha_\zeta} \varepsilon]_{A_\zeta B_\zeta} \{-\zeta [\varepsilon \sigma^{\alpha'_\zeta}]_{A'_\zeta B'_\zeta}\} \\ &= -i\zeta \frac{1}{4} (\sigma)_k A'_\zeta A_\zeta [\sigma^{\alpha_\zeta} \varepsilon]_{A_\zeta B_\zeta} \delta^{B'_\zeta B_\zeta} [\sigma^T \alpha'_\zeta \varepsilon^T]_{B'_\zeta A'_\zeta} \\ &= -i\zeta \frac{1}{4} \text{tr} \{ \sigma_k \sigma^{\alpha_\zeta} \varepsilon I \sigma^T \alpha'_\zeta \varepsilon^T \} \\ &= -i\zeta \frac{1}{4} \text{tr} \{ \sigma_k \sigma^{\alpha_\zeta} \sigma^{\alpha'_\zeta} \} \\ &= i\zeta \frac{1}{4} \text{tr} \{ (\delta_k^{\alpha_\zeta} + i \varepsilon_k^{\alpha_\zeta \beta_\zeta} \sigma_{\beta_\zeta}) \sigma^{\alpha'_\zeta} \} \\ &= -\frac{\zeta}{2} \varepsilon_k^{\alpha_\zeta \beta_\zeta} \delta_{\beta_\zeta \alpha'_\zeta} \\ &= -\frac{\zeta}{2} \varepsilon_k^{\alpha_\zeta \alpha'_\zeta} \quad \square \end{aligned}$$



推论6.3.4.  $\sigma_{\pi\pi}^{\alpha'_s\alpha_s} = \frac{1}{2}\delta^{\alpha_s\alpha'_s}, \sigma_{\alpha_s\alpha'_s}^{\pi\pi} = \frac{1}{2}\delta_{\alpha_s\alpha'_s}$

证明:  $\sigma_{\pi\pi}^{\alpha'_s\alpha_s}$

$$\begin{aligned} &= \frac{i\zeta}{\sqrt{2}}(-i\zeta)_{A'_s A_s} \frac{i\zeta}{\sqrt{2}}(-i\zeta)_{B'_s B_s} \frac{i\zeta}{\sqrt{2}}\sigma_{A_s B_s}^{\alpha_s} \frac{-i\zeta}{\sqrt{2}}\sigma_{A'_s B'_s}^{\alpha'_s} \\ &= \frac{1}{4}\delta_{A'_s A_s} \delta_{B'_s B_s} \sigma_{A_s B_s}^{\alpha_s} \sigma_{A'_s B'_s}^{\alpha'_s} \\ &= \frac{1}{4}\delta_{A'_s A_s} \delta_{B'_s B_s} \zeta[\sigma^{\alpha_s} \varepsilon]_{A_s B_s} \{-\zeta[\varepsilon \sigma^{\alpha'_s}]_{A'_s B'_s}\} \\ &= -\frac{1}{4}\delta_{A'_s A_s} [\sigma^{\alpha_s} \varepsilon]_{A_s B_s} \delta_{B'_s B'_s} [\sigma^{T\alpha'_s} \varepsilon^T]_{B'_s A'_s} \\ &= -\frac{1}{4}tr\{I\sigma^{\alpha_s} \varepsilon I\sigma^{T\alpha'_s} \varepsilon^T\} \\ &= \frac{1}{4}tr\{\sigma^{\alpha_s} \sigma^{\alpha'_s}\} \\ &= \frac{1}{2}\delta^{\alpha_s\alpha'_s} \end{aligned}$$

□

推论6.3.5.  $\begin{cases} \sigma_{\alpha_s\alpha'_s}^{kl} \partial_k \partial_l = \partial_{\alpha_s} \partial_{\alpha'_s} - \frac{1}{2}\delta_{\alpha_s\alpha'_s} \nabla^2, \sigma_{\alpha_s\alpha'_s}^{\pi\pi} \partial_\pi^2 = \frac{1}{2}\delta_{\alpha_s\alpha'_s} \partial_\pi^2 \\ \sigma_{\alpha_s\alpha'_s}^{k\pi} \partial_k \partial_\pi = -\frac{\zeta}{2}\varepsilon^k{}_{\alpha_s\alpha'_s} \partial_k \partial_\pi, \sigma_{\alpha_s\alpha'_s}^{\pi k} \partial_\pi \partial_k = -\frac{\zeta}{2}\varepsilon^k{}_{\alpha_s\alpha'_s} \partial_\pi \partial_k \end{cases}$

推论6.3.6.  $\sigma_{\alpha_s\alpha'_s}^{ab} \partial_a \partial_b = \partial_{\alpha_s} \partial_{\alpha'_s} - \frac{1}{2}\delta_{\alpha_s\alpha'_s} (\nabla^2 - \partial_\pi^2) - \zeta \varepsilon^k{}_{\alpha_s\alpha'_s} \partial_k \partial_\pi$

正交性:

推论6.3.7.  $\sigma_{ab}^{\alpha'_s\alpha_s} \sigma_{\beta_s\beta'_s}^{ab} = \delta^{\alpha_s\beta_s} \delta^{\alpha'_s\beta'_s}$

推论6.3.8.  $\sigma_{ab}^{\alpha'_s\alpha_s} \sigma_{\alpha_s\alpha'_s} cd = \frac{1}{2}\delta_{ac}\delta_{bd} + \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{1}{4}\delta_{ab}\delta_{cd}$

证明方法一:

证明:  $\sigma_{ab}^{\alpha'_s\alpha_s} \sigma_{\alpha_s\alpha'_s} cd$

$$\begin{aligned} &= \frac{1}{4}\delta^{ef} \delta^{gh} (\sigma_{\zeta ae}^{\alpha_s} \sigma_{\zeta \alpha_s cg}) (\sigma_{-\zeta \alpha'_s hd} \sigma_{-\zeta fb}^{\alpha'_s}) \\ &= \frac{1}{4}\delta^{ef} \delta^{gh} (S_{aecg} - \zeta \varepsilon_{aecg}) (S_{hdfb} + \zeta \varepsilon_{hdfb}) \\ &= \frac{1}{4}\delta^{ef} \delta^{gh} (\delta_{ac}\delta_{eg} - \delta_{ag}\delta_{ec} - \zeta \varepsilon_{aecg}) (\delta_{hf}\delta_{db} - \delta_{hb}\delta_{df} + \zeta \varepsilon_{hdfb}) \\ &= \frac{1}{4}[(\delta_{ac}\delta_{eg} - \delta_{ag}\delta_{ec})(\delta_{hf}\delta_{db} - \delta_{hb}\delta_{df}) + (-\zeta \varepsilon_{aecg})(\delta_{hf}\delta_{db} - \delta_{hb}\delta_{df}) + (\delta_{ac}\delta_{eg} - \delta_{ag}\delta_{ec})(\zeta \varepsilon_{hdfb}) + (-\zeta \varepsilon_{aecg})(\zeta \varepsilon_{hdfb})] \\ &= \frac{1}{4}[(2\delta_{ac}\delta_{db} + \delta_{ab}\delta_{cd}) + (-\zeta \varepsilon_{abcd}) + (\zeta \varepsilon_{abcd}) - (\varepsilon_{aecg})\delta^{ef} \delta^{gh} (\varepsilon_{fhbd})] \\ &= \frac{1}{4}[(2\delta_{ac}\delta_{db} + \delta_{ab}\delta_{cd}) + (-2S_{acbd})] \\ &= \frac{1}{4}[(2\delta_{ac}\delta_{db} + \delta_{ab}\delta_{cd}) - 2(\delta_{ab}\delta_{cd} - \delta_{ad}\delta_{cb})] \\ &= \frac{1}{2}\delta_{ac}\delta_{bd} + \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{1}{4}\delta_{ab}\delta_{cd} \end{aligned}$$

□

证明方法二:

证明:

$\sigma_{ab}^{\alpha'_s\alpha_s} \sigma_{\alpha_s\alpha'_s} cd$

$$\begin{aligned} &= \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a A'_s A_s \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b B'_s B_s \frac{i\zeta}{\sqrt{2}}\sigma_{A_s B_s}^{\alpha_s} \frac{-i\zeta}{\sqrt{2}}\sigma_{A'_s B'_s}^{\alpha'_s} \cdot \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{cC_s C'_s} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{dD_s D'_s} \frac{-i\zeta}{\sqrt{2}}\sigma_{C'_s D'_s}^{C'_s D'_s} \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha_s}^{C_s D_s} \\ &= \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a A'_s A_s \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b B'_s B_s \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{cC_s C'_s} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{dD_s D'_s} \cdot \frac{i\zeta}{\sqrt{2}}\sigma_{A_s B_s}^{\alpha_s} \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha_s}^{C_s D_s} \frac{-i\zeta}{\sqrt{2}}\sigma_{A'_s B'_s}^{\alpha'_s} \frac{-i\zeta}{\sqrt{2}}\sigma_{\alpha'_s}^{C'_s D'_s} \\ &= \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a A'_s A_s \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b B'_s B_s \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{cC_s C'_s} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{dD_s D'_s} \cdot \frac{1}{2}(\delta_{A_s}^{C_s} \delta_{B_s}^{D_s} + \delta_{A_s}^{D_s} \delta_{B_s}^{C_s}) \frac{1}{2}(\delta_{A'_s}^{C'_s} \delta_{B'_s}^{D'_s} + \delta_{A'_s}^{D'_s} \delta_{B'_s}^{C'_s}) \\ &= \frac{1}{4} \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a A'_s A_s \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b B'_s B_s \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{cA_s A'_s} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{dB_s B'_s} \\ &+ \frac{1}{4} \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a A'_s A_s \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b B'_s B_s \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{cB_s B'_s} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{dA_s A'_s} \\ &+ \frac{1}{4} \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a A'_s A_s \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b B'_s B_s \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{cA_s B'_s} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{dB_s A'_s} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a A'_\zeta A_\zeta \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_b B'_\zeta B_\zeta \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{c B_\zeta A'_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{d A_\zeta B'_\zeta} \\
& = \frac{1}{4} \delta_{ac} \delta_{bd} + \frac{1}{4} \delta_{ad} \delta_{bc} + \frac{1}{4} \left[ \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{c A_\zeta B'_\zeta} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_b B'_\zeta B_\zeta \right] \left[ \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{d B_\zeta A'_\zeta} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a A'_\zeta A_\zeta \right] \\
& + \frac{1}{4} \left[ \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{c B_\zeta A'_\zeta} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a A'_\zeta A_\zeta \right] \left[ \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{d A_\zeta B'_\zeta} \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_b B'_\zeta B_\zeta \right] \\
& = \frac{1}{4} \delta_{ac} \delta_{bd} + \frac{1}{4} \delta_{ad} \delta_{bc} + \frac{1}{4} \left[ \frac{1}{2} \delta_{cb} \delta_{A_\zeta B_\zeta} + i S_{cb A_\zeta B_\zeta} \right] \left[ \frac{1}{2} \delta_{da} \delta_{B_\zeta A_\zeta} + i S_{da B_\zeta A_\zeta} \right] \\
& + \frac{1}{4} \left[ \frac{1}{2} \delta_{ca} \delta_{B_\zeta A_\zeta} + i S_{ca B_\zeta A_\zeta} \right] \left[ \frac{1}{2} \delta_{db} \delta_{A_\zeta B_\zeta} + i S_{db A_\zeta B_\zeta} \right] \\
& = \frac{3}{8} \delta_{ac} \delta_{bd} + \frac{3}{8} \delta_{ad} \delta_{bc} + \frac{1}{4} i S_{cb A_\zeta B_\zeta} i S_{da B_\zeta A_\zeta} + \frac{1}{4} i S_{ca B_\zeta A_\zeta} i S_{db A_\zeta B_\zeta} \\
& = \frac{3}{8} \delta_{ac} \delta_{bd} + \frac{3}{8} \delta_{ad} \delta_{bc} + \frac{1}{8} \sigma_{\zeta cb}^{\alpha_\zeta} \sigma_{\zeta ca} \sigma_{\zeta da} + \frac{1}{8} \sigma_{\zeta ca}^{\alpha_\zeta} \sigma_{\zeta cb} \sigma_{\zeta db} \\
& = \frac{3}{8} \delta_{ac} \delta_{bd} + \frac{3}{8} \delta_{ad} \delta_{bc} + \frac{1}{8} (-\delta_{cd} \delta_{ba} + \delta_{ca} \delta_{bd} + \zeta \varepsilon_{cbda}) + \frac{1}{8} (-\delta_{cd} \delta_{ab} + \delta_{cb} \delta_{ad} + \zeta \varepsilon_{cabd}) \\
& = \frac{1}{2} \delta_{ac} \delta_{bd} + \frac{1}{2} \delta_{ad} \delta_{bc} - \frac{1}{4} \delta_{ab} \delta_{cd} \quad \square
\end{aligned}$$

### 6.3.2 汇总

定义6.3.1.  $\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} := -\frac{1}{2} (\sigma_{+\zeta \alpha_\zeta} \sigma_{-\zeta \alpha'_\zeta})^{ab} = \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_a A_\zeta A'_\zeta \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_b B_\zeta B'_\zeta \frac{-i\zeta}{\sqrt{2}} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta}$ ,  $\sigma_{\alpha \alpha'}^{ab} \stackrel{P}{=} \frac{1}{2} \sigma_\alpha^{AB} \sigma_{\alpha'}^{A'B'}$

推论6.3.9.  $\sigma_{\alpha_\zeta \alpha'_\zeta}^{kl} = \frac{1}{2} (\delta_{\alpha_\zeta}^k \delta_{\alpha'_\zeta}^l + \delta_{\alpha'_\zeta}^k \delta_{\alpha_\zeta}^l - \delta^{kl} \delta_{\alpha_\zeta \alpha'_\zeta})$ ,  $\sigma_{\alpha_\zeta \alpha'_\zeta}^{k\pi} = \sigma_{\alpha_\zeta \alpha'_\zeta}^{\pi k} = -\frac{\zeta}{2} \varepsilon^k{}_{\alpha_\zeta \alpha'_\zeta}$ ,  $\sigma_{\alpha_\zeta \alpha'_\zeta}^{\pi\pi} = \frac{1}{2} \delta_{\alpha_\zeta \alpha'_\zeta}$

推论6.3.10.  $\sigma_{ab}^{\alpha_\zeta \alpha'_\zeta} \sigma_{\beta_\zeta \beta'_\zeta}^{ab} = \delta^{\alpha_\zeta \beta_\zeta} \delta^{\alpha'_\zeta \beta'_\zeta}$ ,  $\sigma_{ab}^{\alpha_\zeta \alpha'_\zeta} \sigma_{\alpha_\zeta \alpha'_\zeta cd} = \frac{1}{2} \delta_{ac} \delta_{bd} + \frac{1}{2} \delta_{ad} \delta_{bc} - \frac{1}{4} \delta_{ab} \delta_{cd}$

推论6.3.11.  $\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b = \partial_{\alpha_\zeta} \partial_{\alpha'_\zeta} - \frac{1}{2} \delta_{\alpha_\zeta \alpha'_\zeta} (\nabla^2 - \partial_\pi^2) - \zeta \varepsilon^k{}_{\alpha_\zeta \alpha'_\zeta} \partial_k \partial_\pi = \partial_{\alpha_\zeta} \partial_{\alpha'_\zeta} - \frac{1}{2} \delta_{\alpha_\zeta \alpha'_\zeta} (\nabla^2 + \partial_t^2) + i\zeta \varepsilon^k{}_{\alpha_\zeta \alpha'_\zeta} \partial_k \partial_t$

## 6.4 自旋常数不变张量 $S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta)$ , $S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta)$ , $S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta)$ , $S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta)$ 的性质

### 6.4.1 定义

$$S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta) = i \sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta}^{k_\zeta l_\zeta}(s) \quad S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta) = i \sigma_{-\zeta ab}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{k'_\zeta l'_\zeta}(s) \quad (1.262)$$

$$S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta) = i \sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta}^{k_\zeta l_\zeta}(s) \quad S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta) = i \sigma_{-\zeta ab}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{k'_\zeta l'_\zeta}(s) \quad (1.263)$$

### 6.4.2 对称与反对称性

$$S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta) = (-1)^{2s+1} S_{ab}{}^{l_\zeta k_\zeta}(s, \zeta) \quad S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta) = (-1)^{2s+1} S_{ab}{}^{l'_\zeta k'_\zeta}(s, -\zeta) \quad (1.264)$$

$$S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta) = (-1)^{2s+1} S_{ab}{}^{l_\zeta k_\zeta}(s, \zeta) \quad S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta) = (-1)^{2s+1} S_{ab}{}^{l'_\zeta k'_\zeta}(s, -\zeta) \quad (1.265)$$

$$S_{ab}{}^{k'_\zeta m'_\zeta}(s, -\zeta) S_{cd m'_\zeta l'_\zeta}(s, -\zeta) = -S_{ab}{}^{k'_\zeta m'_\zeta}(s, -\zeta) S_{cd}{}^{m'_\zeta l'_\zeta}(s, -\zeta) \quad (1.266)$$

$$S_{ab k_\zeta m_\zeta}(s, \zeta) S_{cd}{}^{m_\zeta l_\zeta}(s, \zeta) = -S_{ab k_\zeta m_\zeta}(s, \zeta) S_{cd m_\zeta}{}^{l_\zeta}(s, \zeta) \quad (1.267)$$

### 6.4.3 对偶性

$$S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta) = -\zeta * S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta) \quad S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta) = \zeta * S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta) \quad (1.268)$$

$$S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta) = -\zeta * S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta) \quad S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta) = \zeta * S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta) \quad (1.269)$$

### 6.4.4 复共轭性

$$[S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta) \partial^a \hat{\partial}^b]^* = (-1)^{2s+1} S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta) \partial^a \hat{\partial}^b \quad [S_{ab}{}^{k_\zeta l_\zeta}(s, \zeta) \partial_a \hat{\partial}_b]^* = (-1)^{2s+1} S_{ab}{}^{k'_\zeta l'_\zeta}(s, -\zeta) \partial_a \hat{\partial}_b \quad (1.270)$$

## 6.5 几个基本常数不变张量之间的重要联系

联系一：

$$\frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_a A_\zeta A'_\zeta \sigma_{\zeta a}^{\alpha\zeta} \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} = \sigma_{\alpha\zeta A_\zeta B_\zeta} \delta_{A'_\zeta B'_\zeta} \quad \sigma_{+a}^{\alpha\zeta b} \stackrel{P}{=} \sigma_{\alpha A B} \delta_{A' B'} \quad (1.271)$$

$$\frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a A'_\zeta A_\zeta \sigma_{-\zeta a}^{\alpha\zeta} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_b^{B_\zeta B'_\zeta} = \sigma_{\alpha'_\zeta A'_\zeta B'_\zeta} \delta_{A_\zeta B_\zeta} \quad \sigma_{-\alpha' a}^{\zeta b} \stackrel{P}{=} \sigma_{\alpha' A' B'} \delta_{A B} \quad (1.272)$$

$$\sigma_{\zeta a}^{\alpha\zeta b} = \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a A'_\zeta A_\zeta \sigma_{\alpha\zeta A_\zeta B_\zeta} \delta_{A'_\zeta B'_\zeta} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_b^{B_\zeta B'_\zeta} \quad (1.273)$$

联系二：

$$\frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a A'_\zeta A_\zeta \sigma_{\zeta a}^{ab} \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} = \zeta \sigma_{\alpha\zeta}^{A_\zeta B_\zeta} \varepsilon_{A'_\zeta B'_\zeta} \quad \sigma_{+\alpha}^{ab} \stackrel{P}{=} \sigma_{\alpha AB} \varepsilon_{A' B'} \quad (1.274)$$

$$\frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a A'_\zeta A_\zeta \sigma_{-\zeta a}^{ab} \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} = -\zeta \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \varepsilon_{A_\zeta B_\zeta} \quad \sigma_{-\alpha'}^{ab} \stackrel{P}{=} -\sigma_{\alpha' A' B'} \varepsilon_{A B} \quad (1.275)$$

$$\frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_a A_\zeta A'_\zeta \sigma_{\zeta a}^{\alpha\zeta} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_b^{B_\zeta B'_\zeta} = \zeta \sigma_{\alpha\zeta}^{A_\zeta B_\zeta} \varepsilon_{A'_\zeta B'_\zeta} \quad \sigma_{+\alpha}^{ab} \stackrel{P}{=} \sigma_{\alpha AB} \varepsilon_{A' B'} \quad (1.276)$$

$$\frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_a A_\zeta A'_\zeta \sigma_{-\zeta a}^{\alpha'_\zeta} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_b^{B_\zeta B'_\zeta} = -\zeta \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \varepsilon_{A_\zeta B_\zeta} \quad \sigma_{-\alpha}^{ab} \stackrel{P}{=} -\sigma_{\alpha' A' B'} \varepsilon_{A B} \quad (1.277)$$

联系三：

$$(\sigma, -i\zeta)_{[a}^{A'_\zeta A_\zeta} (\sigma, i\zeta)_{b]}^{A_\zeta B'_\zeta} = -2\sigma_{-\zeta ab}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \quad (\sigma, -i\zeta)_{\{a}^{A'_\zeta A_\zeta} (\sigma, i\zeta)_{b\}}^{A_\zeta B'_\zeta} = 2\delta_{ab} \delta_{A'_\zeta B'_\zeta} \quad (1.278)$$

$$(\sigma, i\zeta)_{[a}^{A_\zeta A'_\zeta} (\sigma, -i\zeta)_{b]}^{A'_\zeta B_\zeta} = -2\sigma_{-\zeta ab}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \quad (\sigma, i\zeta)_{\{a}^{A_\zeta A'_\zeta} (\sigma, -i\zeta)_{b\}}^{A'_\zeta B_\zeta} = 2\delta_{ab} \delta_{A'_\zeta B'_\zeta} \quad (1.279)$$

## 6.6 自旋常数不变张量 $S_{ab}^{A_\zeta B_\zeta}$ , $S_{ab}^{A'_\zeta B'_\zeta}$ , $S_{ab}^{A_\zeta B'_\zeta}$ , $S_{ab}^{A'_\zeta B_\zeta}$ 的性质

### 6.6.1 定义

$$S_{ab}^{A_\zeta B_\zeta} = \frac{i}{2} \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\alpha\zeta}^{A_\zeta B_\zeta} \quad S_{ab}^{A'_\zeta B'_\zeta} = \frac{i}{2} \sigma_{-\zeta ab}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \quad (1.280)$$

$$S_{ab}^{A_\zeta B'_\zeta} = \frac{i}{2} \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\alpha\zeta}^{A'_\zeta B'_\zeta} \quad S_{ab}^{A'_\zeta B_\zeta} = \frac{i}{2} \sigma_{-\zeta ab}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B_\zeta} \quad (1.281)$$

### 6.6.2 对称与反对称性

$$S_{ab}^{A_\zeta B_\zeta} = S_{ab}^{B_\zeta A_\zeta} \quad S_{ab}^{A'_\zeta B'_\zeta} = S_{ab}^{B'_\zeta A'_\zeta} \quad (1.282)$$

$$S_{ab}^{A_\zeta B'_\zeta} = S_{ab}^{B'_\zeta A_\zeta} \quad S_{ab}^{A'_\zeta B_\zeta} = S_{ab}^{B_\zeta A'_\zeta} \quad (1.283)$$

$$S_{ab}^{A'_\zeta C'_\zeta} S_{cd}^{C'_\zeta B'_\zeta} = -S_{ab}^{A'_\zeta C'_\zeta} S_{cd}^{C'_\zeta B'_\zeta} \quad S_{ab}^{A_\zeta C_\zeta} S_{cd}^{C_\zeta B_\zeta} = -S_{ab}^{A_\zeta C_\zeta} S_{cd}^{C_\zeta B_\zeta} \quad (1.284)$$

### 6.6.3 对偶性

$$S_{ab}^{A_\zeta B_\zeta} = -\zeta * S_{ab}^{A'_\zeta B'_\zeta} \quad S_{ab}^{A'_\zeta B'_\zeta} = \zeta * S_{ab}^{A_\zeta B_\zeta} \quad (1.285)$$

$$S_{ab}^{A_\zeta B'_\zeta} = -\zeta * S_{ab}^{A'_\zeta B_\zeta} \quad S_{ab}^{A'_\zeta B_\zeta} = \zeta * S_{ab}^{A_\zeta B'_\zeta} \quad (1.286)$$

### 6.6.4 复共轭性

$$[S_{ab}^{A_\zeta B_\zeta} \partial^a \hat{\partial}^b]^* = S_{ab}^{A'_\zeta B'_\zeta} \partial^a \hat{\partial}^b \quad [S_{ab}^{A'_\zeta B'_\zeta} \partial_a \hat{\partial}_b]^* = S_{ab}^{A_\zeta B_\zeta} \partial_a \hat{\partial}_b \quad (1.287)$$

## 6.7 不变常数自旋张量之间的重要关系

### 6.7.1 不变常数自旋张量之间的统一关系

$$\begin{cases} S_{ab}^{A'_\zeta B'_\zeta} = -\frac{i}{4}(\sigma, -i\zeta)_{[a}^{A'_\zeta A_\zeta} \delta_{A_\zeta B_\zeta} (\sigma, i\zeta)_{b]}^{A_\zeta B'_\zeta} \\ \delta_{ab} \delta_{A'_\zeta B'_\zeta} = \frac{1}{2}(\sigma, -i\zeta)_{\{a}^{A'_\zeta A_\zeta} \delta_{A_\zeta B_\zeta} (\sigma, i\zeta)_{b\}}^{A_\zeta B'_\zeta} \end{cases} \quad \begin{cases} S_{ab}^{A_\zeta B_\zeta} = -\frac{i}{4}(\sigma, i\zeta)_{[a}^{A_\zeta A'_\zeta} \delta_{A'_\zeta B'_\zeta} (\sigma, -i\zeta)_{b]}^{A'_\zeta B'_\zeta} \\ \delta_{ab} \delta_{A_\zeta B_\zeta} = \frac{1}{2}(\sigma, i\zeta)_{\{a}^{A_\zeta A'_\zeta} \delta_{A'_\zeta B'_\zeta} (\sigma, -i\zeta)_{b\}}^{A'_\zeta B'_\zeta} \end{cases} \quad (1.288)$$

$$\begin{cases} S_{ab}^{A'_i B'_j} = -\frac{i\zeta}{4}(\sigma, -i\zeta)_{[a}^{A'_i A'_k} \varepsilon_{A_k B_c}(\sigma, -i\zeta)_{b]}^{B'_j B'_l} \\ \delta_{ab} \varepsilon^{A'_i B'_j} = -\frac{1}{2}(\sigma, -i\zeta)_{\{a}^{A'_i A'_k} \varepsilon_{A_k B_c}(\sigma, -i\zeta)_{b\}}^{B'_j B'_l} \end{cases} \begin{cases} S_{ab}^{A'_i B'_j} = \frac{i\zeta}{4}(\sigma, i\zeta)_{A'_i A'_k}^{[a} \varepsilon^{A'_k B'_l}(\sigma, i\zeta)_{B'_j B'_l}^{b]} \\ \delta^{ab} \varepsilon_{A'_i B'_j} = -\frac{1}{2}(\sigma, i\zeta)_{A'_i A'_k}^{\{a} \varepsilon^{A'_k B'_l}(\sigma, i\zeta)_{B'_j B'_l}^{b\}} \end{cases} \quad (1.289)$$

$$\begin{cases} S_{ab}^{A'_i B'_j} = -\frac{i\zeta}{4}(\sigma, i\zeta)_{A'_i A'_k}^{[a} \varepsilon^{A'_k B'_l}(\sigma, i\zeta)_{B'_j B'_l}^{b]} \\ \delta_{ab} \varepsilon_{A'_i B'_j} = -\frac{1}{2}(\sigma, i\zeta)_{A'_i A'_k}^{\{a} \varepsilon^{A'_k B'_l}(\sigma, i\zeta)_{B'_j B'_l}^{b\}} \end{cases} \begin{cases} S_{ab}^{A'_i B'_j} = \frac{i\zeta}{4}(\sigma, -i\zeta)_{[a}^{A'_i A'_k} \varepsilon_{A'_k B'_l}(\sigma, -i\zeta)_{b]}^{B'_j B'_l} \\ \delta_{ab} \varepsilon^{A'_i B'_j} = -\frac{1}{2}(\sigma, -i\zeta)_{\{a}^{A'_i A'_k} \varepsilon_{A'_k B'_l}(\sigma, -i\zeta)_{b\}}^{B'_j B'_l} \end{cases} \quad (1.290)$$

### 6.7.2 乘积关系 $S^{ac} \otimes S_{bc}$

$$\text{推论6.7.1.} \begin{cases} S_{ac}^{A'_i C'_j} \delta_c^d S_{bd}^{B'_k D'_l} = -\frac{1}{8}(\delta_{ab} + 2iS_{ab})_{\{A'_i}^{(B'_k} \delta_{C'_j}^{D'_l)} \\ S_{ac}^{A'_i C'_j} \delta_c^d S_{bd}^{B'_k D'_l} = -\frac{1}{8}(\delta_{ab} + 2iS_{ab})_{\{A'_i}^{(B'_k} \delta_{C'_j}^{D'_l)} \end{cases}$$

证明:  $S_{ac}^{A'_i C'_j} \delta_c^d S_{bd}^{B'_k D'_l}$

$$\begin{aligned} &= \frac{i\zeta}{4}(\sigma, i\zeta)_{A'_i A'_k}^{[a} \varepsilon^{A'_k C'_j}(\sigma, i\zeta)_{C'_j C'_l}^{c]} \delta_c^d \frac{i\zeta}{4}(\sigma, -i\zeta)_{[b}^{B'_k B'_l} \varepsilon_{B'_l D'_l}(\sigma, -i\zeta)_{d]}^{D'_l D'_m} \\ &= -\frac{1}{8}(\sigma, i\zeta)_{A'_i A'_k}^a \varepsilon^{A'_k C'_j}(\sigma, -i\zeta)_{b}^{B'_k B'_l} \varepsilon_{B'_l D'_l} \delta_{C'_j}^{D'_l} \delta_{C'_l}^{D'_m} + \dots \\ &= -\frac{1}{8}(\sigma, i\zeta)_{A'_i A'_k}^a (\sigma, -i\zeta)_b^{A'_k B'_l} \delta_{C'_j}^{D'_l} + \dots \\ &= -\frac{1}{8}(\delta_{ab} + 2iS_{ab})_{A'_i}^{B'_k} \delta_{C'_j}^{D'_l} - \frac{1}{8}(\delta_{ab} + 2iS_{ab})_{C'_j}^{B'_k} \delta_{A'_i}^{D'_l} - \frac{1}{8}(\delta_{ab} + 2iS_{ab})_{A'_i}^{D'_l} \delta_{C'_j}^{B'_k} - \frac{1}{8}(\delta_{ab} + 2iS_{ab})_{C'_j}^{D'_l} \delta_{A'_i}^{B'_k} \\ &= -\frac{1}{8}(\delta_{ab} + 2iS_{ab})_{\{A'_i}^{(B'_k} \delta_{C'_j}^{D'_l)} \end{aligned} \quad \square$$

$$\text{推论6.7.2.} \begin{cases} S_{ac}^{A'_i C'_j} \delta_{cd} S_{bd}^{B'_k D'_l} = \frac{1}{8}(\sigma, i\zeta)_{\{A'_i}^{(B'_k} (\sigma, i\zeta)_{C'_j}^{D'_l)} \\ S_{ac}^{A'_i C'_j} \delta_{cd} S_{bd}^{B'_k D'_l} = \frac{1}{8}(\sigma, i\zeta)_{\{A'_i}^{(B'_k} (\sigma, i\zeta)_{C'_j}^{D'_l)} \end{cases}$$

证明:  $S_{ac}^{A'_i C'_j} \delta_{cd} S_{bd}^{B'_k D'_l}$

$$\begin{aligned} &= \frac{i\zeta}{4}(\sigma, i\zeta)_{A'_i A'_k}^{[a} \varepsilon^{A'_k C'_j}(\sigma, i\zeta)_{C'_j C'_l}^{c]} \delta_{cd} (-\frac{i\zeta}{4})(\sigma, i\zeta)_{[b}^{B'_k B'_l} \varepsilon_{B'_l D'_l}(\sigma, i\zeta)_{d]}^{D'_l D'_m} \\ &= -\frac{1}{8} \varepsilon_{C'_j D'_l} \varepsilon_{C'_l D'_l}(\sigma, i\zeta)_{A'_i A'_k}^a \varepsilon^{A'_k C'_j}(\sigma, i\zeta)_{B'_k B'_l}^b \varepsilon^{B'_k D'_l} + \frac{1}{8} \varepsilon_{A'_i D'_l} \varepsilon_{A'_l D'_l}(\sigma, i\zeta)_{C'_j C'_l}^a \varepsilon^{A'_k C'_j}(\sigma, i\zeta)_{B'_k B'_l}^b \varepsilon^{B'_k D'_l} \\ &+ \frac{1}{8} \varepsilon_{C'_j B'_k} \varepsilon_{C'_l B'_k}(\sigma, i\zeta)_{A'_i A'_k}^a \varepsilon^{A'_k C'_j}(\sigma, i\zeta)_{D'_l D'_l}^b \varepsilon^{B'_k D'_l} - \frac{1}{8} \varepsilon_{A'_i B'_k} \varepsilon_{A'_l B'_k}(\sigma, i\zeta)_{C'_j C'_l}^a \varepsilon^{A'_k C'_j}(\sigma, i\zeta)_{D'_l D'_l}^b \varepsilon^{B'_k D'_l} \\ &= \frac{1}{8}(\sigma, i\zeta)_{A'_i D'_l}^a (\sigma, i\zeta)_{C'_j B'_k}^b + \frac{1}{8}(\sigma, i\zeta)_{C'_j D'_l}^a (\sigma, i\zeta)_{A'_i B'_k}^b + \frac{1}{8}(\sigma, i\zeta)_{A'_i B'_k}^a (\sigma, i\zeta)_{C'_j D'_l}^b + \frac{1}{8}(\sigma, i\zeta)_{C'_j B'_k}^a (\sigma, i\zeta)_{A'_i D'_l}^b \\ &= \frac{1}{8}(\sigma, i\zeta)_{\{A'_i}^{(B'_k} (\sigma, i\zeta)_{C'_j}^{D'_l)} \end{aligned} \quad \square$$

$$\text{推论6.7.3.} S_{ac}^{A'_i C'_j} \delta_{cd} S_{bd}^{B'_k D'_l} = \frac{1}{8}(\sigma, i\zeta)_{\{A'_i}^{(B'_k} (\sigma, i\zeta)_{C'_j}^{D'_l)}, S_{ac}^{A'_i C'_j} \delta_{cd} S_{bd}^{A'_k B'_l} = \frac{1}{8}(\sigma, -i\zeta)_a^{\{A'_i} (\sigma, -i\zeta)_b^{\{A'_k} (\sigma, -i\zeta)_{C'_j}^{D'_l} \} B'_l}$$

$$\text{推论6.7.4.} S_{ac}^{A'_i C'_j} \delta_c^d S_{bd}^{B'_k D'_l} \partial_a \partial^b = \frac{1}{8}(\delta_{ab} + 2iS_{ab})_{\{A'_i}^{(B'_k} \delta_{C'_j}^{D'_l)} \partial^a \partial^b = \frac{1}{8} \delta_{\{A'_i}^{(B'_k} \delta_{C'_j}^{D'_l)} \partial^a \partial_a$$

$$\text{推论6.7.5.} S_{ac}^{A'_i C'_j} \delta_c^d S_{bd}^{B'_k D'_l} \partial_a \partial^b \Delta(x - x') = \frac{1}{8} m^2 \delta_{\{A'_i}^{(B'_k} \delta_{C'_j}^{D'_l)} \Delta(x - x')$$

### 6.7.3 乘积关系 $S_{ab} \partial^b \otimes [\sigma_y(\cdot)]^a, [(\cdot)\sigma_y]^a \otimes S_{ab} \partial^b$

推论6.7.6.

$$\begin{cases} S_{ab}^{A'_i B'_j} \partial_b \delta_{aa'} [\sigma_y(\sigma, -i\zeta)]_{B'_j}^{a'} C'_k = -\frac{\zeta}{2} \delta_{\{A'_i}^{C'_k} (\sigma, i\zeta)_{B'_j}^b \} \partial_b \\ S_{ab}^{A'_i B'_j} \partial^b \delta^{aa'} [\sigma_y(\sigma, i\zeta)]_{a'}^{B'_j} C'_k = \frac{\zeta}{2} \delta_{C'_k}^{\{A'_i} (\sigma, -i\zeta)_b^{B'_j} \} \partial^b \\ S_{ab}^{A'_i B'_j} \partial_b \delta_{aa'} [(\sigma, -i\zeta)\sigma_y]_{a'}^{C'_k} B'_l = -\frac{\zeta}{2} \delta_{\{A'_i}^{C'_k} (\sigma, i\zeta)_{B'_j}^b \} \partial_b \\ S_{ab}^{A'_i B'_j} \partial^b \delta^{aa'} [(\sigma, i\zeta)\sigma_y]_{a'}^{C'_k} B'_l = \frac{\zeta}{2} \delta_{C'_k}^{\{A'_i} (\sigma, -i\zeta)_b^{B'_j} \} \partial^b \end{cases}$$

证明:  $S_{ab}^{A'_i B'_j} \partial_b \delta_{aa'} [\sigma_y(\sigma, -i\zeta)]_{B'_j}^{a'} C'_k$

$$\begin{aligned} &= \frac{i\zeta}{4}(\sigma, i\zeta)_{A'_i A'_k}^{[a} \varepsilon^{A'_k C'_j}(\sigma, i\zeta)_{B'_j C'_l}^{b]} \partial_b \delta_{aa'} \sigma_{B'_l D'_l}^y (\sigma, -i\zeta)_{a'}^{D'_l C'_k} \\ &= \frac{\zeta}{2} \delta_{A'_i}^{C'_k} \delta_{A'_l}^{D'_l} \varepsilon^{A'_k C'_j} \varepsilon_{B'_l D'_l}(\sigma, i\zeta)_{B'_j C'_l}^b \partial_b + \dots \\ &= -\frac{\zeta}{2} \delta_{A'_i}^{C'_k} (\sigma, i\zeta)_{B'_j B'_l}^b \partial_b - \frac{\zeta}{2} \delta_{B'_j}^{C'_k} (\sigma, i\zeta)_{A'_i B'_l}^b \partial_b \\ &= -\frac{\zeta}{2} \delta_{\{A'_i}^{C'_k} (\sigma, i\zeta)_{B'_j}^b \} \partial_b \end{aligned} \quad \square$$

$$\begin{aligned}
& \text{证明: } S_{ab}^{A'_\zeta B'_\zeta} \partial^b \delta^{aa'} [\sigma_y(\sigma, i\zeta)]_{a' B'_\zeta C'_\zeta} \\
&= -\frac{i\zeta}{4} (\sigma, -i\zeta)_{[a}^{A'_\zeta A'_\zeta} \varepsilon_{A'_\zeta C'_\zeta} (\sigma, -i\zeta)_{b]}^{B'_\zeta C'_\zeta} \partial^b \delta_a^a \sigma_y^{B'_\zeta D'_\zeta} (\sigma, i\zeta)_{D'_\zeta C'_\zeta}^{a'} \\
&= -\frac{\zeta}{4} (\sigma, -i\zeta)_{[a}^{A'_\zeta A'_\zeta} \varepsilon_{A'_\zeta C'_\zeta} (\sigma, -i\zeta)_{b]}^{B'_\zeta C'_\zeta} \partial^b \delta_a^a \varepsilon^{B'_\zeta D'_\zeta} (\sigma, i\zeta)_{D'_\zeta C'_\zeta}^{a'} \\
&= -\frac{\zeta}{2} \delta_{C'_\zeta}^{A'_\zeta} \delta_{D'_\zeta}^{A'_\zeta} \varepsilon_{A'_\zeta C'_\zeta} \varepsilon^{B'_\zeta D'_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta C'_\zeta} \partial^b + \dots \\
&= \frac{\zeta}{2} \delta_{C'_\zeta}^{\{A'_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta\} B'_\zeta} \partial^b
\end{aligned}$$

□

推论6.7.7.

$$\begin{cases}
S^{ab}{}_{A'_\zeta B'_\zeta} \partial_b \delta_a^{a'} [\sigma_y(\sigma, i\zeta)]_{a' C'_\zeta B'_\zeta} = -\frac{\zeta}{2} \delta_{\{A'_\zeta}^{C'_\zeta} (\sigma, i\zeta)_b^{B'_\zeta\} B'_\zeta} \partial_b \\
S_{ab}{}^{A'_\zeta B'_\zeta} \partial^b \delta_a^{a'} [\sigma_y(\sigma, -i\zeta)]_{a' C'_\zeta B'_\zeta} = \frac{\zeta}{2} \delta_{C'_\zeta}^{\{A'_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta\} B'_\zeta} \partial^b \\
S^{ab}{}_{A'_\zeta B'_\zeta} \partial_b \delta_a^{a'} [(\sigma, i\zeta) \sigma_y]_{a' B'_\zeta C'_\zeta} = -\frac{\zeta}{2} \delta_{\{A'_\zeta}^{C'_\zeta} (\sigma, i\zeta)_b^{B'_\zeta\} B'_\zeta} \partial_b \\
S_{ab}{}^{A'_\zeta B'_\zeta} \partial^b \delta_a^{a'} [(\sigma, -i\zeta) \sigma_y]_{a' B'_\zeta C'_\zeta} = \frac{\zeta}{2} \delta_{C'_\zeta}^{\{A'_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta\} B'_\zeta} \partial^b
\end{cases}$$

$$\begin{aligned}
& \text{证明: } S^{ab}{}_{A'_\zeta B'_\zeta} \partial_b \delta_a^{a'} [\sigma_y(\sigma, i\zeta)]_{a' C'_\zeta B'_\zeta} \\
&= \frac{i\zeta}{4} (\sigma, i\zeta)_{A'_\zeta A'_\zeta}^{[a} \varepsilon_{A'_\zeta C'_\zeta}^{b]} (\sigma, i\zeta)_{B'_\zeta C'_\zeta}^b \partial_b \delta_{aa'} \sigma_y^{C'_\zeta D'_\zeta} (\sigma, i\zeta)_{D'_\zeta B'_\zeta}^{a'} \\
&= \frac{\zeta}{4} (\sigma, i\zeta)_{A'_\zeta A'_\zeta}^{[a} \varepsilon_{A'_\zeta C'_\zeta}^{b]} (\sigma, i\zeta)_{B'_\zeta C'_\zeta}^b \partial_b \delta_{aa'} \varepsilon^{C'_\zeta D'_\zeta} (\sigma, i\zeta)_{D'_\zeta B'_\zeta}^{a'} \\
&= -\frac{\zeta}{2} \varepsilon_{A'_\zeta D'_\zeta} \varepsilon_{A'_\zeta B'_\zeta} \varepsilon_{A'_\zeta C'_\zeta} \varepsilon^{C'_\zeta D'_\zeta} (\sigma, i\zeta)_b^{B'_\zeta C'_\zeta} \partial_b + \dots \\
&= -\frac{\zeta}{2} \delta_{\{A'_\zeta}^{C'_\zeta} (\sigma, i\zeta)_b^{B'_\zeta\} B'_\zeta} \partial_b
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } S_{ab}{}^{A'_\zeta B'_\zeta} \partial^b \delta_a^{a'} [\sigma_y(\sigma, -i\zeta)]_{a' C'_\zeta B'_\zeta} \\
&= -\frac{i\zeta}{4} (\sigma, -i\zeta)_{[a}^{A'_\zeta A'_\zeta} \varepsilon_{A'_\zeta C'_\zeta} (\sigma, -i\zeta)_{b]}^{B'_\zeta C'_\zeta} \partial^b \delta_a^{a'} \sigma_y^{C'_\zeta D'_\zeta} (\sigma, -i\zeta)_{a' D'_\zeta B'_\zeta}^{D'_\zeta} \\
&= \frac{\zeta}{2} \varepsilon_{A'_\zeta D'_\zeta} \varepsilon_{A'_\zeta B'_\zeta} \varepsilon_{A'_\zeta C'_\zeta} \varepsilon^{C'_\zeta D'_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta C'_\zeta} \partial^b + \dots \\
&= \frac{\zeta}{2} \delta_{C'_\zeta}^{\{A'_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta\} B'_\zeta} \partial^b
\end{aligned}$$

□

6.7.4 乘积关系 $[(\sigma_y)]^a \otimes [(\sigma_y)]_a$ 

$$\text{推论6.7.8. } \begin{cases} [(\sigma, i\zeta) \sigma_y]^a{}_{A'_\zeta B'_\zeta} \delta_{aa'} [\sigma_y(\sigma, -i\zeta)]_{a' A'_\zeta B'_\zeta} = 2\delta_{A'_\zeta}^{B'_\zeta} \delta_{A'_\zeta}^{B'_\zeta} \\ [(\sigma, -i\zeta) \sigma_y]_a{}^{A'_\zeta B'_\zeta} \delta^{aa'} [\sigma_y(\sigma, i\zeta)]_{a' A'_\zeta B'_\zeta} = 2\delta_{B'_\zeta}^{A'_\zeta} \delta_{B'_\zeta}^{A'_\zeta} \end{cases}$$

$$\begin{aligned}
& \text{证明: } [(\sigma, i\zeta) \sigma_y]^a{}_{A'_\zeta B'_\zeta} \delta_{aa'} [\sigma_y(\sigma, -i\zeta)]_{a' A'_\zeta B'_\zeta} \\
&= (\sigma, i\zeta)_{A'_\zeta C'_\zeta}^a \sigma_y^{C'_\zeta B'_\zeta} \delta_a^{a'} \sigma_y^{D'_\zeta B'_\zeta} (\sigma, -i\zeta)_{a' D'_\zeta B'_\zeta} \\
&= -2\delta_{A'_\zeta}^{B'_\zeta} \delta_{C'_\zeta}^{D'_\zeta} \varepsilon^{C'_\zeta B'_\zeta} \varepsilon_{A'_\zeta D'_\zeta} \\
&= 2\delta_{A'_\zeta}^{B'_\zeta} \delta_{A'_\zeta}^{B'_\zeta}
\end{aligned}$$

□

$$\text{推论6.7.9. } \begin{cases} [(\sigma, i\zeta) \sigma_y]^a{}_{A'_\zeta B'_\zeta} \delta_a^{a'} [\sigma_y(\sigma, i\zeta)]_{a' B'_\zeta A'_\zeta} = 2\delta_{A'_\zeta}^{B'_\zeta} \delta_{A'_\zeta}^{B'_\zeta} \\ [(\sigma, -i\zeta) \sigma_y]_a{}^{A'_\zeta B'_\zeta} \delta^{aa'} [\sigma_y(\sigma, i\zeta)]_{a' B'_\zeta A'_\zeta} = 2\delta_{B'_\zeta}^{A'_\zeta} \delta_{B'_\zeta}^{A'_\zeta} \end{cases}$$

$$\begin{aligned}
& \text{证明: } [(\sigma, i\zeta) \sigma_y]^a{}_{A'_\zeta B'_\zeta} \delta_a^{a'} [\sigma_y(\sigma, i\zeta)]_{a' B'_\zeta A'_\zeta} \\
&= (\sigma, i\zeta)_{A'_\zeta C'_\zeta}^a \sigma_y^{C'_\zeta B'_\zeta} \delta_{aa'} \sigma_y^{B'_\zeta D'_\zeta} (\sigma, i\zeta)_{D'_\zeta A'_\zeta}^{a'} \\
&= 2\varepsilon_{A'_\zeta D'_\zeta} \varepsilon_{C'_\zeta A'_\zeta} \varepsilon^{C'_\zeta B'_\zeta} \varepsilon^{B'_\zeta D'_\zeta} \\
&= 2\delta_{A'_\zeta}^{B'_\zeta} \delta_{A'_\zeta}^{B'_\zeta}
\end{aligned}$$

□

## 6.7.5 重要定理的详细证明

$$\begin{aligned}
& \text{引理6.7.1. } [(\sigma, i\zeta) \sigma_y]^a{}_{A'_\zeta B'_\zeta} \partial_a [\sigma_y(\sigma, -i\zeta)]_{a' B'_\zeta}^b \partial_b + (\sigma, i\zeta)_{A'_\zeta A'_\zeta}^a \partial_a (\sigma, -i\zeta)_b^{B'_\zeta B'_\zeta} \partial^b \\
&= (i\sigma_z, I, -i\sigma_x, i\zeta \sigma_y)^a{}_{A'_\zeta B'_\zeta} \partial_a (-i\sigma_z, I, i\sigma_x, -i\zeta \sigma_y)^b{}_{A'_\zeta B'_\zeta} \partial_b + (\sigma_x, \sigma_y, \sigma_z, i\zeta)_{A'_\zeta A'_\zeta}^a \partial_a (\sigma_x, \sigma_y, \sigma_z, -i\zeta)_b^{B'_\zeta B'_\zeta} \partial^b
\end{aligned}$$

$$\text{定理6.7.1. } [(\sigma, i\zeta) \sigma_y]^a{}_{A'_\zeta B'_\zeta} \partial_a [\sigma_y(\sigma, -i\zeta)]_{a' B'_\zeta}^b \partial_b + (\sigma, i\zeta)_{A'_\zeta A'_\zeta}^a \partial_a (\sigma, -i\zeta)_b^{B'_\zeta B'_\zeta} \partial^b = \partial^a \partial_a \delta_{A'_\zeta}^{B'_\zeta} \delta_{A'_\zeta}^{B'_\zeta}$$



$$\begin{aligned}
& \text{证明: } (i\sigma_z, I, -i\sigma_x, i\sigma_y)^a_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_a (-i\sigma_z, I, i\sigma_x, -i\sigma_y)^b_{1\zeta} \delta_{1\zeta}^{2\zeta} \partial_b + (\sigma_x, \sigma_y, \sigma_z, i\zeta)^a_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_a (\sigma_x, \sigma_y, \sigma_z, -i\zeta)^b_{1\zeta} \delta_{1\zeta}^{2\zeta} \partial^b \\
& = (-i\partial_z - \zeta\partial_\pi)(i\partial_z - \zeta\partial_\pi) + (\partial_x + i\partial_y)(\partial_x - i\partial_y) \\
& = (\partial_z^2 + \partial_\pi^2) + (\partial_x^2 + \partial_y^2) \\
& = \partial^a \partial_a \delta_{2\zeta}^{2\zeta} \delta_{1\zeta}^{1\zeta}
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } (i\sigma_z, I, -i\sigma_x, i\sigma_y)^a_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_a (-i\sigma_z, I, i\sigma_x, -i\sigma_y)^b_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_b + (\sigma_x, \sigma_y, \sigma_z, i\zeta)^a_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_a (\sigma_x, \sigma_y, \sigma_z, -i\zeta)^b_{1\zeta} \delta_{1\zeta}^{2\zeta} \partial^b \\
& = (-i\partial_z - \zeta\partial_\pi)(i\partial_z + \zeta\partial_\pi) + (-\partial_z + i\zeta\partial_\pi)(\partial_z - i\zeta\partial_\pi) \\
& = 0 \\
& = \partial^a \partial_a \delta_{2\zeta}^{1\zeta} \delta_{2\zeta}^{1\zeta}
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } (i\sigma_z, I, -i\sigma_x, i\sigma_y)^a_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_a (-i\sigma_z, I, i\sigma_x, -i\sigma_y)^b_{2\zeta} \delta_{2\zeta}^{2\zeta} \partial_b + (\sigma_x, \sigma_y, \sigma_z, i\zeta)^a_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_a (\sigma_x, \sigma_y, \sigma_z, -i\zeta)^b_{1\zeta} \delta_{1\zeta}^{2\zeta} \partial^b \\
& = (-i\partial_z - \zeta\partial_\pi)(i\partial_x + \partial_y) + (-\partial_z + i\zeta\partial_\pi)(\partial_x - i\partial_y) \\
& = 0 \\
& = \partial^a \partial_a \delta_{2\zeta}^{2\zeta} \delta_{2\zeta}^{1\zeta}
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } (i\sigma_z, I, -i\sigma_x, i\sigma_y)^a_{2\zeta} \delta_{2\zeta}^{2\zeta} \partial_a (-i\sigma_z, I, i\sigma_x, -i\sigma_y)^b_{1\zeta} \delta_{1\zeta}^{1\zeta} \partial_b + (\sigma_x, \sigma_y, \sigma_z, i\zeta)^a_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_a (\sigma_x, \sigma_y, \sigma_z, -i\zeta)^b_{1\zeta} \delta_{1\zeta}^{2\zeta} \partial^b \\
& = (-i\partial_x + \partial_y)(-i\partial_x + \partial_y) + (\partial_x + i\partial_y)(\partial_x + i\zeta\partial_y) \\
& = 0 \\
& = \partial^a \partial_a \delta_{2\zeta}^{1\zeta} \delta_{1\zeta}^{2\zeta}
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } (i\sigma_z, I, -i\sigma_x, i\sigma_y)^a_{2\zeta} \delta_{2\zeta}^{2\zeta} \partial_a (-i\sigma_z, I, i\sigma_x, -i\sigma_y)^b_{1\zeta} \delta_{1\zeta}^{2\zeta} \partial_b + (\sigma_x, \sigma_y, \sigma_z, i\zeta)^a_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_a (\sigma_x, \sigma_y, \sigma_z, -i\zeta)^b_{1\zeta} \delta_{1\zeta}^{2\zeta} \partial^b \\
& = (-i\partial_x + \partial_y)(i\partial_z - \zeta\partial_\pi) + (\partial_x + i\partial_y)(-\partial_z - i\zeta\partial_\pi) \\
& = 0 \\
& = \partial^a \partial_a \delta_{2\zeta}^{2\zeta} \delta_{1\zeta}^{2\zeta}
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } (i\sigma_z, I, -i\sigma_x, i\sigma_y)^a_{2\zeta} \delta_{2\zeta}^{2\zeta} \partial_a (-i\sigma_z, I, i\sigma_x, -i\sigma_y)^b_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_b + (\sigma_x, \sigma_y, \sigma_z, i\zeta)^a_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_a (\sigma_x, \sigma_y, \sigma_z, -i\zeta)^b_{1\zeta} \delta_{1\zeta}^{2\zeta} \partial^b \\
& = (-i\partial_x + \partial_y)(i\partial_z + \zeta\partial_\pi) + (-\partial_z + i\zeta\partial_\pi)(\partial_x + i\partial_y) \\
& = 0 \\
& = \partial^a \partial_a \delta_{2\zeta}^{1\zeta} \delta_{2\zeta}^{2\zeta}
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } (i\sigma_z, I, -i\sigma_x, i\sigma_y)^a_{2\zeta} \delta_{2\zeta}^{2\zeta} \partial_a (-i\sigma_z, I, i\sigma_x, -i\sigma_y)^b_{2\zeta} \delta_{2\zeta}^{2\zeta} \partial_b + (\sigma_x, \sigma_y, \sigma_z, i\zeta)^a_{2\zeta} \delta_{2\zeta}^{1\zeta} \partial_a (\sigma_x, \sigma_y, \sigma_z, -i\zeta)^b_{1\zeta} \delta_{1\zeta}^{2\zeta} \partial^b \\
& = (-i\partial_x + \partial_y)(i\partial_x + \partial_y) + (-\partial_z + i\zeta\partial_\pi)(-\partial_z - i\zeta\partial_\pi) \\
& = (\partial_x^2 + \partial_y^2) + (\partial_z^2 + \partial_\pi^2) \\
& = \partial^a \partial_a \delta_{2\zeta}^{2\zeta} \delta_{2\zeta}^{2\zeta}
\end{aligned}$$

□

$$\text{定理6.7.2. } (\sigma, i\zeta)^a_{[A_\zeta A'_\zeta]} (\sigma, i\zeta)^b_{[B_\zeta] B'_\zeta} \partial_a \partial_b = -\partial^a \partial_a \varepsilon_{A_\zeta B_\zeta} \varepsilon_{A'_\zeta B'_\zeta}, (\sigma, -i\zeta)^a_{[A'_\zeta A_\zeta]} (\sigma, -i\zeta)^b_{[B'_\zeta] B_\zeta} \partial^a \partial^b = -\partial^a \partial_a \varepsilon_{A'_\zeta B'_\zeta} \varepsilon_{A_\zeta B_\zeta}$$

$$\text{定理6.7.3. } (\sigma, i\zeta)^a_{[A_\zeta A'_\zeta]} (\sigma, i\zeta)^b_{[B_\zeta] B'_\zeta} \delta_{ab} = -\delta^a_a \varepsilon_{A_\zeta B_\zeta} \varepsilon_{A'_\zeta B'_\zeta}, (\sigma, -i\zeta)^a_{[A'_\zeta A_\zeta]} (\sigma, -i\zeta)^b_{[B'_\zeta] B_\zeta} \delta^{ab} = -\delta^a_a \varepsilon_{A'_\zeta B'_\zeta} \varepsilon_{A_\zeta B_\zeta}$$

$$\text{定理6.7.4. } [(\sigma, i\zeta)\sigma_y]^a_{A_\zeta} \delta_{A_\zeta}^{B'_\zeta} [\sigma_y(\sigma, -i\zeta)]^b_{A'_\zeta} \delta_{A'_\zeta}^{B_\zeta} \partial_a \partial_b + (\sigma, i\zeta)^a_{A_\zeta A'_\zeta} (\sigma, -i\zeta)^b_{B'_\zeta B_\zeta} \partial_a \partial^b = \partial^a \partial_a \delta_{A_\zeta}^{B'_\zeta} \delta_{A'_\zeta}^{B_\zeta}$$

$$\text{定理6.7.5. } [(\sigma, i\zeta)\sigma_y]^a_{A_\zeta} \delta_{A_\zeta}^{B'_\zeta} [\sigma_y(\sigma, -i\zeta)]^b_{A'_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta^{ab} + (\sigma, i\zeta)^a_{A_\zeta A'_\zeta} (\sigma, -i\zeta)^b_{B'_\zeta B_\zeta} \delta_a^b = \delta^a_a \delta_{A_\zeta}^{B'_\zeta} \delta_{A'_\zeta}^{B_\zeta}$$

## 6.8 复合常数不变张量 $\Sigma_{k_s l_{s'}}^{k_s l_s}(s, s')$ 的性质

### 6.8.1 定义

$$c(s) = [(-1)^{2s} \frac{8}{3} s(s + \frac{1}{2})(s + 1)]^{-\frac{1}{2}} \quad (1.291)$$

$$\Sigma_{ab}^{k_s l_s}(s) := \begin{cases} c(|s|) S_{ab}^{kl}(|s|, +), & s > 0 \\ c(|s|) S_{abk'l'}(|s|, -), & s < 0 \end{cases} \quad \Sigma_{k_s l_s}^{ab}(s) := \begin{cases} c(|s|) S^{ab}_{kl}(|s|, +), & s > 0 \\ c(|s|) S^{abk'l'}(|s|, -), & s < 0 \end{cases} \quad (1.292)$$

$$\text{定义: } \Sigma_{k_s l_{s'}}^{k_s l_s}(s, s') := \Sigma_{ab}^{k_s l_s}(s) \Sigma_{k_{s'} l_{s'}}^{ab}(s') \quad (1.293)$$

### 6.8.2 传递性

$$\Sigma_{k_s l_{s'}}^{k_s l_s}(s, s') \Sigma_{k_{s'} l_{s''}}^{k_{s'} l_{s'}}(s', s'') = \Sigma_{k_{s'} l_{s''}}^{k_s l_s}(s, s'') \quad (1.294)$$

### 6.8.3 对称性与反对称性

$$\Sigma_{k_s l_{s'}}^{k_s l_s}(s, s') = (-1)^{2s+1} \Sigma_{k_{s'} l_{s'}}^{l_s k_s}(s, s') \quad \Sigma_{k_{s'} l_{s'}}^{l_s k_s}(s, s') = (-1)^{2s'+1} \Sigma_{l_{s'} k_{s'}}^{k_s l_s}(s, s') \quad (1.295)$$

$$\Sigma_{k_s l_{s'}}^{k_s l_s}(s, s') = (-1)^{2(s+s')} \Sigma_{l_{s'} k_{s'}}^{l_s k_s}(s, s') \quad (1.296)$$

## 7 常数不变张量的一般理论

### 7.1 常数不变张量的一般定义

$$\text{定义洛伦兹变换: } \Lambda[L_i] := e^{\frac{i}{2} \vartheta^{ab} S_{ab}[L_i]}, \text{ 定义YM场变换: } \Lambda[Y_j] := e^{i\theta^\alpha T_\alpha[Y_j]} \quad (1.297)$$

常数不变张量的一般定义:

在任何参考系中  $C_{L_1 L_2 \dots L_n}^{Y_1 Y_2 \dots Y_m}$  是常数并且相等, 并满足如下变换:

$$C_{L_1 L_2 \dots L_n}^{Y_1 Y_2 \dots Y_m} = \prod_{i=1}^n \Lambda_{L_i}^{L'_i}[L_i] \prod_{j=1}^m \Lambda_{Y'_j}^{Y_j}[Y_j] C_{L'_1 L'_2 \dots L'_n}^{Y'_1 Y'_2 \dots Y'_m} \quad (1.298)$$

则  $C_{L_1 L_2 \dots L_n}^{Y_1 Y_2 \dots Y_m}$  是常数不变张量,  $L_i \sim e^{\frac{i}{2} \vartheta^{ab} S_{ab}[L_i]}, Y_j \sim e^{i\theta^\alpha T_\alpha[Y_j]}$

无穷小变换

$$0 = \delta C_{L_1 L_2 \dots L_n}^{Y_1 Y_2 \dots Y_m} = \frac{1}{2} \vartheta^{ab} \sum_{i=1}^n S_{ab L_i}^{L'_i}[L_i] C_{L_1 L_2 \dots L'_i \dots L_n}^{Y_1 Y_2 \dots Y_m} + i\theta^\alpha \sum_{j=1}^m T_{\alpha Y'_j}^{Y_j}[L_i] C_{L_1 L_2 \dots L_n}^{Y_1 Y_2 \dots Y'_j \dots Y_m}, \forall \vartheta^{ab}, \forall \theta^\alpha \quad (1.299)$$

$$\Leftrightarrow \sum_{i=1}^n S_{ab L_i}^{L'_i}[L_i] C_{L_1 L_2 \dots L'_i \dots L_n}^{Y_1 Y_2 \dots Y_m} = 0, \sum_{j=1}^m T_{\alpha Y'_j}^{Y_j}[L_i] C_{L_1 L_2 \dots L_n}^{Y_1 Y_2 \dots Y'_j \dots Y_m} = 0 \quad (1.300)$$

### 7.2 常数不变张量的协变导数为零

$$D_u C_{L_1 L_2 \dots L_n}^{Y_1 Y_2 \dots Y_m} = \partial_u C_{L_1 L_2 \dots L_n}^{Y_1 Y_2 \dots Y_m} + \frac{1}{2} \omega_u^{ab} \sum_{i=1}^n S_{ab L_i}^{L'_i}[L_i] C_{L_1 \dots L'_i \dots L_n}^{Y_1 Y_2 \dots Y_m} + iA_u^\alpha \sum_{j=1}^m T_{\alpha Y'_j}^{Y_j}[L_i] C_{L_1 L_2 \dots L_n}^{Y_1 \dots Y'_j \dots Y_m} \quad (1.301)$$

$$D_u C_{L_1 L_2 \dots L_n}^{Y_1 Y_2 \dots Y_m} = 0 + 0 + 0 = 0 \quad (1.302)$$

因此所有常数不变张量的协变导数全为零, 这是一个很好、很方便性质。



## 第二章 完美常数不变张量

自我评述：本章创立的完美常数不变张量，具有一些完美的性质，有很大的广泛性，适用于多种情形。它将全对称的低自旋张量与高自旋量联系起来，是研究一般自旋粒子的一个有力数学工具。我是受全反对称张量的启发，去尝试寻找一个类似的全对称张量，经过不断尝试，最后就找到了这样一个全对称旋量张量。然后结合上一章得到的各种基本常数不变张量，进一步发展得到了多个有用的特殊常数不变张量。

### 1 对称指标排序

自我评述：与之前排序起点从1开始计数不同，本节排序起点从0开始计数，这样更自然，与w+1进制自然吻合。

$$\text{定义1.0.1. } \sigma \otimes I = I \bar{\otimes} \sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$$

#### 1.1 二阶对称指标排序

##### 1.1.1 $A \bar{\otimes} B \bar{\otimes} C \bar{\otimes} D$ 二阶对称指标排列

推论1.1.1.

$$\left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right\} \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \right\} \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \right\} \left\{ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right\}$$

##### 1.1.2 $A \leq B \leq C \leq D$ 二阶对称指标排序规律的推演

推论1.1.2.

$$\begin{cases} c_3 = 0 \\ c_2 = a_2 = 2 \\ c_1 = a_2 + a_1 = 2 + 1 \\ c_n = C_3^2 - C_n^2 \end{cases} \begin{cases} d_0 = c_3 = 0 \\ d_1 = c_2 = 2 \\ d_2 = c_1 = 3 \\ d_n = C_3^2 - C_{3-n}^2 \end{cases}$$

$$\begin{cases} A_0 d_0 = 0 \\ A_0 d_1 = 2 \\ A_0 d_2 = 3 \\ A_0 d_n = (C_3^2 - C_{3-n}^2)u(C_3^2 - C_{5-3}^2) \end{cases} \begin{cases} A_1 d_0 = 0 + C_3^2 \\ A_1 d_1 = 0 + C_3^2 \\ A_1 d_2 = 1 + C_3^2 \\ A_1 d_n = (C_2^2 - C_{3-n}^2)u(C_2^2 - C_{3-n}^2) + C_3^2 \end{cases}$$

以上只是我当初为了寻找二阶全对称指标排序规律而进行试探的原始方法，只是中间产品，无需深究，读者不需看懂。留在这里，只是留住当初自己的灵感，让自己隐约记得当初是如何一步步试探，最终猜测得到以下二阶全对称指标的排序规律而已。

##### 1.1.3 二阶对称指标正向排序规律的归纳小结

推论1.1.3.

$$k_{0 \leq 0 \leq \eta \leq \xi} = (C_3^2 - C_{3-\eta}^2) + \left( \sum_{k=\eta}^{\xi} C_{1-k}^0 - C_{1-\lambda_0}^0 \right)$$

$$k_{0 \leq \mu \leq \eta \leq \xi} = \left( \sum_{k=0}^{\mu} C_{3-k}^2 - C_{3-\eta}^2 \right) + \left( \sum_{k=\eta}^{\xi} C_{1-k}^0 - C_{1-\lambda_0}^0 \right)$$

$$k_{\lambda \leq \mu \leq \eta \leq \xi} = (C_5^4 - C_{5-\lambda}^4) + \left( \sum_{k=\lambda}^{\mu} C_{3-k}^2 - C_{3-\eta}^2 \right) + \left( \sum_{k=\eta}^{\xi} C_{1-k}^0 - C_{1-\lambda_0}^0 \right)$$

$$k_{d \leq \lambda \leq \mu \leq \eta \leq \xi} = \left( \sum_{k=0}^d C_{5-k}^4 - C_{5-\lambda}^4 \right) + \left( \sum_{k=\lambda}^{\mu} C_{3-k}^2 - C_{3-\eta}^2 \right) + \left( \sum_{k=\eta}^{\xi} C_{1-k}^0 - C_{1-\lambda_0}^0 \right)$$

### 1.1.4 二阶对称指标正向排序规律的一般猜想

猜想1.1.1.

$$k_{\lambda_{2s} \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1} = \sum_{l=0}^{[s]} \left( \sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+1-k}^{2l} - C_{2l+1-\lambda_{2l}}^{2l} \right); \lambda_i = 0, i > 2s | i = 0; \lambda_i = (0, 1), 1 \leq i \leq 2s$$

### 1.1.5 二阶对称指标正向排序规律的验证

性质1.1.1. 0 : 0000 → 1 : 0001 → 2 : 0011 → 3 : 0111 → 4 : 1111

## 1.2 四阶对称指标排序

### 1.2.1 $\lambda \otimes \mu \otimes \eta \otimes \xi$ 四阶对称指标排列

定义1.2.1.

$\left\{ \begin{matrix} [0\ 0\ 0\ 0] \\ [0\ 0\ 0\ 1] \\ [0\ 0\ 0\ 2] \\ [0\ 0\ 0\ 3] \\ [0\ 0\ 1\ 0] \\ [0\ 0\ 1\ 1] \\ [0\ 0\ 1\ 2] \\ [0\ 0\ 1\ 3] \\ [0\ 0\ 2\ 0] \\ [0\ 0\ 2\ 1] \\ [0\ 0\ 2\ 2] \\ [0\ 0\ 2\ 3] \\ [0\ 0\ 3\ 0] \\ [0\ 0\ 3\ 1] \\ [0\ 0\ 3\ 2] \\ [0\ 0\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [0\ 1\ 0\ 0] \\ [0\ 1\ 0\ 1] \\ [0\ 1\ 0\ 2] \\ [0\ 1\ 0\ 3] \\ [0\ 1\ 1\ 0] \\ [0\ 1\ 1\ 1] \\ [0\ 1\ 1\ 2] \\ [0\ 1\ 1\ 3] \\ [0\ 1\ 2\ 0] \\ [0\ 1\ 2\ 1] \\ [0\ 1\ 2\ 2] \\ [0\ 1\ 2\ 3] \\ [0\ 1\ 3\ 0] \\ [0\ 1\ 3\ 1] \\ [0\ 1\ 3\ 2] \\ [0\ 1\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [0\ 2\ 0\ 0] \\ [0\ 2\ 0\ 1] \\ [0\ 2\ 0\ 2] \\ [0\ 2\ 0\ 3] \\ [0\ 2\ 1\ 0] \\ [0\ 2\ 1\ 1] \\ [0\ 2\ 1\ 2] \\ [0\ 2\ 1\ 3] \\ [0\ 2\ 2\ 0] \\ [0\ 2\ 2\ 1] \\ [0\ 2\ 2\ 2] \\ [0\ 2\ 2\ 3] \\ [0\ 2\ 3\ 0] \\ [0\ 2\ 3\ 1] \\ [0\ 2\ 3\ 2] \\ [0\ 2\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [0\ 3\ 0\ 0] \\ [0\ 3\ 0\ 1] \\ [0\ 3\ 0\ 2] \\ [0\ 3\ 0\ 3] \\ [0\ 3\ 1\ 0] \\ [0\ 3\ 1\ 1] \\ [0\ 3\ 1\ 2] \\ [0\ 3\ 1\ 3] \\ [0\ 3\ 2\ 0] \\ [0\ 3\ 2\ 1] \\ [0\ 3\ 2\ 2] \\ [0\ 3\ 2\ 3] \\ [0\ 3\ 3\ 0] \\ [0\ 3\ 3\ 1] \\ [0\ 3\ 3\ 2] \\ [0\ 3\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [1\ 0\ 0\ 0] \\ [1\ 0\ 0\ 1] \\ [1\ 0\ 0\ 2] \\ [1\ 0\ 0\ 3] \\ [1\ 0\ 1\ 0] \\ [1\ 0\ 1\ 1] \\ [1\ 0\ 1\ 2] \\ [1\ 0\ 1\ 3] \\ [1\ 0\ 2\ 0] \\ [1\ 0\ 2\ 1] \\ [1\ 0\ 2\ 2] \\ [1\ 0\ 2\ 3] \\ [1\ 0\ 3\ 0] \\ [1\ 0\ 3\ 1] \\ [1\ 0\ 3\ 2] \\ [1\ 0\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [1\ 1\ 0\ 0] \\ [1\ 1\ 0\ 1] \\ [1\ 1\ 0\ 2] \\ [1\ 1\ 0\ 3] \\ [1\ 1\ 1\ 0] \\ [1\ 1\ 1\ 1] \\ [1\ 1\ 1\ 2] \\ [1\ 1\ 1\ 3] \\ [1\ 1\ 2\ 0] \\ [1\ 1\ 2\ 1] \\ [1\ 1\ 2\ 2] \\ [1\ 1\ 2\ 3] \\ [1\ 1\ 3\ 0] \\ [1\ 1\ 3\ 1] \\ [1\ 1\ 3\ 2] \\ [1\ 1\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [1\ 2\ 0\ 0] \\ [1\ 2\ 0\ 1] \\ [1\ 2\ 0\ 2] \\ [1\ 2\ 0\ 3] \\ [1\ 2\ 1\ 0] \\ [1\ 2\ 1\ 1] \\ [1\ 2\ 1\ 2] \\ [1\ 2\ 1\ 3] \\ [1\ 2\ 2\ 0] \\ [1\ 2\ 2\ 1] \\ [1\ 2\ 2\ 2] \\ [1\ 2\ 2\ 3] \\ [1\ 2\ 3\ 0] \\ [1\ 2\ 3\ 1] \\ [1\ 2\ 3\ 2] \\ [1\ 2\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [1\ 3\ 0\ 0] \\ [1\ 3\ 0\ 1] \\ [1\ 3\ 0\ 2] \\ [1\ 3\ 0\ 3] \\ [1\ 3\ 1\ 0] \\ [1\ 3\ 1\ 1] \\ [1\ 3\ 1\ 2] \\ [1\ 3\ 1\ 3] \\ [1\ 3\ 2\ 0] \\ [1\ 3\ 2\ 1] \\ [1\ 3\ 2\ 2] \\ [1\ 3\ 2\ 3] \\ [1\ 3\ 3\ 0] \\ [1\ 3\ 3\ 1] \\ [1\ 3\ 3\ 2] \\ [1\ 3\ 3\ 3] \end{matrix} \right\}$
$\left\{ \begin{matrix} [2\ 0\ 0\ 0] \\ [2\ 0\ 0\ 1] \\ [2\ 0\ 0\ 2] \\ [2\ 0\ 0\ 3] \\ [2\ 0\ 1\ 0] \\ [2\ 0\ 1\ 1] \\ [2\ 0\ 1\ 2] \\ [2\ 0\ 1\ 3] \\ [2\ 0\ 2\ 0] \\ [2\ 0\ 2\ 1] \\ [2\ 0\ 2\ 2] \\ [2\ 0\ 2\ 3] \\ [2\ 0\ 3\ 0] \\ [2\ 0\ 3\ 1] \\ [2\ 0\ 3\ 2] \\ [2\ 0\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [2\ 1\ 0\ 0] \\ [2\ 1\ 0\ 1] \\ [2\ 1\ 0\ 2] \\ [2\ 1\ 0\ 3] \\ [2\ 1\ 1\ 0] \\ [2\ 1\ 1\ 1] \\ [2\ 1\ 1\ 2] \\ [2\ 1\ 1\ 3] \\ [2\ 1\ 2\ 0] \\ [2\ 1\ 2\ 1] \\ [2\ 1\ 2\ 2] \\ [2\ 1\ 2\ 3] \\ [2\ 1\ 3\ 0] \\ [2\ 1\ 3\ 1] \\ [2\ 1\ 3\ 2] \\ [2\ 1\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [2\ 2\ 0\ 0] \\ [2\ 2\ 0\ 1] \\ [2\ 2\ 0\ 2] \\ [2\ 2\ 0\ 3] \\ [2\ 2\ 1\ 0] \\ [2\ 2\ 1\ 1] \\ [2\ 2\ 1\ 2] \\ [2\ 2\ 1\ 3] \\ [2\ 2\ 2\ 0] \\ [2\ 2\ 2\ 1] \\ [2\ 2\ 2\ 2] \\ [2\ 2\ 2\ 3] \\ [2\ 2\ 3\ 0] \\ [2\ 2\ 3\ 1] \\ [2\ 2\ 3\ 2] \\ [2\ 2\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [2\ 3\ 0\ 0] \\ [2\ 3\ 0\ 1] \\ [2\ 3\ 0\ 2] \\ [2\ 3\ 0\ 3] \\ [2\ 3\ 1\ 0] \\ [2\ 3\ 1\ 1] \\ [2\ 3\ 1\ 2] \\ [2\ 3\ 1\ 3] \\ [2\ 3\ 2\ 0] \\ [2\ 3\ 2\ 1] \\ [2\ 3\ 2\ 2] \\ [2\ 3\ 2\ 3] \\ [2\ 3\ 3\ 0] \\ [2\ 3\ 3\ 1] \\ [2\ 3\ 3\ 2] \\ [2\ 3\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [3\ 0\ 0\ 0] \\ [3\ 0\ 0\ 1] \\ [3\ 0\ 0\ 2] \\ [3\ 0\ 0\ 3] \\ [3\ 0\ 1\ 0] \\ [3\ 0\ 1\ 1] \\ [3\ 0\ 1\ 2] \\ [3\ 0\ 1\ 3] \\ [3\ 0\ 2\ 0] \\ [3\ 0\ 2\ 1] \\ [3\ 0\ 2\ 2] \\ [3\ 0\ 2\ 3] \\ [3\ 0\ 3\ 0] \\ [3\ 0\ 3\ 1] \\ [3\ 0\ 3\ 2] \\ [3\ 0\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [3\ 1\ 0\ 0] \\ [3\ 1\ 0\ 1] \\ [3\ 1\ 0\ 2] \\ [3\ 1\ 0\ 3] \\ [3\ 1\ 1\ 0] \\ [3\ 1\ 1\ 1] \\ [3\ 1\ 1\ 2] \\ [3\ 1\ 1\ 3] \\ [3\ 1\ 2\ 0] \\ [3\ 1\ 2\ 1] \\ [3\ 1\ 2\ 2] \\ [3\ 1\ 2\ 3] \\ [3\ 1\ 3\ 0] \\ [3\ 1\ 3\ 1] \\ [3\ 1\ 3\ 2] \\ [3\ 1\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [3\ 2\ 0\ 0] \\ [3\ 2\ 0\ 1] \\ [3\ 2\ 0\ 2] \\ [3\ 2\ 0\ 3] \\ [3\ 2\ 1\ 0] \\ [3\ 2\ 1\ 1] \\ [3\ 2\ 1\ 2] \\ [3\ 2\ 1\ 3] \\ [3\ 2\ 2\ 0] \\ [3\ 2\ 2\ 1] \\ [3\ 2\ 2\ 2] \\ [3\ 2\ 2\ 3] \\ [3\ 2\ 3\ 0] \\ [3\ 2\ 3\ 1] \\ [3\ 2\ 3\ 2] \\ [3\ 2\ 3\ 3] \end{matrix} \right\}$	$\left\{ \begin{matrix} [3\ 3\ 0\ 0] \\ [3\ 3\ 0\ 1] \\ [3\ 3\ 0\ 2] \\ [3\ 3\ 0\ 3] \\ [3\ 3\ 1\ 0] \\ [3\ 3\ 1\ 1] \\ [3\ 3\ 1\ 2] \\ [3\ 3\ 1\ 3] \\ [3\ 3\ 2\ 0] \\ [3\ 3\ 2\ 1] \\ [3\ 3\ 2\ 2] \\ [3\ 3\ 2\ 3] \\ [3\ 3\ 3\ 0] \\ [3\ 3\ 3\ 1] \\ [3\ 3\ 3\ 2] \\ [3\ 3\ 3\ 3] \end{matrix} \right\}$

推论1.2.1.

$$\left\{ \begin{matrix} \text{number}\{\bar{\square}\} : b_4 = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 4 & 4 \\ 2 & 2 & 2 & 4 \\ 3 & 3 & 3 & 3 \\ 4 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} & b_3 = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 1 & 4 & 4 \\ 4 & 2 & 2 & 4 \\ 4 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} & b_2 = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 2 & 4 \\ 4 & 4 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} & b_1 = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & b_n = 4^3 - C_{n+2}^3 \\ \text{number}\{\bar{\square}\} : a_4 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} & a_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} & a_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} & a_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & a_n = C_{n+2}^3 \end{matrix} \right.$$

### 1.2.2 $\lambda \leq \mu \leq \eta \leq \xi$ 四阶对称指标排序规律的推演

推论1.2.2.

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1), \sum_{k=1}^n k^2 = \frac{1}{3}n(n+\frac{1}{2})(n+1), \sum_{k=1}^n k^3 = \frac{1}{4}n^2(n+1)^2$$

$$\left\{ \begin{matrix} c_5 = 0 \\ c_4 = a_4 \\ c_3 = a_4 + a_3 \\ c_2 = a_4 + a_3 + a_2 \\ c_1 = a_4 + a_3 + a_2 + a_1 \\ c_n = C_7^4 - C_{n+2}^4 \end{matrix} \right. \left\{ \begin{matrix} d_0 = c_5 = 0 \\ d_1 = c_4 = 20 \\ d_2 = c_3 = 30 \\ d_3 = c_2 = 34 \\ d_4 = c_1 = 35 \\ d_n = C_7^4 - C_{7-n}^4 \end{matrix} \right.$$

$$\begin{cases}
A_0 d_0 = A_0 c_5 = 0 \\
A_0 d_1 = A_0 c_4 = 4 \\
A_0 d_2 = A_0 c_3 = 7 \\
A_0 d_3 = A_0 c_2 = 9 \\
A_0 d_4 = A_0 c_1 = 10 \\
A_0 d_n = (C_5^2 - C_{5-n}^2)u(C_5^2 - C_{5-n}^2)
\end{cases}
\begin{cases}
A_1 d_0 = A_1 c_5 = 0 + C_5^2 \\
A_1 d_1 = A_1 c_4 = 0 + C_5^2 \\
A_1 d_2 = A_1 c_3 = 3 + C_5^2 \\
A_1 d_3 = A_1 c_2 = 5 + C_5^2 \\
A_1 d_4 = A_1 c_1 = 6 + C_5^2 \\
A_1 d_n = (C_4^2 - C_{5-n}^2)u(C_4^2 - C_{5-n}^2) + C_5^2
\end{cases}$$

$$\begin{cases}
A_2 d_0 = A_2 c_5 = 0 + C_5^2 + C_4^2 \\
A_2 d_1 = A_2 c_4 = 0 + C_5^2 + C_4^2 \\
A_2 d_2 = A_2 c_3 = 0 + C_5^2 + C_4^2 \\
A_2 d_3 = A_2 c_2 = 2 + C_5^2 + C_4^2 \\
A_2 d_4 = A_2 c_1 = 3 + C_5^2 + C_4^2 \\
A_2 d_n = (C_3^2 - C_{5-n}^2)u(C_3^2 - C_{5-n}^2) + C_5^2 + C_4^2
\end{cases}
\begin{cases}
A_3 d_0 = A_3 c_5 = 0 + C_5^2 + C_4^2 + C_3^2 \\
A_3 d_1 = A_3 c_4 = 0 + C_5^2 + C_4^2 + C_3^2 \\
A_3 d_2 = A_3 c_3 = 0 + C_5^2 + C_4^2 + C_3^2 \\
A_3 d_3 = A_3 c_2 = 0 + C_5^2 + C_4^2 + C_3^2 \\
A_3 d_4 = A_3 c_1 = 1 + C_5^2 + C_4^2 + C_3^2 \\
A_3 d_n = (C_2^2 - C_{5-n}^2)u(C_2^2 - C_{5-n}^2) + C_5^2 + C_4^2 + C_3^2
\end{cases}$$

推论1.2.3.

$$\begin{cases}
A_0 d_n = (C_5^2 - C_{5-n}^2)u(C_5^2 - C_{5-n}^2) \\
A_1 d_n = (C_4^2 - C_{5-n}^2)u(C_4^2 - C_{5-n}^2) + C_5^2 \\
A_2 d_n = (C_3^2 - C_{5-n}^2)u(C_3^2 - C_{5-n}^2) + C_5^2 + C_4^2 \\
A_3 d_n = (C_2^2 - C_{5-n}^2)u(C_2^2 - C_{5-n}^2) + C_5^2 + C_4^2 + C_3^2
\end{cases}$$

$$\begin{cases}
B_0 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 20 \\
B_1 d_n = (C_4^2 - C_{5-n}^2)u(C_4^2 - C_{5-n}^2) + 20 \\
B_2 d_n = (C_3^2 - C_{5-n}^2)u(C_3^2 - C_{5-n}^2) + 20 + C_4^2 \\
B_3 d_n = (C_2^2 - C_{5-n}^2)u(C_2^2 - C_{5-n}^2) + 20 + C_4^2 + C_3^2
\end{cases}$$

$$\begin{cases}
C_0 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 30 \\
C_1 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 30 \\
C_2 d_n = (C_3^2 - C_{5-n}^2)u(C_3^2 - C_{5-n}^2) + 30 \\
C_3 d_n = (C_2^2 - C_{5-n}^2)u(C_2^2 - C_{5-n}^2) + 30 + C_3^2
\end{cases}$$

$$\begin{cases}
D_0 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 34 \\
D_1 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 34 \\
D_2 d_n = (0 - C_{5-n}^2)u(0 - C_{5-n}^2) + 34 \\
D_3 d_n = (C_2^2 - C_{5-n}^2)u(C_2^2 - C_{5-n}^2) + 34
\end{cases}$$

推论1.2.4.

$$\begin{cases}
00d_n = (C_5^2 u(0-0) - C_{5-n}^2)u(C_5^2 u(0-0) - C_{5-n}^2) + \sum_{k=0}^{0-1} C_{5-k}^2 + (C_7^4 - C_{7-0}^4) \\
01d_n = (C_4^2 u(1-0) - C_{5-n}^2)u(C_4^2 u(1-0) - C_{5-n}^2) + \sum_{k=0}^{1-1} C_{5-k}^2 + (C_7^4 - C_{7-0}^4) \\
02d_n = (C_3^2 u(2-0) - C_{5-n}^2)u(C_3^2 u(2-0) - C_{5-n}^2) + \sum_{k=0}^{2-1} C_{5-k}^2 + (C_7^4 - C_{7-0}^4) \\
03d_n = (C_2^2 u(3-0) - C_{5-n}^2)u(C_2^2 u(3-0) - C_{5-n}^2) + \sum_{k=0}^{3-1} C_{5-k}^2 + (C_7^4 - C_{7-0}^4)
\end{cases}$$

$$\left\{ \begin{array}{l} 10d_n = (C_5^2 u(0-1) - C_{5-n}^2)u(C_5^2 u(0-1) - C_{5-n}^2) + \sum_{k=1}^{0-1} C_{5-k}^2 + (C_7^4 - C_{7-1}^4) \\ 11d_n = (C_4^2 u(1-1) - C_{5-n}^2)u(C_4^2 u(1-1) - C_{5-n}^2) + \sum_{k=1}^{1-1} C_{5-k}^2 + (C_7^4 - C_{7-1}^4) \\ 12d_n = (C_3^2 u(2-1) - C_{5-n}^2)u(C_3^2 u(2-1) - C_{5-n}^2) + \sum_{k=1}^{2-1} C_{5-k}^2 + (C_7^4 - C_{7-1}^4) \\ 13d_n = (C_2^2 u(3-1) - C_{5-n}^2)u(C_2^2 u(3-1) - C_{5-n}^2) + \sum_{k=1}^{3-1} C_{5-k}^2 + (C_7^4 - C_{7-1}^4) \\ 20d_n = (C_5^2 u(0-2) - C_{5-n}^2)u(C_5^2 u(0-2) - C_{5-n}^2) + \sum_{k=2}^{0-1} C_{5-k}^2 + (C_7^4 - C_{7-2}^4) \\ 21d_n = (C_4^2 u(1-2) - C_{5-n}^2)u(C_4^2 u(1-2) - C_{5-n}^2) + \sum_{k=2}^{1-1} C_{5-k}^2 + (C_7^4 - C_{7-2}^4) \\ 22d_n = (C_3^2 u(2-2) - C_{5-n}^2)u(C_3^2 u(2-2) - C_{5-n}^2) + \sum_{k=2}^{2-1} C_{5-k}^2 + (C_7^4 - C_{7-2}^4) \\ 23d_n = (C_2^2 u(3-2) - C_{5-n}^2)u(C_2^2 u(3-2) - C_{5-n}^2) + \sum_{k=2}^{3-1} C_{5-k}^2 + (C_7^4 - C_{7-2}^4) \\ 30d_n = (C_5^2 u(0-3) - C_{5-n}^2)u(C_5^2 u(0-3) - C_{5-n}^2) + \sum_{k=3}^{0-1} C_{5-k}^2 + (C_7^4 - C_{7-3}^4) \\ 31d_n = (C_4^2 u(1-3) - C_{5-n}^2)u(C_4^2 u(1-3) - C_{5-n}^2) + \sum_{k=3}^{1-1} C_{5-k}^2 + (C_7^4 - C_{7-3}^4) \\ 32d_n = (C_3^2 u(2-3) - C_{5-n}^2)u(C_3^2 u(2-3) - C_{5-n}^2) + \sum_{k=3}^{2-1} C_{5-k}^2 + (C_7^4 - C_{7-3}^4) \\ 33d_n = (C_2^2 u(3-3) - C_{5-n}^2)u(C_2^2 u(3-3) - C_{5-n}^2) + \sum_{k=3}^{3-1} C_{5-k}^2 + (C_7^4 - C_{7-3}^4) \end{array} \right.$$

$$\lambda\mu d_n = (C_{5-\mu}^2 u(\mu-\lambda) - C_{5-n}^2)u(C_{5-\mu}^2 u(\mu-\lambda) - C_{5-n}^2) + \sum_{k=\lambda}^{\mu-1} C_{5-k}^2 + (C_7^4 - C_{7-\lambda}^4)$$

$$\begin{aligned} k_{\lambda \leq \mu \leq \eta \leq \xi} &= (C_{5-\mu}^2 u(\mu-\lambda) - C_{5-\eta}^2)u(C_{5-\mu}^2 u(\mu-\lambda) - C_{5-\eta}^2) + \sum_{k=\lambda}^{\mu-1} C_{5-k}^2 + (C_7^4 - C_{7-\lambda}^4) + (\xi - \eta) \\ &= (C_7^4 - C_{7-\lambda}^4) + \left( \sum_{k=\lambda}^{\mu} C_{5-k}^2 - C_{5-\eta}^2 \right) + (\xi - \eta) \end{aligned}$$

$$k_{0 \leq \mu \leq \eta \leq \xi} = \left( \sum_{k=0}^{\mu} C_{5-k}^2 - C_{5-\eta}^2 \right) + (\xi - \eta)$$

$$k_{0 \leq 0 \leq \eta \leq \xi} = (C_5^2 - C_{5-\eta}^2) + (\xi - \eta)$$

以上是二阶排序向四阶排序试探方法的推广，也只是我当初为了寻找四阶全对称指标排序规律而进行试探的原始方法，只是中间产品，无需深究，读者不需看懂。留在这里，只是留住当初自己的灵感，让自己隐约记得当初是如何一步步试探，最终猜测得到以下四阶全对称指标的排序规律而已。

### 1.3 四阶对称指标正向排序规律的归纳小结

推论1.3.1.

$$k_{0 \leq 0 \leq \eta \leq \xi} = (C_5^2 - C_{5-\eta}^2) + \left( \sum_{k=\eta}^{\xi} C_{3-k}^0 - C_{3-\lambda_0}^0 \right)$$

$$k_{0 \leq \mu \leq \eta \leq \xi} = \left( \sum_{k=0}^{\mu} C_{5-k}^2 - C_{5-\eta}^2 \right) + \left( \sum_{k=\eta}^{\xi} C_{3-k}^0 - C_{3-\lambda_0}^0 \right)$$

$$k_{\lambda \leq \mu \leq \eta \leq \xi} = (C_7^4 - C_{7-\lambda}^4) + \left( \sum_{k=\lambda}^{\mu} C_{5-k}^2 - C_{5-\eta}^2 \right) + \left( \sum_{k=\eta}^{\xi} C_{3-k}^0 - C_{3-\lambda_0}^0 \right)$$

$$k_{d \leq \lambda \leq \mu \leq \eta \leq \xi} = \left( \sum_{k=0}^d C_{7-k}^4 - C_{7-\lambda}^4 \right) + \left( \sum_{k=\lambda}^{\mu} C_{5-k}^2 - C_{5-\eta}^2 \right) + \left( \sum_{k=\eta}^{\xi} C_{3-k}^0 - C_{3-\lambda_0}^0 \right)$$

$$k_{c \leq d \leq \lambda \leq \mu \leq \eta \leq \xi} = (C_9^6 - C_{9-c}^6) + \left( \sum_{k=c}^d C_{7-k}^4 - C_{7-\lambda}^4 \right) + \left( \sum_{k=\lambda}^{\mu} C_{5-k}^2 - C_{5-\eta}^2 \right) + \left( \sum_{k=\eta}^{\xi} C_{3-k}^0 - C_{3-\lambda_0}^0 \right)$$

$$k_{b \leq c \leq d \leq \lambda \leq \mu \leq \eta \leq \xi} = \left( \sum_{k=0}^b C_{9-k}^6 - C_{9-c}^6 \right) + \left( \sum_{k=c}^d C_{7-k}^4 - C_{7-\lambda}^4 \right) + \left( \sum_{k=\lambda}^{\mu} C_{5-k}^2 - C_{5-\eta}^2 \right) + \left( \sum_{k=\eta}^{\xi} C_{3-k}^0 - C_{3-\lambda_0}^0 \right)$$

$$k_{a \leq b \leq c \leq d \leq \lambda \leq \mu \leq \eta \leq \xi} = (C_{11}^8 - C_{11-a}^8) + \left( \sum_{k=a}^b C_{9-k}^6 - C_{9-c}^6 \right) + \left( \sum_{k=c}^d C_{7-k}^4 - C_{7-\lambda}^4 \right) + \left( \sum_{k=\lambda}^{\mu} C_{5-k}^2 - C_{5-\eta}^2 \right) + \left( \sum_{k=\eta}^{\xi} C_{3-k}^0 - C_{3-\lambda_0}^0 \right)$$

### 1.3.1 四阶对称指标正向排序规律的一般猜想

猜想1.3.1.

$$\begin{aligned}
 & k_{\lambda_{2s} \dots \leq \lambda_8 \leq \lambda_7 \leq \lambda_6 \leq \lambda_5 \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1} \\
 &= \sum_{k=0}^0 C_{11-k}^8 + \sum_{k=\lambda_8}^{\lambda_7} C_{9-k}^6 + \sum_{k=\lambda_6}^{\lambda_5} C_{7-k}^4 + \sum_{k=\lambda_4}^{\lambda_3} C_{5-k}^2 + \sum_{k=\lambda_2}^{\lambda_1} C_{3-k}^0 - (C_{11-\lambda_8}^8 + C_{9-\lambda_6}^6 + C_{7-\lambda_4}^4 + C_{5-\lambda_2}^2 + C_{3-\lambda_0}^0) \\
 &= \sum_{l=0}^{[s]} \left( \sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+3-k}^{2l} - C_{2l+3-\lambda_{2l}}^{2l} \right); \lambda_i = 0, i > 2s | i = 0; \lambda_i = (0, 1, 2, 3), 1 \leq i \leq 2s
 \end{aligned}$$

### 1.3.2 四阶对称指标正向排序规律的验证

性质1.3.1.

$$\begin{aligned}
 & 0 : 0000 \rightarrow 1 : 0001 \rightarrow 2 : 0002 \rightarrow 3 : 0003 \rightarrow \\
 & 4 : 0011 \rightarrow 5 : 0012 \rightarrow 6 : 0013 \rightarrow 7 : 0022 \rightarrow 8 : 0023 \rightarrow 9 : 0033 \rightarrow \\
 & 10 : 0111 \rightarrow 11 : 0112 \rightarrow 12 : 0113 \rightarrow 13 : 0122 \rightarrow 14 : 0123 \rightarrow 15 : 0133 \rightarrow \\
 & 16 : 0222 \rightarrow 17 : 0223 \rightarrow 18 : 0233 \rightarrow 19 : 0333 \rightarrow \\
 & 20 : 1111 \rightarrow 21 : 1112 \rightarrow 22 : 1113 \rightarrow 23 : 1122 \rightarrow 24 : 1123 \rightarrow 25 : 1133 \rightarrow \\
 & 26 : 1222 \rightarrow 27 : 1223 \rightarrow 28 : 1233 \rightarrow 29 : 1333 \rightarrow \\
 & 30 : 2222 \rightarrow 31 : 2223 \rightarrow 32 : 2233 \rightarrow 33 : 2333 \rightarrow 34 : 3333
 \end{aligned}$$

## 1.4 $w + 1$ 阶对称指标排序

### 1.4.1 $w + 1$ 阶对称指标正向排序规律的一般猜想

猜想1.4.1.

$$\begin{aligned}
 & k_{\lambda_{2s} \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1} = \sum_{l=0}^{[s]} \left( \sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l-k}^{2l} - C_{2l-\lambda_{2l}}^{2l} \right) = 0, \lambda_i = (0) \\
 & k_{\lambda_{2s} \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1} = \sum_{l=0}^{[s]} \left( \sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+1-k}^{2l} - C_{2l+1-\lambda_{2l}}^{2l} \right), \lambda_i = (0, 1) \\
 & k_{\lambda_{2s} \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1} = \sum_{l=0}^{[s]} \left( \sum_{k=\lambda_{2l+2}}^{\lambda_{2l+2}} C_{2l+1-k}^{2l} - C_{2l+2-\lambda_{2l}}^{2l} \right), \lambda_i = (0, 1, 2) \\
 & k_{\lambda_{2s} \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1} = \sum_{l=0}^{[s]} \left( \sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+3-k}^{2l} - C_{2l+3-\lambda_{2l}}^{2l} \right), \lambda_i = (0, 1, 2, 3) \\
 & k_{\lambda_{2s} \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1} = \sum_{l=0}^{[s]} \left( \sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+w-k}^{2l} - C_{2l+w-\lambda_{2l}}^{2l} \right), \lambda_i = (0, 1, 2, 3, \dots, w) \\
 & k_{\lambda_{2s} \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1} = \sum_{l=0}^{[s]} \left( C_{2l+1+w-\lambda_{2l+2}}^{2l+1} - C_{2l+w-\lambda_{2l+1}}^{2l+1} - C_{2l+w-\lambda_{2l}}^{2l} \right), \lambda_i = (0, 1, 2, 3, \dots, w) \\
 & \underbrace{k_{\dots w}}_{2s} + 1 = \sum_{k=\lambda_{2[s]+2}}^{\lambda_{2[s]+1}} C_{2[s]+w-k}^{2[s]} = \begin{cases} C_{2s+w}^{2s}, [s] = s \\ \sum_{k=0}^w C_{2s-1+w-k}^{2s-1}, [s] = s - \frac{1}{2} \end{cases} = C_{2s+w}^{2s}, \lambda_i = (0, 1, 2, 3, \dots, w)
 \end{aligned}$$

### 1.4.2 $w + 1$ 阶对称指标正向排序规律猜想的严格证明(本质上用了数学归纳法, 从0排起)

定义1.4.1.  $\pi(\lambda_{2s} \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1; w) := k_{\lambda_{2s} \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1}$

定理1.4.1.  $\pi(\lambda_{2s} \leq \lambda_{2s-1} \dots \leq \lambda_1; w) = \sum_{l=0}^{2s-1} (C_{l+1+w-\lambda_{l+2}}^{l+1} - C_{l+1+w-\lambda_{l+1}}^{l+1})$

$$\begin{aligned}
 \text{证明: } & \pi(\lambda_{2s} \leq \lambda_{2s-1} \dots \leq \lambda_1; w) = \pi(\lambda_{2s-1} \dots \lambda_1 - \lambda_{2s} \dots \lambda_{2s}; w - \lambda_{2s}) + \sum_{k=0}^{\lambda_{2s}-1} C_{2s-1+w-k}^{2s-1} \\
 &= \pi(\lambda_{2s-2} \dots \lambda_1 - \lambda_{2s-1} \dots \lambda_{2s-1}; w - \lambda_{2s-1}) + \sum_{k=0}^{\lambda_{2s}-1} C_{2s-1+w-k}^{2s-1} + \sum_{k=0}^{\lambda_{2s-1}-\lambda_{2s}-1} C_{2s-2+w-\lambda_{2s}-k}^{2s-2} \\
 &= \pi(\lambda_1 - \lambda_2; w - \lambda_2) + \sum_{k=0}^{\lambda_{2s}-1} C_{2s-1+w-k}^{2s-1} + \sum_{k=0}^{\lambda_{2s-1}-\lambda_{2s}-1} C_{2s-2+w-\lambda_{2s}-k}^{2s-2} \dots + \sum_{j=0}^{\lambda_2-\lambda_3-1} C_{1+w-\lambda_3-k}^1
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\lambda_{2s}-1} C_{2s-1+w-k}^{2s-1} + \sum_{k=0}^{\lambda_{2s-1}-\lambda_{2s}-1} C_{2s-2+w-\lambda_{2s}-k}^{2s-2} \cdots + \sum_{j=0}^{\lambda_2-\lambda_3-1} C_{1+w-\lambda_3-k}^1 + \lambda_1 - \lambda_2 \\
&= \sum_{l=1}^{2s-1} \sum_{k=0}^{\lambda_{l+1}-\lambda_{l+2}-1} C_{l+w-\lambda_{l+2}-k}^l + (\lambda_1 - \lambda_2) \\
&= \sum_{l=0}^{2s-1} \left( \sum_{k=0}^{\lambda_{l+1}-\lambda_{l+2}} C_{l+w-\lambda_{l+2}-k}^l - C_{l+w-\lambda_{l+1}}^l \right) \\
&= \sum_{l=0}^{2s-1} (C_{l+1+w-\lambda_{l+2}}^{l+1} - C_{l+w-\lambda_{l+1}}^{l+1} - C_{l+w-\lambda_{l+1}}^l) \\
&= \sum_{l=0}^{2s-1} (C_{l+1+w-\lambda_{l+2}}^{l+1} - C_{l+1+w-\lambda_{l+1}}^{l+1})
\end{aligned}$$

□

$$\text{引理1.4.1. } \sum_{l'=2l-1}^{2l} (C_{l'+1+w-\lambda_{l'+2}}^{l'+1} - C_{l'+1+w-\lambda_{l'+1}}^{l'+1}) = C_{2l+1+w-\lambda_{2l+2}}^{2l+1} - C_{2l+w-\lambda_{2l+1}}^{2l+1} - C_{2l+w-\lambda_{2l}}^{2l}$$

$$\text{定理1.4.2. } \sum_{l=0}^{2s-1} (C_{l+1+w-\lambda_{l+2}}^{l+1} - C_{l+1+w-\lambda_{l+1}}^{l+1}) = \sum_{l=0}^{[s]} (C_{2l+1+w-\lambda_{2l+2}}^{2l+1} - C_{2l+w-\lambda_{2l+1}}^{2l+1} - C_{2l+w-\lambda_{2l}}^{2l})$$

### 1.5 $w+1$ 阶反对称指标正向排序规律的严格证明(从0排起)

$$\text{定理1.5.1. } \pi(\lambda_{2s} \leq \lambda_{2s-1} \cdots \leq \lambda_1; w) = \sum_{l=0}^{2s-1} (C_{w-\lambda_{l+2}}^{l+1} - C_{w+1-\lambda_{l+1}}^{l+1}), \lambda_{2s+1} := -1$$

$$\begin{aligned}
\text{证明: } \pi(\lambda_{2s} \leq \lambda_{2s-1} \cdots \leq \lambda_1; w) &= \pi(\lambda_{2s-1} \cdots \lambda_1 - (\lambda_{2s} + 1) \cdots (\lambda_{2s} + 1); w - \lambda_{2s} - 1) + \sum_{k=0}^{\lambda_{2s}-1} C_{w-k}^{2s-1} \\
&= \pi(\lambda_{2s-2} \cdots \lambda_1 - (\lambda_{2s-1} + 1) \cdots (\lambda_{2s-1} + 1); w - \lambda_{2s-1} - 1) + \sum_{k=0}^{\lambda_{2s}-1} C_{w-k}^{2s-1} + \sum_{k=0}^{\lambda_{2s-1}-\lambda_{2s}-2} C_{w-\lambda_{2s}-1-k}^{2s-2} \\
&= \pi(\lambda_1 - (\lambda_2 + 1); w - \lambda_2 - 1) + \sum_{k=0}^{\lambda_{2s}-1} C_{w-k}^{2s-1} + \sum_{k=0}^{\lambda_{2s-1}-\lambda_{2s}-2} C_{w-\lambda_{2s}-1-k}^{2s-2} \cdots + \sum_{k=0}^{\lambda_2-\lambda_3-2} C_{w-\lambda_3-1-k}^1 \\
&= \sum_{k=0}^{\lambda_{2s}-1} C_{w-k}^{2s-1} + \sum_{k=0}^{\lambda_{2s-1}-\lambda_{2s}-2} C_{w-\lambda_{2s}-1-k}^{2s-2} \cdots + \sum_{k=0}^{\lambda_2-\lambda_3-2} C_{w-\lambda_3-1-k}^1 + \lambda_1 - \lambda_2 - 1 \\
&= \sum_{l=1}^{2s-1} \sum_{k=0}^{\lambda_{l+1}-\lambda_{l+2}-2} C_{w-\lambda_{l+2}-1-k}^l + (\lambda_1 - \lambda_2 - 1); \lambda_{2s+1} := -1 \\
&= \sum_{l=1}^{2s-1} \left( \sum_{k=0}^{\lambda_{l+1}-\lambda_{l+2}-1} C_{w-\lambda_{l+2}-1-k}^l - C_{w-\lambda_{l+1}}^l \right) + (\lambda_1 - \lambda_2 - 1) \\
&= \sum_{l=0}^{2s-1} \left( \sum_{k=0}^{\lambda_{l+1}-\lambda_{l+2}-1} C_{w-\lambda_{l+2}-1-k}^l - C_{w-\lambda_{l+1}}^l \right) \\
&= \sum_{l=0}^{2s-1} (C_{w-\lambda_{l+2}}^{l+1} - C_{w+1-\lambda_{l+1}}^{l+1})
\end{aligned}$$

□

$$\text{证明: } \pi[(w-2s+1) \leq (w-2s+2) \cdots \leq w; w]$$

$$\begin{aligned}
&= \sum_{l=0}^{2s-1} (C_{w-\lambda_{l+2}}^{l+1} - C_{w+1-\lambda_{l+1}}^{l+1}) \\
&= (C_{w-\lambda_{2s+1}}^{2s} - C_{w+1-\lambda_{2s}}^{2s}) + \sum_{l=0}^{2s-2} (C_{w-(w-l-1)}^{l+1} - C_{w+1-(w-l)}^{l+1}) \\
&= C_{w-\lambda_{2s+1}}^{2s} - C_{w+1-\lambda_{2s}}^{2s} \\
&= C_{w+1}^{2s} - C_{w+1-(w-2s+1)}^{2s} \\
&= C_{w+1}^{2s} - 1
\end{aligned}$$

□

## 2 完美常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \cdots}^{k_\zeta}(s; w), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}(s; w)$

### 2.1 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \cdots}^{k_\zeta}(s), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}(s)$ 的引入

$$\text{定义2.1.1. } \left\{ \begin{array}{l} \Gamma_{A_\zeta B_\zeta C_\zeta \cdots}^{k_\zeta}(s) = \frac{1}{(2s)!} \Gamma_{(A_\zeta B_\zeta C_\zeta \cdots)}^{k_\zeta}(s), \Gamma_{\underbrace{1 \cdots 1 0 \cdots 0}_{2s}}^k(s) = \sqrt{C_{2s}^{-k}} \delta_{kl}, k, l = 0, 1, \cdots, 2s \\ \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}(s) = \frac{1}{(2s)!} \Gamma_{k_\zeta}^{(A_\zeta B_\zeta C_\zeta \cdots)}(s), \Gamma_{\underbrace{1 \cdots 1 0 \cdots 0}_{l \quad 2s-l}}^k(s) = \sqrt{C_{2s}^{-k}} \delta_{kl}, k, l = 0, 1, \cdots, 2s \end{array} \right.$$

$$\text{定义2.1.2. } \psi^{k_\zeta}(s) = \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s) \overbrace{\psi^{A_\zeta B_\zeta C_\zeta \dots}}^{2s} = \sqrt{C_{2s}^{k_\zeta}} \psi^{\overbrace{2 \dots 2}^{k_\zeta} \overbrace{1 \dots 1}^{2s-k_\zeta}}$$

$$\text{定义2.1.3. } \Gamma_{k_\zeta}^{A_\zeta B_\zeta} = \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)_{k_\zeta}$$

## 2.2 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w)$ , $\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)$ 的引入

$$\text{定义2.2.1. } \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) = \frac{1}{(2s)!} \Gamma_{(A_\zeta B_\zeta C_\zeta \dots)}^{k_\zeta}(s; w)$$

$$\Gamma_{\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}}^{k_\zeta}(s; w) = \sqrt{\frac{l_0! l_1! \dots l_w!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^{2s-1} (C_{l+1+w-\lambda_{l+2}}^{l+1} - C_{l+1+w-\lambda_{l+1}}^{l+1})\}, l_0 + l_1 + \dots + l_w = 2s$$

$$\text{定义2.2.2. } \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w) = \frac{1}{(2s)!} \Gamma_{k_\zeta}^{(A_\zeta B_\zeta C_\zeta \dots)}(s; w)$$

$$\Gamma_{k_\zeta}^{\overbrace{0_\zeta \dots 0_\zeta}^{l_0} \overbrace{1_\zeta \dots 1_\zeta}^{l_1} \dots \overbrace{w_\zeta \dots w_\zeta}^{l_w}}(s; w) = \sqrt{\frac{l_0! l_1! \dots l_w!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^{2s-1} (C_{l+1+w-\lambda_{l+2}}^{l+1} - C_{l+1+w-\lambda_{l+1}}^{l+1})\}, l_0 + l_1 + \dots + l_w = 2s$$

$$\text{推论2.2.1. } [A_\zeta] = w + 1, [k_\zeta(s)] = C_{2s+w}^{2s}$$

$$\text{推论2.2.2. } \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s) = \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 1), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s) = \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 1)$$

自我评述：以上表明  $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w)$ ,  $\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)$  是常数不变张量  $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s)$ ,  $\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s)$  的推广。

## 2.3 定义的解读

上述定义本质上是对所有  $\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}$  先进行排序编号。然后对每一个  $\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}$  赋值，即先排成  $\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}$  的形式，并与之前所有编号进行对比，如果与编号不一致的为零；与编号一致的不为零，并归一化，归一化系数是  $\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}$  全对称化后所有种类数的倒数开方。以上常数不变张量只与旋量全对称代数性质相关，与旋量变换性质无直接相关，是全对称性张量内在固有的代数性质，与  $A_\zeta$  指标阶数直接本质相关。

## 2.4 常数矩阵 $\Gamma(s; w)$ , $\bar{\Gamma}(s; w)$ 的引入

$$\text{定义2.4.1. } \Gamma(s; w) \succ \Gamma_{\underbrace{A_\zeta \otimes B_\zeta \otimes C_\zeta \otimes \dots}_{2s}}^{k_\zeta}(s; w), \bar{\Gamma}(s; w) \succ \Gamma_{k_\zeta}^{\overbrace{A_\zeta \otimes B_\zeta \otimes C_\zeta \otimes \dots}^{2s}}(s; w) \simeq \Gamma^T(s; w)$$

$$\text{推论2.4.1. } [\Gamma(s; w)] = (w + 1)^{2s} \times C_{2s+w}^{2s}, [\bar{\Gamma}(s; w)] = C_{2s+w}^{2s} \times (w + 1)^{2s}$$

$\Gamma(s)$ ,  $\bar{\Gamma}(s)$  的显式表示：

$$\text{推论2.4.2. } \Gamma(s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots) = 1, I, \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 \\ 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{3} \end{bmatrix}, \dots$$

$$\text{推论2.4.3. } \bar{\Gamma}(s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots) = 1, I, \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1} & 0 & \sqrt{1} & \sqrt{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \end{bmatrix}, \dots$$

## 2.5 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w)$ , $\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)$ 的基本性质

相等性:

$$\text{性质2.5.1. } \Gamma_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}^{k'_\zeta}(s; w) \simeq \Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; w) \simeq \Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}^{2s}}(s; w) \simeq \Gamma_{k'_\zeta}^{\overbrace{A'_\zeta B'_\zeta C'_\zeta \dots}^{2s}}(s; w)$$

$$\text{性质2.5.2. } [\Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; w)]^* \simeq \Gamma_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}^{k'_\zeta}(s; w), [\Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}^{2s}}(s; w)]^* \simeq \Gamma_{k'_\zeta}^{\overbrace{A'_\zeta B'_\zeta C'_\zeta \dots}^{2s}}(s; w)$$

$$\text{推论2.5.1. } \Gamma(s; w) = \Gamma^*(s; w), \bar{\Gamma}(s; w) = \bar{\Gamma}^*(s; w), \bar{\Gamma}(s; w) = \Gamma^+(s; w), \Gamma(s; w) = \bar{\Gamma}^+(s; w)$$

正交性:

$$\text{性质2.5.3. } \Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; w) \Gamma_{l_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}^{2s}}(s; w) = \delta^{k_\zeta l_\zeta} [\Leftrightarrow] \bar{\Gamma}(s; w) \Gamma(s; w) = I$$

$$\text{性质2.5.4. } \Gamma_{A_{1\zeta} A_{2\zeta} \dots A_{2s\zeta}}^{k_\zeta}(s; w) \Gamma_{k_\zeta}^{B_{1\zeta} B_{2\zeta} \dots B_{2s\zeta}}(s; w) = \frac{1}{(2s)!} \delta_{A_{1\zeta}}^{(B_{1\zeta}} \delta_{A_{2\zeta}}^{B_{2\zeta}} \dots \delta_{A_{2s\zeta}}^{B_{2s\zeta}}) = \frac{1}{(2s)!} \delta_{(A_{1\zeta}}^{B_{1\zeta}} \delta_{A_{2\zeta}}^{B_{2\zeta}} \dots \delta_{A_{2s\zeta}}^{B_{2s\zeta}})$$

对比性:

$$\text{性质2.5.5. } \varepsilon_{a_1 a_2 \dots a_n} \varepsilon^{b_1 b_2 \dots b_n} = \delta_{a_1}^{[b_1} \delta_{a_2}^{b_2} \dots \delta_{a_n}^{b_n]} = \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_n}^{b_n]}$$

其它性质:

$$\text{性质2.5.6. } \Gamma_{A_\zeta}^{k_\zeta}(\frac{1}{2}; w) = \delta_{A_\zeta}^{k_\zeta}, \Gamma_{k_\zeta}^{A_\zeta}(\frac{1}{2}; w) = \delta_{k_\zeta}^{A_\zeta}; \Gamma(0; w) = 1, \bar{\Gamma}(0; w) = 1$$

性质2.5.7.

$$\begin{cases} \Gamma_{\underbrace{0_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s) = \sqrt{\frac{2s-k_\zeta}{2s}} \Gamma_{\underbrace{B_\zeta C_\zeta \dots}_{2s-1}}^{k_\zeta}(s-\frac{1}{2}) \\ \Gamma_{\underbrace{1_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s) = \sqrt{\frac{k_\zeta}{2s}} \Gamma_{\underbrace{B_\zeta C_\zeta \dots}_{2s-1}}^{k_\zeta-1}(s-\frac{1}{2}) \end{cases} \quad \begin{cases} \Gamma_{k_\zeta}^{\overbrace{0_\zeta B_\zeta C_\zeta \dots}^{2s}}(s) = \sqrt{\frac{2s-k_\zeta}{2s}} \Gamma_{k_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}}(s-\frac{1}{2}), k_\zeta = 0, 1, \dots, 2s-1 \\ \Gamma_{k_\zeta}^{\overbrace{1_\zeta B_\zeta C_\zeta \dots}^{2s}}(s) = \sqrt{\frac{k_\zeta}{2s}} \Gamma_{k_\zeta-1}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}}(s-\frac{1}{2}), k_\zeta = 1, 2, \dots, 2s \end{cases}$$

性质2.5.8.

$$\begin{cases} \Gamma_{\underbrace{1_\zeta \dots 1_\zeta 0_\zeta \dots 0_\zeta}_{l \quad n \quad 2s} B_\zeta C_\zeta \dots}_{k_\zeta}(s) = \sqrt{\frac{C_{2s-l-n}^{(k_\zeta-1)}}{C_{2s}^{k_\zeta}}} \Gamma_{\underbrace{B_\zeta C_\zeta \dots}_{2s-l-n}}^{k_\zeta-l}(s-\frac{l+n}{2}), k_\zeta = l, l+1, \dots, 2s-n \\ \Gamma_{\underbrace{1_\zeta \dots 1_\zeta 0_\zeta \dots 0_\zeta}_{l \quad n \quad 2s-l-n} B_\zeta C_\zeta \dots}_{k_\zeta}(s) = \sqrt{\frac{C_{2s-l-n}^{(k_\zeta-1)}}{C_{2s}^{k_\zeta}}} \Gamma_{\underbrace{B_\zeta C_\zeta \dots}_{2s-l-n}}^{k_\zeta-l}(s-\frac{l+n}{2}), k_\zeta = l, l+1, \dots, 2s-n \end{cases}$$

## 2.6 度规常数不变张量 $\varepsilon_{k_\zeta l_\zeta}(s; w)$ 的引入及其性质(存在 $\varepsilon_{A_\zeta B_\zeta}$ 为前提条件)

$$\text{定义2.6.1. } \varepsilon(\frac{1}{2}; w) \varepsilon^+(\frac{1}{2}; w) = \varepsilon^+(\frac{1}{2}; w) \varepsilon(\frac{1}{2}; w) = 1; \varepsilon(\frac{1}{2}; w) = \varepsilon^*(\frac{1}{2}; w)$$

度规定义:

$$\text{定义2.6.2. } \begin{cases} \varepsilon_{k_\zeta l_\zeta}(s; w) := \Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}^{2s}}(s; w) \underbrace{\varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{l_\zeta}^{\overbrace{E_\zeta F_\zeta G_\zeta \dots}^{2s}}(s; w) \\ \varepsilon^{k_\zeta l_\zeta}(s; w) := \Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; w) \underbrace{\varepsilon^{A_\zeta E_\zeta} \varepsilon^{B_\zeta F_\zeta} \varepsilon^{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{\underbrace{E_\zeta F_\zeta G_\zeta \dots}_{2s}}^{l_\zeta}(s; w) \end{cases}$$

$$\text{性质2.6.1. } \begin{cases} \underbrace{\varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{l_\zeta}^{\overbrace{E_\zeta F_\zeta G_\zeta \dots}^{2s}}(s; w) \text{ 关于 } ABC\dots \text{ 全对称} \\ \varepsilon^{A_\zeta E_\zeta} \varepsilon^{B_\zeta F_\zeta} \varepsilon^{C_\zeta G_\zeta} \dots \Gamma_{\underbrace{E_\zeta F_\zeta G_\zeta \dots}_{2s}}^{l_\zeta}(s; w) \text{ 关于 } ABC\dots \text{ 全对称} \end{cases}$$



推论2.6.1.  $\varepsilon(s; w) := \bar{\Gamma}(s; w) \underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s} \Gamma(s; w); \varepsilon(s; 1) = \varepsilon(\frac{1}{2}; 2s)$

推论2.6.2.  $\varepsilon(s; w)\varepsilon^+(s; w) = \varepsilon^+(s; w)\varepsilon(s; w) = 1; \varepsilon(s; w) = \varepsilon^*(s; w)$

升降指标:

性质2.6.2. 
$$\left\{ \begin{array}{l} \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) = \varepsilon^{k_\zeta l_\zeta}(s; w) \underbrace{\varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots}_{2s} \cdot \underbrace{\Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}}_{2s}(s; w) \\ \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}}_{2s}(s; w) = \varepsilon_{k_\zeta l_\zeta}(s; w) \underbrace{\varepsilon^{A_\zeta E_\zeta} \varepsilon^{B_\zeta F_\zeta} \varepsilon^{C_\zeta G_\zeta} \dots}_{2s} \cdot \underbrace{\Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}}_{2s}(s; w) \end{array} \right.$$

推论2.6.3.  $\Gamma(s; w)\varepsilon(s; w) = \underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s} \Gamma(s; w), \varepsilon(s; w)\bar{\Gamma}(s; w) = \bar{\Gamma}(s; w) \underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s}$

证明: 
$$\begin{aligned} & \varepsilon^{k_\zeta l_\zeta}(s; w) \underbrace{\varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots}_{2s} \cdot \underbrace{\Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}}_{2s}(s; w) \\ &= \underbrace{\Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\varepsilon^{A'_\zeta E'_\zeta} \varepsilon^{B'_\zeta F'_\zeta} \varepsilon^{C'_\zeta G'_\zeta} \dots}_{2s} \cdot \underbrace{\Gamma_{E'_\zeta F'_\zeta G'_\zeta \dots}^{l_\zeta}}_{2s}(s; w) \underbrace{\varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots}_{2s} \cdot \underbrace{\Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}}_{2s}(s; w) \\ &= \frac{1}{(2s)!} \underbrace{\Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\varepsilon^{A'_\zeta E'_\zeta} \varepsilon^{B'_\zeta F'_\zeta} \varepsilon^{C'_\zeta G'_\zeta} \dots}_{2s} \cdot \underbrace{\delta_{(E'_\zeta} \delta_{F'_\zeta} \delta_{G'_\zeta} \dots)}_{2s} \varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots \\ &= \frac{1}{(2s)!} \underbrace{\Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\varepsilon^{A'_\zeta (E_\zeta} \varepsilon^{B'_\zeta F_\zeta} \varepsilon^{C'_\zeta G_\zeta) \dots}}_{2s} \cdot \underbrace{\varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots}_{2s} \\ &= \frac{1}{(2s)!} \underbrace{\Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \delta_{(A'_\zeta}^{A'_\zeta} \delta_{B'_\zeta}^{B'_\zeta} \delta_{C'_\zeta}^{C'_\zeta} \dots) \\ &= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \end{aligned}$$

□

Penrose标准升降规则:

性质2.6.3. 
$$\left\{ \begin{array}{l} \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) = (-1)^{2s} [\zeta^{2s} \varepsilon^{k_\zeta l_\zeta}(s; w)] \underbrace{(-\zeta \varepsilon_{A_\zeta E_\zeta})(-\zeta \varepsilon_{B_\zeta F_\zeta})(-\zeta \varepsilon_{C_\zeta G_\zeta}) \dots}_{2s} \cdot \underbrace{\Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}}_{2s}(s; w) \\ \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}}_{2s}(s; w) = (-1)^{2s} [(-\zeta)^{2s} \varepsilon_{k_\zeta l_\zeta}(s; w)] \underbrace{(\zeta \varepsilon^{A_\zeta E_\zeta})(\zeta \varepsilon^{B_\zeta F_\zeta})(\zeta \varepsilon^{C_\zeta G_\zeta}) \dots}_{2s} \cdot \underbrace{\Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}}_{2s}(s; w) \end{array} \right.$$

2.7 常数不变张量  $\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; w), \Omega(s; w)$  的引入及其基本性质

定义2.7.1.

$$\underbrace{\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}}_{2s}(s; w) := \underbrace{\sigma_{A_\zeta}^{A'_\zeta} \left(\frac{1}{2}; w\right) \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \sigma_{B_\zeta}^{B'_\zeta} \left(\frac{1}{2}; w\right) \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \sigma_{C_\zeta}^{C'_\zeta} \left(\frac{1}{2}; w\right) \dots}_{2s} + \dots$$

⋈

⋈

定义2.7.2.  $\Omega(s; w) := \sigma(\frac{1}{2}; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes \sigma(\frac{1}{2}; w) \otimes I_{(w+1)^{2s-2}} + \dots + I_{(w+1)^{2s-1}} \otimes \sigma(\frac{1}{2}; w)$

定义2.7.3.  $S(s; w) := \overbrace{S_{w+1} \otimes S_{w+1} \otimes S_{w+1} \otimes \cdots \otimes S_{w+1}}^{2s}, S_{w+1} S_{w+1}^+ = S_{w+1}^+ S_{w+1} = I_{w+1}$

推论2.7.1.  $\underbrace{\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}}_{2s}(s; w) := \sigma_{A_\zeta}^{A'_\zeta} \left(\frac{1}{2}; w\right) \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s-1} + \delta_{A_\zeta}^{A'_\zeta} \underbrace{\Omega_{B_\zeta C_\zeta \dots}^{B'_\zeta C'_\zeta \dots}}_{2s-1} \left(s - \frac{1}{2}; w\right)$

[ $\Downarrow$ ][ $\Downarrow$ ]

推论2.7.2.  $\Omega(s; w) = \sigma(\frac{1}{2}; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes \Omega(s - \frac{1}{2}; w)$

推论2.7.3.  $\Omega(s; w) = \Omega(s - s'; w) \otimes I_{(w+1)^{2s'}} + I_{(w+1)^{2(s-s')}} \otimes \Omega(s'; w)$

## 2.8 常数不变张量 $\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; w), \Omega_{ab}(s, \zeta; w)$ 的引入及其基本性质

定义2.8.1.

$$\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; w) := \underbrace{S_{ab A_\zeta}^{A'_\zeta}(\frac{1}{2}; w)}_{2s} \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} S_{ab B_\zeta}^{B'_\zeta}(\frac{1}{2}; w)}_{2s} \underbrace{\delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} S_{ab C_\zeta}^{C'_\zeta}(\frac{1}{2}; w)}_{2s} \dots + \dots$$

[ $\Downarrow$ ][ $\Downarrow$ ]

定义2.8.2.  $\Omega_{ab}(s, \zeta; w) := S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{(w+1)^{2s-2}} + \dots + I_{(w+1)^{2s-1}} \otimes S_{ab}(\frac{1}{2}, \zeta; w)$

$$\text{推论2.8.1. } \Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; w) := S_{ab A_\zeta}^{A'_\zeta}(\frac{1}{2}; w) \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s-1} + \delta_{A_\zeta}^{A'_\zeta} \underbrace{\Omega_{ab B_\zeta C_\zeta \dots}^{B'_\zeta C'_\zeta \dots}(s - \frac{1}{2}; w)}_{2s-1}$$

[ $\Downarrow$ ][ $\Downarrow$ ]

推论2.8.2.  $\Omega_{ab}(s, \zeta; w) = S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes \Omega_{ab}(s - \frac{1}{2}, \zeta; w)$

推论2.8.3.  $\Omega_{ab}(s, \zeta; w) = \Omega_{ab}(s - s', \zeta; w) \otimes I_{(w+1)^{2s'}} + I_{(w+1)^{2(s-s')}} \otimes \Omega_{ab}(s', \zeta; w)$

## 2.9 自旋常数不变张量 $\sigma^{\alpha_\zeta k_\zeta l_\zeta}(s; w), \sigma(s; w)$ 的引入

$$\text{定义2.9.1. } \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w) \sigma^{\alpha_\zeta} \underbrace{Z_\zeta}_{2s} \Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta}(s; w) := \frac{1}{2s} \sigma^{\alpha_\zeta k_\zeta l_\zeta}(s; w) [\Leftrightarrow] \sigma(s; w) := \bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w)$$

## 2.10 自旋常数不变张量 $S_{ab k_\zeta l_\zeta}(s; w), S_{ab}(s, \zeta; w)$ 的引入

定义2.10.1.

$$\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w) \underbrace{S_{ab A_\zeta}^{Z_\zeta}(\frac{1}{2}; w)}_{2s} \Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta}(s; w) := \frac{1}{2s} S_{ab k_\zeta l_\zeta}(s; w) [\Leftrightarrow] S_{ab}(s, \zeta; w) := \bar{\Gamma}(s; w) \Omega_{ab}(s, \zeta; w) \Gamma(s; w)$$

## 2.11 常数矩阵 $\Omega(s; w)$ 的两个重要引理

引理2.11.1.  $\Gamma(s; w) \bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w) = \Omega(s; w) \Gamma(s; w)$

$$\text{证明: } \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}(s; w) \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w) \underbrace{\Omega_{A'_\zeta B'_\zeta C'_\zeta \dots}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)}_{2s} \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) = \underbrace{\Omega_{A'_\zeta B'_\zeta C'_\zeta \dots}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)}_{2s} \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w)$$

$$\Leftrightarrow \Gamma(s; w) \bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w) = \Omega(s; w) \Gamma(s; w) \quad \square$$

引理2.11.2.  $\bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w) \bar{\Gamma}(s; w) = \bar{\Gamma}(s; w) \Omega(s; w)$

$$\text{证明: } \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}(s; w) \underbrace{\Omega_{A'_\zeta B'_\zeta C'_\zeta \dots}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)}_{2s} \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) \Gamma_{l_\zeta}^{A''_\zeta B''_\zeta C''_\zeta \dots}(s; w) = \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w) \underbrace{\Omega_{A''_\zeta B''_\zeta C''_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; w)}_{2s}$$

$$\Leftrightarrow \bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w) \bar{\Gamma}(s; w) = \bar{\Gamma}(s; w) \Omega(s; w) \quad \square$$

## 2.12 常数矩阵 $\Omega_{ab}(s; w)$ 的两个重要引理

引理2.12.1.  $\Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s; w)\Gamma(s; w) = \Omega_{ab}(s; w)\Gamma(s; w)$

$$\begin{aligned} \text{证明: } & \Gamma_{\underbrace{A''_s B''_s C''_s}_{2s}}^{k_s} \dots (s; w) \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) \Omega_{ab} \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}}^{l_s} \dots (s; w) \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}}^{l_s} \dots (s; w) = \Omega_{ab} \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}}^{l_s} \dots (s; w) \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}}^{l_s} \dots (s; w) \\ \Leftrightarrow & \Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s; w)\Gamma(s; w) = \Omega_{ab}(s; w)\Gamma(s; w) \quad \square \end{aligned}$$

引理2.12.2.  $\bar{\Gamma}(s; w)\Omega_{ab}(s; w)\Gamma(s; w)\bar{\Gamma}(s; w) = \bar{\Gamma}(s; w)\Omega_{ab}(s; w)$

$$\begin{aligned} \text{证明: } & \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) \Omega_{ab} \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}}^{l_s} \dots (s; w) \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}}^{l_s} \dots (s; w) = \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) \Omega_{ab} \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) \\ \Leftrightarrow & \bar{\Gamma}(s; w)\Omega_{ab}(s; w)\Gamma(s; w)\bar{\Gamma}(s; w) = \bar{\Gamma}(s; w)\Omega_{ab}(s; w) \quad \square \end{aligned}$$

## 2.13 关于常数矩阵 $\bar{\Gamma}(s; w)$ , $\Omega(s; w)$ , $\sigma(s; w)$ , $\Gamma(s; w)$ 的置换性质及其推论

$$\text{定理2.13.1. } \Omega(s; w)\Gamma(s; w) = \Gamma(s; w)\sigma(s; w) [\Leftrightarrow] \Omega_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}}^{l_s} \dots (s; w) = \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}}^{k_s} \dots (s; w) \sigma_{k_s}^{l_s}(s; w)$$

$$\begin{aligned} \text{证明: } & \bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \sigma(s; w) \\ \Leftrightarrow & \Gamma(s; w)\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \Gamma(s; w)\sigma(s; w) \\ \Leftrightarrow & \Omega(s; w)\Gamma(s; w) = \Gamma(s; w)\sigma(s; w) \\ \Leftrightarrow & \Omega_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}}^{l_s} \dots (s; w) = \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}}^{k_s} \dots (s; w) \sigma_{k_s}^{l_s}(s; w) \quad \square \end{aligned}$$

$$\text{定理2.13.2. } \bar{\Gamma}(s; w)\Omega(s; w) = \sigma(s; w)\bar{\Gamma}(s; w) [\Leftrightarrow] \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) \Omega_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) = \sigma_{k_s}^{l_s}(s; w) \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w)$$

$$\begin{aligned} \text{证明: } & \bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \sigma(s; w) \\ \Leftrightarrow & \bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w)\bar{\Gamma}(s; w) = \sigma(s; w)\bar{\Gamma}(s; w) \\ \Leftrightarrow & \bar{\Gamma}(s; w)\Omega(s; w) = \sigma(s; w)\bar{\Gamma}(s; w) \\ \Leftrightarrow & \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) \Omega_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) = \sigma_{k_s}^{l_s}(s; w) \Gamma_{\underbrace{A'_s B'_s C'_s}_{2s}} \dots (s; w) \quad \square \end{aligned}$$

推论2.13.1.  $\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \sigma(s; w) \Leftrightarrow \Omega(s; w)\Gamma(s; w) = \Gamma(s; w)\sigma(s; w) \Leftrightarrow \bar{\Gamma}(s; w)\Omega(s; w) = \sigma(s; w)\bar{\Gamma}(s; w)$

推论2.13.2.  $\Omega^2(s; w)\Gamma(s; w) = \Gamma(s; w)\sigma^2(s; w)$ ,  $\bar{\Gamma}(s; w)\Omega^2(s; w) = \sigma^2(s; w)\bar{\Gamma}(s; w)$

$$\begin{aligned} \text{证明: } & \Omega^2(s; w)\Gamma(s; w) \\ & = \Omega(s; w) \cdot \Omega(s; w)\Gamma(s; w) = \Omega(s; w) \cdot \Gamma(s; w)\sigma(s; w) \\ & = \Omega(s; w)\Gamma(s; w) \cdot \sigma(s; w) = \Gamma(s; w)\sigma(s; w) \cdot \sigma(s; w) \\ & = \Gamma(s; w)\sigma^2(s; w) \quad \square \end{aligned}$$

$$\begin{aligned} \text{证明: } & \bar{\Gamma}(s; w)\Omega^2(s; w) \\ & = \bar{\Gamma}(s; w)\Omega(s; w) \cdot \Omega(s; w) = \sigma(s; w)\bar{\Gamma}(s; w) \cdot \Omega(s; w) \\ & = \sigma(s; w) \cdot \bar{\Gamma}(s; w)\Omega(s; w) = \sigma(s; w) \cdot \sigma(s; w)\bar{\Gamma}(s; w) \\ & = \sigma^2(s; w)\bar{\Gamma}(s; w) \quad \square \end{aligned}$$

推论2.13.3.  $\Omega^2(s)\Gamma(s) = s(s+1)\Gamma(s)$ ,  $\bar{\Gamma}(s)\Omega^2(s) = \bar{\Gamma}(s)s(s+1)$

## 2.14 关于常数矩阵 $\bar{\Gamma}(s; w)$ , $\Omega_{ab}(s, \varsigma; w)$ , $S_{ab}(s, \varsigma; w)$ , $\Gamma(s; w)$ 的置换性质及其推论

定理2.14.1.  $\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = \Gamma(s; w)S_{ab}(s, \varsigma; w)$

$$[\Leftrightarrow] \underbrace{\Omega_{ab} \Gamma}_{2s} \underbrace{(s; w)}_{2s} \underbrace{\Gamma}_{2s} \underbrace{(s; w)}_{2s} = \underbrace{\Gamma}_{2s} \underbrace{S_{ab} k_{\varsigma}}_{2s} \underbrace{l_{\varsigma}}_{2s}(s; w)$$

证明:  $\bar{\Gamma}(s, \varsigma; w)\Omega_{ab}(s; w)\Gamma(s; w) = S_{ab}(s, \varsigma; w)$

$$\Leftrightarrow \Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = \Gamma(s; w)S_{ab}(s, \varsigma; w)$$

$$\Leftrightarrow \Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = \Gamma(s; w)S_{ab}(s, \varsigma; w)$$

$$\Leftrightarrow \underbrace{\Omega_{ab} \Gamma}_{2s} \underbrace{(s; w)}_{2s} \underbrace{\Gamma}_{2s} \underbrace{(s; w)}_{2s} = \underbrace{\Gamma}_{2s} \underbrace{S_{ab} k_{\varsigma}}_{2s} \underbrace{l_{\varsigma}}_{2s}(s; w) \quad \square$$

定理2.14.2.  $\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w) = S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$

$$[\Leftrightarrow] \underbrace{\bar{\Gamma}}_{k_{\varsigma}} \underbrace{\Omega_{ab}}_{2s} \underbrace{(s; w)}_{2s} \underbrace{\bar{\Gamma}}_{l_{\varsigma}} \underbrace{(s; w)}_{2s} = \underbrace{S_{ab} k_{\varsigma}}_{2s} \underbrace{l_{\varsigma}}_{2s}(s; w) \underbrace{\bar{\Gamma}}_{l_{\varsigma}} \underbrace{(s; w)}_{2s}$$

证明:  $\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = S_{ab}(s, \varsigma; w)$

$$\Leftrightarrow \bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)\bar{\Gamma}(s; w) = S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w) = S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$$

$$\Leftrightarrow \underbrace{\bar{\Gamma}}_{k_{\varsigma}} \underbrace{\Omega_{ab}}_{2s} \underbrace{(s; w)}_{2s} \underbrace{\bar{\Gamma}}_{l_{\varsigma}} \underbrace{(s; w)}_{2s} = \underbrace{S_{ab} k_{\varsigma}}_{2s} \underbrace{l_{\varsigma}}_{2s}(s; w) \underbrace{\bar{\Gamma}}_{l_{\varsigma}} \underbrace{(s; w)}_{2s} \quad \square$$

推论2.14.1.  $\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = S_{ab}(s, \varsigma; w)$

$$\Leftrightarrow \Omega_{ab}(s; w)\Gamma(s, \varsigma; w) = \Gamma(s; w)S_{ab}(s, \varsigma; w) \Leftrightarrow \bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w) = S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$$

## 2.15 关于常数矩阵 $\Omega(s; w)$ , $\sigma(s; w)$ 的重要推论及洛伦兹群表示

推论2.15.1.  $\underbrace{\Gamma}_{2s}^{k_{\varsigma}} \underbrace{(s; w)}_{2s} \sigma_{k_{\varsigma}} \underbrace{l_{\varsigma}}_{2s}(s; w) = \frac{1}{(2s-1)!} \sigma(\frac{1}{2}; w) \underbrace{(A_{\varsigma} \Gamma_{B_{\varsigma} C_{\varsigma}}^{l_{\varsigma}})_{A_{\varsigma}}}_{2s}(s; w)$

$$\begin{aligned} \text{证明: } & \underbrace{\Gamma}_{2s}^{k_{\varsigma}} \underbrace{(s; w)}_{2s} \sigma_{k_{\varsigma}} \underbrace{l_{\varsigma}}_{2s}(s; w) = \underbrace{\Omega}_{2s} \underbrace{(s; w)}_{2s} \underbrace{\Gamma}_{2s}^{l_{\varsigma}} \underbrace{(s; w)}_{2s} \\ & = \sigma(\frac{1}{2}; w)_{A_{\varsigma}} \underbrace{A'_{\varsigma} \Gamma_{A_{\varsigma} B_{\varsigma} C_{\varsigma}}^{l_{\varsigma}}}_{2s}(s; w) + \sigma(\frac{1}{2}; w)_{B_{\varsigma}} \underbrace{B'_{\varsigma} \Gamma_{A_{\varsigma} B_{\varsigma} C_{\varsigma}}^{l_{\varsigma}}}_{2s}(s; w) + \sigma(\frac{1}{2}; w)_{C_{\varsigma}} \underbrace{C'_{\varsigma} \Gamma_{A_{\varsigma} B_{\varsigma} C_{\varsigma}}^{l_{\varsigma}}}_{2s}(s; w) + \cdots \\ & = \sigma(\frac{1}{2}; w)_{A_{\varsigma}} \underbrace{A'_{\varsigma} \Gamma_{B_{\varsigma} C_{\varsigma}}^{l_{\varsigma}}}_{2s}(s; w) + \sigma(\frac{1}{2}; w)_{B_{\varsigma}} \underbrace{A'_{\varsigma} \Gamma_{A_{\varsigma} C_{\varsigma}}^{l_{\varsigma}}}_{2s}(s; w) + \sigma(\frac{1}{2}; w)_{C_{\varsigma}} \underbrace{A'_{\varsigma} \Gamma_{A_{\varsigma} B_{\varsigma}}^{l_{\varsigma}}}_{2s}(s; w) + \cdots \\ & = \frac{1}{(2s-1)!} \sigma(\frac{1}{2}; w)_{A_{\varsigma}} \underbrace{A'_{\varsigma} \Gamma_{(B_{\varsigma} C_{\varsigma})}^{l_{\varsigma}}}_{2s}(s; w) + \frac{1}{(2s-1)!} \sigma(\frac{1}{2}; w)_{B_{\varsigma}} \underbrace{A'_{\varsigma} \Gamma_{(A_{\varsigma} C_{\varsigma})}^{l_{\varsigma}}}_{2s}(s; w) + \frac{1}{(2s-1)!} \sigma(\frac{1}{2}; w)_{C_{\varsigma}} \underbrace{A'_{\varsigma} \Gamma_{(A_{\varsigma} B_{\varsigma})}^{l_{\varsigma}}}_{2s}(s; w) \cdots \\ & \Leftrightarrow \frac{1}{(2s-1)!} \sigma(\frac{1}{2}; w)_{A_{\varsigma}} \underbrace{A'_{\varsigma} \Gamma_{B_{\varsigma} C_{\varsigma}}^{l_{\varsigma}}}_{2s}(s; w) \quad \square \end{aligned}$$

推论2.15.2.  $\sigma(\frac{1}{2}; w) \times \sigma(\frac{1}{2}; w) = i\sigma(\frac{1}{2}; w) [\Rightarrow] \Omega(s; w) \times \Omega(s; w) = i\Omega(s; w) [\Rightarrow] \sigma(s; w) \times \sigma(s; w) = i\sigma(s; w)$

证明:  $\sigma(\frac{1}{2}; w) \times \sigma(\frac{1}{2}; w) = i\sigma(\frac{1}{2}; w)$

$$\Rightarrow \Omega(s; w) \times \Omega(s; w)$$

$$= [\sigma(\frac{1}{2}; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes \sigma(\frac{1}{2}; w) \otimes I_{(w+1)^{2s-2}} + \cdots + I_{(w+1)^{2s-1}} \otimes \sigma(\frac{1}{2}; w)]$$

$$\times [\sigma(\frac{1}{2}; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes \sigma(\frac{1}{2}; w) \otimes I_{(w+1)^{2s-2}} + \cdots + I_{(w+1)^{2s-1}} \otimes \sigma(\frac{1}{2}; w)]$$

$$= [\sigma(\frac{1}{2}; w) \times \sigma(\frac{1}{2}; w)] \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes [\sigma(\frac{1}{2}; w) \times \sigma(\frac{1}{2}; w)] \otimes I_{(w+1)^{2s-2}} + \cdots + I_{(w+1)^{2s-1}} \otimes [\sigma(\frac{1}{2}; w) \times \sigma(\frac{1}{2}; w)]$$

$$= i\sigma(\frac{1}{2}; w) \otimes I_{(w+1)^{2s-1}} + iI_{w+1} \otimes \sigma(\frac{1}{2}; w) \otimes I_{(w+1)^{2s-2}} + \cdots + iI_{(w+1)^{2s-1}} \otimes \sigma(\frac{1}{2}; w)$$

$$= i\Omega(s; w) \quad \square$$

证明:  $\Omega(s; w) \times \Omega(s; w) = i\Omega(s; w)$

$$\Rightarrow \bar{\Gamma}(s; w)\Omega(s; w) \times \Omega(s; w)\Gamma(s; w) = i\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w)\Omega(s; w) \times \Gamma(s; w)\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = i\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) \times \bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = i\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w)$$

$$\Leftrightarrow \sigma(s; w) \times \sigma(s; w) = i\sigma(s; w)$$

□

## 2.16 关于常数矩阵 $\Omega_{ab}(s, \varsigma; w)$ , $S_{ab}(s, \varsigma; w)$ 的重要推论及洛伦兹群表示

推论2.16.1.  $\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}(s; w) S_{abk_\varsigma} l_\varsigma(s; w) = \frac{1}{(2s-1)!} S_{ab}(\frac{1}{2}; w)_{(A_\varsigma A'_\varsigma \Gamma_{B_\varsigma C_\varsigma \dots}^{l_\varsigma})_{A'_\varsigma}}(s; w)$

$$\begin{aligned} \text{证明: } & \Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}(s; w) S_{abk_\varsigma} l_\varsigma(s; w) = \Omega_{ab} \Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots} (s; w) \Gamma_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}^{l_\varsigma}(s; w) \\ & = S_{ab}(\frac{1}{2}; w)_{A_\varsigma} A'_\varsigma \Gamma_{A'_\varsigma B_\varsigma C_\varsigma \dots}^{l_\varsigma}(s; w) + S_{ab}(\frac{1}{2}; w)_{B_\varsigma} B'_\varsigma \Gamma_{A_\varsigma B'_\varsigma C_\varsigma \dots}^{l_\varsigma}(s; w) + S_{ab}(\frac{1}{2}; w)_{C_\varsigma} C'_\varsigma \Gamma_{A_\varsigma B_\varsigma C'_\varsigma \dots}^{l_\varsigma}(s; w) + \dots \\ & = S_{ab}(\frac{1}{2}; w)_{A_\varsigma} A'_\varsigma \Gamma_{B_\varsigma C_\varsigma \dots A'_\varsigma}^{l_\varsigma}(s; w) + S_{ab}(\frac{1}{2}; w)_{B_\varsigma} A'_\varsigma \Gamma_{A_\varsigma C_\varsigma \dots A'_\varsigma}^{l_\varsigma}(s; w) + S_{ab}(\frac{1}{2}; w)_{C_\varsigma} A'_\varsigma \Gamma_{A_\varsigma B_\varsigma \dots A'_\varsigma}^{l_\varsigma}(s; w) + \dots \\ & = \frac{1}{(2s-1)!} S_{ab}(\frac{1}{2}; w)_{A_\varsigma} A'_\varsigma \Gamma_{(B_\varsigma C_\varsigma \dots) A'_\varsigma}^{l_\varsigma}(s; w) + \frac{1}{(2s-1)!} S_{ab}(\frac{1}{2}; w)_{B_\varsigma} A'_\varsigma \Gamma_{(A_\varsigma C_\varsigma \dots) A'_\varsigma}^{l_\varsigma}(s; w) + \dots \\ & \Leftrightarrow \frac{1}{(2s-1)!} S_{ab}(\frac{1}{2}; w)_{(A_\varsigma A'_\varsigma \Gamma_{B_\varsigma C_\varsigma \dots}^{l_\varsigma})_{A'_\varsigma}}(s; w) \end{aligned}$$

□

推论2.16.2.  $i[S_{ab}(\frac{1}{2}, \varsigma; w), S_{cd}(\frac{1}{2}, \varsigma; w)] = \delta_{ad} S_{bc}(\frac{1}{2}, \varsigma; w) - \delta_{ac} S_{bd}(\frac{1}{2}, \varsigma; w) + \delta_{bc} S_{ad}(\frac{1}{2}, \varsigma; w) - \delta_{bd} S_{ac}(\frac{1}{2}, \varsigma; w)$

$$[\Rightarrow] i[\Omega_{ab}(s, \varsigma; w), \Omega_{cd}(s, \varsigma; w)] = \delta_{ad} \Omega_{bc}(s, \varsigma; w) - \delta_{ac} \Omega_{bd}(s, \varsigma; w) + \delta_{bc} \Omega_{ad}(s, \varsigma; w) - \delta_{bd} \Omega_{ac}(s, \varsigma; w)$$

$$[\Rightarrow] i[S_{ab}(s, \varsigma; w), S_{cd}(s, \varsigma; w)] = \delta_{ad} S_{bc}(s, \varsigma; w) - \delta_{ac} S_{bd}(s, \varsigma; w) + \delta_{bc} S_{ad}(s, \varsigma; w) - \delta_{bd} S_{ac}(s, \varsigma; w)$$

证明:  $i[\Omega_{ab}(s, \varsigma; w), \Omega_{cd}(s, \varsigma; w)]$

$$\begin{aligned} & = i[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{(w+1)^{2s-2}} + \dots + I_{(w+1)^{2s-1}} \otimes S_{ab}(\frac{1}{2}, \varsigma; w) \\ & , S_{cd}(\frac{1}{2}, \varsigma; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes S_{cd}(\frac{1}{2}, \varsigma; w) \otimes I_{(w+1)^{2s-2}} + \dots + I_{(w+1)^{2s-1}} \otimes S_{cd}(\frac{1}{2}, \varsigma; w)] \\ & = i\{[S_{ab}(\frac{1}{2}, \varsigma; w), S_{cd}(\frac{1}{2}, \varsigma; w)] \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes [S_{ab}(\frac{1}{2}, \varsigma; w), S_{cd}(\frac{1}{2}, \varsigma; w)] \otimes I_{(w+1)^{2s-2}} \\ & + \dots + I_{(w+1)^{2s-1}} \otimes [S_{ab}(\frac{1}{2}, \varsigma; w), S_{cd}(\frac{1}{2}, \varsigma; w)]\} \\ & = [\delta_{ad} S_{bc}(\frac{1}{2}, \varsigma; w) - \delta_{ac} S_{bd}(\frac{1}{2}, \varsigma; w) + \delta_{bc} S_{ad}(\frac{1}{2}, \varsigma; w) - \delta_{bd} S_{ac}(\frac{1}{2}, \varsigma; w)] \otimes I_{(w+1)^{2s-1}} \\ & + I_{w+1} \otimes [\delta_{ad} S_{bc}(\frac{1}{2}, \varsigma; w) - \delta_{ac} S_{bd}(\frac{1}{2}, \varsigma; w) + \delta_{bc} S_{ad}(\frac{1}{2}, \varsigma; w) - \delta_{bd} S_{ac}(\frac{1}{2}, \varsigma; w)] \otimes I_{(w+1)^{2s-2}} \\ & + \dots + I_{(w+1)^{2s-1}} \otimes [\delta_{ad} S_{bc}(\frac{1}{2}, \varsigma; w) - \delta_{ac} S_{bd}(\frac{1}{2}, \varsigma; w) + \delta_{bc} S_{ad}(\frac{1}{2}, \varsigma; w) - \delta_{bd} S_{ac}(\frac{1}{2}, \varsigma; w)] \\ & = \delta_{ad} \Omega_{bc}(s, \varsigma; w) - \delta_{ac} \Omega_{bd}(s, \varsigma; w) + \delta_{bc} \Omega_{ad}(s, \varsigma; w) - \delta_{bd} \Omega_{ac}(s, \varsigma; w) \end{aligned}$$

□

证明:  $i[\Omega_{ab}(s, \varsigma; w), \Omega_{cd}(s, \varsigma; w)] = \delta_{ad} \Omega_{bc}(s, \varsigma; w) - \delta_{ac} \Omega_{bd}(s, \varsigma; w) + \delta_{bc} \Omega_{ad}(s, \varsigma; w) - \delta_{bd} \Omega_{ac}(s, \varsigma; w)$

$$\Rightarrow \bar{\Gamma}(s; w) i[\Omega_{ab}(s, \varsigma; w), \Omega_{cd}(s, \varsigma; w)] \Gamma(s; w)$$

$$= \bar{\Gamma}(s; w) [\delta_{ad} \Omega_{bc}(s, \varsigma; w) - \delta_{ac} \Omega_{bd}(s, \varsigma; w) + \delta_{bc} \Omega_{ad}(s, \varsigma; w) - \delta_{bd} \Omega_{ac}(s, \varsigma; w)] \Gamma(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w) i[\Omega_{ab}(s, \varsigma; w) \Omega_{cd}(s, \varsigma; w) - \Omega_{cd}(s, \varsigma; w) \Omega_{ab}(s, \varsigma; w)] \Gamma(s; w)$$

$$= \bar{\Gamma}(s; w) [\delta_{ad} \Omega_{bc}(s, \varsigma; w) - \delta_{ac} \Omega_{bd}(s, \varsigma; w) + \delta_{bc} \Omega_{ad}(s, \varsigma; w) - \delta_{bd} \Omega_{ac}(s, \varsigma; w)] \Gamma(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w) i[\Omega_{ab}(s, \varsigma; w) \Gamma(s; w) \bar{\Gamma}(s; w) \Omega_{cd}(s, \varsigma; w) - \Omega_{cd}(s, \varsigma; w) \Gamma(s; w) \bar{\Gamma}(s; w) \Omega_{ab}(s, \varsigma; w)] \Gamma(s; w)$$

$$= \bar{\Gamma}(s; w) [\delta_{ad} \Omega_{bc}(s, \varsigma; w) - \delta_{ac} \Omega_{bd}(s, \varsigma; w) + \delta_{bc} \Omega_{ad}(s, \varsigma; w) - \delta_{bd} \Omega_{ac}(s, \varsigma; w)] \Gamma(s; w)$$

$$\Leftrightarrow i[\bar{\Gamma}(s; w) \Omega_{ab}(s, \varsigma; w) \Gamma(s; w) \bar{\Gamma}(s; w) \Omega_{cd}(s, \varsigma; w) \Gamma(s; w) - \bar{\Gamma}(s; w) \Omega_{cd}(s, \varsigma; w) \Gamma(s; w) \bar{\Gamma}(s; w) \Omega_{ab}(s, \varsigma; w) \Gamma(s; w)]$$

$$= \bar{\Gamma}(s; w) [\delta_{ad} \Omega_{bc}(s, \varsigma; w) - \delta_{ac} \Omega_{bd}(s, \varsigma; w) + \delta_{bc} \Omega_{ad}(s, \varsigma; w) - \delta_{bd} \Omega_{ac}(s, \varsigma; w)] \Gamma(s; w)$$

$$\Leftrightarrow i[S_{ab}(s, \varsigma; w), S_{cd}(s, \varsigma; w)] = \delta_{ad} S_{bc}(s, \varsigma; w) - \delta_{ac} S_{bd}(s, \varsigma; w) + \delta_{bc} S_{ad}(s, \varsigma; w) - \delta_{bd} S_{ac}(s, \varsigma; w)$$

□

## 2.17 常数矩阵 $\Omega_{ab}(s, \varsigma; w)$ , $S_{ab}(s, \varsigma; w)$ 与 $\Omega(s; w)$ , $\sigma(s; w)$ 之间关系的讨论

性质2.17.1.  $S_{ab}(\frac{1}{2}, \varsigma) = \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(\frac{1}{2}) [\Rightarrow] \Omega_{ab}(s, \varsigma) = \sigma_{\varsigma ab}^{\alpha\varsigma} \Omega_{\alpha\varsigma}(s) [\Rightarrow] S_{ab}(s, \varsigma) = \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s)$

第一种可能的关系(四维时空自旋型):

猜想2.17.1.  $S_{ab}(\frac{1}{2}, \varsigma; w) = \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(\frac{1}{2}; w) [\Rightarrow] \Omega_{ab}(s, \varsigma; w) = \sigma_{\varsigma ab}^{\alpha\varsigma} \Omega_{\alpha\varsigma}(s; w) [\Rightarrow] S_{ab}(s, \varsigma; w) = \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s; w)$

第二种可能的关系(任意时空自旋型, 四维时空自旋型是其特例。):

$$\begin{aligned} \text{猜想2.17.2. } S_{ab}(\frac{1}{2}, \varsigma; w) &= \begin{bmatrix} -i[\sigma_{\alpha\varsigma}(\frac{1}{2}; w), \sigma_{\beta\varsigma}(\frac{1}{2}; w)] & -\varsigma \sigma_{\alpha\varsigma}(\frac{1}{2}; w) \\ \varsigma \sigma_{\beta\varsigma}(\frac{1}{2}; w) & 0 \end{bmatrix} \\ [\Rightarrow] \Omega_{ab}(s, \varsigma; w) &= \begin{bmatrix} -i[\Omega_{\alpha\varsigma}(s; w), \Omega_{\beta\varsigma}(s; w)] & -\varsigma \Omega_{\alpha\varsigma}(s; w) \\ \varsigma \Omega_{\beta\varsigma}(s; w) & 0 \end{bmatrix} \\ [\Rightarrow] S_{ab}(s, \varsigma; w) &= \begin{bmatrix} -i[\sigma_{\alpha\varsigma}(s; w), \sigma_{\beta\varsigma}(s; w)] & -\varsigma \sigma_{\alpha\varsigma}(s; w) \\ \varsigma \sigma_{\beta\varsigma}(s; w) & 0 \end{bmatrix} \end{aligned}$$

第三种可能的关系(高维时空型Dirac型):

猜想2.17.3.  $S_{ab}(\frac{1}{2}, \varsigma; w) = -\frac{i}{4}[\gamma_a(\frac{1}{2}, \varsigma; w), \gamma_b(\frac{1}{2}, \varsigma; w)]$   
 $[\Rightarrow] \Omega_{ab}(s, \varsigma; w) = -\frac{i}{4}[\Omega_a(s, \varsigma; w), \Omega_b(s, \varsigma; w)] [\Rightarrow] S_{ab}(s, \varsigma; w) = -\frac{i}{4}[\gamma_a(s, \varsigma; w), \gamma_b(s, \varsigma; w)]$

第四种可能的关系(无关联型): 相对独立。

## 2.18 推论: 常数矩阵 $\Gamma(s; w)$ , $\bar{\Gamma}(s; w)$ 的几个恒等式

$$\text{性质2.18.1. } \begin{cases} \bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w) = \sigma(s; w), [\Gamma(s; w) \bar{\Gamma}(s; w), \Omega(s; w)] = 0 \\ \Gamma(s; w) \sigma(s; w) \bar{\Gamma}(s; w) = \Omega(s; w) \Gamma(s; w) \bar{\Gamma}(s; w) = \Gamma(s; w) \bar{\Gamma}(s; w) \Omega(s; w) \end{cases}$$

$$\text{性质2.18.2. } \begin{cases} \bar{\Gamma}(s; w) \Omega_{ab}(s, \varsigma; w) \Gamma(s; w) = S_{ab}(s, \varsigma; w), [\Gamma(s; w) \bar{\Gamma}(s; w), \Omega_{ab}(s, \varsigma; w)] = 0 \\ \Gamma(s; w) S_{ab}(s, \varsigma; w) \bar{\Gamma}(s; w) = \Omega_{ab}(s, \varsigma; w) \Gamma(s; w) \bar{\Gamma}(s; w) = \Gamma(s; w) \bar{\Gamma}(s; w) \Omega_{ab}(s, \varsigma; w) \end{cases}$$

$$\text{性质2.18.3. } \begin{cases} \bar{\Gamma}(s; w) [\vartheta \cdot \Omega(s; w)]^n \Gamma(s; w) = [\vartheta \cdot \sigma(s; w)]^n, [\Gamma(s; w) \bar{\Gamma}(s; w), [\vartheta \cdot \Omega(s; w)]^n] = 0 \\ \Gamma(s; w) [\vartheta \cdot \sigma(s; w)]^n \bar{\Gamma}(s; w) = [\vartheta \cdot \Omega(s; w)]^n \Gamma(s; w) \bar{\Gamma}(s; w) = \Gamma(s; w) \bar{\Gamma}(s; w) [\vartheta \cdot \Omega(s; w)]^n \end{cases}$$

$$\text{性质2.18.4. } \begin{cases} \bar{\Gamma}(s; w) [\vartheta^{ab} \Omega_{ab}(s, \varsigma; w)]^n \Gamma(s; w) = [\vartheta^{ab} S_{ab}(s, \varsigma; w)]^n, [\Gamma(s; w) \bar{\Gamma}(s; w), [\vartheta^{ab} \Omega_{ab}(s, \varsigma; w)]^n] = 0 \\ \Gamma(s; w) [\vartheta^{ab} S_{ab}(s, \varsigma; w)]^n \bar{\Gamma}(s; w) = [\vartheta^{ab} \Omega_{ab}(s, \varsigma; w)]^n \Gamma(s; w) \bar{\Gamma}(s; w) = \Gamma(s; w) \bar{\Gamma}(s; w) [\vartheta^{ab} \Omega_{ab}(s, \varsigma; w)]^n \end{cases}$$

$$\text{推论2.18.1. } \begin{cases} \bar{\Gamma}(s; w) e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s, \varsigma; w)} \Gamma(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)}, [\Gamma(s; w) \bar{\Gamma}(s; w), e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s, \varsigma; w)}] = 0 \\ \Gamma(s; w) e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \bar{\Gamma}(s; w) = e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s, \varsigma; w)} \Gamma(s; w) \bar{\Gamma}(s; w) = \Gamma(s; w) \bar{\Gamma}(s; w) e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s, \varsigma; w)} \end{cases}$$

## 2.19 常数矩阵 $\Gamma(s; w)$ , $\bar{\Gamma}(s; w)$ 的对称置换性质

$$\text{定义2.19.1. } S_{ex}(s, n) = \left( \overbrace{I_{w+1} \otimes \cdots \otimes I_{w+1}}^{n-1} \otimes S_{ex} \otimes \overbrace{I_{w+1} \otimes \cdots \otimes I}^{2s-n-1} \right)$$

$$\text{推论2.19.1. } \Gamma(s; w) = S_{ex}(s, n) \Gamma(s; w), \bar{\Gamma}(s; w) = \bar{\Gamma}(s; w) S_{ex}(s, n)$$

$$\text{推论2.19.2. } S_{ex}(s, n) \Omega(s; w) S_{ex}(s, n) = \Omega(s; w)$$

$$\text{推论2.19.3. } \hat{\psi}(s, \varsigma; w) = S_{ex}(s, n) \hat{\psi}(s, \varsigma; w), \forall n \in \{1, 2, \dots, 2s+1\}$$

## 2.20 矩阵 $\Gamma(s; w), \bar{\Gamma}(s; w)$ 的常数不变张量性质

**定理2.20.1.**  $\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)}$ ,  $\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}$

**证明:**  $\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = \Gamma(s; w)S_{ab}(s, \varsigma; w)$

$$\Leftrightarrow 0 = \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) - \Gamma(s; w)\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)$$

$$\Leftrightarrow \Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)} \quad \square$$

**定理2.20.2.**  $\bar{\Gamma}(s; w) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)}\bar{\Gamma}(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}$ ,  $\bar{\Gamma}(s; w) = e^{\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}\bar{\Gamma}(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}$

**证明:**  $\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w) = S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$

$$\Leftrightarrow 0 = -\bar{\Gamma}(s; w)\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w) + \frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)}\bar{\Gamma}(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} \quad \square$$

**定理2.20.3.**  $\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}$

$$\Leftrightarrow \Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)\Gamma(s; w)}$$

**证明:**  $\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}$

$$\Leftrightarrow S^+(s; w)S(s; w)\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S^+(s; w)S(s; w)\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}$$

$$\Leftrightarrow S^+(s; w)\Gamma(s; w)\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S^+(s; w)\Gamma(s; w)\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}$$

$$\Leftrightarrow \Gamma(s; w) = [S(s; w)e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S^+(s; w)]\Gamma(s; w)[\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)]e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}[\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)]^+$$

$$\Leftrightarrow \Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)S^+(s; w)\Gamma(s; w)}$$

$$\Leftrightarrow \Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)\Gamma(s; w)}$$

$$\Leftrightarrow \Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)\Gamma(s; w)} \quad \square$$

**推论2.20.1.**

$$e^{\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)\Gamma(s; w)} = [\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)]e^{\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}[\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)]^+$$

**定理2.20.4.**  $S(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)}$

**证明:**  $\Omega_{ab}(s, \varsigma; w)S(s; w) = \Omega_{ab}(s, \varsigma; w)S(s; w), \forall \varsigma = \pm 1$

$$\Leftrightarrow S(s; w)S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w) = \Omega_{ab}(s, \varsigma; w)S(s; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow 0 = -S(s; w)\frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w) + \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)S(s; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = S(s; w)[1 - \frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)] + \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)S(s; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = [1 + \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]S(s; w)[1 - \frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)], \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)}, \forall \vartheta^{ab}, \varsigma = \pm 1 \quad \square$$

**定理2.20.5.**  $S(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}$

**证明:**  $S(s; w)\Omega_{ab}(s, \varsigma; w) = S(s; w)\Omega_{ab}(s, \varsigma; w), \forall \varsigma = \pm 1$

$$\Leftrightarrow S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)S(s; w) = S(s; w)\Omega_{ab}(s, \varsigma; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow 0 = \frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)S(s; w) - S(s; w)\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = [1 + \frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)]S(s; w) - S(s; w)\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = [1 + \frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)]S(s; w)[1 - \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)], \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}, \forall \vartheta^{ab}, \varsigma = \pm 1 \quad \square$$

**推论2.20.2.**

$$\begin{cases} S(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)} \\ S(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} \\ S^+(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S^+(s; w)e^{-\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)} \\ S^+(s; w) = e^{\frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)}S^+(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} \end{cases}$$

## 2.21 矩阵 $\Gamma(s)$ , $\bar{\Gamma}(s)$ 的常数不变张量性质

**定理2.21.1.**  $\Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot\Omega(s)}\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\sigma(s)}$ ,  $\Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot\Omega(s)}\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)\Omega(s)\Gamma(s)}$

**证明:**  $\Omega(s)\Gamma(s) = \Gamma(s)\sigma(s)$

$$\Leftrightarrow 0 = (i\omega + \zeta\epsilon) \cdot \Omega(s)\Gamma(s) - (i\omega + \zeta\epsilon) \cdot \Gamma(s)\sigma(s)$$

$$\Leftrightarrow \Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot\Omega(s)}\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\sigma(s)} \quad \square$$

**定理2.21.2.**  $\bar{\Gamma}(s) = e^{(i\omega+\zeta\epsilon)\cdot\sigma(s)}\bar{\Gamma}(s)e^{-(i\omega+\zeta\epsilon)\cdot\Omega(s)}$ ,  $\bar{\Gamma}(s) = e^{(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)\Omega(s)\Gamma(s)}\bar{\Gamma}(s)e^{-(i\omega+\zeta\epsilon)\cdot\Omega(s)}$

**证明:**  $\bar{\Gamma}(s)\Omega(s) = \sigma(s)\bar{\Gamma}(s)$

$$\Leftrightarrow 0 = -(i\omega + \zeta\epsilon) \cdot \bar{\Gamma}(s)\Omega(s) + (i\omega + \zeta\epsilon) \cdot \sigma(s)\bar{\Gamma}(s)$$

$$\Leftrightarrow \bar{\Gamma}(s) = e^{(i\omega+\zeta\epsilon)\cdot\sigma(s)}\bar{\Gamma}(s)e^{-(i\omega+\zeta\epsilon)\cdot\Omega(s)} \quad \square$$

**定理2.21.3.**  $\Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot\Omega(s)}\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)\Omega(s)\Gamma(s)} \Leftrightarrow \Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot S(s)\Omega(s)S^+(s)}\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)S(s)S^+(s)\Gamma(s)}$

**证明:**  $\Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot\Omega(s)}\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)\Omega(s)\Gamma(s)}$

$$\Leftrightarrow S^+(s)S(s)\Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot\Omega(s)}S^+(s)S(s)\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)\Omega(s)\Gamma(s)}$$

$$\Leftrightarrow S^+(s)\Gamma(s)\bar{\Gamma}(s)S(s)\Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot\Omega(s)}S^+(s)\Gamma(s)\bar{\Gamma}(s)S(s)\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)\Omega(s)\Gamma(s)}$$

$$\Leftrightarrow \Gamma(s) = [S(s)e^{(i\omega+\zeta\epsilon)\cdot\Omega(s)}S^+(s)]\Gamma(s)[\bar{\Gamma}(s)S(s)\Gamma(s)]e^{-(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)\Omega(s)\Gamma(s)}[\bar{\Gamma}(s)S(s)\Gamma(s)]^+$$

$$\Leftrightarrow \Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot S(s)\Omega(s)S^+(s)}\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)S(s)\Gamma(s)\bar{\Gamma}(s)\Omega(s)\Gamma(s)\bar{\Gamma}(s)S^+(s)\Gamma(s)}$$

$$\Leftrightarrow \Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot S(s)\Omega(s)S^+(s)}\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)S(s)\Gamma(s)\bar{\Gamma}(s)\Omega(s)S^+(s)\Gamma(s)}$$

$$\Leftrightarrow \Gamma(s) = e^{(i\omega+\zeta\epsilon)\cdot S(s)\Omega(s)S^+(s)}\Gamma(s)e^{-(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)S(s)\Omega(s)S^+(s)\Gamma(s)} \quad \square$$

**推论2.21.1.**  $e^{(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)S(s)\Omega(s)S^+(s)\Gamma(s)} = [\bar{\Gamma}(s)S(s)\Gamma(s)]e^{(i\omega+\zeta\epsilon)\cdot\bar{\Gamma}(s)\Omega(s)\Gamma(s)}[\bar{\Gamma}(s)S(s)\Gamma(s)]^+$

**推论2.21.2.**

$$\begin{cases} S(s) = e^{(i\omega+\zeta\epsilon)\cdot\Omega(s)}S(s)e^{-(i\omega+\zeta\epsilon)\cdot S^+(s)\Omega(s)S(s)}, S^+(s) = e^{(i\omega+\zeta\epsilon)\cdot\Omega(s)}S^+(s)e^{-(i\omega+\zeta\epsilon)\cdot S(s)\Omega(s)S^+(s)} \\ S(s) = e^{(i\omega+\zeta\epsilon)\cdot S(s)\Omega(s)S^+(s)}S(s)e^{-(i\omega+\zeta\epsilon)\cdot\Omega(s)}, S^+(s) = e^{(i\omega+\zeta\epsilon)\cdot S^+(s)\Omega(s)S(s)}S^+(s)e^{-(i\omega+\zeta\epsilon)\cdot\Omega(s)} \end{cases}$$

## 2.22 常数矩阵 $I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)$ , $I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)$ 的置换性质

**定理2.22.1.**  $\Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]$

**证明:**  $\Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]$

$$= \Omega(s; w)I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)$$

$$= [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}}] + I_{w+1} \otimes [\Gamma(s - \frac{1}{2}; w)\sigma(s - \frac{1}{2}; w)]$$

$$= [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] \quad \square$$

**定理2.22.2.**  $[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w) = [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]$

**证明:**  $[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w)$

$$= [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w)$$

$$= [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}}][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] + I_{w+1} \otimes [\sigma(s - \frac{1}{2}; w)\bar{\Gamma}(s - \frac{1}{2}; w)]$$

$$= [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \quad \square$$

**定理2.22.3.**  $\Omega_{ab}(s, \zeta; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)]$

**证明:**  $\Omega_{ab}(s, \zeta; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]$

$$= \Omega_{ab}(s, \zeta; w)I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)$$

$$= [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}}] + I_{w+1} \otimes [\Gamma(s - \frac{1}{2}; w)S_{ab}(s - \frac{1}{2}, \zeta; w)]$$

$$= [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] \quad \square$$



**定理2.22.4.**  $[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega_{ab}(s, \varsigma; w) = [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]$

**证明:**  $[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega_{ab}(s, \varsigma; w)$

$$= [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega_{ab}(s, \varsigma; w)$$

$$= [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}}][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] + I_{w+1} \otimes [S_{ab}(s - \frac{1}{2}, \varsigma; w)\bar{\Gamma}(s - \frac{1}{2}; w)]$$

$$= [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \quad \square$$

## 2.23 推论：常数矩阵 $\Gamma(s - \frac{1}{2}; w), \bar{\Gamma}(s - \frac{1}{2}; w)$ 的几个恒等式

**性质2.23.1.**

$$\begin{cases} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] \\ [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \\ = \Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w) \\ [[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)], \Omega(s; w)] = 0 \end{cases}$$

**性质2.23.2.**

$$\begin{cases} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega_{ab}(s, \varsigma; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \\ [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \\ = \Omega_{ab}(s, \varsigma; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega_{ab}(s, \varsigma; w) \\ [[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)], \Omega_{ab}(s, \varsigma; w)] = 0 \end{cases}$$

**性质2.23.3.**

$$\begin{cases} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vartheta \cdot \Omega(s; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = \{\vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]\}^n \\ [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]\{\vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]\}^n [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \\ = [\vartheta \cdot \Omega(s; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vartheta \cdot \Omega(s; w)]^n \\ [[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)], [\vartheta \cdot \Omega(s; w)]^n] = 0 \end{cases}$$

**性质2.23.4.**

$$\begin{cases} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = \{\vartheta^{ab}[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]\}^n \\ [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]\{\vartheta^{ab}[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]\}^n [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \\ = [\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n \\ [[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)], [\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n] = 0 \end{cases}$$

**推论2.23.1.**

$$\begin{cases} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]} \\ [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \\ = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} \\ [[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)], e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}] = 0 \end{cases}$$

**推论2.23.2.**

$$\begin{cases} I_{(w+1)2s-1}\Gamma(s - \frac{1}{2}; w) = \Gamma(s - \frac{1}{2}; w)I_{C_{2s-1+w}^{2s-1}}, \bar{\Gamma}(s - \frac{1}{2}; w)I_{(w+1)2s-1} = I_{C_{2s-1+w}^{2s-1}}\bar{\Gamma}(s - \frac{1}{2}; w) \\ [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\sigma(\frac{1}{2}; w) \otimes I_{(w+1)2s-1}[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = \sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} \\ [\sigma(\frac{1}{2}; w) \otimes I_{(w+1)2s-1}[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}}] \\ [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\sigma(\frac{1}{2}; w) \otimes I_{(w+1)2s-1}] = [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}}][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \end{cases}$$

**性质2.23.5.**  $(\sigma \otimes I_{(w+1)2s-1}, -i\varsigma)_a [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]N(s; w) = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]Z_a(s, \varsigma; w)$

## 2.24 矩阵 $I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)$ , $I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)$ 的常数不变张量性质

**定理2.24.1.**  $[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s - \frac{1}{2}, \varsigma; w)}$

**证明:**  $\Omega_{ab}(s, \varsigma; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]$   
 $\Leftrightarrow 0 = \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]$   
 $- \frac{i}{2}\vartheta^{ab}[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]$   
 $\Leftrightarrow [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s - \frac{1}{2}, \varsigma; w)} \quad \square$

**定理2.24.2.**  $[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s - \frac{1}{2}, \varsigma; w)} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}$

**证明:**  $[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega_{ab}(s, \varsigma; w) = [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]$   
 $\Leftrightarrow 0 = -\frac{i}{2}\vartheta^{ab}[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega_{ab}(s, \varsigma; w)$   
 $+ \frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]$   
 $\Leftrightarrow [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s - \frac{1}{2}, \varsigma; w)} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} \quad \square$

## 2.25 矩阵 $I \otimes \bar{\Gamma}(s - \frac{1}{2})$ , $I \otimes \Gamma(s - \frac{1}{2})$ 的常数不变张量性质

**定理2.25.1.**  $[I \otimes \Gamma(s - \frac{1}{2})] = e^{(i\omega + \varsigma\epsilon) \cdot \Omega(s)} [I \otimes \Gamma(s - \frac{1}{2})] e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2})}$

**证明:**  $\Omega(s)[I \otimes \Gamma(s - \frac{1}{2})] = [I \otimes \Gamma(s - \frac{1}{2})][\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})]$   
 $\Leftrightarrow 0 = (i\omega + \varsigma\epsilon) \cdot \Omega(s)[I \otimes \Gamma(s - \frac{1}{2})]$   
 $- (i\omega + \varsigma\epsilon) \cdot [I \otimes \Gamma(s - \frac{1}{2})][\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})]$   
 $\Leftrightarrow [I \otimes \Gamma(s - \frac{1}{2})] = e^{(i\omega + \varsigma\epsilon) \cdot \Omega(s)} [I \otimes \Gamma(s - \frac{1}{2})] e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2})} \quad \square$

**定理2.25.2.**  $[I \otimes \bar{\Gamma}(s - \frac{1}{2})] = e^{(i\omega + \varsigma\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2})} [I \otimes \bar{\Gamma}(s - \frac{1}{2})] e^{-(i\omega + \varsigma\epsilon) \cdot \Omega(s)}$

**证明:**  $[I \otimes \bar{\Gamma}(s - \frac{1}{2})]\Omega(s) = [\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})][I \otimes \bar{\Gamma}(s - \frac{1}{2})]$   
 $\Leftrightarrow 0 = -(i\omega + \varsigma\epsilon) \cdot [I \otimes \bar{\Gamma}(s - \frac{1}{2})]\Omega(s)$   
 $+ (i\omega + \varsigma\epsilon) \cdot [\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})][I \otimes \bar{\Gamma}(s - \frac{1}{2})]$   
 $\Leftrightarrow [I \otimes \bar{\Gamma}(s - \frac{1}{2})] = e^{(i\omega + \varsigma\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2})} [I \otimes \bar{\Gamma}(s - \frac{1}{2})] e^{-(i\omega + \varsigma\epsilon) \cdot \Omega(s)} \quad \square$

## 2.26 两个定理的另外一种证明方法(强调思路启发而不求全)

**定理2.26.1.**  $\Omega(s)\Gamma(s) = \Gamma(s)\sigma(s)$ ,  $\bar{\Gamma}(s)\Omega(s) = \sigma(s)\bar{\Gamma}(s)$

**证明:** 采用数学归纳法

1: 当  $s = \frac{1}{2}$  时,  $\sigma(\frac{1}{2})\Gamma(\frac{1}{2}) = \Gamma(\frac{1}{2})\sigma(\frac{1}{2})$  成立。

2: 假设  $s = k$  时,  $\Omega(k)\Gamma(k) = \Gamma(k)\sigma(k)$  成立。

3: 当  $s = k + \frac{1}{2}$  时

$$\begin{aligned} & \Omega(k + \frac{1}{2})\Gamma(k + \frac{1}{2}) \\ &= [\sigma(\frac{1}{2}) \otimes I_{2k} + I \otimes \Omega(k)][I \otimes \Gamma(k)]N(k + \frac{1}{2}) \\ &= \{[I \otimes \Gamma(k)][\sigma(\frac{1}{2}) \otimes I_{2(k+1)}] + I \otimes [\Gamma(k)\sigma(k)]\}N(k + \frac{1}{2}) \\ &= [I \otimes \Gamma(k)][\sigma(\frac{1}{2}) \otimes I_{2(k+1)} + I \otimes \sigma(k)]N(k + \frac{1}{2}) \\ &= [I \otimes \Gamma(k)]N(k + \frac{1}{2})\sigma(k + \frac{1}{2}) \\ &= \Gamma(k + \frac{1}{2})\sigma(k + \frac{1}{2}) \end{aligned}$$

所以命题成立, 证毕。 □

**定理2.26.2.**  $\bar{\Gamma}(s)\Omega(s) = \sigma(s)\bar{\Gamma}(s)$

证明：采用数学归纳法

1: 当  $s = \frac{1}{2}$  时,  $\bar{\Gamma}(\frac{1}{2})\sigma(\frac{1}{2}) = \sigma(\frac{1}{2})\bar{\Gamma}(\frac{1}{2})$  成立。

2: 假设  $s = k$  时,  $\bar{\Gamma}(k)\Omega(k) = \sigma(k)\bar{\Gamma}(k)$  成立。

3: 当  $s = k + \frac{1}{2}$  时

$$\begin{aligned} & \bar{\Gamma}(k + \frac{1}{2})\Omega(k + \frac{1}{2}) \\ &= \bar{N}(k + \frac{1}{2})[I \otimes \bar{\Gamma}(k)]\{\sigma(\frac{1}{2}) \otimes I_{2^{2k}} + I \otimes \Omega(k)\} \\ &= \bar{N}(k + \frac{1}{2})\{[\sigma(\frac{1}{2}) \otimes I_{2^{(k+1)}}][I \otimes \bar{\Gamma}(k)] + [I \otimes \sigma(k)][I \otimes \bar{\Gamma}(k)]\} \\ &= \bar{N}(k + \frac{1}{2})[\sigma(\frac{1}{2}) \otimes I_{2^{(k+1)}} + I \otimes \sigma(k)][I \otimes \bar{\Gamma}(k)] \\ &= \sigma(k + \frac{1}{2})\bar{N}(k + \frac{1}{2})[I \otimes \bar{\Gamma}(k)] \\ &= \sigma(k + \frac{1}{2})\bar{\Gamma}(k + \frac{1}{2}) \end{aligned}$$

所以命题成立，证毕。 □

### 3 全对称高自旋张量(关于 $w > 1$ 情形的讨论和澄清)

#### 3.1 全对称旋量例子

定义3.1.1.

$$\underbrace{\Omega_{A_\zeta B_\zeta C_\zeta \dots}_{A'_\zeta B'_\zeta C'_\zeta \dots}}_{2s}[s; \sigma(\tau)] := \underbrace{\sigma_{A_\zeta}^{A'_\zeta}(\tau)\delta_{B_\zeta}^{B'_\zeta}\delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta}\sigma_{B_\zeta}^{B'_\zeta}(\tau)\delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta}\delta_{B_\zeta}^{B'_\zeta}\sigma_{C_\zeta}^{C'_\zeta}(\tau) \dots}_{2s} + \dots$$

[⇕] [⇕]

定义3.1.2.  $\Omega[s; \sigma(\tau)] := \sigma(\tau) \otimes I_{(2\tau+1)^{2s-1}} + I_{2\tau+1} \otimes \sigma(\tau) \otimes I_{(2\tau+1)^{2s-2}} + \dots + I_{(2\tau+1)^{2s-1}} \otimes \sigma(\tau)$

定义3.1.3.  $\Omega_{ab}[s, \zeta; \sigma(\tau)] := i\sigma_{\zeta ab}^{\alpha_\zeta}\Omega_{\alpha_\zeta}[s; \sigma(\tau)]$ ,  $S_{ab}[s, \zeta; \sigma(\tau)] := \bar{\Gamma}(s; w)\Omega_{ab}[s, \zeta; \sigma(\tau)]\Gamma(s; w)$

#### 3.2 矩阵 $\Gamma(s; 2\tau)$ , $\bar{\Gamma}(s; 2\tau)$ 无穷种常数不变张量的性质

性质3.2.1.  $\begin{cases} \Gamma(s; 2\tau) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}[s, \zeta; \sigma(\tau)]}\Gamma(s; 2\tau)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}[s, \zeta; \sigma(\tau)]}, \forall \sigma(\tau) \in \text{任意表象} \\ \bar{\Gamma}(s; 2\tau) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}[s, \zeta; \sigma(\tau)]}\bar{\Gamma}(s; 2\tau)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}[s, \zeta; \sigma(\tau)]}, \forall \sigma(\tau) \in \text{任意表象} \end{cases}$

性质3.2.2.  $\begin{cases} \Gamma(s; 2) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \zeta; \gamma)}\Gamma(s; 2)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s, \zeta; \gamma)} \\ \bar{\Gamma}(s; 2) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \zeta; \gamma)}\bar{\Gamma}(s; 2)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \zeta; \gamma)} \end{cases}$

性质3.2.3.  $\begin{cases} \Gamma(s; 2) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}[s, \zeta; \sigma(1)]}\Gamma(s; 2)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}[s, \zeta; \sigma(1)]} \\ \bar{\Gamma}(s; 2) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}[s, \zeta; \sigma(1)]}\bar{\Gamma}(s; 2)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}[s, \zeta; \sigma(1)]} \end{cases}$

#### 3.3 全对称复合旋量例子

性质3.3.1.  $\begin{cases} \Gamma(s; 3) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \zeta; R)}\Gamma(s; 3)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s, \zeta; R)} \\ \bar{\Gamma}(s; 3) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \zeta; R)}\bar{\Gamma}(s; 3)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \zeta; R)} \end{cases}$

性质3.3.2.  $\begin{cases} \Gamma(s; 3) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \zeta; \gamma_a)}\Gamma(s; 3)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s, \zeta; \gamma_a)} \\ \bar{\Gamma}(s; 3) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \zeta; \gamma_a)}\bar{\Gamma}(s; 3)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \zeta; \gamma_a)} \\ \Omega_{ab}(s, \zeta; \gamma_a) := -\frac{i}{4}[\Omega_a(s, \zeta; \gamma_a), \Omega_b(s, \zeta; \gamma_a)], S_{ab}(s, \zeta; \gamma_a) := \bar{\Gamma}(s; w)\Omega_{ab}(s, \zeta; \gamma_a)\Gamma(s; w) \end{cases}$

性质3.3.3.  $\begin{cases} \Gamma(s; 3) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}[s, \zeta; \sigma(\frac{1}{2}) \otimes I]}\Gamma(s; 3)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}[s, \zeta; \sigma(\frac{1}{2}) \otimes I]} \\ \bar{\Gamma}(s; 3) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}[s, \zeta; \sigma(\frac{1}{2}) \otimes I]}\bar{\Gamma}(s; 3)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}[s, \zeta; \sigma(\frac{1}{2}) \otimes I]} \end{cases}$

### 3.4 全对称空间旋量例子

$$\text{性质3.4.1. } \begin{cases} \Gamma(s; 3) = e^{\frac{i}{2}\vartheta^{ij}\Omega_{ij}[s,\varsigma;\sigma(\frac{1}{2})\otimes I]}\Gamma(s; 3)e^{-\frac{i}{2}\vartheta^{ij}S_{ij}[s,\varsigma;\sigma(\frac{1}{2})\otimes I]} \\ \bar{\Gamma}(s; 3) = e^{\frac{i}{2}\vartheta^{ij}S_{ij}[s,\varsigma;\sigma(\frac{1}{2})\otimes I]}\bar{\Gamma}(s; 3)e^{-\frac{i}{2}\vartheta^{ij}\Omega_{ij}[s,\varsigma;\sigma(\frac{1}{2})\otimes I]} \end{cases}$$

### 3.5 自洽性质验证

$$\text{性质3.5.1. } \psi' = S\psi \Rightarrow \hat{\psi}' = \Gamma(s; w)\psi' = \Gamma(s; w)S\psi = \Gamma(s; w)S\bar{\Gamma}(s; w)\hat{\psi}$$

$$\text{性质3.5.2. } \hat{\psi}' = (S \otimes \cdots \otimes S)\hat{\psi} \Rightarrow \psi' = \bar{\Gamma}(s; w)\hat{\psi}' = \bar{\Gamma}(s; w)(S \otimes \cdots \otimes S)\hat{\psi} = \bar{\Gamma}(s; w)(S \otimes \cdots \otimes S)\Gamma(s; w)\psi$$

$$\text{性质3.5.3. } \psi' = S\psi, \hat{\psi}' = \Gamma(s; w)S\bar{\Gamma}(s; w)\hat{\psi}, \hat{\psi} = \Gamma(s; w)\psi \Rightarrow \hat{\psi}' = \Gamma(s; w)\psi'$$

### 3.6 具体例子

$$\text{定义3.6.1. } \Gamma_{A_\varsigma B_\varsigma}^{k_\varsigma}(1; 2) = \frac{1}{2!}\Gamma_{(A_\varsigma B_\varsigma)}^{k_\varsigma}(1; 2)$$

$$\Gamma_{\underbrace{0_\varsigma \cdots 0_\varsigma}_{l_0} \underbrace{1_\varsigma \cdots 1_\varsigma}_{l_1} \underbrace{2_\varsigma \cdots 2_\varsigma}_{l_2}}^{k_\varsigma}(1; 2) = \sqrt{\frac{l_0!l_1!l_2!}{2!}}\delta\{k_\varsigma, \sum_{l=0}^1 (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+2-k}^{2l} - C_{2l+2-\lambda_{2l}}^{2l})\}, l_0 + l_1 + l_2 = 2$$

$$\text{定义3.6.2. } \Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma}(1; 2) = \frac{1}{2!}\Gamma_{k_\varsigma}^{(A_\varsigma B_\varsigma)}(1; 2)$$

$$\Gamma_{k_\varsigma}^{\underbrace{0_\varsigma \cdots 0_\varsigma}_{l_0} \underbrace{1_\varsigma \cdots 1_\varsigma}_{l_1} \underbrace{2_\varsigma \cdots 2_\varsigma}_{l_2}}(1; 2) = \sqrt{\frac{l_0!l_1!l_2!}{2!}}\delta\{k_\varsigma, \sum_{l=0}^1 (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+2-k}^{2l} - C_{2l+2-\lambda_{2l}}^{2l})\}, l_0 + l_1 + l_2 = 2$$

$$\text{性质3.6.1. } \bar{\Gamma}(1; 2)\Gamma(1; 2) = I_6$$

$$\bar{\Gamma}(1; 2) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \Gamma(1; 2) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \Gamma(1; 2)\bar{\Gamma}(1; 2) = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

性质3.6.2.

$$\bar{\Gamma}(1; 2)[\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]\Gamma(1; 2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 1 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{bmatrix}, \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & -1 & -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & \sqrt{2} & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 1 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$[\bar{\Gamma}(1; 2)[\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]\Gamma(1; 2)]^2 = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2\sqrt{2} & 0 & 0 \\ 0 & 0 & 2\sqrt{2} & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

性质3.6.3.

$$\bar{\Gamma}(1; 2)(\gamma \otimes I_3 + I_3 \otimes \gamma)\Gamma(1; 2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & 0 & 0 & 0 & i\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ -i\sqrt{2} & 0 & 0 & 0 & 0 & i\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 \\ i\sqrt{2} & 0 & 0 & -i\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\bar{\Gamma}(1; 2)(\gamma \otimes I_3 + I_3 \otimes \gamma)\Gamma(1; 2)]^2 = \begin{bmatrix} 4 & 0 & 0 & -2 & 0 & -2 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ -2 & 0 & 0 & 4 & 0 & -2 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ -2 & 0 & -2 & 0 & 0 & 4 \end{bmatrix}$$

### 3.7 高维时空中全对称旋量例子

$$\text{性质3.7.1. } \begin{cases} \Gamma(s; \gamma_a) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;\gamma_a)}\Gamma(s; \gamma_a)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;\gamma_a)}, \forall \gamma_a \in \text{任意表象} \\ \bar{\Gamma}(s; \gamma_a) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s,\varsigma;\gamma_a)}\bar{\Gamma}(s; \gamma_a)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\varsigma;\gamma_a)}, \forall \gamma_a \in \text{任意表象} \\ \Omega_{ab}(s, \varsigma; \gamma_a) := -\frac{i}{4}[\Omega_a(s, \varsigma; \gamma_a), \Omega_b(s, \varsigma; \gamma_a)], S_{ab}(s, \varsigma; \gamma_a) := \bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; \gamma_a)\Gamma(s; w) \end{cases}$$

## 4 完美常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$

### 4.1 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的引入

$$\text{定义4.1.1. } N_{A_\zeta l_\zeta}^{k_\zeta}(s) := \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} \underbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}_{2s} \left(s - \frac{1}{2}\right), N_{k_\zeta}^{A_\zeta l_\zeta}(s) := \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots} \underbrace{\Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}}_{2s-1} \left(s - \frac{1}{2}\right)$$

$$\text{定义4.1.2. } N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) := \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} \underbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}_{2s} \left(s - \frac{1}{2}; w\right), N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) := \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots} \underbrace{\Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}}_{2s-1} \left(s - \frac{1}{2}; w\right)$$

$$\text{推论4.1.1. } N(s; w) = [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \Gamma(s; w), \bar{N}(s; w) = \bar{\Gamma}(s; w) [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]$$

$$\text{性质4.1.1. } \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) = N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) \underbrace{\Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}}_{2s-1} \left(s - \frac{1}{2}; w\right), \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}}_{2s}(s; w) = N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) \underbrace{\Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}}_{2s-1} \left(s - \frac{1}{2}; w\right)$$

$$\text{推论4.1.2. } \Gamma(s; w) = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] N(s; w), \bar{\Gamma}(s; w) = \bar{N}(s; w) [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)],$$

$$\text{性质4.1.2. } \Gamma(s; w) = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \Gamma(s; w)$$

$$\text{推论4.1.3. } N_{A_\zeta l_\zeta}^{k_\zeta}(s) = N_{A_\zeta l_\zeta}^{k_\zeta}(s; 1), N_{k_\zeta}^{A_\zeta l_\zeta}(s) = N_{k_\zeta}^{A_\zeta l_\zeta}(s; 1)$$

自我评述：以上表明  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ ,  $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$  是常数不变张量  $N_{A_\zeta l_\zeta}^{k_\zeta}(s)$ ,  $N_{k_\zeta}^{A_\zeta l_\zeta}(s)$  的推广。

### 4.2 常数矩阵 $N_{A_\zeta}(s; w)$ , $N^{A_\zeta}(s; w)$ ; $\bar{N}_{A_\zeta}(s; w)$ , $\bar{N}^{A_\zeta}(s; w)$ ; $N(s; w)$ , $\bar{N}(s; w)$ 的引入

定义4.2.1.

$$\begin{cases} N_{A_\zeta}(s; w) \prec N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), N^{A_\zeta}(s; w) \prec N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) | I_{C_{2s+w}^{2s}} \times I_{C_{2s-1+w}^{2s-1}} \\ \bar{N}_{A_\zeta}(s; w) := N_{A_\zeta}^+(s; w) \succ N_{A_\zeta l_\zeta}^{k_\zeta}(s; w), \bar{N}^{A_\zeta}(s; w) := N^{+A_\zeta}(s; w) \succ N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) | I_{C_{2s-1+w}^{2s-1}} \times I_{C_{2s+w}^{2s}} \\ N(s; w) \prec N_{A_\zeta \otimes l_\zeta}^{k_\zeta}(s; w) | (w+1) I_{C_{2s-1+w}^{2s-1}} \times I_{C_{2s+w}^{2s}}, \bar{N}(s; w) = N^+(s; w) \prec N_{k_\zeta}^{A_\zeta \otimes l_\zeta}(s; w) | I_{C_{2s+w}^{2s}} \times (w+1) I_{C_{2s-1+w}^{2s-1}} \end{cases}$$

$$\text{推论4.2.1. } [N_{A_\zeta}(s; w)] = C_{2s+w}^{2s} \times C_{2s-1+w}^{2s-1}, [\bar{N}_{A_\zeta}(s; w)] = C_{2s-1+w}^{2s-1} \times C_{2s+w}^{2s}$$

$$\text{推论4.2.2. } [N(s; w)] = (w+1) C_{2s-1+w}^{2s-1} \times C_{2s+w}^{2s}, [\bar{N}(s; w)] = C_{2s+w}^{2s} \times (w+1) C_{2s-1+w}^{2s-1}$$

$N_{A_\zeta}(s)$ ,  $\bar{N}_{A_\zeta}(s)$  具体显式表示：

$$\text{推论4.2.3. } N_{A_\zeta}(s) \simeq N^{A_\zeta}(s) = \left\{ \frac{1}{\sqrt{2s}} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2s}} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2s-1} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix} \right\}$$

$$\text{推论4.2.4. } \bar{N}_{A_\zeta}(s) \simeq \bar{N}^{A_\zeta}(s) = \left\{ \frac{1}{\sqrt{2s}} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2s}} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix} \right\}$$

$N(s)$ ,  $\bar{N}(s)$  的显式表示：

$$\text{推论4.2.5. } N^{A_\zeta}(s) \leftrightarrow N^+(s) \simeq \bar{N}(s) = \frac{1}{\sqrt{2s}} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{2s-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{2s-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2s-1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix}$$

$$\text{推论4.2.6. } N^+(s) \simeq \bar{N}(s) = \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{1} \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{1} \\ 0 & 0 & 0 & 0 & \sqrt{3} \end{bmatrix}, \frac{1}{\sqrt{4}} \begin{bmatrix} \sqrt{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & \sqrt{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4} \end{bmatrix}, \dots$$

### 4.3 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的基本性质

相等性:

性质4.3.1.

$$\begin{cases} N_{A'_\zeta l'_\zeta}^{k'_\zeta}(s; w) \simeq N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) \simeq N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) \simeq N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; w) \\ [N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)]^* \simeq N_{A'_\zeta l'_\zeta}^{k'_\zeta}(s; w), [N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)]^* \simeq N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; w) \end{cases}$$

推论4.3.1.

$$\begin{cases} N_{A_\zeta}(s; w) \simeq N^{A_\zeta}(s; w) \simeq N_{A'_\zeta}(s; w) \simeq N^{A'_\zeta}(s; w); \bar{N}_{A_\zeta}(s; w) \simeq \bar{N}^{A_\zeta}(s; w) \simeq \bar{N}_{A'_\zeta}(s; w) \simeq \bar{N}^{A'_\zeta}(s; w) \\ N_{A_\zeta}(s; w) = N_{A_\zeta}^*(s; w), \bar{N}_{A_\zeta}(s; w) = \bar{N}_{A_\zeta}^*(s; w); N(s; w) = N^*(s; w), \bar{N}(s; w) = \bar{N}^*(s; w) \end{cases}$$

### 4.4 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的正交性质

正交性:

引理4.4.1.  $\sum_{k=0}^{2s-1} C_{w+k}^w = C_{w+2s}^{w+1}$

引理4.4.2.  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; w) = \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}_{2s}(s; w)$

证明:  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; w)$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}_{2s}(s; w) \underbrace{\Gamma_{B'_\zeta C'_\zeta \dots}^{l'_\zeta}}_{2s-1}(s - \frac{1}{2}; w)$$

$$= \frac{1}{(2s-1)!} \delta_{(B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots) \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}_{2s}(s; w)$$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}_{2s}(s; w) \underbrace{\delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots}_{2s-1}$$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}_{2s}(s; w)$$

□

定理4.4.1.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{m_\zeta}^{A_\zeta l_\zeta}(s; w) = \delta_{m_\zeta}^{k_\zeta} [\Leftrightarrow] N^{A_\zeta}(s; w) \bar{N}_{A_\zeta}(s; w) = I_{C_{2s+w}^{2s}} [\Leftrightarrow] \bar{N}(s; w) N(s; w) = I_{C_{2s+w}^{2s}} \\ N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{A_\zeta m_\zeta}(s; w) = (1 + \frac{w}{2s}) \delta_{l_\zeta}^{m_\zeta} [\Leftrightarrow] \bar{N}_{A_\zeta}(s; w) N^{A_\zeta}(s; w) = (1 + \frac{w}{2s}) I_{C_{2s-1+w}^{2s-1}} \\ N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{B_\zeta l_\zeta}(s; w) = \frac{1}{w+1} C_{2s+w}^{2s} \delta_{A_\zeta}^{B_\zeta} [\Leftrightarrow] \text{tr}[\bar{N}_{A_\zeta}(s; w) N^{B_\zeta}(s; w)] = \frac{1}{w+1} C_{2s+w}^{2s} \delta_{A_\zeta}^{B_\zeta} \end{cases}$$

证明:  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{m_\zeta}^{A_\zeta l_\zeta}(s; w)$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{m_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}}_{2s}(s; w)$$

$$= \delta_{m_\zeta}^{k_\zeta}$$

□

证明:  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{A_\zeta m_\zeta}(s; w)$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{k_\zeta}^{A_\zeta B'_\zeta C'_\zeta \dots}}_{2s}(s; w) \underbrace{\Gamma_{B'_\zeta C'_\zeta \dots}^{m_\zeta}}_{2s-1}(s - \frac{1}{2}; w)$$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{k_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}_{2s}(s; w) \delta_{A'_\zeta}^{A_\zeta} \underbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{B'_\zeta C'_\zeta \dots}^{m_\zeta}}_{2s-1}(s - \frac{1}{2}; w)$$

$$\begin{aligned}
&= \frac{1}{(2s)!} \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} \delta_{A'_\zeta} \overbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}^{2s-1} (s - \frac{1}{2}; w) \underbrace{\Gamma_{B'_\zeta C'_\zeta \dots}^{m_\zeta}}_{2s-1} (s - \frac{1}{2}; w) \\
&= \frac{1}{(2s)!} \delta_{A_\zeta}^{A'_\zeta} \overbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}^{2s-1} (s - \frac{1}{2}; w) \underbrace{\Gamma_{B_\zeta C_\zeta \dots}^{m_\zeta}}_{2s-1} (s - \frac{1}{2}; w) \\
&= \frac{1}{(2s)!} [\delta_{A_\zeta}^{A'_\zeta} \overbrace{\Gamma_{l_\zeta}^{(B_\zeta C_\zeta \dots)}}^{2s-1} (s - \frac{1}{2}; w) \underbrace{\Gamma_{B_\zeta C_\zeta \dots}^{m_\zeta}}_{2s-1} (s - \frac{1}{2}; w) + (2s-1) \overbrace{\Gamma_{l_\zeta}^{(A_\zeta C_\zeta \dots)}}^{2s-1} (s - \frac{1}{2}; w) \underbrace{\Gamma_{A_\zeta C_\zeta \dots}^{m_\zeta}}_{2s-1} (s - \frac{1}{2}; w)] \\
&= \frac{1}{(2s)!} [(2s-1)! \delta_{A_\zeta}^{A'_\zeta} \overbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}^{2s-1} (s - \frac{1}{2}; w) \underbrace{\Gamma_{B_\zeta C_\zeta \dots}^{m_\zeta}}_{2s-1} (s - \frac{1}{2}; w) + (2s-1)(2s-1)! \overbrace{\Gamma_{l_\zeta}^{A_\zeta C_\zeta \dots}}^{2s-1} (s - \frac{1}{2}; w) \underbrace{\Gamma_{A_\zeta C_\zeta \dots}^{m_\zeta}}_{2s-1} (s - \frac{1}{2}; w)] \\
&= \frac{1}{(2s)!} [(2s-1)! \delta_{A_\zeta}^{A'_\zeta} + (2s-1)(2s-1)!] \delta_{l_\zeta}^{m_\zeta} \\
&= (1 + \frac{w}{2s}) \delta_{l_\zeta}^{m_\zeta} \quad \square
\end{aligned}$$

证明:  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{A'_\zeta l_\zeta}(s; w)$

$$\begin{aligned}
&= \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) \overbrace{\Gamma_{k_\zeta}^{A'_\zeta B_\zeta C_\zeta \dots}}^{2s}(s; w) \\
&= \frac{1}{(2s)!} \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} \\
&= \frac{1}{(2s)!} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} + \overbrace{\delta_{A_\zeta}^{B'_\zeta} \delta_{B_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} + \overbrace{\delta_{A_\zeta}^{C'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{A'_\zeta} \dots}^{2s} + \dots] \\
&= \frac{1}{(2s)!} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} + \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} + \overbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{A_\zeta}^{A'_\zeta} \dots}^{2s} + \dots] \\
&= \frac{1}{(2s)!} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1} + (2s-1) \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1}] \\
&= \frac{1}{(2s)!} [\delta_{A_\zeta}^{A'_\zeta} (2s-1)! \overbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}^{2s-1} (s - \frac{1}{2}; w) \underbrace{\Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}}_{2s-1} (s - \frac{1}{2}; w) + (2s-1) \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1}] \\
&= \frac{1}{(2s)!} [\delta_{A_\zeta}^{A'_\zeta} (2s-1)! \delta_{l_\zeta}^{l_\zeta} (s - \frac{1}{2}; w) + (2s-1) \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1}] \\
&= \frac{1}{(2s)!} [\delta_{A_\zeta}^{A'_\zeta} (2s-1)! C_{2s-1+w}^{2s-1} + (2s-1) \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1}] \\
&= \frac{1}{(2s)!} [\delta_{A_\zeta}^{A'_\zeta} \frac{(2s-1)! (2s-1+w)!}{(2s-1)! w!} + (2s-1) \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1}] \\
&= \frac{1}{(2s)!} [\frac{(2s-1)! (2s-1+w)!}{(2s-1)! w!} + \frac{(2s-1)! (2s-2+w)!}{(2s-2)! w!} + \frac{(2s-1)! (2s-3+w)!}{(2s-3)! w!} + \dots + \frac{(2s-1)! (0+w)!}{0! w!}] \delta_{A_\zeta}^{A'_\zeta} \\
&= \frac{1}{2s} [\frac{(2s-1+w)!}{(2s-1)! w!} + \frac{(2s-2+w)!}{(2s-2)! w!} + \frac{(2s-3+w)!}{(2s-3)! w!} + \dots + \frac{(0+w)!}{0! w!}] \delta_{A_\zeta}^{A'_\zeta} \\
&= \frac{1}{2s} \sum_{k=0}^{2s-1} C_{w+k}^w \delta_{A_\zeta}^{A'_\zeta} = \frac{1}{2s} C_{w+2s}^{w+1} \delta_{A_\zeta}^{A'_\zeta} = \frac{1}{w+1} C_{2s+w}^{2s} \delta_{A_\zeta}^{A'_\zeta} \quad \square
\end{aligned}$$

性质4.4.1. 
$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{B_\zeta m_\zeta}(s; w) = \frac{1}{2s} [\delta_{A_\zeta}^{B_\zeta} \delta_{l_\zeta}^{m_\zeta} + (2s-1) N_{A_\zeta n_\zeta}^{m_\zeta}(s - \frac{1}{2}; w) N_{l_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2}; w)] \\ \bar{N}_{A_\zeta}(s; w) N^{B_\zeta}(s; w) = \frac{1}{2s} [\delta_{A_\zeta}^{B_\zeta} I_{C_{2s-1+w}^{2s-1}} + (2s-1) N^{B_\zeta}(s - \frac{1}{2}; w) \bar{N}_{A_\zeta}(s - \frac{1}{2}; w)] \end{cases}$$

证明:  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{A'_\zeta m_\zeta}(s; w)$

$$= \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) \overbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}^{2s-1}(s - \frac{1}{2}; w) \overbrace{\Gamma_{k_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}^{2s}(s; w) \underbrace{\Gamma_{B'_\zeta C'_\zeta \dots}^{m_\zeta}}_{2s-1}(s - \frac{1}{2}; w)$$

$$\begin{aligned}
&= \frac{1}{(2s)!} \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots} \overbrace{\cdots}^{2s-1} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) \\
&= \frac{1}{(2s)!} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} + \overbrace{\delta_{A_\zeta}^{B'_\zeta} \delta_{B_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} + \overbrace{\delta_{A_\zeta}^{C'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{A'_\zeta} \cdots}^{2s} + \cdots] \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) \\
&= \frac{1}{2s} (\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} + \overbrace{\delta_{A_\zeta}^{B'_\zeta} \delta_{B_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} + \overbrace{\delta_{A_\zeta}^{C'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{A'_\zeta} \cdots}^{2s} + \cdots) \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) \\
&= \frac{1}{2s} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} + (2s-1) \overbrace{\delta_{A_\zeta}^{B'_\zeta} \delta_{B_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s}] \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) \\
&= \frac{1}{2s} [\delta_{A_\zeta}^{A'_\zeta} \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots} (s - \frac{1}{2}; w) \Gamma_{B_\zeta C_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) + (2s-1) \Gamma_{l_\zeta}^{A'_\zeta C_\zeta \cdots} (s - \frac{1}{2}; w) \Gamma_{A_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w)] \\
&= \frac{1}{2s} [\delta_{A_\zeta}^{B'_\zeta} \delta_{l_\zeta}^{m_\zeta} + (2s-1) N_{A_\zeta n_\zeta}^{m_\zeta} (s - \frac{1}{2}; w) N_{l_\zeta}^{A'_\zeta n_\zeta} (s - \frac{1}{2}; w)] \quad \square
\end{aligned}$$

#### 4.5 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta} (s; w)$ , $N_{k_\zeta l_\zeta}^{A_\zeta} (s; w)$ 的升降指标 (存在 $\varepsilon_{A_\zeta B_\zeta}$ 为前提条件)

升降指标:

性质4.5.1.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta} (s; w) = \varepsilon^{k_\zeta m_\zeta} (s; w) \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta} (s - \frac{1}{2}; w) N_{m_\zeta n_\zeta}^{B_\zeta} (s; w) \\ N_{k_\zeta l_\zeta}^{A_\zeta} (s; w) = \varepsilon_{k_\zeta m_\zeta} (s; w) \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta} (s - \frac{1}{2}; w) N_{B_\zeta n_\zeta}^{m_\zeta} (s; w) \end{cases}$$

证明:  $\varepsilon^{k_\zeta m_\zeta} (s; w) \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta} (s - \frac{1}{2}; w) N_{m_\zeta n_\zeta}^{B_\zeta} (s; w)$

$$\begin{aligned}
&= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} (s; w) \overbrace{\varepsilon_{A'_\zeta E'_\zeta} \varepsilon_{B'_\zeta F'_\zeta} \varepsilon_{C'_\zeta G'_\zeta} \cdots}^{2s} \Gamma_{E'_\zeta F'_\zeta G'_\zeta \cdots}^{m_\zeta} (s; w) \varepsilon_{A_\zeta B_\zeta} \\
&\Gamma_{l_\zeta}^{B'_\zeta C'_\zeta \cdots} (s - \frac{1}{2}; w) \overbrace{\varepsilon_{B''_l F''_l} \varepsilon_{C''_l G''_l} \cdots}^{2s-1} \Gamma_{n_\zeta}^{F''_l G''_l \cdots} (s - \frac{1}{2}; w) N_{m_\zeta n_\zeta}^{B_\zeta} (s; w) \\
&= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} (s; w) \overbrace{\varepsilon_{A'_\zeta E'_\zeta} \varepsilon_{B'_\zeta F'_\zeta} \varepsilon_{C'_\zeta G'_\zeta} \cdots}^{2s} \Gamma_{E'_\zeta F'_\zeta G'_\zeta \cdots}^{m_\zeta} (s; w) \varepsilon_{A_\zeta B_\zeta} \Gamma_{l_\zeta}^{B'_\zeta C'_\zeta \cdots} (s - \frac{1}{2}; w) \overbrace{\varepsilon_{B''_l F''_l} \varepsilon_{C''_l G''_l} \cdots}^{2s-1} \Gamma_{m_\zeta}^{B_\zeta F''_l G''_l \cdots} (s; w) \\
&= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} (s; w) \overbrace{\varepsilon_{A'_\zeta E'_\zeta} \varepsilon_{B'_\zeta F'_\zeta} \varepsilon_{C'_\zeta G'_\zeta} \cdots}^{2s} \frac{1}{(2s)!} \overbrace{\delta_{(E'_\zeta} \delta_{F'_\zeta} \delta_{G'_\zeta} \cdots)}^{2s} \varepsilon_{A_\zeta B_\zeta} \overbrace{\varepsilon_{B''_l F''_l} \varepsilon_{C''_l G''_l} \cdots}^{2s-1} \Gamma_{l_\zeta}^{B'_\zeta C'_\zeta \cdots} (s - \frac{1}{2}; w) \\
&= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} (s; w) \overbrace{\varepsilon_{A'_\zeta B_\zeta} \varepsilon_{B'_\zeta F''_l} \varepsilon_{C'_\zeta G''_l} \cdots}^{2s} \varepsilon_{A_\zeta B_\zeta} \overbrace{\varepsilon_{B''_l F''_l} \varepsilon_{C''_l G''_l} \cdots}^{2s-1} \Gamma_{l_\zeta}^{B'_\zeta C'_\zeta \cdots} (s - \frac{1}{2}; w) \\
&= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} (s; w) \overbrace{\delta_{A'_\zeta}^{A'_\zeta} \delta_{B'_\zeta}^{B'_\zeta} \delta_{C'_\zeta}^{C'_\zeta} \cdots}^{2s} \Gamma_{l_\zeta}^{B'_\zeta C'_\zeta \cdots} (s - \frac{1}{2}; w) \\
&= N_{A_\zeta l_\zeta}^{k_\zeta} (s; w) \quad \square
\end{aligned}$$

推论4.5.1.

$$\begin{cases} N_{A_\zeta} (s; w) \varepsilon (s - \frac{1}{2}; w) = \varepsilon_{A_\zeta B_\zeta} \varepsilon (s; w) N^{B_\zeta} (s; w), \varepsilon (s - \frac{1}{2}; w) \bar{N}_{A_\zeta} (s; w) = \bar{N}^{B_\zeta} (s; w) \varepsilon_{B_\zeta A_\zeta} \varepsilon (s; w) \\ N^{A_\zeta} (s; w) \varepsilon (s - \frac{1}{2}; w) = \varepsilon^{A_\zeta B_\zeta} \varepsilon (s; w) N_{B_\zeta} (s; w), \varepsilon (s - \frac{1}{2}; w) \bar{N}^{A_\zeta} (s; w) = \bar{N}_{B_\zeta} (s; w) \varepsilon^{B_\zeta A_\zeta} \varepsilon (s; w) \\ N (s; w) \varepsilon (s; w) = [\varepsilon (\frac{1}{2}; w) \otimes \varepsilon (s - \frac{1}{2}; w)] N (s; w), \varepsilon (s; w) \bar{N} (s; w) = \bar{N} (s; w) [\varepsilon (\frac{1}{2}; w) \otimes \varepsilon (s - \frac{1}{2}; w)] \end{cases}$$



$$\begin{aligned}
& \text{证明: } \Gamma(s; w)\varepsilon(s; w) = \underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s} \Gamma(s; w) \\
& \Leftrightarrow [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Gamma(s; w)\varepsilon(s; w) = [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s} \Gamma(s; w) \\
& \Leftrightarrow N(s; w)\varepsilon(s; w) = [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s} \Gamma(s; w) \\
& \Leftrightarrow N(s; w)\varepsilon(s; w) = \{\varepsilon(\frac{1}{2}; w) \otimes [\bar{\Gamma}(s - \frac{1}{2}; w)\underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s-1}]\}\Gamma(s; w) \\
& \Leftrightarrow N(s; w)\varepsilon(s; w) = [\varepsilon(\frac{1}{2}; w) \otimes \varepsilon(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Gamma(s; w) \\
& \Leftrightarrow N(s; w)\varepsilon(s; w) = [\varepsilon(\frac{1}{2}; w) \otimes \varepsilon(s - \frac{1}{2}; w)]N(s; w) \quad \square
\end{aligned}$$

**Penrose标准升降规则:**

**性质4.5.2.**

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) = (-1)^{2s} [\zeta^{2s} \varepsilon^{k_\zeta m_\zeta}(s; w)] (-\zeta \varepsilon_{A_\zeta B_\zeta}) [(-\zeta)^{2s-1} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; w)] N_{m_\zeta n_\zeta}^{B_\zeta}(s; w) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) = (-1)^{2s} [(-\zeta)^{2s} \varepsilon_{k_\zeta m_\zeta}(s; w)] (\zeta \varepsilon^{A_\zeta B_\zeta}) [\zeta^{2s-1} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; w)] N_{B_\zeta n_\zeta}^{m_\zeta}(s; w) \end{cases}$$

#### 4.6 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的自旋变换

**性质4.6.1.**

$$\begin{cases} N_{k_\zeta}^{A_\zeta m_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) N_{B_\zeta m_\zeta}^{l_\zeta}(s; w) = \frac{1}{2s} \sigma^{\alpha_\zeta}_{k_\zeta l_\zeta}(s; w) [\Leftrightarrow] N^{A_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) = \frac{1}{2s} \sigma^{\alpha_\zeta}(s; w) \\ [\Leftrightarrow] \bar{N}(s; w) \sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}} N(s; w) = \frac{1}{2s} \sigma(s; w) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) N_{B_\zeta m_\zeta}^{k_\zeta}(s; w) = \frac{1}{2s} \sigma^{\alpha_\zeta}_{m_\zeta l_\zeta}(s - \frac{1}{2}; w) [\Leftrightarrow] \bar{N}_{B_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) N^{A_\zeta}(s; w) = \frac{1}{2s} \sigma^{\alpha_\zeta}(s - \frac{1}{2}; w) \end{cases}$$

**证明:**  $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) N_{A'_\zeta m_\zeta}^{k_\zeta}(s; w)$

$$\begin{aligned}
& = \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}}_{2s}(s; w) \underbrace{\Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w) \\
& = \frac{1}{(2s)!} \delta_{(A'_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots) \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w) \\
& = \frac{1}{(2s)!} [\delta_{A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots + \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots + \delta_{C'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{A'_\zeta}^{C_\zeta} \cdots + \cdots] \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w) \\
& = \frac{1}{(2s)!} [\delta_{A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots + (2s-1) \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w) \\
& = \frac{1}{(2s)!} (2s-1) \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w) \\
& = \frac{1}{(2s)!} (2s-1) \sigma^{\alpha_\zeta}_{B'_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{(A'_\zeta C'_\zeta \cdots)}^{l_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w) \\
& = \frac{1}{2s} (2s-1) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w) \sigma^{\alpha_\zeta}_{B'_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{A'_\zeta C'_\zeta \cdots}^{l_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \\
& = \frac{1}{2s} \sigma^{\alpha_\zeta}_{m_\zeta l_\zeta}(s - \frac{1}{2}; w) \quad \square
\end{aligned}$$

**性质4.6.2.**

$$\begin{cases} N_{k_\zeta}^{A_\zeta m_\zeta}(s; w) S_{ab A_\zeta}^{B_\zeta}(\frac{1}{2}; w) N_{B_\zeta m_\zeta}^{l_\zeta}(s; w) = \frac{1}{2s} S_{ab k_\zeta}^{l_\zeta}(s; w) [\Leftrightarrow] N^{A_\zeta}(s; w) S_{ab A_\zeta}^{B_\zeta}(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) = \frac{1}{2s} S_{ab}(s, \zeta; w) \\ [\Leftrightarrow] \bar{N}(s; w) S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}} N(s; w) = \frac{1}{2s} S_{ab}(s, \zeta; w) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) S_{ab A_\zeta}^{B_\zeta}(\frac{1}{2}; w) N_{B_\zeta m_\zeta}^{k_\zeta}(s; w) = \frac{1}{2s} S_{ab m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) [\Leftrightarrow] \bar{N}_{B_\zeta}(s; w) S_{ab A_\zeta}^{B_\zeta}(\frac{1}{2}; w) N^{A_\zeta}(s; w) = \frac{1}{2s} S_{ab}(s - \frac{1}{2}, \zeta; w) \end{cases}$$

证明:  $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) S_{ab A_\zeta}^{A'_\zeta}(\frac{1}{2}; w) N_{A'_\zeta m_\zeta}^{k_\zeta}(s; w)$

$$\begin{aligned} &= \Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}^{2s}}(s; w) \Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}(s - \frac{1}{2}; w) S_{ab A_\zeta}^{A'_\zeta}(\frac{1}{2}; w) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}(s; w) \Gamma_{m_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}}(s - \frac{1}{2}; w) \\ &= \frac{1}{(2s)!} \delta_{(A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots) S_{ab A_\zeta}^{A'_\zeta}(\frac{1}{2}; w) \Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}(s - \frac{1}{2}; w) \Gamma_{m_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}}(s - \frac{1}{2}; w) \\ &= \frac{1}{(2s)!} [\delta_{A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots + \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots + \delta_{C'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{A'_\zeta}^{C_\zeta} \dots + \dots] S_{ab A_\zeta}^{A'_\zeta}(\frac{1}{2}; w) \Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}(s - \frac{1}{2}; w) \Gamma_{m_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}}(s - \frac{1}{2}; w) \\ &= \frac{1}{(2s)!} [\delta_{A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots + (2s-1) \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots] S_{ab A_\zeta}^{A'_\zeta}(\frac{1}{2}; w) \Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}(s - \frac{1}{2}; w) \Gamma_{m_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}}(s - \frac{1}{2}; w) \\ &= \frac{1}{(2s)!} (2s-1) \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots S_{ab A_\zeta}^{A'_\zeta}(\frac{1}{2}; w) \Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}(s - \frac{1}{2}; w) \Gamma_{m_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}}(s - \frac{1}{2}; w) \\ &= \frac{1}{(2s)!} (2s-1) S_{ab B'_\zeta}^{A'_\zeta}(\frac{1}{2}; w) \Gamma_{(A'_\zeta C'_\zeta \dots)}^{l_\zeta}(s - \frac{1}{2}; w) \Gamma_{m_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}}(s - \frac{1}{2}; w) \\ &= \frac{1}{2s} (2s-1) \Gamma_{m_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}}(s - \frac{1}{2}; w) S_{ab B'_\zeta}^{A'_\zeta}(\frac{1}{2}; w) \Gamma_{A'_\zeta C'_\zeta \dots}^{l_\zeta}(s - \frac{1}{2}; w) \\ &= \frac{1}{2s} S_{ab m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) \end{aligned}$$

□

#### 4.7 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的置换性质

定理4.7.1.

$$\begin{cases} \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta}(\frac{1}{2}; w) N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + \sigma^{\alpha_\zeta} l_\zeta^{m_\zeta}(s - \frac{1}{2}; w) N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w) \sigma^{\alpha_\zeta} j_\zeta^{k_\zeta}(s; w) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta}(\frac{1}{2}; w) + N_{k_\zeta}^{B_\zeta m_\zeta}(s; w) \sigma^{\alpha_\zeta} m_\zeta^{l_\zeta}(s - \frac{1}{2}; w) = \sigma^{\alpha_\zeta} k_\zeta^{j_\zeta}(s; w) N_{j_\zeta}^{B_\zeta l_\zeta}(s; w) \\ \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta}(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) + \sigma^{\alpha_\zeta}(s - \frac{1}{2}; w) \bar{N}_{A_\zeta}(s; w) = \bar{N}_{A_\zeta}(s; w) \sigma^{\alpha_\zeta}(s; w) \\ N^{A_\zeta}(s; w) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta}(\frac{1}{2}; w) + N^{B_\zeta}(s; w) \sigma^{\alpha_\zeta}(s - \frac{1}{2}; w) = \sigma^{\alpha_\zeta}(s; w) N^{B_\zeta}(s; w) \\ [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}} + I_{w+1} \otimes \sigma^{\alpha_\zeta}(s - \frac{1}{2}; w)] N(s; w) = N(s; w) \sigma^{\alpha_\zeta}(s; w) \\ \bar{N}(s; w) [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}} + I_{w+1} \otimes \sigma^{\alpha_\zeta}(s - \frac{1}{2}; w)] = \sigma^{\alpha_\zeta}(s; w) \bar{N}(s; w) \end{cases}$$

$$\text{证明: } \Omega_{A_\zeta B_\zeta C_\zeta \dots}^{\overbrace{A'_\zeta B'_\zeta C'_\zeta \dots}^{2s}}(s; w) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) = \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) \sigma_{k_\zeta}^{l_\zeta}(s; w)$$

$$\Rightarrow \Gamma_{j_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}} \Omega_{A_\zeta B_\zeta C_\zeta \dots}^{\overbrace{A'_\zeta B'_\zeta C'_\zeta \dots}^{2s}}(s; w) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) = \Gamma_{j_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) \sigma_{k_\zeta}^{l_\zeta}(s; w)$$

$$\Leftrightarrow \Gamma_{j_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}} [\sigma_{A_\zeta}^{A'_\zeta}(\frac{1}{2}; w) \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots + \delta_{A'_\zeta}^{A_\zeta} \Omega_{B_\zeta C_\zeta \dots}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}}(s - \frac{1}{2}; w)] \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) = \Gamma_{j_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) \sigma_{k_\zeta}^{l_\zeta}(s; w)$$

$$\begin{aligned}
&\Leftrightarrow [\sigma_{A_\zeta} A'_\zeta(\frac{1}{2}; w) \Gamma_{j_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}} + \delta_{A'_\zeta} \sigma_{j_\zeta} n_\zeta(s - \frac{1}{2}; w) \Gamma_{n_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s}}(s; w)] \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) = \Gamma_{j_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) \sigma_{k_\zeta} l_\zeta(s; w) \\
&\Leftrightarrow \sigma_{A_\zeta} A'_\zeta(\frac{1}{2}; w) N_{A'_\zeta j_\zeta}^{l_\zeta}(s; w) + \sigma_{j_\zeta} n_\zeta(s - \frac{1}{2}; w) N_{A_\zeta n_\zeta}^{l_\zeta}(s; w) = N_{A_\zeta j_\zeta}^{k_\zeta}(s; w) \sigma_{k_\zeta} l_\zeta(s; w) \\
&\Leftrightarrow \sigma^{\alpha_\zeta} A_\zeta B_\zeta(\frac{1}{2}; w) N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + \sigma^{\alpha_\zeta} l_\zeta m_\zeta(s - \frac{1}{2}; w) N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w) \sigma^{\alpha_\zeta} j_\zeta k_\zeta(s; w) \quad \square
\end{aligned}$$

定理4.7.2.

$$\begin{cases}
S_{ab A_\zeta} B_\zeta(\frac{1}{2}; w) N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + S_{abl_\zeta} m_\zeta(s - \frac{1}{2}; w) N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w) S_{ab j_\zeta} k_\zeta(s; w) \\
N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) S_{ab A_\zeta} B_\zeta(\frac{1}{2}; w) + N_{k_\zeta}^{B_\zeta m_\zeta}(s; w) S_{ab m_\zeta} l_\zeta(s - \frac{1}{2}; w) = S_{ab k_\zeta} j_\zeta(s; w) N_{j_\zeta}^{B_\zeta l_\zeta}(s; w) \\
S_{ab A_\zeta} B_\zeta(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) + S_{ab}(s - \frac{1}{2}, \zeta; w) \bar{N}_{A_\zeta}(s; w) = \bar{N}_{A_\zeta}(s; w) S_{ab}(s, \zeta; w) \\
N^{A_\zeta}(s; w) S_{ab A_\zeta} B_\zeta(\frac{1}{2}; w) + N^{B_\zeta}(s; w) S_{ab}(s - \frac{1}{2}, \zeta; w) = S_{ab}(s, \zeta; w) N^{B_\zeta}(s; w) \\
[S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] N(s; w) = N(s; w) S_{ab}(s, \zeta; w) \\
\bar{N}(s; w) [S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] = S_{ab}(s, \zeta; w) \bar{N}(s; w)
\end{cases}$$

$$\begin{aligned}
&\text{证明: } \Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{\overbrace{A'_\zeta B'_\zeta C'_\zeta \dots}^{2s}}(s; w) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) = \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) S_{ab k_\zeta} l_\zeta(s; w) \\
&\Rightarrow \Gamma_{j_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}} \Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{\overbrace{A'_\zeta B'_\zeta C'_\zeta \dots}^{2s}}(s; w) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) = \Gamma_{j_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) S_{ab k_\zeta} l_\zeta(s; w) \\
&\Leftrightarrow \Gamma_{j_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}} [S_{ab A_\zeta} A'_\zeta(\frac{1}{2}; w) \underbrace{\delta_{B'_\zeta} \delta_{C'_\zeta} \dots}_{2s-1} + \delta_{A'_\zeta} \Omega_{ab B'_\zeta C'_\zeta \dots}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}}(s - \frac{1}{2}; w)] \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) = \Gamma_{j_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) S_{ab k_\zeta} l_\zeta(s; w) \\
&\Leftrightarrow [S_{ab A_\zeta} A'_\zeta(\frac{1}{2}; w) \Gamma_{j_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s-1}} + \delta_{A'_\zeta} \sigma_{j_\zeta} n_\zeta(s - \frac{1}{2}; w) \Gamma_{n_\zeta}^{\overbrace{B'_\zeta C'_\zeta \dots}^{2s}}(s; w)] \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) = \Gamma_{j_\zeta}^{\overbrace{B_\zeta C_\zeta \dots}^{2s-1}} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) S_{ab k_\zeta} l_\zeta(s; w) \\
&\Leftrightarrow S_{ab A_\zeta} A'_\zeta(\frac{1}{2}; w) N_{A'_\zeta j_\zeta}^{l_\zeta}(s; w) + S_{ab j_\zeta} n_\zeta(s - \frac{1}{2}; w) N_{A_\zeta n_\zeta}^{l_\zeta}(s; w) = N_{A_\zeta j_\zeta}^{k_\zeta}(s; w) S_{ab k_\zeta} l_\zeta(s; w) \\
&\Leftrightarrow S_{ab A_\zeta} B_\zeta(\frac{1}{2}; w) N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + S_{abl_\zeta} m_\zeta(s - \frac{1}{2}; w) N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w) S_{ab j_\zeta} k_\zeta(s; w) \quad \square
\end{aligned}$$

#### 4.8 常数不变张量 $N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; w)$ 的引入及其性质

$$\text{定义4.8.1. } \begin{cases} N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w) := \Gamma_{A_{\zeta 1} \dots A_{\zeta 2s}}^{k_\zeta}(s; w) \Gamma_{l_\zeta}^{A_{\zeta n+1} \dots A_{\zeta 2s}}(s - \frac{n}{2}; w) \\ N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; w) := \Gamma_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta 2s}}(s; w) \Gamma_{A_{\zeta n+1} \dots A_{\zeta 2s}}^{l_\zeta}(s - \frac{n}{2}; w) \end{cases}$$

相等性:

$$\text{性质4.8.1. } N_{A'_{\zeta 1} \dots A'_{\zeta n} l'_\zeta}^{k'_\zeta}(s; w) \simeq N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w) \simeq N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; w) \simeq N_{k'_\zeta}^{A'_{\zeta 1} \dots A'_{\zeta n} l'_\zeta}(s; w)$$

$$\text{性质4.8.2. } [N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w)]^* \simeq N_{A'_{\zeta 1} \dots A'_{\zeta n} l'_\zeta}^{k'_\zeta}(s; w), [N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; w)]^* \simeq N_{k'_\zeta}^{A'_{\zeta 1} \dots A'_{\zeta n} l'_\zeta}(s; w)$$

展开性:

性质4.8.3.

$$\begin{cases} N_{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w) = N_{A_{\zeta 1} l_{\zeta 1}}^{k_\zeta}(s; w) N_{A_{\zeta 2} l_{\zeta 2}}^{l_{\zeta 1}}(s - \frac{1}{2}; w) \dots N_{A_{\zeta n} l_\zeta}^{l_{\zeta n-1}}(s - \frac{n-1}{2}; w) \\ N_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta n} l_\zeta}(s; w) = N_{k_\zeta}^{A_{\zeta 1} l_{\zeta 1}}(s; w) N_{l_{\zeta 1}}^{A_{\zeta 2} l_{\zeta 2}}(s - \frac{1}{2}; w) \dots N_{l_{\zeta n-1}}^{A_{\zeta n} l_\zeta}(s - \frac{n-1}{2}; w) \end{cases}$$

性质4.8.4.

$$\begin{cases} \Gamma_{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta 2s}}^{k_\zeta}(s; w) = N_{A_{\zeta 1} l_{\zeta 1}}^{k_\zeta}(s; w) N_{A_{\zeta 2} l_{\zeta 2}}^{l_{\zeta 1}}(s - \frac{1}{2}; w) \dots N_{A_{\zeta 2s} l_{\zeta 2s}}^{l_{\zeta 2s-1}}(\frac{1}{2}; w) \\ \Gamma_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta 2s}}(s; w) = N_{k_\zeta}^{A_{\zeta 1} l_{\zeta 1}}(s; w) N_{l_{\zeta 1}}^{A_{\zeta 2} l_{\zeta 2}}(s - \frac{1}{2}; w) \dots N_{l_{\zeta 2s-1}}^{A_{\zeta 2s} l_{\zeta 2s}}(\frac{1}{2}; w) \end{cases}$$

$$\begin{cases} \Gamma_{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_{2s}}}(s; w) \succ \Gamma_{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_{2s}}}(s; w) = N_{A_{\zeta_1}}(s; w) N_{A_{\zeta_2}}(s - \frac{1}{2}; w) \cdots N_{A_{\zeta_{2s}}}(\frac{1}{2}; w) \\ \Gamma_{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_{2s}}}(s; w) \succ \Gamma_{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_{2s}}}(s; w) = N^{A_{\zeta_1}}(s; w) N^{A_{\zeta_2}}(s - \frac{1}{2}; w) \cdots N^{A_{\zeta_{2s}}}(\frac{1}{2}; w) \\ \bar{\Gamma}(s; w) = \bar{N}(s; w) [I_{w+1} \otimes \bar{N}(s - \frac{1}{2}; w)] \cdot [I_{(w+1)^{2s-2}} \otimes \bar{N}(1)] [I_{(w+1)^{2s-1}} \otimes \bar{N}(\frac{1}{2}; w)] \\ \Gamma(s; w) = [I_{(w+1)^{2s-1}} \otimes N(\frac{1}{2}; w)] [I_{(w+1)^{2s-2}} \otimes N(1)] \cdot [I_{w+1} \otimes N(s - \frac{1}{2}; w)] N(s; w) \end{cases}$$

#### 4.9 推论1：常数矩阵 $N(s; w)$ , $\bar{N}(s; w)$ 的几个恒等式

性质4.9.1.

$$\begin{cases} \bar{N}(s; w) [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] N(s; w) = \sigma(s; w) \\ N(s; w) \sigma(s; w) \bar{N}(s; w) = [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] N(s; w) \bar{N}(s; w) \\ N(s; w) \sigma(s; w) \bar{N}(s; w) = N(s; w) \bar{N}(s; w) \{ \vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] \}^n \\ [N(s; w) \bar{N}(s; w), \sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] = 0 \end{cases}$$

性质4.9.2.

$$\begin{cases} \bar{N}(s; w) [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] N(s; w) = S_{ab}(s, \varsigma; w) \\ N(s; w) S_{ab}(s, \varsigma; w) \bar{N}(s; w) = [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] N(s; w) \bar{N}(s; w) \\ N(s; w) S_{ab}(s, \varsigma; w) \bar{N}(s; w) = N(s; w) \bar{N}(s; w) [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \\ [N(s; w) \bar{N}(s; w), S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] = 0 \end{cases}$$

性质4.9.3.

$$\begin{cases} \bar{N}(s; w) \{ \vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] \}^n N(s; w) = [\vartheta \cdot \sigma(s; w)]^n \\ N(s; w) [\vartheta \cdot \sigma(s; w)]^n \bar{N}(s; w) = \{ \vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] \}^n N(s; w) \bar{N}(s; w) \\ N(s; w) [\vartheta \cdot \sigma(s; w)]^n \bar{N}(s; w) = N(s; w) \bar{N}(s; w) \{ \vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] \}^n \\ [N(s; w) \bar{N}(s; w), \{ \vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] \}^n] = 0 \end{cases}$$

性质4.9.4.

$$\begin{cases} \bar{N}(s; w) \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \}^n N(s; w) = [\vartheta^{ab} S_{ab}(s, \varsigma; w)]^n \\ N(s; w) [\vartheta^{ab} S_{ab}(s, \varsigma; w)]^n \bar{N}(s; w) = \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \}^n N(s; w) \bar{N}(s; w) \\ N(s; w) [\vartheta^{ab} S_{ab}(s, \varsigma; w)]^n \bar{N}(s; w) = N(s; w) \bar{N}(s; w) \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \}^n \\ [N(s; w) \bar{N}(s; w), \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \}^n] = 0 \end{cases}$$

推论4.9.1.

$$\begin{cases} \bar{N}(s; w) e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]} N(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \\ N(s; w) e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \bar{N}(s; w) = e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]} N(s; w) \bar{N}(s; w) \\ N(s; w) e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \bar{N}(s; w) = N(s; w) \bar{N}(s; w) e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]} \\ [N(s; w) \bar{N}(s; w), e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]}] = 0 \end{cases}$$

#### 4.10 推论2：常数矩阵 $N(s; w)$ , $\bar{N}(s; w)$ 的另外几个恒等式

推论4.10.1.

$$\begin{cases} \bar{N}(s; w) \sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} N(s; w) = \frac{1}{2s} \sigma(s; w) \\ \bar{N}(s; w) I_{w+1} \otimes \sigma(s - \frac{1}{2}; w) N(s; w) = (1 - \frac{1}{2s}) \sigma(s; w) \\ N^{A_{\zeta}}(s; w) \sigma(s - \frac{1}{2}; w) \bar{N}_{A_{\zeta}}(s; w) = (1 - \frac{1}{2s}) \sigma(s; w) \\ \bar{N}_{A_{\zeta}}(s; w) \sigma(s; w) N^{A_{\zeta}}(s; w) = (1 + \frac{w+1}{2s}) \sigma(s - \frac{1}{2}; w) \end{cases}$$

## 推论4.10.2.

$$\begin{cases} \bar{N}(s; w) S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} N(s; w) = \frac{1}{2s} S_{ab}(s, \varsigma; w) \\ \bar{N}(s; w) I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w) N(s; w) = (1 - \frac{1}{2s}) S_{ab}(s, \varsigma; w) \\ N^{A_c}(s; w) S_{ab}(s - \frac{1}{2}, \varsigma; w) \bar{N}_{A_c}(s; w) = (1 - \frac{1}{2s}) S_{ab}(s, \varsigma; w) \\ \bar{N}_{A_c}(s; w) S_{ab}(s, \varsigma; w) N^{A_c}(s; w) = (1 + \frac{w+1}{2s}) S_{ab}(s - \frac{1}{2}, \varsigma; w) \end{cases}$$

## 推论4.10.3.

$$\begin{cases} \bar{N}(1) [\sigma(\frac{1}{2}; w) \otimes I_2 + I_{w+1} \otimes \sigma(\frac{1}{2}; w)] N(1) = \sigma(1) \\ \bar{N}(\frac{3}{2}) \{ \sigma(\frac{1}{2}; w) \otimes I_3 + I_{w+1} \otimes \{ \bar{N}(1) [\sigma(\frac{1}{2}; w) \otimes I_2 + I_{w+1} \otimes \sigma(\frac{1}{2}; w)] N(1) \} \} N(\frac{3}{2}) = \sigma(\frac{3}{2}) \\ \bar{N}(s; w) \cdot \bar{N}(\frac{3}{2}) \{ \sigma(\frac{1}{2}; w) \otimes I_3 + I_{w+1} \otimes \{ \bar{N}(1) [\sigma(\frac{1}{2}; w) \otimes I_2 + I_{w+1} \otimes \sigma(\frac{1}{2}; w)] N(1) \} \} N(\frac{3}{2}) \cdot N(s; w) = \sigma(s; w) \end{cases}$$

4.11 矩阵  $N(s; w)$ ,  $\bar{N}(s; w)$  的常数不变张量性质

$$\text{定理4.11.1. } N(s; w) = e^{\frac{i}{2}\vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{\frac{i}{2}\vartheta^{ab} S_{ab}(s - \frac{1}{2}, \varsigma; w)} N(s; w) e^{-\frac{i}{2}\vartheta^{ab} S_{ab}(s, \varsigma; w)}$$

$$\begin{aligned} \text{证明: } & [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] N(s; w) = N(s; w) S_{ab}(s, \varsigma; w) \\ \Leftrightarrow 0 &= [\frac{i}{2}\vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + \frac{i}{2}\vartheta^{ab} I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] N(s; w) - \frac{i}{2}\vartheta^{ab} N(s; w) S_{ab}(s, \varsigma; w) \\ \Leftrightarrow N(s; w) &= e^{\frac{i}{2}\vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{\frac{i}{2}\vartheta^{ab} S_{ab}(s - \frac{1}{2}, \varsigma; w)} N(s; w) e^{-\frac{i}{2}\vartheta^{ab} S_{ab}(s, \varsigma; w)} \quad \square \end{aligned}$$

$$\text{定理4.11.2. } \bar{N}(s; w) = e^{\frac{i}{2}\vartheta^{ab} S_{ab}(s, \varsigma; w)} \bar{N}(s; w) e^{-\frac{i}{2}\vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2}\vartheta^{ab} S_{ab}(s - \frac{1}{2}, \varsigma; w)}$$

$$\begin{aligned} \text{证明: } & \bar{N}(s; w) [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] = S_{ab}(s, \varsigma; w) \bar{N}(s; w) \\ \Leftrightarrow 0 &= \frac{i}{2}\vartheta^{ab} S_{ab}(s, \varsigma; w) \bar{N}(s; w) - \bar{N}(s; w) [\frac{i}{2}\vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{2s-1+w}^{2s-1}} + \frac{i}{2}\vartheta^{ab} I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \\ \Leftrightarrow \bar{N}(s; w) &= e^{\frac{i}{2}\vartheta^{ab} S_{ab}(s, \varsigma; w)} \bar{N}(s; w) e^{-\frac{i}{2}\vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2}\vartheta^{ab} S_{ab}(s - \frac{1}{2}, \varsigma; w)} \quad \square \end{aligned}$$

4.12 矩阵  $N(s)$ ,  $\bar{N}(s)$  的常数不变张量性质

$$\text{定理4.12.1. } N(s) = e^{(i\omega + \varsigma\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2})} N(s) e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(s)}$$

$$\begin{aligned} \text{证明: } & [\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})] N(s) = N(s) \sigma(s) \\ \Leftrightarrow 0 &= [(i\omega + \varsigma\epsilon) \cdot \sigma(\frac{1}{2}) \otimes I_{2s} + (i\omega + \varsigma\epsilon) \cdot I \otimes \sigma(s - \frac{1}{2})] N(s) - (i\omega + \varsigma\epsilon) \cdot N(s) \sigma(s) \\ \Leftrightarrow N(s) &= e^{(i\omega + \varsigma\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2})} N(s) e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(s)} \quad \square \end{aligned}$$

$$\text{定理4.12.2. } \bar{N}(s) = e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s)} \bar{N}(s) e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2})}$$

$$\begin{aligned} \text{证明: } & \bar{N}(s) [\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})] = \sigma(s) \bar{N}(s) \\ \Leftrightarrow 0 &= (i\omega + \varsigma\epsilon) \cdot \sigma(s) \bar{N}(s) - \bar{N}(s) [(i\omega + \varsigma\epsilon) \cdot \sigma(\frac{1}{2}) \otimes I_{2s} + (i\omega + \varsigma\epsilon) \cdot I \otimes \sigma(s - \frac{1}{2})] \\ \Leftrightarrow \bar{N}(s) &= e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s)} \bar{N}(s) e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2})} \quad \square \end{aligned}$$

4.13 矩阵  $(\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\varsigma)_a$  的常数不变张量性质

## 定理4.13.1.

$$\begin{cases} (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\varsigma)_a \\ = [e^\vartheta]_a^b [e^{\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, -\varsigma; w)} \otimes e^{\frac{i}{2}\vartheta^{cd} S_{cd}(s - \frac{1}{2}, \varsigma; w)}] (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\varsigma)_b [e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(s - \frac{1}{2}, \varsigma; w)}] \\ (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\varsigma)_a \\ = [e^\vartheta]_a^b [e^{\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, -\varsigma; w)} \otimes e^{\frac{i}{2}\vartheta^{cd} S_{cd}(s - \frac{1}{2}, -\varsigma; w)}] (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\varsigma)_b [e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(s - \frac{1}{2}, -\varsigma; w)}] \end{cases}$$

$$\begin{aligned} \text{证明: } & (\sigma, -i\varsigma)_a = [e^\vartheta]_a^b e^{\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, -\varsigma; w)} (\sigma, -i\varsigma)_b e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, \varsigma; w)} \\ \Leftrightarrow (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\varsigma)_a &= [e^\vartheta]_a^b [e^{\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, -\varsigma; w)} \otimes I_{C_{2s-1+w}^{2s-1}}] (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\varsigma)_b [e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, \varsigma; w)} \otimes I_{C_{2s-1+w}^{2s-1}}] \\ \Leftrightarrow (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\varsigma)_a &= [e^\vartheta]_a^b [e^{\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, -\varsigma; w)} \otimes e^{\frac{i}{2}\vartheta^{cd} S_{cd}(s - \frac{1}{2}, \varsigma; w)}] (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\varsigma)_b [e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(s - \frac{1}{2}, \varsigma; w)}] \quad \square \end{aligned}$$

$$\begin{aligned}
& \text{证明: } (\sigma, -i\zeta)_a = [e^\vartheta]_a b e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2}, -\zeta; w)} (\sigma, -i\zeta)_b e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2}, \zeta; w)} \\
& \Leftrightarrow (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)_a = [e^\vartheta]_a b [e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2}, -\zeta; w)} \otimes I_{C_{2s-1+w}^{2s-1}}] (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)_b [e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2}, \zeta; w)} \otimes I_{C_{2s-1+w}^{2s-1}}] \\
& \Leftrightarrow (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)_a \\
& = [e^\vartheta]_a b [e^{\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2}, -\zeta; w)} \otimes e^{\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2}, -\zeta; w)}] (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)_b [e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(\frac{1}{2}, \zeta; w)} \otimes e^{-\frac{i}{2}\vartheta^{cd}S_{cd}(s-\frac{1}{2}, \zeta; w)}] \quad \square
\end{aligned}$$

#### 4.14 矩阵 $(\sigma \otimes I_{2s}, -i\zeta)_a$ 的常数不变张量性质

定理4.14.1.

$$\begin{cases}
(\sigma \otimes I_{2s}, -i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a b e^{(i\omega - \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \zeta \epsilon) \cdot \sigma(s-\frac{1}{2})} (\sigma \otimes I_{2s}, -i\zeta)_b e^{-(i\omega + \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega + \zeta \epsilon) \cdot \sigma(s-\frac{1}{2})} \\
(\sigma \otimes I_{2s}, -i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a b e^{(i\omega - \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega - \zeta \epsilon) \cdot \sigma(s-\frac{1}{2})} (\sigma \otimes I_{2s}, -i\zeta)_b e^{-(i\omega + \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega - \zeta \epsilon) \cdot \sigma(s-\frac{1}{2})}
\end{cases}$$

$$\begin{aligned}
& \text{证明: } (\sigma, -i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a b e^{(i\omega - \zeta \epsilon) \cdot \sigma(\frac{1}{2})} (\sigma, -i\zeta)_b e^{-(i\omega + \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \\
& \Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a b [e^{(i\omega - \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes I_{2s}] (\sigma \otimes I_{2s}, -i\zeta)_b [e^{-(i\omega + \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes I_{2s}] \\
& \Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a b e^{(i\omega - \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \zeta \epsilon) \cdot \sigma(s-\frac{1}{2})} (\sigma \otimes I_{2s}, -i\zeta)_b e^{-(i\omega + \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega + \zeta \epsilon) \cdot \sigma(s-\frac{1}{2})} \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } (\sigma, -i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a b e^{(i\omega - \zeta \epsilon) \cdot \sigma(\frac{1}{2})} (\sigma, -i\zeta)_b e^{-(i\omega + \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \\
& \Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a b [e^{(i\omega - \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes I_{2s}] (\sigma \otimes I_{2s}, -i\zeta)_b [e^{-(i\omega + \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes I_{2s}] \\
& \Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)_a = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a b e^{(i\omega - \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega - \zeta \epsilon) \cdot \sigma(s-\frac{1}{2})} (\sigma \otimes I_{2s}, -i\zeta)_b e^{-(i\omega + \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega - \zeta \epsilon) \cdot \sigma(s-\frac{1}{2})} \quad \square
\end{aligned}$$

## 5 完美常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)$ , $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ (存在 $\varepsilon_{A_\zeta B_\zeta}$ 为前提)

只有满足 $\varepsilon_{A_\zeta B_\zeta} = -\varepsilon_{B_\zeta A_\zeta}$ 反对称条件时, 本章节内容才全部成立, 否则只有部分成立.

### 5.1 完美常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)$ , $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的引入

定义5.1.1.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) := \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{A_\zeta B_\zeta} N_{B_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)$ ,  $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) := \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{A_\zeta B_\zeta} N_{l_\zeta}^{B_\zeta m_\zeta}(s - \frac{1}{2}; w)$

性质5.1.1.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) \simeq X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$

### 5.2 常数矩阵 $X(s; w)$ , $\bar{X}(s; w)$ 的引入

$$\text{定义5.2.1. } \begin{cases} X^{A_\zeta}(s; w) \prec X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta}(s; w) \prec X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) \\ \bar{X}_{A_\zeta}(s; w) \prec X_{A_\zeta l_\zeta}^{m_\zeta}(s; w), \bar{X}^{A_\zeta}(s; w) \prec X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) \\ X(s; w) \prec X_{A_\zeta \otimes l_\zeta}^{m_\zeta}(s; w), \bar{X}(s; w) \prec X_{m_\zeta}^{A_\zeta \otimes l_\zeta}(s; w) = X^+(s; w) \end{cases}$$

$X(s)$ ,  $\bar{X}(s)$ 的显式表示:

$$\text{推论5.2.1. } \bar{X}(s) = \frac{1}{\sqrt{2s}} \begin{bmatrix} 0 & -\sqrt{2s-1} & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2s-2} & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & \sqrt{2s-1} & 0 \end{bmatrix}$$

$$\text{推论5.2.2. } \bar{X}(s=1, \frac{3}{2}, 2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -\sqrt{1} & \sqrt{1} & 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & -\sqrt{2} & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1} & \sqrt{2} & 0 \end{bmatrix}, \frac{1}{\sqrt{4}} \begin{bmatrix} 0 & -\sqrt{3} & \sqrt{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & \sqrt{3} & 0 \end{bmatrix}$$

### 5.3 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)$ , $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的升降指标

性质5.3.1.

$$\begin{cases} X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) = \varepsilon^{A_\zeta B_\zeta} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; w) \varepsilon_{m_\zeta r_\zeta}(s - 1; w) X_{B_\zeta n_\zeta}^{r_\zeta}(s - \frac{1}{2}; w) \\ X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) = \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; w) \varepsilon^{m_\zeta r_\zeta}(s - 1; w) X_{r_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2}; w) \end{cases}$$

$$\begin{aligned}
& \text{证明: } N_{A_\zeta l_\zeta}^{k_\zeta}(s - \frac{1}{2}; w) = \varepsilon^{k_\zeta m_\zeta}(s - \frac{1}{2}; w) \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - 1; w) N_{m_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2}; w) \\
& \Leftrightarrow \varepsilon^{C_\zeta A_\zeta} N_{A_\zeta l_\zeta}^{k_\zeta}(s - \frac{1}{2}; w) = \varepsilon^{C_\zeta A_\zeta} \varepsilon^{k_\zeta m_\zeta}(s - \frac{1}{2}; w) \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - 1; w) N_{m_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2}; w) \\
& \Leftrightarrow X_{l_\zeta}^{C_\zeta k_\zeta}(s; w) = \varepsilon^{C_\zeta A_\zeta} \varepsilon^{k_\zeta m_\zeta}(s - \frac{1}{2}; w) \varepsilon_{l_\zeta n_\zeta}(s - 1; w) X_{A_\zeta m_\zeta}^{n_\zeta}(s - \frac{1}{2}; w) \\
& \Leftrightarrow X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) = \varepsilon^{A_\zeta B_\zeta} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; w) \varepsilon_{m_\zeta r_\zeta}(s - 1; w) X_{B_\zeta n_\zeta}^{r_\zeta}(s - \frac{1}{2}; w) \quad \square
\end{aligned}$$

### 5.4 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)$ , $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的正交性

性质5.4.1.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) = \delta_{m_\zeta}^{n_\zeta} [\Leftrightarrow] X^{A_\zeta}(s; w)\bar{X}_{A_\zeta}(s; w) = I_{C_{2s-2+w}^{2s-2}} [\Leftrightarrow] \bar{X}(s; w)X(s; w) = I_{C_{2s-2+w}^{2s-2}}$

$$\begin{aligned} \text{证明: } & X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) \\ &= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{A_\zeta D_\zeta} N_{l_\zeta}^{D_\zeta n_\zeta}(s - \frac{1}{2}; w) \\ &= \frac{2s-1}{2s-1+w} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) \delta_{D_\zeta}^{C_\zeta} N_{l_\zeta}^{D_\zeta n_\zeta}(s - \frac{1}{2}; w) \\ &= \frac{2s-1}{2s-1+w} (1 + \frac{w}{2s-1}) \delta_{m_\zeta}^{n_\zeta} \\ &= \delta_{m_\zeta}^{n_\zeta} \end{aligned}$$

□

性质5.4.2.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) = 0$

$$[\Leftrightarrow] X^{A_\zeta}(s; w)\bar{N}_{A_\zeta}(s; w) = 0, N_{A_\zeta}(s; w)\bar{X}^{A_\zeta}(s; w) = 0 [\Leftrightarrow] \bar{X}(s; w)N(s; w) = 0, \bar{N}(s; w)X(s; w) = 0$$

$$\begin{aligned} \text{证明: } & X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) \\ &= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) \\ &= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{A_\zeta C_\zeta} \Gamma_{C_\zeta C_\zeta'' D_\zeta''}^{l_\zeta} \dots (s - \frac{1}{2}; w) \Gamma_{m_\zeta}^{C_\zeta'' D_\zeta'' \dots} (s - 1; w) \Gamma_{A_\zeta B_\zeta' C_\zeta' D_\zeta'}^{k_\zeta} \dots (s; w) \Gamma_{l_\zeta}^{B_\zeta' C_\zeta' D_\zeta' \dots} (s - \frac{1}{2}; w) \\ &= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{A_\zeta C_\zeta} \frac{1}{(2s-1)!} \delta_{(C_\zeta' C_\zeta'' D_\zeta'')}^{B_\zeta'} \delta_{(C_\zeta'' D_\zeta'')}^{C_\zeta'} \delta_{D_\zeta'}^{D_\zeta''} \dots \Gamma_{m_\zeta}^{C_\zeta'' D_\zeta'' \dots} (s - 1; w) \Gamma_{A_\zeta B_\zeta' C_\zeta' D_\zeta'}^{k_\zeta} \dots (s; w) \\ &= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{A_\zeta B_\zeta} \Gamma_{A_\zeta B_\zeta C_\zeta'' D_\zeta''}^{k_\zeta} \dots (s; w) \Gamma_{m_\zeta}^{C_\zeta'' D_\zeta'' \dots} (s - 1; w) \\ &= 0 \end{aligned}$$

□

性质5.4.3.  $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)X_{m_\zeta}^{A_\zeta k_\zeta}(s; w) = \frac{2s-1}{2s-1+w} \delta_{l_\zeta}^{k_\zeta} [\Leftrightarrow] \bar{X}_{A_\zeta}(s; w)X^{A_\zeta}(s; w) = \frac{2s-1}{2s-1+w} I_{C_{2s-1+w}^{2s-1}}$

$$\begin{aligned} \text{证明: } & X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)X_{m_\zeta}^{A_\zeta k_\zeta}(s; w) \\ &= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{A_\zeta B_\zeta} N_{l_\zeta}^{B_\zeta m_\zeta}(s - \frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{k_\zeta}(s - \frac{1}{2}; w) \\ &= \frac{2s-1}{2s-1+w} N_{l_\zeta}^{B_\zeta m_\zeta}(s - \frac{1}{2}; w) N_{B_\zeta m_\zeta}^{k_\zeta}(s - \frac{1}{2}; w) \\ &= \frac{2s-1}{2s-1+w} \delta_{l_\zeta}^{k_\zeta} \end{aligned}$$

□

性质5.4.4.  $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)X_{m_\zeta}^{B_\zeta l_\zeta}(s; w) = \frac{1}{w+1} C_{2s-2+w}^{2s-2} \delta_{A_\zeta}^{B_\zeta} [\Leftrightarrow] \text{tr}[\bar{X}_{A_\zeta}(s; w)X^{B_\zeta}(s; w)] = \frac{1}{w+1} C_{2s-2+w}^{2s-2} \delta_{A_\zeta}^{B_\zeta}$

$$\begin{aligned} \text{证明: } & X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)X_{m_\zeta}^{B_\zeta l_\zeta}(s; w) \\ &= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{A_\zeta C_\zeta} N_{l_\zeta}^{C_\zeta m_\zeta}(s - \frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{B_\zeta D_\zeta} N_{D_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) \\ &= \frac{2s-1}{2s-1+w} \varepsilon_{A_\zeta C_\zeta} \varepsilon^{B_\zeta D_\zeta} N_{l_\zeta}^{C_\zeta m_\zeta}(s - \frac{1}{2}; w) N_{D_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) \\ &= \frac{2s-1}{2s-1+w} \varepsilon_{A_\zeta C_\zeta} \varepsilon^{B_\zeta D_\zeta} \frac{1}{w+1} C_{2s-1+w}^{2s-1} \delta_{D_\zeta}^{C_\zeta} \\ &= \frac{1}{w+1} C_{2s-2+w}^{2s-2} \delta_{A_\zeta}^{B_\zeta} \end{aligned}$$

□

推论5.4.1.  $\bar{N}(s; w)N(s; w) = I_{C_{2s+w}^{2s}}$ ,  $\bar{X}(s; w)X(s; w) = I_{C_{2s-2+w}^{2s-2}}$ ,  $\bar{N}(s; w)X(s; w) = 0$ ,  $\bar{X}(s; w)N(s; w) = 0$

### 5.5 矩阵 $N(s)$ , $\bar{N}(s)$ , $X(s)$ , $\bar{X}(s)$ 的联合正交性质

$$\text{性质5.5.1. } \begin{cases} X_{A_\zeta l_\zeta}^{n_\zeta}(s)X_{n_\zeta}^{B_\zeta m_\zeta}(s) = \frac{1}{2s} [(2s-1)\delta_{A_\zeta}^{B_\zeta} \delta_{l_\zeta}^{m_\zeta} - (2s-1)N_{A_\zeta n_\zeta}^{m_\zeta}(s - \frac{1}{2})N_{l_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2})] \\ \bar{X}_{A_\zeta}(s)X^{B_\zeta}(s) = \frac{1}{2s} [(2s-1)\delta_{A_\zeta}^{B_\zeta} I_{2s} - (2s-1)N^{B_\zeta}(s - \frac{1}{2})\bar{N}_{A_\zeta}(s - \frac{1}{2}; w)] \end{cases}$$

$$\begin{aligned} \text{证明: } & X_{A_\zeta l_\zeta}^{n_\zeta}(s)X_{n_\zeta}^{A_\zeta m_\zeta}(s) = \frac{2s-1}{2s} \varepsilon_{A_\zeta E_\zeta} \varepsilon^{A_\zeta' E_\zeta'} N_{l_\zeta}^{E_\zeta n_\zeta}(s - \frac{1}{2}) N_{E_\zeta' n_\zeta}^{m_\zeta}(s - \frac{1}{2}) \\ &= \frac{2s-1}{2s} \varepsilon_{A_\zeta E_\zeta} \varepsilon^{A_\zeta' E_\zeta'} \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots} (s - \frac{1}{2}) \Gamma_{n_\zeta}^{F_\zeta G_\zeta \dots} (s - 1) \Gamma_{E_\zeta' F_\zeta' G_\zeta'}^{m_\zeta} \dots (s - \frac{1}{2}) \Gamma_{n_\zeta}^{F_\zeta' G_\zeta' \dots} (s - 1) \end{aligned}$$

$$\begin{aligned}
&= \frac{2s-1}{2s} \varepsilon_{A_\zeta E_\zeta} \varepsilon_{A'_\zeta E'_\zeta} \frac{1}{(2s-2)!} \overbrace{\delta_{F'_\zeta}^{(F'_\zeta)} \delta_{G'_\zeta}^{(G'_\zeta)} \dots}^{2s-2} \Gamma_{l'_\zeta}^{E_\zeta F_\zeta G_\zeta \dots} \overbrace{\Gamma_{l'_\zeta}^{E'_\zeta F'_\zeta G'_\zeta \dots}}^{2s-1} (s - \frac{1}{2}) \Gamma_{E'_\zeta F'_\zeta G'_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}) \\
&= \frac{2s-1}{2s} \varepsilon_{A_\zeta E_\zeta} \varepsilon_{A'_\zeta E'_\zeta} \Gamma_{l'_\zeta}^{E_\zeta B_\zeta C_\zeta \dots} (s - \frac{1}{2}) \Gamma_{E'_\zeta B_\zeta C_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}) \\
&= \frac{2s-1}{2s} \delta_{A'_\zeta}^{[A'_\zeta} \delta_{E'_\zeta}^{E'_\zeta]} \Gamma_{l'_\zeta}^{E_\zeta B_\zeta C_\zeta \dots} (s - \frac{1}{2}) \Gamma_{E'_\zeta B_\zeta C_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}) \\
&= \frac{2s-1}{2s} \delta_{A'_\zeta}^{A'_\zeta} \Gamma_{l'_\zeta}^{E_\zeta B_\zeta C_\zeta \dots} (s - \frac{1}{2}) \Gamma_{E'_\zeta B_\zeta C_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}) - \frac{2s-1}{2s} \Gamma_{l'_\zeta}^{A'_\zeta B_\zeta C_\zeta \dots} (s - \frac{1}{2}) \Gamma_{A'_\zeta B_\zeta C_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}) \\
&= \frac{1}{2s} [(2s-1) \delta_{A'_\zeta}^{A'_\zeta} \delta_{l'_\zeta}^{m_\zeta} - (2s-1) N_{A'_\zeta n_\zeta}^{m_\zeta} (s - \frac{1}{2}) N_{l'_\zeta}^{A'_\zeta n_\zeta} (s - \frac{1}{2})]
\end{aligned}$$

□

推论5.5.1.  $N_{A'_\zeta l'_\zeta}^{k_\zeta} (s) N_{k_\zeta}^{B_\zeta m_\zeta} (s) + X_{A'_\zeta l'_\zeta}^{n_\zeta} (s) X_{n_\zeta}^{B_\zeta m_\zeta} (s) = \delta_{A'_\zeta}^{B_\zeta} \delta_{l'_\zeta}^{m_\zeta}$   
 $[\Leftrightarrow] \bar{N}_{A'_\zeta} (s) N^{B_\zeta} (s) + \bar{X}_{A'_\zeta} (s) X^{B_\zeta} (s) = \delta_{A'_\zeta}^{B_\zeta} I_{2s} [\Leftrightarrow] N(s) \bar{N}(s) + X(s) \bar{X}(s) = I_{4s}$

推论5.5.2.  $\begin{bmatrix} \bar{N}(s) \\ \bar{X}(s) \end{bmatrix} [N(s), X(s)] = [N(s), X(s)] \begin{bmatrix} \bar{N}(s) \\ \bar{X}(s) \end{bmatrix} = I_{4s} [\Leftrightarrow] \begin{cases} N(s) \bar{N}(s) + X(s) \bar{X}(s) = I_{4s} \\ \bar{N}(s) N(s) = I_{2s+1}, \bar{X}(s) X(s) = I_{2s-1} \\ \bar{N}(s) X(s) = 0, \bar{X}(s) N(s) = 0 \end{cases}$

## 5.6 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta} (s; w)$ , $X_{A_\zeta l_\zeta}^{m_\zeta} (s; w)$ 的自旋变换

推论5.6.1.  $X_{m_\zeta}^{A_\zeta l_\zeta} (s; w) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta} (\frac{1}{2}; w) X_{B_\zeta l_\zeta}^{n_\zeta} (s; w) = -\frac{1}{2s-1+w} \sigma^{\alpha_\zeta} m_\zeta^{n_\zeta} (s-1; w)$   
 $[\Leftrightarrow] X^{A_\zeta} (s; w) \sigma_{A_\zeta}^{B_\zeta} (\frac{1}{2}; w) \bar{X}_{B_\zeta} (s; w) = -\frac{1}{2s-1+w} \sigma (s-1; w)$   
 $[\Leftrightarrow] \bar{X} (s; w) \sigma (\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} X (s; w) = -\frac{1}{2s-1+w} \sigma (s-1; w)$

证明:  $X_{m_\zeta}^{A_\zeta l_\zeta} (s; w) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta} (\frac{1}{2}; w) X_{B_\zeta l_\zeta}^{n_\zeta} (s; w)$   
 $= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta} (s - \frac{1}{2}; w) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta} (\frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{B_\zeta D_\zeta} N_{l_\zeta}^{D_\zeta n_\zeta} (s - \frac{1}{2}; w)$   
 $= -\frac{2s-1}{2s-1+w} N_{C_\zeta m_\zeta}^{l_\zeta} (s - \frac{1}{2}; w) \sigma^{\alpha_\zeta} D_\zeta^{C_\zeta} (\frac{1}{2}; w) N_{l_\zeta}^{D_\zeta n_\zeta} (s - \frac{1}{2}; w)$   
 $= -\frac{1}{2s-1+w} \sigma^{\alpha_\zeta} m_\zeta^{n_\zeta} (s-1; w)$

□

推论5.6.2.  $X_{m_\zeta}^{A_\zeta l_\zeta} (s; w) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta} (\frac{1}{2}; w) X_{B_\zeta k_\zeta}^{m_\zeta} (s; w) = -\frac{1}{2s-1+w} \sigma^{\alpha_\zeta} k_\zeta^{l_\zeta} (s - \frac{1}{2}; w)$   
 $[\Leftrightarrow] \bar{X}^{A_\zeta} (s; w) \sigma_{A_\zeta}^{B_\zeta} (\frac{1}{2}; w) X_{B_\zeta} (s; w) = -\frac{1}{2s-1+w} \sigma^{\alpha_\zeta} (s - \frac{1}{2}; w)$

证明:  $X_{m_\zeta}^{A_\zeta l_\zeta} (s; w) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta} (\frac{1}{2}; w) X_{B_\zeta k_\zeta}^{m_\zeta} (s; w)$   
 $= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta} (s - \frac{1}{2}; w) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta} (\frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{B_\zeta D_\zeta} N_{k_\zeta}^{D_\zeta m_\zeta} (s - \frac{1}{2}; w)$   
 $= -\frac{2s-1}{2s-1+w} N_{C_\zeta m_\zeta}^{l_\zeta} (s - \frac{1}{2}; w) \sigma^{\alpha_\zeta} D_\zeta^{C_\zeta} (\frac{1}{2}; w) N_{k_\zeta}^{D_\zeta m_\zeta} (s - \frac{1}{2}; w)$   
 $= -\frac{1}{2s-1+w} \sigma^{\alpha_\zeta} k_\zeta^{l_\zeta} (s - \frac{1}{2}; w)$

□

推论5.6.3.  $X_{m_\zeta}^{A_\zeta l_\zeta} (s; w) S_{ab A_\zeta}^{B_\zeta} (\frac{1}{2}; w) X_{B_\zeta l_\zeta}^{n_\zeta} (s; w) = -\frac{1}{2s-1+w} S_{ab m_\zeta}^{n_\zeta} (s-1; w)$   
 $[\Leftrightarrow] X^{A_\zeta} (s; w) S_{ab} (\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-2+w}^{2s-2}} \bar{X}_{A_\zeta} (s; w) = -\frac{1}{2s-1+w} S_{ab} (s-1, \zeta; w)$   
 $[\Leftrightarrow] \bar{X} (s; w) S_{ab} (\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} X (s; w) = -\frac{1}{2s-1+w} S_{ab} (s-1, \zeta; w)$

证明:  $X_{m_\zeta}^{A_\zeta l_\zeta} (s; w) S_{ab A_\zeta}^{B_\zeta} (\frac{1}{2}; w) X_{B_\zeta l_\zeta}^{n_\zeta} (s; w)$   
 $= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta} (s - \frac{1}{2}; w) S_{ab A_\zeta}^{B_\zeta} (\frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{B_\zeta D_\zeta} N_{l_\zeta}^{D_\zeta n_\zeta} (s - \frac{1}{2}; w)$   
 $= -\frac{2s-1}{2s-1+w} N_{C_\zeta m_\zeta}^{l_\zeta} (s - \frac{1}{2}; w) S_{ab D_\zeta}^{C_\zeta} (\frac{1}{2}; w) N_{l_\zeta}^{D_\zeta n_\zeta} (s - \frac{1}{2}; w)$   
 $= -\frac{1}{2s-1+w} S_{ab m_\zeta}^{n_\zeta} (s-1; w)$

□

推论5.6.4.  $X_{m_\zeta}^{A_\zeta l_\zeta} (s; w) S_{ab A_\zeta}^{B_\zeta} (\frac{1}{2}; w) X_{B_\zeta k_\zeta}^{m_\zeta} (s; w) = -\frac{1}{2s} S_{ab k_\zeta}^{l_\zeta} (s - \frac{1}{2}; w)$   
 $[\Leftrightarrow] \bar{X}^{A_\zeta} (s; w) S_{ab A_\zeta}^{B_\zeta} (\frac{1}{2}; w) X_{B_\zeta} (s; w) = -\frac{1}{2s-1+w} S_{ab} (s - \frac{1}{2}, \zeta; w)$



$$\begin{aligned}
& \text{证明: } X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) S_{ab A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) X_{B_\zeta k_\zeta}^{m_\zeta}(s; w) \\
&= \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w) S_{ab A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon_{B_\zeta D_\zeta} N_{k_\zeta}^{D_\zeta m_\zeta}(s - \tfrac{1}{2}; w) \\
&= -\frac{2s-1}{2s-1+w} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w) S_{ab D_\zeta}^{C_\zeta}(\tfrac{1}{2}; w) N_{k_\zeta}^{D_\zeta m_\zeta}(s - \tfrac{1}{2}; w) \\
&= -\frac{1}{2s-1+w} S_{ab k_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w)
\end{aligned}$$

□

## 5.7 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)$ , $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的置换性质

定理5.7.1.

$$\begin{cases}
X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) [\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{l_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - 1; w) X_{n_\zeta}^{B_\zeta k_\zeta}(s; w) \\
[\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{l_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w)] X_{B_\zeta k_\zeta}^{n_\zeta}(s; w) = X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) \sigma_{m_\zeta}^{n_\zeta}(s - 1; w) \\
X^{A_\zeta}(s; w) [\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta} \sigma(s - \tfrac{1}{2}; w)] = \sigma(s - 1; w) X^{B_\zeta}(s; w) \\
[\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta} \sigma(s - \tfrac{1}{2}; w)] \bar{X}_{B_\zeta}(s; w) = \bar{X}_{A_\zeta}(s; w) \sigma(s - 1; w) \\
\bar{X}(s; w) [\sigma(\tfrac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \tfrac{1}{2}; w)] = \sigma(s - 1; w) \bar{X}(s; w) \\
[\sigma(\tfrac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \tfrac{1}{2}; w)] X(s; w) = X(s; w) \sigma(s - 1; w)
\end{cases}$$

$$\begin{aligned}
& \text{证明: } X_{m_\zeta}^{A_\zeta k_\zeta}(s; w) [\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{k_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - 1; w) X_{n_\zeta}^{B_\zeta l_\zeta}(s; w) \\
&\Leftrightarrow \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w) [\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{k_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - 1; w) \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{B_\zeta D_\zeta} N_{D_\zeta n_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w) \\
&\Leftrightarrow \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w) [\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{k_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - 1; w) \varepsilon^{B_\zeta D_\zeta} N_{D_\zeta n_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w) \\
&\Leftrightarrow \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{k_\zeta}(s; w) [\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{k_\zeta}^{l_\zeta}(s; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - \tfrac{1}{2}; w) \varepsilon^{B_\zeta D_\zeta} N_{D_\zeta n_\zeta}^{l_\zeta}(s; w) \\
&\Leftrightarrow \varepsilon_{E_\zeta B_\zeta} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{k_\zeta}(s; w) [\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{k_\zeta}^{l_\zeta}(s; w)] = \varepsilon_{E_\zeta B_\zeta} \sigma_{m_\zeta}^{n_\zeta}(s - \tfrac{1}{2}; w) \varepsilon^{B_\zeta D_\zeta} N_{D_\zeta n_\zeta}^{l_\zeta}(s; w) \\
&\Leftrightarrow N_{C_\zeta m_\zeta}^{k_\zeta}(s; w) [\sigma(\tfrac{1}{2}; w) \varepsilon^{C_\zeta} \delta_{k_\zeta}^{l_\zeta} - \delta_{E_\zeta}^{C_\zeta} \sigma_{k_\zeta}^{l_\zeta}(s; w)] = -\sigma_{m_\zeta}^{n_\zeta}(s - \tfrac{1}{2}; w) N_{E_\zeta n_\zeta}^{l_\zeta}(s; w) \\
&\Leftrightarrow [\sigma(\tfrac{1}{2}; w) \varepsilon^{C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s; w) + \sigma_{m_\zeta}^{n_\zeta}(s - \tfrac{1}{2}; w) N_{E_\zeta n_\zeta}^{l_\zeta}(s; w)] = N_{E_\zeta m_\zeta}^{k_\zeta}(s; w) \sigma_{k_\zeta}^{l_\zeta}(s; w) \\
&\Leftrightarrow \sigma^{\alpha_\zeta} \varepsilon^{B_\zeta}(\tfrac{1}{2}; w) N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + \sigma^{\alpha_\zeta} \varepsilon^{m_\zeta}(s - \tfrac{1}{2}; w) N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w) \sigma^{\alpha_\zeta} \varepsilon^{k_\zeta}(s; w)
\end{aligned}$$

□

定理5.7.2.

$$\begin{cases}
X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) [S_{ab A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{abl_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w)] = S_{ab m_\zeta}^{n_\zeta}(s - 1; w) X_{n_\zeta}^{B_\zeta \otimes k_\zeta}(s; w) \\
[S_{ab A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{abl_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w)] X_{B_\zeta k_\zeta}^{n_\zeta}(s; w) = X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) S_{ab m_\zeta}^{n_\zeta}(s - 1; w) \\
X^{A_\zeta}(s; w) [S_{ab A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta} S_{ab}(s - \tfrac{1}{2}, \zeta; w)] = S_{ab}(s - 1, \zeta; w) X^{B_\zeta}(s; w) \\
[S_{ab A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta} S_{ab}(s - \tfrac{1}{2}, \zeta; w)] \bar{X}_{B_\zeta}(s; w) = \bar{X}_{A_\zeta}(s; w) S_{ab}(s - 1, \zeta; w) \\
\bar{X}(s; w) [S_{ab}(\tfrac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \tfrac{1}{2}, \zeta; w)] = S_{ab}(s - 1, \zeta; w) \bar{X}(s; w) \\
[S_{ab}(\tfrac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \tfrac{1}{2}, \zeta; w)] X(s; w) = X(s; w) S_{ab}(s - 1, \zeta; w)
\end{cases}$$

$$\begin{aligned}
& \text{证明: } X_{m_\zeta}^{A_\zeta k_\zeta}(s; w) [S_{ab A_\zeta}^{B_\zeta} \delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{ab k_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w)] = S_{ab m_\zeta}^{n_\zeta}(s - 1; w) X_{n_\zeta}^{B_\zeta l_\zeta}(s; w) \\
&\Leftrightarrow \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w) [S_{ab A_\zeta}^{B_\zeta} \delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{ab k_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w)] \\
&= S_{ab m_\zeta}^{n_\zeta}(s - 1; w) \frac{\sqrt{2s-1}}{\sqrt{2s-1+w}} \varepsilon^{B_\zeta D_\zeta} N_{D_\zeta n_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w) \\
&\Leftrightarrow \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w) [S_{ab A_\zeta}^{B_\zeta} \delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{ab k_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w)] = S_{ab m_\zeta}^{n_\zeta}(s - 1; w) \varepsilon^{B_\zeta D_\zeta} N_{D_\zeta n_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w) \\
&\Leftrightarrow \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{k_\zeta}(s; w) [S_{ab A_\zeta}^{B_\zeta} \delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{ab k_\zeta}^{l_\zeta}(s; w)] = S_{ab m_\zeta}^{n_\zeta}(s - \tfrac{1}{2}; w) \varepsilon^{B_\zeta D_\zeta} N_{D_\zeta n_\zeta}^{l_\zeta}(s; w) \\
&\Leftrightarrow \varepsilon_{E_\zeta B_\zeta} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{k_\zeta}(s; w) [S_{ab A_\zeta}^{B_\zeta} \delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{ab k_\zeta}^{l_\zeta}(s; w)] = \varepsilon_{E_\zeta B_\zeta} S_{ab m_\zeta}^{n_\zeta}(s - \tfrac{1}{2}; w) \varepsilon^{B_\zeta D_\zeta} N_{D_\zeta n_\zeta}^{l_\zeta}(s; w) \\
&\Leftrightarrow N_{C_\zeta m_\zeta}^{k_\zeta}(s; w) [S_{ab E_\zeta}^{C_\zeta} \delta_{k_\zeta}^{l_\zeta} - \delta_{E_\zeta}^{C_\zeta} S_{ab k_\zeta}^{l_\zeta}(s; w)] = -S_{ab m_\zeta}^{n_\zeta}(s - \tfrac{1}{2}; w) N_{E_\zeta n_\zeta}^{l_\zeta}(s; w) \\
&\Leftrightarrow [S_{ab E_\zeta}^{C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s; w) + S_{ab m_\zeta}^{n_\zeta}(s - \tfrac{1}{2}; w) N_{E_\zeta n_\zeta}^{l_\zeta}(s; w)] = N_{E_\zeta m_\zeta}^{k_\zeta}(s; w) S_{ab k_\zeta}^{l_\zeta}(s; w) \\
&\Leftrightarrow S_{ab A_\zeta}^{B_\zeta} N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + S_{abl_\zeta}^{m_\zeta}(s - \tfrac{1}{2}; w) N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w) S_{ab j_\zeta}^{k_\zeta}(s; w)
\end{aligned}$$

□

推论5.7.1.

$$\begin{cases}
N_{C_\zeta m_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w) \varepsilon^{C_\zeta A_\zeta} [\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{l_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - 1; w) N_{D_\zeta n_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w) \varepsilon^{D_\zeta B_\zeta} \\
[\sigma_{A_\zeta}^{B_\zeta}(\tfrac{1}{2}; w) \delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{l_\zeta}^{k_\zeta}(s - \tfrac{1}{2}; w)] \varepsilon_{B_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \tfrac{1}{2}; w) = \varepsilon_{A_\zeta D_\zeta} N_{l_\zeta}^{D_\zeta m_\zeta}(s - \tfrac{1}{2}; w) \sigma_{m_\zeta}^{n_\zeta}(s - 1; w)
\end{cases}$$

$$\begin{cases} N_{C_\zeta}(s - \frac{1}{2}; w) \varepsilon^{C_\zeta A_\zeta} [\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta} \sigma(s - \frac{1}{2}; w)] = \sigma(s - 1; w) N_{D_\zeta}(s - \frac{1}{2}; w) \varepsilon^{D_\zeta B_\zeta} \\ [\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta} \sigma(s - \frac{1}{2}; w)] \varepsilon_{B_\zeta C_\zeta} N^{C_\zeta}(s - \frac{1}{2}; w) = \varepsilon_{A_\zeta D_\zeta} N^{D_\zeta}(s - \frac{1}{2}; w) \sigma(s - 1; w) \end{cases}$$

推论5.7.2.

$$\begin{cases} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) \varepsilon^{C_\zeta A_\zeta} [S_{ab A_\zeta}^{B_\zeta}(\frac{1}{2}; w) \delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)] = S_{ab m_\zeta}^{n_\zeta}(s - 1; w) N_{D_\zeta n_\zeta}^{k_\zeta}(s - \frac{1}{2}; w) \varepsilon^{D_\zeta B_\zeta} \\ [S_{ab A_\zeta}^{B_\zeta}(\frac{1}{2}; w) \delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)] \varepsilon_{B_\zeta C_\zeta} N_{k_\zeta}^{C_\zeta n_\zeta}(s - \frac{1}{2}; w) = \varepsilon_{A_\zeta D_\zeta} N_{l_\zeta}^{D_\zeta m_\zeta}(s - \frac{1}{2}; w) S_{ab m_\zeta}^{n_\zeta}(s - 1; w) \\ N_{C_\zeta}(s - \frac{1}{2}; w) \varepsilon^{C_\zeta A_\zeta} [S_{ab A_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta} S_{ab}(s - \frac{1}{2}, \zeta; w)] = S_{ab}(s - 1, \zeta; w) N_{D_\zeta}(s - \frac{1}{2}; w) \varepsilon^{D_\zeta B_\zeta} \\ [S_{ab A_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta} S_{ab}(s - \frac{1}{2}, \zeta; w)] \varepsilon_{B_\zeta C_\zeta} N^{C_\zeta}(s - \frac{1}{2}; w) = \varepsilon_{A_\zeta D_\zeta} N^{D_\zeta}(s - \frac{1}{2}; w) S_{ab}(s - 1, \zeta; w) \end{cases}$$

## 5.8 推论：关于常数矩阵 $X(s; w)$ , $\bar{X}(s; w)$ 的重要性质

推论5.8.1.

$$\begin{cases} \bar{X}(s; w) [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] X(s; w) = \sigma(s - 1; w) \\ X(s; w) \sigma(s - 1; w) \bar{X}(s; w) = [\sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] X(s; w) \bar{X}(s; w) \\ [X(s; w) \bar{X}(s; w), \sigma(\frac{1}{2}; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] = 0 \end{cases}$$

推论5.8.2.

$$\begin{cases} \bar{X}(s; w) [S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] X(s; w) = S_{ab}(s - 1, \zeta; w) \\ X(s; w) S_{ab}(s, \zeta - 1, \zeta; w) \bar{X}(s; w) = [S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] X(s; w) \bar{X}(s; w) \\ [X(s; w) \bar{X}(s; w), S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] = 0 \end{cases}$$

推论5.8.3.  $X^{A_\zeta}(s; w) \sigma(s - \frac{1}{2}; w) \bar{X}_{A_\zeta}(s; w) = \frac{2s+w}{2s-1+w} \sigma(s - 1; w)$

$$[\Leftrightarrow] \bar{X}(s; w) I_{w+1} \otimes \sigma(s - \frac{1}{2}; w) X(s; w) = \frac{2s+w}{2s-1+w} \sigma(s - 1; w)$$

推论5.8.4.  $X^{A_\zeta}(s; w) I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w) \bar{X}_{A_\zeta}(s; w) = \frac{2s+w}{2s-1+w} S_{ab}(s - 1, \zeta; w)$

$$[\Leftrightarrow] \bar{X}(s; w) I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w) X(s; w) = \frac{2s+w}{2s-1+w} S_{ab}(s - 1, \zeta; w)$$

## 5.9 矩阵 $X(s; w)$ , $\bar{X}(s; w)$ 的常数不变张量性质

定理5.9.1.  $X(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \zeta; w)} \otimes e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s - \frac{1}{2}, \zeta; w)} X(s; w) e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(s - 1, \zeta; w)}$

证明:  $[S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] X(s; w) = X(s; w) S_{ab}(s - 1, \zeta; w)$

$$\Leftrightarrow 0 = [\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + \frac{i}{2} \vartheta^{ab} I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] X(s; w) - \frac{i}{2} \vartheta^{ab} X(s; w) S_{ab}(s - 1, \zeta; w)$$

$$\Leftrightarrow X(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \zeta; w)} \otimes e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s - \frac{1}{2}, \zeta; w)} X(s; w) e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(s - 1, \zeta; w)} \quad \square$$

定理5.9.2.  $\bar{X}(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s - 1, \zeta; w)} \bar{X}(s; w) e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \zeta; w)} \otimes e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(s - \frac{1}{2}, \zeta; w)}$

证明:  $\bar{X}(s; w) [S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] = S_{ab}(s - 1, \zeta; w) \bar{X}(s; w)$

$$\Leftrightarrow 0 = \frac{i}{2} \vartheta^{ab} S_{ab}(s - 1, \zeta; w) \bar{X}(s; w) - \bar{X}(s; w) [\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}} + \frac{i}{2} \vartheta^{ab} I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)]$$

$$\Leftrightarrow \bar{X}(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s - 1, \zeta; w)} \bar{X}(s; w) e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \zeta; w)} \otimes e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(s - \frac{1}{2}, \zeta; w)} \quad \square$$

## 5.10 矩阵 $X(s)$ , $\bar{X}(s)$ 的常数不变张量性质

定理5.10.1.  $X(s) = e^{(i\omega + \zeta\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \zeta\epsilon) \cdot \sigma(s - \frac{1}{2})} X(s) e^{-(i\omega + \zeta\epsilon) \cdot \sigma(s - 1)}$

证明:  $[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})] X(s) = X(s) \sigma(s - 1)$

$$\Leftrightarrow 0 = [(i\omega + \zeta\epsilon) \cdot \sigma(\frac{1}{2}) \otimes I_{2s} + (i\omega + \zeta\epsilon) \cdot I \otimes \sigma(s - \frac{1}{2})] X(s) - (i\omega + \zeta\epsilon) \cdot X(s) \sigma(s - 1)$$

$$\Leftrightarrow X(s) = e^{(i\omega + \zeta\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \zeta\epsilon) \cdot \sigma(s - \frac{1}{2})} X(s) e^{-(i\omega + \zeta\epsilon) \cdot \sigma(s - 1)} \quad \square$$

定理5.10.2.  $\bar{X}(s) = e^{(i\omega + \zeta\epsilon) \cdot \sigma(s - 1)} \bar{X}(s) e^{-(i\omega + \zeta\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega + \zeta\epsilon) \cdot \sigma(s - \frac{1}{2})}$

证明:  $\bar{X}(s)[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})] = \sigma(s - 1)\bar{X}(s)$

$$\Leftrightarrow 0 = (i\omega + \zeta\epsilon) \cdot \sigma(s - 1)\bar{X}(s) - \bar{X}(s)[(i\omega + \zeta\epsilon) \cdot \sigma(\frac{1}{2}) \otimes I_{2s} + (i\omega + \zeta\epsilon) \cdot I \otimes \sigma(s - \frac{1}{2})]$$

$$\Leftrightarrow \bar{X}(s) = e^{(i\omega + \zeta\epsilon) \cdot \sigma(s-1)} \bar{X}(s) e^{-(i\omega + \zeta\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega + \zeta\epsilon) \cdot \sigma(s-\frac{1}{2})} \quad \square$$

推论5.10.1.  $[N(s), X(s)] = e^{(i\omega + \zeta\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \zeta\epsilon) \cdot \sigma(s-\frac{1}{2})} [N(s), X(s)] [e^{-(i\omega + \zeta\epsilon) \cdot \sigma(s)} \oplus e^{-(i\omega + \zeta\epsilon) \cdot \sigma(s-1)}]$

推论5.10.2.  $\begin{bmatrix} \bar{N}(s) \\ \bar{X}(s) \end{bmatrix} = [e^{(i\omega + \zeta\epsilon) \cdot \sigma(s)} \oplus e^{(i\omega + \zeta\epsilon) \cdot \sigma(s-1)}] \begin{bmatrix} \bar{N}(s) \\ \bar{X}(s) \end{bmatrix} e^{-(i\omega + \zeta\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega + \zeta\epsilon) \cdot \sigma(s-\frac{1}{2})}$

### 5.11 常数矩阵 $\Omega(s; w), \sigma(s - 1; w)$ 的置换性质

推论5.11.1.

$$\begin{cases} \Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]X(s; w) = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]X(s; w)\sigma(s - 1; w) \\ \bar{X}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w) = \sigma(s - 1; w)\bar{X}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \end{cases}$$

推论5.11.2.

$$\begin{cases} \sigma(s; w) = \bar{N}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]N(s; w) \\ \sigma(s - 1; w) = \bar{X}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]X(s; w) \end{cases}$$

推论5.11.3.

$$\begin{cases} [\vec{\vartheta} \cdot \sigma(s; w)]^n = \bar{N}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vec{\vartheta} \cdot \Omega(s; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]N(s; w) \\ [\vec{\vartheta} \cdot \sigma(s - 1; w)]^n = \bar{X}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vec{\vartheta} \cdot \Omega(s; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]X(s; w) \end{cases}$$

推论5.11.4.

$$\begin{cases} e^{\vec{\vartheta} \cdot \sigma(s; w)} = \bar{N}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]e^{\vec{\vartheta} \cdot \Omega(s; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]N(s; w) \\ e^{\vec{\vartheta} \cdot \sigma(s-1; w)} = \bar{X}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]e^{\vec{\vartheta} \cdot \Omega(s; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]X(s; w) \end{cases}$$

### 5.12 常数矩阵 $\Omega(s - l; w), [\vec{\vartheta} \cdot \Omega(s - l; w)]^n, e^{\vec{\vartheta} \cdot \Omega(s-l; w)}$ 的同构性表示

推论5.12.1.  $\Omega(s; w) = \Omega(s - 1; w) \otimes I_{(w+1)^2} + I_{(w+1)^{2s-2}} \otimes \Omega(1; w)$

推论5.12.2.

$$\begin{cases} \Omega(s; w)I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\} = I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\}\Omega(s - 1; w) \\ I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\}\Omega(s; w) = \Omega(s - 1; w)I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\} \end{cases}$$

推论5.12.3.

$$\begin{cases} \Omega(s - 1; w) = I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\}\Omega(s; w)I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\} \\ [\vec{\vartheta} \cdot \Omega(s - 1; w)]^n = I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\}[\vec{\vartheta} \cdot \Omega(s; w)]^n I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\} \\ e^{\vec{\vartheta} \cdot \Omega(s-1; w)} = I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\}e^{\vec{\vartheta} \cdot \Omega(s; w)} I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\} \end{cases}$$

定义5.12.1.

$$\begin{cases} T(s; w) := I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\} \\ \bar{T}(s; w) := I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\} = T^+(s; w) \end{cases}$$

推论5.12.4.

$$\begin{cases} \Omega(s - l; w) = \bar{T}(s - l + 1; w) \cdots \bar{T}(s - 1; w)\bar{T}(s; w)\Omega(s; w)T(s; w)T(s - 1; w) \cdots T(s - l + 1; w) \\ [\vec{\vartheta} \cdot \Omega(s - l; w)]^n = \bar{T}(s - l + 1; w) \cdots \bar{T}(s - 1; w)\bar{T}(s; w)[\vec{\vartheta} \cdot \Omega(s; w)]^n T(s; w)T(s - 1; w) \cdots T(s - l + 1; w) \\ e^{\vec{\vartheta} \cdot \Omega(s-l; w)} = \bar{T}(s - l + 1; w) \cdots \bar{T}(s - 1; w)\bar{T}(s; w)e^{\vec{\vartheta} \cdot \Omega(s; w)} T(s; w)T(s - 1; w) \cdots T(s - l + 1; w) \end{cases}$$

推论5.12.5.

$$\begin{cases} \sigma(s - l; w) = \bar{\Gamma}(s - l; w)\bar{T}(s - l + 1; w) \cdots \bar{T}(s; w)\Omega(s; w)T(s; w) \cdots T(s - l + 1; w)\Gamma(s - l; w) \\ [\vec{\vartheta} \cdot \sigma(s - l; w)]^n = \bar{\Gamma}(s - l; w)\bar{T}(s - l + 1; w) \cdots \bar{T}(s; w)[\vec{\vartheta} \cdot \Omega(s; w)]^n T(s; w) \cdots T(s - l + 1; w)\Gamma(s - l; w) \\ e^{\vec{\vartheta} \cdot \sigma(s-l; w)} = \bar{\Gamma}(s - l; w)\bar{T}(s - l + 1; w) \cdots \bar{T}(s; w)e^{\vec{\vartheta} \cdot \Omega(s; w)} T(s; w) \cdots T(s - l + 1; w)\Gamma(s - l; w) \end{cases}$$

# 第三章 重要的复合常数不变张量

了自我评述：本章对多个重要的复合常数不变张量进行了深入分析，对复合常数不变张量进行了分解，部分解决求得了其中二元数列的通项公式，并结合线性代数方法得到一些十分有用的结论，是后续研究高自旋粒子的一个有力数学工具。

## 1 本章需满足的代数同构前提条件

定义1.0.2.

$$\begin{cases} N^{A_\zeta}(s; w) \bar{N}_{A_\zeta}(s; w) = 1, N^{A_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta}{}^{B_\zeta}(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) = \frac{1}{2s} \sigma^{\alpha_\zeta}(s; w) \\ N^{A_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta}{}^{B_\zeta}(\frac{1}{2}; w) + N^{B_\zeta}(s; w) \sigma^{\alpha_\zeta}(s - \frac{1}{2}; w) = \sigma^{\alpha_\zeta}(s; w) N^{B_\zeta}(s; w) \\ \sigma^{\alpha_\zeta}_{A_\zeta}{}^{B_\zeta}(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) + \sigma^{\alpha_\zeta}(s - \frac{1}{2}; w) \bar{N}_{A_\zeta}(s; w) = \bar{N}_{A_\zeta}(s; w) \sigma^{\alpha_\zeta}(s; w) \\ \{\sigma^{\alpha_\zeta}(\frac{1}{2}; w), \sigma^{\beta_\zeta}(\frac{1}{2}; w)\} = \frac{1}{2} \delta^{\alpha_\zeta \beta_\zeta} \end{cases}$$

## 2 复合常数不变张量 $X^{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}}(s, n, l; w)$

### 2.1 复合常数不变张量 $X^{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}}(s, n, l; w)$ 的引入

#### 2.1.1 复合常数不变张量 $X^{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}}(s, n, l; w)$ 的定义

定义2.1.1.

$$\begin{cases} X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, n, l; w) := N^{A_\zeta}(s; w) [\sigma^{\alpha_{1\zeta}}(\frac{1}{2}; w) \cdots \sigma^{\alpha_{n\zeta}}(\frac{1}{2}; w)]_{A_\zeta}{}^{B_\zeta} [\sigma^{\beta_{1\zeta}}(s - \frac{1}{2}; w) \cdots \sigma^{\beta_{l\zeta}}(s - \frac{1}{2}; w)] \bar{N}_{B_\zeta}(s; w) \\ X(s, 0, 0; w) := 1, X^{\alpha_{1\zeta}}(s, 1, 0; w) = \frac{1}{2s} \sigma^{\alpha_{1\zeta}}(s; w), X^{\beta_{1\zeta}}(s, 0, 1; w) = (1 - \frac{1}{2s}) \sigma^{\beta_{1\zeta}}(s; w); s \geq 1, n \geq 0, l \geq 0 \end{cases}$$

推论2.1.1.

$$\begin{cases} X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, n, l; w) := N^{A_\zeta}(s; w) [\sigma^{\{\alpha_{1\zeta}\}}(\frac{1}{2}; w) \cdots \sigma^{\{\alpha_{n\zeta}\}}(\frac{1}{2}; w)]_{A_\zeta}{}^{B_\zeta} [\sigma^{\{\beta_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\{\beta_{l\zeta}\}}(s - \frac{1}{2}; w)] \bar{N}_{B_\zeta}(s; w) \\ X^{\{\}}(s, 0, 0; w) := 1, X^{\{\alpha_{1\zeta}\}}(s, 1, 0; w) = \frac{1}{2s} \sigma^{\alpha_{1\zeta}}(s; w), X^{\{\beta_{1\zeta}\}}(s, 0, 1; w) = (1 - \frac{1}{2s}) \sigma^{\beta_{1\zeta}}(s; w); s \geq 1, n \geq 0, l \geq 0 \end{cases}$$

#### 2.1.2 复合常数不变张量 $X^{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}}(s, n, l; w)$ 的递推关系

定理2.1.1.  $X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, n, l; w) = \frac{1}{4} \frac{1}{(n+l-2)!} X^{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, n-2, l; w) \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}}; s \geq 1, n \geq 2, l \geq 0$

$$\begin{aligned} \text{证明: } X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, n, l; w) &= N^{A_\zeta}(s; w) [\sigma^{\{\alpha_{1\zeta}\}}(\frac{1}{2}; w) \cdots \sigma^{\{\alpha_{n\zeta}\}}(\frac{1}{2}; w)]_{A_\zeta}{}^{B_\zeta} [\sigma^{\{\beta_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\{\beta_{l\zeta}\}}(s - \frac{1}{2}; w)] \bar{N}_{B_\zeta}(s; w) \\ &= N^{A_\zeta}(s; w) [\sigma^{\{\alpha_{1\zeta}\}}(\frac{1}{2}; w) \cdots \sigma^{\{\alpha_{(n-2)\zeta}\}}(\frac{1}{2}; w) \sigma^{\{\alpha_{(n-1)\zeta}\}}(\frac{1}{2}; w) \sigma^{\{\alpha_{n\zeta}\}}(\frac{1}{2}; w)]_{A_\zeta}{}^{B_\zeta} [\sigma^{\{\beta_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\{\beta_{l\zeta}\}}(s - \frac{1}{2}; w)] \bar{N}_{B_\zeta}(s; w) \\ &= N^{A_\zeta}(s; w) \{\sigma^{\{\alpha_{1\zeta}\}}(\frac{1}{2}; w) \cdots \sigma^{\{\alpha_{(n-2)\zeta}\}}(\frac{1}{2}; w) [\frac{1}{4} \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}}]\}_{A_\zeta}{}^{B_\zeta} [\sigma^{\{\beta_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\{\beta_{l\zeta}\}}(s - \frac{1}{2}; w)] \bar{N}_{B_\zeta}(s; w) \\ &= \frac{1}{4} \frac{1}{(n+l-2)!} X^{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, n-2, l; w) \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}} \quad \square \end{aligned}$$

推论2.1.2.  $s \geq 1, l \geq 0$

$$X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, n, l; w) = \begin{cases} \frac{1}{2^n} \frac{1}{l!} \delta^{\{\alpha_{1\zeta} \alpha_{2\zeta} \cdots \alpha_{(n-1)\zeta} \alpha_{n\zeta}\}} X^{\{\beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, 0, l; w), n = 2k \geq 0 \\ \frac{1}{2^{(n-1)}} \frac{1}{(l+1)!} \delta^{\{\alpha_{1\zeta} \alpha_{2\zeta} \cdots \alpha_{(n-2)\zeta} \alpha_{(n-1)\zeta}\}} X^{\{\alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, 1, l; w), n = 2k + 1 \geq 0 \end{cases}$$

定理2.1.2.  $X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, n, l; w); s \geq 1, n \geq 1, l \geq 1$

$$= \frac{1}{(n+l-1)!} X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{(l-1)\zeta}\}}(s, n, l-1; w) \sigma^{\beta_{l\zeta}}(s; w) - \frac{1}{4} \frac{1}{(n+l-2)!} X^{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta} \beta_{1\zeta} \cdots \beta_{(l-1)\zeta}\}}(s, n-1, l-1; w) \delta^{\alpha_{n\zeta} \beta_{l\zeta}}$$

$$\begin{aligned} \text{证明: } X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, n, l; w) &= N^{A_\zeta}(s; w) [\sigma^{\{\alpha_{1\zeta}\}}(\frac{1}{2}; w) \cdots \sigma^{\{\alpha_{n\zeta}\}}(\frac{1}{2}; w)]_{A_\zeta}{}^{B_\zeta} [\sigma^{\{\beta_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\{\beta_{l\zeta}\}}(s - \frac{1}{2}; w)] \bar{N}_{B_\zeta}(s; w) \\ &= N^{A_\zeta}(s; w) [\sigma^{\{\alpha_{1\zeta}\}}(\frac{1}{2}; w) \cdots \sigma^{\{\alpha_{n\zeta}\}}(\frac{1}{2}; w)]_{A_\zeta}{}^{B_\zeta} [\sigma^{\{\beta_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\{\beta_{(l-1)\zeta}\}}(s - \frac{1}{2}; w)] [\bar{N}_{B_\zeta}(s; w) \sigma^{\beta_{l\zeta}}(s; w) - \sigma^{\beta_{l\zeta}}]_{B_\zeta}{}^{C_\zeta}(\frac{1}{2}; w) \bar{N}_{C_\zeta}(s; w)] \\ &= N^{A_\zeta}(s; w) [\sigma^{\{\alpha_{1\zeta}\}}(\frac{1}{2}; w) \cdots \sigma^{\{\alpha_{n\zeta}\}}(\frac{1}{2}; w)]_{A_\zeta}{}^{B_\zeta} [\sigma^{\{\beta_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\{\beta_{(l-1)\zeta}\}}(s - \frac{1}{2}; w)] \bar{N}_{B_\zeta}(s; w) \sigma^{\beta_{l\zeta}}(s; w) \\ &\quad - N^{A_\zeta}(s; w) [\sigma^{\{\alpha_{1\zeta}\}}(\frac{1}{2}; w) \cdots \sigma^{\{\alpha_{n\zeta}\}}(\frac{1}{2}; w) \sigma^{\beta_{l\zeta}}(\frac{1}{2}; w)]_{A_\zeta}{}^{B_\zeta} [\sigma^{\{\beta_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\{\beta_{(l-1)\zeta}\}}(s - \frac{1}{2}; w)] \bar{N}_{B_\zeta}(s; w) \\ &= \frac{1}{(n+l-1)!} X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{(l-1)\zeta}\}}(s, n, l-1; w) \sigma^{\beta_{l\zeta}}(s; w) - X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{l\zeta}\}}(s, n+1, l-1; w) \\ &= \frac{1}{(n+l-1)!} X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta} \beta_{1\zeta} \cdots \beta_{(l-1)\zeta}\}}(s, n, l-1; w) \sigma^{\beta_{l\zeta}}(s; w) - \frac{1}{4} \frac{1}{(n+l-2)!} X^{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta} \beta_{1\zeta} \cdots \beta_{(l-1)\zeta}\}}(s, n-1, l-1; w) \delta^{\alpha_{n\zeta} \beta_{l\zeta}} \quad \square \end{aligned}$$

## 2.2 复合常数不变张量 $M^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w)$ 和 $N^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w)$

### 2.2.1 复合常数不变张量 $M^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w)$ 和 $N^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w)$ 的引入

定义2.2.1.

$$\begin{cases} M^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w) := N^{A_\zeta}(s; w) \sigma^{\alpha_{1\zeta}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{n\zeta}}(s - \frac{1}{2}; w) \bar{N}_{A_\zeta}(s; w) = X^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, 0, n; w) \\ M(s, 0; w) = 1, M^{\alpha_{1\zeta}}(s, 1; w) = (1 - \frac{1}{2s}) \sigma^{\alpha_{1\zeta}}(s; w); s \geq 1, n \geq 0 \end{cases}$$

推论2.2.1.

$$\begin{cases} M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) := N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\{\alpha_{n\zeta}\}}(s - \frac{1}{2}; w) \bar{N}_{A_\zeta}(s; w) = X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, 0, n; w) \\ M^{\{\}}(s, 0; w) = 1, M^{\{\alpha_{1\zeta}\}}(s, 1; w) = (1 - \frac{1}{2s}) \sigma^{\alpha_{1\zeta}}(s; w); s \geq 1, n \geq 0 \end{cases}$$

定义2.2.2.

$$\begin{cases} N^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w) := N^{A_\zeta}(s; w) \sigma^{\alpha_{1\zeta}}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\alpha_{2\zeta}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{n\zeta}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) = X^{\alpha_{1\zeta}; \alpha_{2\zeta} \cdots \alpha_{n\zeta}}(s, 1, n - 1; w) \\ N(s, 0; w) = 1, N^{\alpha_{1\zeta}}(s, 1; w) = \frac{1}{2s} \sigma^{\alpha_{1\zeta}}(s; w); s \geq 1, n \geq 0 \end{cases}$$

推论2.2.2.

$$\begin{cases} N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) := N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta}\}}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\alpha_{2\zeta}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{n\zeta}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) = X^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, 1, n - 1; w) \\ N^{\{\}}(s, 0; w) = 1, N^{\{\alpha_{1\zeta}\}}(s, 1; w) = \frac{1}{2s} \sigma^{\alpha_{1\zeta}}(s; w); s \geq 1, n \geq 0 \end{cases}$$

### 2.2.2 复合常数不变张量 $M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w)$ 和 $N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w)$ 的递推关系

定理2.2.1.  $s \geq 1, n \geq 2$

$$N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) = \frac{1}{(n-1)!} N^{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}}(s, n-1; w) \sigma^{\alpha_{n\zeta}}(s; w) - \frac{1}{4} \frac{1}{(n-2)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta}\}}}(s, n-2; w) \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}}$$

证明:  $N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta}\}}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\alpha_{2\zeta}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{n\zeta}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w)$   
 $= N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta}\}}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\alpha_{2\zeta}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{(n-1)\zeta}}(s - \frac{1}{2}; w) [\bar{N}_{B_\zeta}(s; w) \sigma^{\alpha_{n\zeta}}(s; w) - \sigma^{\alpha_{n\zeta}}_{B_\zeta C_\zeta}(\frac{1}{2}; w) \bar{N}_{C_\zeta}(s; w)]$   
 $= N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta}\}}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\alpha_{2\zeta}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{(n-1)\zeta}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) \sigma^{\alpha_{n\zeta}}(s; w)$   
 $- N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta}\}}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\alpha_{2\zeta}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{(n-1)\zeta}}(s - \frac{1}{2}; w) \sigma^{\alpha_{n\zeta}}_{B_\zeta C_\zeta}(\frac{1}{2}; w) \bar{N}_{C_\zeta}(s; w)$   
 $= \frac{1}{(n-1)!} N^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}}}(s, n-1; w) \sigma^{\alpha_{n\zeta}}(s; w) - \frac{1}{4} \frac{1}{(n-2)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta}\}}}(s, n-2; w) \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}} \quad \square$

定理2.2.2.  $N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) = \frac{1}{(n-1)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}}}(s, n-1; w) \sigma^{\alpha_{n\zeta}}(s; w) - M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w); s \geq 1, n \geq 1$

证明:  $M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w)$

$$\begin{aligned} &= N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{n\zeta}}(s - \frac{1}{2}; w) \bar{N}_{A_\zeta}(s; w) \\ &= N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{(n-1)\zeta}}(s - \frac{1}{2}; w) [\bar{N}_{A_\zeta}(s; w) \sigma^{\alpha_{n\zeta}}(s; w) - \sigma^{\alpha_{n\zeta}}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w)] \\ &= N^{A_\zeta}(s; w) \frac{1}{(n-1)!} \sigma^{\{\{\alpha_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{(n-1)\zeta}}(s - \frac{1}{2}; w) \bar{N}_{A_\zeta}(s; w) \sigma^{\alpha_{n\zeta}}(s; w) \\ &- N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta}\}}(s - \frac{1}{2}; w) \cdots \sigma^{\alpha_{(n-1)\zeta}}(s - \frac{1}{2}; w) \sigma^{\alpha_{n\zeta}}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) \\ &= \frac{1}{(n-1)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}}}(s, n-1; w) \sigma^{\alpha_{n\zeta}}(s; w) - N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) \quad \square \end{aligned}$$

推论2.2.3.

$$\begin{aligned} &M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) \\ &= \frac{2}{(n-1)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}}}(s, n-1; w) \sigma^{\alpha_{n\zeta}}(s; w) - \frac{1}{(n-2)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta}\}}}(s, n-2; w) \sigma^{\alpha_{(n-1)\zeta}}(s; w) \sigma^{\alpha_{n\zeta}}(s; w) \\ &+ \frac{1}{4} \frac{1}{(n-2)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta}\}}}(s, n-2; w) \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}}; s \geq 1, n \geq 2 \end{aligned}$$

证明:  $N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w)$

$$\begin{aligned} &= \frac{1}{(n-1)!} N^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}}}(s, n-1; w) \sigma^{\alpha_{n\zeta}}(s; w) - \frac{1}{4} \frac{1}{(n-2)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta}\}}}(s, n-2; w) \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}} \\ &\Leftrightarrow \frac{1}{(n-1)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}}}(s, n-1; w) \sigma^{\alpha_{n\zeta}}(s; w) - M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) \\ &= \frac{1}{(n-1)!} [\frac{1}{(n-2)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta}\}}}(s, n-2; w) \sigma^{\alpha_{(n-1)\zeta}}(s; w) - M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}}}(s, n-1; w)] \sigma^{\alpha_{n\zeta}}(s; w) \\ &- \frac{1}{4} \frac{1}{(n-2)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta}\}}}(s, n-2; w) \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}} \\ &\Leftrightarrow \frac{1}{(n-1)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}}}(s, n-1; w) \sigma^{\alpha_{n\zeta}}(s; w) - M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) \end{aligned}$$



$$\begin{aligned}
&= \frac{2}{(n-3)!} M^{\{\alpha_{1\zeta} \cdots \alpha_{(n-3)\zeta}\}}(s, n-3; w) \sigma^{\alpha_{(n-2)\zeta}}(s; w) \left[ \frac{1}{4} \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}} + 3 \sigma^{\alpha_{(n-1)\zeta}}(s; w) \sigma^{\alpha_{n\zeta}}(s; w) \right] \\
&+ \frac{1}{(n-4)!} M^{\{\alpha_{1\zeta} \cdots \alpha_{(n-4)\zeta}\}}(s, n-4; w) \left[ \frac{1}{4} \delta^{\alpha_{(n-3)\zeta} \alpha_{(n-2)\zeta}} - \sigma^{\alpha_{(n-3)\zeta}}(s; w) \sigma^{\alpha_{(n-2)\zeta}}(s; w) \right] \left[ \frac{1}{4} \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}} + 3 \sigma^{\alpha_{(n-1)\zeta}}(s; w) \sigma^{\alpha_{n\zeta}}(s; w) \right] \\
&+ \frac{1}{(n-3)!} M^{\{\alpha_{1\zeta} \cdots \alpha_{(n-3)\zeta}\}}(s, n-3; w) \left[ \frac{1}{2} \delta^{\alpha_{(n-2)\zeta} \alpha_{(n-1)\zeta}} - 2 \sigma^{\alpha_{(n-2)\zeta}}(s; w) \sigma^{\alpha_{(n-1)\zeta}}(s; w) \right] \sigma^{\alpha_{n\zeta}}(s; w) \\
&= \frac{1}{(n-3)!} M^{\{\alpha_{1\zeta} \cdots \alpha_{(n-3)\zeta}\}}(s, n-3; w) \left[ \delta^{\alpha_{(n-2)\zeta} \alpha_{(n-1)\zeta}} + 4 \sigma^{\alpha_{(n-2)\zeta}}(s; w) \sigma^{\alpha_{(n-1)\zeta}}(s; w) \right] \sigma^{\alpha_{n\zeta}}(s; w) \\
&+ \frac{1}{(n-4)!} M^{\{\alpha_{1\zeta} \cdots \alpha_{(n-4)\zeta}\}}(s, n-4; w) \left[ \frac{1}{4} \delta^{\alpha_{(n-3)\zeta} \alpha_{(n-2)\zeta}} - \sigma^{\alpha_{(n-3)\zeta}}(s; w) \sigma^{\alpha_{(n-2)\zeta}}(s; w) \right] \left[ \frac{1}{4} \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}} + 3 \sigma^{\alpha_{(n-1)\zeta}}(s; w) \sigma^{\alpha_{n\zeta}}(s; w) \right]
\end{aligned}$$

### 2.2.4 利用全对称技巧对常数不变张量 $N^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w)$ 前几项的直接计算

性质2.2.1.  $N^{\{\alpha_\zeta\}}(s, 1) = N^{A_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) = \frac{1}{2s} \sigma^{\alpha_\zeta}(s; w)$

性质2.2.2.  $N^{\{\alpha_\zeta \beta_\zeta\}}(s, 2) = N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w)$   
 $= \frac{1}{2s} [\sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) - \frac{s}{2} \delta^{\{\alpha_\zeta \beta_\zeta\}}]$

证明:  $N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w)$   
 $= N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) [\bar{N}_{B_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) - \sigma^{\beta_\zeta\}}_{B_\zeta C_\zeta}(\frac{1}{2}; w) \bar{N}_{C_\zeta}(s; w)]$   
 $= N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) - N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\beta_\zeta\}}_{B_\zeta C_\zeta}(\frac{1}{2}; w) \bar{N}_{C_\zeta}(s; w)$   
 $= \frac{1}{2s} \sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) - N^{A_\zeta}(s; w) \frac{1}{2} \delta_{A_\zeta C_\zeta} \delta^{\alpha_\zeta \beta_\zeta} \bar{N}_{C_\zeta}(s; w)$   
 $= \frac{1}{2s} [\sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) - \frac{s}{2} \delta^{\{\alpha_\zeta \beta_\zeta\}}]$  □

性质2.2.3.  $N^{\{\alpha_\zeta \beta_\zeta \gamma_\zeta\}}(s, 3) = N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) \sigma^{\gamma_\zeta\}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w)$   
 $= \frac{1}{2s} [\sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) \sigma^{\gamma_\zeta\}}(s; w) + \frac{1-4s}{4} \sigma^{\{\alpha_\zeta}(s; w) \delta^{\beta_\zeta \gamma_\zeta\}}]$

证明:  $N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) \sigma^{\gamma_\zeta\}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w)$   
 $= N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) [\bar{N}_{B_\zeta}(s; w) \sigma^{\gamma_\zeta\}}(s; w) - \sigma^{\gamma_\zeta\}}_{B_\zeta C_\zeta}(\frac{1}{2}; w) \bar{N}_{C_\zeta}(s; w)]$   
 $= N^{A_\zeta}(s; w) \frac{1}{2!} \sigma^{\{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) \sigma^{\gamma_\zeta\}}(s; w)$   
 $- N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\gamma_\zeta\}}_{B_\zeta C_\zeta}(\frac{1}{2}; w) \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) \bar{N}_{C_\zeta}(s; w)$   
 $= \frac{1}{2s} \frac{1}{2!} [\sigma^{\{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) - \frac{s}{2} \delta^{\{\{\alpha_\zeta \beta_\zeta\}}\}} \sigma^{\gamma_\zeta\}}(s; w)$   
 $- N^{A_\zeta}(s; w) \frac{1}{2!} \sigma^{\{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\gamma_\zeta\}}_{B_\zeta C_\zeta}(\frac{1}{2}; w) \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) \bar{N}_{C_\zeta}(s; w)$   
 $= \frac{1}{2s} [\sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) - \frac{s}{2} \delta^{\{\{\alpha_\zeta \beta_\zeta\}}\}} \sigma^{\gamma_\zeta\}}(s; w) - N^{A_\zeta}(s; w) \frac{1}{2!} \frac{1}{2} \delta_{A_\zeta C_\zeta} \delta^{\{\alpha_\zeta \gamma_\zeta \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) \bar{N}_{C_\zeta}(s; w)$   
 $= \frac{1}{2s} [\sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) - \frac{s}{2} \delta^{\{\{\alpha_\zeta \beta_\zeta\}}\}} \sigma^{\gamma_\zeta\}}(s; w) - \frac{1}{2!} \frac{1}{2} \delta^{\{\alpha_\zeta \gamma_\zeta\}} N^{A_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) \bar{N}_{A_\zeta}(s; w)$   
 $= \frac{1}{2s} [\sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) \sigma^{\gamma_\zeta\}}(s; w) + \frac{1-4s}{4} \sigma^{\{\alpha_\zeta}(s; w) \delta^{\beta_\zeta \gamma_\zeta\}}]$  □

性质2.2.4.  $N^{\{\alpha_\zeta \beta_\zeta \gamma_\zeta \eta_\zeta\}}(s, 4; w) = N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \sigma^{\beta_\zeta\}}(s - \frac{1}{2}; w) \sigma^{\gamma_\zeta\}}(s - \frac{1}{2}; w) \sigma^{\eta_\zeta\}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w)$   
 $= \frac{1}{2s} [\sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) \sigma^{\gamma_\zeta\}}(s; w) \sigma^{\eta_\zeta\}}(s; w) + \frac{3}{4} (1-2s) \sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) \delta^{\gamma_\zeta \eta_\zeta\}} - \frac{s}{8} \delta^{\{\alpha_\zeta \beta_\zeta \delta^{\gamma_\zeta \eta_\zeta\}}}]$

#### 推论2.2.9.

$$\begin{cases}
N^{\{\alpha_\zeta\}}(s, 1; w) = \frac{1}{2s} \sigma^{\alpha_\zeta}(s; w) \\
N^{\{\alpha_\zeta \beta_\zeta\}}(s, 2; w) = \frac{1}{2s} [\sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) - \frac{s}{2} \delta^{\{\alpha_\zeta \beta_\zeta\}}] \\
N^{\{\alpha_\zeta \beta_\zeta \gamma_\zeta\}}(s, 3; w) = \frac{1}{2s} [\sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) \sigma^{\gamma_\zeta\}}(s; w) - \frac{1+2s}{4} \sigma^{\{\alpha_\zeta}(s; w) \delta^{\beta_\zeta \gamma_\zeta\}}] \\
N^{\{\alpha_\zeta \beta_\zeta \gamma_\zeta \eta_\zeta\}}(s, 4; w) = \frac{1}{2s} [\sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) \sigma^{\gamma_\zeta\}}(s; w) \sigma^{\eta_\zeta\}}(s; w) + \frac{3}{4} (1-2s) \sigma^{\{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta\}}(s; w) \delta^{\gamma_\zeta \eta_\zeta\}} - \frac{s}{8} \delta^{\{\alpha_\zeta \beta_\zeta \delta^{\gamma_\zeta \eta_\zeta\}}}]
\end{cases}$$

### 2.2.5 常数不变张量 $N^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n)$ 前几项的直接计算

性质2.2.5.  $N^{\alpha_\zeta}(s, 1) = N^{A_\zeta}(s) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}) \bar{N}_{B_\zeta}(s) = \frac{1}{2s} \sigma^{\alpha_\zeta}(s)$

性质2.2.6.  $N^{\alpha_\zeta \beta_\zeta}(s, 2) = N^{A_\zeta}(s) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}) \sigma^{\beta_\zeta}(s - \frac{1}{2}) \bar{N}_{B_\zeta}(s) = \frac{1}{4s} [\sigma^{\{\alpha_\zeta}(s) \sigma^{\beta_\zeta\}}(s) - s \delta^{\alpha_\zeta \beta_\zeta}]$

证明:  $N^{A_\zeta}(s) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}) \sigma^{\beta_\zeta}(s - \frac{1}{2}) \bar{N}_{B_\zeta}(s)$   
 $= N^{A_\zeta}(s) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}) [-\sigma^{\beta_\zeta}_{B_\zeta C_\zeta}(\frac{1}{2}) \bar{N}_{C_\zeta}(s) + \bar{N}_{B_\zeta}(s) \sigma^{\beta_\zeta}(s)]$   
 $= -\frac{1}{4s} [s \delta^{\alpha_\zeta \beta_\zeta} + \sigma^{\{\alpha_\zeta}(s) \sigma^{\beta_\zeta\}}(s)] + \frac{1}{2s} \sigma^{\alpha_\zeta}(s) \sigma^{\beta_\zeta}(s)$   
 $= \frac{1}{4s} [\sigma^{\{\alpha_\zeta}(s) \sigma^{\beta_\zeta\}}(s) - s \delta^{\alpha_\zeta \beta_\zeta}]$  □





$$\begin{aligned}
& -\frac{1}{4}[\delta^{\alpha_\zeta \eta_\zeta} \sigma^{\beta_\zeta}(s) \sigma^{\gamma_\zeta}(s) + \delta^{\alpha_\zeta \{\beta_\zeta \sigma^{\gamma_\zeta}\}}(s) \sigma^{\eta_\zeta}(s)] \\
& + \frac{1}{4s}[\sigma^{\alpha_\zeta}(s) \sigma^{\{\beta_\zeta \sigma^{\gamma_\zeta}\}}(s) \sigma^{\eta_\zeta}(s) + \sigma^{\{\beta_\zeta \sigma^{\alpha_\zeta}\}}(s) \sigma^{\gamma_\zeta}(s) \sigma^{\eta_\zeta}(s) - \sigma^{\{\alpha_\zeta \sigma^{\eta_\zeta}\}}(s) \sigma^{\beta_\zeta}(s) \sigma^{\gamma_\zeta}(s)]
\end{aligned}$$

□

$$\begin{aligned}
& \text{推论2.2.11. } N^{A_\zeta}(s) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta} \left(\frac{1}{2}\right) \sigma^{\{\gamma_\zeta(s - \frac{1}{2})\}} [\sigma^{\beta_\zeta(s - \frac{1}{2})}] \sigma^{\eta_\zeta} \left(s - \frac{1}{2}\right) \bar{N}_{B_\zeta}(s) \\
& = \frac{1}{8} (-\delta^{\alpha_\zeta \eta_\zeta} \delta^{\gamma_\zeta \beta_\zeta} + \delta^{\alpha_\zeta \beta_\zeta} \delta^{\gamma_\zeta \eta_\zeta} - \delta^{\alpha_\zeta \gamma_\zeta} \delta^{\beta_\zeta \eta_\zeta}) - \frac{i}{16s} [\varepsilon^{\gamma_\zeta \beta_\zeta \alpha_\zeta} \sigma^{\eta_\zeta}(s) + \varepsilon^{\eta_\zeta \beta_\zeta \alpha_\zeta} \sigma^{\gamma_\zeta}(s)] \\
& \frac{1}{16s} [\delta^{\alpha_\zeta \eta_\zeta} \sigma^{\{\gamma_\zeta \sigma^{\beta_\zeta}\}}(s) + \delta^{\alpha_\zeta \gamma_\zeta} \sigma^{\{\eta_\zeta \sigma^{\beta_\zeta}\}}(s) - \sigma^{\{\alpha_\zeta \sigma^{\eta_\zeta}\}}(s) \delta^{\gamma_\zeta \beta_\zeta} - \sigma^{\{\alpha_\zeta \sigma^{\gamma_\zeta}\}}(s) \delta^{\eta_\zeta \beta_\zeta}] \\
& + \frac{1}{4s} [\sigma^{\alpha_\zeta}(s) \delta^{\eta_\zeta \gamma_\zeta} \sigma^{\beta_\zeta}(s) + \frac{1}{4s} [\sigma^{\alpha_\zeta}(s) \delta^{\eta_\zeta \beta_\zeta} \sigma^{\gamma_\zeta}(s) + \frac{1}{4s} [\sigma^{\alpha_\zeta}(s) \delta^{\beta_\zeta \gamma_\zeta} \sigma^{\eta_\zeta}(s) \\
& - \frac{1}{4} [\delta^{\alpha_\zeta \{\beta_\zeta \sigma^{\gamma_\zeta}\}}(s) \sigma^{\eta_\zeta}(s)]] \\
& + \frac{1}{4s} [\sigma^{\{\beta_\zeta \sigma^{\alpha_\zeta}\}}(s) \sigma^{\gamma_\zeta}(s) \sigma^{\eta_\zeta}(s) - \sigma^{\alpha_\zeta}(s) \sigma^{\{\beta_\zeta \sigma^{\gamma_\zeta}\}}(s) \sigma^{\eta_\zeta}(s) + 2\sigma^{\alpha_\zeta}(s) \sigma^{\{\gamma_\zeta \sigma^{\beta_\zeta}\}}(s) \sigma^{\eta_\zeta}(s)]
\end{aligned}$$

$$\begin{aligned}
& \text{推论2.2.12. } N^{A_\zeta}(s) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta} \left(\frac{1}{2}\right) [(s - \frac{3}{2}) \sigma^{\beta_\zeta}(s - \frac{1}{2}) \delta^{\gamma_\zeta \eta_\zeta} + (s - \frac{1}{2}) \delta^{\beta_\zeta \{\gamma_\zeta \sigma^{\eta_\zeta}\}}(s - \frac{1}{2})] \bar{N}_{B_\zeta}(s) \\
& = (s - \frac{3}{2}) \frac{1}{4s} [\sigma^{\{\alpha_\zeta \sigma^{\beta_\zeta}\}}(s) - s \delta^{\alpha_\zeta \beta_\zeta}] \delta^{\gamma_\zeta \eta_\zeta} + (s - \frac{1}{2}) \frac{1}{4s} [\sigma^{\{\alpha_\zeta \sigma^{\gamma_\zeta}\}}(s) \delta^{\eta_\zeta \beta_\zeta} + \sigma^{\{\alpha_\zeta \sigma^{\eta_\zeta}\}}(s) \delta^{\gamma_\zeta \beta_\zeta} - s \delta^{\alpha_\zeta \{\gamma_\zeta \delta^{\eta_\zeta \beta_\zeta}\}}]
\end{aligned}$$

$$\text{性质2.2.9. } \sigma^{\alpha_\zeta}(s) \delta^{\beta_\zeta \gamma_\zeta} + \sigma^{\{\beta_\zeta \sigma^{\alpha_\zeta}\}}(s) \sigma^{\gamma_\zeta}(s) = \frac{1}{3!} [\sigma^{\{\alpha_\zeta \sigma^{\delta^{\beta_\zeta \gamma_\zeta}\}}(s) + 2\sigma^{\{\alpha_\zeta \sigma^{\beta_\zeta}\}}(s) \sigma^{\gamma_\zeta}(s)]$$

## 2.3 复合常数不变张量 $\Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}} l_\zeta(s; w)$ 的引入

### 2.3.1 复合常数不变张量 $\Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}} l_\zeta(s, n; w)$ 的定义

定义2.3.1.

$$\begin{cases} \Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}} l_\zeta(s, n; w) := \Gamma_{k_\zeta}^{A_{1\zeta} \dots A_{n\zeta} A_{(n+1)\zeta} \dots A_{(2s)\zeta}}(s; w) \prod_{i=1}^n \sigma^{\alpha_{i\zeta}} A_{i\zeta}^{B_{i\zeta}} \left(\frac{1}{2}; w\right) \Gamma_{B_{1\zeta} \dots B_{n\zeta} A_{(n+1)\zeta} \dots A_{(2s)\zeta}}^{l_\zeta}(s; w); n \geq 0, s \geq \frac{n}{2} \Big| \frac{1}{2} \\ \Gamma^{\alpha'_{1\zeta} \dots \alpha'_{n\zeta}} k'_\zeta l'_\zeta(s, n; w) := \Gamma_{A'_{1\zeta} \dots A'_{n\zeta} A'_{(n+1)\zeta} \dots A'_{(2s)\zeta}}(s; w) \prod_{i=1}^n \sigma^{\alpha'_{i\zeta}} A'_{i\zeta}^{B'_{i\zeta}} \left(\frac{1}{2}; w\right) \Gamma_{B'_{1\zeta} \dots B'_{n\zeta} A'_{(n+1)\zeta} \dots A'_{(2s)\zeta}}^{k'_\zeta l'_\zeta}(s; w); n \geq 0, s \geq \frac{n}{2} \Big| \frac{1}{2} \end{cases}$$

$$\text{定义2.3.2. } \Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}}(s, n; w) := \Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}} l_\zeta(s, n; w), \Gamma^{\alpha'_{1\zeta} \dots \alpha'_{n\zeta}}(s, n; w) := \Gamma^{\alpha'_{1\zeta} \dots \alpha'_{n\zeta}} k'_\zeta l'_\zeta(s, n; w); n \geq 0, s \geq \frac{n}{2} \Big| \frac{1}{2}$$

$$\text{推论2.3.1. } \Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}} l_\zeta(s, n; w) \simeq \Gamma^{\alpha'_{1\zeta} \dots \alpha'_{n\zeta}} k'_\zeta l'_\zeta(s, n; w), \Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}}(s, n; w) \simeq \Gamma^{\alpha'_{1\zeta} \dots \alpha'_{n\zeta}}(s, n; w); n \geq 0, s \geq \frac{n}{2} \Big| \frac{1}{2}$$

### 2.3.2 复合常数不变张量 $\Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}} l_\zeta(s, n; w)$ 的递推公式

$$\text{定理2.3.1. } \Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}}(s, n; w) = N^{A_{1\zeta}}(s; w) \sigma^{\alpha_{1\zeta}} A_{1\zeta}^{B_{1\zeta}} \left(\frac{1}{2}; w\right) \Gamma^{\alpha_{2\zeta} \dots \alpha_{n\zeta}}(s, n-1; w) \bar{N}_{B_{1\zeta}}(s; w); n \geq 1, s \geq \frac{n}{2} \Big| \frac{1}{2}$$

证明:  $\Gamma^{\{\alpha_{1\zeta} \dots \alpha_{n\zeta}\}}(s, n; w)$

$$\begin{aligned}
& = \Gamma_{k_{1\zeta}}^{A_{1\zeta} \dots A_{n\zeta} A_{(n+1)\zeta} \dots A_{(2s)\zeta}}(s; w) \prod_{i=1}^n \sigma^{\alpha_{i\zeta}} A_{i\zeta}^{B_{i\zeta}} \left(\frac{1}{2}; w\right) \Gamma_{B_{1\zeta} \dots B_{n\zeta} A_{(n+1)\zeta} \dots A_{(2s)\zeta}}^{l_{1\zeta}}(s; w) \\
& = N_{k_{1\zeta}}^{A_{1\zeta} k_{2\zeta}}(s; w) \Gamma_{k_{2\zeta}}^{A_{2\zeta} \dots A_{n\zeta} A_{(n+1)\zeta} \dots A_{(2s)\zeta}}(s - \frac{1}{2}; w) \prod_{i=1}^n \sigma^{\alpha_{i\zeta}} A_{i\zeta}^{B_{i\zeta}} \left(\frac{1}{2}; w\right) N_{B_{2\zeta} l_{2\zeta}}^{l_{1\zeta}} \Gamma_{B_{2\zeta} \dots B_{n\zeta} A_{(n+1)\zeta} \dots A_{(2s)\zeta}}^{l_{2\zeta}}(s; w) \\
& = N_{k_{1\zeta}}^{A_{1\zeta} k_{2\zeta}}(s; w) \sigma^{\alpha_{1\zeta}} A_{1\zeta}^{B_{1\zeta}} \left(\frac{1}{2}; w\right) \\
& [\Gamma_{k_{2\zeta}}^{A_{2\zeta} \dots A_{n\zeta} A_{(n+1)\zeta} \dots A_{(2s)\zeta}}(s - \frac{1}{2}; w) \prod_{i=2}^n \sigma^{\alpha_{i\zeta}} A_{i\zeta}^{B_{i\zeta}} \left(\frac{1}{2}; w\right) \Gamma_{B_{2\zeta} \dots B_{n\zeta} A_{(n+1)\zeta} \dots A_{(2s)\zeta}}^{l_{2\zeta}}(s - \frac{1}{2}; w)] N_{B_{1\zeta} l_{1\zeta}}^{l_{1\zeta}}(s; w) \\
& = N_{k_{1\zeta}}^{A_{1\zeta} k_{2\zeta}}(s; w) \sigma^{\alpha_{1\zeta}} A_{1\zeta}^{B_{1\zeta}} \left(\frac{1}{2}; w\right) \Gamma^{\alpha_{2\zeta} \dots \alpha_{n\zeta}} l_{2\zeta}(s, n-1; w) N_{B_{1\zeta} l_{1\zeta}}^{l_{1\zeta}}(s; w) \\
& = N^{A_{1\zeta}}(s; w) \sigma^{\alpha_{1\zeta}} A_{1\zeta}^{B_{1\zeta}} \left(\frac{1}{2}; w\right) \Gamma^{\alpha_{2\zeta} \dots \alpha_{n\zeta}}(s, n-1; w) \bar{N}_{B_{1\zeta}}(s; w)
\end{aligned}$$

□

$$\text{推论2.3.2. } \Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}}(s, n; w); n \geq 1, s \geq \frac{n}{2} \Big| \frac{1}{2}$$

$$= N^{A_{1\zeta}}(s; w) \cdot N^{A_{n\zeta}}(s - \frac{n-1}{2}; w) \sigma^{\alpha_{1\zeta}} A_{1\zeta}^{B_{1\zeta}} \left(\frac{1}{2}; w\right) \cdot \sigma^{\alpha_{n\zeta}} A_{n\zeta}^{B_{n\zeta}} \left(\frac{1}{2}; w\right) \bar{N}_{B_{n\zeta}}(s - \frac{n-1}{2}; w) \cdot \bar{N}_{B_{1\zeta}}(s; w)$$

$$\text{推论2.3.3. } \Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}}(s, n; w) = N^{A_{1\zeta} \dots A_{n\zeta}}(s, n; w) \sigma^{\alpha_{1\zeta}} A_{1\zeta}^{B_{1\zeta}} \left(\frac{1}{2}; w\right) \cdot \sigma^{\alpha_{n\zeta}} A_{n\zeta}^{B_{n\zeta}} \left(\frac{1}{2}; w\right) \bar{N}_{B_{1\zeta} \dots B_{n\zeta}}(s, n; w)$$

### 2.3.3 利用前几节公式对常数不变张量 $\Gamma^{\alpha_{1\zeta} \dots \alpha_{n\zeta}} l_\zeta(s, n; w)$ 前几项的直接计算

$$\text{性质2.3.1. } N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta A_\zeta^{B_\zeta} \left(\frac{1}{2}; w\right) \sigma^{\beta_\zeta} B_\zeta^{C_\zeta} \left(\frac{1}{2}; w\right) \sigma^{\gamma_\zeta}\}}(s - \frac{1}{2}; w) \bar{N}_{C_\zeta}(s; w) = \frac{1}{4} (1 - \frac{1}{2s}) \sigma^{\{\alpha_\zeta \sigma^{\beta_\zeta \gamma_\zeta}\}}(s; w) \delta^{\beta_\zeta \gamma_\zeta}$$

$$\begin{aligned}
& \text{性质2.3.2. } \Gamma^{\alpha_\zeta}_{k_\zeta} l_\zeta(s, 1) = \Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}^{2s}}(s; w) \sigma^{\alpha_\zeta} A_\zeta^{I_\zeta} \left(\frac{1}{2}; w\right) \Gamma_{\underbrace{I_\zeta B_\zeta C_\zeta \dots}_{2s}}^{l_\zeta}(s; w) = \frac{1}{2s} \sigma^{\alpha_\zeta}_{k_\zeta} l_\zeta(s; w) \\
& = \frac{1}{(1!)^2} C_{2s}^{-1} \sigma^{\alpha_\zeta}_{k_\zeta} l_\zeta(s; w)
\end{aligned}$$

$$\begin{aligned} \text{性质2.3.3. } \Gamma^{\alpha_\zeta \beta_\zeta}_{k_\zeta}{}^{l_\zeta}(s, 2) &= \Gamma^{\overbrace{A_\zeta B_\zeta C_\zeta \cdots}^{2s}}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta} I_\zeta(\tfrac{1}{2}; w) \sigma^{\beta_\zeta}_{B_\zeta} J_\zeta(\tfrac{1}{2}; w) \Gamma^{\underbrace{l_\zeta}_{2s}}_{I_\zeta J_\zeta C_\zeta \cdots}(s; w) \\ &= \frac{1}{(2!)^2} C_{2s}^{-2} [\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) - \frac{s}{2} \delta^{\{\alpha_\zeta \beta_\zeta\}}]_{k_\zeta}{}^{l_\zeta} \end{aligned}$$

$$\begin{aligned} \text{证明: } \Gamma^{\overbrace{A_\zeta B_\zeta C_\zeta \cdots}^{2s}}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta} I_\zeta(\tfrac{1}{2}; w) \sigma^{\beta_\zeta}_{B_\zeta} J_\zeta(\tfrac{1}{2}; w) \Gamma^{\underbrace{l_\zeta}_{2s}}_{I_\zeta J_\zeta C_\zeta \cdots}(s; w) \\ &= \frac{1}{2!} \Gamma^{\overbrace{A_\zeta B_\zeta C_\zeta \cdots}^{2s}}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta} I_\zeta(\tfrac{1}{2}; w) \sigma^{\beta_\zeta\}}_{B_\zeta} J_\zeta(\tfrac{1}{2}; w) \Gamma^{\underbrace{l_\zeta}_{2s}}_{I_\zeta J_\zeta C_\zeta \cdots}(s; w) \\ &= \frac{1}{2!} [N^{A_\zeta}(s; w) \sigma^{\{\alpha_\zeta}_{A_\zeta} B_\zeta(\tfrac{1}{2}; w) \frac{1}{2s-1} \sigma^{\beta_\zeta\}}(s - \tfrac{1}{2}; w) \bar{N}_{B_\zeta}(s; w)]_{k_\zeta}{}^{l_\zeta} \\ &= \frac{1}{2s-1} \frac{1}{4s} [\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) - s \delta^{\{\alpha_\zeta \beta_\zeta\}}]_{k_\zeta}{}^{l_\zeta} \\ &= \frac{1}{(2!)^2} C_{2s}^{-2} [\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) - \frac{s}{2} \delta^{\{\alpha_\zeta \beta_\zeta\}}]_{k_\zeta}{}^{l_\zeta} \quad \square \end{aligned}$$

$$\begin{aligned} \text{性质2.3.4. } \Gamma^{\alpha_\zeta \beta_\zeta \gamma_\zeta}_{k_\zeta}{}^{l_\zeta}(s, 3) &= \Gamma^{\overbrace{A_\zeta B_\zeta C_\zeta D_\zeta \cdots}^{2s}}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta} I_\zeta(\tfrac{1}{2}; w) \sigma^{\beta_\zeta}_{B_\zeta} J_\zeta(\tfrac{1}{2}; w) \sigma^{\gamma_\zeta}_{C_\zeta} K_\zeta(\tfrac{1}{2}; w) \Gamma^{\underbrace{l_\zeta}_{2s}}_{I_\zeta J_\zeta K_\zeta D_\zeta \cdots}(s; w) \\ &= \frac{1}{(3!)^2} C_{2s}^{-3} [\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) \sigma^{\gamma_\zeta}(s; w) + \frac{1-3s}{2} \sigma^{\{\alpha_\zeta\}}(s; w) \delta^{\beta_\zeta \gamma_\zeta}] \end{aligned}$$

$$\begin{aligned} \text{证明: } \Gamma^{\overbrace{A_\zeta B_\zeta C_\zeta D_\zeta \cdots}^{2s}}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta} I_\zeta(\tfrac{1}{2}; w) \sigma^{\beta_\zeta}_{B_\zeta} J_\zeta(\tfrac{1}{2}; w) \sigma^{\gamma_\zeta}_{C_\zeta} K_\zeta(\tfrac{1}{2}; w) \Gamma^{\underbrace{l_\zeta}_{2s}}_{I_\zeta J_\zeta K_\zeta D_\zeta \cdots}(s; w) \\ &= N^{A_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta} B_\zeta(\tfrac{1}{2}; w) \frac{1}{2s-2} [\frac{1}{4s-2} \sigma^{\{\beta_\zeta\}}(s - \tfrac{1}{2}; w) \sigma^{\gamma_\zeta}(s - \tfrac{1}{2}; w) - \frac{1}{4} \delta^{\beta_\zeta \gamma_\zeta}] \bar{N}_{B_\zeta}(s; w) \\ &= \frac{1}{4(s-1)(2s-1)} N^{A_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta} B_\zeta(\tfrac{1}{2}; w) \sigma^{\{\beta_\zeta\}}(s - \tfrac{1}{2}; w) \sigma^{\gamma_\zeta}(s - \tfrac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) - \frac{1}{16s(s-1)} \sigma^{\alpha_\zeta}(s; w) \delta^{\beta_\zeta \gamma_\zeta} \\ &= \frac{1}{4(s-1)(2s-1)} \{ \frac{1}{2s} [\sigma^{\alpha_\zeta}(s; w) \delta^{\beta_\zeta \gamma_\zeta} + \sigma^{\beta_\zeta}(s; w) \sigma^{\alpha_\zeta}(s; w) \sigma^{\gamma_\zeta}(s; w) + \sigma^{\gamma_\zeta}(s; w) \sigma^{\alpha_\zeta}(s; w) \sigma^{\beta_\zeta}(s; w)] - \frac{1}{4} \sigma^{\{\alpha_\zeta\}}(s; w) \delta^{\beta_\zeta \gamma_\zeta} \} \\ &= \frac{1}{4(s-1)(2s-1)} \{ \frac{1}{2s} \frac{1}{3!} [\sigma^{\{\alpha_\zeta\}}(s; w) \delta^{\beta_\zeta \gamma_\zeta} + 2\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) \sigma^{\gamma_\zeta}(s; w)] - \frac{1}{4} \sigma^{\{\alpha_\zeta\}}(s; w) \delta^{\beta_\zeta \gamma_\zeta} \} \\ &= \frac{1}{48s(s-1)(2s-1)} [2\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) \sigma^{\gamma_\zeta}(s; w) + (1-3s) \sigma^{\{\alpha_\zeta\}}(s; w) \delta^{\beta_\zeta \gamma_\zeta}] \\ &= \frac{1}{16s(s-\frac{1}{2})(s-1)} \{ \sigma^{\{\beta_\zeta\}}(s; w) [\sigma^{\alpha_\zeta}(s; w)] \sigma^{\gamma_\zeta}(s; w) - [(s-1) \sigma^{\alpha_\zeta}(s; w) \delta^{\beta_\zeta \gamma_\zeta} + s \delta^{\alpha_\zeta \{\beta_\zeta \gamma_\zeta\}}(s; w)] \} \\ &= \frac{1}{(3!)^2} C_{2s}^{-3} [\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) \sigma^{\gamma_\zeta}(s; w) + \frac{1-3s}{2} \sigma^{\{\alpha_\zeta\}}(s; w) \delta^{\beta_\zeta \gamma_\zeta}] \quad \square \end{aligned}$$

$$\text{性质2.3.5. } \Gamma^{\alpha_\zeta \beta_\zeta \gamma_\zeta \eta_\zeta}_{k_\zeta}{}^{l_\zeta}(s, 4)$$

$$\begin{aligned} &= \Gamma^{\overbrace{A_\zeta B_\zeta C_\zeta D_\zeta \cdots}^{2s}}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta} I_\zeta(\tfrac{1}{2}; w) \sigma^{\beta_\zeta}_{B_\zeta} J_\zeta(\tfrac{1}{2}; w) \sigma^{\gamma_\zeta}_{C_\zeta} K_\zeta(\tfrac{1}{2}; w) \sigma^{\eta_\zeta}_{D_\zeta} L_\zeta(\tfrac{1}{2}; w) \Gamma^{\underbrace{l_\zeta}_{2s}}_{I_\zeta J_\zeta K_\zeta L_\zeta \cdots}(s; w) \\ &= \frac{1}{(4!)^2} C_{2s}^{-4} [\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) \sigma^{\gamma_\zeta}(s; w) \sigma^{\eta_\zeta}(s; w) + (2-3s) \sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) \delta^{\gamma_\zeta \eta_\zeta} + \frac{3}{4} s(s-1) \delta^{\{\alpha_\zeta \beta_\zeta \gamma_\zeta \eta_\zeta\}}] \end{aligned}$$

$$\begin{aligned} \text{证明: } \Gamma^{\overbrace{A_\zeta B_\zeta C_\zeta D_\zeta \cdots}^{2s}}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta} I_\zeta(\tfrac{1}{2}; w) \sigma^{\beta_\zeta}_{B_\zeta} J_\zeta(\tfrac{1}{2}; w) \sigma^{\gamma_\zeta}_{C_\zeta} K_\zeta(\tfrac{1}{2}; w) \sigma^{\eta_\zeta}_{D_\zeta} L_\zeta(\tfrac{1}{2}; w) \Gamma^{\underbrace{l_\zeta}_{2s}}_{I_\zeta J_\zeta K_\zeta L_\zeta \cdots}(s; w) \\ &= \frac{1}{16(s-\frac{1}{2})(s-1)(s-\frac{3}{2})} N^{A_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta} B_\zeta(\tfrac{1}{2}; w) \\ &\{ \sigma^{\{\gamma_\zeta\}}(s - \tfrac{1}{2}; w) [\sigma^{\beta_\zeta}(s - \tfrac{1}{2}; w)] \sigma^{\eta_\zeta}(s - \tfrac{1}{2}; w) - [(s - \tfrac{3}{2}) \sigma^{\beta_\zeta}(s - \tfrac{1}{2}; w) \delta^{\gamma_\zeta \eta_\zeta} + (s - \tfrac{1}{2}) \delta^{\beta_\zeta \{\gamma_\zeta \eta_\zeta\}}(s - \tfrac{1}{2}; w)] \} \bar{N}_{B_\zeta}(s; w) \\ &= \frac{1}{16(s-\frac{1}{2})(s-1)(s-\frac{3}{2})} \\ &\{ \frac{1}{8} (-\delta^{\alpha_\zeta \eta_\zeta} \delta^{\gamma_\zeta \beta_\zeta} + \delta^{\alpha_\zeta \beta_\zeta} \delta^{\gamma_\zeta \eta_\zeta} - \delta^{\alpha_\zeta \gamma_\zeta} \delta^{\beta_\zeta \eta_\zeta}) - \frac{i}{16s} [\varepsilon^{\gamma_\zeta \beta_\zeta \alpha_\zeta} \sigma^{\eta_\zeta}(s; w) + \varepsilon^{\eta_\zeta \beta_\zeta \alpha_\zeta} \sigma^{\gamma_\zeta}(s; w)] \\ &\frac{1}{16s} [\delta^{\alpha_\zeta \eta_\zeta} \sigma^{\{\gamma_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) + \delta^{\alpha_\zeta \gamma_\zeta} \sigma^{\{\eta_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) - \sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\eta_\zeta}(s; w) \delta^{\gamma_\zeta \beta_\zeta} - \sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\gamma_\zeta}(s; w) \delta^{\eta_\zeta \beta_\zeta}] \\ &+ \frac{1}{4s} [\sigma^{\alpha_\zeta}(s; w) \delta^{\eta_\zeta \gamma_\zeta} \sigma^{\beta_\zeta}(s; w) + \frac{1}{4s} [\sigma^{\alpha_\zeta}(s; w) \delta^{\eta_\zeta \beta_\zeta} \sigma^{\gamma_\zeta}(s; w) + \frac{1}{4s} [\sigma^{\alpha_\zeta}(s; w) \delta^{\beta_\zeta \gamma_\zeta} \sigma^{\eta_\zeta}(s; w) - \frac{1}{4} [\delta^{\alpha_\zeta \{\beta_\zeta \gamma_\zeta\}} \sigma^{\eta_\zeta}(s; w) \sigma^{\eta_\zeta}(s; w)]] \\ &+ \frac{1}{4s} [\sigma^{\{\beta_\zeta\}}(s; w) [\sigma^{\alpha_\zeta}(s; w)] \sigma^{\gamma_\zeta}(s; w) \sigma^{\eta_\zeta}(s; w) - \sigma^{\alpha_\zeta}(s; w) \sigma^{\{\beta_\zeta\}}(s; w) \sigma^{\gamma_\zeta}(s; w) \sigma^{\eta_\zeta}(s; w) + 2\sigma^{\alpha_\zeta}(s; w) \sigma^{\{\gamma_\zeta\}}(s; w) [\sigma^{\beta_\zeta}(s; w)] \sigma^{\eta_\zeta}(s; w) \\ &- (s - \tfrac{3}{2}) \frac{1}{4s} [\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) - s \delta^{\alpha_\zeta \beta_\zeta}] \delta^{\gamma_\zeta \eta_\zeta} - (s - \tfrac{1}{2}) \frac{1}{4s} [\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\gamma_\zeta}(s; w) \delta^{\eta_\zeta \beta_\zeta} + \sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\eta_\zeta}(s; w) \delta^{\gamma_\zeta \beta_\zeta} - \\ &s \delta^{\alpha_\zeta \{\gamma_\zeta \eta_\zeta\} \beta_\zeta} \} \\ &= \frac{1}{(4!)^2} C_{2s}^{-4} [\sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) \sigma^{\gamma_\zeta}(s; w) \sigma^{\eta_\zeta}(s; w) + (2-3s) \sigma^{\{\alpha_\zeta\}}(s; w) \sigma^{\beta_\zeta}(s; w) \delta^{\gamma_\zeta \eta_\zeta} + \frac{3}{4} s(s-1) \delta^{\{\alpha_\zeta \beta_\zeta \gamma_\zeta \eta_\zeta\}}] \quad \square \end{aligned}$$

## 2.4 $M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w)$ , $N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w)$ , $\Gamma^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n; w)$ 的展开式

### 2.4.1 $M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w)$ 的展开式及其递推关系

定义2.4.1.  $\Omega^i(s, n; w) := \sigma^{\{\alpha_{1\zeta}(s; w)\}} \cdot \sigma^{\alpha_{i\zeta}(s; w)} \delta^{\alpha_{(i+1)\zeta} \alpha_{(i+1)\zeta}} \cdots \delta^{\{\alpha_{(n-1)\zeta}\} \alpha_{n\zeta}}$

定义2.4.2.  $\Omega^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n, n-2k; w) := \delta^{\{\alpha_{1\zeta} \alpha_{2\zeta}\}} \cdots \delta^{\alpha_{(2k-1)\zeta} \alpha_{(2k)\zeta}} \sigma^{\alpha_{(2k+1)\zeta}}(s; w) \cdots \sigma^{\alpha_{n\zeta}}(s; w)$ ,  $1 \leq k \leq [n/2]$

性质2.4.1.  $\sum_{k=0}^{[n/2]} c_{n-2k} \Omega^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n, n-2k; w) = 0 \Leftrightarrow c_{n-2k} = 0, 1 \leq k \leq [n/2]; \forall \alpha_{i\zeta}$

证明:  $\sum_{k=0}^{[n/2]} c_{n-2k} \Omega^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n, n-2k; w) = 0$

$$\Leftrightarrow c_n \sigma^{\{\alpha_{1\zeta}(s; w)\}} \cdot \sigma^{\alpha_{n\zeta}}(s; w) + \delta^{\{\alpha_{1\zeta} \alpha_{2\zeta}\}} \sum_{k=0}^{[n/2]-1} c_{n-2-2k} \Omega^{\alpha_{1\zeta} \cdots \alpha_{n\zeta}}(s, n-2, n-2-2k; w) = 0 \quad \square$$

猜想2.4.1.  $M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) = \sum_{i=0}^n m(s, n; i) \Omega^i(s, n; w) = \sum_{k=0}^{[n/2]} m(s, n; n-2k) \Omega^{n-2k}(s, n; w)$

以上猜想可以采用数学归纳法严格证明，以后有时间再补上。

定理2.4.1.

$$\begin{cases} m(s, n; 0) = \frac{1}{4} m(s, n-2; 0), n \geq 2 \\ m(s, n; 1) = 2m(s, n-1; 0), n = 2 \\ m(s, n; 1) = 2m(s, n-1; 0) + \frac{1}{4} m(s, n-2; 1), n \geq 3 \\ m(s, n; i) = 2m(s, n-1; i-1) - m(s, n-2; i-2) + \frac{1}{4} m(s, n-2; i), 2 \leq i \leq n-2, n \geq 2 \\ m(s, n; i) = 2m(s, n-1; i-1) - m(s, n-2; i-2), i \geq 2, n-1 \leq i \leq n, n \geq 2 \end{cases} \quad \begin{cases} m(s, 0; 0) = 1 \\ m(s, 1; 0) = 0 \\ m(s, 1; 1) = 1 - \frac{1}{2s} \end{cases}$$

证明:

$$M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) = \frac{2}{(n-1)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}\}}(s, n-1; w) \sigma^{\alpha_{n\zeta}}(s; w) - \frac{1}{(n-2)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta}\}\}}(s, n-2; w) \sigma^{\alpha_{(n-1)\zeta}}(s; w) \sigma^{\alpha_{n\zeta}}(s; w) \\ + \frac{1}{4} \frac{1}{(n-2)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-2)\zeta}\}\}}(s, n-2; w) \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}}, n \geq 2$$

$$\Leftrightarrow \sum_{i=0}^n m(s, n; i) \Omega^i(s; w) = 2 \sum_{i=0}^{n-1} m(s, n-1; i) \Omega^{i+1}(s; w) - \sum_{i=0}^{n-2} m(s, n-2; i) \Omega^{i+2}(s; w) + \frac{1}{4} \sum_{i=0}^{n-2} m(s, n-2; i) \Omega^i(s; w), n \geq 2$$

$$\Leftrightarrow \sum_{i=0}^n m(s, n; i) \Omega^i(s; w) = 2 \sum_{i=1}^n m(s, n-1; i-1) \Omega^i(s; w) - \sum_{i=2}^n m(s, n-2; i-2) \Omega^i(s; w) + \frac{1}{4} \sum_{i=0}^{n-2} m(s, n-2; i) \Omega^i(s; w), n \geq 2$$

$$\Leftrightarrow \begin{cases} m(s, n; 0) = \frac{1}{4} m(s, n-2; 0), n \geq 2 \\ m(s, n; 1) = 2m(s, n-1; 0), n = 2 \\ m(s, n; 1) = 2m(s, n-1; 0) + \frac{1}{4} m(s, n-2; 1), n \geq 3 \\ m(s, n; i) = 2m(s, n-1; i-1) - m(s, n-2; i-2) + \frac{1}{4} m(s, n-2; i), 2 \leq i \leq n-2, n \geq 2 \\ m(s, n; i) = 2m(s, n-1; i-1) - m(s, n-2; i-2), i \geq 2, n-1 \leq i \leq n, n \geq 2 \end{cases}$$

$$\Leftrightarrow \begin{cases} m(s, n; i) = 2m(s, n-1; i-1) - m(s, n-2; i-2) + \frac{1}{4} m(s, n-2; i), 0 \leq i \leq n, n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \quad \square$$

定理2.4.2.

$$\begin{cases} m(s, n; i) = 2m(s, n-1; i-1) - m(s, n-2; i-2) + \frac{1}{4} m(s, n-2; i), 0 \leq i \leq n, n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

以上展开系数递推关系和初始条件均与 $w$ 独立无关，所以通项展开系数也与 $w$ 独立无关。即对某些 $w$ 代数同构，具有同样的展开系数，只需求任何一个 $w$ 的展开系数即可，哪个方便用哪个。好像不对，是否对 $w$ 代数同构，取决于 $\{\sigma^{\alpha\zeta}(\frac{1}{2}; w), \sigma^{\beta\zeta}(\frac{1}{2}; w)\} = \frac{1}{4} \delta^{\alpha\zeta \beta\zeta}$ 是否一致成立。

### 2.4.2 $N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w)$ 的展开式及其递推关系

猜想2.4.2.  $N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) = \sum_{i=0}^n n(s, n; i) \Omega^i(s, n; w)$

以上猜想也可以采用数学归纳法严格证明，以后有时间再补上。

定理2.4.3. 
$$\begin{cases} n(s, n; 0) = -m(s, n; 0), n \geq 1 \\ n(s, n; i) = m(s, n-1; i-1) - m(s, n; i), 1 \leq i \leq n \end{cases}$$

证明:  $N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w) = \frac{1}{(n-1)!} M^{\{\{\alpha_{1\zeta} \cdots \alpha_{(n-1)\zeta}\}}(s, n-1; w) \sigma^{\alpha_{n\zeta}}(s; w) - M^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w)$

$$\begin{aligned} &\Leftrightarrow \sum_{i=0}^n n(s, n; i) \Omega^i(s; w) = \sum_{i=0}^{n-1} m(s, n-1; i) \Omega^{i+1}(s; w) - \sum_{i=0}^n m(s, n; i) \Omega^i(s; w) \\ &\Leftrightarrow \sum_{i=0}^n n(s, n; i) \Omega^i(s; w) = \sum_{i=1}^n m(s, n-1; i-1) \Omega^i(s; w) - \sum_{i=0}^n m(s, n; i) \Omega^i(s; w) \\ &\Leftrightarrow \begin{cases} n(s, n; 0) = -m(s, n; 0), n \geq 1 \\ n(s, n; i) = m(s, n-1; i-1) - m(s, n; i), 1 \leq i \leq n \end{cases} \quad \square \end{aligned}$$

由于 $m(s, n; i)$ 某些情况下与 $w$ 独立无关，对某些 $w$ 代数同构，具有同样的展开系数。故 $n(s, n; i)$ 也与 $w$ 独立无关，对某些 $w$ 代数同构，具有同样的展开系数。也只需求任何一个 $w$ 的展开系数即可，哪个方便用哪个。好像不对，是否对 $w$ 代数同构，取决于 $\{\sigma^{\alpha_\zeta}(\frac{1}{2}; w), \sigma^{\beta_\zeta}(\frac{1}{2}; w)\} = \frac{1}{4} \delta^{\alpha_\zeta \beta_\zeta}$ 是否一致成立。

### 2.4.3 $\Gamma^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s; w)$ 的展开式及其递推关系

猜想2.4.3.  $\Gamma^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s; w) = \sum_{i=0}^n c(s, n; i) \Omega^i(s, n; w)$

以上猜想也可以采用数学归纳法严格证明，以后有时间再补上。

推论2.4.1.  $\Gamma^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s; w) = N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta} B_\zeta(\frac{1}{2}; w)\}} \Gamma^{\alpha_{2\zeta} \cdots \alpha_{n\zeta}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w)$

定理2.4.4.  $c(s, n; i) = \sum_{j=i-1}^{n-1} c(s - \frac{1}{2}, n-1; j) n(s, j+1; i), n \geq 1$

$$\begin{aligned} &\text{证明: } \Gamma^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s; w) = N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta} B_\zeta(\frac{1}{2}; w)\}} \Gamma^{\alpha_{2\zeta} \cdots \alpha_{n\zeta}}(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) \\ &\Leftrightarrow \sum_{i=0}^n c(s, n; i) \Omega^i(s; w) = N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta} B_\zeta(\frac{1}{2}; w)\}} \sum_{i=0}^{n-1} c(s - \frac{1}{2}, n-1; i) \Omega^i(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) \\ &\Leftrightarrow \sum_{i=0}^n c(s, n; i) \Omega^i(s; w) = \sum_{i=0}^{n-1} c(s - \frac{1}{2}, n-1; i) N^{A_\zeta}(s; w) \sigma^{\{\alpha_{1\zeta} B_\zeta(\frac{1}{2}; w)\}} \Omega^i(s - \frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) \\ &\Leftrightarrow \sum_{i=0}^n c(s, n; i) \Omega^i(s; w) = \sum_{i=0}^{n-1} c(s - \frac{1}{2}, n-1; i) \frac{1}{(i+1)!} N^{\{\{\alpha_{1\zeta} \cdots \alpha_{(i+1)\zeta}\}}(s, i+1) \delta^{\alpha_{(i+2)\zeta} \alpha_{(i+3)\zeta}} \cdots \delta^{\alpha_{(n-1)\zeta} \alpha_{n\zeta}} \\ &\Leftrightarrow \sum_{i=0}^n c(s, n; i) \Omega^i(s; w) = \sum_{i=0}^{n-1} c(s - \frac{1}{2}, n-1; i) \sum_{j=0}^{i+1} n(s, i+1; j) \Omega^j(s; w) \\ &\Leftrightarrow \sum_{i=0}^n c(s, n; i) \Omega^i(s; w) = \sum_{j=0}^{n-1} \sum_{i=0}^{j+1} c(s - \frac{1}{2}, n-1; j) n(s, j+1; i) \Omega^i(s; w) \\ &\Leftrightarrow c(s, n; i) = \sum_{j=i-1}^{n-1} c(s - \frac{1}{2}, n-1; j) n(s, j+1; i) \quad \square \end{aligned}$$

由于 $n(s, n; i)$ 某些情况下与 $w$ 独立无关，对某些 $w$ 代数同构，具有同样的展开系数。故 $c(s, n; i)$ 也与 $w$ 独立无关，对某些 $w$ 代数同构，具有同样的展开系数。也只需求任何一个 $w$ 的展开系数即可，哪个方便用哪个。好像不对，是否对 $w$ 代数同构，取决于 $\{\sigma^{\alpha_\zeta}(\frac{1}{2}; w), \sigma^{\beta_\zeta}(\frac{1}{2}; w)\} = \frac{1}{4} \delta^{\alpha_\zeta \beta_\zeta}$ 是否一致成立。

## 2.5 应用迭代法对前几项展开系数进行具体计算

推论2.5.1.

$$\begin{cases} m(s, 0; 0) = 1 \\ m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s} \\ m(s, 2; 0) = \frac{1}{4}, m(s, 2; 1) = 0, m(s, 2; 2) = 1 - \frac{2}{2s} \\ m(s, 3; 0) = 0, m(s, 3; 1) = \frac{3}{4} - \frac{1}{8s}, m(s, 3; 2) = 0, m(s, 3; 3) = 1 - \frac{3}{2s} \\ m(s, 4; 0) = \frac{1}{16}, m(s, 4; 1) = 0, m(s, 4; 2) = \frac{3}{2} - \frac{1}{2s}, m(s, 4; 3) = 0, m(s, 4; 4) = 1 - \frac{4}{2s} \end{cases}$$

推论2.5.2.

$$\begin{cases} n(s, 0; 0) := \frac{1}{2s} \\ n(s, 1; 0) = 0, n(s, 1; 1) = \frac{1}{2s} \\ n(s, 2; 0) = -\frac{1}{4}, n(s, 2; 1) = 0, n(s, 2; 2) = \frac{1}{2s} \\ n(s, 3; 0) = 0, n(s, 3; 1) = -\frac{1}{2} + \frac{1}{8s}, n(s, 3; 2) = 0, n(s, 3; 3) = \frac{1}{2s} \\ n(s, 4; 0) = -\frac{1}{16}, n(s, 4; 1) = 0, n(s, 4; 2) = -\frac{3}{4} + \frac{3}{8s}, n(s, 4; 3) = 0, n(s, 4; 4) = \frac{1}{2s} \end{cases}$$

推论2.5.3.  $c(s, n; i) = \sum_{j=i-1}^{n-1} c(s - \frac{1}{2}, n-1; j)n(s, j+1; i); c(s, 0; 0) = 1, c(s, 1; 0) = 0, c(s, 1; 1) = \frac{1}{2s}$

$$\Rightarrow \begin{cases} c(s, 0; 0) = 1 \\ c(s, 1; 0) = 0, c(s, 1; 1) = \frac{(2s-1)!}{(2s)!} \\ c(s, 2; 0) = \frac{(2s-2)!}{(2s)!} \frac{-s}{2}, c(s, 2; 1) = 0, c(s, 2; 2) = \frac{(2s-2)!}{(2s)!} \\ c(s, 3; 0) = 0, c(s, 3; 1) = \frac{(2s-3)!}{(2s)!} \frac{1-3s}{2}, c(s, 3; 2) = 0, c(s, 3; 3) = \frac{(2s-3)!}{(2s)!} \\ c(s, 4; 0) = \frac{(2s-4)!}{(2s)!} \frac{3s(s-1)}{4}, c(s, 4; 1) = 0, c(s, 4; 2) = \frac{(2s-4)!}{(2s)!} (2-3s), c(s, 4; 3) = 0, c(s, 4; 4) = \frac{(2s-4)!}{(2s)!} \end{cases}$$

以上计算结果可以与前几节的直接计算方法相互验证，结果完全相同，说明此解析方法是正确有效的。

## 2.6 应用迭代法求解一般展开系数

推论2.6.1.

$$\begin{cases} m(s, 2k; 0) = \frac{1}{2^{2k}}, m(s, 2k+1; 0) = 0 \\ m(s, 2k; 1) = 0, m(s, 2k+1; 1) = \frac{2}{2^{2k+1}}(2k+1 - \frac{1}{2s}) \\ m(s, n; i) = 2m(s, n-1; i-1) - m(s, n-2; i-2) + \frac{1}{4}m(s, n-2; i), 2 \leq i \leq n-2 \\ m(s, n; n-1) = 0, m(s, n; n) = 1 - \frac{n}{2s} \end{cases}$$

推论2.6.2.

$$\begin{cases} m(s, n; 2) - \frac{1}{4}m(s, n-2; 2) = 2m(s, n-1; 1) - m(s, n-2; 0), 2 \leq 2 \leq n-2 \\ m(s, n; 3) - \frac{1}{4}m(s, n-2; 3) = 2m(s, n-1; 2) - m(s, n-2; 1), 2 \leq 3 \leq n-2 \\ m(s, n; 4) - \frac{1}{4}m(s, n-2; 4) = 2m(s, n-1; 3) - m(s, n-2; 2), 2 \leq 4 \leq n-2 \end{cases}$$

推论2.6.3.  $m(s, 2k; 2) - \frac{1}{4}m(s, 2k-2; 2) = \frac{k}{4^{k-2}} - \frac{1}{4^{k-1}}(3 + \frac{1}{s})$

$$\begin{aligned} \text{证明: } & m(s, 2k; 2) - \frac{1}{4}m(s, 2k-2; 2) \\ &= 2m(s, 2k-1; 1) - m(s, 2k-2; 0), 2 \leq 2 \leq 2k-2 \\ &= 2 \frac{2}{2^{2k-1}}(2k-1 - \frac{1}{2s}) - \frac{1}{2^{2k-2}} \\ &= \frac{1}{2^{2k-2}}(4k-3 - \frac{1}{s}) \\ &= \frac{k}{4^{k-2}} - \frac{1}{4^{k-1}}(3 + \frac{1}{s}) \end{aligned}$$

□

推论2.6.4.  $m(s, 2k; 2) = \frac{1}{4^{k-1}}[2k^2 - (1 + \frac{1}{s})k], k \geq 1$

证明:

$$\begin{cases} m(s, 2k; 2) - \frac{1}{4}m(s, 2(k-1); 2) = \frac{k}{4^{k-2}} - \frac{1}{4^{k-1}}(3 + \frac{1}{s}) \\ \frac{1}{4}m(s, 2(k-1); 2) - (\frac{1}{4})^2m(s, 2(k-2); 2) = \frac{k-1}{4^{k-2}} - \frac{1}{4^{k-1}}(3 + \frac{1}{s}) \\ \dots \\ (\frac{1}{4})^{k-2}m(s, 4; 2) - (\frac{1}{4})^{k-1}m(s, 2; 2) = \frac{2}{4^{k-2}} - \frac{1}{4^{k-1}}(3 + \frac{1}{s}) \end{cases}$$

$$\Rightarrow m(s, 2k; 2) - (\frac{1}{4})^{k-1}m(s, 2; 2) = \frac{k(k+1)-2}{2 \cdot 4^{k-2}} - \frac{k-1}{4^{k-1}}(3 + \frac{1}{s})$$

$$\Rightarrow m(s, 2k; 2) = \frac{k(k+1)-2}{2 \cdot 4^{k-2}} - \frac{k-1}{4^{k-1}}(3 + \frac{1}{s}) + \frac{1}{4^{k-1}}(1 - \frac{1}{s}) = \frac{1}{4^{k-1}}[2k^2 - (1 + \frac{1}{s})k] \quad \square$$

推论2.6.5.  $m(s, 2k+1; 3) = \frac{1}{4^{k-1}}[\frac{4}{3}k^3 - \frac{2}{2s}k^2 - (\frac{1}{3} + \frac{1}{2s})k^1], k \geq 1$

证明:  $m(s, 2k+1; 3) - \frac{1}{4}m(s, 2k-1; 3)$

$$= 2m(s, 2k; 2) - m(s, 2k-1; 1), k \geq 2$$

$$= 2 \cdot \frac{1}{4^{k-1}}[2k^2 - (1 + \frac{2}{2s})k] - \frac{1}{4^{k-1}}[2k^1 - (1 + \frac{1}{2s})]$$

$$\Rightarrow m(s, 2k+1; 3) - \frac{1}{4^{k-1}}m(s, 3; 3) = \sum_{i=2}^k 2 \cdot \frac{1}{4^{k-1}}[2i^2 - (1 + \frac{2}{2s})i] - \frac{1}{4^{k-1}}[2i^1 - (1 + \frac{1}{2s})]$$

$$\Rightarrow m(s, 2k+1; 3) = \frac{1}{4^{k-1}}[\frac{4}{3}k^3 - \frac{2}{2s}k^2 - (\frac{1}{3} + \frac{1}{2s})k^1] \quad \square$$

推论2.6.6.

$$m(s, n; n) = 1 - \frac{n}{2s}, n \geq 0$$

$$\begin{cases} m(s, 2k-2; 0) = \frac{1}{4^{k-1}}, k \geq 1 \\ \sum_{i=2}^k 4^{i-1}m(s, 2i-2; 0) = k-1 \\ m(s, 2k-1; 1) = \frac{1}{4^{k-1}}[2k^1 - (1 + \frac{1}{2s})], k \geq 1 \\ \sum_{i=2}^k 4^{i-1}m(s, 2i-1; 1) = k^2 - \frac{1}{2s}k - (1 - \frac{1}{2s}) \\ m(s, 2k-0; 2) = \frac{1}{4^{k-1}}[2k^2 - (1 + \frac{2}{2s})k], k \geq 1 \\ \sum_{i=2}^k 4^{i-1}m(s, 2i-0; 2) = \frac{2}{3}k^3 + \frac{1}{2}(1 - \frac{2}{2s})k^2 - (\frac{1}{6} + \frac{1}{2s})k - (1 - \frac{2}{2s}) \\ m(s, 2k+1; 3) = \frac{1}{4^{k-1}}[\frac{4}{3}k^3 - \frac{2}{2s}k^2 - \frac{1}{3}(1 + \frac{3}{2s})k^1], k \geq 1 \end{cases}$$

推论2.6.7.

$$\begin{cases} n(s, 2k; 0) = -\frac{1}{2^{2k}}, n(s, 2k+1; 0) = 0 \\ n(s, 2k; 1) = 0, n(s, 2k+1; 1) = -\frac{2}{2^{2k+1}}(2k+1 - 1 - \frac{1}{2s}) \\ n(s, n; i) = n(s, n-1; i-1) - \frac{1}{4}m(s, n-2; i), 2 \leq i \leq n-2 \\ n(s, n; n-1) = 0, n(s, n; n) = \frac{1}{2s} \end{cases}$$

按以上的方法可以递推地求出所有的 $m(s, n; i)$ 和 $n(s, n; i)$ , 其中必定会出现伯努利数 $B_k$ 。

性质2.6.1.  $\begin{cases} \sum_{i=0}^n i^p = \frac{1}{p+1} \sum_{k=0}^p (-1)^k C_{p+1}^k B_k n^{p+1-k}, B_k = \delta_{k0} - \frac{1}{k+1} \sum_{j=0}^{k-1} C_{k+1}^j B_j, \frac{z}{e^z-1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \\ B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{2k+1} = 0(k \geq 1) \end{cases}$

### 3 解析延拓后数列的纯数学递推关系和初始条件

#### 3.1 解析延拓后几个数列的重新定义

定义3.1.1.

$$\begin{cases} m(s, n; i) = 2m(s, n-1; i-1) - m(s, n-2; i-2) + \frac{1}{4}m(s, n-2; i); n \geq 2, n \in \mathbb{Z}; i \in \mathbb{Z}; s \neq 0, s \in \mathbb{C} \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, n \geq 0; i < 0 | i > n) := 0; i \in \mathbb{Z}, s \in \mathbb{C} \\ m(s, 1; 1) = 1 - \frac{1}{2s}; s \neq 0, s \in \mathbb{C} \end{cases}$$

定义3.1.2.

$$\begin{cases} n(s, n; i) := m(s, n-1; i-1) - m(s, n; i); n \geq 1, n \in \mathbb{Z}; i \in \mathbb{Z}; s \neq 0, s \in \mathbb{C} \\ n(s, 0; 0) := \frac{1}{2s}; s \neq 0, s \in \mathbb{C} \\ n(s, 0; i < 0 | i > 0) := 0; i \in \mathbb{Z}, s \in \mathbb{C} \end{cases}$$

推论3.1.1.  $n(s, n; i < 0 | i > n) = 0; n \geq 0, n \in \mathbb{Z}; i \in \mathbb{Z}, s \in \mathbb{C}$

定义3.1.3.

$$\begin{cases} c(s, n; i) = \sum_{j=1}^n c(s - \frac{1}{2}, n-1; j-1)n(s, j; i); n \geq 1, n \in \mathbb{Z}; i \in \mathbb{Z}; s \neq 0, \frac{1}{2}, 1, \dots, \frac{n}{2}, s \in \mathbb{C} \\ c(s, 0; 0) = 1, c(s, 0; i < 0 | i > 0) := 0; i \in \mathbb{Z}, s \in \mathbb{C} \end{cases}$$

推论3.1.2.  $c(s, n; i < 0 | i > n) = 0; n \geq 0, n \in \mathbb{Z}; i \in \mathbb{Z}, s \in \mathbb{C}$

## 3.2 解析延拓后等价的归一化定义

定义3.2.1.  $\bar{m}(s, n; i) := \frac{2^n 2s}{2^i} m(s, n; i); \bar{n}(s, n; i) := \frac{2^n 2s}{2^i} n(s, n; i); n \geq 0, n \in \mathbb{Z}; i \in \mathbb{Z}, s \in \mathbb{C}$

定义3.2.2.  $\bar{c}(s, n; i) := \frac{(2s)! 2^{n-i}}{(2s-n)!} c(s, n; i); n \geq 0, n \in \mathbb{Z}; i \in \mathbb{Z}, s \in \mathbb{C}$

定义3.2.3.

$$\begin{cases} [\bar{m}(s, n; i) = 2\bar{m}(s, n-1; i-1) - \bar{m}(s, n-2; i-2) + \frac{1}{4}\bar{m}(s, n-2; i), n \geq 2, n \in \mathbb{Z}; i \in \mathbb{Z}, s \in \mathbb{C}; \\ \bar{m}(s, 0; 0) = 2s, \bar{m}(s, 1; 0) = 0, \bar{m}(s, n \geq 0; i < 0 | i > n) := 0; i \in \mathbb{Z}, s \in \mathbb{C} \\ \bar{m}(s, 1; 1) = 2s - 1; s \in \mathbb{C} \end{cases}$$

定义3.2.4.

$$\begin{cases} \bar{n}(s, n; i) := \bar{m}(s, n-1; i-1) - \bar{m}(s, n; i); n \geq 1, n \in \mathbb{Z}; i \in \mathbb{Z}, s \in \mathbb{C} \\ \bar{n}(s, 0; 0) := 1; s \in \mathbb{C} \\ \bar{n}(s, 0; i < 0 | i > 0) := 0; i \in \mathbb{Z}, s \in \mathbb{C} \end{cases}$$

推论3.2.1.  $\bar{n}(s, n; i < 0 | i > n) = 0; n \geq 0, n \in \mathbb{Z}; i \in \mathbb{Z}, s \in \mathbb{C}$

定义3.2.5.

$$\begin{cases} \bar{c}(s, n; i) = \sum_{j=1}^n \bar{c}(s - \frac{1}{2}, n-1; j-1)\bar{n}(s, j; i), n \geq 1, n \in \mathbb{Z}; i \in \mathbb{Z}, s \in \mathbb{C} \\ \bar{c}(s, 0; 0) = 1, \bar{c}(s, 0; i < 0 | i > 0) := 0; i \in \mathbb{Z}, s \in \mathbb{C} \end{cases}$$

推论3.2.2.  $\bar{c}(s, n; i < 0 | i > n) = 0; n \geq 0, n \in \mathbb{Z}; i \in \mathbb{Z}, s \in \mathbb{C}$

## 4 解析延拓后几个数列通项公式的求解

### 4.1 $m(s, n; i)$ 通项公式的正向推导

#### 4.1.1 新约定下解析延拓后的离散方程

定理4.1.1.

$$\begin{cases} [m(s, n; i) - m(s, n-1; i-1)] - [m(s, n-1; i-1) - m(s, n-2; i-2)] = \frac{1}{4}m(s, n-2; i), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s} \\ m(s, n; i) := 2^{-n}a(s, n; i), m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

[ $\Leftrightarrow$ ]

$$\begin{cases} a(s, n; i) - a(s, n-2; i) = 4a(s, n-1; i-1) - 4a(s, n-2; i-2), 0 \leq i \leq n, n \geq 2 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \\ a(s, n; i) := 2^n m(s, n; i), a(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

推论4.1.1.

$$\begin{cases} a(s, 2k; i) - a(s, 2k - 2; i) = 4a(s, 2k - 1; i - 1) - 4a(s, 2k - 2; i - 2), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

推论4.1.2.

$$\begin{cases} a(s, 2k + 1; i) - a(s, 2k - 1; i) = 4a(s, 2k; i - 1) - 4a(s, 2k - 1; i - 2), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

#### 4.1.2 $n$ 偶数项和奇数项的递推解

定理4.1.2.

$$\begin{cases} a(s, 2k; i) = a(s, 0; i) + 4 \sum_{j=1}^k a(s, 2j - 1; i - 1) - 4 \sum_{j=1}^k a(s, 2j - 2; i - 2), k \geq 1 \\ a(s, 2k + 1; i) = a(s, 1; i) + 4 \sum_{j=1}^k a(s, 2j; i - 1) - 4 \sum_{j=1}^k a(s, 2j - 1; i - 2), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s}, a(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

证明:  $k \geq 1$

$$\begin{aligned} a(s, 2k - 0; i) - a(s, 2k - 2; i) &= 4a(s, 2k - 1; i - 1) - 4a(s, 2k - 2; i - 2) \\ a(s, 2k - 2; i) - a(s, 2k - 4; i) &= 4a(s, 2k - 3; i - 1) - 4a(s, 2k - 4; i - 2) \\ a(s, 2k - 4; i) - a(s, 2k - 6; i) &= 4a(s, 2k - 5; i - 1) - 4a(s, 2k - 6; i - 2) \\ &\dots\dots \\ a(s, 4; i) - a(s, 2; i) &= 4a(s, 3; i - 1) - 4a(s, 2; i - 2) \\ a(s, 2; i) - a(s, 0; i) &= 4a(s, 1; i - 1) - 4a(s, 0; i - 2) \end{aligned}$$

$$\Rightarrow a(s, 2k; i) = a(s, 0; i) + 4 \sum_{j=1}^k a(s, 2j - 1; i - 1) - 4 \sum_{j=1}^k a(s, 2j - 2; i - 2) \quad \square$$

证明:  $k \geq 1$

$$\begin{aligned} a(s, 2k + 1; i) - a(s, 2k - 1; i) &= 4a(s, 2k; i - 1) - 4a(s, 2k - 1; i - 2) \\ a(s, 2k - 1; i) - a(s, 2k - 3; i) &= 4a(s, 2k - 2; i - 1) - 4a(s, 2k - 3; i - 2) \\ a(s, 2k - 3; i) - a(s, 2k - 5; i) &= 4a(s, 2k - 4; i - 1) - 4a(s, 2k - 5; i - 2) \\ &\dots\dots \\ a(s, 5; i) - a(s, 3; i) &= 4a(s, 4; i - 1) - 4a(s, 3; i - 2) \\ a(s, 3; i) - a(s, 1; i) &= 4a(s, 2; i - 1) - 4a(s, 1; i - 2) \end{aligned}$$

$$\Rightarrow a(s, 2k + 1; i) = a(s, 1; i) + 4 \sum_{j=1}^k a(s, 2j; i - 1) - 4 \sum_{j=1}^k a(s, 2j - 1; i - 2) \quad \square$$

#### 4.1.3 $i = 0$ 情形下关于 $n$ 的通解

推论4.1.3.

$$\begin{cases} a(s, 2k; 0) = 1, k \geq 0 \\ 4 \sum_{j=1}^k a(s, 2j; 0) = 4k, 4 \sum_{j=1}^k a(s, 2j - 2; 0) = 4k, k \geq 1 \end{cases} \quad \begin{cases} a(s, 2k + 1; 0) = 0, k \geq 0 \\ 4 \sum_{j=1}^k a(s, 2j - 1; 0) = 0, k \geq 1 \end{cases}$$

证明:

$$\begin{cases} a(s, 2k; 0) - a(s, 0; 0) = 4 \sum_{j=1}^k a(s, 2j - 1; 0 - 1) - 4 \sum_{j=1}^k a(s, 2j - 2; 0 - 2), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases} \\ \Leftrightarrow \begin{cases} a(s, 2k; 0) = a(s, 0; 0), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$



$$\Leftrightarrow \begin{cases} a(s, 2k; 0) = 1, k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \\ \Rightarrow a(s, 2k; 0) = 1, k \geq 0 \end{cases}$$

$$\Rightarrow 4 \sum_{j=1}^k a(s, 2j; 0) = 4k, 4 \sum_{j=1}^k a(s, 2j - 2; 0) = 4k, k \geq 1 \quad \square$$

证明:

$$\begin{cases} a(s, 2k + 1; 0) - a(s, 1; 0) = 4 \sum_{j=1}^k a(s, 2j; 0 - 1) - 4 \sum_{j=1}^k a(s, 2j - 1; 0 - 2), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

$$\Leftrightarrow \begin{cases} a(s, 2k + 1; 0) = a(s, 1; 0), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

$$\Leftrightarrow \begin{cases} a(s, 2k + 1; 0) = 0, k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

$$\Rightarrow a(s, 2k + 1; 0) = 0, k \geq 0$$

$$\Rightarrow 4 \sum_{j=1}^k a(s, 2j - 1; 0) = 0, k \geq 1 \quad \square$$

#### 4.1.4 $i = 1$ 情形下关于 $n$ 的通解

推论4.1.4.

$$\begin{cases} a(s, 2k; 1) = 0, k \geq 0 \\ 4 \sum_{j=1}^k a(s, 2j; 1) = 0, 4 \sum_{j=1}^k a(s, 2j - 2; 1) = 0, k \geq 1 \end{cases} \quad \begin{cases} a(s, 2k + 1; 1) = 4k + 2 - \frac{1}{s}, k \geq 0 \\ 4 \sum_{j=1}^k a(s, 2j - 1; 0) = 4k(2k - \frac{1}{s}), k \geq 1 \end{cases}$$

证明:

$$\begin{cases} a(s, 2k; 1) - a(s, 0; 1) = 4 \sum_{j=1}^k a(s, 2j - 1; 1 - 1) - 4 \sum_{j=1}^k a(s, 2j - 2; 1 - 2), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

$$\Leftrightarrow \begin{cases} a(s, 2k; 1) - a(s, 0; 1) = 4 \sum_{j=1}^k a(s, 2j - 1; 0), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

$$\Leftrightarrow \begin{cases} a(s, 2k; 1) = 0, k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

$$\Rightarrow a(s, 2k; 1) = 0, k \geq 0$$

$$\Rightarrow 4 \sum_{j=1}^k a(s, 2j; 1) = 0, 4 \sum_{j=1}^k a(s, 2j - 2; 1) = 0, k \geq 1 \quad \square$$

证明:

$$\begin{cases} a(s, 2k + 1; 1) - a(s, 1; 1) = 4 \sum_{j=1}^k a(s, 2j; 1 - 1) - 4 \sum_{j=1}^k a(s, 2j - 1; 1 - 2), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

$$\Leftrightarrow \begin{cases} a(s, 2k + 1; 1) - a(s, 1; 1) = 4 \sum_{j=1}^k a(s, 2j; 0), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

$$\Leftrightarrow \begin{cases} a(s, 2k + 1; 1) = 0, k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases}$$

$$\begin{cases} a(s, 2k+1; 1) - a(s, 1; 1) = 4k, k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases} \\
\Leftrightarrow \\
\begin{cases} a(s, 2k+1; 1) = 4k + 2 - \frac{1}{s}, k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 0) = 0, a(s, 1; 1) = 2 - \frac{1}{s} \end{cases} \\
\Rightarrow a(s, 2k+1; 1) = 4k + 2 - \frac{1}{s}, k \geq 0 \\
\Rightarrow 4 \sum_{j=1}^k a(s, 2j-1; 0) = 4k(2k - \frac{1}{s}), k \geq 1$$

□

## 4.2 $m(s, n; i)$ 通项公式的反向推导

### 4.2.1 一般情形下关于 $n$ 的通解

定理4.2.1.

$$\begin{cases} m(s, n; i) = nm(s, 1; i-n+1) - (n-1)m(s, 0; i-n) + \frac{1}{4} \sum_{j=1}^{n-1} jm(s, n-j-1; i-j+1), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

证明:

$$\begin{cases} [m(s, n; i) - m(s, n-1; i-1)] - [m(s, n-1; i-1) - m(s, n-2; i-2)] = \frac{1}{4}m(s, n-2; i), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\
\Leftrightarrow \\
\begin{cases} [m(s, n; i) - m(s, n-1; i-1)] - [m(s, 1; i-n+1) - m(s, 0; i-n)] = \frac{1}{4} \sum_{j=2}^n m(s, j-2; j-n+i), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\
\Leftrightarrow \\
\begin{cases} [m(s, n; i) - m(s, n-1; i-1)] = [m(s, 1; i-n+1) - m(s, 0; i-n)] + \frac{1}{4} \sum_{j=2}^n m(s, j-2; j-n+i), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\
\Leftrightarrow \\
\begin{cases} m(s, n; i) = m(s, 1; i-n+1) + \sum_{r=2}^n \{ [m(s, 1; i-n+1) - m(s, 0; i-n)] + \frac{1}{4} \sum_{j=2}^r m(s, j-2; j-n+i) \}, n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\
\Leftrightarrow \\
\begin{cases} m(s, n; i) = nm(s, 1; i-n+1) - (n-1)m(s, 0; i-n) + \frac{1}{4} \sum_{r=2}^n \sum_{j=2}^r m(s, j-2; j-n+i), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\
\Leftrightarrow \\
\begin{cases} m(s, n; i) = nm(s, 1; i-n+1) - (n-1)m(s, 0; i-n) + \frac{1}{4} \sum_{j=2}^n (n-j+1)m(s, j-2; j-n+i), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\
\Leftrightarrow \\
\begin{cases} m(s, n; i) = nm(s, 1; i-n+1) - (n-1)m(s, 0; i-n) + \frac{1}{4} \sum_{j=1}^{n-1} (n-j)m(s, j-1; j+1-n+i), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\
\Leftrightarrow \\
\begin{cases} m(s, n; i) = nm(s, 1; i-n+1) - (n-1)m(s, 0; i-n) + \frac{1}{4} \sum_{j=1}^{n-1} jm(s, n-j-1; i-j+1), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

□

推论4.2.1.

$$\begin{cases} m(s, n; n-l) = nm(s, 1; 1-l) - (n-1)m(s, 0; 0-l) + \frac{1}{4} \sum_{j=1}^{n-1} jm(s, n-j-1; n-l-j+1), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

推论4.2.2.

$$\begin{cases} m(s, n; n-l) = \frac{1}{4} \sum_{j=1}^{n-1} jm(s, n-j-1; n-l-j+1) = \frac{1}{4} \sum_{j=1}^{n-1} (n-j)m(s, j-1; j-l+1), l \geq 2, n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

4.2.2  $i = n$ 情形下关于 $n$ 的通解

推论4.2.3.  $m(s, n; n) = 1 - \frac{n}{2s}, n \geq 0$

$$\begin{aligned} \text{证明: } & \begin{cases} m(s, n; n) = nm(s, 1; 1) - (n-1)m(s, 0; 0), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\ \Leftrightarrow & \begin{cases} m(s, n; n) = nm(s, 1; 1) - (n-1)m(s, 0; 0) = 1 - \frac{n}{2s}, n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\ \Rightarrow & m(s, n; n) = 1 - \frac{n}{2s}, n \geq 0 \end{aligned} \quad \square$$

4.2.3  $i = n-1$ 情形下关于 $n$ 的通解

推论4.2.4.  $m(s, n; n-1) = 0, n \geq 0$

$$\begin{aligned} \text{证明: } & \begin{cases} m(s, n; n-1) = nm(s, 1; 0) - (n-1)m(s, 0; -1), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\ \Leftrightarrow & \begin{cases} m(s, n; n-1) = nm(s, 1; 0) - (n-1)m(s, 0; -1) = 0, n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\ \Rightarrow & m(s, n; n-1) = 0, n \geq 0 \end{aligned} \quad \square$$

4.2.4  $i = n-2$ 情形下关于 $n$ 的通解

推论4.2.5.  $m(s, n; n-2) = \frac{1}{2^3 3! s} n(n-1)(6s+2-n) = -\frac{1}{48s} n^3 + (\frac{1}{8} + \frac{1}{16s}) n^2 - (\frac{1}{8} + \frac{1}{24s}) n, m(s, 4; 2) = \frac{3}{2} - \frac{1}{2s}, n \geq 0$

$$\begin{aligned} \text{证明: } & \begin{cases} m(s, n; n-2) = nm(s, 1; -1) - (n-1)m(s, 0; -2) + \frac{1}{4} \sum_{j=1}^{n-1} jm(s, n-j-1; n-j-1), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\ \Leftrightarrow & \begin{cases} m(s, n; n-2) = \frac{1}{4} \sum_{j=1}^{n-1} jm(s, n-j-1; n-j-1), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\ \Leftrightarrow & \begin{cases} m(s, n; n-2) = \frac{1}{4} \sum_{j=1}^{n-1} j(1 - \frac{n-j-1}{2s}) = \frac{1}{48s} n(n-1)(6s+2-n), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases} \\ \Rightarrow & m(s, n; n-2) = \frac{1}{48s} n(n-1)(6s+2-n) = -\frac{1}{48s} n^3 + (\frac{1}{8} + \frac{1}{16s}) n^2 - (\frac{1}{8} + \frac{1}{24s}) n, n \geq 0 \end{aligned} \quad \square$$

$$\begin{aligned} \text{证明: } & m(s, n; n-2) = \frac{1}{4} \sum_{j=1}^{n-1} j(1 - \frac{n-j-1}{2s}) = \frac{1}{4} \sum_{j=1}^{n-1} j[(1 + \frac{1}{2s}) - \frac{n}{2s} + \frac{j}{2s}] \\ & = \frac{1}{4} [(\frac{1}{2} + \frac{1}{4s}) n(n-1) - \frac{n^2(n-1)}{4s} + \frac{(n-1)n(2n-1)}{12s}] = \frac{1}{16s} [(2s+1)n(n-1) - \frac{(n-1)n(n+1)}{3}] \\ & = \frac{1}{48s} n(n-1)(6s+2-n) = \frac{n(n-1)}{3} - \frac{n(n-1)(n-2)}{48s} = -\frac{1}{48s} n^3 + (\frac{1}{8} + \frac{1}{16s}) n^2 - (\frac{1}{8} + \frac{1}{24s}) n \end{aligned} \quad \square$$

定理4.2.2.

$$\begin{cases} m(s, n; n-3) = nm(s, 1; -2) - (n-1)m(s, 0; -3) + \frac{1}{4} \sum_{j=1}^{n-1} jm(s, n-j-1; n-j-2) = 0, n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

定理4.2.3.  $m(s, n; n-2k-1) = 0, n \geq 0$

定理4.2.4.

$$\begin{cases} m(s, n; i) = nm(s, 1; i - n + 1) - (n - 1)m(s, 0; i - n) + \frac{1}{4} \sum_{j=1}^{n-1} jm(s, n - j - 1; i - j + 1), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 0) = 0, m(s, 1; 1) = 1 - \frac{1}{2s}, m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

#### 4.2.5 等价最小离散方程及其通解

定理4.2.5.

$$\begin{cases} m(s, n; n - 2l) = 2m(s, n - 1; n - 1 - 2l) - m(s, n - 2; n - 2 - 2l) + \frac{1}{4}m(s, n - 2; n - 2l), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 1) = 1 - \frac{1}{2s}; m(s, n; i < 0 | i > n) := 0, n \geq 0 \\ m(s, 2k; 0) = 2^{-2k}, k \geq 0 \end{cases}$$

[ $\Leftrightarrow$ ]

$$\begin{cases} m(s, n; n - 2l) = nm(s, 1; 1 - 2l) - (n - 1)m(s, 0; -2l) + \frac{1}{4} \sum_{j=1}^{n-1} jm(s, n - j - 1; n - j + 1 - 2l), n \geq 2 \\ m(s, 0; 0) = 1, m(s, 1; 1) = 1 - \frac{1}{2s}; m(s, n; i < 0 | i > n) := 0, n \geq 0 \end{cases}$$

[ $\Leftrightarrow$ ]

$$\begin{cases} a(s, 2k; 2k - 2l) = a(s, 0; 2k - 2l) + 4 \sum_{j=1}^k a(s, 2j - 1; 2k - 1 - 2l) - 4 \sum_{j=1}^k a(s, 2j - 2; 2k - 2 - 2l), k \geq 1 \\ a(s, 2k + 1; 2k + 1 - 2l) = a(s, 1; 2k + 1 - 2l) + 4 \sum_{j=1}^k a(s, 2j; 2k - 2l) - 4 \sum_{j=1}^k a(s, 2j - 1; 2k - 1 - 2l), k \geq 1 \\ a(s, 0; 0) = 1, a(s, 1; 1) = 2 - \frac{1}{s}; a(s, n; i < 0 | i > n) := 0, n \geq 0 \\ a(s, 2k; 0) = 1, k \geq 0 \end{cases}$$

### 4.3 $m(s, n; n - 2l)$ 的通项公式

#### 4.3.1 $i = n - 2l$ 情形下通项公式的试探和猜想

推论4.3.1.

$$\begin{aligned} m(s, n; n - 0) &= \frac{1}{2^1 1! s} (2s \cdot 1 + 0 - n), n \geq 0 \\ m(s, n; n - 2) &= \frac{1}{2^3 3! s} n(n - 1)(2s \cdot 3 + 2 - n), n \geq 0 \\ m(s, n; n - 4) &= \frac{1}{2^5 5! s} n(n - 1)(n - 2)(n - 3)(2s \cdot 5 + 4 - n), n \geq 0 \\ m(s, n; n - 6) &= \frac{1}{2^7 7! s} n(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)(2s \cdot 7 + 6 - n), n \geq 0 \end{aligned}$$

推论4.3.2.

$$\begin{aligned} m(s, n; n - 0) &= \frac{1}{2^1 1! s} C_n^0 [1(2s + 1) - (n + 1)], n \geq 0 \\ m(s, n; n - 2) &= \frac{1}{2^3 3! s} C_n^2 [3(2s + 1) - (n + 1)], n \geq 0 \\ m(s, n; n - 4) &= \frac{1}{2^5 5! s} C_n^4 [5(2s + 1) - (n + 1)], n \geq 0 \\ m(s, n; n - 6) &= \frac{1}{2^7 7! s} C_n^6 [7(2s + 1) - (n + 1)], n \geq 0 \end{aligned}$$

$$\text{猜想4.3.1. } m(s, n; n - 2l) = \frac{C_n^{2l}}{2^{2l+1}(2l+1)s} [(2l+1)(2s+1) - (n+1)] = \frac{1}{2^{2l+1}s} [(2s+1)C_n^{2l} - C_{n+1}^{2l+1}], n \geq 0$$

#### 4.3.2 $i = n - 2l$ 情形下通项公式猜想的证明

$$\text{引理4.3.1. } \sum_{a+b=k} C_a^c C_b^d = C_{k+1}^{c+d+1} [\Rightarrow] \sum_{j=1}^{n-1} C_{n-j}^1 C_{j-1}^{2l} = C_n^{2l+2}, \sum_{j=1}^{n-1} C_{n-j}^1 C_j^{2l+1} = C_{n+1}^{2l+3}$$

以上引理肯定是正确的, 我在书上看见过这个公式, 但还没有看到合适易懂的证明方法。

$$\text{定理4.3.1. } m(s, n; n - 2l) = \frac{1}{2^{2l+1}s} [(2s+1)C_n^{2l} - C_{n+1}^{2l+1}], l \geq 0, n \geq 2$$

证明: 采用数学归纳法证明此定理。

第一步:  $l = 0$ 时成立:

$$m(s, n; n - 0) = \frac{1}{2^1 s} [(2s+1)C_n^0 - C_{n+1}^1]$$

第二步：假设  $l' = l$  时成立：

$$m(s, n; n - 2l) = \frac{1}{2^{2l+1}s} [(2s+1)C_n^{2l} - C_{n+1}^{2l+1}]$$

第三步： $l' = l + 1 \geq 1$  时：

$$\Rightarrow m(s, n; n - 2l - 2) = \frac{1}{4} \sum_{j=1}^{n-1} (n-j)m(s, j-1; j-1-2l)$$

$$= \frac{1}{4} \sum_{j=1}^{n-1} \frac{n-j}{2^{2l+1}s} [(2s+1)C_{j-1}^{2l} - C_j^{2l+1}]$$

$$= \frac{1}{2^{2l+3}s} \sum_{j=1}^{n-1} [(2s+1)C_{n-j}^1 C_{j-1}^{2l} - C_{n-j}^1 C_j^{2l+1}]$$

$$= \frac{1}{2^{2l+3}s} [(2s+1)C_n^{2l+2} - C_{n+1}^{2l+3}]$$

此步证明了  $l' = l + 1$  时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

$$\text{推论4.3.3. } m(s, n; n - 2l) = \frac{1}{2^{2l+1}s} [(2s+1)C_n^{2l} - C_{n+1}^{2l+1}] = \frac{1}{2^{2l+1}s} (2sC_n^{2l} - C_n^{2l+1}) = \frac{C_n^{2l}}{2^{2l}} - \frac{1}{s} \frac{C_n^{2l+1}}{2^{2l+1}}, n \geq 0$$

$$\text{推论4.3.4. } m(s, n; n - 2l) = \frac{2^{n-2l}}{2^n} \frac{1}{2s} (2sC_n^{n-2l} - C_n^{n-2l-1}), n \geq 0$$

$$\text{定理4.3.2. } M^{\{\alpha_1, \dots, \alpha_n\}}(s, n; w) = \sum_{l=0}^{[n/2]} \left( \frac{C_n^{2l}}{2^{2l}} - \frac{1}{s} \frac{C_n^{2l+1}}{2^{2l+1}} \right) \Omega^{n-2l}(s, n; w); n \geq 0, s \geq 1$$

本质上依赖于对应的猜想，需先证明猜想。但对应的猜想应该能被证明的，不存在原则性的困难，只是有点繁琐，有空我再彻底证明它。所以此定理应该是成立的，下面章节就直接应用此定理了。

#### 4.4 $n(s, n; n - 2l)$ 的通项公式

$$\text{定理4.4.1. } n(s, n; n - 2l - 1) = m(s, n - 1; n - 1 - 2l - 1) - m(s, n; n - 2l - 1) = 0, n \geq 1$$

$$\text{引理4.4.1. } n(s, n; n - 2l) = m(s, n - 1; n - 1 - 2l) - m(s, n; n - 2l), n \geq 1$$

$$\text{定理4.4.2. } n(s, n; n - 2l) = -\frac{1}{2^{2l+1}s} [(2s+1)C_{n-1}^{2l-1} - C_n^{2l}], n \geq 1$$

$$\text{证明: } n(s, n; n - 2l) = m(s, n - 1; n - 1 - 2l) - m(s, n; n - 2l), n \geq 1$$

$$= \frac{1}{2^{2l+1}s} [(2s+1)C_{n-1}^{2l} - C_n^{2l+1}] - \frac{1}{2^{2l+1}s} [(2s+1)C_n^{2l} - C_{n+1}^{2l+1}]$$

$$= -\frac{1}{2^{2l+1}s} [(2s+1)C_{n-1}^{2l-1} - C_n^{2l}]$$

□

$$\text{推论4.4.1. } n(s, n; n - 2l) = -\frac{1}{2^{2l+1}s} [(2s+1)C_{n-1}^{2l-1} - C_n^{2l}], n \geq 0$$

$$\text{推论4.4.2. } n(s, n; n - 2l) = -\frac{1}{2^{2l+1}s} (2sC_{n-1}^{2l-1} - C_{n-1}^{2l}) = -\frac{C_{n-1}^{2l-1}}{2^{2l}} + \frac{1}{s} \frac{C_{n-1}^{2l}}{2^{2l+1}}, n \geq 1$$

$$\text{推论4.4.3. } n(s, n; n - 2l) = \frac{2^{n-2l}}{2^n} \frac{1}{2s} (C_{n-1}^{n-2l-1} - 2sC_{n-1}^{n-2l}), n \geq 1$$

推论4.4.4.

$$\left\{ \begin{array}{l} n(s, 0; 0) := \frac{1}{2s} \\ n(s, 1; 1) = \frac{1}{2s} \\ n(s, 2; 0) = -\frac{1}{2^2}, n(s, 2; 2) = \frac{1}{2s} \\ n(s, 3; 1) = \frac{1-4s}{8s}, n(s, 3; 3) = \frac{1}{2s} \\ n(s, 4; 0) = -\frac{1}{2^4}, n(s, 4; 2) = \frac{3-6s}{8s}, n(s, 4; 4) = \frac{1}{2s} \\ n(s, 5; 1) = \frac{1-8s}{32s}, n(s, 5; 3) = \frac{6-8s}{8s}, n(s, 5; 5) = \frac{1}{2s} \\ n(s, 6; 0) = -\frac{1}{2^6}, n(s, 6; 2) = \frac{5-20s}{32s}, n(s, 6; 4) = \frac{10-10s}{8s}, n(s, 6; 6) = \frac{1}{2s} \\ n(s, 7; 1) = -\frac{1-12s}{2^6 \cdot 2s}, n(s, 7; 3) = \frac{15-40s}{32s}, n(s, 7; 5) = \frac{15-12s}{8s}, n(s, 7; 7) = \frac{1}{2s} \end{array} \right.$$

$$\text{定理4.4.3. } N^{\{\alpha_1, \dots, \alpha_n\}}(s, n; w) = -\frac{1}{2} \sum_{l=0}^{[n/2]} \left( \frac{C_{n-1}^{2l-1}}{2^{2l-1}} - \frac{1}{s} \frac{C_{n-1}^{2l}}{2^{2l}} \right) \Omega^{n-2l}(s, n; w), n \geq 1, s \geq 1$$

本质上依赖于对应的猜想，需先证明猜想。但对应的猜想应该能被证明的，不存在原则性的困难，只是有点繁琐，有空我再来彻底证明它。所以此定理应该是成立的，下面章节就直接应用此定理了。对于求解 $m(s, n; i), n(s, n; i)$ 的通项公式，我已思考了近三年，一直没得到彻底解决。终于在2023年8月15日那天灵感突发，一下子就猜测得到了 $m(s, n; i)$ 的通项公式，然后采用数学归纳法严格证明了它。就这样长久坚持了三年，问题终于彻底解决了。但是同样的技巧对求解 $c(s, n; n-2l)$ 的通项公式却毫无帮助、无能为力，仍需努力！

#### 4.5 $\bar{n}(s, n-2k; n-2l)$ 的定义

定义4.5.1.  $\bar{n}(s, n; n-2l) := \frac{2^n 2s}{2^{n-2l}} n(s, n; n-2l) = C_{n-1}^{2l} - 2sC_{n-1}^{2l-1}, n \geq 1$

定义4.5.2.  $\bar{n}(s, n-2k; n-2l) := \frac{2^{n-2k} 2s}{2^{n-2l}} n(s, n-2k; n-2l) = C_{n-1-2k}^{n-1-2l} - 2sC_{n-1-2k}^{n-2l}, n-2k \geq 1$

性质4.5.1.  $\bar{n}(s, n-2k; n-2k) = 1, n-2k \geq 0$

#### 4.6 $c(s, n; n-2l)$ 通项公式的寻找(未获成功)

##### 4.6.1 $c(s, n; n-2l)$ 的递推公式

推论4.6.1.

$$\begin{cases} c(s, n; i) = \sum_{j=i-1}^{n-1} c(s - \frac{1}{2}, n-1; j) n(s, j+1; i), n \geq 1; c(s, 0; 0) = 1 \\ c(s, n; i) = \sum_{j=i}^n c(s - \frac{1}{2}, n-1; j-1) n(s, j; i), n \geq 1; c(s, 0; 0) = 1 \end{cases}$$

推论4.6.2.

$$\begin{cases} c(s, n; i) = \sum_{j=1}^n c(s - \frac{1}{2}, n-1; j-1) n(s, j; i), n \geq 1 \\ c(s, 0; 0) = 1, n(s, n; i < 0 | i > n) := 0, c(s, n; i < 0 | i > n) := 0 \end{cases}$$

推论4.6.3.

$$\begin{cases} c(s, n; n-2l-1) = \sum_{j=1}^n c(s - \frac{1}{2}, n-1; j-1) n(s, j; n-2l-1) = 0, n \geq 1; c(s, 1; 0) = 0 \\ c(s, n; n-2l) = \sum_{j=1}^n c(s - \frac{1}{2}, n-1; j-1) n(s, j; n-2l), n \geq 1; c(s, 0; 0) = 1 \end{cases}$$

定义4.6.1.  $\bar{c}(s, n; n-2l) := \frac{(2s)! 2^{2l}}{(2s-n)!} c(s, n; n-2l), n \geq 0$

##### 4.6.2 $c(s, n; n-2l)$ 按 $s \in \mathbb{C}$ 解析延拓后的递推公式

推论4.6.4.

$$\begin{cases} c(s, n; n-2l-1) = \sum_{j=1}^n c(s - \frac{1}{2}, n-1; j-1) n(s, j; n-2l-1) = 0, c(s, 1; 0) = 0; n \geq 1 \\ c(s, n; n-2l) = \sum_{j=1}^n c(s - \frac{1}{2}, n-1; j-1) n(s, j; n-2l), c(s, 0; 0) = 1; n \geq 1 \end{cases}$$

##### 4.6.3 $c(s, n; n-2l)$ 前几项公式

推论4.6.5.  $c(\frac{1}{2}, 0; 0) = 1, c(\frac{1}{2}, 1; 1) = 1$

推论4.6.6.  $c(1, 1; 1) = \frac{1}{2}, c(1, 2; 2) = \frac{1}{4}, c(1, 2; 0) = -\frac{1}{4}$

推论4.6.7.

$$\begin{cases} c(s, 0; 0) := \frac{(2s-0)!}{(2s)!} \\ c(s, 1; 1) = \frac{(2s-1)!}{(2s)!} \\ c(s, 2; 0) = \frac{(2s-2)!}{(2s)!} \frac{s}{2}, c(s, 2; 2) = \frac{(2s-2)!}{(2s)!} \\ c(s, 3; 1) = \frac{(2s-3)!}{(2s)!} \frac{1-3s}{2}, c(s, 3; 3) = \frac{(2s-3)!}{(2s)!} \\ c(s, 4; 0) = \frac{(2s-4)!}{(2s)!} \frac{3s(s-1)}{4}, c(s, 4; 2) = \frac{(2s-4)!}{(2s)!} (2-3s), c(s, 4; 4) = \frac{(2s-4)!}{(2s)!} \end{cases}$$

## 推论4.6.8.

$$\begin{cases} c(s, 0; 0) = 1 \\ c(s, 1; 1) = \frac{1}{1!C_{2s}^1} \\ c(s, 2; 0) = \frac{1}{2!C_{2s}^2} \frac{C_2^{-1}-sC_2^0}{2}, c(s, 2; 2) = \frac{1}{2!C_{2s}^2} \\ c(s, 3; 1) = \frac{1}{3!C_{2s}^3} \frac{C_3^0-sC_3^1}{2}, c(s, 3; 3) = \frac{1}{3!C_{2s}^3} \\ c(s, 4; 0) = \frac{1}{4!C_{2s}^4} \frac{3s^2-3s}{4}, c(s, 4; 2) = \frac{1}{4!C_{2s}^4} \frac{C_4^{-1}-sC_4^2}{2}, c(s, 4; 4) = \frac{1}{4!C_{2s}^4} \\ c(s, 5; 1) = \frac{1}{5!C_{2s}^5} \frac{(15s^2-25s+6)}{4}, c(s, 5; 3) = \frac{1}{5!C_{2s}^5} \frac{C_5^2-sC_5^3}{2}, c(s, 5; 5) = \frac{1}{5!C_{2s}^5} \end{cases}$$

## 推论4.6.9.

$$\begin{cases} c(s, 0; 0) = 1 \\ c(s, 1; 1) = \frac{1}{1!C_{2s}^1} \\ c(s, 2; 0) = \frac{1}{2!C_{2s}^2} \frac{-(2s)}{4}, c(s, 2; 2) = \frac{1}{2!C_{2s}^2} \\ c(s, 3; 1) = \frac{1}{3!C_{2s}^3} \frac{2-3(2s)}{4}, c(s, 3; 3) = \frac{1}{3!C_{2s}^3} \\ c(s, 4; 0) = \frac{1}{4!C_{2s}^4} \frac{0-6(2s)+3(2s)^2}{16}, c(s, 4; 2) = \frac{1}{4!C_{2s}^4} \frac{8-6(2s)}{4}, c(s, 4; 4) = \frac{1}{4!C_{2s}^4} \\ c(s, 5; 1) = \frac{1}{5!C_{2s}^5} \frac{24-50(2s)+15(2s)^2}{16}, c(s, 5; 3) = \frac{1}{5!C_{2s}^5} \frac{20-10(2s)}{4}, c(s, 5; 5) = \frac{1}{5!C_{2s}^5} \\ c(s, 6; 2) = \frac{1}{6!C_{2s}^6} \frac{184-210(2s)+45(2s)^2}{16}, c(s, 6; 4) = \frac{1}{6!C_{2s}^6} \frac{40-15(2s)}{4}, c(s, 6; 6) = \frac{1}{6!C_{2s}^6} \end{cases}$$

4.6.4  $c(s, n; n-2l)$ 的迭代通项公式一

$$\text{定理4.6.1. } c(s, n; n-2k_0) = \frac{(2s-i)!}{(2s)!} \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_i=0}^{k_{i-1}} 2^{2k_i-2k_0} c(s-\frac{i}{2}, n-i; n-i-2k_i) \bar{n}(s, n-2k_1; n-2k_0) \\ \bar{n}(s-\frac{1}{2}, n-1-2k_2; n-1-2k_1) \bar{n}(s-1, n-2-2k_3; n-2-2k_2) \cdots \bar{n}(s-\frac{i}{2}, n-i-2k_i; n-i-2k_{i-1}); n \geq 1$$

定理4.6.2.  $c(s, n; n-2k_0)$ 

$$\begin{aligned} &= \frac{(2s-n)!}{(2s)!} \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-1}=0}^{k_{n-2}} \sum_{k_n=0}^{k_{n-1}} 2^{2k_n-2k_0} c(s-\frac{n}{2}, 0; 0-2k_n) \bar{n}(s, n-2k_1; n-2k_0) \bar{n}(s-\frac{1}{2}, n-1-2k_2; n-1-2k_1) \\ &\bar{n}(s-1, n-2-2k_3; n-2-2k_2) \cdots \bar{n}(s-\frac{n-2}{2}, 2-2k_{n-1}; 2-2k_{n-2}) \bar{n}(s-\frac{n-1}{2}, 1-2k_n; 1-2k_{n-1}) \\ &= \frac{(2s-n)!}{(2s)! 2^{2k_0}} \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-1}=0}^{k_{n-2}} \bar{n}(s, n-2k_1; n-2k_0) \bar{n}(s-\frac{1}{2}, n-1-2k_2; n-1-2k_1) \\ &\bar{n}(s-1, n-2-2k_3; n-2-2k_2) \cdots \bar{n}(s-\frac{n-2}{2}, 2-2k_{n-1}; 2-2k_{n-2}) \bar{n}(s-\frac{n-1}{2}, 1; 1-2k_{n-1}) \\ &= \frac{(2s-n)!}{(2s)! 2^{2k_0}} \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-3}=0}^{k_{n-2}} \bar{n}(s, n-2k_1; n-2k_0) \bar{n}(s-\frac{1}{2}, n-1-2k_2; n-1-2k_1) \\ &\bar{n}(s-1, n-2-2k_3; n-2-2k_2) \cdots \bar{n}(s-\frac{n-2}{2}, 2; 2-2k_{n-2}); n \geq 1 \end{aligned}$$

推论4.6.10.  $\bar{c}(s, n; n-2k_0) = \frac{(2s)! 2^{2k_0}}{(2s-n)!} c(s, n; n-2k_0)$ 

$$\begin{aligned} &= \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-2}=0}^{k_{n-3}} \bar{n}(s, n-2k_1; n-2k_0) \bar{n}(s-\frac{1}{2}, n-1-2k_2; n-1-2k_1) \bar{n}(s-1, n-2-2k_3; n-2-2k_2) \cdots \\ &\bar{n}(s-\frac{n-2}{2}, 2; 2-2k_{n-2}); n \geq 1 \end{aligned}$$

4.6.5  $c(s, n; n-2l)$ 的迭代通项公式二

$$\text{定理4.6.3. } c(s, n; n-2k_0) = \frac{(2s-i)!}{(2s)!} \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_i=0}^{k_{i-1}} 2^{2k_i-2k_0} c(s-\frac{i}{2}, n-i; n-i-2k_i) (C_{n-1-2k_1}^{n-1-2k_0} - 2sC_{n-1-2k_1}^{n-2k_0}) \\ [C_{n-2-2k_2}^{n-2-2k_1} - (2s-1)C_{n-2-2k_2}^{n-1-2k_1}] [C_{n-3-2k_3}^{n-3-2k_2} - (2s-2)C_{n-3-2k_3}^{n-2-2k_2}] \cdots [C_{n-i-2k_i}^{n-i-2k_{i-1}} - (2s-i+1)C_{n-i-2k_i}^{n-i+1-2k_{i-1}}]; n \geq 1$$

定理4.6.4.  $c(s, n; n-2k_0)$ 

$$= \frac{(2s-n)!}{(2s)!} \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-1}=0}^{k_{n-2}} \sum_{k_n=0}^{k_{n-1}} 2^{2k_n-2k_0} c(s-\frac{n}{2}, 0; 0-2k_n) (C_{n-1-2k_1}^{n-1-2k_0} - 2sC_{n-1-2k_1}^{n-2k_0}) [C_{n-2-2k_2}^{n-2-2k_1} - (2s-1)C_{n-2-2k_2}^{n-1-2k_1}]$$

$$\begin{aligned}
& [C_{n-3-2k_3}^{m-3-2k_2} - (2s-2)C_{n-3-2k_3}^{m-2-2k_2}] \cdot [C_{1-2k_{n-1}}^{1-2k_{n-2}} - (2s-n+2)C_{1-2k_{n-1}}^{2-2k_{n-2}}][C_{0-2k_n}^{0-2k_{n-1}} - (2s-n+1)C_{0-2k_n}^{1-2k_{n-1}}] \\
&= \frac{(2s-n)!}{(2s)!2^{2k_0}} \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-1}=0}^{k_{n-2}} (C_{n-1-2k_1}^{m-1-2k_0} - 2sC_{n-1-2k_1}^{m-2k_0}) [C_{n-2-2k_2}^{m-2-2k_1} - (2s-1)C_{n-2-2k_2}^{m-1-2k_1}] \\
& [C_{n-3-2k_3}^{m-3-2k_2} - (2s-2)C_{n-3-2k_3}^{m-2-2k_2}] \cdot [C_{1-2k_{n-1}}^{1-2k_{n-2}} - (2s-n+2)C_{1-2k_{n-1}}^{2-2k_{n-2}}][C_{0-2k_n}^{0-2k_{n-1}} - (2s-n+1)C_{0-2k_n}^{1-2k_{n-1}}] \\
&= \frac{(2s-n)!}{(2s)!2^{2k_0}} \sum_{k_1=0}^{k_0} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-2}=0}^{k_{n-3}} (C_{n-1-2k_1}^{m-1-2k_0} - 2sC_{n-1-2k_1}^{m-2k_0}) [C_{n-2-2k_2}^{m-2-2k_1} - (2s-1)C_{n-2-2k_2}^{m-1-2k_1}] \\
& [C_{n-3-2k_3}^{m-3-2k_2} - (2s-2)C_{n-3-2k_3}^{m-2-2k_2}] \cdot [C_{1-2k_{n-2}}^{1-2k_{n-2}} - (2s-n+2)C_{1-2k_{n-2}}^{2-2k_{n-2}}]; n \geq 1
\end{aligned}$$

#### 4.6.6 $c(s, n; n), \bar{c}(s, n; n)$ 的通项公式

**定理4.6.5.**  $c(s, n; n) = \frac{(2s-n)!}{(2s)!}, \bar{c}(s, n; n) = 1; n \geq 1$

**证明:**  $n \geq 1$

$$\begin{aligned}
c(s, n; n) &= \frac{(2s-n)!}{(2s)!} \sum_{k_1=0}^0 \sum_{k_2=0}^0 \sum_{k_3=0}^0 \cdots \sum_{k_{n-2}=0}^0 (C_{n-1-2k_1}^{m-1-2k_0} - 2sC_{n-1-2k_1}^{m-2k_0}) [C_{n-2-2k_2}^{m-2-2k_1} - (2s-1)C_{n-2-2k_2}^{m-1-2k_1}] \\
& [C_{n-3-2k_3}^{m-3-2k_2} - (2s-2)C_{n-3-2k_3}^{m-2-2k_2}] \cdot [C_{1-2k_{n-2}}^{1-2k_{n-2}} - (2s-n+2)C_{1-2k_{n-2}}^{2-2k_{n-2}}] \\
&= \frac{(2s-n)!}{(2s)!} (C_{n-1}^{m-1} - 2sC_{n-1}^m) [C_{n-2}^{m-2-2k_1} - (2s-1)C_{n-2}^{m-1}] [C_{n-3}^{m-3} - (2s-2)C_{n-3}^{m-2}] \cdots [C_1^1 - (2s-n+2)C_1^2] \\
&= \frac{(2s-n)!}{(2s)!}
\end{aligned}$$

□

**推论4.6.11.**  $c(s, n; n) = \frac{(2s-n)!}{(2s)!}, \bar{c}(s, n; n) = 1; n \geq 0$

#### 4.6.7 $c(s, n; n-2), \bar{c}(s, n; n-2)$ 的通项公式

**引理4.6.1.**

$$\begin{cases} C_{n-1}^{p+1} + C_{n-1}^p = C_n^{p+1}, C_{n-1}^p + C_{n-2}^p + \cdots + C_p^p = C_n^{p+1} \\ pC_{n-2}^p + (p+1)C_{n-3}^p + \cdots + (n-2)C_p^p = C_n^{p+2} + (p-1)C_{n-1}^{p+1} = C_n^{p+2} + pC_{n-1}^{p+1} \end{cases}$$

**定理4.6.6.**  $c(s, n; n-2) = \frac{(2s-n)!}{(2s)!2} (C_n^3 - sC_n^2), \bar{c}(s, n; n-2) = 2C_n^3 - 2sC_n^2; n \geq 1$

**证明:**  $n \geq 1$

$$\begin{aligned}
c(s, n; n-2) &= \frac{2^{-2}(2s-n)!}{(2s)!} \sum_{k_1=0}^1 \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-2}=0}^{k_{n-3}} (C_{n-1-2k_1}^{m-1-2} - 2sC_{n-1-2k_1}^{m-2}) [C_{n-2-2k_2}^{m-2-2k_1} - (2s-1)C_{n-2-2k_2}^{m-1-2k_1}] \\
& [C_{n-3-2k_3}^{m-3-2k_2} - (2s-2)C_{n-3-2k_3}^{m-2-2k_2}] \cdot [C_{1-2k_{n-2}}^{1-2k_{n-2}} - (2s-n+2)C_{1-2k_{n-2}}^{2-2k_{n-2}}] \\
&= \frac{2^{-2}(2s-n)!}{(2s)!} \left\{ \sum_{k_1=0}^0 \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-2}=0}^{k_{n-3}} (C_{n-1-2k_1}^{m-1-2} - 2sC_{n-1-2k_1}^{m-2}) [C_{n-2-2k_2}^{m-2-2k_1} - (2s-1)C_{n-2-2k_2}^{m-1-2k_1}] \right. \\
& [C_{n-3-2k_3}^{m-3-2k_2} - (2s-2)C_{n-3-2k_3}^{m-2-2k_2}] \cdot [C_{1-2k_{n-2}}^{1-2k_{n-2}} - (2s-n+2)C_{1-2k_{n-2}}^{2-2k_{n-2}}] \\
& \left. + \sum_{k_1=1}^1 \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-2}=0}^{k_{n-3}} (C_{n-1-2k_1}^{m-1-2} - 2sC_{n-1-2k_1}^{m-2}) [C_{n-2-2k_2}^{m-2-2k_1} - (2s-1)C_{n-2-2k_2}^{m-1-2k_1}] \right. \\
& [C_{n-3-2k_3}^{m-3-2k_2} - (2s-2)C_{n-3-2k_3}^{m-2-2k_2}] \cdot [C_{1-2k_{n-2}}^{1-2k_{n-2}} - (2s-n+2)C_{1-2k_{n-2}}^{2-2k_{n-2}}] \left. \right\} \\
&= \frac{2^{-2}(2s-n)!}{(2s)!} \left\{ (C_{n-1}^{m-3} - 2sC_{n-1}^{m-2}) + \sum_{k_1=1}^1 \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-2}=0}^{k_{n-3}} (C_{n-1-2k_1}^{m-1-2} - 2sC_{n-1-2k_1}^{m-2}) [C_{n-2-2k_2}^{m-2-2k_1} - (2s-1)C_{n-2-2k_2}^{m-1-2k_1}] \right. \\
& [C_{n-3-2k_3}^{m-3-2k_2} - (2s-2)C_{n-3-2k_3}^{m-2-2k_2}] \cdot [C_{1-2k_{n-2}}^{1-2k_{n-2}} - (2s-n+2)C_{1-2k_{n-2}}^{2-2k_{n-2}}] \left. \right\} \\
&= \frac{2^{-2}(2s-n)!}{(2s)!} \left\{ (C_{n-1}^{m-3} - 2sC_{n-1}^{m-2}) + \sum_{k_2=0}^1 \sum_{k_3=0}^{k_2} \cdots \sum_{k_{n-2}=0}^{k_{n-3}} [C_{n-2-2k_2}^{m-2-2} - (2s-1)C_{n-2-2k_2}^{m-1-2}] [C_{n-3-2k_3}^{m-3-2k_2} - (2s-2)C_{n-3-2k_3}^{m-2-2k_2}] \right. \\
& \left. \cdots [C_{2-2k_{n-3}}^{2-2k_{n-3}} - (2s-n+3)C_{2-2k_{n-3}}^3] [C_{1-2k_{n-2}}^{1-2k_{n-2}} - (2s-n+2)C_{1-2k_{n-2}}^{2-2k_{n-2}}] \right\} \\
&= \frac{2^{-2}(2s-n)!}{(2s)!} \left\{ (C_{n-1}^{m-3} - 2sC_{n-1}^{m-2}) + [C_{n-2}^{m-4} - (2s-1)C_{n-2}^{m-3}] + [C_{n-3}^{m-5} - (2s-2)C_{n-3}^{m-4}] + \cdots \right. \\
& \left. + [C_2^0 - (2s-n+3)C_2^1] + [C_1^{-1} - (2s-n+2)C_1^0] \right\} \\
&= \frac{2^{-2}(2s-n)!}{(2s)!} \left\{ (C_{n-1}^2 - 2sC_{n-1}^1) + [C_{n-2}^2 - (2s-1)C_{n-2}^1] + [C_{n-3}^2 - (2s-2)C_{n-3}^1] + \cdots \right. \\
& \left. + [C_2^2 - (2s-n+3)C_2^1] + [C_1^2 - (2s-n+2)C_1^1] \right\} \\
&= \frac{2^{-2}(2s-n)!}{(2s)!} \left\{ (C_{n-1}^2 + C_{n-2}^2 + \cdots + C_2^2) - 2s(C_{n-1}^1 + C_{n-2}^1 + \cdots + C_1^1) + [1C_{n-2}^1 + 2C_{n-3}^1 + \cdots + (n-2)C_1^1] \right\} \\
&= \frac{2^{-2}(2s-n)!}{(2s)!} (C_n^3 - 2sC_n^2 + C_n^3) = \frac{(2s-n)!}{(2s)!2} (C_n^3 - sC_n^2)
\end{aligned}$$

□



推论4.6.12.  $c(s, n; n-2) = \frac{(2s-n)!}{(2s)!2} (C_n^3 - sC_n^2)$ ,  $\bar{c}(s, n; n-2) = 2C_n^3 - 2sC_n^2; n \geq 0$

#### 4.6.8 $\bar{c}(s, n; n-2l)$ 的递推通项公式

推论4.6.13.  $c(s, n; n-2l) = \sum_{j=1}^n c(s - \frac{1}{2}, n-1; j-1)n(s, j; n-2l), n \geq 1; c(s, 0; 0) = 1$

$\Leftrightarrow \bar{c}(s, n; n-2l) = \sum_{j=1}^n \bar{c}(s - \frac{1}{2}, n-1; j-1)\bar{n}(s, j; n-2l), n \geq 1; \bar{c}(s, 0; 0) = 1, \bar{n}(s, 0; 0) = 1$

证明:  $c(s, n; n-2l) = \sum_{j=1}^n c(s - \frac{1}{2}, n-1; j-1)n(s, j; n-2l), n \geq 1; c(s, 0; 0) = 1$

$\Leftrightarrow \frac{(2s-n)!}{(2s)!2^{2l}} \bar{c}(s, n; n-2l) = \sum_{j=1}^n \frac{(2s-n)!}{(2s-1)!2^{n-j}} \bar{c}(s - \frac{1}{2}, n-1; j-1) \frac{2^{n-2l}}{2^j 2^s} \bar{n}(s, j; n-2l), n \geq 1; c(s, 0; 0) = 1$

$\Leftrightarrow \bar{c}(s, n; n-2l) = \sum_{j=1}^n \bar{c}(s - \frac{1}{2}, n-1; j-1)\bar{n}(s, j; n-2l), n \geq 1; \bar{c}(s, 0; 0) = 1, \bar{n}(s, 0; 0) = 1$  □

定理4.6.7.  $\bar{c}(s, n; n-2l) = \bar{c}(s - \frac{n-2l}{2}, 2l; 0) + \sum_{i=2l+1}^n \sum_{k=0}^{l-1} \bar{c}(s - \frac{n-i+1}{2}, i-1; i-1-2k)\bar{n}(s - \frac{n-i}{2}, i-2k; i-2l)$   
 $n \geq 1, l \geq 0; \bar{c}(s, 0; 0) = 1, \bar{n}(s, 0; 0) = 1$

证明:  $n \geq 1, l \geq 0; \bar{c}(s, 0; 0) = 1, \bar{n}(s, 0; 0) = 1$

$\bar{c}(s, n; n-2l) = \sum_{j=n-2l}^n \bar{c}(s - \frac{1}{2}, n-1; j-1)\bar{n}(s, j; n-2l)$

$= \sum_{n-2k=n-2l}^n \bar{c}(s - \frac{1}{2}, n-1; n-1-2k)\bar{n}(s, n-2k; n-2l)$

$= \sum_{k=0}^l \bar{c}(s - \frac{1}{2}, n-1; n-1-2k)\bar{n}(s, n-2k; n-2l)$

$\Leftrightarrow \bar{c}(s, n; n-2l) - \bar{c}(s - \frac{1}{2}, n-1; n-1-2l) = \sum_{k=0}^{l-1} \bar{c}(s - \frac{1}{2}, n-1; n-1-2k)\bar{n}(s, n-2k; n-2l)$

$\Leftrightarrow \sum_{i=2l+1}^n [\bar{c}(s - \frac{n-i}{2}, i; i-2l) - \bar{c}(s - \frac{n-i+1}{2}, i-1; i-1-2l)] = \sum_{i=2l+1}^n \sum_{k=0}^{l-1} \bar{c}(s - \frac{n-i+1}{2}, i-1; i-1-2k)\bar{n}(s - \frac{n-i}{2}, i-2k; i-2l)$

$\Leftrightarrow \bar{c}(s, n; n-2l) = \bar{c}(s - \frac{n-2l}{2}, 2l; 0) + \sum_{i=2l+1}^n \sum_{k=0}^{l-1} \bar{c}(s - \frac{n-i+1}{2}, i-1; i-1-2k)\bar{n}(s - \frac{n-i}{2}, i-2k; i-2l)$  □

推论4.6.14.  $\bar{c}(s, n; n-2l) = \sum_{i=2l}^n \sum_{k=0}^{l-1} \bar{c}(s - \frac{n-i+1}{2}, i-1; i-1-2k)\bar{n}(s - \frac{n-i}{2}, i-2k; i-2l)$

$n \geq 1, l \geq 1; \bar{c}(s, 0; 0) = 1, \bar{n}(s, 0; 0) = 1$

定理4.6.8.  $\bar{c}(s - \frac{n-2l}{2}, 2l; 0) = \sum_{j=1}^{2l} \bar{c}(s - \frac{n-2l+1}{2}, 2l-1; j-1)\bar{n}(s - \frac{n-2l}{2}, j; 0), 2l \geq 1$

证明:  $2l \geq 1$

$\bar{c}(s - \frac{n-2l}{2}, 2l; 0) = \sum_{j=1}^{2l} \bar{c}(s - \frac{n-2l+1}{2}, 2l-1; j-1)\bar{n}(s - \frac{n-2l}{2}, j; 0)$

$= - \sum_{j=1}^{2l} (2s-n+2l)\bar{c}(s - \frac{n-2l+1}{2}, 2l-1; j-1)$

$= -(2s-n+2l) \sum_{j=1}^{2l} \bar{c}(s - \frac{n-2l+1}{2}, 2l-1; j-1)$

$= -(2s-n+2l) \sum_{k=1}^l \bar{c}(s - \frac{n-2l+1}{2}, 2l-1; 2k-1)$  □

定理4.6.9.  $\bar{c}(s, 2l; 0) = \sum_{j=1}^{2l} \bar{c}(s - \frac{1}{2}, 2l-1; j-1)\bar{n}(s, j; 0) = -2s \sum_{k=1}^l \bar{c}(s - \frac{1}{2}, 2l-1; 2k-1), l \geq 1$

推论4.6.15.  $\bar{c}(s, 4; 0) = -2s \sum_{k=1}^2 \bar{c}(s - \frac{1}{2}, 3; 2k-1) = -2s[\bar{c}(s - \frac{1}{2}, 3; 3) + \bar{c}(s - \frac{1}{2}, 3; 1)] = -2s[1 + 2C_3^3 - (2s-1)C_3^2]$

4.6.9  $c(s, n; n-4)$ 的通项公式

引理4.6.2.

$$\begin{cases} C_{n-1}^{p+1} + C_{n-1}^p = C_n^{p+1}, C_{n-1}^p + C_{n-2}^p + \cdots + C_p^p = C_n^{p+1} \\ pC_{n-2}^p + (p+1)C_{n-3}^p + \cdots + (n-2)C_p^p = C_n^{p+2} + (p-1)C_{n-1}^{p+1} = C_{n-1}^{p+2} + pC_{n-1}^{p+1} \end{cases}$$

引理4.6.3.  $C_{i-1}^2 C_{i-3}^1 = 3C_{i-1}^3, C_{i-1}^2 C_{i-3}^2 = 6C_{i-1}^4$

引理4.6.4.  $\bar{c}(s, n; n-2) = 2C_n^3 - 2sC_n^2, n \geq 0$

定理4.6.10.  $\bar{c}(s, n; n-4) = (2s)^2 c_3^2 C_n^4 - (2s)^1 2(c_3^2 C_n^4 + c_5^2 C_n^5) + 4(c_4^2 C_n^5 + c_5^2 C_n^6), n \geq 1$

定理4.6.11.  $\bar{c}(s, n; n-4) = (2s)^2 3C_n^4 - (2s)^1 (6C_n^4 + 20C_n^5) + (24C_n^5 + 40C_n^6), n \geq 1$

推论4.6.16.  $c(s, n; n) = \frac{(2s-n)!}{(2s)!}, \bar{c}(s, n; n) = C_n^0; n \geq 0$

推论4.6.17.  $c(s, n; n-2) = \frac{(2s-n)!}{(2s)! 2} (C_n^3 - sC_n^2), \bar{c}(s, n; n-2) = 2C_n^3 - 2sC_n^2; n \geq 0$

定理4.6.12.  $\bar{c}(s, n; n-4) = 12s(s-1)C_n^4 + 4(-10s+6)C_n^5 + 40C_n^6, n \geq 1$

证明:  $n \geq 1$ 

$$\begin{aligned} \bar{c}(s, n; n-4) &= \sum_{i=4}^n \sum_{k=0}^1 \bar{c}(s - \frac{n-i+1}{2}, i-1; i-1-2k) \bar{n}(s - \frac{n-i}{2}, i-2k; i-4) \\ &= \sum_{i=4}^n [\bar{c}(s - \frac{n-i+1}{2}, i-1; i-1) \bar{n}(s - \frac{n-i}{2}, i; i-4) + \bar{c}(s - \frac{n-i+1}{2}, i-1; i-3) \bar{n}(s - \frac{n-i}{2}, i-2; i-4)] \\ &= \sum_{i=4}^n \{ [C_{i-1}^4 - (2s-n+i)C_{i-1}^3] + [2C_{i-1}^3 - (2s-1-n+i)C_{i-1}^2] [C_{i-3}^2 - (2s-n+i)C_{i-3}^1] \} \\ &= \sum_{i=4}^n (2s-1-n+i)(2s-n+i)C_{i-1}^2 C_{i-3}^1 - (2s-n+i)(C_{i-1}^3 + 2C_{i-1}^3 C_{i-3}^1 + C_{i-1}^2 C_{i-3}^2) + (C_{i-1}^4 + 2C_{i-1}^3 C_{i-3}^2 + C_{i-1}^2 C_{i-3}^2) \\ &= \sum_{i=4}^n (2s-1-n+i)(2s-n+i)3C_{i-1}^3 - (2s-n+i)(C_{i-1}^3 + 2C_{i-1}^3 C_{i-3}^1 + 6C_{i-1}^4) + (7C_{i-1}^4 + 4C_{i-1}^4 C_{i-3}^1) \\ &= (2s)^2 \sum_{i=4}^n 3C_{i-1}^3 - (2s)^1 \sum_{i=4}^n [2(n-i)3C_{i-1}^3 + (4C_{i-1}^3 + 2C_{i-1}^3 C_{i-3}^1 + 6C_{i-1}^4)] \\ &+ \sum_{i=4}^n [3C_{i-1}^3 (n+1-i)(n-i)3C_{i-1}^3 + (n-i)(C_{i-1}^3 + 2C_{i-1}^3 C_{i-3}^1 + 6C_{i-1}^4) + (7C_{i-1}^4 + 4C_{i-1}^4 C_{i-3}^1)] \\ &= (2s)^2 3C_n^4 - (2s)^1 \sum_{i=4}^n [6(n-i)C_{i-1}^3 + (4C_{i-1}^3 + 2C_{i-1}^3 C_{i-3}^1 + 6C_{i-1}^4)] \\ &+ \sum_{i=4}^n [(n+1-i)(n-i)3C_{i-1}^3 + (n-i)(C_{i-1}^3 + 2C_{i-1}^3 C_{i-3}^1 + 6C_{i-1}^4) + (7C_{i-1}^4 + 4C_{i-1}^4 C_{i-3}^1)] \\ &= (2s)^2 3C_n^4 - (2s)^1 \sum_{i=4}^n [6(n-i)C_{i-1}^3 + (4C_{i-1}^3 + 2C_{i-1}^3 (i-n+n-3) + 6C_{i-1}^4)] \\ &+ \sum_{i=4}^n [(n+1-i)(n-i)3C_{i-1}^3 + (n-i)(C_{i-1}^3 + 2C_{i-1}^3 (i-n-1+n+1-3) + 6C_{i-1}^4) + (7C_{i-1}^4 + 4C_{i-1}^4 (i-n+n-3))] \\ &= (2s)^2 3C_n^4 - (2s)^1 \sum_{i=4}^n [4(n-i)C_{i-1}^3 + (4C_{i-1}^3 + 2(n-3)C_{i-1}^3 + 6C_{i-1}^4)] \\ &+ \sum_{i=4}^n [(n+1-i)(n-i)C_{i-1}^3 + (n-i)(C_{i-1}^3 + 2(n-2)C_{i-1}^3 + 2C_{i-1}^4) + 7C_{i-1}^4 + 4(n-3)C_{i-1}^4] \\ &= (2s)^2 3C_n^4 - (2s)^1 \sum_{i=4}^n [4C_{n-i}^1 C_{i-1}^3 + (2n-2)C_{i-1}^3 + 6C_{i-1}^4] \\ &+ \sum_{i=4}^n \{ C_{n+1-i}^2 2C_{i-1}^3 + C_{n-i}^1 [(2n-3)C_{i-1}^3 + 2C_{i-1}^4] + (4n-5)C_{i-1}^4 \} \\ &= (2s)^2 3C_n^4 - (2s)^1 [4C_n^5 + (2n-2)C_n^4 + 6C_n^5] + \{ 2C_{n+1}^6 + [(2n-3)C_n^5 + 2C_n^6] + (4n-5)C_n^5 \} \\ &= (2s)^2 3C_n^4 - (2s)^1 [10C_n^5 + (2n-2)C_n^4] + [2C_{n+1}^6 + 2C_n^6 + (6n-8)C_n^5] \\ &= (2s)^2 3C_n^4 - (2s)^1 (6C_n^4 + 20C_n^5) + (24C_n^5 + 40C_n^6) \\ &= 12s(s-1)C_n^4 + 4(-10s+6)C_n^5 + 40C_n^6, n \geq 1 \end{aligned}$$

□

推论4.6.18.  $\bar{c}(s, n; n-4) = 12s(s-1)C_n^4 + 4(-10s+6)C_n^5 + 40C_n^6, n \geq 0$

4.6.10  $c(s, n; n-6)$ 的通项公式(越来越复杂)

定义4.6.2.  $\bar{n}(s, n; n-2l) = [C_{n-1}^{2l} - 2sC_{n-1}^{2l-1}], n \geq 1$

定理4.6.13.  $\bar{c}(s, n; n-6) = ?C_n^6 + ?C_n^7 + ?C_n^8 + ?C_n^9, n \geq 1$

证明:  $n \geq 1$

$$\begin{aligned}
\bar{c}(s, n; n-6) &= \sum_{i=6}^n \sum_{k=0}^2 \bar{c}(s - \frac{n-i+1}{2}, i-1; i-1-2k) \bar{n}(s - \frac{n-i}{2}, i-2k; i-6) \\
&= \sum_{i=6}^n [\bar{c}(s - \frac{n-i+1}{2}, i-1; i-1) \bar{n}(s - \frac{n-i}{2}, i; i-6) \\
&+ \bar{c}(s - \frac{n-i+1}{2}, i-1; i-3) \bar{n}(s - \frac{n-i}{2}, i-2; i-6) \\
&+ \bar{c}(s - \frac{n-i+1}{2}, i-1; i-5) \bar{n}(s - \frac{n-i}{2}, i-4; i-6)] \\
&= \sum_{i=6}^n \{ [C_{i-1}^6 - (2s-n+i)C_{i-1}^5] \\
&+ [2C_{i-1}^3 - (2s-1-n+i)C_{i-1}^2] [C_{i-3}^4 - (2s-n+i)C_{i-3}^3] \\
&+ [(2s-1-n+i)^2 3C_{i-1}^4 - (2s-1-n+i)(6C_{i-1}^4 + 20C_{i-1}^5) + (24C_{i-1}^5 + 40C_{i-1}^6)] [C_{i-5}^2 - (2s-n+i)C_{i-5}^1] \} \\
&= \sum_{i=6}^n \{ [-5C_{i-1}^6 - (2s-n+6)C_{i-1}^5] \\
&+ [-C_{i-1}^3 - (2s-1-n+3)C_{i-1}^2] [-3C_{i-3}^4 - (2s-n+4)C_{i-3}^3] \\
&+ \{3(2s-1-n+i)[5C_{i-1}^5 + (2s-1-n+5)C_{i-1}^4] - [6(2s-1-n+5)C_{i-1}^4 + 30C_{i-1}^5 + 20(2s-1-n+6)C_{i-1}^5 + 120C_{i-1}^6] + (24C_{i-1}^5 + 40C_{i-1}^6) \} [-C_{i-5}^2 - (2s-n+2)C_{i-5}^1] \} \\
&= \sum_{i=6}^n \{ [-5C_{i-1}^6 + (2s-n+6)C_{i-1}^5] \\
&+ [C_{i-1}^3 + (2s-n+2)C_{i-1}^2] [3C_{i-3}^4 + (2s-n+4)C_{i-3}^3] \\
&+ \{15(2s-1-n+6)C_{i-1}^5 + 90C_{i-1}^6 + 3(2s-n+4)(2s-1-n+5)C_{i-1}^4 + 15(2s-n+4)C_{i-1}^5\} \\
&- [6(2s-n+4)C_{i-1}^4 + 30C_{i-1}^5 + 20(2s-n+5)C_{i-1}^5 + 120C_{i-1}^6] + (24C_{i-1}^5 + 40C_{i-1}^6) \} [-C_{i-5}^2 - (2s-n+2)C_{i-5}^1] \} \\
&= \sum_{i=6}^n \{ [-5C_{i-1}^6 + (2s-n+6)C_{i-1}^5] \\
&+ [C_{i-1}^3 + (2s-n+2)C_{i-1}^2] [3C_{i-3}^4 + (2s-n+4)C_{i-3}^3] \\
&+ \{15(2s-n+5)C_{i-1}^5 + 90C_{i-1}^6 + 3(2s-n+4)(2s-n+4)C_{i-1}^4 + 15(2s-n+4)C_{i-1}^5\} \\
&- [6(2s-n+4)C_{i-1}^4 + 30C_{i-1}^5 + 20(2s-n+5)C_{i-1}^5 + 120C_{i-1}^6] + (24C_{i-1}^5 + 40C_{i-1}^6) \} [-C_{i-5}^2 - (2s-n+2)C_{i-5}^1] \} \\
&= \sum_{i=6}^n \{ [-5C_{i-1}^6 + (2s-n+6)C_{i-1}^5] \\
&+ [C_{i-1}^3 + (2s-n+2)C_{i-1}^2] [3C_{i-3}^4 + (2s-n+4)C_{i-3}^3] \\
&+ \{3(2s-n+3)^2 C_{i-1}^4 + 10(2s-n+3)C_{i-1}^5 + (-3C_{i-1}^4 - C_{i-1}^5 + 10C_{i-1}^6) \} [-C_{i-5}^2 - (2s-n+2)C_{i-5}^1] \} \quad \square
\end{aligned}$$

引理4.6.5.

$$\begin{cases} C_{n-1}^{p+1} + C_{n-1}^p = C_n^{p+1}, C_{n-1}^p + C_{n-2}^p + \dots + C_p^p = C_n^{p+1} \\ pC_{n-2}^p + (p+1)C_{n-3}^p + \dots + (n-2)C_p^p = C_n^{p+2} + (p-1)C_{n-1}^{p+1} = C_{n-1}^{p+2} + pC_{n-1}^{p+1} \end{cases}$$

## 4.7 展开系数的线性代数解法

本节线性代数解法体现了数学的全息原理, 只需通过求解一个投影方向, 便可以得到整个空间的解, 即一个投影方向就包含了整个空间的信息, 体现了全息原理。

4.7.1  $M^{\{\alpha_1, \dots, \alpha_n\}}(s, n; w)$ 展开系数的线性代数解法

由于 $m(s, n; i)$ 某些情况下与 $w$ 独立无关, 对某些 $w$ 代数同构, 具有同样的展开系数。也只需求任何一个 $w$ 的展开系数即可, 哪个方便用哪个, 这里取 $w = 1$ 。

$$\text{定理4.7.1. } 2s \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \dots & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \dots & (s-1)^{n-2[n/2]} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^n & (1-s)^{n-2} & (1-s)^{n-4} & \dots & (1-s)^{n-2[n/2]} \\ (-s)^n & (-s)^{n-2} & (-s)^{n-4} & \dots & (-s)^{n-2[n/2]} \end{bmatrix} \begin{bmatrix} m(s, n; n) \\ m(s, n; n-2) \\ m(s, n; n-4) \\ \dots \\ m(s, n; n-2[n/2]) \end{bmatrix} = \begin{bmatrix} 2s(s-1/2)^n \\ (2s-1)(s-3/2)^n + 1(s-1/2)^n \\ (2s-2)(s-5/2)^n + 2(s-3/2)^n \\ \dots \\ 2s(1/2-s)^n \end{bmatrix}$$

$$\begin{aligned}
& \text{证明: } \frac{1}{n!} M^{\{z_1 \zeta \cdots z_n \zeta\}}(s, n) = N^{A_\zeta}(s) \sigma_{z_\zeta}^n (s - \frac{1}{2}) \bar{N}_{A_\zeta}(s) = \sum_{k=0}^{[n/2]} m(s, n; n-2k) \sigma_{z_\zeta}^{n-2k}(s) \\
& \Leftrightarrow N^{A_\zeta}(s) \begin{bmatrix} (s-1/2)^n & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^n & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^n & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^n \end{bmatrix} \bar{N}_{A_\zeta}(s) = \sum_{k=0}^{[n/2]} m(s, n; n-2k) \begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix} \\
& \Leftrightarrow \frac{1}{2s} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} \end{bmatrix} \begin{bmatrix} (s-1/2)^n & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^n & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^n & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^n \end{bmatrix} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} \end{bmatrix} \\
& + \frac{1}{2s} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2s-1} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix} \begin{bmatrix} (s-1/2)^n & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^n & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^n & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^n \end{bmatrix} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s-1} \\ 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix} \\
& = \sum_{k=0}^{[n/2]} m(s, n; n-2k) \begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix} \\
& \Leftrightarrow \frac{1}{2s} \begin{bmatrix} 2s(s-1/2)^n & 0 & 0 & 0 & 0 \\ 0 & (2s-1)(s-3/2)^n & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 2(3/2-s)^n & 0 \\ 0 & 0 & 0 & 0 & 1(1/2-s)^n \end{bmatrix} + \frac{1}{2s} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1(s-1/2)^n & 0 & 0 & 0 \\ 0 & 0 & 2(s-3/2)^n & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & (2s-1)(3/2-s)^n \\ 0 & 0 & 0 & 0 & 2s(1/2-s)^n \end{bmatrix} \\
& = \sum_{k=0}^{[n/2]} m(s, n; n-2k) \begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix} \\
& \Leftrightarrow \begin{cases} 2s \sum_{k=0}^{[n/2]} m(s, n; n-2k) s^{n-2k} = 2s(s-1/2)^n \\ 2s \sum_{k=0}^{[n/2]} m(s, n; n-2k) (s-1)^{n-2k} = (2s-1)(s-3/2)^n + 1(s-1/2)^n \\ \dots \\ 2s \sum_{k=0}^{[n/2]} m(s, n; n-2k) (1-s)^{n-2k} = 1(1/2-s)^n + (2s-1)(3/2-s)^n \\ 2s \sum_{k=0}^{[n/2]} m(s, n; n-2k) (-s)^{n-2k} = 2s(1/2-s)^n \end{cases} \\
& \Leftrightarrow 2s \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdots & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdots & (s-1)^{n-2[n/2]} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (1-s)^n & (1-s)^{n-2} & (1-s)^{n-4} & \cdots & (1-s)^{n-2[n/2]} \\ (-s)^n & (-s)^{n-2} & (-s)^{n-4} & \cdots & (-s)^{n-2[n/2]} \end{bmatrix} \begin{bmatrix} m(s, n; n) \\ m(s, n; n-2) \\ \cdots \\ m(s, n; n-2[n/2]) \end{bmatrix} = \begin{bmatrix} 2s(s-1/2)^n \\ (2s-1)(s-3/2)^n + 1(s-1/2)^n \\ \cdots \\ (2s-2)(s-5/2)^n + 2(s-3/2)^n \\ 2s(1/2-s)^n \end{bmatrix} \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{推论4.7.1. } \begin{bmatrix} m(s, n; n) \\ m(s, n; n-2) \\ m(s, n; n-4) \\ \cdots \\ m(s, n; n-2[n/2]) \end{bmatrix} \\
& = \frac{1}{2s} \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdots & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdots & (s-1)^{n-2[n/2]} \\ (s-2)^n & (s-2)^{n-2} & (s-2)^{n-4} & \cdots & (s-2)^{n-2[n/2]} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (s-[n/2])^n & (s-[n/2])^{n-2} & (s-[n/2])^{n-4} & \cdots & (s-[n/2])^{n-2[n/2]} \end{bmatrix}^{-1} \begin{bmatrix} 2s(s-1/2)^n \\ (2s-1)(s-3/2)^n + 1(s-1/2)^n \\ (2s-2)(s-5/2)^n + 2(s-3/2)^n \\ \cdots \\ (2s-[n/2])(s-1/2-[n/2])^n + [n/2](s+1/2-[n/2])^n \end{bmatrix}
\end{aligned}$$

$$\text{推论4.7.2. } 2s \sum_{k=0}^{[n/2]} m(s, n; n-2k) (s-h)^{n-2k} = (2s-h)(s-h-\frac{1}{2})^n + h(s-h+\frac{1}{2})^n, 0 \leq h \leq 2s, n \geq 0, s \geq 1$$

$$\text{推论4.7.3. } 2s \sum_{k=0}^{[n/2]} \left( \frac{C_n^{2k}}{2^{2k}} - \frac{1}{s} \frac{C_n^{2k+1}}{2^{2k+1}} \right) (s-h)^{n-2k} = (2s-h)(s-h-\frac{1}{2})^n + h(s-h+\frac{1}{2})^n, 0 \leq h \leq 2s, n \geq 0, s \geq 1$$

$$\text{推论4.7.4. } 2s \sum_{k=0}^{[n/2]} m(s, n; n-2k) h^{n-2k} = (s+h)(h-\frac{1}{2})^n + (s-h)(h+\frac{1}{2})^n, -s \leq h \leq s, n \geq 0, s \geq 1$$

$$\text{推论4.7.5. } 2s \sum_{k=0}^{[n/2]} \left( \frac{C_n^{2k}}{2^{2k}} - \frac{1}{s} \frac{C_n^{2k+1}}{2^{2k+1}} \right) h^{n-2k} = (s+h)(h-\frac{1}{2})^n + (s-h)(h+\frac{1}{2})^n, -s \leq h \leq s, n \geq 0, s \geq 1$$

### 4.7.2 $N^{\{\alpha_{1\zeta} \cdots \alpha_{n\zeta}\}}(s, n; w)$ 展开系数的线性代数解法

由于  $n(s, n; i)$  某些情况下与  $w$  独立无关, 对某些  $w$  代数同构, 具有同样的展开系数。也只需求任何一个  $w$  的展开系数即可, 哪个方便用哪个, 这里取  $w = 1$ 。

$$\text{定理4.7.2. } 4s \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdots & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdots & (s-1)^{n-2[n/2]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-s)^n & (1-s)^{n-2} & (1-s)^{n-4} & \cdots & (1-s)^{n-2[n/2]} \\ (-s)^n & (-s)^{n-2} & (-s)^{n-4} & \cdots & (-s)^{n-2[n/2]} \end{bmatrix} \begin{bmatrix} n(s, n; n) \\ n(s, n; n-2) \\ n(s, n; n-4) \\ \vdots \\ n(s, n; n-2[n/2]) \end{bmatrix} = \begin{bmatrix} 2s(s-1/2)^{n-1} \\ (2s-1)(s-3/2)^{n-1} - 1(s-1/2)^{n-1} \\ (2s-2)(s-5/2)^{n-1} - 2(s-3/2)^{n-1} \\ \vdots \\ -2s(1/2-s)^{n-1} \end{bmatrix}$$

$$\text{证明: } \frac{1}{n!} N^{\{z_{1\zeta} \cdots z_{n\zeta}\}}(s, n) = N^{A_\zeta}(s) \sigma^{z_\zeta} A_\zeta^{B_\zeta} \left(\frac{1}{2}\right) \sigma_{z_\zeta}^{n-1} \left(s - \frac{1}{2}\right) \bar{N}_{A_\zeta}(s) = \sum_{k=0}^{[n/2]} n(s, n; n-2k) \sigma_{z_\zeta}^{n-2k}(s), n \geq 1$$

$$\begin{aligned} &\Leftrightarrow N^{A_\zeta}(s) \sigma^{z_\zeta} A_\zeta^{B_\zeta} \begin{bmatrix} (s-1/2)^{n-1} & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^{n-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^{n-1} & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^{n-1} \end{bmatrix} \bar{N}_{A_\zeta}(s) \\ &= \sum_{k=0}^{[n/2]} n(s, n; n-2k) \begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix} \\ &\Leftrightarrow \frac{1}{4s} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} \end{bmatrix} \begin{bmatrix} (s-1/2)^{n-1} & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^{n-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^{n-1} & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^{n-1} \end{bmatrix} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1} \end{bmatrix} \\ &- \frac{1}{4s} \begin{bmatrix} \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2s-1} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix} \begin{bmatrix} (s-1/2)^{n-1} & 0 & 0 & 0 & 0 \\ 0 & (s-3/2)^{n-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (3/2-s)^{n-1} & 0 \\ 0 & 0 & 0 & 0 & (1/2-s)^{n-1} \end{bmatrix} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s-1} \\ 0 & 0 & 0 & 0 & \sqrt{2s} \end{bmatrix} \\ &= \sum_{k=0}^{[n/2]} n(s, n; n-2k) \begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix} \\ &\Leftrightarrow \frac{1}{4s} \begin{bmatrix} 2s(s-1/2)^{n-1} & 0 & 0 & 0 & 0 \\ 0 & (2s-1)(s-3/2)^{n-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 2(3/2-s)^{n-1} & 0 \\ 0 & 0 & 0 & 0 & 1(1/2-s)^{n-1} \end{bmatrix} \\ &- \frac{1}{4s} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1(s-1/2)^{n-1} & 0 & 0 & 0 \\ 0 & 0 & 2(s-3/2)^{n-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & (2s-1)(3/2-s)^{n-1} \\ 0 & 0 & 0 & 0 & 2s(1/2-s)^{n-1} \end{bmatrix} \\ &= \sum_{k=0}^{[n/2]} n(s, n; n-2k) \begin{bmatrix} s^{n-2k} & 0 & 0 & 0 & 0 \\ 0 & (s-1)^{n-2k} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & (1-s)^{n-2k} & 0 \\ 0 & 0 & 0 & 0 & (-s)^{n-2k} \end{bmatrix} \\ &\Leftrightarrow \begin{cases} 4s \sum_{k=0}^{[n/2]} n(s, n; n-2k) s^{n-2k} = 2s(s-1/2)^{n-1} \\ 4s \sum_{k=0}^{[n/2]} n(s, n; n-2k) (s-1)^{n-2k} = (2s-1)(s-3/2)^{n-1} - 1(s-1/2)^{n-1} \\ \dots \\ 4s \sum_{k=0}^{[n/2]} n(s, n; n-2k) (1-s)^{n-2k} = 1(1/2-s)^{n-1} - (2s-1)(3/2-s)^{n-1} \\ 4s \sum_{k=0}^{[n/2]} n(s, n; n-2k) (-s)^{n-2k} = -2s(1/2-s)^{n-1} \end{cases} \\ &\Leftrightarrow 4s \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdots & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdots & (s-1)^{n-2[n/2]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-s)^n & (1-s)^{n-2} & (1-s)^{n-4} & \cdots & (1-s)^{n-2[n/2]} \\ (-s)^n & (-s)^{n-2} & (-s)^{n-4} & \cdots & (-s)^{n-2[n/2]} \end{bmatrix} \begin{bmatrix} n(s, n; n) \\ n(s, n; n-2) \\ n(s, n; n-4) \\ \vdots \\ n(s, n; n-2[n/2]) \end{bmatrix} = \begin{bmatrix} 2s(s-1/2)^{n-1} \\ (2s-1)(s-3/2)^{n-1} - 1(s-1/2)^{n-1} \\ (2s-2)(s-5/2)^{n-1} - 2(s-3/2)^{n-1} \\ \vdots \\ -2s(1/2-s)^{n-1} \end{bmatrix} \quad \square \end{aligned}$$

$$\text{推论4.7.6. } \begin{bmatrix} n(s, n; n) \\ n(s, n; n-2) \\ n(s, n; n-4) \\ \vdots \\ n(s, n; n-2[n/2]) \end{bmatrix}$$

$$= \frac{1}{4s} \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \dots & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \dots & (s-1)^{n-2[n/2]} \\ (s-2)^n & (s-2)^{n-2} & (s-2)^{n-4} & \dots & (s-2)^{n-2[n/2]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (s-[n/2])^n & (s-[n/2])^{n-2} & (s-[n/2])^{n-4} & \dots & (s-[n/2])^{n-2[n/2]} \end{bmatrix}^{-1} \begin{bmatrix} 2s(s-1/2)^{n-1} \\ (2s-1)(s-3/2)^{n-1} - 1(s-1/2)^{n-1} \\ (2s-2)(s-5/2)^{n-1} - 2(s-3/2)^{n-1} \\ \vdots \\ (2s-[n/2])(s-1/2-[n/2])^n - [n/2](s+1/2-[n/2])^n \end{bmatrix}$$

$$\text{推论4.7.7. } 2s \sum_{k=0}^{[n/2]} n(s, n; n-2k)(s-h)^{n-2k} = (2s-h)(s-h-\frac{1}{2})^{n-1} - h(s-h+\frac{1}{2})^{n-1}, 0 \leq h \leq 2s, n \geq 1, s \geq 1$$

$$\text{推论4.7.8. } -2s \sum_{k=0}^{[n/2]} (\frac{C_{n-1}^{2k-1}}{2^{2k-1}} - \frac{1}{s} \frac{C_n^{2k}}{2^{2k}})(s-h)^{n-2k} = (2s-h)(s-h-\frac{1}{2})^{n-1} - h(s-h+\frac{1}{2})^{n-1}, 0 \leq h \leq 2s, n \geq 1, s \geq 1$$

$$\text{推论4.7.9. } 2s \sum_{k=0}^{[n/2]} n(s, n; n-2k)h^{n-2k} = (s+h)(h-\frac{1}{2})^{n-1} - (s-h)(h+\frac{1}{2})^{n-1}, -s \leq h \leq s, n \geq 1, s \geq 1$$

$$\text{推论4.7.10. } -2s \sum_{k=0}^{[n/2]} (\frac{C_{n-1}^{2k-1}}{2^{2k-1}} - \frac{1}{s} \frac{C_n^{2k}}{2^{2k}})h^{n-2k} = (s+h)(h-\frac{1}{2})^{n-1} - (s-h)(h+\frac{1}{2})^{n-1}, -s \leq h \leq s, n \geq 1, s \geq 1$$

### 4.7.3 两个线性代数方程解的性质与验证

$$\text{推论4.7.11. } 2s \sum_{k=0}^{[n/2]} (\frac{C_n^{2k}}{2^{2k}} - \frac{1}{s} \frac{C_n^{2k+1}}{2^{2k+1}})s^{n-2k} = 2s(s-\frac{1}{2})^n \Leftrightarrow \sum_{i=0}^n C_n^i (-\frac{1}{2s})^i = (1-\frac{1}{2s})^n, h = s, n \geq 0$$

$$\text{推论4.7.12. } -2s \sum_{k=0}^{[(n+1)/2]} (\frac{C_n^{2k-1}}{2^{2k-1}} - \frac{1}{s} \frac{C_n^{2k}}{2^{2k}})s^{n+1-2k} = 2s(s-\frac{1}{2})^n \Leftrightarrow \sum_{i=0}^n C_n^i (-\frac{1}{2s})^i = (1-\frac{1}{2s})^n, h = s, n \geq 0$$

$$\text{推论4.7.13. } 2s \sum_{k=0}^{[n/2]} (\frac{C_n^{2k}}{2^{2k}} - \frac{1}{s} \frac{C_n^{2k+1}}{2^{2k+1}})h^{n-2k} = (s+h)(h-\frac{1}{2})^n + (s-h)(h+\frac{1}{2})^n, -s \leq h \leq s, n \geq 0, s \geq 1$$

$$\text{推论4.7.14. } -2s \sum_{k=0}^{[(n+1)/2]} (\frac{C_n^{2k-1}}{2^{2k-1}} - \frac{1}{s} \frac{C_n^{2k}}{2^{2k}})h^{n+1-2k} = (s+h)(h-\frac{1}{2})^n - (s-h)(h+\frac{1}{2})^n, -s \leq h \leq s, n \geq 0, s \geq 1$$

#### 定理4.7.3.

$$\begin{cases} 2s \sum_{k=0}^{[n/2]} (\frac{C_n^{2k}}{2^{2k}} - \frac{1}{s} \frac{C_n^{2k+1}}{2^{2k+1}})h^{n-2k} = (s+h)(h-\frac{1}{2})^n + (s-h)(h+\frac{1}{2})^n, n \geq 0 \\ -2s \sum_{k=0}^{[(n+1)/2]} (\frac{C_n^{2k-1}}{2^{2k-1}} - \frac{1}{s} \frac{C_n^{2k}}{2^{2k}})h^{n+1-2k} = (s+h)(h-\frac{1}{2})^n - (s-h)(h+\frac{1}{2})^n, n \geq 0 \end{cases}$$

$$\Leftrightarrow 2s \sum_{i=0}^n C_n^i (-\frac{1}{2})^i h^{n-i} + 2h \sum_{i=0}^n C_n^i (-\frac{1}{2})^i h^{n-i} = 2(s+h)(h-\frac{1}{2})^n, n \geq 0$$

$$\text{证明: } 2s \sum_{k=0}^{[n/2]} (\frac{C_n^{2k}}{2^{2k}} - \frac{1}{s} \frac{C_n^{2k+1}}{2^{2k+1}})h^{n-2k} - 2s \sum_{k=0}^{[(n+1)/2]} (\frac{C_n^{2k-1}}{2^{2k-1}} - \frac{1}{s} \frac{C_n^{2k}}{2^{2k}})h^{n+1-2k} = 2(s+h)(h-\frac{1}{2})^n, n \geq 0$$

$$\Leftrightarrow 2s (\sum_{k=0}^{[n/2]} \frac{C_n^{2k}}{2^{2k}} h^{n-2k} - \sum_{k=0}^{[(n+1)/2]} \frac{C_n^{2k-1}}{2^{2k-1}} h^{n+1-2k}) - 2 (\sum_{k=0}^{[n/2]} \frac{C_n^{2k+1}}{2^{2k+1}} h^{n-2k} - \sum_{k=0}^{[(n+1)/2]} \frac{C_n^{2k}}{2^{2k}} h^{n+1-2k}) = 2(s+h)(h-\frac{1}{2})^n, n \geq 0$$

$$\Leftrightarrow 2s \sum_{i=0}^n C_n^i (-\frac{1}{2})^i h^{n-i} + 2h \sum_{i=0}^n C_n^i (-\frac{1}{2})^i h^{n-i} = 2(s+h)(h-\frac{1}{2})^n, n \geq 0 \quad \square$$

以上定理表明, 将 $m(s, n; n-2k)$ ,  $n(s, n; n-2k)$ 代入方程后的结果, 就是变形的二项式展开恒等式, 因而也不会提供新的额外信息与技巧。对于 $\forall h, \forall s, m(s, n; n-2k), n(s, n; n-2k)$ 分别是下节定义域解析延拓后的解, 形式上与原来一样, 但含义不同。

### 4.7.4 两个线性代数方程的解析延拓

$$\text{推论4.7.15. } 2s \sum_{k=0}^{[n/2]} m(s, n; n-2k)h^{n-2k} = (s+h)(h-\frac{1}{2})^n + (s-h)(h+\frac{1}{2})^n, \forall h \in \mathbb{C}, \forall s \in \mathbb{C} \neq 0, n \in \mathbb{N} \geq 0$$

$$\Leftrightarrow m(s, n; n-2k) = \frac{C_n^{2k}}{2^{2k}} - \frac{1}{s} \frac{C_n^{2k+1}}{2^{2k+1}}, \forall h \in \mathbb{C}, \forall s \in \mathbb{C} \neq 0, n \in \mathbb{N} \geq 0$$

$$\text{推论4.7.16. } -2s \sum_{k=0}^{[n/2]} n(s, n; n-2k)h^{n-2k} = (s+h)(h-\frac{1}{2})^{n-1} - (s-h)(h+\frac{1}{2})^{n-1}, \forall h \in \mathbb{C}, \forall s \in \mathbb{C} \neq 0, n \in \mathbb{N} \geq 1$$

$$\Leftrightarrow n(s, n; n-2k) = -\frac{1}{2} (\frac{C_{n-1}^{2k-1}}{2^{2k-1}} - \frac{1}{s} \frac{C_n^{2k}}{2^{2k}}), \forall h \in \mathbb{C}, \forall s \in \mathbb{C} \neq 0, n \in \mathbb{N} \geq 1$$

4.7.5  $\Gamma^{z_1\zeta \cdots z_n\zeta} k_\zeta^{l_\zeta}(s, n)$ 的性质

推论4.7.17.  $\Gamma^{z_1\zeta \cdots z_n\zeta} k_\zeta^{l_\zeta}(s, n) := \frac{1}{2^n} \Gamma_{k_\zeta}^{A_{1\zeta} \cdots A_{n\zeta} A_{(n+1)\zeta} \cdots A_{(2s)\zeta}}(s) \prod_{i=1}^n \sigma_{A_{i\zeta}}^{z_i} B_{i\zeta} \Gamma_{B_{1\zeta} \cdots B_{n\zeta} A_{(n+1)\zeta} \cdots A_{(2s)\zeta}}^{l_\zeta}(s)$

引理4.7.1.

$$\Gamma^{z_1\zeta \cdots z_n\zeta} k_\zeta^{l_\zeta}(s, n) = \frac{1}{2^n} \begin{bmatrix} (C_{2s}^0)^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{0-i} & 0 & \cdots & 0 & 0 \\ 0 & (C_{2s}^1)^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{1-i} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & C_{2s}^{1-2s} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{2s-1-i} & 0 \\ 0 & 0 & \cdots & 0 & (C_{2s}^{2s})^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{2s-i} \end{bmatrix}$$

4.7.6  $\Gamma^{\alpha_1\zeta \cdots \alpha_n\zeta}(s, n; w)$ 展开系数的线性代数解法

猜想4.7.1.  $\Gamma^{\alpha_1\zeta \cdots \alpha_n\zeta}(s, n; w) = \frac{1}{n!} \sum_k^{[n/2]} c(s, n; n-2k) \Omega^{n-2k}(s; w), c(s, n; n-2k-1) = 0$

由于 $c(s, n; i)$ 某些情况下与 $w$ 独立无关, 对某些 $w$ 代数同构, 具有同样的展开系数。也只需求任何一个 $w$ 的展开系数即可, 哪个方便用哪个, 这里取 $w = 1$ 。怎么推出来还需严格仔细写出来, 久了自己都不记得了。

推论4.7.18.  $\Gamma^{z_1\zeta \cdots z_n\zeta}(s, n; w = 1) = \sum_k^{[n/2]} c(s, n; n-2k) \sigma_z^{n-2k}(s; w = 1) = \frac{1}{2^n}$

$$\begin{bmatrix} (C_{2s}^0)^{-1} (-1)^0 C_n^0 C_{2s-n}^{0-0} & 0 & \cdots & 0 & 0 \\ 0 & (C_{2s}^1)^{-1} [(-1)^0 C_n^0 C_{2s-n}^{1-0} + (-1)^1 C_n^1 C_{2s-n}^{0-1}] & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & (C_{2s}^{2s-1})^{-1} [(-1)^{n-1} C_n^{n-1} C_{2s-n}^{2s-n} + (-1)^n C_n^n C_{2s-n}^{2s-n-1}] & 0 \\ 0 & 0 & \cdots & 0 & (C_{2s}^{2s})^{-1} (-1)^n C_n^n C_{2s-n}^{2s-n} \end{bmatrix} \Rightarrow \frac{1}{2^n} (C_{2s}^{s-h})^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{s-h-i} = \sum_k^{[n/2]} c(s, n; n-2k) h^{n-2k}, h = s, s-1, \cdots, -(s-1), -s$$

推论4.7.19.

$$2^n \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdots & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdots & (s-1)^{n-2[n/2]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-s)^n & (1-s)^{n-2} & (1-s)^{n-4} & \cdots & (1-s)^{n-2[n/2]} \\ (-s)^n & (-s)^{n-2} & (-s)^{n-4} & \cdots & (-s)^{n-2[n/2]} \end{bmatrix} \begin{bmatrix} c(s, n; n) \\ c(s, n; n-2) \\ c(s, n; n-4) \\ \vdots \\ c(s, n; n-2[n/2]) \end{bmatrix} = \begin{bmatrix} (C_{2s}^0)^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{0-i} \\ (C_{2s}^1)^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{1-i} \\ \vdots \\ (C_{2s}^{2s-1})^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{2s-1-i} \\ (C_{2s}^{2s})^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{2s-i} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - \frac{n}{s} \\ 1 - \frac{n(2s-n)}{s(s-1/2)} \\ 1 - \frac{n(6s^2-6ns-3s+2n^2+1)}{2s(s-1/2)(s-1)} \\ \vdots \end{bmatrix}$$

推论4.7.20.

$$\begin{bmatrix} c(s, n; n) \\ c(s, n; n-2) \\ c(s, n; n-4) \\ \vdots \\ c(s, n; n-2[n/2]) \end{bmatrix} = \frac{1}{2^n} \begin{bmatrix} s^n & s^{n-2} & s^{n-4} & \cdots & s^{n-2[n/2]} \\ (s-1)^n & (s-1)^{n-2} & (s-1)^{n-4} & \cdots & (s-1)^{n-2[n/2]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (s-2)^n & (s-2)^{n-2} & (s-2)^{n-4} & \cdots & (s-2)^{n-2[n/2]} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (s-[n/2])^n & (s-[n/2])^{n-2} & (s-[n/2])^{n-4} & \cdots & (s-[n/2])^{n-2[n/2]} \end{bmatrix}^{-1} \begin{bmatrix} (C_{2s}^0)^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{0-i} \\ (C_{2s}^1)^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{1-i} \\ (C_{2s}^2)^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{2-i} \\ \vdots \\ (C_{2s}^{[n/2]})^{-1} \sum_{i=0}^n (-1)^i C_n^i C_{2s-n}^{[n/2]-i} \end{bmatrix}$$

4.7.7  $\Gamma^{\alpha_1\zeta \cdots \alpha_n\zeta} k_\zeta^{l_\zeta}(s, n; w)$ 前几项的线性代数解法验证

推论4.7.21.

$$\begin{bmatrix} s^0 \\ (s-1)^0 \\ \vdots \\ (1-s)^0 \\ (-s)^0 \end{bmatrix} [c(s, 0; 0)] = \begin{bmatrix} (C_{2s}^0)^{-1} (-1)^0 C_0^0 C_{2s-0}^{0-0} \\ (C_{2s}^1)^{-1} (-1)^0 C_0^0 C_{2s-0}^{1-0} \\ \vdots \\ (C_{2s}^{2s-1})^{-1} (-1)^0 C_0^0 C_{2s-0}^{2s-1-0} \\ (C_{2s}^{2s})^{-1} (-1)^0 C_0^0 C_{2s-0}^{2s-0} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \Leftrightarrow c(s, 0; 0) = 1$$

$s^0 c(s, 0; 0) = 1 \Leftrightarrow c(s, 0; 0) = 1, s \geq \frac{1}{2}$

推论4.7.22.

$$\begin{bmatrix} s^1 \\ (s-1)^1 \\ \vdots \\ (1-s)^1 \\ (-s)^1 \end{bmatrix} [c(s, 1; 1)] = \frac{1}{2^1} \begin{bmatrix} (C_{2s}^0)^{-1} [(-1)^0 C_1^0 C_{2s-1}^{0-0} + (-1)^1 C_1^1 C_{2s-1}^{0-1}] \\ (C_{2s}^1)^{-1} [(-1)^0 C_1^0 C_{2s-1}^{1-0} + (-1)^1 C_1^1 C_{2s-1}^{1-1}] \\ \vdots \\ (C_{2s}^{2s-1})^{-1} [(-1)^0 C_1^0 C_{2s-1}^{2s-1-0} + (-1)^1 C_1^1 C_{2s-1}^{2s-1-1}] \\ (C_{2s}^{2s})^{-1} [(-1)^0 C_1^0 C_{2s-1}^{2s-0} + (-1)^1 C_1^1 C_{2s-1}^{2s-1}] \end{bmatrix} \Leftrightarrow c(s, 1; 1) = \frac{1}{2s}$$

$s^1 c(s, 1; 1) = \frac{1}{2} \Leftrightarrow c(s, 1; 1) = \frac{1}{2s}, s \geq \frac{1}{2}$

推论4.7.23.

$$\begin{aligned} \begin{bmatrix} s^2 & s^0 \\ (s-1)^2 & (s-1)^0 \end{bmatrix} \begin{bmatrix} c(s,2;2) \\ c(s,2;0) \end{bmatrix} &= \frac{1}{2^2} \begin{bmatrix} (C_{2s}^0)^{-1} \sum_{i=0}^2 (-1)^i C_2^i C_{2s-2}^{0-i} \\ (C_{2s}^1)^{-1} \sum_{i=0}^2 (-1)^i C_2^i C_{2s-2}^{1-i} \end{bmatrix} = \frac{1}{2^2} \begin{bmatrix} (C_{2s}^0)^{-1} (-1)^0 C_2^0 C_{2s-2}^{0-0} \\ (C_{2s}^1)^{-1} [(-1)^0 C_2^0 C_{2s-2}^{1-0} + (-1)^1 C_2^1 C_{2s-2}^{1-1}] \end{bmatrix} = \frac{1}{2^2} \begin{bmatrix} 1 \\ 1 - \frac{2}{s} \end{bmatrix} \\ \Leftrightarrow \begin{bmatrix} c(s,2;2) \\ c(s,2;0) \end{bmatrix} &= \frac{1}{2^2} \begin{bmatrix} \frac{1}{s(s-1/2)} \\ 1 - \frac{s^2}{s(s-1/2)} \end{bmatrix} = \frac{(2s-2)!}{(2s)!} \begin{bmatrix} -\frac{1}{2s} \\ -\frac{1}{2s} \end{bmatrix} = \frac{(2s-2)!}{(2s)!} \begin{bmatrix} 1 \\ \frac{1}{2}(C_2^3 - C_2^2 s) \end{bmatrix}, s \geq 1 \end{aligned}$$

推论4.7.24.

$$\begin{aligned} \begin{bmatrix} s^3 & s^1 \\ (s-1)^3 & (s-1)^1 \end{bmatrix} \begin{bmatrix} c(s,3;3) \\ c(s,3;1) \end{bmatrix} &= \frac{1}{2^3} \begin{bmatrix} (C_{2s}^0)^{-1} \sum_{i=0}^3 (-1)^i C_3^i C_{2s-3}^{0-i} \\ (C_{2s}^1)^{-1} \sum_{i=0}^3 (-1)^i C_3^i C_{2s-3}^{1-i} \end{bmatrix} = \frac{1}{2^3} \begin{bmatrix} (C_{2s}^0)^{-1} (-1)^0 C_3^0 C_{2s-3}^{0-0} \\ (C_{2s}^1)^{-1} [(-1)^0 C_3^0 C_{2s-3}^{1-0} + (-1)^1 C_3^1 C_{2s-3}^{1-1}] \end{bmatrix} = \frac{1}{2^3} \begin{bmatrix} 1 \\ 1 - \frac{3}{s} \end{bmatrix} \\ \Leftrightarrow \begin{bmatrix} c(s,3;3) \\ c(s,3;1) \end{bmatrix} &= \frac{1}{2^3} \begin{bmatrix} \frac{1}{s(s-1/2)(s-1)} \\ \frac{1}{1 - \frac{s^2}{s(s-1/2)(s-1)}} \end{bmatrix} = \frac{(2s-3)!}{(2s)!} \begin{bmatrix} 1 \\ \frac{1}{2}(1-3s) \end{bmatrix} = \frac{(2s-3)!}{(2s)!} \begin{bmatrix} 1 \\ \frac{1}{2}(C_3^3 - C_3^2 s) \end{bmatrix}, s \geq \frac{3}{2} \end{aligned}$$

推论4.7.25.

$$\begin{aligned} \begin{bmatrix} s^4 & s^2 & s^0 \\ (s-1)^4 & (s-1)^2 & (s-1)^0 \\ (s-2)^4 & (s-2)^2 & (s-2)^0 \end{bmatrix} \begin{bmatrix} c(s,4;4) \\ c(s,4;2) \\ c(s,4;0) \end{bmatrix} &= \frac{1}{2^4} \begin{bmatrix} (C_{2s}^0)^{-1} \sum_{i=0}^4 (-1)^i C_4^i C_{2s-4}^{0-i} \\ (C_{2s}^1)^{-1} \sum_{i=0}^4 (-1)^i C_4^i C_{2s-4}^{1-i} \\ (C_{2s}^2)^{-1} \sum_{i=0}^4 (-1)^i C_4^i C_{2s-4}^{2-i} \end{bmatrix} = \frac{1}{2^4} \begin{bmatrix} 1 \\ 1 - \frac{4}{s} \\ 1 - \frac{4(2s-4)}{s(s-1/2)} \end{bmatrix} \\ \Leftrightarrow \begin{bmatrix} c(s,4;4) \\ c(s,4;2) \\ c(s,4;0) \end{bmatrix} &= \frac{1}{2^4} \begin{bmatrix} \frac{1}{s(s-1/2)(s-1)(s-3/2)} \\ \frac{2}{s(s-1/2) - \frac{s^2 + (s-1)^2}{s(s-1/2)(s-1)(s-3/2)}} \\ 1 - \frac{2s^2}{s(s-1/2)} + \frac{s^2(s-1)^2}{s(s-1/2)(s-1)(s-3/2)} \end{bmatrix} = \frac{(2s-4)!}{(2s)!} \begin{bmatrix} 1 \\ 2-3s \\ \frac{3}{4}s(s-1) \end{bmatrix} = \frac{(2s-4)!}{(2s)!} \begin{bmatrix} 1 \\ \frac{1}{2}(C_4^3 - C_4^2 s) \\ \frac{3}{4}s(s-1) \end{bmatrix}, s \geq 2 \end{aligned}$$

推论4.7.26.

$$\begin{aligned} \begin{bmatrix} s^5 & s^3 & s^1 \\ (s-1)^5 & (s-1)^3 & (s-1)^1 \\ (s-2)^5 & (s-2)^3 & (s-2)^1 \end{bmatrix} \begin{bmatrix} c(s,5;5) \\ c(s,5;3) \\ c(s,5;1) \end{bmatrix} &= \frac{1}{2^5} \begin{bmatrix} (C_{2s}^0)^{-1} \sum_{i=0}^5 (-1)^i C_5^i C_{2s-5}^{0-i} \\ (C_{2s}^1)^{-1} \sum_{i=0}^5 (-1)^i C_5^i C_{2s-5}^{1-i} \\ (C_{2s}^2)^{-1} \sum_{i=0}^5 (-1)^i C_5^i C_{2s-5}^{2-i} \end{bmatrix} = \frac{1}{2^5} \begin{bmatrix} 1 \\ 1 - \frac{5}{s} \\ 1 - \frac{5(2s-5)}{s(s-1/2)} \end{bmatrix} \\ \Leftrightarrow \begin{bmatrix} c(s,5;5) \\ c(s,5;3) \\ c(s,5;1) \end{bmatrix} &= \frac{1}{2^5} \begin{bmatrix} \frac{1}{s(s-1/2)(s-1)(s-3/2)(s-2)} \\ \frac{2}{s(s-1/2)(s-1) - \frac{s^2 + (s-1)^2}{s(s-1/2)(s-1)(s-3/2)(s-2)}} \\ \frac{1}{s - \frac{2s^2}{s(s-1/2)(s-1)} + \frac{s^2(s-1)^2}{s(s-1/2)(s-1)(s-3/2)(s-2)}} \end{bmatrix} = \frac{(2s-5)!}{(2s)!} \begin{bmatrix} 1 \\ \frac{1}{2}(C_5^3 - C_5^2 s) \\ \frac{1}{4}(15s^2 - 25s + 6) \end{bmatrix}, s \geq \frac{5}{2} \end{aligned}$$

推论4.7.27.

$$\begin{aligned} \begin{bmatrix} s^6 & s^4 & s^2 & s^0 \\ (s-1)^6 & (s-1)^4 & (s-1)^2 & (s-1)^0 \\ (s-2)^6 & (s-2)^4 & (s-2)^2 & (s-2)^0 \\ (s-3)^6 & (s-3)^4 & (s-3)^2 & (s-3)^0 \end{bmatrix} \begin{bmatrix} c(s,6;6) \\ c(s,6;4) \\ c(s,6;2) \\ c(s,6;0) \end{bmatrix} &= \frac{1}{2^6} \begin{bmatrix} (C_{2s}^0)^{-1} \sum_{i=0}^6 (-1)^i C_6^i C_{2s-6}^{0-i} \\ (C_{2s}^1)^{-1} \sum_{i=0}^6 (-1)^i C_6^i C_{2s-6}^{1-i} \\ (C_{2s}^2)^{-1} \sum_{i=0}^6 (-1)^i C_6^i C_{2s-6}^{2-i} \\ (C_{2s}^3)^{-1} \sum_{i=0}^6 (-1)^i C_6^i C_{2s-6}^{3-i} \end{bmatrix} = \frac{1}{2^6} \begin{bmatrix} 1 \\ 1 - \frac{6}{s} \\ 1 - \frac{6(2s-6)}{s(s-1/2)} \\ 1 - \frac{3(6s^2 - 39s + 73)}{s(s-1/2)(s-1)} \end{bmatrix}, s \geq 3 \end{aligned}$$

推论4.7.28.

$$\begin{aligned} \begin{bmatrix} s^7 & s^5 & s^3 & s^1 \\ (s-1)^7 & (s-1)^5 & (s-1)^3 & (s-1)^1 \\ (s-2)^7 & (s-2)^5 & (s-2)^3 & (s-2)^1 \\ (s-3)^7 & (s-3)^5 & (s-3)^3 & (s-3)^1 \end{bmatrix} \begin{bmatrix} c(s,7;7) \\ c(s,7;5) \\ c(s,7;3) \\ c(s,7;1) \end{bmatrix} &= \frac{1}{2^7} \begin{bmatrix} (C_{2s}^0)^{-1} \sum_{i=0}^7 (-1)^i C_7^i C_{2s-7}^{0-i} \\ (C_{2s}^1)^{-1} \sum_{i=0}^7 (-1)^i C_7^i C_{2s-7}^{1-i} \\ (C_{2s}^2)^{-1} \sum_{i=0}^7 (-1)^i C_7^i C_{2s-7}^{2-i} \\ (C_{2s}^3)^{-1} \sum_{i=0}^7 (-1)^i C_7^i C_{2s-7}^{3-i} \end{bmatrix} = \frac{1}{2^7} \begin{bmatrix} 1 \\ 1 - \frac{7}{s} \\ 1 - \frac{7(2s-7)}{s(s-1/2)} \\ 1 - \frac{7(6s^2 - 45s + 99)}{2s(s-1/2)(s-1)} \end{bmatrix}, s \geq \frac{7}{2} \end{aligned}$$

#### 4.7.8 复合常数不变张量 $\Gamma^{\alpha_1 \varsigma \cdots \alpha_{n\varsigma}}(s, n; w) \hat{p}_{\alpha_1 \varsigma} \cdots \hat{p}_{\alpha_{n\varsigma}}$ 的展开式

$$\text{推论4.7.29. } \Gamma^{\alpha_1 \varsigma \cdots \alpha_{n\varsigma}}(s, n; w) \hat{p}_{\alpha_1 \varsigma} \cdots \hat{p}_{\alpha_{n\varsigma}} = \sum_k^{\lfloor n/2 \rfloor} c(s, n; n-2k) [\sigma(s; w) \cdot \hat{p}]^{n-2k}$$

$$\text{推论4.7.30. } \Gamma^{\alpha_1 \varsigma \cdots \alpha_{n\varsigma}}(s, n; w) \hat{\partial}_{\alpha_1 \varsigma} \cdots \hat{\partial}_{\alpha_{n\varsigma}} = \sum_k^{\lfloor n/2 \rfloor} c(s, n; n-2k) [\sigma(s; w) \cdot \hat{\nabla}]^{n-2k}$$



## 5 常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w)$ , $Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w)$

### 5.1 常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w)$ , $Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w)$ 的引入

定义5.1.1.  $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w) := \frac{i\zeta}{\sqrt{2}}(\sigma\langle w \rangle, -i\zeta)_a^{A'_\zeta A_\zeta} N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ ,  $Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w) := \frac{-i\zeta}{\sqrt{2}}(\sigma\langle w \rangle, i\zeta)_{A_\zeta A'_\zeta}^a N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$

$$\text{性质5.1.1. } \begin{cases} Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w) = \delta^{ab} \varepsilon_{k_\zeta m_\zeta}(s; w) \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; w) \varepsilon_{A'_\zeta B'_\zeta} Z_{bn_\zeta}^{B'_\zeta m_\zeta}(s, \zeta; w) \\ Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w) = \delta_{ab} \varepsilon^{k_\zeta m_\zeta}(s; w) \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; w) \varepsilon_{A'_\zeta B'_\zeta} Z_{B'_\zeta m_\zeta}^{bn_\zeta}(s, \zeta; w) \end{cases}$$

$$\text{性质5.1.2. } \begin{cases} Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w) = (-1)^{2s+1} \delta^{ab} [(-\zeta)^{2s} \varepsilon_{k_\zeta m_\zeta}(s; w)] [\zeta^{2s-1} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; w)] [-\zeta \varepsilon_{A'_\zeta B'_\zeta}] Z_{bn_\zeta}^{B'_\zeta m_\zeta}(s, \zeta; w) \\ Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w) = (-1)^{2s+1} \delta_{ab} [(\zeta)^{2s} \varepsilon^{k_\zeta m_\zeta}(s; w)] [(-\zeta)^{2s-1} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; w)] [\zeta \varepsilon_{A'_\zeta B'_\zeta}] Z_{B'_\zeta m_\zeta}^{bn_\zeta}(s, \zeta; w) \end{cases}$$

### 5.2 常数矩阵 $Z_a^{A'_\zeta}(s, \zeta; w)$ , $Z_{A'_\zeta}^a(s, \zeta; w)$ 的引入

$$\text{定义5.2.1. } \begin{cases} Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w) \succ Z_a^{A'_\zeta}(s, \zeta; w) := \frac{i\zeta}{\sqrt{2}}(\sigma\langle w \rangle, -i\zeta)_a^{A'_\zeta A_\zeta} \bar{N}_{A_\zeta}(s; w) \\ Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w) \succ Z_{A'_\zeta}^a(s, \zeta; w) := \frac{-i\zeta}{\sqrt{2}}(\sigma\langle w \rangle, i\zeta)_{A_\zeta A'_\zeta}^a \bar{N}^{A_\zeta}(s; w) \end{cases}$$

$$\text{定义5.2.2. } \begin{cases} \bar{Z}_a^{A'_\zeta}(s, \zeta; w) := Z_a^{TA'_\zeta}(s, \zeta; w) = \frac{i\zeta}{\sqrt{2}}(\sigma\langle w \rangle, -i\zeta)_a^{A'_\zeta A_\zeta} N_{A_\zeta}(s; w) \\ \bar{Z}_{A'_\zeta}^a(s, \zeta; w) := Z_{A'_\zeta}^{Ta}(s, \zeta; w) = \frac{-i\zeta}{\sqrt{2}}(\sigma\langle w \rangle, i\zeta)_{A_\zeta A'_\zeta}^a N^{A_\zeta}(s; w) \end{cases}$$

### 5.3 常数不变张量矩阵 $Z_a(s, \zeta; w)$ , $\bar{Z}_a(s, \zeta; w)$ 的引入

$$\text{定义5.3.1. } \begin{cases} Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w) |_{A'_\zeta \otimes l_\zeta}^{k_\zeta} \succ Z_a(s, \zeta; w) := \frac{i\zeta}{\sqrt{2}}(\sigma \otimes I_{C_{2s-1+w}}^{2s-1}, -i\zeta)_a N(s; w) \\ Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w) |_{k_\zeta A'_\zeta}^{\otimes l_\zeta} \succ \bar{Z}_a(s, \zeta; w) := \frac{-i\zeta}{\sqrt{2}} \bar{N}(s; w) (\sigma \otimes I_{C_{2s-1+w}}^{2s-1}, i\zeta)_a \simeq Z_a^+(s, \zeta; w) \end{cases}$$

### 5.4 矩阵 $Z_a(s, \zeta; w)$ , $\bar{Z}_a(s, \zeta; w)$ 的常数不变张量性质

$$\text{性质5.4.1. } Z_a(s, \zeta; w) = [e^\vartheta]_a^b [e^{\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, -\zeta; w)} \otimes e^{\frac{i}{2}\vartheta^{cd} S_{cd}(s-\frac{1}{2}, \zeta; w)}] Z_b(s, \zeta; w) e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(s, \zeta; w)}$$

$$\text{性质5.4.2. } \bar{Z}_a(s, \zeta; w) = [e^\vartheta]_a^b e^{\frac{i}{2}\vartheta^{cd} S_{cd}(s, \zeta; w)} \bar{Z}_b(s, \zeta; w) [e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(\frac{1}{2}, -\zeta; w)} \otimes e^{-\frac{i}{2}\vartheta^{cd} S_{cd}(s-\frac{1}{2}, \zeta; w)}]$$

### 5.5 常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w)$ , $Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w)$ 的性质

#### 1、缩减两对指标 $A'_\zeta, l_\zeta$ :

$$\text{性质5.5.1. } Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w) Z_{bl_\zeta}^{A'_\zeta m_\zeta}(s, \zeta; w) = \frac{1}{2s} [s \delta^a_b \delta_{k_\zeta}^{m_\zeta} + i S^a_{bk_\zeta} m_\zeta(s, \zeta; w)]$$

$$[\Leftrightarrow] \bar{Z}_{A'_\zeta}^a(s, \zeta; w) Z_b^{A'_\zeta}(s, \zeta; w) = \frac{1}{2s} [s \delta^a_b + i S^a_b(s, \zeta; w)] [\Leftrightarrow] \bar{Z}_a(s, \zeta; w) Z_b(s, \zeta; w) = \frac{1}{2s} [s \delta_{ab} + i S_{ab}(s, \zeta; w)]$$

$$\begin{aligned} \text{证明: } & Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w) Z_{bl_\zeta}^{A'_\zeta m_\zeta}(s, \zeta; w) \\ &= \frac{-i\zeta}{\sqrt{2}}(\sigma\langle w \rangle, i\zeta)_{A_\zeta A'_\zeta}^a N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) \frac{i\zeta}{\sqrt{2}}(\sigma\langle w \rangle, -i\zeta)_b^{A'_\zeta B_\zeta} N_{B_\zeta l_\zeta}^{m_\zeta}(s; w) \\ &= \frac{1}{2} N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) (\sigma\langle w \rangle, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma\langle w \rangle, -i\zeta)_b^{A'_\zeta B_\zeta} N_{B_\zeta l_\zeta}^{m_\zeta}(s; w) \\ &= \frac{1}{2} N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) (\delta^a_b \delta_{A_\zeta}^{B_\zeta} + 2i S^a_{bA_\zeta} B_\zeta) N_{B_\zeta l_\zeta}^{m_\zeta}(s; w) \\ &= \frac{1}{2s} [s \delta^a_b \delta_{k_\zeta}^{m_\zeta} + i S^a_{bk_\zeta} m_\zeta(s, \zeta; w)] \end{aligned}$$

□

$$\text{推论5.5.1. } Z_{aA'_\zeta k_\zeta}^{l_\zeta}(s, \zeta; w) Z_{al_\zeta}^{A'_\zeta m_\zeta}(s, \zeta; w) = \frac{1}{2} \delta_{k_\zeta}^{m_\zeta}$$

$$[\Leftrightarrow] \bar{Z}_{aA'_\zeta}(s, \zeta; w) Z_a^{A'_\zeta}(s, \zeta; w) = \frac{1}{2} I_{C_{2s+w}}^{2s} [\Leftrightarrow] \bar{Z}_a(s, \zeta; w) Z_a(s, \zeta; w) = \frac{1}{2} I_{C_{2s+w}}^{2s}$$

#### 2、缩减两对指标 $A'_\zeta, k_\zeta$ :

$$\text{性质5.5.2. } Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta; w) Z_{bm_\zeta}^{A'_\zeta k_\zeta}(s, \zeta; w) = \frac{1}{2s} [(s + \frac{w}{2}) \delta^a_b \delta_{m_\zeta}^{l_\zeta} + i S^a_{bm_\zeta} l_\zeta(s - \frac{1}{2}, \zeta; w)]$$

$$[\Leftrightarrow] Z_b^{A'_\zeta}(s, \zeta; w) \bar{Z}_{A'_\zeta}^a(s, \zeta; w) = \frac{1}{2s} [(s + \frac{w}{2}) \delta^a_b + i S^a_b(s - \frac{1}{2}, \zeta; w)]$$

$$\begin{aligned}
& \text{证明: } Z_{A'_k k_\zeta}^{A'_k} (s, \zeta; w) Z_{B_m k_\zeta}^{A'_k k_\zeta} (s, \zeta; w) \\
&= \frac{-i\zeta}{\sqrt{2}} (\sigma \langle w \rangle, i\zeta)_{A'_k A'_k}^a N_{k_\zeta}^{A'_k l_\zeta} (s; w) \frac{i\zeta}{\sqrt{2}} (\sigma \langle w \rangle, -i\zeta)_b^{A'_k B_\zeta} N_{B_\zeta m_\zeta}^{k_\zeta} (s; w) \\
&= \frac{1}{2} N_{k_\zeta}^{A'_k l_\zeta} (s; w) (\sigma \langle w \rangle, i\zeta)_{A'_k A'_k}^a (\sigma \langle w \rangle, -i\zeta)_b^{A'_k B_\zeta} N_{B_\zeta m_\zeta}^{k_\zeta} (s; w) \\
&= \frac{1}{2} N_{k_\zeta}^{A'_k l_\zeta} (s; w) (\delta_b^a \delta_{A'_k}^{B_\zeta} + 2i S_b^a{}_{A'_k}{}^{B_\zeta}) N_{B_\zeta m_\zeta}^{k_\zeta} (s; w) \\
&= \frac{1}{2s} [(s + \frac{w}{2}) \delta_b^a \delta_{m_\zeta}^{l_\zeta} + i S_b^a{}_{m_\zeta}{}^{l_\zeta} (s - \frac{1}{2}, \zeta; w)]
\end{aligned}$$

□

$$\text{推论5.5.2. } Z_{A'_k k_\zeta}^{l_\zeta} (s, \zeta; w) Z_{A_m k_\zeta}^{A'_k} (s, \zeta; w) = \frac{1}{2} (1 + \frac{w}{2s}) \delta_{m_\zeta}^{l_\zeta} [\Leftrightarrow] Z_{A'_k}^{A'_k} (s, \zeta; w) \bar{Z}_{A_m k_\zeta} (s, \zeta; w) = \frac{1}{2} (1 + \frac{w}{2s})$$

### 3、缩减两对指标 $k_\zeta, l_\zeta$ :

$$\begin{aligned}
& \text{性质5.5.3. } Z_{A'_k k_\zeta}^{l_\zeta} (s, \zeta; w) Z_{B_l k_\zeta}^{B'_k k_\zeta} (s, \zeta; w) = \frac{1}{w+1} C_{2s+w}^{2s} (\frac{1}{2} \delta_b^a \delta_{A'_k}^{B'_k} + i S_b^a{}_{A'_k}{}^{B'_k}) \\
& [\Leftrightarrow] \text{tr} [\bar{Z}_{A'_k}^a (s, \zeta; w) Z_{B'_k}^{B'_k} (s, \zeta; w)] = \frac{1}{w+1} C_{2s+w}^{2s} (\frac{1}{2} \delta_b^a \delta_{A'_k}^{B'_k} + i S_b^a{}_{A'_k}{}^{B'_k})
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } Z_{A'_k k_\zeta}^{l_\zeta} (s, \zeta; w) Z_{B_l k_\zeta}^{B'_k k_\zeta} (s, \zeta; w) \\
&= \frac{-i\zeta}{\sqrt{2}} (\sigma \langle w \rangle, i\zeta)_{A'_k A'_k}^a N_{k_\zeta}^{A'_k l_\zeta} (s; w) \frac{i\zeta}{\sqrt{2}} (\sigma \langle w \rangle, -i\zeta)_b^{B'_k B_\zeta} N_{B_\zeta l_\zeta}^{k_\zeta} (s; w) \\
&= \frac{1}{2} \frac{1}{w+1} C_{2s+w}^{2s} (\sigma \langle w \rangle, i\zeta)_{A'_k A'_k}^a (\sigma \langle w \rangle, -i\zeta)_b^{B'_k A'_k} \\
&= \frac{1}{w+1} C_{2s+w}^{2s} (\frac{1}{2} \delta_b^a \delta_{A'_k}^{B'_k} + i S_b^a{}_{A'_k}{}^{B'_k})
\end{aligned}$$

□

$$\text{推论5.5.3. } Z_{A'_k k_\zeta}^{l_\zeta} (s, \zeta; w) Z_{A_l k_\zeta}^{B'_k k_\zeta} (s, \zeta; w) = \frac{1}{2(w+1)} C_{2s+w}^{2s} \delta_{A'_k}^{B'_k} [\Leftrightarrow] \text{tr} [\bar{Z}_{A'_k}^a (s, \zeta; w) Z_b^{B'_k} (s, \zeta; w)] = \frac{1}{2(w+1)} C_{2s+w}^{2s} \delta_{A'_k}^{B'_k}$$

## 5.6 猜想(不具一般性)

$$\text{猜想5.6.1. } \frac{i\zeta}{\sqrt{2}} (\sigma \langle w \rangle, -i\zeta)_a^{A'_k A'_k} \frac{-i\zeta}{\sqrt{2}} (\sigma \langle w \rangle, i\zeta)_{B_\zeta B_\zeta}^a = \delta_{B_\zeta}^{A'_k} \delta_{B'_k}^{A'_k}$$

### 4、缩减两对指标 $a, l_\zeta$ :

$$\text{性质5.6.1. } Z_{A'_k k_\zeta}^{l_\zeta} (s, \zeta; w) Z_{A_l k_\zeta}^{B'_k m_\zeta} (s, \zeta; w) = \delta_{A'_k}^{B'_k} \delta_{k_\zeta}^{m_\zeta} [\Leftrightarrow] \bar{Z}_{A'_k}^a (s, \zeta; w) Z_a^{B'_k} (s, \zeta; w) = \delta_{A'_k}^{B'_k} I_{C_{2s+w}^{2s}}$$

### 5、缩减两对指标 $a, k_\zeta$ :

$$\begin{aligned}
& \text{性质5.6.2. } Z_{A'_k k_\zeta}^{l_\zeta} (s, \zeta; w) Z_{B'_k k_\zeta}^{a m_\zeta} (s, \zeta; w) = (1 + \frac{w}{2s}) \delta_{B'_k}^{A'_k} \delta_{l_\zeta}^{m_\zeta} \\
& [\Leftrightarrow] Z_a^{A'_k} (s, \zeta; w) \bar{Z}_{B'_k}^a (s, \zeta; w) = (1 + \frac{w}{2s}) \delta_{B'_k}^{A'_k} I_{C_{2s-1+w}^{2s-1}} [\Leftrightarrow] Z_a (s, \zeta; w) \bar{Z}^a (s, \zeta; w) = (1 + \frac{w}{2s}) I_{(w+1) C_{2s-1+w}^{2s-1}}
\end{aligned}$$

## 5.7 常数不变张量矩阵 $Z_a(s, \zeta; w), \bar{Z}_a(s, \zeta; w)$ 的性质(不具一般性)

$$\text{性质5.7.1. } \begin{cases} \bar{Z}_a (s, \zeta; w) Z_b (s, \zeta; w) = \frac{1}{2s} [s \delta_{ab} + i S_{ab} (s, \zeta; w)] \\ Z^a (s, \zeta; w) \bar{Z}_a (s, \zeta; w) = (1 + \frac{w}{2s}) I_{(w+1) C_{2s-1+w}^{2s-1}} \end{cases}$$

$$\text{性质5.7.2. } \begin{cases} (s+w) Z_b (s, \zeta; w) = Z^a (s, \zeta; w) i S_{ab} (s, \zeta; w), (s+w) \bar{Z}_a (s, \zeta; w) = i S_{ab} (s, \zeta; w) \bar{Z}^b (s, \zeta; w) \\ Z^a (s, \zeta; w) i S_{ab} (s, \zeta; w) \bar{Z}^b (s, \zeta; w) = (s+w) (1 + \frac{w}{2s}), Z_a (s, \zeta; w) \bar{Z}_a (s, \zeta; w) \neq k I_{(w+1) C_{2s-1+w}^{2s-1}} \end{cases}$$

$$\text{性质5.7.3. } \begin{cases} -S_{ac} (s, \zeta; w) S_b^c (s, \zeta; w) = s(s+w) \delta_{ab} + iw S_{ab} (s, \zeta; w) \\ \bar{Z}_a (s, \zeta; w) Z_b (s, \zeta; w) = -\frac{1}{2sw} [s^2 \delta_{ab} + S_{ac} (s, \zeta; w) S_b^c (s, \zeta; w)] \end{cases}$$

$$\text{性质5.7.4. } [\sigma(s; w), i\zeta(s+w)]^a \bar{Z}_a (s, \zeta; w) = 0, Z_a (s, \zeta; w) [\sigma(s; w), -i\zeta(s+w)]^a = 0$$

$$\begin{aligned}
& \text{证明: } [\sigma(s; w), i\zeta(s+w)]^a \bar{Z}_a (s, \zeta; w) \\
&= \frac{-i\zeta}{\sqrt{2}} [\sigma(s; w), i\zeta(s+w)]^a \bar{N}(s; w) (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, i\zeta)_a \\
&= \frac{-i\zeta}{\sqrt{2}} \bar{N}(s; w) [s \sigma \otimes I_{C_{2s-1+w}^{2s-1}}, i\zeta(s+w)]^a N(s; w) \bar{N}(s; w) (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, i\zeta)_a \\
&= \frac{-i\zeta}{\sqrt{2}} [\bar{N}(s; w) (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)^a N(s; w) s \bar{N}(s; w) (\sigma \otimes I_{C_{2s-1+w}^{2s-1}}, i\zeta)_a - (2s+w) \bar{N}(s; w)] \\
&= \frac{-i\zeta}{\sqrt{2}} [\bar{N}(s; w) Z^a (s, \zeta; w) 2s \bar{Z}_a (s, \zeta; w) - (2s+w) \bar{N}(s; w)] \\
&= \frac{-i\zeta}{\sqrt{2}} [\bar{N}(s; w) (2s+w) - (2s+w) \bar{N}(s; w)] \\
&= 0
\end{aligned}$$

□

## 6 常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta)$ , $Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta)$

### 6.1 常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta)$ , $Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta)$ 的引入

定义6.1.1.  $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta) := \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a^{A'_\zeta} N_{A'_\zeta l_\zeta}^{k_\zeta}(s)$ ,  $Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta) := \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{A'_\zeta}^a N_{k_\zeta}^{A'_\zeta l_\zeta}(s)$

$$\text{性质6.1.1. } \begin{cases} Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta) = \delta^{ab} \varepsilon_{k_\zeta m_\zeta}(s) \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}) \varepsilon_{A'_\zeta B'_\zeta} Z_{bn_\zeta}^{B'_\zeta m_\zeta}(s, \zeta) \\ Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta) = \delta_{ab} \varepsilon^{k_\zeta m_\zeta}(s) \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}) \varepsilon^{A'_\zeta B'_\zeta} Z_{B'_\zeta m_\zeta}^{bn_\zeta}(s, \zeta) \end{cases}$$

$$\text{性质6.1.2. } \begin{cases} Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta) = (-1)^{2s+1} \delta^{ab} [(-\zeta)^{2s} \varepsilon_{k_\zeta m_\zeta}(s)] [\zeta^{2s-1} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2})] [-\zeta \varepsilon_{A'_\zeta B'_\zeta}] Z_{bn_\zeta}^{B'_\zeta m_\zeta}(s, \zeta) \\ Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta) = (-1)^{2s+1} \delta_{ab} [(\zeta)^{2s} \varepsilon^{k_\zeta m_\zeta}(s)] [(-\zeta)^{2s-1} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2})] [\zeta \varepsilon^{A'_\zeta B'_\zeta}] Z_{B'_\zeta m_\zeta}^{bn_\zeta}(s, \zeta) \end{cases}$$

### 6.2 常数矩阵 $Z_a^{A'_\zeta}(s, \zeta)$ , $Z_{A'_\zeta}^a(s, \zeta)$ 的引入

$$\text{定义6.2.1. } \begin{cases} Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta) \succ Z_a^{A'_\zeta}(s, \zeta) := \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a^{A'_\zeta} \bar{N}_{A'_\zeta}(s) \\ Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta) \succ Z_{A'_\zeta}^a(s, \zeta) := \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{A'_\zeta}^a \bar{N}^{A'_\zeta}(s) \end{cases}$$

$$\text{定义6.2.2. } \begin{cases} \bar{Z}_a^{A'_\zeta}(s, \zeta) := Z_a^{TA'_\zeta}(s, \zeta) = \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a^{A'_\zeta} N_{A'_\zeta}(s) \\ \bar{Z}_{A'_\zeta}^a(s, \zeta) := Z_{A'_\zeta}^{Ta}(s, \zeta) = \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{A'_\zeta}^a N^{A'_\zeta}(s) \end{cases}$$

### 6.3 常数不变张量矩阵 $Z_a(s, \zeta)$ , $\bar{Z}_a(s, \zeta)$ 的引入

$$\text{定义6.3.1. } \begin{cases} Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta) |_{A'_\zeta \otimes l_\zeta}^{k_\zeta} \succ Z_a(s, \zeta) := \frac{i\zeta}{\sqrt{2}}(\sigma \otimes I_{2s}, -i\zeta)_a N(s) \\ Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta) |_{k_\zeta A'_\zeta}^{\otimes l_\zeta} \succ \bar{Z}_a(s, \zeta) := \frac{-i\zeta}{\sqrt{2}} \bar{N}(s) (\sigma \otimes I_{2s}, i\zeta)_a \simeq Z_a^+(s, \zeta) \end{cases}$$

### 6.4 矩阵 $Z_a(s, \zeta)$ , $\bar{Z}_a(s, \zeta)$ 的常数不变张量性质

$$\text{性质6.4.1. } Z_a(s, \zeta) = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a^b e^{(i\omega - \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \zeta \epsilon) \cdot \sigma(s - \frac{1}{2})} Z_b(s, \zeta) e^{-(i\omega + \zeta \epsilon) \cdot \sigma(s)}$$

$$\text{性质6.4.2. } \bar{Z}_a(s, \zeta) = [e^{(i\omega \cdot R + \epsilon \cdot L)}]_a^b e^{(i\omega + \zeta \epsilon) \cdot \sigma(s)} \bar{Z}_b(s, \zeta) e^{-(i\omega - \zeta \epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{-(i\omega + \zeta \epsilon) \cdot \sigma(s - \frac{1}{2})}$$

### 6.5 常数不变张量 $Z_{al_\zeta}^{A'_\zeta k_\zeta}(s, \zeta)$ , $Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta)$ 的性质I

#### 1、缩减两对指标 $A'_\zeta, l_\zeta$ :

$$\text{性质6.5.1. } Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta) Z_{bl_\zeta}^{A'_\zeta m_\zeta}(s, \zeta) = \frac{1}{2s} [s \delta^a_b \delta_{k_\zeta}^{m_\zeta} + i S^a_{bk_\zeta} m_\zeta(s, \zeta)]$$

$$[\Leftrightarrow] \bar{Z}_a^{A'_\zeta}(s, \zeta) Z_b^{A'_\zeta}(s, \zeta) = \frac{1}{2s} [s \delta^a_b + i S^a_b(s, \zeta)] [\Leftrightarrow] \bar{Z}_a(s, \zeta) Z_b(s, \zeta) = \frac{1}{2s} [s \delta_{ab} + i S_{ab}(s, \zeta)]$$

$$\begin{aligned} \text{证明: } & Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta) Z_{bl_\zeta}^{A'_\zeta m_\zeta}(s, \zeta) \\ &= \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{A'_\zeta}^a N_{k_\zeta}^{A'_\zeta l_\zeta}(s) \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b^{A'_\zeta} N_{B'_\zeta l_\zeta}^{m_\zeta}(s) \\ &= \frac{1}{2} N_{k_\zeta}^{A'_\zeta l_\zeta}(s) (\sigma, i\zeta)_{A'_\zeta}^a (\sigma, -i\zeta)_b^{A'_\zeta} N_{B'_\zeta l_\zeta}^{m_\zeta}(s) \\ &= \frac{1}{2} N_{k_\zeta}^{A'_\zeta l_\zeta}(s) (\delta^a_b \delta_{A'_\zeta}^{B'_\zeta} + 2i S^a_{b A'_\zeta} B'_\zeta) N_{B'_\zeta l_\zeta}^{m_\zeta}(s) \\ &= \frac{1}{2s} [s \delta^a_b \delta_{k_\zeta}^{m_\zeta} + i S^a_{bk_\zeta} m_\zeta(s, \zeta)] \end{aligned}$$

□

$$\text{推论6.5.1. } Z_a^{l_\zeta}(s, \zeta) Z_{al_\zeta}^{A'_\zeta m_\zeta}(s, \zeta) = \frac{1}{2} \delta_{k_\zeta}^{m_\zeta}$$

$$[\Leftrightarrow] \bar{Z}_a^{A'_\zeta}(s, \zeta) Z_a^{A'_\zeta}(s, \zeta) = \frac{1}{2} I_{2s+1} [\Leftrightarrow] \bar{Z}_a(s, \zeta) Z_a(s, \zeta) = \frac{1}{2} I_{2s+1}$$

#### 2、缩减两对指标 $A'_\zeta, k_\zeta$ :

$$\text{性质6.5.2. } Z_{A'_\zeta k_\zeta}^{al_\zeta}(s, \zeta) Z_{bm_\zeta}^{A'_\zeta k_\zeta}(s, \zeta) = \frac{1}{2s} [(s + \frac{1}{2}) \delta^a_b \delta_{m_\zeta}^{l_\zeta} + i S^a_{bm_\zeta} l_\zeta(s - \frac{1}{2}, \zeta)]$$

$$[\Leftrightarrow] Z_b^{A'_\zeta}(s, \zeta) \bar{Z}_a^{A'_\zeta}(s, \zeta) = \frac{1}{2s} [(s + \frac{1}{2}) \delta^a_b + i S^a_b(s - \frac{1}{2}, \zeta)]$$

证明:  $Z_{A'_k \zeta}^{al_\zeta}(s, \zeta) Z_{bm_\zeta}^{A'_k \zeta}(s, \zeta)$

$$= \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a N_{k_\zeta}^{A_\zeta l_\zeta}(s) \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_{b^{A'_\zeta B_\zeta}}^{A'_\zeta B_\zeta} N_{B_\zeta m_\zeta}^{k_\zeta}(s)$$

$$= \frac{1}{2} N_{k_\zeta}^{A_\zeta l_\zeta}(s) (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, -i\zeta)_{b^{A'_\zeta B_\zeta}}^{A'_\zeta B_\zeta} N_{B_\zeta m_\zeta}^{k_\zeta}(s)$$

$$= \frac{1}{2} N_{k_\zeta}^{A_\zeta l_\zeta}(s) (\delta_b^a \delta_{A_\zeta}^{B_\zeta} + 2i S_{b A_\zeta}^{a B_\zeta}) N_{B_\zeta m_\zeta}^{k_\zeta}(s)$$

$$= \frac{1}{2s} [(s + \frac{1}{2}) \delta_b^a \delta_{m_\zeta}^{l_\zeta} + i S_{b m_\zeta}^{a l_\zeta} (s - \frac{1}{2}, \zeta)]$$

□

推论6.5.2.  $Z_{a A'_k \zeta}^{l_\zeta}(s, \zeta) Z_{am_\zeta}^{A'_k \zeta}(s, \zeta) = \frac{1}{2} (1 + \frac{1}{2s}) \delta_{m_\zeta}^{l_\zeta} [\Leftrightarrow] Z_{a \zeta}^{A'_k}(s, \zeta) \bar{Z}_{a A'_\zeta}(s, \zeta) = \frac{1}{2} (1 + \frac{1}{2s}) I_{2s}$

### 3、缩减两对指标 $k_\zeta, l_\zeta$ :

性质6.5.3.  $Z_{A'_k \zeta}^{al_\zeta}(s, \zeta) Z_{bl_\zeta}^{B'_k \zeta}(s, \zeta) = (s + \frac{1}{2}) (\frac{1}{2} \delta_b^a \delta_{A'_\zeta}^{B'_\zeta} + i S_{b^{A'_\zeta} A'_\zeta}^{a B'_\zeta})$

$$[\Leftrightarrow] \text{tr}[\bar{Z}_{A'_\zeta}^a(s, \zeta) Z_b^{B'_\zeta}(s, \zeta)] = (s + \frac{1}{2}) (\frac{1}{2} \delta_b^a \delta_{A'_\zeta}^{B'_\zeta} + i S_{b^{A'_\zeta} A'_\zeta}^{a B'_\zeta})$$

证明:  $Z_{A'_k \zeta}^{al_\zeta}(s, \zeta) Z_{bl_\zeta}^{B'_k \zeta}(s, \zeta)$

$$= \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a N_{k_\zeta}^{A_\zeta l_\zeta}(s) \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_{b^{B'_\zeta B_\zeta}}^{B'_\zeta B_\zeta} N_{B_\zeta l_\zeta}^{k_\zeta}(s)$$

$$= \frac{1}{2} (s + \frac{1}{2}) (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, -i\zeta)_{b^{B'_\zeta B_\zeta}}^{B'_\zeta B_\zeta}$$

$$= (s + \frac{1}{2}) (\frac{1}{2} \delta_b^a \delta_{A'_\zeta}^{B'_\zeta} + i S_{b^{A'_\zeta} A'_\zeta}^{a B'_\zeta})$$

□

推论6.5.3.  $Z_{a A'_k \zeta}^{l_\zeta}(s, \zeta) Z_{al_\zeta}^{B'_k \zeta}(s, \zeta) = \frac{1}{2} (s + \frac{1}{2}) \delta_{A'_\zeta}^{B'_\zeta} [\Leftrightarrow] \text{tr}[\bar{Z}_{A'_\zeta}^a(s, \zeta) Z_b^{B'_\zeta}(s, \zeta)] = \frac{1}{2} (s + \frac{1}{2}) \delta_{A'_\zeta}^{B'_\zeta}$

## 6.6 常数不变张量 $Z_{al_\zeta}^{A'_k \zeta}(s, \zeta), Z_{A'_k \zeta}^{al_\zeta}(s, \zeta)$ 的性质II

性质6.6.1.  $\frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{B_\zeta B'_\zeta}^a = \delta_{B_\zeta}^{A'_\zeta} \delta_{B'_\zeta}^{A_\zeta}$

### 4、缩减两对指标 $a, l_\zeta$ :

性质6.6.2.  $Z_{A'_k \zeta}^{al_\zeta}(s, \zeta) Z_{al_\zeta}^{B'_k m_\zeta}(s, \zeta) = \delta_{A'_\zeta}^{B'_\zeta} \delta_{k_\zeta}^{m_\zeta} [\Leftrightarrow] \bar{Z}_{A'_\zeta}^a(s, \zeta) Z_a^{B'_\zeta}(s, \zeta) = \delta_{A'_\zeta}^{B'_\zeta} I_{2s+1}$

### 5、缩减两对指标 $a, k_\zeta$ :

性质6.6.3.  $Z_{al_\zeta}^{A'_k \zeta}(s, \zeta) Z_{B'_k \zeta}^{am_\zeta}(s, \zeta) = (1 + \frac{1}{2s}) \delta_{B'_\zeta}^{A'_\zeta} \delta_{l_\zeta}^{m_\zeta}$

$$[\Leftrightarrow] Z_a^{A'_\zeta}(s, \zeta) \bar{Z}_{B'_\zeta}^a(s, \zeta) = (1 + \frac{1}{2s}) \delta_{B'_\zeta}^{A'_\zeta} I_{2s} [\Leftrightarrow] Z_a(s, \zeta) \bar{Z}^a(s, \zeta) = (1 + \frac{1}{2s}) I_{4s}$$

## 6.7 常数不变张量矩阵 $Z_a(s, \zeta), \bar{Z}_a(s, \zeta)$ 的性质

性质6.7.1.  $\begin{cases} \bar{Z}_a(s, \zeta) Z_b(s, \zeta) = \frac{1}{2s} [s \delta_{ab} + i S_{ab}(s, \zeta)] \\ Z^a(s, \zeta) \bar{Z}_a(s, \zeta) = (1 + \frac{1}{2s}) I_{4s} \end{cases}$

性质6.7.2.  $\begin{cases} (s+1) Z_b(s, \zeta) = Z^a(s, \zeta) i S_{ab}(s, \zeta), (s+1) \bar{Z}_a(s, \zeta) = i S_{ab}(s, \zeta) \bar{Z}^b(s, \zeta) \\ Z^a(s, \zeta) i S_{ab}(s, \zeta) \bar{Z}^b(s, \zeta) = (s+1) (1 + \frac{1}{2s}), Z_a(s, \zeta) \bar{Z}_a(s, \zeta) \neq k I_{4s} \end{cases}$

性质6.7.3.  $\begin{cases} -S_{ac}(s, \zeta) S^c_b(s, \zeta) = s(s+1) \delta_{ab} + i S_{ab}(s, \zeta) \\ \bar{Z}_a(s, \zeta) Z_b(s, \zeta) = -\frac{1}{2s} [s^2 \delta_{ab} + S_{ac}(s, \zeta) S^c_b(s, \zeta)] \end{cases}$

性质6.7.4.  $[\sigma(s), i\zeta(s+1)]^a \bar{Z}_a(s, \zeta) = 0, Z_a(s, \zeta) [\sigma(s), -i\zeta(s+1)]^a = 0$

证明:  $[\sigma(s), i\zeta(s+1)]^a \bar{Z}_a(s, \zeta)$

$$= \frac{-i\zeta}{\sqrt{2}} [\sigma(s), i\zeta(s+1)]^a \bar{N}(s) (\sigma \otimes I_{2s}, i\zeta)_a$$

$$= \frac{-i\zeta}{\sqrt{2}} \bar{N}(s) [s \sigma \otimes I_{2s}, i\zeta(s+1)]^a N(s) \bar{N}(s) (\sigma \otimes I_{2s}, i\zeta)_a$$

$$= \frac{-i\zeta}{\sqrt{2}} [\bar{N}(s) (\sigma \otimes I_{2s}, -i\zeta)^a N(s) s \bar{N}(s) (\sigma \otimes I_{2s}, i\zeta)_a - (2s+1) \bar{N}(s)]$$

$$= \frac{-i\zeta}{\sqrt{2}} [\bar{N}(s) Z^a(s, \zeta) 2s \bar{Z}_a(s, \zeta) - (2s+1) \bar{N}(s)]$$

$$= \frac{-i\zeta}{\sqrt{2}} [\bar{N}(s) (2s+1) - (2s+1) \bar{N}(s)]$$

$$= 0$$

□

## 7 几个重要的复合常数不变张量

### 7.1 复合常数不变张量 $\Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}, \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n)$

定义7.1.1.

$$\Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n) := \left(\frac{i_\zeta}{\sqrt{2}}\right)^n \underbrace{\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \dots}_{n} \cdot \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}^{k_\zeta}}_{2n}(n), \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) := \left(\frac{i_\zeta}{\sqrt{2}}\right)^n \underbrace{\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots}_{n} \cdot \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n}(n)$$

等价性:

推论7.1.1.

$$\Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) = \left(\frac{i_\zeta}{\sqrt{2}}\right)^n \underbrace{\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots}_{n} \cdot \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n}(n) \Leftrightarrow \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n}(n) = \left(\frac{i_\zeta}{\sqrt{2}}\right)^n \underbrace{\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \dots}_{n} \cdot \underbrace{\Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}}_{n}(n)$$

推论7.1.2.

$$\Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n) = \left(\frac{i_\zeta}{\sqrt{2}}\right)^n \underbrace{\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \dots}_{n} \cdot \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}^{k_\zeta}}_{2n}(n) \Leftrightarrow \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}^{k_\zeta}}_{2n}(n) = \left(\frac{i_\zeta}{\sqrt{2}}\right)^n \underbrace{\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots}_{n} \cdot \underbrace{\Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}}_{n}(n)$$

相等性:

$$\text{推论7.1.3. } \Gamma_{k'_\zeta}^{\alpha'_\zeta \beta'_\zeta \dots}(n) = [\Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n)]^* \simeq \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n), \Gamma_{\alpha'_\zeta \beta'_\zeta \dots}^{k'_\zeta}(n) = [\Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n)]^* \simeq \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n)$$

全对称性:

$$\text{推论7.1.4. } \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) = \frac{1}{n!} \Gamma_{k_\zeta}^{(\alpha_\zeta \beta_\zeta \dots)}(n), \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n) = \frac{1}{n!} \Gamma_{(\alpha_\zeta \beta_\zeta \dots)}^{k_\zeta}(n)$$

无迹性:

$$\text{推论7.1.5. } \delta_{\alpha_\zeta \beta_\zeta} \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) = 0, \delta^{\alpha_\zeta \beta_\zeta} \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n) = 0$$

类Penrose对应:

$$\Gamma_k^{\alpha \beta \dots}(n) \stackrel{P}{=} \Gamma_k^{ABCD \dots}(2n) \quad \Gamma_{k'}^{\alpha' \beta' \dots}(n) \stackrel{P}{=} \Gamma_{k'}^{A'B'C'D' \dots}(2n) \quad (3.1)$$

$$\Gamma_{\alpha \beta \dots}^k(n) \stackrel{P}{=} \Gamma_{2n}^{ABCD \dots}(n) \quad \Gamma_{\alpha' \beta' \dots}^{k'}(n) \stackrel{P}{=} \Gamma_{2n}^{A'B'C'D' \dots}(n) \quad (3.2)$$

正交性:

$$\text{推论7.1.6. } \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n) \Gamma_{l_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) = \delta^{k_\zeta l_\zeta}$$

$$\text{推论7.1.7. } \Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Gamma_{l_\zeta}^{\alpha_\zeta}(1) = \delta_{l_\zeta}^{k_\zeta}, \Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Gamma_{k_\zeta}^{\beta_\zeta}(1) = \delta_{\alpha_\zeta}^{\beta_\zeta}$$

$$\text{推论7.1.8. } \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1) = \frac{i_\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \Gamma_{k_\zeta}^{\alpha_\zeta}(1) \Leftrightarrow \frac{i_\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} = \Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1)$$

$$\text{推论7.1.9. } \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1) = \frac{i_\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Leftrightarrow \frac{i_\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} = \Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)$$

$$\text{推论7.1.10. } \Gamma_{\alpha_{1\zeta} \alpha_{2\zeta}}^{k_\zeta}(2) \Gamma_{k_\zeta}^{\beta_{1\zeta} \beta_{2\zeta}}(2) = \frac{1}{2!} \delta_{\alpha_{1\zeta}}^{(\beta_{1\zeta}} \delta_{\alpha_{2\zeta}}^{\beta_{2\zeta})} - \frac{1}{3!} \delta^{(\beta_{1\zeta} \beta_{2\zeta})} \delta_{\alpha_{1\zeta} \alpha_{2\zeta}}$$

$$\begin{aligned} \text{证明: } & \Gamma_{\alpha_{1\zeta} \alpha_{2\zeta}}^{k_\zeta}(2) \Gamma_{k_\zeta}^{\beta_{1\zeta} \beta_{2\zeta}}(2) \\ &= \left(\frac{i_\zeta}{\sqrt{2}}\right)^2 \sigma_{\alpha_{1\zeta}}^{A_{1\zeta} A_{2\zeta}} \sigma_{\alpha_{2\zeta}}^{A_{3\zeta} A_{4\zeta}} \left(\frac{i_\zeta}{\sqrt{2}}\right)^2 \sigma_{B_{1\zeta} B_{2\zeta}}^{\beta_{1\zeta}} \sigma_{B_{3\zeta} B_{4\zeta}}^{\beta_{2\zeta}} \Gamma_{A_{1\zeta} A_{2\zeta} \dots A_{2s\zeta}}^{k_\zeta}(s) \Gamma_{k_\zeta}^{B_{1\zeta} B_{2\zeta} \dots B_{2s\zeta}}(s) \\ &= \left(\frac{i_\zeta}{\sqrt{2}}\right)^2 \sigma_{\alpha_{1\zeta}}^{A_{1\zeta} A_{2\zeta}} \sigma_{\alpha_{2\zeta}}^{A_{3\zeta} A_{4\zeta}} \left(\frac{i_\zeta}{\sqrt{2}}\right)^2 \sigma_{B_{1\zeta} B_{2\zeta}}^{\beta_{1\zeta}} \sigma_{B_{3\zeta} B_{4\zeta}}^{\beta_{2\zeta}} \frac{1}{4!} \delta_{(A_{1\zeta} A_{2\zeta}}^{B_{1\zeta} B_{2\zeta}} \delta_{A_{3\zeta} A_{4\zeta}}^{B_{3\zeta} B_{4\zeta}} \\ &= \frac{1}{4!} [12 \delta_{\alpha_{1\zeta}}^{(\beta_{1\zeta}} \delta_{\alpha_{2\zeta}}^{\beta_{2\zeta})} - 8 \delta^{\beta_{1\zeta} \beta_{2\zeta}} \delta_{\alpha_{1\zeta} \alpha_{2\zeta}}] \\ &= \frac{1}{2!} \delta_{\alpha_{1\zeta} \alpha_{2\zeta}}^{(\beta_{1\zeta} \beta_{2\zeta})} - \frac{1}{3!} \delta^{(\beta_{1\zeta} \beta_{2\zeta})} \delta_{\alpha_{1\zeta} \alpha_{2\zeta}} \quad \square \end{aligned}$$

## 7.2 复合常数不变张量 $\Gamma_{abcd..}^{k_\zeta}(n), \Gamma_{k_\zeta}^{abcd..}(n)$

$$\text{定义7.2.1. } \Gamma_{k_\zeta}^{abcd..}(n) := \left(\frac{i}{2}\right)^n \underbrace{\sigma_{\zeta\alpha_\zeta}^{ab} \sigma_{\zeta\beta_\zeta}^{cd} \cdots}_{n} \cdot \Gamma_{k_\zeta}^{\alpha_\zeta\beta_\zeta \cdots}(n), \Gamma_{abcd..}^{k_\zeta}(n) := \left(\frac{i}{2}\right)^n \underbrace{\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta cd}^{\beta_\zeta} \cdots}_{2n} \cdot \Gamma_{\alpha_\zeta\beta_\zeta \cdots}^{k_\zeta}(n)$$

推论7.2.1.

$$\Gamma_{k_\zeta}^{abcd..}(n) = \left(\frac{i}{2}\right)^n \underbrace{\sigma_{\zeta\alpha_\zeta}^{ab} \sigma_{\zeta\beta_\zeta}^{cd} \cdots}_{n} \cdot \Gamma_{k_\zeta}^{\alpha_\zeta\beta_\zeta \cdots}(n) \Rightarrow \Gamma_{k_\zeta}^{\alpha_\zeta\beta_\zeta \cdots}(n) = \left(\frac{i}{2}\right)^n \underbrace{\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta cd}^{\beta_\zeta} \cdots}_{2n} \cdot \Gamma_{k_\zeta}^{abcd..}(n)$$

$$\Gamma_{abcd..}^{k_\zeta}(n) = \left(\frac{i}{2}\right)^n \underbrace{\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta cd}^{\beta_\zeta} \cdots}_{2n} \cdot \Gamma_{\alpha_\zeta\beta_\zeta \cdots}^{k_\zeta}(n) \Rightarrow \Gamma_{\alpha_\zeta\beta_\zeta \cdots}^{k_\zeta}(n) = \left(\frac{i}{2}\right)^n \underbrace{\sigma_{\zeta\alpha_\zeta}^{ab} \sigma_{\zeta\beta_\zeta}^{cd} \cdots}_{n} \cdot \Gamma_{abcd..}^{k_\zeta}(n)$$

推论7.2.2. 以下两个等式可以互推，相互等价。

$$\left\{ \begin{array}{l} \Gamma_{k_\zeta}^{abcd..}(n) = \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2n} \underbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c (\sigma, i\zeta)_{D_\zeta D'_\zeta}^d}_{2n} \cdot \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \cdots}(n) \varepsilon_{A'_\zeta B'_\zeta C'_\zeta D'_\zeta \cdots} \\ \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \cdots}(n) \varepsilon_{A'_\zeta B'_\zeta C'_\zeta D'_\zeta \cdots} = \left(\frac{i\zeta}{\sqrt{2}}\right)^{2n} \underbrace{(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} (\sigma, -i\zeta)_c^{C'_\zeta C_\zeta} (\sigma, -i\zeta)_d^{D'_\zeta D_\zeta}}_{2n} \cdot \Gamma_{k_\zeta}^{abcd..}(n) \end{array} \right.$$

推论7.2.3. 以下两个等式可以互推，相互等价。

$$\left\{ \begin{array}{l} \Gamma_{abcd..}^{k_\zeta}(n) = \left(\frac{i\zeta}{\sqrt{2}}\right)^{2n} \underbrace{(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} (\sigma, -i\zeta)_c^{C'_\zeta C_\zeta} (\sigma, -i\zeta)_d^{D'_\zeta D_\zeta}}_{2n} \cdot \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta \cdots}^{k_\zeta}(n) \varepsilon_{A'_\zeta B'_\zeta C'_\zeta D'_\zeta \cdots} \\ \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta \cdots}^{k_\zeta}(n) \varepsilon_{A'_\zeta B'_\zeta C'_\zeta D'_\zeta \cdots} = \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2n} \underbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c (\sigma, i\zeta)_{D_\zeta D'_\zeta}^d}_{2n} \cdot \Gamma_{abcd..}^{k_\zeta}(n) \end{array} \right.$$

对称性：

$$\text{推论7.2.4. } \Gamma_{k_\zeta}^{abcd..}(n) = \frac{1}{n!} \frac{1}{2^n} \Gamma_{k_\zeta}^{([ab][cd] \cdots)}(n), \Gamma_{abcd..}^{k_\zeta}(n) = \frac{1}{n!} \frac{1}{2^n} \Gamma_{([ab][cd] \cdots)}^{k_\zeta}(n)$$

对偶性：

$$\text{推论7.2.5. } \Gamma_{k_\zeta}^{abcd..}(n) = [-\zeta]^n \Gamma_{k_\zeta}^{*ab * cd \cdots}(n), \Gamma_{abcd..}^{k_\zeta}(n) = [-\zeta]^n \Gamma_{*ab * cd \cdots}^{k_\zeta}(n)$$

无迹性：

$$\text{推论7.2.6. } \delta_{ab} \Gamma_{k_\zeta}^{abcd..}(n) = 0, \delta_{ac} \Gamma_{k_\zeta}^{abcd..}(n) = 0, \delta^{ab} \Gamma_{abcd..}^{k_\zeta}(n) = 0, \delta^{ac} \Gamma_{abcd..}^{k_\zeta}(n) = 0$$

Penrose对应：

$$\left\{ \begin{array}{l} \Gamma_k^{abcd..}(n) \stackrel{P}{=} \left(\frac{1}{\sqrt{2}}\right)^n \Gamma_k^{ABCD..}(n) \varepsilon^{A'B'} \varepsilon^{C'D'} \cdots, \Gamma_{k'}^{abcd..}(n) \stackrel{P}{=} \left(\frac{1}{\sqrt{2}}\right)^n \Gamma_{k'}^{A'B'C'D'..}(n) \varepsilon^{AB} \varepsilon^{CD} \cdots \\ \Gamma_{2n}^k \stackrel{P}{=} \left(\frac{1}{\sqrt{2}}\right)^n \Gamma_{2n}^k \varepsilon^{A'B'} \varepsilon^{C'D'} \cdots, \Gamma_{2n}^{k'} \stackrel{P}{=} \left(\frac{1}{\sqrt{2}}\right)^n \Gamma_{2n}^{k'} \varepsilon^{A'B'C'D'..}(n) \varepsilon^{AB} \varepsilon^{CD} \cdots \end{array} \right. \quad (3.3)$$

类Penrose对应：

$$\left\{ \begin{array}{l} \Gamma_k^{abcd..}(n) \stackrel{P}{=} \left(\frac{1}{\sqrt{2}}\right)^n \Gamma_k^{\alpha\beta \cdots}(n) \varepsilon^{A'B'} \varepsilon^{C'D'} \cdots, \Gamma_{k'}^{abcd..}(n) \stackrel{P}{=} \left(\frac{1}{\sqrt{2}}\right)^n \Gamma_{k'}^{\alpha'\beta' \cdots}(n) \varepsilon^{AB} \varepsilon^{CD} \cdots \\ \Gamma_{2n}^k \stackrel{P}{=} \left(\frac{1}{\sqrt{2}}\right)^n \Gamma_{2n}^k \varepsilon^{A'B'} \varepsilon^{C'D'} \cdots, \Gamma_{2n}^{k'} \stackrel{P}{=} \left(\frac{1}{\sqrt{2}}\right)^n \Gamma_{2n}^{k'} \varepsilon^{\alpha'\beta' \cdots}(n) \varepsilon^{AB} \varepsilon^{CD} \cdots \end{array} \right. \quad (3.4)$$

一一对应关系：

$$\begin{cases} \Gamma_{\overset{2n}{k}}^{abcd\dots}(n) \leftrightarrow \Gamma_{\overset{n}{k}}^{\alpha\beta\dots}(n) \leftrightarrow \Gamma_{\overset{2n}{k}}^{ABCD\dots}(n), \Gamma_{\overset{2n}{k'}}^{abcd\dots}(n) \leftrightarrow \Gamma_{\overset{n}{k'}}^{\alpha'\beta'\dots}(n) \leftrightarrow \Gamma_{\overset{2n}{k'}}^{A'B'C'D'\dots}(n) \\ \Gamma_{\underset{2n}{k}}^{abcd\dots}(n) \leftrightarrow \Gamma_{\underset{n}{k}}^{\alpha\beta\dots}(n) \leftrightarrow \Gamma_{\underset{2n}{k}}^{ABCD\dots}(n), \Gamma_{\underset{2n}{k'}}^{abcd\dots}(n) \leftrightarrow \Gamma_{\underset{n}{k'}}^{\alpha'\beta'\dots}(n) \leftrightarrow \Gamma_{\underset{2n}{k'}}^{A'B'C'D'\dots}(n) \end{cases} \quad (3.5)$$

正交性：

$$\text{推论7.2.7. } \Gamma_{\overset{2n}{abcd\dots}}^{k_\zeta}(n) \Gamma_{\overset{2n}{l_\zeta}}^{abcd\dots}(n) = 2^n \delta^{k_\zeta l_\zeta}$$

### 7.3 复合常数不变张量 $N_{l_\zeta l'_\zeta a}^{k_\zeta k'_\zeta}(s)$ , $N_{k'_\zeta k_\zeta}^{l'_\zeta l_\zeta a}(s)$ 的引入

定义7.3.1.  $N_{l_\zeta l'_\zeta a}^{k_\zeta k'_\zeta}(s) := \frac{i_\zeta}{\sqrt{2}} N_{A_\zeta l_\zeta}^{k_\zeta}(s) N_{A'_\zeta l'_\zeta}^{k'_\zeta}(s) (\sigma, -i_\zeta)_a^{A'_\zeta A_\zeta}$ ,  $N_{k'_\zeta k_\zeta}^{l'_\zeta l_\zeta a}(s) := \frac{-i_\zeta}{\sqrt{2}} N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s) N_{k_\zeta}^{A_\zeta l_\zeta}(s) (\sigma, i_\zeta)_a^{A_\zeta A'_\zeta}$

推论7.3.1.  $N_{A_\zeta l_\zeta}^{k_\zeta}(s) N_{A'_\zeta l'_\zeta}^{k'_\zeta}(s) = \frac{-i_\zeta}{\sqrt{2}} N_{l_\zeta l'_\zeta a}^{k_\zeta k'_\zeta}(s) (\sigma, i_\zeta)_a^{A_\zeta A'_\zeta}$ ,  $N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s) N_{k_\zeta}^{A_\zeta l_\zeta}(s) = \frac{i_\zeta}{\sqrt{2}} N_{k'_\zeta k_\zeta}^{l'_\zeta l_\zeta a}(s) (\sigma, -i_\zeta)_a^{A'_\zeta A_\zeta}$

Penrose记法：

$$N_{l'_\zeta l_\zeta a}^{k k'} \stackrel{P}{=} N_{Al}^k(s) N_{A'l'}^{k'}(s) \quad N_{k'_\zeta k_\zeta}^{l'_\zeta l_\zeta a} \stackrel{P}{=} N_{k'l'}^{A'l'}(s) N_{k}^{Al}(s) \quad (3.6)$$

### 7.4 复合常数不变张量 $\Gamma_{abcd\dots}^{a_1 b_1 c_1 d_1 \dots a_2 b_2 c_2 d_2 \dots}(n)$ 的引入

定义7.4.1.  $\Gamma_{\overset{2n}{abcd\dots}}^{a_1 b_1 c_1 d_1 \dots a_2 b_2 c_2 d_2 \dots}(n) := \Gamma_{\overset{2n}{abcd\dots}}^{k_\zeta k'_\zeta}(n) \Gamma_{\overset{2n}{k_\zeta}}^{a_1 b_1 c_1 d_1 \dots}(n) \Gamma_{\overset{2n}{k'_\zeta}}^{a_2 b_2 c_2 d_2 \dots}(n)$

### 7.5 复合常数不变张量 $\Gamma_{\alpha_\zeta \beta_\zeta \dots}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots}(n)$ , $\Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}^{\alpha_\zeta \beta_\zeta \dots}(n)$ 的引入

定义7.5.1.

$$\Gamma_{\overset{2n}{\alpha_\zeta \beta_\zeta \dots}}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots}(n) := \Gamma_{\overset{2n}{\alpha_\zeta \beta_\zeta \dots}}^{k_\zeta}(n) \Gamma_{\overset{2n}{k_\zeta}}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots}(n) = \left(\frac{i_\zeta}{\sqrt{2}}\right)^n \frac{1}{(2n)!} \underbrace{\sigma_{\alpha_\zeta}^{(A_\zeta B_\zeta C_\zeta D_\zeta \dots)}}_n$$

$$\Gamma_{\overset{2n}{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}^{\alpha_\zeta \beta_\zeta \dots}(n) := \Gamma_{\overset{2n}{\alpha_\zeta \beta_\zeta \dots}}^{k_\zeta}(n) \Gamma_{\overset{2n}{k_\zeta}}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots}(n) = \left(\frac{i_\zeta}{\sqrt{2}}\right)^n \frac{1}{(2n)!} \underbrace{\sigma_{(A_\zeta B_\zeta C_\zeta D_\zeta \dots)}^{\alpha_\zeta \beta_\zeta \dots}}_n$$

## 8 复合常数不变张量 $\Gamma_{abc\dots}^{k_\zeta k'_\zeta}(s)$ , $\Gamma_{k'_\zeta k_\zeta}^{abc\dots}(s)$

### 8.1 复合常数不变张量 $\Gamma_{abc\dots}^{k_\zeta k'_\zeta}(s)$ , $\Gamma_{k'_\zeta k_\zeta}^{abc\dots}(s)$ 的引入

定义8.1.1.

$$\begin{cases} \Gamma_{\overset{2s}{abc\dots}}^{k'_\zeta k_\zeta}(s) := \left(\frac{i_\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma, -i_\zeta)_a^{A'_\zeta A_\zeta} (\sigma, -i_\zeta)_b^{B'_\zeta B_\zeta} (\sigma, -i_\zeta)_c^{C'_\zeta C_\zeta} \dots}^{2s} \Gamma_{\overset{2s}{A'_\zeta B'_\zeta C'_\zeta \dots}}^{k'_\zeta}(s) \Gamma_{\overset{2s}{A_\zeta B_\zeta C_\zeta \dots}}^{k_\zeta}(s) \\ \Gamma_{\overset{2s}{k'_\zeta k_\zeta}}^{abc\dots}(s) := \left(\frac{-i_\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma, i_\zeta)_a^{A_\zeta A'_\zeta} (\sigma, i_\zeta)_b^{B_\zeta B'_\zeta} (\sigma, i_\zeta)_c^{C_\zeta C'_\zeta} \dots}^{2s} \Gamma_{\overset{2s}{A_\zeta B_\zeta C_\zeta \dots}}^{k_\zeta}(s) \Gamma_{\overset{2s}{A'_\zeta B'_\zeta C'_\zeta \dots}}^{k'_\zeta}(s) \end{cases}$$

$\Leftrightarrow$

推论8.1.1.

$$\begin{cases} \Gamma_{\overset{2s}{A'_\zeta B'_\zeta C'_\zeta \dots}}^{k'_\zeta}(s) \Gamma_{\overset{2s}{A_\zeta B_\zeta C_\zeta \dots}}^{k_\zeta}(s) = \left(\frac{-i_\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma, i_\zeta)_a^{A_\zeta A'_\zeta} (\sigma, i_\zeta)_b^{B_\zeta B'_\zeta} (\sigma, i_\zeta)_c^{C_\zeta C'_\zeta} \dots}^{2s} \Gamma_{\overset{2s}{abc\dots}}^{k'_\zeta k_\zeta}(s) \\ \Gamma_{\overset{2s}{A_\zeta B_\zeta C_\zeta \dots}}^{k_\zeta}(s) \Gamma_{\overset{2s}{A'_\zeta B'_\zeta C'_\zeta \dots}}^{k'_\zeta}(s) = \left(\frac{i_\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma, -i_\zeta)_a^{A'_\zeta A_\zeta} (\sigma, -i_\zeta)_b^{B'_\zeta B_\zeta} (\sigma, -i_\zeta)_c^{C'_\zeta C_\zeta} \dots}^{2s} \Gamma_{\overset{2s}{k'_\zeta k_\zeta}}^{abc\dots}(s) \end{cases}$$

非协变关系：

推论8.1.2.

$$\begin{cases} \Gamma_{abc \dots}^{k_\zeta k'_\zeta} (s) \overbrace{\partial^a \partial^b \partial^c \dots}^{2s} \simeq (-1)^{2s} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\partial_a^+ \partial_b^+ \partial_c^+ \dots}^{2s} \\ \Gamma_{k'_\zeta k_\zeta}^{abc \dots} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s} \simeq (-1)^{2s} \Gamma_{abc \dots}^{k'_\zeta k_\zeta} (s) \overbrace{\partial^{+a} \partial^{+b} \partial^{+c} \dots}^{2s} \end{cases}$$

全对称性:

$$\text{推论8.1.3. } \Gamma_{abc \dots}^{k'_\zeta k_\zeta} (s) = \frac{1}{(2s)!} \Gamma_{abc \dots}^{k'_\zeta k_\zeta} (s), \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) = \frac{1}{(2s)!} \Gamma_{k_\zeta k'_\zeta}^{(abc \dots)} (s)$$

无迹性:

$$\text{推论8.1.4. } \delta^{ab} \Gamma_{abc \dots}^{k'_\zeta k_\zeta} (s) = 0, \delta_{ab} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) = 0$$

Penrose记法:

$$\Gamma_{abc \dots}^{k' k} (s) \stackrel{P}{=} \Gamma_{A' B' C' \dots}^{k'} (s) \Gamma_{ABC \dots}^k (s) \quad \Gamma_{kk'}^{abc \dots} (s) \stackrel{P}{=} \Gamma_k^{ABC \dots} (s) \Gamma_{k'}^{A' B' C' \dots} (s) \quad (3.7)$$

正交性:

$$\text{推论8.1.5. } \Gamma_{abc \dots}^{k'_\zeta k_\zeta} (s) \Gamma_{l'_\zeta l'_\zeta}^{abc \dots} (s) = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta}$$

## 8.2 复合常数不变张量 $\Gamma_{abc \dots}^{k_\zeta k'_\zeta} (s, w)$ , $\Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s, w)$ 的引入

定义8.2.1.

$$\begin{cases} \Gamma_{abc \dots}^{k'_\zeta k_\zeta} (s, w) := \left(\frac{i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} (\sigma, -i\zeta)_c^{C'_\zeta C_\zeta} \dots}^{2s, w} \cdot \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k'_\zeta} (s, w) \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s, w) \\ \Gamma_{k'_\zeta k_\zeta}^{abc \dots} (s, w) := \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma, i\zeta)_{A'_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B'_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C'_\zeta C'_\zeta}^c \dots}^{2s, w} \cdot \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s, w) \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots} (s, w) \end{cases}$$

$\Leftrightarrow$

定义8.2.2.

$$\begin{cases} \Gamma_{abc \dots} (s, -\zeta; w) := \left(\frac{i\zeta}{\sqrt{2}}\right)^{2s} \bar{\Gamma}(s) \overbrace{(\sigma, -i\zeta)_a \otimes (\sigma, -i\zeta)_b \otimes (\sigma, -i\zeta)_c \otimes \dots}^{2s} \Gamma(s) \\ \Gamma_{abc \dots} (s, \zeta; w) := \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \bar{\Gamma}(s) \overbrace{(\sigma, i\zeta)^a \otimes (\sigma, i\zeta)^b \otimes (\sigma, i\zeta)^c \otimes \dots}^{2s} \Gamma(s) \end{cases}$$

$$\text{推论8.2.1. } \Gamma_{abc \dots} (s, \zeta; w) = \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \bar{N}(s; w) [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \overbrace{(\sigma, i\zeta)^a \otimes (\sigma, i\zeta)^b \otimes (\sigma, i\zeta)^c \otimes \dots}^{2s} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] N(s; w)$$

$$\text{推论8.2.2. } \Gamma_{abc \dots} (s, \zeta; w) = \left(\frac{-i\zeta}{\sqrt{2}}\right) \bar{N}(s; w) [(\sigma, i\zeta)^a \otimes \Gamma_{bc \dots}^{2s-1} (s - \frac{1}{2}, \zeta; w)] N(s; w)$$

## 8.3 复合常数不变张量 $\Gamma_{abcd \dots}^{\alpha_\zeta \alpha'_\zeta \beta_\zeta \beta'_\zeta} (n)$ , $\Gamma_{\alpha_\zeta \alpha'_\zeta \beta_\zeta \beta'_\zeta}^{abcd \dots} (n)$ 的引入

定义8.3.1.

$$\Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta} (n) := \left(\frac{i\zeta}{\sqrt{2}}\right)^n \underbrace{\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \dots}_{n} \cdot \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}^{k_\zeta} (n), \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots} (n) := \left(\frac{i\zeta}{\sqrt{2}}\right)^n \overbrace{\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots}_{n} \cdot \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots} (n)$$



定义8.3.2.

$$\left\{ \begin{array}{l} \Gamma_{abc \dots}^{k'_\zeta k_\zeta}(s) := \overbrace{\left( \frac{i\zeta}{\sqrt{2}} \right)^{2s} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} (\sigma, -i\zeta)_c^{C'_\zeta C_\zeta} \dots}^{2s} \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k'_\zeta}(s) \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s) \\ \Gamma_{k'_\zeta k_\zeta}^{abc \dots}(s) := \overbrace{\left( \frac{-i\zeta}{\sqrt{2}} \right)^{2s} (\sigma, i\zeta)_{A'_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B'_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C'_\zeta C'_\zeta}^c \dots}^{2s} \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s) \end{array} \right.$$

性质8.3.1.  $\Gamma_{A_{1\zeta} A_{2\zeta} \dots A_{2s\zeta}}^{k_\zeta}(s) \Gamma_{k_\zeta}^{B_{1\zeta} B_{2\zeta} \dots B_{2s\zeta}}(s) = \frac{1}{(2s)!} \delta_{A_{1\zeta}}^{(B_{1\zeta}} \delta_{A_{2\zeta}}^{B_{2\zeta}} \dots \delta_{A_{2s\zeta}}^{B_{2s\zeta}}) = \frac{1}{(2s)!} \delta_{(A_{1\zeta}}^{B_{1\zeta}} \delta_{A_{2\zeta}}^{B_{2\zeta}} \dots \delta_{A_{2s\zeta}}^{B_{2s\zeta}})$ 

定义8.3.3.

$$\left\{ \begin{array}{l} \Gamma_{abc \dots}^{\alpha'_\zeta \alpha_\zeta \beta'_\zeta \beta_\zeta \dots}(n) := \Gamma_{k'_\zeta}^{\alpha'_\zeta \beta'_\zeta \dots}(n) \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) \Gamma_{abc \dots}^{k'_\zeta k_\zeta}(n) \\ = \left( -\frac{1}{4} \right)^n \overbrace{\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \sigma_{C'_\zeta D'_\zeta}^{\beta'_\zeta} \dots}^n \overbrace{(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} (\sigma, -i\zeta)_c^{C'_\zeta C_\zeta} \dots}^{2n} \\ \Gamma_{\alpha'_\zeta \alpha_\zeta \beta'_\zeta \beta_\zeta \dots}^{abc \dots}(n) := \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n) \Gamma_{\alpha'_\zeta \beta'_\zeta \dots}^{k'_\zeta}(n) \Gamma_{abc \dots}^{k'_\zeta k_\zeta}(n) \\ = \left( -\frac{1}{4} \right)^n \overbrace{\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \dots \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \sigma_{\beta'_\zeta}^{C'_\zeta D'_\zeta} \dots}^n \overbrace{(\sigma, i\zeta)_{A'_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B'_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C'_\zeta C'_\zeta}^c \dots}^{2n} \end{array} \right.$$

⇒

推论8.3.1.

$$\left\{ \begin{array}{l} \Gamma_{abc \dots}^{k'_\zeta k_\zeta}(n) := \Gamma_{\alpha'_\zeta \beta'_\zeta \dots}^{k'_\zeta}(n) \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n) \Gamma_{abc \dots}^{\alpha'_\zeta \alpha_\zeta \beta'_\zeta \beta_\zeta \dots}(n) \\ \Gamma_{k'_\zeta k_\zeta}^{abc \dots}(n) := \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n) \Gamma_{\alpha'_\zeta \beta'_\zeta \dots}^{k'_\zeta}(n) \Gamma_{\alpha'_\zeta \alpha_\zeta \beta'_\zeta \beta_\zeta \dots}^{abc \dots}(n) \end{array} \right.$$

推论8.3.2.

$$\left\{ \begin{array}{l} \Gamma_{ab}^{\alpha'_\zeta \alpha_\zeta}(1) := \Gamma_{k'_\zeta}^{\alpha'_\zeta}(1) \Gamma_{k_\zeta}^{\alpha_\zeta}(1) \Gamma_{ab}^{k'_\zeta k_\zeta}(1) \\ \Gamma_{\alpha'_\zeta \alpha_\zeta}^{ab}(1) := \Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Gamma_{\alpha'_\zeta}^{k'_\zeta}(1) \Gamma_{k'_\zeta k_\zeta}^{ab}(1) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \Gamma_{ab}^{k'_\zeta k_\zeta}(1) := \Gamma_{\alpha'_\zeta}^{k'_\zeta}(1) \Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Gamma_{ab}^{\alpha'_\zeta \alpha_\zeta}(1) \\ \Gamma_{k'_\zeta k_\zeta}^{ab}(1) := \Gamma_{k_\zeta}^{\alpha_\zeta}(1) \Gamma_{k'_\zeta}^{\alpha'_\zeta}(1) \Gamma_{\alpha'_\zeta \alpha_\zeta}^{ab}(1) \end{array} \right\} \left\{ \begin{array}{l} \Gamma_{ab}^{\alpha'_\zeta \alpha_\zeta}(1) = \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} \\ \Gamma_{\alpha'_\zeta \alpha_\zeta}^{ab}(1) = \sigma_{\alpha'_\zeta \alpha_\zeta}^{ab} \end{array} \right.$$

8.4 复合常数不变张量  $\Gamma_{abc \dots}^{k'_\zeta k_\zeta}(s), \Gamma_{k'_\zeta k_\zeta}^{abc \dots}(s)$  前几项的具体展开证明:  $\Gamma_{k'_\zeta k_\zeta}^{\pi \pi \pi \dots}(s)$ 

$$\begin{aligned} &= \left( \frac{-i\zeta}{\sqrt{2}} \right)^{2s} \overbrace{(i\zeta)_{A'_\zeta A'_\zeta} (i\zeta)_{B'_\zeta B'_\zeta} (i\zeta)_{C'_\zeta C'_\zeta} \dots}^{2s} \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s) \\ &= \left( \frac{1}{\sqrt{2}} \right)^{2s} \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s) \overbrace{\delta_{A'_\zeta A'_\zeta} \delta_{B'_\zeta B'_\zeta} \delta_{C'_\zeta C'_\zeta} \dots}^{2s} \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s) \\ &= \left( \frac{1}{\sqrt{2}} \right)^{2s} \delta_{k'_\zeta k_\zeta} \end{aligned} \quad \square$$

证明:  $\Gamma_{k'_\zeta k_\zeta}^{i \pi \pi \dots}(s)$ 

$$\begin{aligned} &= \left( \frac{-i\zeta}{\sqrt{2}} \right)^{2s} \overbrace{(\sigma)_{A'_\zeta A'_\zeta}^i (i\zeta)_{B'_\zeta B'_\zeta} (i\zeta)_{C'_\zeta C'_\zeta} \dots}^{2s} \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s) \\ &= -i\zeta \left( \frac{1}{\sqrt{2}} \right)^{2s} \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s) \overbrace{(\sigma)_{A'_\zeta A'_\zeta}^i \delta_{B'_\zeta B'_\zeta} \delta_{C'_\zeta C'_\zeta} \dots}^{2s} \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s) \\ &= -i\zeta \left( \frac{1}{\sqrt{2}} \right)^{2s} \frac{1}{s} \sigma^i(s)_{k'_\zeta k_\zeta} \end{aligned} \quad \square$$

$$\begin{aligned}
& \text{证明: } \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij\pi\cdots}}^{2s}(s) \\
&= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma)_{A_\zeta A'_\zeta}^i (\sigma)_{B_\zeta B'_\zeta}^j (i\zeta)_{C_\zeta C'_\zeta} \cdots}^{2s} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}}^{2s}(s) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \cdots}}^{2s}(s) \\
&= -\left(\frac{1}{\sqrt{2}}\right)^{2s} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}}^{2s}(s) \overbrace{(\sigma)_{A_\zeta A'_\zeta}^i (\sigma)_{B_\zeta B'_\zeta}^j \delta_{C_\zeta C'_\zeta} \cdots}^{2s} \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \cdots}}^{2s}(s) \\
&= -\left(\frac{1}{\sqrt{2}}\right)^{2s} \frac{1}{2s(s-\frac{1}{2})} [\{\sigma^i(s), \sigma^j(s)\} - s\delta^{ij}]_{k_\zeta k'_\zeta}
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ijk\pi\cdots}}^{2s}(s) \\
&= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma)_{A_\zeta A'_\zeta}^i (\sigma)_{B_\zeta B'_\zeta}^j (\sigma)_{C_\zeta C'_\zeta}^k (i\zeta)_{D_\zeta D'_\zeta} \cdots}^{2s} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \cdots}}^{2s}(s) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta D'_\zeta \cdots}}^{2s}(s) \\
&= \left(\frac{1}{\sqrt{2}}\right)^{2s} \frac{i\zeta}{2s(s-\frac{1}{2})(s-1)} \{\sigma^i(s) [\sigma^j(s) \sigma^k(s)] - [(s-1)\sigma^i(s) \delta^{jk} + s\delta^{ij} \sigma^k(s)]\}_{k_\zeta k'_\zeta}
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ijkl\cdots}}^{2s}(s) \partial_i \partial_j \partial_k \partial_l \\
&= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma)_{A_\zeta A'_\zeta}^i (\sigma)_{B_\zeta B'_\zeta}^j (\sigma)_{C_\zeta C'_\zeta}^k (\sigma)_{D_\zeta D'_\zeta}^l \cdots}^{2s} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \cdots}}^{2s}(s) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta D'_\zeta \cdots}}^{2s}(s) \\
&= \left(\frac{1}{\sqrt{2}}\right)^{2s} \frac{1}{s(s-\frac{1}{2})(s-1)(s-\frac{3}{2})} [\sigma^i(s) \sigma^j(s) \sigma^k(s) \sigma^l(s) + (2-3s)\sigma^i(s) \sigma^j(s) \delta^{kl} + \frac{3s(s-1)}{4} \delta^{ij} \delta^{kl}]_{k_\zeta k'_\zeta} \partial_i \partial_j \partial_k \partial_l
\end{aligned}$$

□

### 8.5 $\Gamma_+^{abc\cdots}(s)$ 和 $\Gamma_-^{abc\cdots}(s)$ 的定义

定义8.5.1.  $odd := -, even := +$

$$\text{定义8.5.2. } \begin{cases} \overbrace{\Gamma_+^{abc\cdots}}^{2s}(s) := 1 \cdot \overbrace{\Gamma^{ij\cdots\pi\cdots\pi}}^{2s-2l-1, 2l}(s), 1 \cdot \overbrace{\Gamma^{ij\cdots\pi\cdots\pi}}^{2s-2l-1, 2l+1}(s), l = 0, \dots, 2s \\ \overbrace{\Gamma_-^{abc\cdots}}^{2s}(s) := 0 \cdot \overbrace{\Gamma^{ij\cdots\pi\cdots\pi}}^{2s-2l-1, 2l}(s), 1 \cdot \overbrace{\Gamma^{ij\cdots\pi\cdots\pi}}^{2s-2l-1, 2l+1}(s), l = 0, \dots, 2s \end{cases}$$

$$\text{推论8.5.1. } \overbrace{\Gamma^{abc\cdots}}^{2s}(s) = \overbrace{\Gamma_+^{abc\cdots}}^{2s}(s) + \overbrace{\Gamma_-^{abc\cdots}}^{2s}(s)$$

### 8.6 算符 $\Gamma_\pm^{abc\cdots}(s) p_a p_b p_c \cdots$ 和 $\Gamma_\pm^{abc\cdots}(s) \partial_a \partial_b \partial_c \cdots$ 的基本性质

$$\text{性质8.6.1. } \begin{cases} \overbrace{\Gamma^{abc\cdots}}^{2s}(s) \overbrace{p_a p_b p_c \cdots}^{2s} = \sum_{n=0}^{2s} C_{2s}^n \overbrace{\Gamma^{ij\cdots\pi\cdots\pi}}^{2s-n, n}(s) \overbrace{p_i p_j \cdots}^{2s-n} p_\pi^n \\ \overbrace{\Gamma^{abc\cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} = \sum_{n=0}^{2s} C_{2s}^n \overbrace{\Gamma^{ij\cdots\pi\cdots\pi}}^{2s-n, n}(s) \overbrace{\partial_i \partial_j \cdots}^{2s-n} \partial_\pi^n \end{cases}$$

$$\text{性质8.6.2. } \begin{cases} \overbrace{\Gamma_+^{abc\cdots}}^{2s}(s) \overbrace{p_a p_b p_c \cdots}^{2s} := \sum_{l=0}^{[s]} C_{2s}^{2l} \overbrace{\Gamma^{ij\cdots\pi\cdots\pi}}^{2s-2l, 2l}(s) \overbrace{p_i p_j \cdots}^{2s-2l} p_\pi^{2l} \\ \overbrace{\Gamma_+^{abc\cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} := \sum_{l=0}^{[s]} C_{2s}^{2l} \overbrace{\Gamma^{ij\cdots\pi\cdots\pi}}^{2s-2l, 2l}(s) \overbrace{\partial_i \partial_j \cdots}^{2s-2l} \partial_\pi^{2l} \end{cases}$$

$$\text{性质8.6.3. } \begin{cases} \overbrace{\Gamma_-^{abc\cdots}}^{2s}(s) \overbrace{p_a p_b p_c \cdots}^{2s} := \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \overbrace{\Gamma^{ij\cdots\pi\cdots\pi}}^{2s-2l-1, 2l+1}(s) \overbrace{p_i p_j \cdots}^{2s-2l-1} p_\pi^{2l+1} \\ \overbrace{\Gamma_-^{abc\cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} := \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \overbrace{\Gamma^{ij\cdots\pi\cdots\pi}}^{2s-2l-1, 2l+1}(s) \overbrace{\partial_i \partial_j \cdots}^{2s-2l-1} \partial_\pi^{2l+1} \end{cases}$$

$$\text{性质8.6.4. } \begin{cases} \overbrace{\Gamma^{abc\cdots}}^{2s}(s) \overbrace{p_a p_b p_c \cdots}^{2s} = \overbrace{\Gamma_+^{abc\cdots}}^{2s}(s) \overbrace{p_a p_b p_c \cdots}^{2s} + \overbrace{\Gamma_-^{abc\cdots}}^{2s}(s) \overbrace{p_a p_b p_c \cdots}^{2s} \\ \overbrace{\Gamma^{abc\cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} = \overbrace{\Gamma_+^{abc\cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} + \overbrace{\Gamma_-^{abc\cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \end{cases}$$

### 8.7 算符 $\hat{p}_a$ 和 $\hat{\partial}_a$ 的定义

定义8.7.1.  $\hat{p}_a := \frac{p_a}{|\vec{p}|} = (\hat{p}, i); \hat{p} = \frac{\vec{p}}{|\vec{p}|}, \hat{p}_\pi = \frac{p_\pi}{|\vec{p}|} = i; \hat{p}^2 = 1, \hat{p}_\pi^2 = i^2$

定义8.7.2.  $\hat{\partial}_a := \frac{\partial_a}{i\sqrt{-\nabla^2}} = \frac{-i\partial_a}{\sqrt{-\nabla^2}} = \frac{(-i\nabla, -\partial_t)}{\sqrt{-\nabla^2}}; \hat{\nabla} = \frac{\nabla}{i\sqrt{-\nabla^2}} = \frac{-i\nabla}{\sqrt{-\nabla^2}}; \hat{\nabla}^2 = 1, \hat{\nabla}_\pi^2 = i^2$

推论8.7.1.  $p_a \simeq -i\partial_a, |\vec{p}| \simeq \sqrt{-\nabla^2}, \hat{p}_a \simeq \hat{\partial}_a, p_a = |\vec{p}|\hat{p}_a, \partial_a = (i\sqrt{-\nabla^2})\hat{\partial}_a$

### 8.8 算符 $\Gamma^{abc\dots}(s)\hat{p}_a\hat{p}_b\hat{p}_c\dots$ 和 $\Gamma^{abc\dots}(s)\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots$ 的基本性质

$$\begin{aligned} \text{推论8.8.1.} & \left\{ \begin{aligned} \Gamma^{abc\dots}(s) \overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} &= |\vec{p}|^{2s} \Gamma^{abc\dots}(s) \overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} \\ \Gamma^{abc\dots}(s) \overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots}^{2s} &= (i\sqrt{-\nabla^2})^{2s} \Gamma^{abc\dots}(s) \overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots}^{2s} \end{aligned} \right. \\ \text{性质8.8.1.} & \left\{ \begin{aligned} \Gamma^{abc\dots}(s) \overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} &= \sum_{n=0}^{2s} i^n C_{2s}^n \Gamma^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{p}_i\hat{p}_j\dots}^{2s-n} = \sum_{n=0}^{2s} i^{2s-n} C_{2s}^n \Gamma^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{p}_i\hat{p}_j\dots}^n \\ \Gamma^{abc\dots}(s) \overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots}^{2s, \hat{\partial}_\pi \rightarrow i} &= \sum_{n=0}^{2s} i^n C_{2s}^n \Gamma^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{\partial}_i\hat{\partial}_j\dots}^{2s-n} = \sum_{n=0}^{2s} i^{2s-n} C_{2s}^n \Gamma^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{\partial}_i\hat{\partial}_j\dots}^n \end{aligned} \right. \\ \text{性质8.8.2.} & \left\{ \begin{aligned} \Gamma_+^{abc\dots}(s) \overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} &= \sum_{l=0}^{[s]} (-1)^l C_{2s}^{2l} \Gamma^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{p}_i\hat{p}_j\dots}^{2s-2l} \\ \Gamma_+^{abc\dots}(s) \overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots}^{2s, \hat{\partial}_\pi \rightarrow i} &= \sum_{l=0}^{[s]} (-1)^l C_{2s}^{2l} \Gamma^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{\partial}_i\hat{\partial}_j\dots}^{2s-2l} \end{aligned} \right. \\ \text{性质8.8.3.} & \left\{ \begin{aligned} \Gamma_-^{abc\dots}(s) \overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} &= i \sum_{l=0}^{[s-\frac{1}{2}]} (-1)^l C_{2s}^{2l+1} \Gamma^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{p}_i\hat{p}_j\dots}^{2s-2l-1} \\ \Gamma_-^{abc\dots}(s) \overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots}^{2s, \hat{\partial}_\pi \rightarrow i} &= i \sum_{l=0}^{[s-\frac{1}{2}]} (-1)^l C_{2s}^{2l+1} \Gamma^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{\partial}_i\hat{\partial}_j\dots}^{2s-2l-1} \end{aligned} \right. \end{aligned}$$

### 8.9 算符 $\Gamma_{k_\zeta k'_\zeta}^{ij\dots\pi\dots\pi}(s)\hat{p}_i\hat{p}_j\dots$ 和 $\Gamma_{k_\zeta k'_\zeta}^{ij\dots\pi\dots\pi}(s)\hat{\partial}_i\hat{\partial}_j\dots$ 的展开式

推论8.9.1.  $\Gamma_{k_\zeta k'_\zeta}^{ij\dots\pi\dots\pi}(s) = (-i\zeta)^n 2^{n-s} \Gamma_{k_\zeta k'_\zeta}^{ij\dots\pi\dots\pi}(s, n) = \frac{(-i\zeta)^n}{2^{s-n}} \frac{1}{n!} \sum_k^{[n/2]} c(s, n; n-2k) \Omega^{n-2k}(s)$

$$\begin{aligned} \text{证明: } & \Gamma_{k_\zeta k'_\zeta}^{ij\dots\pi\dots\pi}(s) \\ &= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j \dots (i\zeta)_{P_\zeta P'_\zeta} (i\zeta)_{Q_\zeta Q'_\zeta} \dots \Gamma_{k_\zeta}^{A_\zeta B_\zeta \dots P_\zeta Q_\zeta \dots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta \dots P'_\zeta Q'_\zeta \dots}(s)}^n \\ &= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} 2^n (i\zeta)^{2s-n} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta \dots P_\zeta Q_\zeta \dots}(s) \sigma(\frac{1}{2})_{A_\zeta A'_\zeta}^i \sigma(\frac{1}{2})_{B_\zeta B'_\zeta}^j \dots \delta_{P_\zeta P'_\zeta} \delta_{Q_\zeta Q'_\zeta} \dots \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta \dots P'_\zeta Q'_\zeta \dots}(s)}^n \\ &= (-i\zeta)^n 2^{n-s} \Gamma_{k_\zeta k'_\zeta}^{ij\dots\pi\dots\pi}(s, n) = \frac{(-i\zeta)^n}{2^{s-n}} \frac{1}{n!} \sum_{k=0}^{[n/2]} c(s, n; n-2k) \Omega^{n-2k}(s) \quad \square \end{aligned}$$

$$\text{推论8.9.2.} \left\{ \begin{aligned} \Gamma_{k_\zeta k'_\zeta}^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{p}_i\hat{p}_j\dots}^n &= \frac{(-i\zeta)^n}{2^{s-n}} \sum_{k=0}^{[n/2]} c(s, n; n-2k) [\sigma(s) \cdot \hat{p}]^{n-2k} \\ \Gamma_{k_\zeta k'_\zeta}^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{\partial}_i\hat{\partial}_j\dots}^n &= \frac{(-i\zeta)^n}{2^{s-n}} \sum_{k=0}^{[n/2]} c(s, n; n-2k) [\sigma(s) \cdot \hat{\nabla}]^{n-2k} \end{aligned} \right.$$

$$\text{推论8.9.3.} \left\{ \begin{aligned} \Gamma_{k_\zeta k'_\zeta}^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{p}_i\hat{p}_j\dots}^{2s-n} &= \frac{(-i\zeta)^{2s-n}}{2^{n-s}} \sum_{k=0}^{[(2s-n)/2]} c(s, 2s-n; 2s-n-2k) [\sigma(s) \cdot \hat{p}]^{2s-n-2k} \\ \Gamma_{k_\zeta k'_\zeta}^{ij\dots\pi\dots\pi}(s) \overbrace{\hat{\partial}_i\hat{\partial}_j\dots}^{2s-n} &= \frac{(-i\zeta)^{2s-n}}{2^{n-s}} \sum_{k=0}^{[(2s-n)/2]} c(s, 2s-n; 2s-n-2k) [\sigma(s) \cdot \hat{\nabla}]^{2s-n-2k} \end{aligned} \right.$$

8.10 算符 $\Gamma^{abc\dots}(s)\hat{p}_a\hat{p}_b\hat{p}_c\dots$ 和 $\Gamma^{abc\dots}(s)\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots$ 的展开式

$$\text{性质8.10.1.} \quad \begin{cases} \Gamma^{abc\dots}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} = \frac{i^{2s}}{2^s} \sum_{n=0}^{2s} \sum_{k=0}^{[n/2]} C_{2s}^n(-2\zeta)^n c(s, n; n-2k) [\sigma(s) \cdot \hat{p}]^{n-2k}(s) \\ \Gamma^{abc\dots}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots}^{2s, \hat{\partial}_\pi \rightarrow i} = \frac{i^{2s}}{2^s} \sum_{n=0}^{2s} \sum_{k=0}^{[n/2]} C_{2s}^n(-2\zeta)^n c(s, n; n-2k) [\sigma(s) \cdot \hat{\nabla}]^{n-2k}(s) \end{cases}$$

$$\text{性质8.10.2.} \quad \begin{cases} \Gamma^{abc\dots}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} = \frac{i^{2s}}{2^s} \sum_{n=0}^{2s} \sum_{k=0}^{[(2s-n)/2]} C_{2s}^n(-2\zeta)^{2s-n} c(s, 2s-n; 2s-n-2k) [\sigma(s) \cdot \hat{p}]^{2s-n-2k}(s) \\ \Gamma^{abc\dots}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots}^{2s, \hat{\partial}_\pi \rightarrow i} = \frac{i^{2s}}{2^s} \sum_{n=0}^{2s} \sum_{k=0}^{[(2s-n)/2]} C_{2s}^n(-2\zeta)^{2s-n} c(s, 2s-n; 2s-n-2k) [\sigma(s) \cdot \hat{\nabla}]^{2s-n-2k}(s) \end{cases}$$

$$\text{性质8.10.3.} \quad \begin{cases} \Gamma_+^{abc\dots}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} = \frac{(-i\zeta)^{2s}}{2^{-s}} \sum_{l=0}^{[s]} \sum_{k=0}^{[s-l]} C_{2s}^{2l} 2^{-2l} c(s, 2s-2l; 2s-2l-2k) [\sigma(s) \cdot \hat{p}]^{2s-2l-2k} \\ \Gamma_+^{abc\dots}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots}^{2s, \hat{\partial}_\pi \rightarrow i} = \frac{(-i\zeta)^{2s}}{2^{-s}} \sum_{l=0}^{[s]} \sum_{k=0}^{[s-l]} C_{2s}^{2l} 2^{-2l} c(s, 2s-2l; 2s-2l-2k) [\sigma(s) \cdot \hat{\nabla}]^{2s-2l-2k} \end{cases}$$

$$\text{性质8.10.4.} \quad \begin{cases} \Gamma_-^{abc\dots}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} = i \frac{(-i\zeta)^{2s-1}}{2^{1-s}} \sum_{l=0}^{[s-\frac{1}{2}]} \sum_{k=0}^{[s-\frac{1}{2}-l]} 2^{-2l} C_{2s}^{2l+1} c(s, 2s-2l-1; 2s-2l-1-2k) [\sigma(s) \cdot \hat{p}]^{2s-2l-1-2k} \\ \Gamma_-^{abc\dots}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots}^{2s, \hat{\partial}_\pi \rightarrow i} = i \frac{(-i\zeta)^{2s-1}}{2^{1-s}} \sum_{l=0}^{[s-\frac{1}{2}]} \sum_{k=0}^{[s-\frac{1}{2}-l]} 2^{-2l} C_{2s}^{2l+1} c(s, 2s-2l-1; 2s-2l-1-2k) [\sigma(s) \cdot \hat{\nabla}]^{2s-2l-1-2k} \end{cases}$$

8.11 算符 $\Gamma^{abc\dots}(s)\hat{p}_a\hat{p}_b\hat{p}_c\dots$ 和 $\Gamma^{abc\dots}(s)\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots$ 展开系数的线性代数解法

$$\text{性质8.11.1.} \quad \Gamma^{abc\dots}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} = \sum_{n=0}^{2s} C_n [\sigma(s) \cdot \hat{p}]^n, \quad \Gamma^{abc\dots}(s)\overbrace{\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c\dots}^{2s, \hat{\partial}_\pi \rightarrow i} = \sum_{n=0}^{2s} C_n [\sigma(s) \cdot \hat{\nabla}]^n$$

$$\text{证明:} \quad \lambda^+(\hat{p}, h; s) \Gamma^{abc\dots}(s)\overbrace{\hat{p}_a\hat{p}_b\hat{p}_c\dots}^{2s} \lambda(\hat{p}, h; s) = \lambda^+(\hat{p}, h; s) \sum_{n=0}^{2s} C_n(\zeta) [\sigma(s) \cdot \hat{p}]^n(s) \lambda(\hat{p}, h; s)$$

$$\Leftrightarrow \lambda^+(\hat{p}, h; s) (i\sqrt{2})^{2s} \lambda(\hat{p}, -s\zeta) \lambda^+(\hat{p}, -s\zeta) \lambda(\hat{p}, h; s) = \sum_{n=0}^{2s} C_n(\zeta) \lambda^+(\hat{p}, h; s) [\sigma(s) \cdot \hat{p}]^n(s) \lambda(\hat{p}, h; s)$$

$$\Leftrightarrow (i\sqrt{2})^{2s} \delta(-s\zeta, h) = \sum_{n=0}^{2s} C_n(\zeta) h^n, \quad h = -s\zeta, \dots, s\zeta$$

$$\Leftrightarrow \begin{bmatrix} s^0 & s^1 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s-1} & (s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix} \begin{bmatrix} C_0(-1) \\ C_1(-1) \\ \dots \\ C_{2s-1}(-1) \\ C_{2s}(-1) \end{bmatrix} = (i\sqrt{2})^{2s} \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} C_0(-1) \\ C_1(-1) \\ \dots \\ C_{2s-1}(-1) \\ C_{2s}(-1) \end{bmatrix} = (i\sqrt{2})^{2s} \begin{bmatrix} s^0 & s^1 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s-1} & (s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} C_0(1) \\ C_1(1) \\ \dots \\ C_{2s-1}(1) \\ C_{2s}(1) \end{bmatrix} = (i\sqrt{2})^{2s} \begin{bmatrix} s^0 & s^1 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s-1} & (s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$$

□

$$\text{性质8.11.2.} \quad \begin{cases} \Gamma_{+}^{abc\dots}(s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \dots}^{2s} = \sum_{l=0}^{[s]} C_{2s-2l} [\sigma(s) \cdot \hat{p}]^{2s-2l} \\ \Gamma_{+}^{abc\dots}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s, \hat{\partial}_\pi \rightarrow i} = \sum_{l=0}^{[s]} C_{2s-2l} [\sigma(s) \cdot \hat{\nabla}]^{2s-2l} \end{cases}$$

$$\text{性质8.11.3.} \quad \begin{cases} \Gamma_{-}^{abc\dots}(s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \dots}^{2s} = \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s-2l-1} [\sigma(s) \cdot \hat{p}]^{2s-2l-1} \\ \Gamma_{-}^{abc\dots}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s, \hat{\partial}_\pi \rightarrow i} = \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s-2l-1} [\sigma(s) \cdot \hat{\nabla}]^{2s-2l-1} \end{cases}$$

$$\text{定理8.11.1.} \quad \sum_{n=0}^{2s} k_n [\sigma(s) \cdot \hat{p}]^n = 0 \Leftrightarrow k_n = 0$$

$$\text{证明:} \quad \sum_{n=0}^{2s} k_n [\sigma(s) \cdot \hat{p}]^n = 0 (\Rightarrow \sum_{n=0}^{2s} k_n [\sigma_z(s)]^n = 0)$$

$$\Rightarrow \lambda^+(\hat{p}, h; s) \sum_{n=0}^{2s} k_n [\sigma(s) \cdot \hat{p}]^n \lambda(\hat{p}, h; s) = 0$$

$$\Leftrightarrow \sum_{n=0}^{2s} k_n h^n = 0, h = -s, \dots, s$$

$$\Leftrightarrow \begin{bmatrix} s^0 & s^1 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s-1} & (s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ \dots \\ k_{2s-1} \\ k_{2s} \end{bmatrix} = 0$$

$$\Leftrightarrow k_n = 0$$

□

## 8.12 特殊复合常数不变张量之间的关系

$$\text{推论8.12.1.} \quad \begin{cases} \Gamma_{k_\zeta}^{a\bar{a}c\bar{c}\dots}(n) \overbrace{\delta_{\bar{a}b} \delta_{\bar{c}d}}^n \dots \Gamma_{k'_\zeta}^{b\bar{b}d\bar{d}\dots}(n) = \left(\frac{1}{2}\right)^n \overbrace{\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\beta_\zeta \beta'_\zeta}^{cd}}^n \dots \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) \Gamma_{k'_\zeta}^{\alpha'_\zeta \beta'_\zeta \dots}(n) \\ \Gamma_{\frac{2n}{k'_\zeta}}^{k'_\zeta a\bar{a}c\bar{c}\dots}(n) \overbrace{\delta_{\bar{a}b} \delta_{\bar{c}d}}^n \dots \Gamma_{\frac{2n}{k_\zeta}}^{k_\zeta b\bar{b}d\bar{d}\dots}(n) = \left(\frac{1}{2}\right)^n \overbrace{\sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} \sigma_{cd}^{\beta'_\zeta \beta_\zeta}}^n \dots \Gamma_{\frac{n}{k'_\zeta}}^{\alpha'_\zeta \beta'_\zeta \dots}(n) \Gamma_{\frac{n}{k_\zeta}}^{\alpha_\zeta \beta_\zeta \dots}(n) \end{cases}$$

$$\text{证明:} \quad \Gamma_{k_\zeta}^{a\bar{a}c\bar{c}\dots}(n) \overbrace{\delta_{\bar{a}b} \delta_{\bar{c}d}}^n \dots \Gamma_{k'_\zeta}^{b\bar{b}d\bar{d}\dots}(n) \\ = \left(\frac{i}{2}\right)^n \underbrace{\sigma_{\alpha_\zeta \alpha'_\zeta}^{a\bar{a}} \sigma_{\beta_\zeta \beta'_\zeta}^{c\bar{c}}}_{n} \dots \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) \overbrace{\delta_{\bar{a}b} \delta_{\bar{c}d}}^n \dots \left(\frac{i}{2}\right)^n \underbrace{\sigma_{-\alpha'_\zeta}^{b\bar{b}} \sigma_{-\beta'_\zeta}^{d\bar{d}}}_{n} \dots \Gamma_{k'_\zeta}^{\alpha'_\zeta \beta'_\zeta \dots}(n)$$

$$= \left(\frac{1}{2}\right)^n \underbrace{\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\beta_\zeta \beta'_\zeta}^{cd}}_n \dots \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) \Gamma_{k'_\zeta}^{\alpha'_\zeta \beta'_\zeta \dots}(n)$$

□

$$\text{推论8.12.2.} \quad \begin{cases} \Gamma_{k_\zeta k'_\zeta}^{abc\dots}(n) = \underbrace{\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\beta_\zeta \beta'_\zeta}^{cd}}_n \dots \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) \Gamma_{k'_\zeta}^{\alpha'_\zeta \beta'_\zeta \dots}(n) = 2^n \Gamma_{k_\zeta}^{a\bar{a}c\bar{c}\dots}(n) \overbrace{\delta_{\bar{a}b} \delta_{\bar{c}d}}^n \dots \Gamma_{k'_\zeta}^{b\bar{b}d\bar{d}\dots}(n) \\ \Gamma_{\frac{2n}{k'_\zeta}}^{k'_\zeta abc\dots}(n) = \underbrace{\sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} \sigma_{cd}^{\beta'_\zeta \beta_\zeta}}_n \dots \Gamma_{\frac{n}{k'_\zeta}}^{\alpha'_\zeta \beta'_\zeta \dots}(n) \Gamma_{\frac{n}{k_\zeta}}^{\alpha_\zeta \beta_\zeta \dots}(n) = 2^n \Gamma_{\frac{2n}{k_\zeta}}^{k_\zeta a\bar{a}c\bar{c}\dots}(n) \overbrace{\delta_{\bar{a}b} \delta_{\bar{c}d}}^n \dots \Gamma_{\frac{2n}{k'_\zeta}}^{k'_\zeta b\bar{b}d\bar{d}\dots}(n) \end{cases}$$

$$\text{证明:} \quad \Gamma_{k_\zeta k'_\zeta}^{abc\dots}(n) = \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2n} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c (\sigma, i\zeta)_{D_\zeta D'_\zeta}^d}_{2n} \dots \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s)$$

$$= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2n} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c}_{2n} \dots \left(\frac{i\zeta}{\sqrt{2}}\right)^n \underbrace{\sigma_{\alpha_\zeta B_\zeta}^{A_\zeta C_\zeta} \sigma_{\beta_\zeta D_\zeta}^{C'_\zeta D'_\zeta}}_n \dots \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) \left(\frac{-i\zeta}{\sqrt{2}}\right)^n \underbrace{\sigma_{\alpha'_\zeta B'_\zeta}^{A'_\zeta C'_\zeta} \sigma_{\beta'_\zeta D'_\zeta}^{C'_\zeta D'_\zeta}}_n \dots \Gamma_{k'_\zeta}^{\alpha'_\zeta \beta'_\zeta \dots}(n)$$

$$\begin{aligned}
&= \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)^a_{A_\zeta A'_\zeta} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)^b_{B_\zeta B'_\zeta} \frac{-i\zeta}{\sqrt{2}}\sigma_{\alpha'_\zeta}^{A_\zeta B'_\zeta} \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)^c_{C_\zeta C'_\zeta} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)^d_{D_\zeta D'_\zeta} \frac{-i\zeta}{\sqrt{2}}\sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} \dots \\
&\Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots} \cdot \Gamma_{k'_\zeta}^{\alpha'_\zeta \beta'_\zeta \dots} (n) \\
&= \underbrace{\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\beta_\zeta \beta'_\zeta}^{cd} \dots}_{n} \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots} \cdot \Gamma_{k'_\zeta}^{\alpha'_\zeta \beta'_\zeta \dots} (n) \\
&= 2^n \Gamma_{k_\zeta}^{a\bar{a}c\bar{c}\dots} \cdot \Gamma_{\bar{a}\bar{b}}^{\alpha_\zeta \beta_\zeta \dots} \delta_{\bar{c}\bar{d}} \cdot \Gamma_{k'_\zeta}^{b\bar{b}d\bar{d}\dots} (n)
\end{aligned}$$

□

## 9 最基本的常数不变张量

最基本的常数不变张量：

$$\varepsilon_{AB}, \varepsilon^{AB}; \delta_{ab}, \delta^{ab}; \delta_{\alpha\beta}, \delta^{\alpha\beta}; \varepsilon_{\alpha\beta\gamma}; (\sigma, -i)_a^{A'A}; \sigma^{\alpha A}{}^B; N_{Al}^k(s), N_k^{Al}(s) \quad (3.8)$$

以上是最基本的常数不变张量，本章与上一章其它所有的常数不变张量均可以由它们导出，所以只要以上几个常数不变张量的协变性得到证明，由它们导出的常数不变张量的协变性自然成立。

# 第四章 高维常数不变张量初步

自我评述：本章是在前面章节基础上，将常数不变张量全面系统地往高阶、甚至无穷阶和高低维时空两个方向发展，得到了高低维时空中的各种常数不变张量，从而为研究高低维时空粒子物理提供了一个有力的数学工具。

## 1 N+1维时空的洛伦兹群表示

### 1.1 N+1维时空Dirac矩阵的递推表象

$$\text{定义1.1.1.1. } \begin{cases} \gamma_a(1) = (1) \\ \gamma_1(1) = 1 \end{cases}$$

$$\text{定义1.1.2. } \begin{cases} \gamma_a(2) := (\gamma_a(1) \otimes \sigma_x, 1 \otimes \sigma_y) = (\sigma_x, \sigma_y) \\ \Gamma^a(2) := [\gamma_a(1), i\zeta] = (1, i\zeta) \\ \gamma_1(2)\gamma_2(2) = i\sigma_z \end{cases} \begin{cases} C_1(2) := \gamma_1(2) = \sigma_x, C_2(2) := \gamma_2(2) = \sigma_y \\ C_1^+(2)\gamma_a(2)C_1(2) = \gamma_a^T(2), C_1^T(2) = C_1(2) \\ C_2^+(2)\gamma_a(2)C_2(2) = -\gamma_a^T(2), C_2^T(2) = -C_2(2) \end{cases}$$

$$\text{定义1.1.3. } \begin{cases} \gamma_a(3) = [\gamma_a(2), 1 \otimes \sigma_z] = (\sigma_x, \sigma_y, \sigma_z) \\ \gamma_1(3) \cdots \gamma_3(3) = i \end{cases} \begin{cases} C(3) := \gamma_2(3) = \sigma_y, C(3) = C_2(2) \\ C^+(3)\gamma_a(3)C(3) = -\gamma_a^T(3), C^T(3) = -C(3) \end{cases}$$

$$\text{定义1.1.4. } \begin{cases} \gamma_a(4) = [\gamma_a(3) \otimes \sigma_x, I \otimes \sigma_y] = (\sigma \otimes \sigma_x, I \otimes \sigma_y) \\ \Gamma^a(4) = [\gamma_a(3), i\zeta] \\ \gamma_1(4) \cdots \gamma_4(4) = -I \otimes \sigma_z \end{cases} \begin{cases} C_1(4) := \gamma_1(4)\gamma_3(4) = -i\sigma_y \otimes I \\ C_2(4) := \gamma_2(4)\gamma_4(4) = i\sigma_y \otimes \sigma_z \\ C_1^+(4)\gamma_a(4)C_1(4) = -\gamma_a^T(4), C_1^T(4) = -C_1(4) \\ C_2^+(4)\gamma_a(4)C_2(4) = \gamma_a^T(4), C_2^T(4) = -C_2(4) \end{cases}$$

$$\text{定义1.1.5. } \begin{cases} \gamma_a(5) = [\gamma_a(4), I \otimes \sigma_z] \\ \gamma_1(5) \cdots \gamma_5(5) = -1 \end{cases} \begin{cases} C(5) := \gamma_2(4)\gamma_4(4) = i\sigma_y \otimes \sigma_z, C(5) = C_2(4) \\ C^+(5)\gamma_a(5)C(5) = \gamma_a^T(5), C^T(5) = -C(5) \end{cases}$$

$$\text{定义1.1.6. } \begin{cases} \gamma_a(6) = [\gamma_a(5) \otimes \sigma_x, I_4 \otimes \sigma_y] \\ \Gamma^a(6) = [\gamma_a(5), i\zeta] \\ \gamma_1(6) \cdots \gamma_6(6) = -iI_4 \otimes \sigma_z \end{cases} \begin{cases} C_1(6) := \gamma_1(6)\gamma_3(6)\gamma_5(6) = -i\sigma_y \otimes \sigma_z \otimes \sigma_x \\ C_2(6) := \gamma_2(6)\gamma_4(6)\gamma_6(6) = i\sigma_y \otimes \sigma_z \otimes \sigma_y \\ C_1^+(6)\gamma_a(6)C_1(6) = \gamma_a^T(6), C_1^T(6) = -C_1(6) \\ C_2^+(6)\gamma_a(6)C_2(6) = -\gamma_a^T(6), C_2^T(6) = C_2(6) \end{cases}$$

$$\text{定义1.1.7. } \begin{cases} \gamma_a(7) = [\gamma_a(6), I_4 \otimes \sigma_z] \\ \gamma_1(7) \cdots \gamma_7(7) = -i \end{cases} \begin{cases} C(7) := \gamma_2(7)\gamma_4(7)\gamma_6(7) = i\sigma_y \otimes \sigma_z \otimes \sigma_y, C(7) = C_2(6) \\ C^+(7)\gamma_a(7)C(7) = -\gamma_a^T(7), C^T(7) = C(7) \end{cases}$$

$$\text{定义1.1.8. } \begin{cases} \gamma_a(8) = [\gamma_a(7) \otimes \sigma_x, I_8 \otimes \sigma_y] \\ \Gamma^a(8) = [\gamma_a(7), i\zeta] \\ \gamma_1(8) \cdots \gamma_8(8) = I_8 \otimes \sigma_z \end{cases} \begin{cases} C_1(8) := \gamma_1(8)\gamma_3(8)\gamma_5(8)\gamma_7(8) = -\sigma_y \otimes \sigma_z \otimes \sigma_y \otimes I \\ C_2(8) := \gamma_2(8)\gamma_4(8)\gamma_6(8)\gamma_8(8) = -\sigma_y \otimes \sigma_z \otimes \sigma_y \otimes \sigma_z \\ C_1^+(8)\gamma_a(8)C_1(8) = -\gamma_a^T(8), C_1^T(8) = C_1(8) \\ C_2^+(8)\gamma_a(8)C_2(8) = \gamma_a^T(8), C_2^T(8) = C_2(8) \end{cases}$$

$$\text{定义1.1.9. } \begin{cases} \gamma_a(9) = [\gamma_a(8), I_8 \otimes \sigma_z] \\ \gamma_1(9) \cdots \gamma_9(9) = 1 \end{cases} \begin{cases} C(9) := \gamma_2(8)\gamma_4(8)\gamma_6(8)\gamma_8(8) = -\sigma_y \otimes \sigma_z \otimes \sigma_y \otimes \sigma_z, C(9) = C_2(8) \\ C^+(9)\gamma_a(9)C(9) = \gamma_a^T(9), C^T(9) = C(9) \end{cases}$$

$$\text{定义1.1.10.} \begin{cases} \gamma_a(10) = [\gamma_a(9) \otimes \sigma_x, I_{16} \otimes \sigma_y] \\ \Gamma^a(10) = [\gamma_a(9), i\zeta] \\ \gamma_1(10) \cdots \gamma_{10}(10) = iI_{16} \otimes \sigma_z \end{cases} \begin{cases} C_1(10) := \gamma_1(10)\gamma_3(10)\gamma_5(10)\gamma_7(10)\gamma_9(10) \\ = -\sigma_y \otimes \sigma_z \otimes \sigma_y \otimes \sigma_z \otimes \sigma_x \\ C_2(10) := \gamma_2(10)\gamma_4(10)\gamma_6(10)\gamma_8(10)\gamma_{10}(10) \\ = -\sigma_y \otimes \sigma_z \otimes \sigma_y \otimes \sigma_z \otimes \sigma_y \\ C_2^+(10)\gamma_a(10)C_2(10) = -\gamma_a^T(10), C_2^T(10) = -C_2(10) \\ C_1^+(10)\gamma_a(10)C_1(10) = \gamma_a^T(10), C_1^T(10) = C_1(10) \end{cases}$$

$$\text{定义1.1.11.} \begin{cases} \gamma_a(11) = [\gamma_a(10), I_{16} \otimes \sigma_z] \\ \gamma_1(11) \cdots \gamma_{11}(11) = i \end{cases} \begin{cases} C(11) := \gamma_2(11)\gamma_4(11)\gamma_6(11)\gamma_8(11)\gamma_{10}(11) \\ = -\sigma_y \otimes \sigma_z \otimes \sigma_y \otimes \sigma_z \otimes \sigma_y, C(11) = C_2(10) \\ C^+(11)\gamma_a(10)C(11) = -\gamma_a^T(11), C^T(11) = -C(11) \end{cases}$$

## 1.2 N+1维时空Dirac矩阵、最小旋量和实表象形式

$$\text{定义1.2.1.} \begin{cases} \gamma_a(2) = (\sigma_x, \sigma_y), 2^1 \times 2^1 \\ \Gamma^a(2) = (1, i\zeta), 2^0 \times 2^0 \\ \gamma_1(2)\gamma_2(2) = i\sigma_z \end{cases}$$

$$\text{定义1.2.2.} \begin{cases} \gamma_a(3) = (\sigma_x, \sigma_y, \sigma_z) \rightarrow (\sigma_z, \sigma_x, \sigma_y), 2^1 \times 2^1 \\ \gamma_1(3) \cdots \gamma_3(3) = i \end{cases}$$

$$\text{定义1.2.3.} \begin{cases} \gamma_a(4) = (\sigma \otimes \sigma_x, I \otimes \sigma_y) \rightarrow (\sigma_+ \sigma_{-x}, \sigma_{-y}), 2^2 \times 2^2 \\ \Gamma^a(4) = [\gamma_a(3), i\zeta] \rightarrow \text{Null}, 2^1 \times 2^1 \\ \gamma_1(4) \cdots \gamma_4(4) = -I \otimes \sigma_z \end{cases}$$

$$\text{定义1.2.4.} \begin{cases} \gamma_a(5) = [\sigma \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z] \rightarrow \text{Null}, 2^2 \times 2^2 \\ \gamma_1(5) \cdots \gamma_5(5) = -1 \end{cases}$$

$$\text{定义1.2.5.} \begin{cases} \gamma_a(6) = [(\sigma \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z) \otimes \sigma_x, I_4 \otimes \sigma_y], 2^3 \times 2^3 \\ \Gamma^a(6) = [(\sigma \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z), i\zeta], 2^2 \times 2^2 \\ \gamma_1(6) \cdots \gamma_6(6) = -iI_4 \otimes \sigma_z \end{cases}$$

$$\text{定义1.2.6.} \begin{cases} \gamma_a(7) = [(\sigma \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z) \otimes \sigma_x, I_4 \otimes \sigma_y, I_4 \otimes \sigma_z], 2^3 \times 2^3 \\ \gamma_1(7) \cdots \gamma_7(7) = -i \end{cases}$$

$$\text{定义1.2.7.} \begin{cases} \gamma_a(8) = [[(\sigma \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z) \otimes \sigma_x, I_4 \otimes \sigma_y, I_4 \otimes \sigma_z] \otimes \sigma_x, I_8 \otimes \sigma_y], 2^4 \times 2^4 \\ \Gamma^a(8) = [[(\sigma \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z) \otimes \sigma_x, I_4 \otimes \sigma_y, I_4 \otimes \sigma_z], i\zeta] \rightarrow \text{Null}, 2^3 \times 2^3 \\ \gamma_1(8) \cdots \gamma_8(8) = I_8 \otimes \sigma_z \end{cases}$$

$$\text{定义1.2.8.} \begin{cases} \gamma_a(9) = [[(\sigma \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z) \otimes \sigma_x, I_4 \otimes \sigma_y, I_4 \otimes \sigma_z] \otimes \sigma_x, I_8 \otimes \sigma_y, I_8 \otimes \sigma_z] \rightarrow \text{Null}, 2^4 \times 2^4 \\ \gamma_1(9) \cdots \gamma_9(9) = 1 \\ \gamma_s^a(9) = S\gamma^a(9)S^+, S = [\sigma_z \otimes \sigma_y \otimes I \otimes \sigma_y][S_{ex}S_{em}(-1) \otimes I_4][I \otimes S_{em}(-1) \otimes S_c(\frac{1}{2})] \\ = -\{[(\sigma_x \otimes I, \sigma_y \otimes \sigma_y, \sigma_z \otimes I) \otimes \sigma_y, \sigma_y \otimes \sigma_x \otimes I, I \otimes \sigma_y \otimes \sigma_z, I \otimes \sigma_y \otimes \sigma_x, \sigma_y \otimes \sigma_z \otimes I] \otimes \sigma_y \\ , I \otimes I \otimes I \otimes \sigma_z, I \otimes I \otimes I \otimes \sigma_x\} \in R \end{cases}$$



定义1.2.9.

$$\begin{cases} \gamma_a(10) = [[(\sigma \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z) \otimes \sigma_x, I_4 \otimes \sigma_y, I_4 \otimes \sigma_z] \otimes \sigma_x, I_8 \otimes \sigma_y, I_8 \otimes \sigma_z] \otimes \sigma_x, I_{16} \otimes \sigma_y], 2^5 \times 2^5 \\ \Gamma^a(10) = [[(\sigma \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z) \otimes \sigma_x, I_4 \otimes \sigma_y, I_4 \otimes \sigma_z] \otimes \sigma_x, I_8 \otimes \sigma_y, I_8 \otimes \sigma_z], i\zeta], 2^4 \times 2^4 \\ \gamma_1(10) \cdots \gamma_{10}(10) = iI_{16} \otimes \sigma_z \\ \Gamma_s^a(10) = [\gamma_s(9), i\zeta], \gamma_s^a(10) = [\gamma_s(9) \otimes \sigma_x, I_{16} \otimes \sigma_y] \end{cases}$$

定义1.2.10.

$$\begin{cases} \gamma_a(11) = [[(\sigma \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z) \otimes \sigma_x, I_4 \otimes \sigma_y, I_4 \otimes \sigma_z] \otimes \sigma_x, I_8 \otimes \sigma_y, I_8 \otimes \sigma_z] \otimes \sigma_x, I_{16} \otimes \sigma_y, I_{16} \otimes \sigma_z] \\ \gamma_1(11) \cdots \gamma_{11}(11) = i \\ \gamma_s^a(11) = [\Gamma_s^i(10) \otimes \sigma_x, I_{16} \otimes \sigma_z, I_{16} \otimes \sigma_y] \end{cases}$$

### 1.3 N+1维时空Dirac矩阵简洁构造

$$\text{定义1.3.1.} \begin{cases} \gamma_{n+1}(n) := i^{-[n/2]} \gamma_1(n) \cdots \gamma_n(n) \\ \gamma_a(4) = [\gamma_a(2) \otimes I, \gamma_3(2) \otimes \gamma_a(2)] \\ \gamma_a(5) = [\gamma_a(2) \otimes I, \gamma_3(2) \otimes \gamma_a(3)] = [\gamma_a(2) \otimes I, \gamma_3(2) \otimes \gamma_a(2), \gamma_3(2) \otimes \gamma_3(2)] \\ \gamma_a(6) = [\gamma_a(2) \otimes I_4, \gamma_3(2) \otimes \gamma_a(4)] \\ \gamma_a(7) = [\gamma_a(2) \otimes I_4, \gamma_3(2) \otimes \gamma_a(5)] = [\gamma_a(2) \otimes I_4, \gamma_3(2) \otimes \gamma_a(4), \gamma_3(2) \otimes \gamma_5(4)] \\ \gamma_a(8) = [\gamma_a(4) \otimes I_4, \gamma_5(4) \otimes \gamma_a(4)] \\ \gamma_a(9) = [\gamma_a(4) \otimes I_4, \gamma_5(4) \otimes \gamma_a(5)] = [\gamma_a(4) \otimes I_4, \gamma_5(4) \otimes \gamma_a(4), \gamma_5(4) \otimes \gamma_5(4)] \\ \gamma_a(10) = [\gamma_a(4) \otimes I_8, \gamma_5(4) \otimes \gamma_a(6)] \\ \gamma_a(11) = [\gamma_a(4) \otimes I_8, \gamma_5(4) \otimes \gamma_a(6), \gamma_5(4) \otimes \gamma_7(6)] \end{cases}$$

$$\text{定义1.3.2.} \begin{cases} \gamma_s^a(10) = [\gamma_s^a(9) \otimes \sigma_x, I_{16} \otimes \sigma_y] \\ \gamma_s^a(11) = [\gamma_s^a(9) \otimes \sigma_x, I_{16} \otimes \sigma_z, I_{16} \otimes \sigma_y] \end{cases}$$

### 1.4 N+1=2n维时空通过Dirac矩阵构造费米算子 [35]

$$\text{定义1.4.1.} \begin{cases} a_1(2n) = \frac{1}{2}[\gamma_1(2n) + i\gamma_2(2n)] \\ a_2(2n) = \frac{1}{2}[\gamma_3(2n) + i\gamma_4(2n)] \\ \dots\dots \\ a_n(2n) = \frac{1}{2}[\gamma_{2n-1}(2n) + i\gamma_{2n}(2n)] \end{cases} \Rightarrow \begin{cases} \{a_j(2n), a_k^+(2n)\} = \delta_{jk} \\ \{a_j(2n), a_k(2n)\} = 0 \\ \{a_j^+(2n), a_k^+(2n)\} = 0 \end{cases}$$

### 1.5 N+1维时空Dirac矩阵的递推关系

$$\text{定义1.5.1.} \begin{cases} \gamma_a(2n) = [\gamma_a(2n-1) \otimes \sigma_x, I_{2n} \otimes \sigma_y] \\ \Gamma^a(2n) = [\gamma_a(2n-1), i\zeta] \\ \gamma_1(2n) \cdots \gamma_{2n}(2n) = i^n I_{2n} \otimes \sigma_z = i^n \gamma_{2n+1}(2n+1) \end{cases} \begin{cases} C_1(2n) := \gamma_1(2n)\gamma_3(2n) \cdots \gamma_{2n-1}(2n) \\ C_2(2n) := \gamma_2(2n)\gamma_4(2n) \cdots \gamma_{2n}(2n) \\ C_r^+(2n)\gamma_a(2n)C_r(2n) = (-1)^{n+r}\gamma_a^*(2n) \end{cases}$$

$$\text{定义1.5.2.} \begin{cases} \gamma_a(2n+1) = [\gamma_a(2n), I_{2n} \otimes \sigma_z] \\ \gamma_1(2n+1) \cdots \gamma_{2n+1}(2n+1) = i^n \end{cases} \begin{cases} C(2n+1) = C_2(2n) \\ C^+(2n+1)\gamma_a(2n+1)C(2n+1) = (-1)^n \gamma_a^*(2n+1) \end{cases}$$

### 1.6 N+1维时空的自旋张量

$$\text{定义1.6.1.} \begin{cases} S_{ab}(v; 2n) := \begin{bmatrix} S_{ij}(e; 2n-1) - \frac{1}{2}\bar{\gamma}(2n-1) \\ \frac{1}{2}\bar{\gamma}(2n-1) & 0 \end{bmatrix} = -\frac{i}{4}[\gamma(2n-1), i]_{[a}[\gamma(2n-1), -i]_{b]} \\ S_{ab}(\bar{v}; 2n) := \begin{bmatrix} S_{ij}(e; 2n-1) & \frac{1}{2}\bar{\gamma}(2n-1) \\ -\frac{1}{2}\bar{\gamma}(2n-1) & 0 \end{bmatrix} = -\frac{i}{4}[\gamma(2n-1), -i]_{[a}[\gamma(2n-1), i]_{b]} \end{cases}$$

推论1.6.1.  $S_{ab}(e; 2n) = S_{ab}(\nu; 2n) \oplus S_{ab}(\bar{\nu}; 2n) = -\frac{i}{4}[\gamma_a(2n), \gamma_b(2n)]$

推论1.6.2.  $S_{ab}(\varsigma; 2n) = -\frac{i}{4}[\gamma(2n-1), i\varsigma]_a[\gamma(2n-1), -i\varsigma]_b \Leftrightarrow S_{ab}(\varsigma; 2n) := \begin{bmatrix} S_{ij}(e; 2n-1) - \frac{\varsigma}{2}\bar{\gamma}(2n-1) \\ \frac{\varsigma}{2}\bar{\gamma}(2n-1) & 0 \end{bmatrix}$

推论1.6.3.  $S_{ab}(e; 2n+1) = -\frac{i}{4}[\gamma_a(2n+1), \gamma_b(2n+1)] = -\frac{i}{4}[i\varsigma\gamma(2n)\gamma_0(2n), -i\varsigma]_a[i\varsigma\gamma(2n)\gamma_0(2n), i\varsigma]_b$

推论1.6.4.  $\frac{i}{2}\vartheta^{ab}S_{ab}(e; 2n+1) = \frac{i}{2}\vartheta^{ab}[\gamma_a(2n+1), \gamma_b(2n+1)]$

推论1.6.5.  $\frac{i}{2}\vartheta^{ab}S_{ab}(e; 2n) = \frac{i}{2}\vartheta^{ab}S_{ab}(\nu; 2n) \oplus \frac{i}{2}\vartheta^{ab}S_{ab}(\bar{\nu}; 2n)$

证明:  $\frac{i}{2}\vartheta^{ab}S_{ab}(e; 2n) = \frac{1}{8}\vartheta^{ab}[\gamma_a(2n), \gamma_b(2n)] = \frac{1}{4}\vartheta^{i<j}[\gamma_i(2n), \gamma_j(2n)] + \frac{1}{4}\vartheta^{i\pi}[\gamma_i(2n), \gamma_\pi(2n)]$   
 $= \frac{1}{4}\vartheta^{i<j}[\gamma_i(2n-1), \gamma_j(2n-1)] \otimes I - \frac{i}{2}\vartheta^{i\pi}\gamma_i(2n-1) \otimes \sigma_z$   
 $= i\vartheta^{i<j}S_{ij}(e; 2n-1) \otimes I - \frac{i}{2}\vartheta^{i\pi}\gamma_i(2n-1) \otimes \sigma_z$   
 $= [i\vartheta^{i<j}S_{ij}(e; 2n-1) - \frac{i}{2}\vartheta^{i\pi}\gamma_i(2n-1)] \oplus [i\vartheta^{i<j}S_{ij}(e; 2n-1) + \frac{i}{2}\vartheta^{i\pi}\gamma_i(2n-1)]$   
 $= [i\vartheta^{i<j}S_{ij}(e; 2n-1) + \epsilon \cdot \frac{1}{2}\gamma(2n-1)] \oplus [i\vartheta^{i<j}S_{ij}(e; 2n-1) - \epsilon \cdot \frac{1}{2}\gamma(2n-1)]$   
 $= \frac{i}{2}\vartheta^{ab}S_{ab}(\nu; 2n) \oplus \frac{i}{2}\vartheta^{ab}S_{ab}(\bar{\nu}; 2n)$  □

## 1.7 N+1维时空的洛伦兹群表示

推论1.7.1.  $\{\gamma_a, \gamma_b\} = 2g_{ab} \Rightarrow i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}; S_{ab} := -\frac{i}{4}[\gamma_a, \gamma_b]$

证明:  $i[S_{ab}, S_{cd}]$   
 $= -\frac{i}{16}[[\gamma_a, \gamma_b], [\gamma_c, \gamma_d]]$   
 $= -\frac{i}{16}\{[[\gamma_a, \gamma_b], \gamma_c\gamma_d] - [[\gamma_a, \gamma_b], \gamma_d\gamma_c]\}$   
 $= -\frac{i}{16}\{[[\gamma_a, \gamma_b], \gamma_c]\gamma_d + \gamma_c[[\gamma_a, \gamma_b], \gamma_d] - [[\gamma_a, \gamma_b], \gamma_d]\gamma_c - \gamma_d[[\gamma_a, \gamma_b], \gamma_c]\}$   
 $= -\frac{i}{16}\{-\{\{\gamma_c, \gamma_a\}, \gamma_b\}\gamma_d + \{\gamma_a, \{\gamma_c, \gamma_b\}\}\gamma_d - \gamma_c\{\{\gamma_d, \gamma_a\}, \gamma_b\} + \gamma_c\{\gamma_a, \{\gamma_d, \gamma_b\}\}\}$   
 $\quad + \{\{\gamma_d, \gamma_a\}, \gamma_b\}\gamma_c - \{\gamma_a, \{\gamma_d, \gamma_b\}\}\gamma_c + \gamma_d\{\{\gamma_c, \gamma_a\}, \gamma_b\} - \gamma_d\{\gamma_a, \{\gamma_c, \gamma_b\}\}\}$   
 $= -\frac{i}{16}\{-4\delta_{ca}\gamma_b\gamma_d + 4\delta_{cb}\gamma_a\gamma_d - 4\delta_{da}\gamma_c\gamma_b + 4\delta_{db}\gamma_c\gamma_a + 4\delta_{da}\gamma_b\gamma_c - 4\delta_{db}\gamma_a\gamma_c + 4\delta_{ca}\gamma_d\gamma_b - 4\delta_{cb}\gamma_d\gamma_a\}$   
 $= -\frac{i}{4}\{\delta_{da}[\gamma_b, \gamma_c] - \delta_{db}[\gamma_a, \gamma_c] - \delta_{ca}[\gamma_b, \gamma_d] + \delta_{cb}[\gamma_a, \gamma_d]\}$   
 $= g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}$  □

推论1.7.2.  $\begin{cases} i[S_{ab}(\nu; 2n), S_{cd}(\nu; 2n)] = g_{ad}S_{bc}(\nu; 2n) - g_{ac}S_{bd}(\nu; 2n) + g_{bc}S_{ad}(\nu; 2n) - g_{bd}S_{ac}(\nu; 2n) \\ i[S_{ab}(\bar{\nu}; 2n), S_{cd}(\bar{\nu}; 2n)] = g_{ad}S_{bc}(\bar{\nu}; 2n) - g_{ac}S_{bd}(\bar{\nu}; 2n) + g_{bc}S_{ad}(\bar{\nu}; 2n) - g_{bd}S_{ac}(\bar{\nu}; 2n) \\ i[S_{ab}(\varsigma; 2n), S_{cd}(\varsigma; 2n)] = g_{ad}S_{bc}(\varsigma; 2n) - g_{ac}S_{bd}(\varsigma; 2n) + g_{bc}S_{ad}(\varsigma; 2n) - g_{bd}S_{ac}(\varsigma; 2n) \end{cases}$

推论1.7.3.  $\vec{S}_{ab} := -iS_{ab|cd} = -i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \Rightarrow i[\vec{S}_{ab}, \vec{S}_{cd}] = g_{ad}\vec{S}_{bc} - g_{ac}\vec{S}_{bd} + g_{bc}\vec{S}_{ad} - g_{bd}\vec{S}_{ac}$

推论1.7.4.  $e^{\frac{i}{2}\theta^{ab}\vec{S}_{ab}} = e^\theta$

## 1.8 N+1维时空的度规张量和电荷共轭矩阵

定义1.8.1.  $C^+\gamma_a(\varsigma)C = -\gamma_a^T(\varsigma), C^T = -C, C^+C = CC^+ = I$

推论1.8.1.

$$\begin{cases} C(n)S_{ab}(e; n) = [C(n)S_{ab}(e; n)]^T, C(2n-1)S_{ab}(\varsigma; 2n) = [C(2n-1)S_{ab}(\varsigma; 2n)]^T \\ C(n)\gamma_a(e; n) = [C(n)\gamma_a(e; n)]^T, C(2n-1)\gamma_a(2n-1) = [C(2n-1)\gamma_a(2n-1)]^T \end{cases}$$

推论1.8.2.

$$\begin{cases} \varepsilon(2n)S_{ab}(e; 2n) = -S_{ab}^T(e; 2n)\varepsilon(2n), \varepsilon(2n-1)S_{ab}(\varsigma; 2n) = -S_{ab}^T(\varsigma; 2n)\varepsilon(2n-1) \\ \varepsilon(2n-1)S_{ab}(e; 2n-1) = -S_{ab}^T(e; 2n-1)\varepsilon(2n-1), \varepsilon(2n-1)\gamma_a(2n-1) = -\gamma_a^T(2n-1)\varepsilon(2n-1) \end{cases}$$

## 1.9 N+1维时空中的常数不变张量(终于成功推广了)

在任意 $n=N+1$ 维时空中存在如下定理:

$$\text{定理1.9.1. } [\Gamma(N), i\zeta]^a = [e^\vartheta]_b^a e^{\frac{1}{8}\vartheta^{ij}[\Gamma_i(N), \Gamma_j(N)] + \zeta \epsilon \cdot \frac{1}{2}\Gamma(N)} [\Gamma(N), i\zeta]^b e^{-\frac{1}{8}\vartheta^{ij}[\Gamma_i(N), \Gamma_j(N)] + \zeta \epsilon \cdot \frac{1}{2}\Gamma(N)}$$

自我评述: 所以 $[\Gamma(N), i\zeta]_{A_\zeta A'_\zeta}^a$ 和 $[\Gamma(N), -i\zeta]_{A'_\zeta A_\zeta}^a$ 是常数不变张量, 是Penrose旋量在高低维时空的推广。  
在任意 $n=N+1$ 维时空中存在如下定理:

$$\text{定理1.9.2. } \Gamma_{ab} = [e^\vartheta]_a^c [e^\vartheta]_b^d e^{\frac{i}{2}\vartheta^{ef}\Gamma_{ef}} \Gamma_{cd} e^{-\frac{i}{2}\vartheta^{ef}\Gamma_{ef}} \Leftrightarrow i[\Gamma_{ab}, \Gamma_{cd}] = \delta_{ad}\Gamma_{bc} - \delta_{ac}\Gamma_{bd} + \delta_{bc}\Gamma_{ad} - \delta_{bd}\Gamma_{ac}$$

所以 $S_{ab\lambda_\zeta}{}^{\mu_\zeta}(e, \zeta; n)$ ,  $S_{ab}{}^{\lambda'_\zeta}{}_{\mu'_\zeta}(e, -\zeta; n)$ ,  $S_{abA_\zeta}{}^{B_\zeta}(\frac{1}{2}, \zeta; 2n)$ ,  $S_{ab}{}^{A'_\zeta}{}_{B'_\zeta}(\frac{1}{2}, -\zeta; 2n)$ 是常数不变张量。

$$\text{定理1.9.3. } \Gamma_a = [e^\vartheta]_a^b e^{\frac{i}{2}\vartheta^{cd}\Gamma_{cd}} \Gamma_b e^{-\frac{i}{2}\vartheta^{cd}\Gamma_{cd}} \Leftrightarrow i[\Gamma_a, \Gamma_{cd}] = \delta_{a[c}\Gamma_{d]}$$

$$\text{定理1.9.4. } \begin{cases} \Gamma_0 = e^{\frac{i}{2}\vartheta^{ab}S_{ab}} \Gamma_0 e^{-\frac{i}{2}\vartheta^{ab}S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b], \Gamma_0 = \Gamma_1 \cdots \Gamma_{N+1} \\ \Gamma_0 = e^{\frac{i}{2}\vartheta^{*ab}S_{ab}} \Gamma_0 e^{-\frac{i}{2}\vartheta^{*ab}S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b], \Gamma_0 = \Gamma_1 \cdots \Gamma_{N+1} \end{cases}$$

所以 $\gamma^a{}_{\lambda_\zeta}{}^{\mu_\zeta}(n)$ ,  $\gamma_a{}^{\lambda'_\zeta}{}_{\mu'_\zeta}(n)$ ,  $\gamma^0{}_{\lambda_\zeta}{}^{\mu_\zeta}(n)$ ,  $\gamma_0{}^{\lambda'_\zeta}{}_{\mu'_\zeta}(n)$ 是常数不变张量。

$$\text{定理1.9.5. } \begin{cases} \Gamma_{N+1} = e^{\frac{i}{2}\vartheta^{*ab}S_{ab}} \Gamma_{N+1} e^{-\frac{i}{2}\vartheta^{*ab}S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] \\ \Gamma_{N+1} = e^{\frac{i}{2}\vartheta^{ab}S_{ab}} \Gamma_{N+1} e^{-\frac{i}{2}\vartheta^{ab}S_{ab}}, S_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] \end{cases}$$

所以 $\gamma_n^{\lambda'_\zeta}{}_{\lambda_\zeta}(n)$ ,  $\gamma_{\lambda_\zeta}{}^{\lambda'_\zeta}(n)$ ,  $(\gamma_n \gamma_a)^{\lambda'_\zeta}{}_{\lambda_\zeta}(n)$ ,  $(\gamma^a \gamma_n)_{\lambda_\zeta}{}^{\lambda'_\zeta}(n)$ ,  $(\gamma^n \gamma^a)_{\lambda_\zeta}{}^{\lambda'_\zeta}(n)$ ,  $(\gamma_a \gamma_n)^{\lambda'_\zeta}{}_{\lambda_\zeta}(n)$ 是常数不变张量。

## 1.10 n=N+1维时空中的常数不变张量性质

$$\text{定理1.10.1. } \begin{cases} S_{ab}(e; n) = -\frac{i}{4}[\gamma_a(n), \gamma_b(n)] = -\frac{i}{4}[i\zeta\gamma(N)\gamma_0(N), -i\zeta]_a [i\zeta\gamma(N)\gamma_0(N), i\zeta]_b \\ 2\delta_{ab} = \{\gamma_a(n), \gamma_b(n)\} = [i\zeta\gamma(N)\gamma_0(N), -i\zeta]_a [i\zeta\gamma(N)\gamma_0(N), i\zeta]_b \\ \gamma_a(n)\gamma_b(n) = [i\zeta\gamma(N)\gamma_0(N), -i\zeta]_a [i\zeta\gamma(N)\gamma_0(N), i\zeta]_b = \delta_{ab} + 2iS_{ab}(e, \zeta; n) \end{cases}$$

$$\text{定理1.10.2. } \begin{cases} S_{ab}(\frac{1}{2}, \zeta; n) = -\frac{i}{4}[\Gamma(N), i\zeta]_a [\Gamma(N), -i\zeta]_b \\ 2\delta_{ab} = [\Gamma(N), i\zeta]_a [\Gamma(N), -i\zeta]_b \\ [\Gamma(N), i\zeta]_a [\Gamma(N), -i\zeta]_b = \delta_{ab} + 2iS_{ab}(\frac{1}{2}, \zeta; n) \end{cases}$$

## 1.11 n=N+1维时空中的Penrose变换

$$\text{定理1.11.1. } x^{A'_\zeta A_\zeta}(n) := [\Gamma(N), -i\zeta]_a^{A'_\zeta A_\zeta} x^a \Rightarrow x^a = \frac{1}{2^{[N/2]}} [\Gamma(N), i\zeta]_{A'_\zeta A_\zeta}^a x^{A'_\zeta A_\zeta}(n)$$

$$\text{定理1.11.2. } x_{\lambda_\zeta}{}^{\mu_\zeta}(n) := \gamma^a{}_{\lambda_\zeta}{}^{\mu_\zeta}(n) x_a \Rightarrow x^a = \frac{1}{2^{[n/2]}} \gamma^a{}_{\mu_\zeta}{}^{\lambda_\zeta}(n) x_{\lambda_\zeta}{}^{\mu_\zeta}(n), n \geq 2$$

## 2 外部时空对称变换 [26, 27]

### 2.1 彭加莱变换

$$\text{推论2.1.1. } x' = e^\varepsilon x + \theta \Leftrightarrow x' = e^{-\frac{i}{2}\varepsilon^{ab}L_{ab} + i\theta^a p_a} x, L_{ab} = -i(x_a \partial_b - x_b \partial_a), p_a = -i\partial_a$$

证明:  $x' = e^\varepsilon x + \theta$

$$\Leftrightarrow x'_a = x_a + \varepsilon_{ab} x^b + \theta_a$$

$$\Leftrightarrow x'_a = [1 - \frac{1}{2}\varepsilon^{bc}(x_b \partial_c - x_c \partial_b) + \theta^b \partial_b] x_a$$

$$\Leftrightarrow x' = [1 - \frac{1}{2}\varepsilon^{ab}(x_a \partial_b - x_b \partial_a) + \theta^a \partial_a] x$$

$$\Leftrightarrow x' = [1 - \frac{i}{2}\varepsilon^{ab}(x_a p_b - x_b p_a) + i\theta^a p_a] x, p_a = -i\partial_a$$

$$\Leftrightarrow x' = (1 - \frac{i}{2}\varepsilon^{ab}L_{ab} + i\theta^a p_a) x, L_{ab} := -i(x_a \partial_b - x_b \partial_a)$$

$$\Leftrightarrow x' = e^{-\frac{i}{2}\varepsilon^{ab}L_{ab} + i\theta^a p_a} x$$

□

推论2.1.2.  $e^{-\frac{i}{2}\varepsilon^{ab}L_{ab}+i\theta^a p_a}x = e^\varepsilon x + \theta$

彭加莱变换包含两种含义，一种是常规直观的含义；另一种是包含彭加莱部分生成元的含义，实际上就是把变换看作算子，是不容易想到的。

## 2.2 洛伦兹变换

推论2.2.1.  $x' = e^\varepsilon x \Leftrightarrow x' = e^{-\frac{i}{2}\varepsilon^{ab}L_{ab}}x, L_{ab} = -i(x_a\partial_b - x_b\partial_a)$

推论2.2.2.  $\varphi'(x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\varphi(e^{-\varepsilon}x) \Leftrightarrow \varphi'(x) = e^{\frac{i}{2}\varepsilon^{ab}M_{ab}}\varphi(x), M_{ab} = L_{ab} - iS_{ab}$

证明:  $\varphi'(x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\varphi(e^{-\varepsilon}x)$

$\Leftrightarrow \varphi'(x) = (1 + \frac{i}{2}\varepsilon^{ab}S_{ab})\varphi((1 - \varepsilon)x)$

$\Leftrightarrow \varphi'(x) = (1 + \frac{i}{2}\varepsilon^{ab}S_{ab})[\varphi(x) + \frac{1}{2}\varepsilon^{ab}(x_a\partial_b - x_b\partial_a)\varphi(x)]$

$\Leftrightarrow \varphi'(x) = \varphi(x) + \frac{i}{2}\varepsilon^{ab}[-i(x_a\partial_b - x_b\partial_a) + S_{ab}]\varphi(x)$

$\Leftrightarrow \delta\varphi(x) = \frac{i}{2}\varepsilon^{ab}[-i(x_a\partial_b - x_b\partial_a) + S_{ab}]\varphi(x)$

$\Leftrightarrow \varphi'(x) = e^{\frac{i}{2}\varepsilon^{ab}[-i(x_a\partial_b - x_b\partial_a) + S_{ab}]} \varphi(x)$

$\Leftrightarrow \varphi'(x) = e^{\frac{i}{2}\varepsilon^{ab}M_{ab}}\varphi(x), M_{ab} = L_{ab} + S_{ab}$  □

推论2.2.3.  $e^{\frac{i}{2}\varepsilon^{ab}M_{ab}}\varphi(x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\varphi(e^{-\varepsilon}x)$

推论2.2.4.  $x = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}(e^{-\varepsilon}x) \Leftrightarrow x = e^{\frac{i}{2}\varepsilon^{ab}M_{ab}}x$

在态变换和坐标恒等式变换中隐含了洛伦兹群生成元： $M_{ab}$ ，生成元的含义是自变量不变，函数发生改变，它是一个算子。

## 2.3 平移变换

推论2.3.1.  $\varphi'(x) = \varphi(x + \theta) \Leftrightarrow \varphi'(x) = e^{\theta^a\partial_a}\varphi(x) \Leftrightarrow \varphi'(x) = e^{i\theta^a p_a}\varphi(x)$

推论2.3.2.  $e^{i\theta^a p_a}\varphi(x) = \varphi(x + \theta)$

简单的平移变换中隐含了平移生成元： $p_a$ 及其算子变换。

## 2.4 彭加莱群生成元对易关系

彭加莱群生成元 $M_{ab}, p_a$ 对易关系：

$$M_{ab} = L_{ab} + S_{ab}, L_{ab} = x_a p_b - x_b p_a, g_{ab} = \delta_{ab} \quad (4.1)$$

$$\begin{cases} i[M_{ab}, M_{cd}] = g_{ad}M_{bc} - g_{ac}M_{bd} + g_{bc}M_{ad} - g_{bd}M_{ac} \\ i[M_{ab}, p_c] = g_{bc}p_a - g_{ac}p_b, [p_a, p_b] = 0 \end{cases} \quad (4.2)$$

彭加莱群生成元 $L_{ab}, S_{ab}, p_a$ 对易关系：

$$\begin{cases} L_{ab}, L_{cd} = g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac} \\ i[L_{ab}, p_c] = g_{bc}p_a - g_{ac}p_b, [p_a, p_b] = 0 \end{cases} \quad (4.3)$$

$$i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \quad (4.4)$$

$$[S_{ab}, L_{cd}] = 0, [S_{ab}, p_c] = 0 \quad (4.5)$$

## 2.5 生成元的意义

生成元可以生成相应的对称变换，生成元一般也是系统的守恒量，反过来，守恒量也是系统的生成元，并且生成元也形成一定的封闭代数。

# 3 无穷维常数不变张量

## 3.1 量子力学意义上的无穷维不变张量 [26, 27]

定义3.1.1.  $\hat{M}_{ab} := \hat{L}_{ab} + S_{ab}, \hat{L}_{ab} := x_a \hat{p}_b - x_b \hat{p}_a, \hat{p}_a := -i\partial_a, g_{ab} = \delta_{ab}$

$$\text{推论3.1.1.} \begin{cases} i[\hat{M}_{ab}, \hat{M}_{cd}] = g_{ad}\hat{M}_{bc} - g_{ac}\hat{M}_{bd} + g_{bc}\hat{M}_{ad} - g_{bd}\hat{M}_{ac} \Leftrightarrow \hat{M}_{ab} = [e^\vartheta]_a^c [e^\vartheta]_b^d e^{\frac{i}{2}\vartheta^{ef}\hat{M}_{ef}} \hat{M}_{cd} e^{-\frac{i}{2}\vartheta^{ef}\hat{M}_{ef}} \\ i[\hat{L}_{ab}, \hat{L}_{cd}] = g_{ad}\hat{L}_{bc} - g_{ac}\hat{L}_{bd} + g_{bc}\hat{L}_{ad} - g_{bd}\hat{L}_{ac} \Leftrightarrow \hat{L}_{ab} = [e^\vartheta]_a^c [e^\vartheta]_b^d e^{\frac{i}{2}\vartheta^{ef}\hat{L}_{ef}} \hat{L}_{cd} e^{-\frac{i}{2}\vartheta^{ef}\hat{L}_{ef}} \\ i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Leftrightarrow S_{ab} = [e^\vartheta]_a^c [e^\vartheta]_b^d e^{\frac{i}{2}\vartheta^{ef}S_{ef}} S_{cd} e^{-\frac{i}{2}\vartheta^{ef}S_{ef}} \end{cases}$$

$$\text{推论3.1.2.} \begin{cases} i[\hat{p}_a, \hat{M}_{cd}] = \delta_{a[c}\hat{p}_{d]} \Leftrightarrow \hat{p}_a = [e^\vartheta]_a^b e^{\frac{i}{2}\vartheta^{cd}\hat{M}_{cd}} \hat{p}_b e^{-\frac{i}{2}\vartheta^{cd}\hat{M}_{cd}} \\ i[\hat{p}_a, \hat{L}_{cd}] = \delta_{a[c}\hat{p}_{d]} \Leftrightarrow \hat{p}_a = [e^\vartheta]_a^b e^{\frac{i}{2}\vartheta^{cd}\hat{L}_{cd}} \hat{p}_b e^{-\frac{i}{2}\vartheta^{cd}\hat{L}_{cd}} \end{cases}$$

从上可知 $\hat{M}_{ab}, \hat{p}_a$ 是不变张量，但显然不是常数张量。换一种等价写法，可以将相对论中的洛伦兹变换和量子力学中的么正变换联系起来。

$$\text{推论3.1.3.} \begin{cases} i[\hat{M}_{ab}, \hat{M}_{cd}] = g_{ad}\hat{M}_{bc} - g_{ac}\hat{M}_{bd} + g_{bc}\hat{M}_{ad} - g_{bd}\hat{M}_{ac} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{ef}\hat{M}_{ef}} \hat{M}_{ab} e^{\frac{i}{2}\vartheta^{ef}\hat{M}_{ef}} = [e^\vartheta]_a^c [e^\vartheta]_b^d \hat{M}_{cd} \\ i[\hat{L}_{ab}, \hat{L}_{cd}] = g_{ad}\hat{L}_{bc} - g_{ac}\hat{L}_{bd} + g_{bc}\hat{L}_{ad} - g_{bd}\hat{L}_{ac} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{ef}\hat{L}_{ef}} \hat{L}_{ab} e^{\frac{i}{2}\vartheta^{ef}\hat{L}_{ef}} = [e^\vartheta]_a^c [e^\vartheta]_b^d \hat{L}_{cd} \\ i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{ef}S_{ef}} S_{ab} e^{\frac{i}{2}\vartheta^{ef}S_{ef}} = [e^\vartheta]_a^c [e^\vartheta]_b^d S_{cd} \end{cases}$$

$$\text{推论3.1.4.} \begin{cases} i[\hat{p}_a, \hat{M}_{cd}] = \delta_{a[c}\hat{p}_{d]} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{cd}\hat{M}_{cd}} \hat{p}_a e^{\frac{i}{2}\vartheta^{cd}\hat{M}_{cd}} = [e^\vartheta]_a^b \hat{p}_b \\ i[\hat{p}_a, \hat{L}_{cd}] = \delta_{a[c}\hat{p}_{d]} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{cd}\hat{L}_{cd}} \hat{p}_a e^{\frac{i}{2}\vartheta^{cd}\hat{L}_{cd}} = [e^\vartheta]_a^b \hat{p}_b \end{cases}$$

$$\text{推论3.1.5.} e^{\frac{i}{2}\varepsilon^{ab}M_{ab}}\varphi(x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\varphi(e^{-\varepsilon}x), e^{i\theta^a p_a}\varphi(x) = \varphi(x + \theta)$$

$$\text{推论3.1.6.} \langle'|\hat{P}_a|'\rangle = [e^\vartheta]_a^b \langle|\hat{P}_b|\rangle, \langle'|\hat{J}_{ab}|'\rangle = [e^\vartheta]_a^c [e^\vartheta]_b^d \langle|\hat{J}_{cd}|\rangle, \langle'|S_{ab}|'\rangle = [e^\vartheta]_a^c [e^\vartheta]_b^d \langle|S_{cd}|\rangle$$

$$\text{推论3.1.7.} \langle'|\hat{P}_a|'\rangle = [e^\vartheta]_a^b \langle|\hat{P}_b|\rangle, \langle'|\hat{J}_{\alpha\zeta}|'\rangle = [e^{(i\omega + \zeta\varepsilon)\cdot\sigma(s)}]_{\alpha\zeta}^{\beta\varsigma} \langle|\hat{J}_{\beta\varsigma}|\rangle$$

从上可知右边是洛伦兹变换，左边可以看作么正变换，即洛伦兹变换等价于么正变换。

$$\text{推论3.1.8.} [\hat{p}_a, \hat{p}_b] = 0 \Leftrightarrow \hat{p}_a = e^{-i\vartheta^b \hat{p}_b} \hat{p}_a e^{i\vartheta^b \hat{p}_b}$$

### 3.2 量子场论意义上的无穷维不变张量 [26, 27]

$$\text{推论3.2.1.} \begin{cases} i[\hat{J}_{ab}, \hat{J}_{cd}] = g_{ad}\hat{J}_{bc} - g_{ac}\hat{J}_{bd} + g_{bc}\hat{J}_{ad} - g_{bd}\hat{J}_{ac} \Leftrightarrow \hat{J}_{ab} = [e^\vartheta]_a^c [e^\vartheta]_b^d e^{\frac{i}{2}\vartheta^{ef}\hat{J}_{ef}} \hat{J}_{cd} e^{-\frac{i}{2}\vartheta^{ef}\hat{J}_{ef}} \\ i[\hat{J}_{ab}, \hat{J}_{cd}] = g_{ad}\hat{J}_{bc} - g_{ac}\hat{J}_{bd} + g_{bc}\hat{J}_{ad} - g_{bd}\hat{J}_{ac} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{ef}\hat{J}_{ef}} \hat{J}_{ab} e^{\frac{i}{2}\vartheta^{ef}\hat{J}_{ef}} = [e^\vartheta]_a^c [e^\vartheta]_b^d \hat{J}_{cd} \end{cases}$$

$$\text{推论3.2.2.} \begin{cases} i[\hat{P}_a, \hat{J}_{cd}] = \delta_{a[c}\hat{P}_{d]} \Leftrightarrow \hat{P}_a = [e^\vartheta]_a^b e^{\frac{i}{2}\vartheta^{cd}\hat{J}_{cd}} \hat{P}_b e^{-\frac{i}{2}\vartheta^{cd}\hat{J}_{cd}} \\ i[\hat{P}_a, \hat{J}_{cd}] = \delta_{a[c}\hat{P}_{d]} \Leftrightarrow e^{-\frac{i}{2}\vartheta^{cd}\hat{J}_{cd}} \hat{P}_a e^{\frac{i}{2}\vartheta^{cd}\hat{J}_{cd}} = [e^\vartheta]_a^b \hat{P}_b \end{cases}$$

$$\text{推论3.2.3.} [\hat{P}_a, \hat{P}_b] = 0 \Leftrightarrow \hat{P}_a = e^{-i\vartheta^b \hat{P}_b} \hat{P}_a e^{i\vartheta^b \hat{P}_b}$$

$$\text{推论3.2.4.} \langle'|\hat{P}_a|'\rangle = [e^\vartheta]_a^b \langle|\hat{P}_b|\rangle, \langle'|\hat{J}_{ab}|'\rangle = [e^\vartheta]_a^c [e^\vartheta]_b^d \langle|\hat{J}_{cd}|\rangle, \langle'|S_{ab}|'\rangle = [e^\vartheta]_a^c [e^\vartheta]_b^d \langle|S_{cd}|\rangle$$

$$\text{推论3.2.5.} \langle'|\hat{P}_a|'\rangle = [e^\vartheta]_a^b \langle|\hat{P}_b|\rangle, \langle'|\hat{J}_{\alpha\zeta}|'\rangle = [e^{(i\omega + \zeta\varepsilon)\cdot\sigma(s)}]_{\alpha\zeta}^{\beta\varsigma} \langle|\hat{J}_{\beta\varsigma}|\rangle$$

猜测的协变方程：

$$\text{推论3.2.6.} [(s + \phi)\hat{P}_a + i\hat{J}_{ab}\hat{P}^b]\psi(s, \varsigma) = 0$$

$$\text{推论3.2.7.} [(s + \phi)\hat{\partial}_a + i\hat{J}_{ab}\hat{\partial}^b]\Psi(x, F[\varphi(y)]) = 0$$

$$\text{推论3.2.8.} (\hat{P}^a \partial_a + m)\Psi(x, F[\varphi(y)]) = 0$$

$$\text{推论3.2.9.} \partial_a \Psi(x, F[\varphi(y)]) = \hat{P}_a \Psi(x, F[\varphi(y)])$$

### 3.3 量子场论意义上的一般不变张量 [26, 27]

$$\text{推论3.3.1. } \begin{cases} i[\hat{J}_{ab}, \hat{J}_{cd}] = g_{ad}\hat{J}_{bc} - g_{ac}\hat{J}_{bd} + g_{bc}\hat{J}_{ad} - g_{bd}\hat{J}_{ac} \Leftrightarrow \psi_\lambda = \Lambda_\lambda^\mu U^+ \psi_\mu U \\ i[\hat{J}_{ab}, \hat{J}_{cd}] = g_{ad}\hat{J}_{bc} - g_{ac}\hat{J}_{bd} + g_{bc}\hat{J}_{ad} - g_{bd}\hat{J}_{ac} \Leftrightarrow U\psi_\lambda U^+ = \Lambda_\lambda^\mu \psi_\mu \end{cases}$$

$$\text{推论3.3.2. } U\psi_\lambda U^+ = \Lambda_\lambda^\mu \psi_\mu, U = e^{iX}$$

### 3.4 量子场论算子代数

$$\text{推论3.4.1. } [x, \hat{p}_x] = i \Leftrightarrow \hat{p}_x \equiv -i\frac{\partial}{\partial x}, \Psi = \psi(x)$$

↓

$$\text{推论3.4.2. } [x_i, \hat{p}_j] = i\delta_{ij} \Leftrightarrow \hat{p}_i \equiv -i\frac{\partial}{\partial x_i}, \Psi = \psi(x_1, x_2, \dots, x_n)$$

↓

$$\text{推论3.4.3. } [\psi(x_i), \pi(x_j)] = i\delta_{ij} \Leftrightarrow \pi(x_i) \equiv -i\frac{\partial}{\partial \psi(x_i)}, \Psi = F[\psi(x_1), \psi(x_2), \dots, \psi(x_n), \dots, \psi(x_\infty)]$$

↓

$$\text{推论3.4.4. } [\psi(x), \pi(x')] = i\delta(x - x') \Leftrightarrow \pi(x) \equiv -i\frac{\delta}{\delta \psi(x)}$$

$$\Psi = \int dx F[\psi(x)] = \sum_i \Delta x_i F[\psi(x_i)] = \varepsilon F[\psi(x_1), \psi(x_2), \dots, \psi(x_n), \dots, \psi(x_\infty)]$$

### 3.5 任意算子的本征态和本征值如何求

# 第五章 常数不变张量与表象变换

自我评述：本章理清了常数不变张量与表象变换之间的对应关系，指出了表象变换矩阵就是一个常数不变张量。

## 1 一般 $s$ -自旋场各种旋量的定义

### 1.1 Penrose抽象符号规则 [1, 2]

推论1.1.1.  $g^{A_\zeta B_\zeta} \psi_{B_\zeta} = \psi^{A_\zeta} = [\psi_{A_\zeta}]^*$ ,  $g_{A_\zeta B_\zeta} \psi^{B_\zeta} = \psi_{A_\zeta} = [\psi^{A_\zeta}]^*$

以上说明 $\psi_{A_\zeta}, \psi^{A_\zeta}$ 是完全相关的，真正独立的只是其中一个，不妨选定 $\psi_{A_\zeta}$ 作为基本量。

### 1.2 场旋量 $\psi(s, \zeta; w), \tilde{\psi}(s, \zeta; w), \hat{\psi}(s, \zeta; w)$ 的引入

定义1.2.1.  $\psi(s, \zeta; w) \prec \psi_{k_\zeta}(s; w) \Leftrightarrow \psi^*(s, -\zeta; w) \prec \psi^{k_\zeta}(s; w)$

定义1.2.2.  $\tilde{\psi}(s, \zeta; w) := \psi_{A_\zeta \otimes l_\zeta}(s; w) \Leftrightarrow \tilde{\psi}^*(s, -\zeta; w) := \psi^{A_\zeta \otimes l_\zeta}(s; w)$

定义1.2.3.  $\hat{\psi}(s, \zeta; w) := \underbrace{\psi_{A_\zeta \otimes B_\zeta \otimes C_\zeta \otimes \dots}}_{2s}(s; w) \Leftrightarrow \hat{\psi}^*(s, -\zeta; w) := \overbrace{\psi^{A_\zeta \otimes B_\zeta \otimes C_\zeta \otimes \dots}}^{2s}(s; w)$

### 1.3 源旋量 $\tilde{J}(s, \zeta; w), \hat{J}(s, \zeta; w)$ 的引入

定义1.3.1.  $\tilde{J}(s, \zeta; w) := J^{A_\zeta \otimes l_\zeta}(s; w) \Leftrightarrow \tilde{J}^*(s, -\zeta; w) := J_{A_\zeta \otimes l_\zeta}(s; w)$

定义1.3.2.  $\hat{J}(s, \zeta; w) := J^{A_\zeta \otimes \underbrace{B_\zeta \otimes C_\zeta \otimes \dots}_{2s-1}}(s; w) \Leftrightarrow \hat{J}^*(s, -\zeta; w) := J_{A_\zeta \otimes \overbrace{B_\zeta \otimes C_\zeta \otimes \dots}^{2s-1}}(s; w)$

### 1.4 旋量 $\psi_{k_\zeta}(s; w), \psi^{k_\zeta}(s; w)$ 和 $\psi_{A_\zeta l_\zeta}(s; w), \psi^{A_\zeta l_\zeta}(s; w)$ 的引入

#### 1.4.1 $\psi(s, \zeta; w), \hat{\psi}(s, \zeta; w)$ 之间的关系

定义1.4.1. 
$$\left\{ \begin{array}{l} \psi_{k_\zeta}(s; w) := \Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta \dots}^{2s}}(s; w) \underbrace{\psi_{A_\zeta B_\zeta \dots}}_{2s}(s; w) \Leftrightarrow \psi(s, \zeta; w) = \bar{\Gamma}(s; w) \hat{\psi}(s, \zeta; w) \\ \updownarrow \qquad \qquad \qquad \updownarrow \\ \psi^{k_\zeta}(s; w) = \Gamma_{A_\zeta B_\zeta \dots}^{k_\zeta}(s; w) \underbrace{\psi^{A_\zeta B_\zeta \dots}}_{2s}(s; w) \Leftrightarrow \psi^*(s, -\zeta; w) = \bar{\Gamma}(s; w) \hat{\psi}^*(s, -\zeta; w) \end{array} \right.$$

推论1.4.1.  $\psi_{k_\zeta}(s; w) = \Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta \dots}^{2s}}(s; w) \underbrace{\psi_{A_\zeta B_\zeta \dots}}_{2s}(s; w) \Leftrightarrow \frac{1}{(2s)!} \underbrace{\psi_{(A_\zeta B_\zeta \dots)}}_{2s}(s; w) = \Gamma_{A_\zeta B_\zeta \dots}^{k_\zeta}(s; w) \underbrace{\psi_{k_\zeta}}_{2s}(s; w)$

推论1.4.2. 
$$\left\{ \begin{array}{l} \underbrace{\psi_{A_\zeta B_\zeta \dots}}_{2s}(s; w) = \frac{1}{(2s)!} \underbrace{\psi_{(A_\zeta B_\zeta \dots)}}_{2s}(s; w) = \Gamma_{A_\zeta B_\zeta \dots}^{k_\zeta}(s; w) \underbrace{\psi_{k_\zeta}}_{2s}(s; w) \Leftrightarrow \hat{\psi}(s, \zeta; w) = \Gamma(s; w) \psi(s, \zeta; w) \\ \updownarrow \qquad \qquad \qquad \updownarrow \\ \underbrace{\psi^{A_\zeta B_\zeta \dots}}_{2s}(s; w) = \frac{1}{(2s)!} \underbrace{\psi^{(A_\zeta B_\zeta \dots)}}_{2s}(s; w) = \Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta \dots}^{2s}}(s; w) \underbrace{\psi^{k_\zeta}}_{2s}(s; w) \Leftrightarrow \hat{\psi}^*(s, -\zeta; w) = \Gamma(s; w) \psi^*(s, -\zeta; w) \end{array} \right.$$

1.4.2  $\tilde{\psi}(s, \varsigma; w), \hat{\psi}(s, \varsigma; w)$ 之间的关系

$$\text{定义1.4.2.} \left\{ \begin{array}{l} \psi_{A_\varsigma l_\varsigma}(s; w) := \Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma \cdots}^{2s-1}}(s - \frac{1}{2}; w) \underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \cdots}}_{2s}(s; w) \Leftrightarrow \tilde{\psi}(s, \varsigma; w) = [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \hat{\psi}(s, \varsigma; w) \\ \updownarrow \\ \psi^{A_\varsigma l_\varsigma}(s; w) = \Gamma_{B_\varsigma C_\varsigma \cdots}^{l_\varsigma}(s - \frac{1}{2}; w) \underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \cdots}}_{2s}(s; w) \Leftrightarrow \tilde{\psi}^*(s, -\varsigma; w) = [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \hat{\psi}^*(s, -\varsigma; w) \end{array} \right.$$

## 推论1.4.3.

$$\left\{ \begin{array}{l} \underbrace{\psi_{A_\varsigma B_\varsigma \cdots}}_{2s}(s; w) = \frac{1}{(2s-1)!} \underbrace{\psi_{A_\varsigma(B_\varsigma C_\varsigma \cdots)}}_{2s}(s; w) = \Gamma_{B_\varsigma C_\varsigma \cdots}^{l_\varsigma}(s - \frac{1}{2}; w) \psi_{A_\varsigma l_\varsigma}(s; w) [\Leftrightarrow] \hat{\psi}(s, \varsigma; w) = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{\psi}(s, \varsigma; w) \\ \updownarrow \qquad \qquad \qquad \updownarrow \\ \underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \cdots}}_{2s}(s; w) = \frac{1}{(2s-1)!} \underbrace{\psi^{A_\varsigma(B_\varsigma C_\varsigma \cdots)}}_{2s}(s; w) = \Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma \cdots}^{2s-1}}(s - \frac{1}{2}; w) \psi^{A_\varsigma l_\varsigma}(s; w) [\Leftrightarrow] \hat{\psi}^*(s, -\varsigma; w) = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{\psi}^*(s, -\varsigma; w) \end{array} \right.$$

1.4.3  $\psi(s, \varsigma; w), \tilde{\psi}(s, \varsigma; w)$ 之间的关系

$$\text{推论1.4.4.} \left\{ \begin{array}{l} \psi_{k_\varsigma}(s; w) = N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; w) \psi_{A_\varsigma l_\varsigma}(s; w) [\Leftrightarrow] \psi(s, \varsigma; w) = \bar{N}(s; w) \tilde{\psi}(s, \varsigma; w) \\ \updownarrow \qquad \qquad \qquad \updownarrow \\ \psi^{k_\varsigma}(s; w) = N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; w) \psi^{A_\varsigma l_\varsigma}(s; w) [\Leftrightarrow] \psi^*(s, -\varsigma; w) = \bar{N}(s; w) \tilde{\psi}^*(s, -\varsigma; w) \end{array} \right.$$

$$\text{推论1.4.5.} \left\{ \begin{array}{l} \psi_{A_\varsigma l_\varsigma}(s; w) = N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; w) \psi_{k_\varsigma}(s; w), \underbrace{\psi_{A_\varsigma B_\varsigma \cdots}}_{2s}(s; w) = \frac{1}{(2s)!} \underbrace{\psi_{(A_\varsigma B_\varsigma \cdots)}}_{2s}(s; w) [\Leftrightarrow] \tilde{\psi}(s, \varsigma; w) = N(s; w) \psi(s, \varsigma; w) \\ \updownarrow \qquad \qquad \qquad \updownarrow \\ \psi^{A_\varsigma l_\varsigma}(s; w) = N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; w) \psi^{k_\varsigma}(s; w), \underbrace{\psi_{A_\varsigma B_\varsigma \cdots}}_{2s}(s; w) = \frac{1}{(2s)!} \underbrace{\psi^{(A_\varsigma B_\varsigma \cdots)}}_{2s}(s; w) [\Leftrightarrow] \tilde{\psi}^*(s, -\varsigma; w) = N(s; w) \psi^*(s, -\varsigma; w) \end{array} \right.$$

1.5 旋量  $J_{A'_\varsigma l_\varsigma}(s; w), J_{A'_\varsigma} l_\varsigma(s; w)$ 的引入及其  $\tilde{J}(s, \varsigma; w), \hat{J}(s, \varsigma; w)$ 之间的关系

$$\text{定义1.5.1.} \left\{ \begin{array}{l} J_{A'_\varsigma l_\varsigma}(s; w) := \Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma \cdots}^{2s-1}}(s - \frac{1}{2}; w) J_{A'_\varsigma} \underbrace{l_\varsigma}_{2s}(s; w) \Leftrightarrow \tilde{J}(s, \varsigma; w) = [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \hat{J}(s, \varsigma; w) \\ \updownarrow \\ J_{A'_\varsigma} l_\varsigma(s; w) = \Gamma_{B_\varsigma C_\varsigma \cdots}^{l_\varsigma}(s - \frac{1}{2}; w) J_{A'_\varsigma} \underbrace{l_\varsigma}_{2s}(s; w) \Leftrightarrow \tilde{J}^*(s, -\varsigma; w) = [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \hat{J}^*(s, -\varsigma; w) \end{array} \right.$$

## 推论1.5.1.

$$\left\{ \begin{array}{l} \underbrace{J_{A'_\varsigma} l_\varsigma}_{2s}(s; w) = \frac{1}{(2s-1)!} J_{A'_\varsigma} \underbrace{l_\varsigma}_{2s}(s; w) = \Gamma_{B_\varsigma C_\varsigma \cdots}^{l_\varsigma}(s - \frac{1}{2}; w) J_{A'_\varsigma l_\varsigma}(s; w) [\Leftrightarrow] \hat{J}(s, \varsigma; w) = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{J}(s, \varsigma; w) \\ \updownarrow \qquad \qquad \qquad \updownarrow \\ \underbrace{J_{A'_\varsigma} l_\varsigma}_{2s}(s; w) = \frac{1}{(2s-1)!} J_{A'_\varsigma} \underbrace{l_\varsigma}_{2s}(s; w) = \Gamma_{l_\varsigma}^{\overbrace{B_\varsigma C_\varsigma \cdots}^{2s-1}}(s - \frac{1}{2}; w) J_{A'_\varsigma l_\varsigma}(s; w) [\Leftrightarrow] \hat{J}^*(s, -\varsigma; w) = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{J}^*(s, -\varsigma; w) \end{array} \right.$$

1.6 旋量  $J_{A'_\varsigma B_\varsigma C_\varsigma D_\varsigma \cdots}$ 的引入及其  $\hat{\mathcal{J}}(n), \hat{J}(n, \varsigma)$ 之间的关系

## 推论1.6.1.

$$\left\{ \begin{array}{l} \underbrace{J_{A'_\varsigma B_\varsigma C_\varsigma D_\varsigma \cdots}}_{2n-1} := (\frac{i\varsigma}{\sqrt{2}})^n \varsigma (\sigma \varepsilon, -i\varsigma \varepsilon) a_{A'_\varsigma B_\varsigma} \underbrace{\sigma_{C_\varsigma D_\varsigma}^{\beta_\varsigma} \cdots}_{n-1} J_{a\beta_\varsigma \cdots} \underbrace{\quad}_n [\Leftrightarrow] \hat{\mathcal{J}}(n, \varsigma) = \overbrace{S_{em}^+(\varsigma) \otimes S_{em}^+(\pm\varsigma) \cdots}^n \hat{J}(n) \\ \updownarrow \qquad \qquad \qquad \updownarrow \\ \underbrace{J_{a\beta_\varsigma \cdots}}_n = (\frac{i\varsigma}{\sqrt{2}})^n \varsigma (\varepsilon \sigma, -i\varsigma \varepsilon) a_{A'_\varsigma B_\varsigma} \underbrace{\sigma_{C_\varsigma D_\varsigma}^{\beta_\varsigma} \cdots}_{n-1} J_{A'_\varsigma} \underbrace{l_\varsigma}_{2n-1} [\Leftrightarrow] \hat{\mathcal{J}}(n) = \overbrace{S_{em}(\varsigma) \otimes S_{em}(\pm\varsigma) \cdots}^n \hat{J}(n, \varsigma) \end{array} \right.$$



$$[\Downarrow]$$

推论1.6.2.

$$\left\{ \begin{array}{l} \overbrace{J_{A'_\zeta}^{B_\zeta C_\zeta D_\zeta \dots}}^{2n-1} := \left(\frac{i\zeta}{\sqrt{2}}\right)^n \zeta (\varepsilon \sigma, -i\zeta \varepsilon)_{aA'_\zeta} B_\zeta \overbrace{\sigma_{C_\zeta D_\zeta}^{n-1}} \dots \overbrace{J^{a\beta_\zeta \dots}}^n [\Leftrightarrow] \hat{J}^*(n, -\zeta) = \overbrace{S_{em}^T(\zeta) \otimes S_{em}^T(\pm\zeta) \dots}^n \hat{J}(n) \\ \overbrace{J^{a\beta_\zeta \dots}}^n = \left(\frac{i\zeta}{\sqrt{2}}\right)^n \zeta (\sigma \varepsilon, -i\zeta \varepsilon)^{aA'_\zeta} B_\zeta \overbrace{\sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots}^{n-1} \overbrace{J_{A'_\zeta}^{B_\zeta C_\zeta D_\zeta \dots}}^{2n-1} [\Leftrightarrow] \hat{J}(n) = \overbrace{S_{em}^*(\zeta) \otimes S_{em}^*(\pm\zeta) \dots}^n \hat{J}^*(n, -\zeta) \end{array} \right. \quad \begin{array}{l} [\Downarrow] \\ [\Downarrow] \end{array}$$

1.7  $\psi_{\alpha_\zeta \beta_\zeta \dots}(n), \psi^{\alpha_\zeta \beta_\zeta \dots}(n), \Psi(n, \zeta), \hat{\Psi}(n, \zeta)$  的定义

$$\text{定义1.7.1. } \hat{\Psi}(n, \zeta) \prec \underbrace{\psi_{\alpha_\zeta \otimes \beta_\zeta \otimes \dots}}_n(n) \Leftrightarrow \hat{\Psi}^*(n, -\zeta) \prec \overbrace{\psi^{\alpha_\zeta \otimes \beta_\zeta \otimes \dots}}^n(n)$$

$$\text{定义1.7.2. } \left\{ \begin{array}{l} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n(n) := \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n) \psi_{k_\zeta}(n) \Rightarrow \psi_{k_\zeta}(n) = \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n(n) \\ \underbrace{\psi^{\alpha_\zeta \beta_\zeta \dots}}_n(n) := \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots}(n) \psi^{k_\zeta}(n) \Rightarrow \psi^{k_\zeta}(n) = \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta}(n) \underbrace{\psi^{\alpha_\zeta \beta_\zeta \dots}}_n(n) \end{array} \right. \quad \begin{array}{l} \Downarrow \\ \Downarrow \end{array}$$

$$\text{推论1.7.1. } \left\{ \begin{array}{l} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n(n) = \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{A_\zeta B_\zeta \dots}(n) \underbrace{\psi_{A_\zeta B_\zeta \dots}}_{2n}(n) [\Leftrightarrow] \frac{1}{(2n)!} \underbrace{\psi_{(A_\zeta B_\zeta \dots)}}_{2n}(n) = \Gamma_{A_\zeta B_\zeta \dots}^{\alpha_\zeta \beta_\zeta \dots}(n) \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n(n) \\ \underbrace{\psi^{\alpha_\zeta \beta_\zeta \dots}}_n(n) = \Gamma_{A_\zeta B_\zeta \dots}^{\alpha_\zeta \beta_\zeta \dots}(n) \underbrace{\psi^{A_\zeta B_\zeta \dots}}_{2n}(n) [\Leftrightarrow] \frac{1}{(2n)!} \underbrace{\psi^{(A_\zeta B_\zeta \dots)}}_{2n}(n) = \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{A_\zeta B_\zeta \dots}(n) \underbrace{\psi^{\alpha_\zeta \beta_\zeta \dots}}_n(n) \end{array} \right. \quad \begin{array}{l} [\Downarrow] \\ [\Downarrow] \end{array}$$

$$\text{推论1.7.2. } \left\{ \begin{array}{l} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n(n) = \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{A_\zeta B_\zeta \dots}(n) \underbrace{\psi_{A_\zeta B_\zeta \dots}}_{2n}(n) [\Leftrightarrow] \hat{\Psi}(n, \zeta) = \overbrace{[S_{em}(\pm\zeta) \otimes S_{em}(\pm\zeta) \dots]}^n \hat{\psi}(n, \zeta) \\ \underbrace{\psi^{\alpha_\zeta \beta_\zeta \dots}}_n(n) = \Gamma_{A_\zeta B_\zeta \dots}^{\alpha_\zeta \beta_\zeta \dots}(n) \underbrace{\psi^{A_\zeta B_\zeta \dots}}_{2n}(n) [\Leftrightarrow] \hat{\Psi}^*(n, -\zeta) = \overbrace{[S_{em}^*(\mp\zeta) \otimes S_{em}^*(\mp\zeta) \dots]}^n \hat{\psi}^*(n, -\zeta) \end{array} \right. \quad \begin{array}{l} [\Downarrow] \\ [\Downarrow] \end{array}$$

推论1.7.3.

$$\left\{ \begin{array}{l} \underbrace{\psi_{A_\zeta B_\zeta \dots}}_{2n}(n) = \frac{1}{(2n)!} \underbrace{\psi_{(A_\zeta B_\zeta \dots)}}_{2n}(n) = \Gamma_{A_\zeta B_\zeta \dots}^{\alpha_\zeta \beta_\zeta \dots}(n) \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n(n) [\Leftrightarrow] \hat{\psi}(n, \zeta) = \overbrace{[S_{em}^+(\pm\zeta) \otimes S_{em}^+(\pm\zeta) \dots]}^n \hat{\Psi}(n, \zeta) \\ \underbrace{\psi^{A_\zeta B_\zeta \dots}}_{2n}(n) = \frac{1}{(2n)!} \underbrace{\psi^{(A_\zeta B_\zeta \dots)}}_{2n}(n) = \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{A_\zeta B_\zeta \dots}(n) \underbrace{\psi^{\alpha_\zeta \beta_\zeta \dots}}_n(n) [\Leftrightarrow] \hat{\psi}^*(n, -\zeta) = \overbrace{[S_{em}^T(\mp\zeta) \otimes S_{em}^T(\mp\zeta) \dots]}^n \hat{\Psi}^*(n, -\zeta) \end{array} \right. \quad \begin{array}{l} [\Downarrow] \\ [\Downarrow] \end{array}$$

$$\text{推论1.7.4. } \underbrace{\psi_{A_\zeta B_\zeta \dots}}_{2n}(n) = \frac{1}{(2n)!} \underbrace{\psi_{(A_\zeta B_\zeta \dots)}}_{2n}(n) [\Leftrightarrow] \underbrace{\psi^{\alpha_\zeta \beta_\zeta \dots}}_n(n) = \frac{1}{n!} \underbrace{\psi^{(\alpha_\zeta \beta_\zeta \dots)}}_n(n), \delta_{\alpha_\zeta \beta_\zeta} \underbrace{\psi^{\alpha_\zeta \beta_\zeta \dots}}_n(n) = 0$$

## 1.8 旋量 $J_{B_\zeta C_\zeta D_\zeta \dots}^{A_\zeta}$ 的引入及其 $\hat{\mathcal{J}}(n)$ , $\hat{J}(n, \zeta)$ 之间的关系

$$\text{推论1.8.1.} \left\{ \begin{array}{l} \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} := \left(\frac{i\zeta}{\sqrt{2}}\right)^n \overbrace{\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots}^n \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n [\Leftrightarrow] \hat{\psi}(n, \zeta) = \overbrace{S_{em}^+(\pm\zeta) \otimes S_{em}^+(\pm\zeta) \dots}^n \hat{\Psi}(n) \\ \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n = \left(\frac{i\zeta}{\sqrt{2}}\right)^n \overbrace{\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \dots}^n \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} [\Leftrightarrow] \hat{\Psi}(n) = \overbrace{S_{em}(\pm\zeta) \otimes S_{em}(\pm\zeta) \dots}^n \hat{\psi}(n, \zeta) \end{array} \right. \quad \begin{array}{l} [\Downarrow] \\ [\Downarrow] \end{array}$$

推论1.8.2.

$$\left\{ \begin{array}{l} \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} := \left(\frac{i\zeta}{\sqrt{2}}\right)^n \overbrace{\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \dots}^n \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n [\Leftrightarrow] \hat{\psi}^*(n, -\zeta) = \overbrace{S_{em}^T(\mp\zeta) \otimes S_{em}^T(\mp\zeta) \dots}^n \hat{\Psi}(n) \\ \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n = \left(\frac{i\zeta}{\sqrt{2}}\right)^n \overbrace{\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots}^n \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} [\Leftrightarrow] \hat{\Psi}(n) = \overbrace{S_{em}^*(\mp\zeta) \otimes S_{em}^*(\mp\zeta) \dots}^n \hat{\psi}^*(n, -\zeta) \end{array} \right. \quad \begin{array}{l} [\Downarrow] \\ [\Downarrow] \end{array}$$

## 2 全对称性条件分析

### 2.1 场量全对称性条件分析

定义2.1.1.

$$\underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} Z_\zeta := \left(\frac{i\zeta}{\sqrt{2}}\right)^n \overbrace{(\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots)}^n \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta \Leftrightarrow \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} Z_\zeta = \left(\frac{i\zeta}{\sqrt{2}}\right)^n \overbrace{(\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \dots)}^n \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta \quad \begin{array}{l} [\Downarrow] \\ [\Downarrow] \end{array}$$

推论2.1.1.

$$\underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta = \left(\frac{i\zeta}{\sqrt{2}}\right)^n \overbrace{(\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \dots)}^n \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} Z_\zeta [\Leftrightarrow] \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta = \left(\frac{i\zeta}{\sqrt{2}}\right)^n \overbrace{(\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots)}^n \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} Z_\zeta$$

推论2.1.2.  $\underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} Z_\zeta$  关于  $(A_\zeta B_\zeta), (C_\zeta D_\zeta), \dots$  () 内全对称, () 间全对称  $\Leftrightarrow \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta = \frac{1}{n!} \underbrace{\psi_{(\alpha_\zeta \beta_\zeta \dots)}}_n Z_\zeta$

推论2.1.3.  $\underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} Z_\zeta$  关于  $(A_\zeta B_\zeta), (C_\zeta D_\zeta), \dots$  () 内全对称, () 间全对称  $\Leftrightarrow \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta = \frac{1}{n!} \underbrace{\psi_{(\alpha_\zeta \beta_\zeta \dots)}}_n Z_\zeta$

推论2.1.4.  $\underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} Z_\zeta = \underbrace{\psi_{A_\zeta C_\zeta B_\zeta D_\zeta \dots}}_{2n} Z_\zeta \Leftrightarrow \delta^{\alpha_\zeta \beta_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta = 0$

$$[\Downarrow] \quad [\Downarrow]$$

推论2.1.5.  $\underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} Z_\zeta = \underbrace{\psi_{A_\zeta C_\zeta B_\zeta D_\zeta \dots}}_{2n} Z_\zeta \Leftrightarrow \delta_{\alpha_\zeta \beta_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta = 0$

证明:  $\underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} Z_\zeta = \underbrace{\psi_{A_\zeta C_\zeta B_\zeta D_\zeta \dots}}_{2n} Z_\zeta$

$$\Leftrightarrow \varepsilon^{B_\zeta C_\zeta} \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}}_{2n} Z_\zeta = 0$$

$$\Leftrightarrow \varepsilon^{B_\zeta C_\zeta} \overbrace{(\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \dots)}^n \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta = 0$$

$$\Leftrightarrow \varepsilon^{B_\zeta C_\zeta} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta = 0$$

$$\Leftrightarrow \sigma_{A_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta}^{\beta_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \dots}}_n Z_\zeta = 0$$

$$\Leftrightarrow \sigma^{\alpha_\zeta} \sigma^{\beta_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = 0$$

$$\Leftrightarrow \delta^{\alpha_\zeta \beta_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = 0$$

□

$$\text{推论2.1.6. } \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}}_{2n} = \underbrace{\psi_{A_\zeta Z_\zeta C_\zeta D_\zeta \cdots B_\zeta}}_{2n} \Leftrightarrow \sigma^{\alpha_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots [Z_\zeta]}}_n = 0$$

[⇔]

[⇔]

$$\text{推论2.1.7. } \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}}_{2n} = \underbrace{\psi_{A_\zeta Z_\zeta C_\zeta D_\zeta \cdots B_\zeta}}_{2n} \Leftrightarrow \sigma_{\alpha_\zeta}^* \underbrace{\psi^{\alpha_\zeta \beta_\zeta \cdots [Z_\zeta]}}_n = 0$$

$$\text{证明: } \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}}_{2n} = \underbrace{\psi_{A_\zeta Z_\zeta C_\zeta D_\zeta \cdots B_\zeta}}_{2n}$$

$$\Leftrightarrow \varepsilon^{B_\zeta Z_\zeta} \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}}_{2n} = 0$$

$$\Leftrightarrow \varepsilon^{B_\zeta Z_\zeta} (\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \cdots) \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = 0$$

$$\Leftrightarrow \varepsilon^{B_\zeta Z_\zeta} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = 0$$

$$\Leftrightarrow \sigma^{\alpha_\zeta} A_\zeta \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = 0$$

$$\Leftrightarrow \sigma^{\alpha_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots [Z_\zeta]}}_n = 0$$

□

$$\text{推论2.1.8. } \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}}_{2n} = \frac{1}{(2n)!} \underbrace{\psi_{(A_\zeta B_\zeta C_\zeta D_\zeta \cdots)}}_{2n} Z_\zeta \Leftrightarrow \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = \frac{1}{n!} \underbrace{\psi_{(\alpha_\zeta \beta_\zeta \cdots)}}_n Z_\zeta, \delta^{\alpha_\zeta \beta_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = 0$$

[⇔]

[⇔]

$$\text{推论2.1.9. } \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}}_{2n} = \frac{1}{(2n)!} \underbrace{\psi_{(A_\zeta B_\zeta C_\zeta D_\zeta \cdots)}}_{2n} Z_\zeta \Leftrightarrow \underbrace{\psi^{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = \frac{1}{n!} \underbrace{\psi^{(\alpha_\zeta \beta_\zeta \cdots)}}_n Z_\zeta, \delta_{\alpha_\zeta \beta_\zeta} \underbrace{\psi^{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = 0$$

$$\text{推论2.1.10. } \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}}_{2n+1} = \frac{1}{(2n+1)!} \underbrace{\psi_{(A_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta)}}_{2n+1} \Leftrightarrow \begin{cases} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = \frac{1}{n!} \underbrace{\psi_{(\alpha_\zeta \beta_\zeta \cdots)}}_n Z_\zeta \\ \delta^{\alpha_\zeta \beta_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = 0, \sigma^{\alpha_\zeta} \underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots [Z_\zeta]}}_n = 0 \end{cases}$$

[⇔]

[⇔]

$$\text{推论2.1.11. } \underbrace{\psi_{A_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}}_{2n+1} = \frac{1}{(2n+1)!} \underbrace{\psi_{(A_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta)}}_{2n+1} \Leftrightarrow \begin{cases} \underbrace{\psi^{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = \frac{1}{n!} \underbrace{\psi^{(\alpha_\zeta \beta_\zeta \cdots)}}_n Z_\zeta \\ \delta_{\alpha_\zeta \beta_\zeta} \underbrace{\psi^{\alpha_\zeta \beta_\zeta \cdots Z_\zeta}}_n = 0, \sigma_{\alpha_\zeta} \underbrace{\psi^{\alpha_\zeta \beta_\zeta \cdots [Z_\zeta]}}_n = 0 \end{cases}$$

## 2.2 源对称性条件分析

$$\text{定义2.2.1. } J_{\underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'_\zeta} := \left(\frac{i\zeta}{\sqrt{2}}\right)^n \zeta [(\sigma\varepsilon, -i\zeta\varepsilon)^{aA'_\zeta} \underbrace{\sigma_{B_\zeta C_\zeta D_\zeta \cdots}^{\beta_\zeta}}_n] \underbrace{J_{a\beta_\zeta \cdots Z_\zeta}}_n$$

$$\text{推论2.2.1. } \underbrace{J_{a\beta_\zeta \cdots Z_\zeta}}_n = \left(\frac{i\zeta}{\sqrt{2}}\right)^n \zeta [(\varepsilon\sigma, -i\zeta\varepsilon)_{aA'_\zeta} \underbrace{\sigma_{B_\zeta C_\zeta D_\zeta \cdots}^{\beta_\zeta}}_n] \underbrace{J_{A'_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}}_{2n}$$

$$\text{推论2.2.2. } J_{\underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'_\zeta} \text{ 关于 } (C_\zeta D_\zeta), (E_\zeta F_\zeta), \cdots () \text{ 内全对称, } () \text{ 间全对称} \Leftrightarrow \underbrace{J_{a\beta_\zeta \cdots Z_\zeta}}_n = \frac{1}{(n-1)!} \underbrace{J_{a(\beta_\zeta \cdots)}}_n Z_\zeta$$

$$\text{推论2.2.3. } J_{\underbrace{A'_\zeta B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}} \text{ 关于 } (C_\zeta D_\zeta), (E_\zeta F_\zeta), \cdots () \text{ 内全对称, } () \text{ 间全对称} \Leftrightarrow \underbrace{J^{a\beta_\zeta \cdots Z_\zeta}}_n = \frac{1}{(n-1)!} \underbrace{J^{a(\beta_\zeta \cdots)}}_n Z_\zeta$$

$$\text{推论2.2.4. } J_{\underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'} = J_{\underbrace{C_\zeta B_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'} \Leftrightarrow (\sigma, -i\zeta)^a \sigma^{\beta_\zeta} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} = 0$$

$$[\Leftrightarrow] \quad [\Leftrightarrow]$$

$$\text{推论2.2.5. } J_{A'_\zeta \underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}} = J_{A'_\zeta \underbrace{C_\zeta B_\zeta D_\zeta \cdots Z_\zeta}_{2n}} \Leftrightarrow (\sigma, -i\zeta)_a \sigma_{\beta_\zeta} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} = 0$$

$$\begin{aligned} \text{证明: } J_{\underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'} &= J_{\underbrace{C_\zeta B_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'} \\ \Leftrightarrow \varepsilon^{B_\zeta C_\zeta} J_{\underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'} &= 0 \\ \Leftrightarrow \varepsilon^{B_\zeta C_\zeta} [(\sigma \varepsilon, -i\zeta \varepsilon)^{aA'_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \cdots] J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} &= 0 \\ \Leftrightarrow \varepsilon^{B_\zeta C_\zeta} (\sigma \varepsilon, -i\zeta \varepsilon)^{aA'_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} &= 0 \\ \Leftrightarrow (\sigma, -i\zeta)^{aA'_\zeta C_\zeta} \sigma^{\beta_\zeta} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} &= 0 \\ \Leftrightarrow (\sigma, -i\zeta)^a \sigma^{\beta_\zeta} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} &= 0 \end{aligned}$$

□

$$\text{推论2.2.6. } J_{\underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'} = J_{\underbrace{Z_\zeta C_\zeta D_\zeta \cdots B_\zeta}_{2n}}^{A'} \Leftrightarrow (\sigma, -i\zeta)^a J_{\underbrace{a\beta_\zeta \cdots [Z_\zeta]}_n} = 0$$

$$[\Leftrightarrow] \quad [\Leftrightarrow]$$

$$\text{推论2.2.7. } J_{A'_\zeta \underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}} = J_{A'_\zeta \underbrace{Z_\zeta C_\zeta D_\zeta \cdots B_\zeta}_{2n}} \Leftrightarrow (\sigma^*, i\zeta)_a J_{\underbrace{a\beta_\zeta \cdots [Z_\zeta]}_n} = 0$$

$$\begin{aligned} \text{证明: } J_{\underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'} &= J_{\underbrace{Z_\zeta C_\zeta D_\zeta \cdots B_\zeta}_{2n}}^{A'} \\ \Leftrightarrow \varepsilon^{B_\zeta Z_\zeta} J_{\underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'} &= 0 \\ \Leftrightarrow \varepsilon^{B_\zeta Z_\zeta} [(\sigma \varepsilon, -i\zeta \varepsilon)^{aA'_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \cdots] J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} &= 0 \\ \Leftrightarrow \varepsilon^{B_\zeta Z_\zeta} (\sigma \varepsilon, -i\zeta \varepsilon)^{aA'_\zeta} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} &= 0 \\ \Leftrightarrow (\sigma, -i\zeta)^{aA'_\zeta Z_\zeta} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} &= 0 \\ \Leftrightarrow (\sigma, -i\zeta)^a J_{\underbrace{a\beta_\zeta \cdots [Z_\zeta]}_n} &= 0 \end{aligned}$$

□

$$\text{推论2.2.8. } J_{\underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n-1}}^{A'} = \frac{1}{(2n-1)!} J_{\underbrace{(B_\zeta C_\zeta D_\zeta \cdots) Z_\zeta}_{2n-1}}^{A'} \Leftrightarrow \begin{cases} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} = \frac{1}{(n-1)!} J_{\underbrace{a(\beta_\zeta \cdots) Z_\zeta}_n} \\ (\sigma, -i\zeta)^a \sigma^{\beta_\zeta} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} = 0 \end{cases}$$

$$[\Leftrightarrow] \quad [\Leftrightarrow]$$

$$\text{推论2.2.9. } J_{A'_\zeta \underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n-1}} = \frac{1}{(2n-1)!} J_{A'_\zeta \underbrace{(B_\zeta C_\zeta D_\zeta \cdots) Z_\zeta}_{2n-1}} \Leftrightarrow \begin{cases} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} = \frac{1}{(n-1)!} J_{\underbrace{a(\beta_\zeta \cdots) Z_\zeta}_n} \\ (\sigma, -i\zeta)_a \sigma_{\beta_\zeta} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} = 0 \end{cases}$$

$$\text{推论2.2.10. } J_{\underbrace{B_\zeta C_\zeta D_\zeta \cdots Z_\zeta}_{2n}}^{A'} = \frac{1}{(2n)!} J_{\underbrace{(B_\zeta C_\zeta D_\zeta \cdots) Z_\zeta}_{2n}}^{A'} \Leftrightarrow \begin{cases} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} = \frac{1}{(n-1)!} J_{\underbrace{a(\beta_\zeta \cdots) Z_\zeta}_n} \\ (\sigma, -i\zeta)^a \sigma^{\beta_\zeta} J_{\underbrace{a\beta_\zeta \cdots Z_\zeta}_n} = 0, (\sigma, -i\zeta)^a J_{\underbrace{a\beta_\zeta \cdots [Z_\zeta]}_n} = 0 \end{cases}$$

$$\text{推论2.2.11. } J_{A'_c} \overbrace{B_c C_c D_c \cdots Z_c}^{2n} = \frac{1}{(2n)!} J_{A'_c} \overbrace{(B_c C_c D_c \cdots Z_c)}^{2n} \Leftrightarrow \begin{cases} \overbrace{J^{a\beta\gamma \cdots} Z_c}^n = \frac{1}{(n-1)!} \overbrace{J^{a(\beta\gamma \cdots)} Z_c}^n \\ (\sigma, -i\zeta)_a \sigma_{\beta\gamma} \overbrace{J^{a\beta\gamma \cdots} Z_c}^n = 0, (\sigma^*, i\zeta)_a \overbrace{J^{a\beta\gamma \cdots} Z_c}^n = 0 \end{cases}$$

### 3 电磁场的全对称旋量与表象变换

#### 3.1 电磁场的全对称旋量 $\psi_{A_c B_c}$

定义3.1.1.  $\psi_{A_c B_c} = \psi_{B_c A_c} \Leftrightarrow \hat{\psi}(1, \varsigma) = S_{ex} \hat{\psi}(1, \varsigma)$

定义3.1.2.  $\hat{\Psi}(1, \varsigma) = \tilde{\Psi}(1, \varsigma) := [\psi_{x_c}, \psi_{y_c}, \psi_{z_c}, 0]^T, \hat{\psi}(1, \varsigma) = \tilde{\psi}(1, \varsigma) := [\psi_{1_c 1_c}, \psi_{1_c 2_c}, \psi_{1_c 2_c}, \psi_{2_c 2_c}]^T$

定义3.1.3.  $\psi_{\alpha_c} \succ \Psi(1, \varsigma) := [\psi_{x_c}, \psi_{y_c}, \psi_{z_c}]^T, \psi(1, \varsigma) := [\psi_{1_c 1_c}, \sqrt{C_2^1} \psi_{1_c 2_c}, \psi_{2_c 2_c}]^T$

#### 3.2 $\psi(1, \varsigma), \hat{\psi}(1, \varsigma)$ 之间的关系

$$\text{推论3.2.1. } \begin{cases} \psi_{k_c}(1) = \Gamma_{k_c}^{A_c B_c}(1) \psi_{A_c B_c} \Leftrightarrow \psi(1, \varsigma) = \bar{\Gamma}(\frac{3}{2}) \hat{\psi}(1, \varsigma) \\ \psi^{k_c}(1) = \Gamma_{A_c B_c}^{k_c}(1) \psi^{A_c B_c} \Leftrightarrow \psi^*(1, -\varsigma) = \bar{\Gamma}(\frac{3}{2}) \hat{\psi}^*(1, -\varsigma) \end{cases}$$

$$\text{推论3.2.2. } \begin{cases} \psi_{A_c B_c} = \Gamma_{A_c B_c}^{k_c}(1) \psi_{k_c}(1) \Leftrightarrow \hat{\psi}(1, \varsigma) = \Gamma(\frac{3}{2}) \psi(1, \varsigma) \\ \psi^{A_c B_c} = \Gamma_{k_c}^{A_c B_c}(1) \psi^{k_c}(1) \Leftrightarrow \hat{\psi}^*(1, -\varsigma) = \Gamma(\frac{3}{2}) \psi^*(1, -\varsigma) \end{cases}$$

#### 3.3 $\tilde{\psi}(1, \varsigma), \hat{\psi}(1, \varsigma)$ 之间的关系

推论3.3.1.  $\tilde{\psi}(1, \varsigma) = \hat{\psi}(1, \varsigma)$

#### 3.4 $\psi(1, \varsigma), \tilde{\psi}(1, \varsigma)$ 之间的关系

$$\text{推论3.4.1. } \begin{cases} \psi_{k_c}(1) = N_{k_c}^{A_c B_c}(1) \psi_{A_c B_c} \Leftrightarrow \psi(1, \varsigma) = \bar{N}(1) \tilde{\psi}(1, \varsigma) \\ \psi^{k_c}(1) = N_{A_c B_c}^{k_c}(1) \psi^{A_c B_c} \Leftrightarrow \psi^*(1, -\varsigma) = \bar{N}(1) \tilde{\psi}^*(1, -\varsigma) \end{cases}$$

$$\text{推论3.4.2. } \begin{cases} \psi_{A_c B_c} = N_{A_c B_c}^{k_c}(1) \psi_{k_c}(1) \Leftrightarrow \tilde{\psi}(1, \varsigma) = N(1) \psi(1, \varsigma) \\ \psi^{A_c B_c} = N_{k_c}^{A_c B_c}(1) \psi^{k_c}(1) \Leftrightarrow \tilde{\psi}^*(1, -\varsigma) = N(1) \psi^*(1, -\varsigma) \end{cases}$$

#### 3.5 表象变换矩阵是常数不变张量

定义3.5.1.  $J_{A'_c A_c} := \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{A'_c A_c}^a J_a \Leftrightarrow J_a = \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_c A_c} J_{A'_c A_c}$

推论3.5.1.  $J_a = \frac{i}{\sqrt{2}} (\varepsilon\sigma, -i\zeta\varepsilon)_{aA'_c} B_c J^{A'_c B_c}$

证明:  $J_a = \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_c A_c} J_{A'_c A_c} = \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_c A_c} (-\zeta\varepsilon A'_c B'_c) J^{B'_c A_c} = \frac{i}{\sqrt{2}} (\varepsilon\sigma, -i\zeta\varepsilon)_{aA'_c} B_c J^{A'_c B_c}$  □

$$\text{推论3.5.2. } \begin{cases} \frac{i}{\sqrt{2}} (\varepsilon\sigma, -i\zeta\varepsilon)_{aA'_c} B_c \succ S_{em}(\varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & -i & -i & 0 \\ 0 & -\zeta & \zeta & 0 \end{bmatrix} \\ \frac{i}{\sqrt{2}} (\sigma\varepsilon, -i\zeta\varepsilon)_{aA'_c} B_c \succ S_{em}^*(\varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & i & 0 \\ 0 & -\zeta & \zeta & 0 \end{bmatrix} \end{cases}$$

$$\text{推论3.5.3. } \begin{cases} \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha\zeta}^{A_\zeta B_\zeta} = \frac{i}{\sqrt{2}}[\varepsilon\sigma]_{\alpha\zeta}^{A_\zeta B_\zeta} \rightarrow \frac{i}{\sqrt{2}}(\varepsilon\sigma, -i\zeta\varepsilon)_{\alpha\zeta}^{A_\zeta \otimes B_\zeta} \succ S_{em}(\zeta) \\ \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta} = \frac{i}{\sqrt{2}}[\sigma\varepsilon]_{\alpha\zeta}^{A_\zeta B_\zeta} \rightarrow \frac{i}{\sqrt{2}}(\sigma\varepsilon, -i\zeta\varepsilon)_{A_\zeta B_\zeta}^{\alpha\zeta} \succ S_{em}^*(\zeta) \end{cases}$$

$$\text{推论3.5.4. } \begin{cases} \Gamma_{\alpha\zeta}^{k_\zeta}(1) \succ S_m(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & -i \\ -1 & 0 & -1 \\ 0 & -i\sqrt{2} & 0 \end{bmatrix} & \left\{ \begin{array}{l} \Gamma^{\alpha\zeta}_{k_\zeta}(1) \succ S_m^*(1) \\ \Gamma^{k_\zeta}_{\alpha\zeta}(1) \succ S_m^T(1) \end{array} \right. \\ \Gamma_{k_\zeta}^{\alpha\zeta}(1) \succ S_m^+(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 \\ 0 & 0 & i\sqrt{2} \\ i & -1 & 0 \end{bmatrix} & \end{cases}$$

$$\text{推论3.5.5. } S_{em}(\zeta) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & -i & i & 0 \\ 0 & -\zeta & \zeta & 0 \end{bmatrix}, S_{em}^+(\zeta) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 & 0 \\ 0 & 0 & i & -\zeta \\ 0 & 0 & i & \zeta \\ i & -1 & 0 & 0 \end{bmatrix}, S_{em}(\zeta)S_{em}^+(\zeta) = S_{em}^+(\zeta)S_{em}(\zeta) = I_4$$

$$\text{推论3.5.6. } S_m(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & -i \\ -1 & 0 & -1 \\ 0 & -i\sqrt{2} & 0 \end{bmatrix}, S_m^+(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 \\ 0 & 0 & i\sqrt{2} \\ i & -1 & 0 \end{bmatrix}, S_m(1)S_m^+(1) = S_m^+(1)S_m(1) = I_3$$

### 3.6 电磁场量的旋量关系与表象变换

推论3.6.1.

$$\begin{cases} \psi_{\alpha\zeta} = \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta} [\Leftrightarrow] \hat{\Psi}(1, \zeta) = S_{em}(\pm\zeta) \hat{\psi}(1, \zeta) & \left\{ \begin{array}{l} \psi^{\alpha\zeta} = \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta} \psi^{A_\zeta B_\zeta} [\Leftrightarrow] \hat{\Psi}^*(1, -\zeta) = S_{em}^*(\mp\zeta) \hat{\psi}^*(1, -\zeta) \\ [\Downarrow] & [\Downarrow] \end{array} \right. \\ \psi_{A_\zeta B_\zeta} = \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta} \psi_{\alpha\zeta} [\Leftrightarrow] \hat{\psi}(1, \zeta) = S_{em}^+(\pm\zeta) \hat{\Psi}(1, \zeta) & \left\{ \begin{array}{l} \psi^{A_\zeta B_\zeta} = \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha\zeta}^{A_\zeta B_\zeta} \psi^{\alpha\zeta} [\Leftrightarrow] \hat{\psi}^*(1, -\zeta) = S_{em}^T(\mp\zeta) \hat{\Psi}^*(1, -\zeta) \\ [\Downarrow] & [\Downarrow] \end{array} \right. \end{cases} \quad [\Downarrow]$$

推论3.6.2.

$$\begin{cases} \psi_{\alpha\zeta} = \Gamma_{\alpha\zeta}^{k_\zeta}(1) \psi_{k_\zeta} [\Leftrightarrow] \Psi(1, \zeta) = S_m(1) \psi(1, \zeta) & \left\{ \begin{array}{l} \psi^{\alpha\zeta} = \Gamma_{k_\zeta}^{\alpha\zeta}(1) \psi^{k_\zeta} [\Leftrightarrow] \Psi^*(1, -\zeta) = S_m^*(1) \psi^*(1, -\zeta) \\ [\Downarrow] & [\Downarrow] \end{array} \right. \\ \psi_{k_\zeta} = \Gamma_{k_\zeta}^{\alpha\zeta}(1) \psi_{\alpha\zeta} [\Leftrightarrow] \psi(1, \zeta) = S_m^+(1) \Psi(1, \zeta) & \left\{ \begin{array}{l} \psi^{k_\zeta} = \Gamma_{\alpha\zeta}^{k_\zeta}(1) \psi^{\alpha\zeta} [\Leftrightarrow] \psi^*(1, -\zeta) = S_m^T(1) \Psi^*(1, -\zeta) \\ [\Downarrow] & [\Downarrow] \end{array} \right. \end{cases} \quad [\Downarrow]$$

### 3.7 电磁场源的旋量关系与表象变换

$$\text{推论3.7.1. } \tilde{J}(1, \zeta) = \hat{J}(1, \zeta) = (J^{1'_{1\zeta}}, J^{2'_{1\zeta}}, J^{1'_{2\zeta}}, J^{2'_{2\zeta}})$$

$$\text{推论3.7.2. } \begin{cases} J_a = \frac{i}{\sqrt{2}}(\varepsilon\sigma, -i\zeta\varepsilon)_{aA'_\zeta}{}^{B_\zeta} J^{A'_\zeta B_\zeta} [\Leftrightarrow] \tilde{J}(1) = S_{em}(\zeta) \hat{J}(1, \zeta) \\ [\Downarrow] & [\Downarrow] \\ J^{A'_\zeta B_\zeta} = \frac{i}{\sqrt{2}}(\sigma\varepsilon, -i\zeta\varepsilon)^{aA'_\zeta}{}_{B_\zeta} J_a [\Leftrightarrow] \hat{J}(1, \zeta) = S_{em}^+(\zeta) \tilde{J}(1) \\ [\Downarrow] & [\Downarrow] \end{cases}$$

$$\text{推论3.7.3. } \begin{cases} J^a = \frac{i}{\sqrt{2}}(\sigma\varepsilon, -i\zeta\varepsilon)^{aA'_\zeta}{}_{B_\zeta} J_{A'_\zeta B_\zeta} [\Leftrightarrow] \hat{J}(1) = S_{em}^*(\zeta) \tilde{J}^*(1, -\zeta) \\ [\Downarrow] & [\Downarrow] \\ J_{A'_\zeta B_\zeta} = \frac{i}{\sqrt{2}}(\varepsilon\sigma, -i\zeta\varepsilon)_{aA'_\zeta}{}^{B_\zeta} J_a [\Leftrightarrow] \tilde{J}^*(1, -\zeta) = S_{em}^T(\zeta) \hat{J}(1) \\ [\Downarrow] & [\Downarrow] \end{cases}$$

推论3.7.4.

$$\Psi(1, \zeta) \sim e^{(i\omega+\zeta\varepsilon)\cdot\gamma} \Leftrightarrow \tilde{\Psi}(1, \zeta) \sim e^{(i\omega+\zeta\varepsilon)\cdot R} \Leftrightarrow \tilde{\psi}(1, \zeta) \sim e^{(i\omega+\zeta\varepsilon)\cdot\frac{1}{2}\sigma} \otimes e^{(i\omega+\zeta\varepsilon)\cdot\frac{1}{2}\sigma} \Leftrightarrow \psi(1, \zeta) \sim e^{(i\omega+\zeta\varepsilon)\cdot\sigma(1)}$$

## 4 引力场的全对称旋量与表象变换

### 4.1 引力场的全对称条件

$$\text{推论4.1.1. } \psi_{A_\zeta B_\zeta C_\zeta D_\zeta} = \frac{1}{4!} \psi_{(A_\zeta B_\zeta C_\zeta D_\zeta)} \Leftrightarrow \psi_{\alpha\zeta\beta\zeta} = \frac{1}{2!} \psi_{(\alpha\zeta\beta\zeta)}, \delta^{\alpha\zeta\beta\zeta} \psi_{\alpha\zeta\beta\zeta} = 0$$

$$\text{推论4.1.2. } J^{A'_\zeta B_\zeta C_\zeta D_\zeta} = \frac{1}{3!} J^{A'_\zeta(B_\zeta C_\zeta D_\zeta)} \Leftrightarrow (\sigma, -i\zeta)^a \sigma^{\beta\zeta} J_{a\beta\zeta} = 0$$

## 4.2 引力场的全对称旋量

定义4.2.1.  $\psi^{\alpha\zeta\beta\zeta} = C^{\alpha\zeta\beta\zeta}, \psi^{A\zeta B\zeta C\zeta D\zeta} = C^{A\zeta B\zeta C\zeta D\zeta}$

定义4.2.2.  $\psi_{A\zeta B\zeta C\zeta D\zeta} = \psi_{B\zeta A\zeta} \Leftrightarrow \hat{\psi}(2, \zeta) = S_{ex} \hat{\psi}(2, \zeta)$

定义4.2.3.  $\hat{\Psi}(2, \zeta) \equiv [\psi_{x\zeta x\zeta}, \psi_{y\zeta x\zeta}, \psi_{z\zeta x\zeta}, 0, |\psi_{x\zeta y\zeta}, \psi_{y\zeta y\zeta}, \psi_{z\zeta y\zeta}, 0, |\psi_{x\zeta z\zeta}, \psi_{y\zeta z\zeta}, \psi_{z\zeta z\zeta}, 0, |0, 0, 0, 0]^T$

定义4.2.4.  $\hat{\psi}(2, \zeta) \equiv [\psi_{1\zeta 1\zeta 1\zeta 1\zeta}, \psi_{1\zeta 1\zeta 1\zeta 2\zeta}, \psi_{1\zeta 1\zeta 1\zeta 2\zeta}, \psi_{1\zeta 1\zeta 2\zeta 2\zeta}, |\psi_{1\zeta 1\zeta 1\zeta 2\zeta}, \psi_{1\zeta 1\zeta 2\zeta 2\zeta}, \psi_{1\zeta 1\zeta 2\zeta 2\zeta}, \psi_{1\zeta 2\zeta 2\zeta 2\zeta}, |$   
 $\psi_{1\zeta 1\zeta 1\zeta 2\zeta}, \psi_{1\zeta 1\zeta 2\zeta 2\zeta}, \psi_{1\zeta 1\zeta 2\zeta 2\zeta}, \psi_{1\zeta 2\zeta 2\zeta 2\zeta}, |\psi_{1\zeta 1\zeta 2\zeta 2\zeta}, \psi_{1\zeta 2\zeta 2\zeta 2\zeta}, \psi_{1\zeta 2\zeta 2\zeta 2\zeta}, \psi_{2\zeta 2\zeta 2\zeta 2\zeta}]^T$

定义4.2.5.  $\Psi(2, \zeta) \equiv [\psi_{x\zeta x\zeta}, \psi_{y\zeta x\zeta}, \psi_{z\zeta x\zeta}, \psi_{y\zeta y\zeta}, \psi_{z\zeta y\zeta}]^T$

定义4.2.6.  $\psi(2, \zeta) := [\psi_{1\zeta 1\zeta 1\zeta 1\zeta}, \sqrt{C_4^1} \psi_{1\zeta 1\zeta 1\zeta 2\zeta}, \sqrt{C_4^2} \psi_{1\zeta 1\zeta 2\zeta 2\zeta}, \sqrt{C_4^3} \psi_{1\zeta 2\zeta 2\zeta 2\zeta}, \psi_{2\zeta 2\zeta 2\zeta 2\zeta}]^T$

定义4.2.7.  $\bar{\psi}(2, \zeta) := [\psi_{1\zeta 1\zeta 1\zeta 1\zeta}, \psi_{1\zeta 1\zeta 1\zeta 2\zeta}, \psi_{1\zeta 1\zeta 2\zeta 2\zeta}, \psi_{1\zeta 2\zeta 2\zeta 2\zeta}, \psi_{2\zeta 2\zeta 2\zeta 2\zeta}]^T$

定义4.2.8.  $\tilde{\Psi}(2, \zeta) \equiv [\psi_{x\zeta x\zeta}, \psi_{y\zeta x\zeta}, \psi_{z\zeta x\zeta}, 0, |\psi_{x\zeta y\zeta}, \psi_{y\zeta y\zeta}, \psi_{z\zeta y\zeta}, 0]^T$

定义4.2.9.  $\tilde{C}(2, \zeta) \equiv [C_{x\zeta x\zeta}, C_{y\zeta x\zeta}, C_{z\zeta x\zeta}, 0, |C_{x\zeta y\zeta}, C_{y\zeta y\zeta}, C_{z\zeta y\zeta}, 0]^T$

定义4.2.10.

$\tilde{\tilde{\psi}}(2, \zeta) := [(\psi_{1\zeta 1\zeta 1\zeta 1\zeta}, \psi_{1\zeta 1\zeta 1\zeta 2\zeta}), \sqrt{C_3^1}(\psi_{1\zeta 1\zeta 1\zeta 2\zeta}, \psi_{1\zeta 1\zeta 2\zeta 2\zeta}), \sqrt{C_3^2}(\psi_{1\zeta 1\zeta 2\zeta 2\zeta}, \psi_{1\zeta 2\zeta 2\zeta 2\zeta}), (\psi_{1\zeta 2\zeta 2\zeta 2\zeta}, \psi_{2\zeta 2\zeta 2\zeta 2\zeta})]^T$

定义4.2.11.

$\bar{\bar{\psi}}(2, \zeta) := [(\psi_{1\zeta 1\zeta 1\zeta 1\zeta}, \psi_{1\zeta 1\zeta 1\zeta 2\zeta}), (\psi_{1\zeta 1\zeta 1\zeta 2\zeta}, \psi_{1\zeta 1\zeta 2\zeta 2\zeta}), (\psi_{1\zeta 1\zeta 2\zeta 2\zeta}, \psi_{1\zeta 2\zeta 2\zeta 2\zeta}), (\psi_{1\zeta 2\zeta 2\zeta 2\zeta}, \psi_{2\zeta 2\zeta 2\zeta 2\zeta})]^T$

## 4.3 $\psi(2, \zeta), \hat{\psi}(2, \zeta)$ 之间的关系

$$\text{推论4.3.1.} \begin{cases} \psi_{k\zeta}(2) = \Gamma_{k\zeta}^{A\zeta B\zeta C\zeta D\zeta}(2) \psi_{A\zeta B\zeta C\zeta D\zeta}[\Leftrightarrow] \hat{\psi}(2, \zeta) = \bar{\Gamma}(\frac{5}{2}) \hat{\psi}(2, \zeta) \\ \quad [\Updownarrow] \quad [\Downarrow] \\ \psi^{k\zeta}(2) = \Gamma_{A\zeta B\zeta C\zeta D\zeta}^{k\zeta}(2) \psi^{A\zeta B\zeta C\zeta D\zeta}[\Leftrightarrow] \hat{\psi}^*(2, -\zeta) = \bar{\Gamma}(\frac{5}{2}) \hat{\psi}^*(2, -\zeta) \\ \quad [\Downarrow] \end{cases}$$

$$\text{推论4.3.2.} \begin{cases} \psi_{A\zeta B\zeta C\zeta D\zeta} = \Gamma_{A\zeta B\zeta C\zeta D\zeta}^{k\zeta}(2) \psi_{k\zeta}(2) [\Leftrightarrow] \hat{\psi}(2, \zeta) = \Gamma(\frac{5}{2}) \psi(2, \zeta) \\ \quad [\Updownarrow] \quad [\Downarrow] \\ \psi^{A\zeta B\zeta C\zeta D\zeta} = \Gamma_{k\zeta}^{A\zeta B\zeta C\zeta D\zeta}(2) \psi^{k\zeta}(2) [\Leftrightarrow] \hat{\psi}^*(2, -\zeta) = \Gamma(\frac{5}{2}) \psi^*(2, -\zeta) \end{cases}$$

## 4.4 $\tilde{\psi}(2, \zeta), \hat{\psi}(2, \zeta)$ 之间的关系

$$\text{定义4.4.1.} \begin{cases} \psi_{A\zeta l\zeta}(2) := \Gamma_{l\zeta}^{B\zeta C\zeta D\zeta}(\frac{3}{2}) \psi_{A\zeta B\zeta C\zeta D\zeta} \Leftrightarrow \tilde{\psi}(2, \zeta) = [I_{w+1} \otimes \bar{\Gamma}(2)] \hat{\psi}(2, \zeta) \\ \quad \Updownarrow \quad \Updownarrow \\ \psi^{A\zeta l\zeta}(2) = \Gamma_{B\zeta C\zeta D\zeta}^{l\zeta}(\frac{3}{2}) \psi^{A\zeta B\zeta C\zeta D\zeta} \Leftrightarrow \tilde{\psi}^*(2, -\zeta) = [I_{w+1} \otimes \bar{\Gamma}(2)] \hat{\psi}^*(2, -\zeta) \\ \quad [\Downarrow] \end{cases}$$

$$\text{推论4.4.1.} \begin{cases} \psi_{A\zeta B\zeta C\zeta D\zeta} = \Gamma_{B\zeta C\zeta D\zeta}^{l\zeta}(\frac{3}{2}) \psi_{A\zeta l\zeta}(2) [\Leftrightarrow] \tilde{\psi}(2, \zeta) = [I_{w+1} \otimes \Gamma(2)] \hat{\psi}(2, \zeta) \\ \quad [\Downarrow] \quad [\Downarrow] \\ \psi^{A\zeta B\zeta C\zeta D\zeta} = \Gamma_{l\zeta}^{B\zeta C\zeta D\zeta}(\frac{3}{2}) \psi^{A\zeta l\zeta}(2) [\Leftrightarrow] \tilde{\psi}^*(2, -\zeta) = [I_{w+1} \otimes \Gamma(2)] \hat{\psi}^*(2, -\zeta) \end{cases}$$





## 4.8 表象变换间关系

推论4.8.1.  $\hat{\Psi}(2, \varsigma) = S_{em}(\pm\varsigma) \otimes S_{em}(\pm\varsigma) \hat{\psi}(2, \varsigma) [\Leftrightarrow] \tilde{\Psi}(2, \varsigma) = S_{em}(\pm\varsigma) \otimes S_{em}(\frac{1}{2}) \tilde{\psi}(2, \varsigma) [\Leftrightarrow] \Psi(2, \varsigma) = S_m(2) \bar{\psi}(2, \varsigma)$

推论4.8.2.  $\hat{\mathcal{J}}(2) = S_{em}(\varsigma) \otimes S_{em}(\pm\varsigma) \hat{J}(2, \varsigma) [\Leftrightarrow] \tilde{\mathcal{J}}(2) = S_{em}(\varsigma) \otimes S_{em}(\frac{1}{2}) \tilde{J}(2, \varsigma)$

推论4.8.3.  $S_{em}(\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}, S_{em}^+(\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$

## 4.9 纯虚相似变换

定义4.9.1.  $\Psi(2, \varsigma) \equiv [\psi_{x_\varsigma x_\varsigma}, \psi_{y_\varsigma x_\varsigma}, \psi_{z_\varsigma x_\varsigma}, \psi_{y_\varsigma y_\varsigma}, \psi_{z_\varsigma y_\varsigma}]^T$

推论4.9.1.  $\Psi(2, \varsigma) = S_m(2) \bar{\psi}(2, \varsigma)$

$$S_m(2) = -\frac{1}{2} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 \\ -i & 0 & 0 & 0 & i \\ 0 & 2 & 0 & -2 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2i & 0 & 2i & 0 \end{bmatrix}, S_m^-(2) = -\frac{1}{2} \begin{bmatrix} -1 & 2i & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -i \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -i \\ -1 & -2i & 0 & 1 & 0 \end{bmatrix} \quad (5.1a)$$

$$G_m = S_m(2) \tau(2) S_m^-(2) \quad S_m(2) S_m^-(2) = S_m^-(2) S_m(2) = I \quad (5.1b)$$

$$G_m = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2i \\ i & 0 & 0 & 2i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ -2i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2i & 0 & 0 & 0 \\ i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{bmatrix} \right\} \quad (5.1c)$$

$$\text{推论4.9.2.} \quad \begin{cases} [G_m, -2i\varsigma]^a \partial_a \Psi = 0 \\ \partial_x \Psi_1 + \partial_y \Psi_2 + \partial_z \Psi_3 = 0 \\ \partial_x \Psi_2 + \partial_y \Psi_4 + \partial_z \Psi_5 = 0 \\ \partial_x \Psi_3 + \partial_y \Psi_5 - \partial_z (\Psi_1 + \Psi_4) = 0 \end{cases} \Leftrightarrow \begin{cases} [G_m, -2i\varsigma]^a \partial_a \Psi = 0 \\ \nabla \cdot \vec{\psi}^{\beta\varsigma} = 0 \end{cases}$$

## 4.10 纯虚表象变换

推论4.10.1.  $\Psi_{im}(2, \varsigma) \equiv -\sqrt{2} [\psi^{y_\varsigma x_\varsigma}, -\frac{1}{2}(\psi^{x_\varsigma x_\varsigma} - \psi^{y_\varsigma y_\varsigma}), \psi^{z_\varsigma y_\varsigma}, \varsigma \psi^{z_\varsigma x_\varsigma}, \frac{\sqrt{3}}{2}(\psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma})]^T$

推论4.10.2.  $\Psi_{im}(2, \varsigma) = S_{im}(2, \varsigma) \psi(2, \varsigma)$

$$S_{im}(2, \varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & 0 & -i \\ -1 & 0 & 0 & 0 & -1 \\ 0 & -i & 0 & -i & 0 \\ 0 & -\varsigma & 0 & \varsigma & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 \end{bmatrix}, S_{im}^+(2, \varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 & 0 & 0 \\ 0 & 0 & i & -\varsigma & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & i & \varsigma & 0 \\ i & -1 & 0 & 0 & 0 \end{bmatrix} \quad (5.2)$$

$$G_{im}(\varsigma) = S_{im}(2, \varsigma) \sigma(2) S_{im}^+(2, \varsigma) \quad S_{im}(2, \varsigma) S_{im}^+(2, \varsigma) = S_{im}^+(2, \varsigma) S_{im}(2, \varsigma) = I \quad (5.3)$$

$$G_{im}(\varsigma) = \left\{ \begin{bmatrix} 0 & 0 & 0 & -i\varsigma & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & i\sqrt{3} \\ i\varsigma & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{3} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i\varsigma & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & i\varsigma & 0 & 0 & -i\varsigma\sqrt{3} \\ 0 & 0 & 0 & i\varsigma\sqrt{3} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2i & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\varsigma & 0 \\ 0 & 0 & -i\varsigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\} \quad (5.4)$$

$$G_{im}(+) = \left\{ \begin{bmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & i\sqrt{3} \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{3} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & -i\sqrt{3} \\ 0 & 0 & 0 & i\sqrt{3} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2i & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\} \quad (5.5)$$

推论4.10.3.

$$\begin{cases} \varsigma \partial_\pi \Psi_1 = \frac{1}{2} \partial_y \Psi_3 - \partial_z \Psi_2 - \frac{1}{2} \partial_x \Psi_4 \\ \varsigma \partial_\pi \Psi_2 = \partial_z \Psi_1 - \frac{1}{2} \partial_x \Psi_3 - \frac{1}{2} \partial_y \Psi_4 \\ \varsigma \partial_\pi \Psi_3 = -\frac{1}{2} \partial_y \Psi_1 + \frac{1}{2} \partial_z \Psi_4 + \partial_x (\frac{1}{2} \Psi_2 + \frac{\sqrt{3}}{2} \Psi_5) \Leftrightarrow [G_{im}(+), -2i\varsigma]^a \partial_a \Psi = 0 \\ \varsigma \partial_\pi \Psi_4 = \frac{1}{2} \partial_x \Psi_1 - \frac{1}{2} \partial_z \Psi_3 + \partial_y (\frac{1}{2} \Psi_2 - \frac{\sqrt{3}}{2} \Psi_5) \\ \varsigma \partial_\pi \Psi_5 = -\frac{\sqrt{3}}{2} \partial_x \Psi_3 + \frac{\sqrt{3}}{2} \partial_y \Psi_4 \\ \partial_x (-\Psi_3 + \frac{1}{\sqrt{3}} \Psi_5) + \partial_y \Psi_1 + \partial_z \Psi_4 = 0 \\ \partial_x \Psi_1 + \partial_y (\Psi_2 + \frac{1}{\sqrt{3}} \Psi_5) + \partial_z \Psi_3 = 0 \Leftrightarrow \nabla \cdot \vec{\psi}^{\beta\varsigma} = 0 \\ \partial_x \Psi_4 + \partial_y \Psi_3 - \frac{2}{\sqrt{3}} \partial_z \Psi_5 = 0 \end{cases}$$

## 5 s-自旋粒子各种场量的关系

### 5.1 场旋量 $\hat{\psi}(s, \varsigma; w)$ , $\tilde{\psi}(s, \varsigma; w)$ , $\psi(s, \varsigma; w)$ 之间的恒等表象变换关系

推论5.1.1.

$$\begin{cases} \hat{\psi}(s, \varsigma; w) \equiv \Gamma(s; w) \psi(s, \varsigma; w) \\ \psi(s, \varsigma; w) \equiv \bar{\Gamma}(s; w) \hat{\psi}(s, \varsigma; w) \end{cases} \quad \begin{cases} \tilde{\psi}(s, \varsigma; w) \equiv N(s; w) \psi(s, \varsigma; w) \\ \psi(s, \varsigma; w) \equiv \bar{N}(s; w) \tilde{\psi}(s, \varsigma; w) \end{cases}$$

推论5.1.2.

$$\begin{cases} \hat{\psi}(s, \varsigma; w) \equiv \Gamma(s; w) \bar{\Gamma}(s; w) \hat{\psi}(s, \varsigma; w) \\ \psi(s, \varsigma; w) \equiv \bar{\Gamma}(s; w) \Gamma(s; w) \psi(s, \varsigma; w) \end{cases} \quad \begin{cases} \tilde{\psi}(s, \varsigma; w) \equiv N(s; w) \bar{N}(s; w) \tilde{\psi}(s, \varsigma; w) \\ \psi(s, \varsigma; w) \equiv \bar{N}(s; w) N(s; w) \psi(s, \varsigma; w) \end{cases}$$

推论5.1.3.

$$\begin{cases} \hat{\psi}(s, \varsigma; w) \equiv [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{\psi}(s, \varsigma; w) \\ \tilde{\psi}(s, \varsigma; w) \equiv [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \hat{\psi}(s, \varsigma; w) \end{cases}$$

推论5.1.4.

$$\begin{cases} \hat{\psi}(s, \varsigma; w) \equiv [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \hat{\psi}(s, \varsigma; w) \\ \tilde{\psi}(s, \varsigma; w) \equiv [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{\psi}(s, \varsigma; w) \end{cases}$$

### 5.2 源旋量 $\hat{J}(s, \varsigma; w)$ , $\tilde{J}(s, \varsigma; w)$ 之间的恒等表象变换关系

推论5.2.1.

$$\begin{cases} \hat{J}(s, \varsigma; w) \equiv [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{J}(s, \varsigma; w) \\ \tilde{J}(s, \varsigma; w) \equiv [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \hat{J}(s, \varsigma; w) \end{cases} \quad \begin{cases} \hat{J}(s, \varsigma; w) \equiv [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \hat{J}(s, \varsigma; w) \\ \tilde{J}(s, \varsigma; w) \equiv [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{J}(s, \varsigma; w) \end{cases}$$

### 5.3 s-自旋粒子各种场量之间的变换关系

定理5.3.1.  $\psi(s, \varsigma) \sim e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s)} \Leftrightarrow \hat{\psi}(s, \varsigma) \sim e^{(i\omega + \varsigma\epsilon) \cdot \bar{\Omega}(s)} \Leftrightarrow \tilde{\psi}(s, \varsigma) \sim e^{(i\omega + \varsigma\epsilon) \cdot [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(s - \frac{1}{2})]}$

证明:  $\psi'(s, \varsigma; w)$

$$\begin{aligned} &= \bar{\Gamma}(s; w) \hat{\psi}'(s, \varsigma; w) \\ &= \bar{\Gamma}(s; w) e^{\frac{i}{2} \theta^{ab} \Omega_{ab}(s; w)} \hat{\psi}(s, \varsigma; w) \end{aligned}$$

$$\begin{aligned}
&= \bar{\Gamma}(s; w) e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s; w)} \Gamma(s; w) \psi(s, \varsigma; w) \\
&= e^{\frac{i}{2} \vartheta^{ab} \bar{\Gamma}(s; w) \Omega_{ab}(s; w) \Gamma(s; w)} \psi(s, \varsigma; w) \\
&= e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \psi(s, \varsigma; w)
\end{aligned}$$

□

证明:  $\tilde{\psi}'(s, \varsigma; w)$

$$\begin{aligned}
&= [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \hat{\psi}'(s, \varsigma; w) \\
&= [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s; w)} \hat{\psi}(s, \varsigma; w) \\
&= [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{\psi}(s, \varsigma; w) \\
&= e^{\frac{i}{2} \vartheta^{ab} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \Omega_{ab}(s; w) [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]} \tilde{\psi}(s, \varsigma; w) \\
&= e^{\frac{i}{2} \vartheta^{ab} [S_{ab} \otimes I_{4^{2s-1}} + I_{w+1} \otimes S_{ab}(s; s - \frac{1}{2})]} \tilde{\psi}(s, \varsigma; w)
\end{aligned}$$

□

证明:  $\psi'(s, \varsigma; w) = \bar{N}(s; w) \tilde{\psi}'(s, \varsigma; w)$

$$= \bar{N}(s; w) e^{\frac{i}{2} \vartheta^{ab} [S_{ab} \otimes I_{4^{2s-1}} + I_{w+1} \otimes S_{ab}(s; s - \frac{1}{2})]} \tilde{\psi}(s, \varsigma; w)$$

□

证明:  $\bar{N}(s; w) [S_{ab} \otimes I_{4^{2s-1}} + I_{w+1} \otimes S_{ab}(s; s - \frac{1}{2})] N(s; w) = S_{ab}(s, \varsigma; w)$

$$\bar{N}(s; w) [S_{ab} \otimes I_{4^{2s-1}}] N(s; w) = \frac{1}{2s} S_{ab}(s, \varsigma; w)$$

$$\bar{N}(s; w) [I_{w+1} \otimes S_{ab}(e, \varsigma; s - \frac{1}{2})] N(s; w) = (1 - \frac{1}{2s}) S_{ab}(s, \varsigma; w)$$

□

## 5.4 同步表象变换

推论5.4.1.  $\sigma' = S\sigma S^+ = c^k \sigma_k$

$$\Leftrightarrow \sigma'(s; w) = [\bar{\Gamma}(s; w) (S \otimes S \otimes \cdots \otimes S) \Gamma(s; w)] \sigma(s; w) [\bar{\Gamma}(s; w) (S \otimes S \otimes \cdots \otimes S) \Gamma(s; w)]^+ = c^k \sigma_k(s; w)$$

证明:  $\sigma' = S\sigma S^+ \Leftrightarrow \sigma'(s; w) = c^k \sigma_k(s; w) = c^k \sigma_k$

$$\begin{aligned}
&\Leftrightarrow (S \otimes S \otimes \cdots \otimes S) \Omega(s; w) (S^+ \otimes S^+ \otimes \cdots \otimes S^+) \Gamma(s; w) = \Gamma(s; w) \sigma'(s; w) \\
&\Leftrightarrow (S \otimes S \otimes \cdots \otimes S) \Omega(s; w) \Gamma(s; w) \bar{\Gamma}(s; w) (S^+ \otimes S^+ \otimes \cdots \otimes S^+) \Gamma(s; w) = \Gamma(s; w) \sigma'(s; w) \\
&\Leftrightarrow \sigma'(s; w) = \bar{\Gamma}(s; w) (S \otimes S \otimes \cdots \otimes S) \Gamma(s; w) \sigma(s; w) \bar{\Gamma}(s; w) (S^+ \otimes S^+ \otimes \cdots \otimes S^+) \Gamma(s; w) \\
&\Leftrightarrow \sigma'(s; w) = [\bar{\Gamma}(s; w) (S \otimes S \otimes \cdots \otimes S) \Gamma(s; w)] \sigma(s; w) [\bar{\Gamma}(s; w) (S \otimes S \otimes \cdots \otimes S) \Gamma(s; w)]^+ \\
&\Leftrightarrow \sigma' = S\sigma S^+ = c^k \sigma_k \Leftrightarrow \sigma'(s; w) = S' \sigma(s; w) S'^+ = c^k \sigma_k(s; w), S' = [\bar{\Gamma}(s; w) (S \otimes S \otimes \cdots \otimes S) \Gamma(s; w)]
\end{aligned}$$

□

## 6 常用矩阵汇总

### 6.1 自旋矩阵的一个非厄密表象

从全对称二分量子魏尔<sup>[5]</sup>自旋张量的洛伦兹变换性质出发, 可以得到一个特殊表象的自旋矩阵。

$$\tau(s) = \left( \frac{1}{2} \begin{bmatrix} 0 & 2s & 0 & 0 & 0 \\ 1 & 0 & 2s-1 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 2s & 0 \end{bmatrix}, \frac{i}{2} \begin{bmatrix} 0 & -2s & 0 & 0 & 0 \\ 1 & 0 & -(2s-1) & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 2s & 0 \end{bmatrix}, \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix} \right) \quad (5.6a)$$

$$\sigma(s) = \mathbb{S}(s) \tau(s) \mathbb{S}^{-1}(s), [\tau_{\alpha_\varsigma}(s), \tau_{\beta_\varsigma}(s)] = i \varepsilon_{\alpha_\varsigma \beta_\varsigma} \gamma_\varsigma \tau_{\gamma_\varsigma}(s), s = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots \quad (5.6b)$$

$$\tau^2(s) = s(s+1) \quad (5.6c)$$

$$\tau_{\alpha_\varsigma}(s) \prec \tau_{\alpha_\varsigma}^{A_\varsigma}{}_{B_\varsigma}(s), \alpha_\varsigma \sim e^{(i\omega + \varsigma \epsilon) \cdot \gamma}, A_\varsigma \sim e^{(i\omega + \varsigma \epsilon) \cdot \tau(s)}, B_\varsigma \sim e^{-(i\omega + \varsigma \epsilon) \cdot \tau^T(s)} \quad (5.6d)$$

与此自旋矩阵相对应的度规张量： $\epsilon(s)\bar{\epsilon}(s) = \bar{\epsilon}(s)\epsilon(s) = I$

$$\epsilon_{A_\zeta B_\zeta}(s) \succ \epsilon(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & (-1)^0 C_n^0 \\ 0 & 0 & 0 & (-1)^1 C_n^1 & 0 \\ 0 & 0 & (-1)^2 C_n^2 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ (-1)^n C_n^n & 0 & 0 & 0 & 0 \end{bmatrix}, C_n^{-k} \equiv (C_n^k)^{-1} \quad (5.7a)$$

$$\bar{\epsilon}^{A_\zeta B_\zeta}(s) \succ \bar{\epsilon}(s) = \begin{bmatrix} 0 & 0 & 0 & 0 & (-1)^n C_n^{-0} \\ 0 & 0 & 0 & (-1)^{n-1} C_n^{-1} & 0 \\ 0 & 0 & (-1)^{n-2} C_n^{-2} & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ (-1)^0 C_n^{-n} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.7b)$$

$$\epsilon(s) = \mathbb{S}^T(s)\epsilon(s)\mathbb{S}(s), \bar{\epsilon}(s) = \mathbb{S}^{-T}(s)\bar{\epsilon}(s)\mathbb{S}^{-1}(s), \bar{\epsilon}(s) = (-1)^{2s}\mathbb{S}^{2T}(s)\epsilon(s)\mathbb{S}^2(s) \quad (5.7c)$$

$$\mathbb{S}(s) = \begin{bmatrix} \sqrt{C_{2s}^0} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{C_{2s}^1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{C_{2s}^2} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \sqrt{C_{2s}^{2s}} \end{bmatrix}, \mathbb{S}^{-1}(s) = \begin{bmatrix} \sqrt{C_{2s}^{-0}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{C_{2s}^{-1}} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{C_{2s}^{-2}} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \sqrt{C_{2s}^{-2s}} \end{bmatrix} \quad (5.7d)$$

## 6.2 $\sigma$ 轮换表象变换矩阵

推论6.2.1.

$$S_c(\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix}, S_c^+(\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix}, S_c(\frac{1}{2})S_c^+(\frac{1}{2}) = S_c^+(\frac{1}{2})S_c(\frac{1}{2}) = I, S_c(\frac{1}{2}) = k e^{i\frac{\pi}{4}\sigma_y(\frac{1}{2})} e^{i\frac{\pi}{4}\sigma_z(\frac{1}{2})}$$

$$\text{推论6.2.2. } S_c(\frac{1}{2}) = e^{i\varphi} e^{-i\frac{\pi}{2}\sigma_y(\frac{1}{2})} e^{-i\frac{\pi}{2}\sigma_z(\frac{1}{2})}, S_c(\frac{1}{2}) = e^{-i\varphi} e^{i\frac{\pi}{2}\sigma_z(\frac{1}{2})} e^{i\frac{\pi}{2}\sigma_y(\frac{1}{2})}$$

$$\text{推论6.2.3. } S_c(\frac{1}{2})(\sigma_x, \sigma_y, \sigma_z)S_c^+(\frac{1}{2}) = (\sigma_y, \sigma_z, \sigma_x), S_c^+(\frac{1}{2})(\sigma_x, \sigma_y, \sigma_z)S_c(\frac{1}{2}) = (\sigma_z, \sigma_x, \sigma_y)$$

$$\text{推论6.2.4. } S_{em}(\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}, S_{em}^+(\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}, S_{em}(\frac{1}{2})S_{em}^+(\frac{1}{2}) = S_{em}^+(\frac{1}{2})S_{em}(\frac{1}{2}) = I$$

$$\text{推论6.2.5. } S_{em}(\frac{1}{2})(\sigma_x, \sigma_y, \sigma_z)S_{em}^+(\frac{1}{2}) = (-\sigma_z, -\sigma_x, \sigma_y)$$

$$\text{推论6.2.6. } S_{xy}(\sigma_x, \sigma_y, \sigma_z)S_{xy}^+ = (-\sigma_y, \sigma_x, \sigma_z), S_{xy} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}, S_{xy}^+ = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

## 6.3 $\sigma(s)$ 轮换表象变换矩阵

推论6.3.1.  $\sigma^{\alpha_\zeta}(s) = [e^{(i\omega+\zeta\epsilon)\cdot\gamma}]^{\alpha_\zeta}_{\beta_\zeta} e^{(i\omega+\zeta\epsilon)\cdot\sigma(s)} \sigma^{\beta_\zeta}(s) e^{-(i\omega+\zeta\epsilon)\cdot\sigma(s)}$ , 即  $\sigma^{\alpha_\zeta}_{\beta_\zeta}(s)$  是常数不变张量。

[ $\Downarrow$ ]

$$\text{推论6.3.2. } S_c(s) = e^{i\varphi} e^{-i\frac{\pi}{2}\sigma_y(s)} e^{-i\frac{\pi}{2}\sigma_z(s)}, S_c^+(s) = e^{-i\varphi} e^{i\frac{\pi}{2}\sigma_z(s)} e^{i\frac{\pi}{2}\sigma_y(s)}$$

$$\text{推论6.3.3. } S_c^+(s)[\sigma_x(s), \sigma_y(s), \sigma_z(s)]S_c(s) = [\sigma_z(s), \sigma_x(s), \sigma_y(s)]$$

$$\text{推论6.3.4. } S_c^+(s)[\sigma_x(s), \sigma_y(s), \sigma_z(s)]S_c(s) = [\sigma_z(s), \sigma_x(s), \sigma_y(s)]$$

$$\text{推论6.3.5. } [\sigma_x(s), \sigma_y(s), \sigma_z(s)] \simeq [\hat{e}_x, \hat{e}_y, \hat{e}_z]$$

## 6.4 电磁纯虚表象变换矩阵和交换矩阵

$$\text{推论6.4.1. } S_{em}(\zeta) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & -i & -i & 0 \\ 0 & -\zeta & \zeta & 0 \end{bmatrix}, S_{em}^+(\zeta) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 & 0 \\ 0 & 0 & i & -\zeta \\ 0 & 0 & -i & \zeta \\ i & -1 & 0 & 0 \end{bmatrix}, S_{em}(\zeta)S_{em}^+(\zeta) = S_{em}^+(\zeta)S_{em}(\zeta) = I_4$$

$$\text{推论6.4.2. } S_{em}(\zeta) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & -i & -i & 0 \\ 0 & -\zeta & \zeta & 0 \end{bmatrix}, S_{em}^T(\zeta) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 & 0 & 0 \\ 0 & 0 & -i & -\zeta \\ 0 & 0 & -i & \zeta \\ -i & -1 & 0 & 0 \end{bmatrix}, S_{em}^T(\zeta)S_{em}(\zeta) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = -\sigma_y \otimes \sigma_y$$

$$\text{推论6.4.3. } S_m(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & -i \\ -1 & 0 & -1 \\ 0 & -i\sqrt{2} & 0 \end{bmatrix}, S_m^+(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 \\ 0 & 0 & i\sqrt{2} \\ i & -1 & 0 \end{bmatrix}, S_m(1)S_m^+(1) = S_m^+(1)S_m(1) = I_3$$

$$\text{推论6.4.4. } S_{ex} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, S_{ex}^2 = I, S_{em}(\zeta)S_{ex} = S_{em}(-\zeta), S_{ex}S_{em}^+(\zeta) = S_{em}^+(-\zeta)$$

$$\text{推论6.4.5. } (\sigma \otimes I) = S_{ex}(I \otimes \sigma)S_{ex}, (I \otimes \sigma) = S_{ex}(\sigma \otimes I)S_{ex}$$

$$\text{推论6.4.6. } \sigma_{-\zeta} = S_{em}(\zeta)(\sigma \otimes I)S_{em}^+(\zeta), \sigma_{+\zeta} = S_{em}(\zeta)(I \otimes \sigma)S_{em}^+(\zeta), \gamma = S_m(1)\sigma(1)S_m^-(1)$$

## 6.5 引力纯虚表象变换

定义6.5.1.  $\Psi_{im}(2, \varsigma) := S_{im}(2, \varsigma)\psi(2, \varsigma)$

推论6.5.1.  $\Psi_{im}(2, \varsigma) = -\sqrt{2}[\psi^{y_\varsigma x_\varsigma}, -\frac{1}{2}(\psi^{x_\varsigma x_\varsigma} - \psi^{y_\varsigma y_\varsigma}), \psi^{z_\varsigma y_\varsigma}, \varsigma\psi^{z_\varsigma x_\varsigma}, \frac{\sqrt{3}}{2}(\psi^{x_\varsigma x_\varsigma} + \psi^{y_\varsigma y_\varsigma})]^T$

$$S_{im}(2, \varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & 0 & -i \\ -1 & 0 & 0 & 0 & -1 \\ 0 & -i & 0 & -i & 0 \\ 0 & -\varsigma & 0 & \varsigma & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 \end{bmatrix}, S_{im}^+(2, \varsigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 & 0 & 0 \\ 0 & 0 & i & -\varsigma & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & i & \varsigma & 0 \\ i & -1 & 0 & 0 & 0 \end{bmatrix} \quad (5.8)$$

$$G_{im}(\varsigma) = S_{im}(2, \varsigma)\sigma(2)S_{im}^+(2, \varsigma) \quad S_{im}(2, \varsigma)S_{im}^+(2, \varsigma) = S_{im}^+(2, \varsigma)S_{im}(2, \varsigma) = I \quad (5.9)$$

$$G_{im}(\varsigma) = \left\{ \begin{bmatrix} 0 & 0 & 0 & -i\varsigma & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & i\sqrt{3} \\ i\varsigma & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{3} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i\varsigma & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & i\varsigma & 0 & 0 & -i\varsigma\sqrt{3} \\ 0 & 0 & 0 & i\varsigma\sqrt{3} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2i & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\varsigma & 0 \\ 0 & 0 & -i\varsigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\} \quad (5.10)$$

## 6.6 引力纯虚相似变换矩阵

定义6.6.1.  $\Psi(2, \varsigma) \equiv [\psi_{x_\varsigma x_\varsigma}, \psi_{y_\varsigma x_\varsigma}, \psi_{z_\varsigma x_\varsigma}, \psi_{y_\varsigma y_\varsigma}, \psi_{z_\varsigma y_\varsigma}]^T$

推论6.6.1.  $\Psi(2, \varsigma) = S_m(2)\bar{\psi}(2, \varsigma)$

$$S_m(2) = -\frac{1}{2} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 \\ -i & 0 & 0 & 0 & i \\ 0 & 2 & 0 & -2 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2i & 0 & 2i & 0 \end{bmatrix}, S_m^-(2) = -\frac{1}{2} \begin{bmatrix} -1 & 2i & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -i \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -i \\ -1 & -2i & 0 & 1 & 0 \end{bmatrix} \quad (5.11a)$$

$$G_m = S_m(2)\tau(2)S_m^-(2) \quad S_m(2)S_m^-(2) = S_m^-(2)S_m(2) = I \quad (5.11b)$$

$$G_m = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2i \\ i & 0 & 0 & 2i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ -2i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2i & 0 & 0 & 0 \\ i & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{bmatrix} \right\} \quad (5.11c)$$

## 6.7 全对称旋量条件矩阵

$$T(s) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, 2s - 1 \uparrow \tau, \tau = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tau^n = \tau, T^n(s) = T(s) \quad (5.12a)$$

$$\text{全对称旋量条件: } \tilde{\psi} = T(s)\tilde{\psi} \text{ (类似于马约拉纳条件}^{[4]} \text{和魏尔条件}^{[5]}) \quad (5.12b)$$

# 第六章 电磁场方程的重新表述

自我评述：本章提出了电磁场方程的多种等价表述形式，并严格证明了它们之间的等价性。

## 1 应用常数张量定义电磁场 [7] 的各种旋量

### 1.1 电磁场强的经典描述

$$\text{电磁张量: } F_{ab} = \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix}, \text{对偶张量: } *F_{ab} = \begin{bmatrix} 0 & -iE_z & iE_y & B_x \\ iE_z & 0 & -iE_x & B_y \\ -iE_y & iE_x & 0 & B_z \\ -B_x & -B_y & -B_z & 0 \end{bmatrix} \quad (6.1)$$

### 1.2 电磁场强的复矢量描述

第一种定义，本章采用第一种定义。

定义1.2.1. 电磁场复矢量第一种定义  $\psi_{\alpha\zeta} := \frac{i}{2}\sigma_{\zeta\alpha\zeta}^{ab} F_{ab} = i\zeta(E - i\zeta B)_{\alpha\zeta} = (i\zeta E + B)_{\alpha\zeta}$

第二种定义，后面电磁场单独量子化的章节采用第二种定义。

定义1.2.2. 电磁场复矢量第二种定义  $\Psi_{\alpha\zeta} := \frac{\zeta}{2\sqrt{2}}\sigma_{\zeta\alpha\zeta}^{ab} F_{ab} = \frac{1}{\sqrt{2}}(E - i\zeta B)_{\alpha\zeta}$

第三种定义，后面B-G量子化的章节采用第三种定义。

定义1.2.3. 电磁场复矢量第三种定义  $\psi_{\alpha\zeta} := -\frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha\zeta}^{ab} F_{ab} = -\frac{\zeta}{\sqrt{2}}(E - i\zeta B)_{\alpha\zeta}$

以后有时间全部统一成第二种定义，后面量子化通用Penrose和B-G对易规则是采用第三种定义的。

### 1.3 电磁场强的基本性质

推论1.3.1.  $\frac{1}{2}(F_{ab} - \zeta * F_{ab}) = \frac{i}{2}\sigma_{\zeta ab}^{\alpha\zeta} \psi_{\alpha\zeta}$

证明:  $F_{ab} = -F_{ba}$

$\Leftrightarrow F_{ab} = \frac{1}{2}S_{abcd}F^{cd}, *F_{ab} := \frac{1}{2}\varepsilon_{abcd}F^{cd}$

$\Leftrightarrow F_{ab} - \zeta * F_{ab} = \frac{1}{2}(S_{abcd} - \zeta\varepsilon_{abcd})F^{cd}$

$\Leftrightarrow F_{ab} - \zeta * F_{ab} = -\frac{1}{2}\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta}^{cd}F^{cd}$

$\Leftrightarrow F_{ab} - \zeta * F_{ab} = i\sigma_{\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}$

$\Leftrightarrow \frac{1}{2}(F_{ab} - \zeta * F_{ab}) = \frac{i}{2}\sigma_{\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}$  □

推论1.3.2.  $\psi_{\alpha\zeta} = \frac{i}{2}\sigma_{\zeta\alpha\zeta}^{ab}\frac{1}{2}(F_{ab} - \zeta * F_{ab})$

推论1.3.3.  $\psi_{\alpha\zeta} = -\frac{i}{2}\zeta\sigma_{\zeta\alpha\zeta}^{ab} * F_{ab}$

推论1.3.4.  $\sigma_{\zeta\alpha\zeta}^{ab}(F_{ab} + \zeta * F_{ab}) = 0$

推论1.3.5.  $F_{ab} - \zeta * F_{ab} = -\frac{1}{4}\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta}^{cd}(F_{cd} - \zeta * F_{cd})$

推论1.3.6.  $F_{ab} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'} + \sigma_{+ab}^{\alpha}\psi_{\alpha}), *F_{ab} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'} - \sigma_{+ab}^{\alpha}\psi_{\alpha})$

证明:  $F_{ab} - \zeta * F_{ab} = i\sigma_{\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}$

$\Leftrightarrow F_{ab} - *F_{ab} = i\sigma_{+ab}^{\alpha}\psi_{\alpha}, F_{ab} + *F_{ab} = i\sigma_{-ab}^{\alpha'}\psi_{\alpha'}$

$\Leftrightarrow F_{ab} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'} + \sigma_{+ab}^{\alpha}\psi_{\alpha}), *F_{ab} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'} - \sigma_{+ab}^{\alpha}\psi_{\alpha})$

$\Leftrightarrow F_{ab} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'} + \sigma_{+ab}^{\alpha}\psi_{\alpha})$

$\Leftrightarrow *F_{ab} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'} - \sigma_{+ab}^{\alpha}\psi_{\alpha})$  □

推论1.3.7.  $F_{ab} = -F_{ba} \Leftrightarrow F_{ab} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'} + \sigma_{+ab}^{\alpha}\psi_{\alpha})$

## 1.4 电磁场强的 $\frac{1}{2}$ -旋量描述 [1, 2]

定义1.4.1. 电磁场 $\frac{1}{2}$ -旋量张量:  $\psi_{A_\zeta B_\zeta} := \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\psi_{\alpha\zeta} = \frac{i\zeta}{\sqrt{2}}S^{ab}{}_{A_\zeta B_\zeta}F_{ab}$

推论1.4.1.  $\psi_{A_\zeta B_\zeta} = \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\psi_{\alpha\zeta} \Leftrightarrow \psi_{\alpha\zeta} = \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha\zeta}^{A_\zeta B_\zeta}\psi_{A_\zeta B_\zeta}$

推论1.4.2.  $\psi_{A_\zeta B_\zeta} = \psi_{B_\zeta A_\zeta}$

推论1.4.3.  $\psi_{A_\zeta B_\zeta} = \frac{-i}{\sqrt{2}}S^{ab}{}_{A_\zeta B_\zeta} * F_{ab}$

推论1.4.4.  $\frac{1}{2}(F_{ab} - \zeta * F_{ab}) = \frac{i\zeta}{\sqrt{2}}S_{ab}{}^{A_\zeta B_\zeta}\psi_{A_\zeta B_\zeta} \Leftrightarrow \psi_{A_\zeta B_\zeta} = \frac{i\zeta}{\sqrt{2}}S^{ab}{}_{A_\zeta B_\zeta}\frac{1}{2}(F_{ab} - \zeta * F_{ab})$

推论1.4.5.  $F_{ab} - \zeta * F_{ab} = -\frac{1}{2}S_{ab}{}^{A_\zeta B_\zeta}S^{cd}{}_{A_\zeta B_\zeta}(F_{cd} - \zeta * F_{cd})$

推论1.4.6.  $F_{ab} = \frac{i\zeta}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'} + S_{ab}{}^{AB}\psi_{AB}), *F_{ab} = \frac{i\zeta}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'} - S_{ab}{}^{AB}\psi_{AB})$

推论1.4.7.  $F_{ab} = -F_{ba} \Leftrightarrow F_{ab} = \frac{i\zeta}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'} + S_{ab}{}^{AB}\psi_{AB})$

结合推论1.3.6和(1.274), (1.275) 式可得Penrose对应记法:

推论1.4.8.  $F_{ab} \stackrel{P}{=} \frac{1}{\sqrt{2}}(\psi_{A'B'}\varepsilon_{AB} + \psi_{AB}\varepsilon_{A'B'}), *F_{ab} \stackrel{P}{=} \frac{1}{\sqrt{2}}(\psi_{A'B'}\varepsilon_{AB} - \psi_{AB}\varepsilon_{A'B'})$

## 1.5 电磁场强的1-旋量描述

定义1.5.1. 电磁场1-旋量 $\psi_{k_\zeta}(1) := \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1)\psi_{A_\zeta B_\zeta} = \Gamma_{k_\zeta}^{\alpha\zeta}(1)\psi_{\alpha\zeta}, \psi^{k_\zeta}(1) := \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\psi^{A_\zeta B_\zeta} = \Gamma_{\alpha\zeta}^{k_\zeta}(1)\psi^{\alpha\zeta}$

推论1.5.1.  $\psi_{A_\zeta B_\zeta} = \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\psi_{k_\zeta}(1), \psi_{\alpha\zeta} = \Gamma_{\alpha\zeta}^{k_\zeta}(1)\psi_{k_\zeta}(1)$

推论1.5.2.  $\psi^{A_\zeta B_\zeta} = \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\psi^{k_\zeta}(1), \psi^{\alpha\zeta} = \Gamma_{\alpha\zeta}^{k_\zeta}(1)\psi^{k_\zeta}(1)$

从推论1.246可得:  $[\Gamma_{\alpha\zeta}^{k_\zeta}(1)]^* \simeq \Gamma_{k_\zeta}^{\alpha\zeta}(1)$

推论1.5.3.  $\Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1) \simeq \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1) \succ \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \left\{ \frac{1}{2}(\sigma_z + I), \frac{1}{\sqrt{2}}\sigma_x, \frac{1}{2}(-\sigma_z + I) \right\}$

结合上式与(1.246)式可得:

推论1.5.4.  $\Gamma_{k_\zeta}^{\alpha\zeta}(1) \succ \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 & 0 \\ 0 & 0 & -i\sqrt{2} \\ -i & -1 & 0 \end{bmatrix}, \Gamma_{\alpha\zeta}^{k_\zeta}(1) \succ \frac{1}{\sqrt{2}} \begin{bmatrix} -i & 0 & i \\ -1 & 0 & -1 \\ 0 & i\sqrt{2} & 0 \end{bmatrix}$

## 1.6 电磁场源的 $\frac{1}{2}$ -旋量描述 [1, 2]

定义1.6.1. 电磁场源 $\frac{1}{2}$ -旋量张量 $J_{A_\zeta A'_\zeta} := \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a J_a, J^{A'_\zeta A_\zeta} := \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_{A'_\zeta A_\zeta}^{a'} J_{a'}$

Penrose记法:  $J_a \stackrel{P}{=} J_{AA'}, J^a \stackrel{P}{=} J^{A'A}$

# 2 电磁场方程的多种等价表述形式

## 2.1 电磁场规范理论的标准描述

$$F_{uv} = \partial_u A_v - \partial_v A_u \quad (6.2)$$

规范变换:

$$\begin{cases} \psi \rightarrow U(\theta)\psi, U(\theta) = e^{ig\theta}, \psi \text{带电荷} \mathbf{g} \\ A_u \rightarrow U(\theta)A_u U^{-1}(\theta) + \frac{i}{g}[\partial_u U(\theta)]U^{-1}(\theta) = A_u - \partial_u \theta \end{cases} \quad (6.3)$$

推论2.1.1.  $D_u\psi \rightarrow U D_u\psi, D_u = \partial_u + igA_u$

证明:  $D_u\psi = (\partial_u + igA_u)\psi \rightarrow [\partial_u + UigA_u U^{-1} - (\partial_u U)U^{-1}](U\psi)$

$\Leftrightarrow D_u\psi \rightarrow [\partial_u(U\psi) + UigA_u\psi - (\partial_u U)\psi]$

$\Leftrightarrow D_u\psi \rightarrow U(\partial_u + igA_u)\psi$

$\Leftrightarrow D_u\psi \rightarrow U D_u\psi, D_u = \partial_u + igA_u$  □

推论2.1.2.  $F_{uv} \rightarrow U F_{uv} U^{-1} = F_{uv}$

推论2.1.3.  $D_w F_{uv} \rightarrow U D_w F_{uv} U^{-1} = D_w F_{uv}, D_w = \partial_w + ig[A_w, \ ] = \partial_w$

## 2.2 电磁场方程的经典形式

$$\begin{cases} \nabla \cdot \vec{E} = \rho, \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = \vec{J} + \partial_t \vec{E} \end{cases} \Leftrightarrow \begin{cases} \partial^a F_{ab} = -J_b, \partial^a * F_{ab} \equiv 0 \\ F_{ab} = \partial_a A_b - \partial_b A_a \end{cases} \quad (6.4)$$

## 2.3 电磁场方程的复矢量表述形式

复矢量张量形式：

$$\text{定理2.3.1. } \partial^a F_{ab} = -J_b \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a{}_{b\alpha_\zeta} \partial_a \tilde{\Psi}^{\alpha_\zeta} = iJ_b; F_{ab} = \partial_a A_b - \partial_b A_a, \tilde{\Psi}^{\alpha_\zeta} = \left[ \psi^{\alpha_\zeta} = \frac{i}{2} \sigma_{ab}^{\alpha_\zeta} F^{ab} \right]$$

证明： $\partial^a F_{ab} = -J_b$

$$\Leftrightarrow \partial^a F_{ab} = -J_b, \partial^a * F_{ab} \equiv 0$$

$$\Leftrightarrow \partial^a (F_{ab} - \zeta * F_{ab}) = -J_b$$

$$\Leftrightarrow \partial^a (i\sigma_{sab}^{\alpha_\zeta} \psi_{\alpha_\zeta}) = -J_b, \alpha_\zeta = 1, 2, 3$$

$$\Leftrightarrow \partial^a [(\sigma_\zeta, -i\zeta)^{\alpha_\zeta} |_{ab} \tilde{\Psi}_{\alpha_\zeta}] = iJ_b, \alpha_\zeta = 1, 2, 3, 4$$

$$\Leftrightarrow \partial^a [(\sigma_{-\zeta}, -i\zeta)_a |_{b\alpha_\zeta} \tilde{\Psi}_{\alpha_\zeta}] = iJ_b, \alpha_\zeta = 1, 2, 3, 4$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)_a |_{b\alpha_\zeta} \partial^a \tilde{\Psi}_{\alpha_\zeta} = iJ_b, \alpha_\zeta = 1, 2, 3, 4$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a{}_{b\alpha_\zeta} \partial_a \tilde{\Psi}^{\alpha_\zeta} = iJ_b, \alpha_\zeta = 1, 2, 3, 4$$

□

复矢量矩阵形式：

$$\text{推论2.3.1. } (\sigma_{-\zeta}, -i\zeta)^a{}_{b\alpha_\zeta} \partial_a \tilde{\Psi}^{\alpha_\zeta} = iJ_b \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a \partial_a \tilde{\Psi}(1, \zeta) = iJ$$

表象变换：

$$\text{推论2.3.2. } (\sigma_{-\zeta}, -i\zeta)^a \partial_a \tilde{\Psi}(1, \zeta) = iJ \Leftrightarrow (\sigma \otimes I, -i\zeta)^a \partial_a \tilde{\psi}(1, \zeta) = i\tilde{J}(1, \zeta)$$

## 2.4 电磁场方程的 $\frac{1}{2}$ -旋量表述形式

$\frac{1}{2}$ -旋量Penrose抽象指标形式<sup>[1, 2]</sup>：

$$\text{定理2.4.1. } (\sigma_{-\zeta}, -i\zeta)^a{}_{b\alpha_\zeta} \partial_a \tilde{\Psi}^{\alpha_\zeta} = iJ_b \Leftrightarrow \nabla^{A'_\zeta A_\zeta} \psi_{A_\zeta B_\zeta} = \frac{-\zeta}{\sqrt{2}} J^{A'_\zeta B_\zeta}, \nabla^{A'_\zeta A_\zeta} = \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a$$

证明： $(\sigma_{-\zeta}, -i\zeta)^a{}_{b\alpha_\zeta} \partial_a \tilde{\Psi}^{\alpha_\zeta} = iJ_b$

$$\Leftrightarrow \partial^a (i\sigma_{sab}^{\alpha_\zeta} \psi_{\alpha_\zeta}) = -J_b$$

$$\Leftrightarrow \partial^a (i\sigma_{sab}^{\alpha_\zeta} \cdot \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta}) = -J_b$$

$$\Leftrightarrow iS_{ab}^{A_\zeta B_\zeta} \partial^a \psi_{A_\zeta B_\zeta} = \frac{-\zeta}{\sqrt{2}} J_b$$

$$\Leftrightarrow (\frac{\zeta}{2} \delta_{ab} \varepsilon^{A_\zeta B_\zeta} + iS_{ab}^{A_\zeta B_\zeta}) \partial^a \psi_{A_\zeta B_\zeta} = \frac{-\zeta}{\sqrt{2}} J_b$$

$$\Leftrightarrow \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \varepsilon_{A'_\zeta B'_\zeta} \cdot \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} \partial^a \psi_{A_\zeta B_\zeta} = \frac{-1}{\sqrt{2}} J_b$$

$$\Leftrightarrow \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \varepsilon_{A'_\zeta B'_\zeta} \partial^a \psi_{A_\zeta B_\zeta} = \frac{-1}{\sqrt{2}} J_b \cdot \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{B'_\zeta B_\zeta}^b$$

$$\Leftrightarrow \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \psi_{A_\zeta B_\zeta} = \frac{-\zeta}{\sqrt{2}} \zeta \varepsilon^{A'_\zeta B'_\zeta} J_{B'_\zeta B_\zeta}$$

$$\Leftrightarrow \nabla^{A'_\zeta A_\zeta} \psi_{A_\zeta B_\zeta} = \frac{-\zeta}{\sqrt{2}} J^{A'_\zeta B_\zeta}, \nabla^{A'_\zeta A_\zeta} = \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a$$

□

$\frac{1}{2}$ -旋量张量形式：

$$\text{推论2.4.1. } \nabla^{A'_\zeta A_\zeta} \psi_{A_\zeta B_\zeta} = \frac{-\zeta}{\sqrt{2}} J^{A'_\zeta B_\zeta} \Leftrightarrow (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \psi_{A_\zeta B_\zeta} = iJ^{A'_\zeta B_\zeta}$$

$\frac{1}{2}$ -旋量矩阵形式：

$$\text{推论2.4.2. } (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \psi_{A_\zeta B_\zeta} = iJ^{A'_\zeta B_\zeta} \Leftrightarrow (\sigma \otimes I, -i\zeta)_a \partial^a \tilde{\psi}(1, \zeta) = i\tilde{J}(1, \zeta)$$

$\frac{1}{2}$ -旋量方矩阵形式：

$$\text{推论2.4.3. } (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \psi_{A_\zeta B_\zeta} = iJ^{A'_\zeta B_\zeta} \Leftrightarrow (\sigma, -i\zeta)_a \partial^a [\psi] = i[J]$$



$\frac{1}{2}$ -自旋张量表述形式:(证明留待以后)

$$\text{推论2.4.4. } (\sigma, -i\varsigma)_{a}^{A' A \varsigma} D^a \psi_{A \varsigma B \varsigma} = iJ_{A \varsigma B \varsigma}^{A'} \Leftrightarrow [\partial_a + iS_{ab}(1, \varsigma)\partial^b]_{k \varsigma}^{l \varsigma} \psi_{l \varsigma}(1, \varsigma) = \mathbb{J}_{ak \varsigma}(1, \varsigma)$$

$$\text{推论2.4.5. } \begin{cases} \partial^a F_{ab} = -J_b \\ \partial^a * F_{ab} \equiv 0 \end{cases} \Leftrightarrow [\partial_a + iS_{ab}(1, \varsigma)\partial^b]_{k \varsigma}^{l \varsigma} \psi_{l \varsigma}(1, \varsigma) = \mathbb{J}_{ak \varsigma}(1, \varsigma)$$

## 2.5 猜测

$$\text{定理2.5.1. } \partial^a * F_{ab} = 0 \Leftrightarrow F_{ab} = \partial_a A_b - \partial_b A_a \Leftrightarrow \partial^a * F_{ab} \equiv 0$$

$$\text{定理2.5.2. } \partial^a F_{ab} = -J_b, \partial^a * F_{ab} = 0 \Leftrightarrow \partial^a F_{ab} = -J_b, F_{ab} = \partial_a A_b - \partial_b A_a$$

2.6 电磁场方程<sup>[7]</sup>的自旋张量表述形式

$$\text{电磁场的自旋张量矩阵: } S_{ab} = i\sigma_{\varsigma ab}^{\alpha \varsigma} \gamma_{\alpha \varsigma} \succ \begin{bmatrix} 0 & \gamma_z & -\gamma_y & -\varsigma\gamma_x \\ -\gamma_z & 0 & \gamma_x & -\varsigma\gamma_y \\ \gamma_y & -\gamma_x & 0 & -\varsigma\gamma_z \\ \varsigma\gamma_x & \varsigma\gamma_y & \varsigma\gamma_z & 0 \end{bmatrix} \quad (6.5)$$

$$\text{定理2.6.1. } (\partial_a + iS_{ab}\partial^b)^{\beta \varsigma} \gamma_{\varsigma} \psi^{\gamma \varsigma}(1, \varsigma) = -i\sigma_{\varsigma ab}^{\beta \varsigma} J^b, S_{ab} = i\sigma_{\varsigma ab}^{\alpha \varsigma} \gamma_{\alpha \varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a \partial_a \tilde{\Psi}(1, \varsigma) = iJ$$

一种直观证法如下:

$$\text{证明: } (\partial_a + iS_{ab}\partial^b)^{\beta \varsigma} \gamma_{\varsigma} \psi^{\gamma \varsigma} = -i\sigma_{\varsigma ab}^{\beta \varsigma} J^b, S_{ab} = i\sigma_{\varsigma ab}^{\alpha \varsigma} \gamma_{\alpha \varsigma}$$

$$\Leftrightarrow \begin{cases} (\partial_x + i\gamma_z\partial_y - i\gamma_y\partial_z - i\varsigma\gamma_x\partial_\pi)^{\beta \varsigma} \gamma_{\varsigma} \psi^{\gamma \varsigma} = -i\sigma_{\varsigma xb}^{\beta \varsigma} J^b \\ (\partial_y + i\gamma_x\partial_z - i\gamma_z\partial_x - i\varsigma\gamma_y\partial_\pi)^{\beta \varsigma} \gamma_{\varsigma} \psi^{\gamma \varsigma} = -i\sigma_{\varsigma yb}^{\beta \varsigma} J^b \\ (\partial_z + i\gamma_y\partial_x - i\gamma_x\partial_y - i\varsigma\gamma_z\partial_\pi)^{\beta \varsigma} \gamma_{\varsigma} \psi^{\gamma \varsigma} = -i\sigma_{\varsigma zb}^{\beta \varsigma} J^b \\ (\partial_\pi + i\varsigma\gamma_x\partial_x + i\varsigma\gamma_y\partial_y + i\varsigma\gamma_z\partial_z)^{\beta \varsigma} \gamma_{\varsigma} \psi^{\gamma \varsigma} = -i\sigma_{\varsigma \pi b}^{\beta \varsigma} J^b \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} \partial_x & \partial_y & \partial_z \\ -\partial_y & \partial_x & -\varsigma\partial_\pi \\ -\partial_z & \varsigma\partial_\pi & \partial_x \end{bmatrix} \begin{bmatrix} \psi^{x\varsigma} \\ \psi^{y\varsigma} \\ \psi^{z\varsigma} \end{bmatrix} = \begin{bmatrix} \varsigma J^\pi \\ J^z \\ -J^y \end{bmatrix}, \begin{bmatrix} \partial_y & -\partial_x & \varsigma\partial_\pi \\ \partial_x & \partial_y & \partial_z \\ -\varsigma\partial_\pi & -\partial_z & \partial_y \end{bmatrix} \begin{bmatrix} \psi^{x\varsigma} \\ \psi^{y\varsigma} \\ \psi^{z\varsigma} \end{bmatrix} = \begin{bmatrix} -J^z \\ \varsigma J^\pi \\ J^x \end{bmatrix} \\ \begin{bmatrix} \partial_z & -\varsigma\partial_\pi & -\partial_x \\ \varsigma\partial_\pi & \partial_z & -\partial_y \\ \partial_x & \partial_y & \partial_z \end{bmatrix} \begin{bmatrix} \psi^{x\varsigma} \\ \psi^{y\varsigma} \\ \psi^{z\varsigma} \end{bmatrix} = \begin{bmatrix} J^y \\ -J^x \\ \varsigma J^\pi \end{bmatrix}, i\partial_\pi \Psi(1, \varsigma) = \varsigma\gamma \cdot \nabla \Psi(1, \varsigma) - i\varsigma \vec{J} \end{cases}$$

$$\Leftrightarrow \begin{cases} i\partial_\pi \Psi(1, \varsigma) = i\varsigma \nabla \times \Psi(1, \varsigma) - i\varsigma \vec{J} \\ \nabla \cdot \Psi(1, \varsigma) = \varsigma J^\pi \end{cases}$$

$$\Leftrightarrow \begin{cases} i\partial_\pi \Psi(1, \varsigma) = \varsigma\gamma \cdot \nabla \Psi(1, \varsigma) - i\varsigma \vec{J} \\ \nabla \cdot \Psi(1, \varsigma) = \varsigma J^\pi \end{cases}$$

$$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a \partial_a \tilde{\Psi}(1, \varsigma) = iJ \quad \square$$

另一种更解析更抽象的证法如下:

$$\text{证明: } (\partial_a + iS_{ab}\partial^b)^{\beta \varsigma} \gamma_{\varsigma} \psi^{\gamma \varsigma} = -i\sigma_{\varsigma ab}^{\beta \varsigma} J^b, S_{ab} = i\sigma_{\varsigma ab}^{\alpha \varsigma} \gamma_{\alpha \varsigma}$$

$$\Leftrightarrow \sigma_{\varsigma a}^{\beta \varsigma c} \sigma_{\varsigma \gamma \varsigma cb} \partial^b \psi^{\gamma \varsigma} = -i\sigma_{\varsigma ab}^{\beta \varsigma} J^b$$

$$\Leftrightarrow \sigma_{\varsigma ac}^{\beta \varsigma} \sigma_{\varsigma \gamma \varsigma}^{cb} \partial_b \psi^{\gamma \varsigma} = -i\sigma_{\varsigma ab}^{\beta \varsigma} J^b$$

$$\Leftrightarrow \sigma_{\beta \varsigma}^{\varsigma ad} \sigma_{\varsigma ac}^{\beta \varsigma} \sigma_{\varsigma \gamma \varsigma}^{cb} \partial_b \psi^{\gamma \varsigma} = -i\sigma_{\beta \varsigma}^{\varsigma ad} \sigma_{\varsigma ab}^{\beta \varsigma} J^b$$

$$\Leftrightarrow \sigma_{\varsigma \gamma \varsigma}^{db} \partial_b \psi^{\gamma \varsigma} = -iJ^d$$

$$\Leftrightarrow \sigma_{\varsigma \alpha \varsigma}^{ab} \partial_a \psi^{\alpha \varsigma} = iJ^b, \alpha \varsigma = 1, 2, 3$$

$$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a \partial_a \tilde{\Psi}^{\alpha \varsigma} = iJ^b, \alpha \varsigma = 1, 2, 3, 4 \quad \square$$

此方程(3.3.2)完全等价于电磁场方程，它就是电磁场方程的自旋张量表述形式。

$$\text{引理2.6.1. } \mathbb{J}_a^{\beta\zeta} = -i\sigma_{\zeta ab}^{\beta\zeta} J^b \Leftrightarrow \begin{cases} \mathbb{J}_y^{z\zeta} = -\mathbb{J}_z^{y\zeta} = -\zeta\mathbb{J}_\pi^{x\zeta} = J^x \\ \mathbb{J}_z^{x\zeta} = -\mathbb{J}_x^{z\zeta} = -\zeta\mathbb{J}_\pi^{y\zeta} = J^y \\ \mathbb{J}_x^{y\zeta} = -\mathbb{J}_y^{x\zeta} = -\zeta\mathbb{J}_\pi^{z\zeta} = J^z \\ \mathbb{J}_x^{x\zeta} = \mathbb{J}_y^{y\zeta} = \mathbb{J}_z^{z\zeta} = \zeta J^\pi \end{cases}$$

展开即可证明。以上自旋方程是关于特殊的源项，那么对于一般的源项又会怎样呢？请看下面的定理。

$$\text{定理2.6.2. } (\partial_a + iS_{ab}\partial^b)^{\beta\zeta} \psi^{\gamma\zeta} = \mathbb{J}_a^{\beta\zeta}, S_{ab} = i\sigma_{\zeta ab}^{\alpha\zeta} \gamma_{\alpha\zeta} \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a \partial_a \tilde{\Psi}(1, \zeta) = iJ, \mathbb{J}_a^{\beta\zeta} = -i\sigma_{\zeta ab}^{\beta\zeta} J^b$$

$$\text{证明: } (\partial_a + iS_{ab}\partial^b)^{\beta\zeta} \psi^{\gamma\zeta} = \mathbb{J}_a^{\beta\zeta}, S_{ab} = i\sigma_{\zeta ab}^{\alpha\zeta} \gamma_{\alpha\zeta}$$

$$\Leftrightarrow \begin{cases} (\partial_x + i\gamma_z\partial_y - i\gamma_y\partial_z - i\zeta\gamma_x\partial_\pi)^{\beta\zeta} \psi^{\gamma\zeta} = \mathbb{J}_x^{\beta\zeta} \\ (\partial_y + i\gamma_x\partial_z - i\gamma_z\partial_x - i\zeta\gamma_y\partial_\pi)^{\beta\zeta} \psi^{\gamma\zeta} = \mathbb{J}_y^{\beta\zeta} \\ (\partial_z + i\gamma_y\partial_x - i\gamma_x\partial_y - i\zeta\gamma_z\partial_\pi)^{\beta\zeta} \psi^{\gamma\zeta} = \mathbb{J}_z^{\beta\zeta} \\ (\partial_\pi + i\zeta\gamma_x\partial_x + i\zeta\gamma_y\partial_y + i\zeta\gamma_z\partial_z)^{\beta\zeta} \psi^{\gamma\zeta} = \mathbb{J}_\pi^{\beta\zeta} \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} \partial_x & \partial_y & \partial_z \\ -\partial_y & \partial_x & -\zeta\partial_\pi \\ -\partial_z & \zeta\partial_\pi & \partial_x \end{bmatrix} \begin{bmatrix} \psi^{x\zeta} \\ \psi^{y\zeta} \\ \psi^{z\zeta} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_x^{x\zeta} \\ \mathbb{J}_x^{y\zeta} \\ \mathbb{J}_x^{z\zeta} \end{bmatrix} \Leftrightarrow \begin{cases} \nabla \cdot \Psi(1, \zeta) = \mathbb{J}_x^{x\zeta} \\ [\nabla \times \Psi(1, \zeta)]^{z\zeta} - \zeta\partial_\pi \psi^{z\zeta}(1, \zeta) = \mathbb{J}_x^{y\zeta} \\ -[\nabla \times \Psi(1, \zeta)]^{y\zeta} + \zeta\partial_\pi \psi^{y\zeta}(1, \zeta) = \mathbb{J}_x^{z\zeta} \end{cases} \\ \begin{bmatrix} \partial_y & -\partial_x & \zeta\partial_\pi \\ \partial_x & \partial_y & \partial_z \\ -\zeta\partial_\pi & -\partial_z & \partial_y \end{bmatrix} \begin{bmatrix} \psi^{x\zeta} \\ \psi^{y\zeta} \\ \psi^{z\zeta} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_y^{x\zeta} \\ \mathbb{J}_y^{y\zeta} \\ \mathbb{J}_y^{z\zeta} \end{bmatrix} \Leftrightarrow \begin{cases} -[\nabla \times \Psi(1, \zeta)]^{z\zeta} + \zeta\partial_\pi \psi^{z\zeta}(1, \zeta) = \mathbb{J}_y^{x\zeta} \\ \nabla \cdot \Psi(1, \zeta) = \mathbb{J}_y^{y\zeta} \\ [\nabla \times \Psi(1, \zeta)]^{x\zeta} - \zeta\partial_\pi \psi^{x\zeta}(1, \zeta) = \mathbb{J}_y^{z\zeta} \end{cases} \\ \begin{bmatrix} \partial_z & -\zeta\partial_\pi & -\partial_x \\ \zeta\partial_\pi & \partial_z & -\partial_y \\ \partial_x & \partial_y & \partial_z \end{bmatrix} \begin{bmatrix} \psi^{x\zeta} \\ \psi^{y\zeta} \\ \psi^{z\zeta} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_z^{x\zeta} \\ \mathbb{J}_z^{y\zeta} \\ \mathbb{J}_z^{z\zeta} \end{bmatrix} \Leftrightarrow \begin{cases} [\nabla \times \Psi(1, \zeta)]^{y\zeta} - \zeta\partial_\pi \psi^{y\zeta}(1, \zeta) = \mathbb{J}_z^{x\zeta} \\ -[\nabla \times \Psi(1, \zeta)]^{x\zeta} + \zeta\partial_\pi \psi^{x\zeta}(1, \zeta) = \mathbb{J}_z^{y\zeta} \\ \nabla \cdot \Psi(1, \zeta) = \mathbb{J}_z^{z\zeta} \end{cases} \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial_\pi \Psi(1, \zeta) + i\zeta\gamma \cdot \nabla \psi = \mathbb{J}_\pi \Leftrightarrow \partial_\pi \Psi(1, \zeta) - \zeta\nabla \times \Psi(1, \zeta) = \mathbb{J}_\pi \\ \mathbb{J}_y^{z\zeta} = -\mathbb{J}_z^{y\zeta} = -\zeta\mathbb{J}_\pi^{x\zeta} := J^x \\ \mathbb{J}_z^{x\zeta} = -\mathbb{J}_x^{z\zeta} = -\zeta\mathbb{J}_\pi^{y\zeta} := J^y \\ \mathbb{J}_x^{y\zeta} = -\mathbb{J}_y^{x\zeta} = -\zeta\mathbb{J}_\pi^{z\zeta} := J^z \\ \mathbb{J}_x^{x\zeta} = \mathbb{J}_y^{y\zeta} = \mathbb{J}_z^{z\zeta} := \zeta J^\pi \\ \partial_\pi \Psi(1, \zeta) - \zeta\nabla \times \Psi(1, \zeta) = i\vec{J} \\ \nabla \cdot \Psi(1, \zeta) = -iJ^\pi \end{cases}$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a \partial_a \tilde{\Psi}(1, \zeta) = iJ, \mathbb{J}_a^{\beta\zeta} = -i\sigma_{\zeta ab}^{\beta\zeta} J^b$$

□

另一种更解析更抽象的证法如下：

$$\text{定理2.6.3. } (\partial_a + iS_{ab}\partial^b)^{\beta\zeta} \psi^{\gamma\zeta} = \mathbb{J}_a^{\beta\zeta}, S_{ab} = i\sigma_{\zeta ab}^{\alpha\zeta} \gamma_{\alpha\zeta} \Leftrightarrow \mathbb{J}_a^{\beta\zeta} = \sigma_{\zeta ab}^{\beta\zeta} \sigma_{\zeta\gamma\zeta}^{bc} \partial_c \psi^{\gamma\zeta}$$

$$\text{证明: } (\partial_a + iS_{ab}\partial^b)^{\beta\zeta} \psi^{\gamma\zeta} = \mathbb{J}_a^{\beta\zeta}, S_{ab} = i\sigma_{\zeta ab}^{\alpha\zeta} \gamma_{\alpha\zeta}$$

$$\Leftrightarrow \sigma_{\zeta a}^{\beta\zeta c} \sigma_{\zeta\gamma\zeta}^{cb} \partial^b \psi^{\gamma\zeta} = \mathbb{J}_a^{\beta\zeta}$$

$$\Leftrightarrow \mathbb{J}_a^{\beta\zeta} = \sigma_{\zeta ab}^{\beta\zeta} \sigma_{\zeta\alpha\zeta}^{bc} \partial_c \psi^{\alpha\zeta}$$

$$\Leftrightarrow \begin{cases} \mathbb{J}_y^{z\zeta} = -\mathbb{J}_z^{y\zeta} = -\zeta \mathbb{J}_\pi^{x\zeta} = i\sigma_{\zeta\alpha\zeta}^{xb} \partial_b \psi^{\alpha\zeta} \\ \mathbb{J}_z^{x\zeta} = -\mathbb{J}_x^{z\zeta} = -\zeta \mathbb{J}_\pi^{y\zeta} = i\sigma_{\zeta\alpha\zeta}^{yb} \partial_b \psi^{\alpha\zeta} \\ \mathbb{J}_x^{y\zeta} = -\mathbb{J}_y^{x\zeta} = -\zeta \mathbb{J}_\pi^{z\zeta} = i\sigma_{\zeta\alpha\zeta}^{zb} \partial_b \psi^{\alpha\zeta} \\ \mathbb{J}_x^{x\zeta} = \mathbb{J}_y^{y\zeta} = \mathbb{J}_z^{z\zeta} = i\zeta \sigma_{\zeta\alpha\zeta}^{\pi b} \partial_b \psi^{\alpha\zeta} \end{cases} \quad \square$$

这个定理表明此自旋方程源项受到一定限制，不是随意的，只有前一定理描述的源项情形才有解，其他情形全无解。

推论2.6.1.  $(\partial_a + iS_{ab}\partial^b)^{\beta\zeta} \gamma_\zeta \psi^{\alpha\zeta} = \mathbb{J}_a^{\beta\zeta}, S_{ab} = i\sigma_{\zeta\alpha\zeta}^{ab} \gamma_{\alpha\zeta}$  有解  $\Leftrightarrow \mathbb{J}_a^{\beta\zeta} = -i\sigma_{\zeta\alpha\zeta}^{\beta b} J^b, \exists J^b$

## 2.7 电磁场方程的经典分离形式

推论2.7.1.  $(\sigma_{-,\zeta} - i\zeta)^a \partial_a \tilde{\Psi}(1, \zeta) = iJ \Leftrightarrow (\gamma, -i\zeta)^a \partial_a \Psi(1, \zeta) = i\vec{J}, i\zeta \nabla \cdot \Psi(1, \zeta) = iJ_\pi$

推论2.7.2.  $S := \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & -\zeta\sqrt{1} & \zeta\sqrt{1} & 0 \end{bmatrix}$

推论2.7.3.  $\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & -\zeta\sqrt{1} & \zeta\sqrt{1} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & -\zeta\sqrt{1} \\ 0 & \sqrt{1} & 0 & \zeta\sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & -\zeta \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \zeta \\ -\zeta & 0 & \zeta & 0 \end{bmatrix}$

推论2.7.4.  $\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & -\zeta\sqrt{1} & \zeta\sqrt{1} & 0 \end{bmatrix} i \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & -\zeta\sqrt{1} \\ 0 & \sqrt{1} & 0 & \zeta\sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix} = \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 & \zeta \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & \zeta \\ -\zeta & 0 & -\zeta & 0 \end{bmatrix}$

推论2.7.5.  $\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & -\zeta\sqrt{1} & \zeta\sqrt{1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & -\zeta\sqrt{1} \\ 0 & \sqrt{1} & 0 & \zeta\sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta \\ 0 & 0 & -1 & 0 \\ 0 & \zeta & 0 & 0 \end{bmatrix}$

推论2.7.6.

$$(\sigma \otimes I, -i\zeta)^a \partial_a \tilde{\psi}(1, \zeta) = i\tilde{J}(1, \zeta) \Leftrightarrow \begin{cases} [\sigma(1), -i\zeta]^a \partial_a \psi(1, \zeta) = i\bar{N}(1)\tilde{J}(1, \zeta) \\ i\zeta \nabla \cdot S_m(1)\psi(1, \zeta) = iJ_\pi \end{cases} \Leftrightarrow \begin{cases} \left[ \begin{matrix} \bar{N}(1)\tilde{J}(1, \zeta) \\ J_\pi \end{matrix} \right] = S\tilde{J}(1, \zeta) \\ \left[ \begin{matrix} \psi(1, \zeta) \\ 0 \end{matrix} \right] = S\tilde{\psi}(1, \zeta) \end{cases}$$

推论2.7.7.  $S_m(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & -i \\ -1 & 0 & -1 \\ 0 & -i\sqrt{2} & 0 \end{bmatrix}, S_m^+(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 \\ 0 & 0 & i\sqrt{2} \\ i & -1 & 0 \end{bmatrix}, S_m(1)S_m^+(1) = S_m^+(1)S_m(1) = I_3$

推论2.7.8.  $\begin{cases} [\sigma(1), -i\zeta]^a \partial_a \psi(1, \zeta) = i\bar{N}(1)\tilde{J}(1, \zeta) \\ i\zeta \nabla \cdot S_m(1)\psi(1, \zeta) = iJ_\pi \end{cases} \Leftrightarrow_{S_m^+(1)} \begin{cases} (\gamma, -i\zeta)^a \partial_a \Psi(1, \zeta) = i\vec{J}, \vec{J} = S_m(1)\bar{N}(1)\tilde{J}(1, \zeta) \\ i\zeta \nabla \cdot \Psi(1, \zeta) = iJ_\pi, \Psi(1, \zeta) = S_m(1)\bar{N}(1)\tilde{\psi}(1, \zeta) \end{cases}$

# 第七章 Yang-Mills场方程的重新表述

## 1 应用常数张量定义Yang-Mills场 [7]的各种旋量

自我评述：本章提出了Yang-Mills场方程的多种等价表述形式，并严格证明了它们之间的等价性。

### 1.1 Yang-Mills场强的经典描述

$$\text{电磁张量: } F_{ab}^\sigma = \begin{bmatrix} 0 & B_z^\sigma & -B_y^\sigma & -iE_x^\sigma \\ -B_z^\sigma & 0 & B_x^\sigma & -iE_y^\sigma \\ B_y^\sigma & -B_x^\sigma & 0 & -iE_z^\sigma \\ iE_x^\sigma & iE_y^\sigma & iE_z^\sigma & 0 \end{bmatrix}, \text{对偶张量: } *F_{ab}^\sigma = \begin{bmatrix} 0 & -iE_z^\sigma & iE_y^\sigma & B_x^\sigma \\ iE_z^\sigma & 0 & -iE_x^\sigma & B_y^\sigma \\ -iE_y^\sigma & iE_x^\sigma & 0 & B_z^\sigma \\ -B_x^\sigma & -B_y^\sigma & -B_z^\sigma & 0 \end{bmatrix} \quad (7.1)$$

### 1.2 Yang-Mills场强的复矢量描述

定义1.2.1. Yang-Mills场复矢量  $\psi_{\alpha\zeta}^\sigma := \frac{i}{2}\sigma_{\zeta\alpha\zeta}^{ab}F_{ab}^\sigma = i\zeta(E - i\zeta B)_{\alpha\zeta}^\sigma = (i\zeta E + B)_{\alpha\zeta}^\sigma$

推论1.2.1.  $\frac{1}{2}(F_{ab}^\sigma - \zeta * F_{ab}^\sigma) = \frac{i}{2}\sigma_{\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}^\sigma$

证明:  $F_{ab}^\sigma = -F_{ba}^\sigma$

$$\Leftrightarrow F_{ab}^\sigma = \frac{1}{2}S_{abcd}F^{cd}, *F_{ab}^\sigma := \frac{1}{2}\varepsilon_{abcd}F^{cd}$$

$$\Leftrightarrow F_{ab}^\sigma - \zeta * F_{ab}^\sigma = \frac{1}{2}(S_{abcd} - \zeta\varepsilon_{abcd})F^{cd}$$

$$\Leftrightarrow F_{ab}^\sigma - \zeta * F_{ab}^\sigma = -\frac{1}{2}\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta}^{cd}F^{cd}$$

$$\Leftrightarrow F_{ab}^\sigma - \zeta * F_{ab}^\sigma = i\sigma_{\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}^\sigma$$

$$\Leftrightarrow \frac{1}{2}(F_{ab}^\sigma - \zeta * F_{ab}^\sigma) = \frac{i}{2}\sigma_{\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}^\sigma \quad \square$$

推论1.2.2.  $\psi_{\alpha\zeta}^\sigma = \frac{i}{2}\sigma_{\zeta\alpha\zeta}^{ab}\frac{1}{2}(F_{ab}^\sigma - \zeta * F_{ab}^\sigma)$

推论1.2.3.  $\psi_{\alpha\zeta}^\sigma = -\frac{i}{2}\zeta\sigma_{\zeta\alpha\zeta}^{ab} * F_{ab}^\sigma$

推论1.2.4.  $\sigma_{\zeta\alpha\zeta}^{ab}(F_{ab}^\sigma + \zeta * F_{ab}^\sigma) = 0$

推论1.2.5.  $F_{ab}^\sigma - \zeta * F_{ab}^\sigma = -\frac{1}{4}\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta}^{cd}(F_{cd}^\sigma - \zeta * F_{cd}^\sigma)$

推论1.2.6.  $F_{ab}^\sigma = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^\sigma + \sigma_{+ab}^\alpha\psi_\alpha^\sigma), *F_{ab}^\sigma = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^\sigma - \sigma_{+ab}^\alpha\psi_\alpha^\sigma)$

证明:  $F_{ab}^\sigma - \zeta * F_{ab}^\sigma = i\sigma_{\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}^\sigma$

$$\Leftrightarrow F_{ab}^\sigma - *F_{ab}^\sigma = i\sigma_{+ab}^\alpha\psi_\alpha^\sigma, F_{ab}^\sigma + *F_{ab}^\sigma = i\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^\sigma$$

$$\Leftrightarrow F_{ab}^\sigma = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^\sigma + \sigma_{+ab}^\alpha\psi_\alpha^\sigma), *F_{ab}^\sigma = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^\sigma - \sigma_{+ab}^\alpha\psi_\alpha^\sigma)$$

$$\Leftrightarrow F_{ab}^\sigma = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^\sigma + \sigma_{+ab}^\alpha\psi_\alpha^\sigma)$$

$$\Leftrightarrow *F_{ab}^\sigma = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^\sigma - \sigma_{+ab}^\alpha\psi_\alpha^\sigma) \quad \square$$

推论1.2.7.  $F_{ab}^\sigma = -F_{ba}^\sigma \Leftrightarrow F_{ab}^\sigma = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^\sigma + \sigma_{+ab}^\alpha\psi_\alpha^\sigma)$

### 1.3 Yang-Mills场强的 $\frac{1}{2}$ -旋量描述 [1, 2]

定义1.3.1. Yang-Mills场 $\frac{1}{2}$ -旋量张量:  $\psi_{A_\zeta B_\zeta}^\sigma := \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\psi_{\alpha\zeta}^\sigma = \frac{i\zeta}{\sqrt{2}}S^{ab}{}_{A_\zeta B_\zeta}F_{ab}^\sigma$

推论1.3.1.  $\psi_{A_\zeta B_\zeta}^\sigma = \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\psi_{\alpha\zeta}^\sigma \Leftrightarrow \psi_{\alpha\zeta}^\sigma = \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha\zeta}^{A_\zeta B_\zeta}\psi_{A_\zeta B_\zeta}^\sigma$

推论1.3.2.  $\psi_{A_\zeta B_\zeta}^\sigma = \psi_{B_\zeta A_\zeta}^\sigma$

推论1.3.3.  $\psi_{A_\zeta B_\zeta}^\sigma = \frac{-i}{\sqrt{2}}S^{ab}{}_{A_\zeta B_\zeta} * F_{ab}^\sigma$

推论1.3.4.  $\frac{1}{2}(F_{ab}^\sigma - \zeta * F_{ab}^\sigma) = \frac{i\zeta}{\sqrt{2}}S^{ab}{}_{A_\zeta B_\zeta}\psi_{A_\zeta B_\zeta}^\sigma \Leftrightarrow \psi_{A_\zeta B_\zeta}^\sigma = \frac{i\zeta}{\sqrt{2}}S^{ab}{}_{A_\zeta B_\zeta}\frac{1}{2}(F_{ab}^\sigma - \zeta * F_{ab}^\sigma)$

推论1.3.5.  $F_{ab}^\sigma - \varsigma * F_{ab}^\sigma = -\frac{1}{2}S_{ab}{}^{A_\varsigma B_\varsigma} S^{cd}{}_{A_\varsigma B_\varsigma} (F_{cd}^\sigma - \varsigma * F_{cd}^\sigma)$

推论1.3.6.  $F_{ab}^\sigma = \frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'}^\sigma + S_{ab}{}^{AB}\psi_{AB}^\sigma), *F_{ab}^\sigma = \frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'}^\sigma - S_{ab}{}^{AB}\psi_{AB}^\sigma)$

推论1.3.7.  $F_{ab}^\sigma = -F_{ba}^\sigma \Leftrightarrow F_{ab}^\sigma = \frac{i\varsigma}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'}^\sigma + S_{ab}{}^{AB}\psi_{AB}^\sigma)$

结合推论1.3.6和(1.274), (1.275) 式可得Penrose对应记法<sup>[1, 2]</sup>:

推论1.3.8.  $F_{ab}^\sigma \stackrel{P}{=} \frac{1}{\sqrt{2}}(\psi_{A'B'}^\sigma \varepsilon_{AB} + \psi_{AB}^\sigma \varepsilon_{A'B'}), *F_{ab}^\sigma \stackrel{P}{=} \frac{1}{\sqrt{2}}(\psi_{A'B'}^\sigma \varepsilon_{AB} - \psi_{AB}^\sigma \varepsilon_{A'B'})$

## 1.4 Yang-Mills场强的1-旋量描述

定义1.4.1. Yang-Mills场1-旋量 $\psi_{k_\varsigma}^\sigma(1) := \Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma}(1)\psi_{A_\varsigma B_\varsigma}^\sigma = \Gamma_{k_\varsigma}^{\alpha_\varsigma}(1)\psi_{\alpha_\varsigma}^\sigma$

推论1.4.1.  $\psi_{A_\varsigma B_\varsigma}^\sigma = \Gamma_{A_\varsigma B_\varsigma}^{k_\varsigma}(1)\psi_{k_\varsigma}^\sigma(1), \psi_{\alpha_\varsigma}^\sigma = \Gamma_{\alpha_\varsigma}^{k_\varsigma}(1)\psi_{k_\varsigma}^\sigma(1)$

## 1.5 Yang-Mills场源的 $\frac{1}{2}$ -旋量描述<sup>[1, 2]</sup>

定义1.5.1. Yang-Mills场源 $\frac{1}{2}$ -旋量张量 $J^{A'_\varsigma A_\varsigma \sigma} := \frac{i\varsigma}{\sqrt{2}}(\sigma, -i\varsigma)_{a'_\varsigma a_\varsigma} J^{a\sigma}, J_{A_\varsigma A'_\varsigma}^\sigma := \frac{-i\varsigma}{\sqrt{2}}(\sigma, i\varsigma)_{A_\varsigma A'_\varsigma} J_a^\sigma$

Penrose记法:  $J^{a\sigma} \stackrel{P}{=} J^{A'A\sigma}, J_a \stackrel{P}{=} J_{AA'}$

## 2 Yang-Mills场方程的多种等价表述形式

### 2.1 Yang-Mills理论的标准描述

$$\begin{cases} F_{uv}^\sigma T_\sigma = \partial_u A_v^\sigma T_\sigma - \partial_v A_u^\sigma T_\sigma + ig[A_u^\tau T_\tau, A_v^\rho T_\rho] \\ [T_\tau, T_\rho] = if_{\tau\rho}^\sigma T_\sigma, c^\sigma T_\sigma = 0 \Leftrightarrow c^\sigma = 0 \end{cases} \quad (7.2)$$

规范变换:

$$\begin{cases} \psi \rightarrow U(\theta)\psi, U(\theta) = e^{ig\theta^\sigma T_\sigma}, \psi \text{带YM荷g} \\ A_u^\sigma T_\sigma \rightarrow U(\theta)A_u^\sigma T_\sigma U^{-1}(\theta) + \frac{i}{g}[\partial_u U(\theta)]U^{-1}(\theta) \end{cases} \quad (7.3)$$

推论2.1.1.  $D_u\psi \rightarrow UD_u\psi, D_u = \partial_u + igA_u^\sigma T_\sigma$

证明:  $D_u\psi = (\partial_u + igA_u^\sigma T_\sigma)\psi \rightarrow [\partial_u + UigA_u^\sigma T_\sigma U^{-1} - (\partial_u U)U^{-1}](U\psi)$

$\Leftrightarrow D_u\psi \rightarrow [\partial_u(U\psi) + UigA_u^\sigma T_\sigma\psi - (\partial_u U)\psi]$

$\Leftrightarrow D_u\psi \rightarrow U(\partial_u + igA_u^\sigma T_\sigma)\psi$

$\Leftrightarrow D_u\psi \rightarrow UD_u\psi, D_u = \partial_u + igA_u^\sigma T_\sigma$  □

引理2.1.1.  $\partial_u(U^{-1}) = -U^{-1}\partial_u(U)U^{-1}$

证明:  $\partial_u(UU^{-1}) = \partial_u(I)$

$\Leftrightarrow \partial_u(U)U^{-1} + U\partial_u(U^{-1}) = 0$

$\Leftrightarrow U\partial_u(U^{-1}) = -\partial_u(U)U^{-1}$

$\Leftrightarrow \partial_u(U^{-1}) = -U^{-1}\partial_u(U)U^{-1}$  □

推论2.1.2.  $F_{uv}^\sigma T_\sigma \rightarrow UF_{uv}^\sigma T_\sigma U^{-1}$

证明:  $F_{uv}^\sigma T_\sigma = \partial_u A_v^\sigma T_\sigma - \partial_v A_u^\sigma T_\sigma + ig[A_u^\rho T_\rho, A_v^\tau T_\tau]$

$\rightarrow \partial_u[U A_v^\sigma T_\sigma U^{-1} + \frac{i}{g}(\partial_v U)U^{-1}] - \partial_v[U A_u^\sigma T_\sigma U^{-1} + \frac{i}{g}(\partial_u U)U^{-1}]$

$+ ig[U A_u^\rho T_\rho U^{-1} + \frac{i}{g}(\partial_u U)U^{-1}, U A_v^\tau T_\tau U^{-1} + \frac{i}{g}(\partial_v U)U^{-1}]$

$\Leftrightarrow F_{uv}^\sigma T_\sigma \rightarrow U(\partial_u A_v^\sigma T_\sigma - \partial_v A_u^\sigma T_\sigma + ig[A_u^\rho T_\rho, A_v^\tau T_\tau])U^{-1}$

$\Leftrightarrow F_{uv}^\sigma T_\sigma \rightarrow UF_{uv}^\sigma T_\sigma U^{-1}$  □

推论2.1.3.  $D_w F_{uv}^\sigma T_\sigma \rightarrow UD_w F_{uv}^\sigma T_\sigma U^{-1}, D_w = \nabla_w + ig[A_w^\sigma T_\sigma, \ ]$

$$\begin{aligned}
& \text{证明: } D_w F_{uv}^\sigma T_\sigma = \partial_w F_{uv}^\sigma T_\sigma + ig[A_w^\rho T_\rho, F_{uv}^\tau T_\tau] \\
& \rightarrow \partial_w (U F_{uv}^\sigma T_\sigma U^{-1}) + ig[U A_w^\rho T_\rho U^{-1} + \frac{i}{g}(\partial_w U)U^{-1}, U F_{uv}^\tau T_\tau U^{-1}] \\
& \Leftrightarrow D_w F_{uv}^\sigma T_\sigma \rightarrow U(\partial_w F_{uv}^\sigma T_\sigma + ig[A_w^\rho T_\rho, F_{uv}^\tau T_\tau])U^{-1} \\
& \Leftrightarrow D_w F_{uv}^\sigma T_\sigma \rightarrow U D_w F_{uv}^\sigma T_\sigma U^{-1}
\end{aligned}$$

□

$$\text{推论2.1.4. } D_w F_{uv}^\sigma = \nabla_w F_{uv}^\sigma - gf_{\rho\tau}^\sigma A_w^\rho F_{uv}^\tau$$

$$\text{推论2.1.5. } D_w F_{uv}^\sigma = \nabla_w F_{uv}^\sigma + igA_w^\rho (-if_\rho^\sigma) F_{uv}^\tau$$

$$\text{推论2.1.6. } D_w F_{uv} = [\nabla_w + igA_w^\rho (-if_\rho)] F_{uv}, D_w = \nabla_w + igA_w^\rho (-if_\rho)$$

## 2.2 Yang-Mills理论的分量描述

$$\text{推论2.2.1. } F_{uv}^\sigma T_\sigma = \partial_u A_v^\sigma T_\sigma - \partial_v A_u^\sigma T_\sigma + ig[A_u^\rho T_\rho, A_v^\tau T_\tau] \Leftrightarrow F_{uv}^\sigma = \partial_u A_v^\sigma - \partial_v A_u^\sigma - gf_{\rho\tau}^\sigma A_u^\rho A_v^\tau$$

推论2.2.2.

$$F_{uv}^\sigma T_\sigma = (\partial_u + igA_u^\rho T_\rho) A_v^\sigma T_\sigma - (\partial_v + igA_v^\rho T_\rho) A_u^\sigma T_\sigma \Leftrightarrow F_{uv} = [\partial_u + \frac{1}{2}igA_u^\rho (-if_\rho)] A_v - [\partial_v + \frac{1}{2}igA_v^\rho (-if_\rho)] A_u$$

$$\text{推论2.2.3. 规范变换: } \delta\psi = ig\theta^\sigma T_\sigma \psi, \delta A_u = ig\theta^\rho (-if_\rho) A_u - \partial_u \theta$$

$$\text{推论2.2.4. } \delta F_{uv} = ig\theta^\rho (-if_\rho) F_{uv}$$

## 2.3 Yang-Mills方程的标架描述

$$\text{定义2.3.1. } F_{ab}^\sigma := e_a^u e_b^v F_{uv}^\sigma, A_a^\sigma := e_a^u A_u^\sigma$$

Yang-Mills方程的标架描述

$$D^a F_{ab}^\sigma = -J_b^\sigma, D^a * F_{ab}^\sigma \equiv 0 \quad (7.4)$$

## 2.4 Yang-Mills场方程的经典形式

$$\begin{cases} \nabla_d \cdot \vec{E}^\sigma = \rho^\sigma, \nabla_d \times \vec{E}^\sigma = -D_t \vec{B}^\sigma \\ \nabla_d \cdot \vec{B}^\sigma = 0, \nabla_d \times \vec{B}^\sigma = \vec{J}^\sigma + D_t \vec{E}^\sigma \end{cases} \Leftrightarrow D^a F_{ab}^\sigma = -J_b^\sigma, D^a * F_{ab}^\sigma \equiv 0 \quad (7.5)$$

## 2.5 Yang-Mills场方程的复矢量表述形式

复矢量张量形式:

$$\text{定理2.5.1. } D^a F_{ab}^\sigma = -J_b^\sigma \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ_b^\sigma; F_{ab}^\sigma = D_a A_b - D_b A_a, \tilde{\Psi}^{\alpha_\varsigma\sigma} = \left[ \psi^{\alpha_\varsigma\sigma} = \frac{i}{2} \sigma_{\varsigma ab}^{\alpha_\varsigma} F^{ab\sigma} \right]$$

证明:  $D^a F_{ab}^\sigma = -J_b^\sigma$

$$\Leftrightarrow D^a F_{ab}^\sigma = -J_b^\sigma, D^a * F_{ab}^\sigma \equiv 0$$

$$\Leftrightarrow D^a (F_{ab}^\sigma - \varsigma * F_{ab}^\sigma) = -J_b^\sigma$$

$$\Leftrightarrow D^a (i\sigma_{\varsigma ab}^{\alpha_\varsigma} \psi_{\alpha_\varsigma}^\sigma) = -J_b^\sigma, \alpha_\varsigma = 1, 2, 3$$

$$\Leftrightarrow D^a [(\sigma_\varsigma, -i\varsigma)^{\alpha_\varsigma} |_{ab} \tilde{\Psi}^{\alpha_\varsigma\sigma}] = iJ_b^\sigma, \alpha_\varsigma = 1, 2, 3, 4$$

$$\Leftrightarrow D^a [(\sigma_{-\varsigma}, -i\varsigma)_a |_{b\alpha_\varsigma} \tilde{\Psi}^{\alpha_\varsigma\sigma}] = iJ_b^\sigma, \alpha_\varsigma = 1, 2, 3, 4$$

$$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ_b^\sigma, \alpha_\varsigma = 1, 2, 3, 4$$

□

复矢量矩阵形式:

$$\text{推论2.5.1. } (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ_b^\sigma \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1, \varsigma) = iJ^\sigma$$

表象变换:

$$\text{推论2.5.2. } (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1, \varsigma) = iJ^\sigma \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1, \varsigma) = i\tilde{J}^\sigma(1, \varsigma)$$

## 2.6 Yang-Mills场方程的 $\frac{1}{2}$ -旋量表述形式

$\frac{1}{2}$ -旋量Penrose抽象指标形式<sup>[1,2]</sup>:

**定理2.6.1.**  $(\sigma_{-\varsigma}, -i\varsigma)_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ_b^\sigma \Leftrightarrow \nabla_d^{A'_\varsigma A_\varsigma} \psi_{A_\varsigma B_\varsigma}^\sigma = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma}, \nabla_d^{A'_\varsigma A_\varsigma} = \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D^a$

**证明:**  $(\sigma_{-\varsigma}, -i\varsigma)_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ_b^\sigma$

$$\Leftrightarrow D^a (i\sigma_{\varsigma ab}^{\alpha_\varsigma} \psi_{\alpha_\varsigma}^\sigma) = -J_b^\sigma$$

$$\Leftrightarrow D^a (i\sigma_{\varsigma ab}^{\alpha_\varsigma} \cdot \frac{i\varsigma}{\sqrt{2}} \sigma_{\alpha_\varsigma}^{A'_\varsigma B'_\varsigma} \psi_{A'_\varsigma B'_\varsigma}^\sigma) = -J_b^\sigma$$

$$\Leftrightarrow iS_{ab}^{A'_\varsigma B'_\varsigma} D^a \psi_{A'_\varsigma B'_\varsigma}^\sigma = \frac{-\varsigma}{\sqrt{2}} J_b^\sigma$$

$$\Leftrightarrow (\frac{\varsigma}{2} \delta_{ab} \varepsilon^{A'_\varsigma B'_\varsigma} + iS_{ab}^{A'_\varsigma B'_\varsigma}) D^a \psi_{A'_\varsigma B'_\varsigma}^\sigma = \frac{-\varsigma}{\sqrt{2}} J_b^\sigma$$

$$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A'_\varsigma} \varepsilon_{A'_\varsigma B'_\varsigma} \cdot \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_b^{B'_\varsigma B'_\varsigma} D^a \psi_{A'_\varsigma B'_\varsigma}^\sigma = \frac{-1}{\sqrt{2}} J_b^\sigma$$

$$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A'_\varsigma} \varepsilon_{A'_\varsigma B'_\varsigma} D^a \psi_{A'_\varsigma B'_\varsigma}^\sigma = \frac{-1}{\sqrt{2}} J_b^\sigma \cdot \frac{-i\varsigma}{\sqrt{2}} (\sigma, i\varsigma)_b^{B'_\varsigma B'_\varsigma}$$

$$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A'_\varsigma} D^a \psi_{A'_\varsigma B'_\varsigma}^\sigma = \frac{-\varsigma}{\sqrt{2}} \varsigma \varepsilon^{A'_\varsigma B'_\varsigma} J_{B'_\varsigma B'_\varsigma}$$

$$\Leftrightarrow \nabla_d^{A'_\varsigma A_\varsigma} \psi_{A_\varsigma B_\varsigma}^\sigma = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma}, \nabla_d^{A'_\varsigma A_\varsigma} = \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D^a$$

□

$\frac{1}{2}$ -旋量张量形式:

**推论2.6.1.**  $\nabla_d^{A'_\varsigma A_\varsigma} \psi_{A_\varsigma B_\varsigma}^\sigma = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma} \Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D_a \psi_{A_\varsigma B_\varsigma}^\sigma = iJ_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma}$

$\frac{1}{2}$ -旋量矩阵形式:

**推论2.6.2.**  $(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D_a \psi_{A_\varsigma B_\varsigma}^\sigma = iJ_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1, \varsigma) = i\tilde{J}^\sigma(1, \varsigma)$

$\frac{1}{2}$ -旋量方阵形式:

**推论2.6.3.**  $(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D_a \psi_{A_\varsigma B_\varsigma}^\sigma = iJ_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma} \Leftrightarrow (\sigma, -i\varsigma)^a D_a [\psi]^\sigma = i[J]^\sigma$

$\frac{1}{2}$ -自旋张量表述形式:(证明留待以后)

**推论2.6.4.**  $(\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D^a \psi_{A_\varsigma B_\varsigma}^\sigma = iJ_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma} \Leftrightarrow [\partial_a + iS_{ab}(1, \varsigma) \partial^b]_{k_\varsigma}{}^{l_\varsigma} \psi_{l_\varsigma}^\sigma(1, \varsigma) = \mathbb{J}_{ak_\varsigma}^\sigma(1, \varsigma)$

**推论2.6.5.**  $\begin{cases} \partial^a F_{ab}^\sigma = -J_b^\sigma \\ \partial^a * F_{ab}^\sigma \equiv 0 \end{cases} \Leftrightarrow [\partial_a + iS_{ab}(1, \varsigma) \partial^b]_{k_\varsigma}{}^{l_\varsigma} \psi_{l_\varsigma}^\sigma(1, \varsigma) = \mathbb{J}_{ak_\varsigma}^\sigma(1, \varsigma)$

## 2.7 猜测

**定理2.7.1.**  $D^a * F_{ab}^\sigma = 0 \Leftrightarrow F_{ab}^\sigma = D_a A_b - D_b A_a \Leftrightarrow D^a * F_{ab}^\sigma \equiv 0$

**定理2.7.2.**  $D^a F_{ab}^\sigma = -J_b^\sigma, D^a * F_{ab}^\sigma = 0 \Leftrightarrow D^a F_{ab}^\sigma = -J_b^\sigma, F_{ab}^\sigma = D_a A_b - D_b A_a$

## 2.8 Yang-Mills场方程<sup>[7]</sup>的自旋张量表述形式

Yang-Mills场的自旋张量矩阵:  $S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \succ \begin{bmatrix} 0 & \gamma_z & -\gamma_y & -\varsigma\gamma_x \\ -\gamma_z & 0 & \gamma_x & -\varsigma\gamma_y \\ \gamma_y & -\gamma_x & 0 & -\varsigma\gamma_z \\ \varsigma\gamma_x & \varsigma\gamma_y & \varsigma\gamma_z & 0 \end{bmatrix}$  (7.6)

**定理2.8.1.**  $(D_a + iS_{ab} D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma} \psi^{\gamma_\varsigma\sigma}(1, \varsigma) = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\sigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1, \varsigma) = iJ^\sigma$

一种直观证法如下:

**证明:**  $(D_a + iS_{ab} D^b)^{\beta_\varsigma}{}_{\gamma_\varsigma} \psi^{\gamma_\varsigma\sigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\sigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (D_x + i\gamma_z D_y - i\gamma_y D_z - i\varsigma\gamma_x D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma} \psi^{\gamma_\varsigma\sigma} = -i\sigma_{\varsigma xb}^{\beta_\varsigma} J^{b\sigma} \\ (D_y + i\gamma_x D_z - i\gamma_z D_x - i\varsigma\gamma_y D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma} \psi^{\gamma_\varsigma\sigma} = -i\sigma_{\varsigma yb}^{\beta_\varsigma} J^{b\sigma} \\ (D_z + i\gamma_y D_x - i\gamma_x D_y - i\varsigma\gamma_z D_\pi)^{\beta_\varsigma}{}_{\gamma_\varsigma} \psi^{\gamma_\varsigma\sigma} = -i\sigma_{\varsigma zb}^{\beta_\varsigma} J^{b\sigma} \\ (D_\pi + i\varsigma\gamma_x D_x + i\varsigma\gamma_y D_y + i\varsigma\gamma_z D_z)^{\beta_\varsigma}{}_{\gamma_\varsigma} \psi^{\gamma_\varsigma\sigma} = -i\sigma_{\varsigma \pi b}^{\beta_\varsigma} J^{b\sigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} D_x & D_y & D_z \\ -D_y & D_x & -\varsigma D_\pi \\ -D_z & \varsigma D_\pi & D_x \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma\sigma} \\ \psi^{y_\varsigma\sigma} \\ \psi^{z_\varsigma\sigma} \end{bmatrix} = \begin{bmatrix} \varsigma J^{\pi\sigma} \\ J^{z\sigma} \\ -J^{y\sigma} \end{bmatrix}, & \begin{bmatrix} D_y & -D_x & \varsigma D_\pi \\ D_x & D_y & D_z \\ -\varsigma D_\pi & -D_z & D_y \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma\sigma} \\ \psi^{y_\varsigma\sigma} \\ \psi^{z_\varsigma\sigma} \end{bmatrix} = \begin{bmatrix} -J^{z\sigma} \\ \varsigma J^{\pi\sigma} \\ J^{x\sigma} \end{bmatrix} \\ \begin{bmatrix} D_z & -\varsigma D_\pi & -D_x \\ \varsigma D_\pi & D_z & -D_y \\ D_x & D_y & D_z \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma\sigma} \\ \psi^{y_\varsigma\sigma} \\ \psi^{z_\varsigma\sigma} \end{bmatrix} = \begin{bmatrix} J^{y\sigma} \\ -J^{x\sigma} \\ \varsigma J^{\pi\sigma} \end{bmatrix}, & iD_\pi \Psi^\sigma(1, \varsigma) = \varsigma \gamma \cdot \nabla_d \Psi^\sigma(1, \varsigma) - i\varsigma \vec{J}^\sigma \\ iD_\pi \Psi^\sigma(1, \varsigma) = i\varsigma \nabla_d \times \Psi^\sigma(1, \varsigma) - i\varsigma \vec{J}^\sigma \\ \nabla_d \cdot \Psi^\sigma(1, \varsigma) = \varsigma J^{\pi\sigma} \\ iD_\pi \Psi^\sigma(1, \varsigma) = \varsigma \gamma \cdot \nabla_d \Psi^\sigma(1, \varsigma) - i\varsigma \vec{J}^\sigma \\ \nabla_d \cdot \Psi^\sigma(1, \varsigma) = \varsigma J^{\pi\sigma} \\ (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1, \varsigma) = iJ \end{cases}$$

□

另一种更解析更抽象的证法如下：

$$\begin{aligned} \text{证明: } & (D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma\sigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\sigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \\ \Leftrightarrow & \sigma_{\varsigma a}^{\beta_\varsigma} \sigma_{\varsigma \gamma_\varsigma cb} D^b \psi^{\gamma_\varsigma\sigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\sigma} \\ \Leftrightarrow & \sigma_{\varsigma ac}^{\beta_\varsigma} \sigma_{\varsigma \gamma_\varsigma}^{cb} D_b \psi^{\gamma_\varsigma\sigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\sigma} \\ \Leftrightarrow & \sigma_{\beta_\varsigma}^{\varsigma ad} \sigma_{\varsigma ac}^{\beta_\varsigma} \sigma_{\varsigma \gamma_\varsigma}^{cb} D_b \psi^{\gamma_\varsigma\sigma} = -i\sigma_{\beta_\varsigma}^{\varsigma ad} \sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\sigma} \\ \Leftrightarrow & \sigma_{\varsigma \gamma_\varsigma}^{db} D_b \psi^{\gamma_\varsigma\sigma} = -iJ^{d\sigma} \\ \Leftrightarrow & \sigma_{\varsigma \alpha_\varsigma}^{ab} D_a \psi^{\alpha_\varsigma\sigma} = iJ^{b\sigma}, \alpha_\varsigma = 1, 2, 3 \\ \Leftrightarrow & (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma\sigma} = iJ_b^\sigma, \alpha_\varsigma = 1, 2, 3, 4 \end{aligned}$$

□

此方程(3.3.2)完全等价于Yang-Mills场方程，它就是Yang-Mills场方程的自旋张量表述形式。

$$\text{引理2.8.1. } \mathbb{J}_a^{\beta_\varsigma\sigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\sigma} \Leftrightarrow \begin{cases} \mathbb{J}_y^{z_\varsigma\sigma} = -\mathbb{J}_z^{y_\varsigma\sigma} = -\varsigma \mathbb{J}_\pi^{x_\varsigma\sigma} = J^{x\sigma} \\ \mathbb{J}_z^{x_\varsigma\sigma} = -\mathbb{J}_x^{z_\varsigma\sigma} = -\varsigma \mathbb{J}_\pi^{y_\varsigma\sigma} = J^{y\sigma} \\ \mathbb{J}_x^{y_\varsigma\sigma} = -\mathbb{J}_y^{x_\varsigma\sigma} = -\varsigma \mathbb{J}_\pi^{z_\varsigma\sigma} = J^{z\sigma} \\ \mathbb{J}_x^{x_\varsigma\sigma} = \mathbb{J}_y^{y_\varsigma\sigma} = \mathbb{J}_z^{z_\varsigma\sigma} = \varsigma J^{\pi\sigma} \end{cases}$$

展开即可证明。以上自旋方程是关于特殊的源项，那么对于一般的源项又会怎样呢？请看下面的定理。

$$\text{定理2.8.2. } (D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1, \varsigma) = iJ^\sigma, \mathbb{J}_a^{\beta_\varsigma\sigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\sigma}$$

$$\begin{aligned} \text{证明: } & (D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \\ \Leftrightarrow & \begin{cases} (D_x + i\gamma_z D_y - i\gamma_y D_z - i\varsigma \gamma_x D_\pi)^{\beta_\varsigma} \psi^{\gamma_\varsigma\sigma} = \mathbb{J}_x^{\beta_\varsigma\sigma} \\ (D_y + i\gamma_x D_z - i\gamma_z D_x - i\varsigma \gamma_y D_\pi)^{\beta_\varsigma} \psi^{\gamma_\varsigma\sigma} = \mathbb{J}_y^{\beta_\varsigma\sigma} \\ (D_z + i\gamma_y D_x - i\gamma_x D_y - i\varsigma \gamma_z D_\pi)^{\beta_\varsigma} \psi^{\gamma_\varsigma\sigma} = \mathbb{J}_z^{\beta_\varsigma\sigma} \\ (D_\pi + i\varsigma \gamma_x D_x + i\varsigma \gamma_y D_y + i\varsigma \gamma_z D_z)^{\beta_\varsigma} \psi^{\gamma_\varsigma\sigma} = \mathbb{J}_\pi^{\beta_\varsigma\sigma} \end{cases} \end{aligned}$$



$$\begin{aligned}
& \Leftrightarrow \left\{ \begin{array}{l} \begin{bmatrix} D_x & D_y & D_z \\ -D_y & D_x & -\varsigma D_\pi \\ -D_z & \varsigma D_\pi & D_x \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma\sigma} \\ \psi^{y_\varsigma\sigma} \\ \psi^{z_\varsigma\sigma} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_x^{x_\varsigma\sigma} \\ \mathbb{J}_x^{y_\varsigma\sigma} \\ \mathbb{J}_x^{z_\varsigma\sigma} \end{bmatrix} \\ \begin{bmatrix} D_y & -D_x & \varsigma D_\pi \\ D_x & D_y & D_z \\ -\varsigma D_\pi & -D_z & D_y \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma\sigma} \\ \psi^{y_\varsigma\sigma} \\ \psi^{z_\varsigma\sigma} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_y^{x_\varsigma\sigma} \\ \mathbb{J}_y^{y_\varsigma\sigma} \\ \mathbb{J}_y^{z_\varsigma\sigma} \end{bmatrix} \\ \begin{bmatrix} D_z & -\varsigma D_\pi & -D_x \\ \varsigma D_\pi & D_z & -D_y \\ D_x & D_y & D_z \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma\sigma} \\ \psi^{y_\varsigma\sigma} \\ \psi^{z_\varsigma\sigma} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_z^{x_\varsigma\sigma} \\ \mathbb{J}_z^{y_\varsigma\sigma} \\ \mathbb{J}_z^{z_\varsigma\sigma} \end{bmatrix} \end{array} \Leftrightarrow \begin{cases} \nabla_d \cdot \Psi^\sigma(1, \varsigma) = \mathbb{J}_x^{x_\varsigma\sigma} \\ [\nabla_d \times \Psi^\sigma(1, \varsigma)]^{z_\varsigma\sigma} - \varsigma D_\pi \psi^{z_\varsigma\sigma}(1, \varsigma) = \mathbb{J}_x^{y_\varsigma\sigma} \\ -[\nabla_d \times \Psi^\sigma(1, \varsigma)]^{y_\varsigma\sigma} + \varsigma D_\pi \psi^{y_\varsigma\sigma}(1, \varsigma) = \mathbb{J}_x^{z_\varsigma\sigma} \\ -[\nabla_d \times \Psi^\sigma(1, \varsigma)]^{z_\varsigma\sigma} + \varsigma D_\pi \psi^{z_\varsigma\sigma}(1, \varsigma) = \mathbb{J}_y^{x_\varsigma\sigma} \\ \nabla_d \cdot \Psi^\sigma(1, \varsigma) = \mathbb{J}_y^{y_\varsigma\sigma} \\ [\nabla_d \times \Psi^\sigma(1, \varsigma)]^{x_\varsigma\sigma} - \varsigma D_\pi \psi^{x_\varsigma\sigma}(1, \varsigma) = \mathbb{J}_y^{z_\varsigma\sigma} \\ [\nabla_d \times \Psi^\sigma(1, \varsigma)]^{y_\varsigma\sigma} - \varsigma D_\pi \psi^{y_\varsigma\sigma}(1, \varsigma) = \mathbb{J}_z^{x_\varsigma\sigma} \\ -[\nabla_d \times \Psi^\sigma(1, \varsigma)]^{x_\varsigma\sigma} + \varsigma D_\pi \psi^{x_\varsigma\sigma}(1, \varsigma) = \mathbb{J}_z^{y_\varsigma\sigma} \\ \nabla_d \cdot \Psi^\sigma(1, \varsigma) = \mathbb{J}_z^{z_\varsigma\sigma} \end{cases} \\ & D_\pi \Psi^\sigma(1, \varsigma) + i\varsigma \gamma \cdot \nabla_d \psi^\sigma = \mathbb{J}_\pi^\sigma \Leftrightarrow D_\pi \Psi^\sigma(1, \varsigma) - \varsigma \nabla_d \times \Psi^\sigma(1, \varsigma) = \mathbb{J}_\pi^\sigma \\ & \Leftrightarrow \begin{cases} \mathbb{J}_y^{z_\varsigma\sigma} = -\mathbb{J}_z^{y_\varsigma\sigma} = -\varsigma \mathbb{J}_\pi^{x_\varsigma\sigma} := J^{x\sigma} \\ \mathbb{J}_z^{x_\varsigma\sigma} = -\mathbb{J}_x^{z_\varsigma\sigma} = -\varsigma \mathbb{J}_\pi^{y_\varsigma\sigma} := J^{y\sigma} \\ \mathbb{J}_x^{y_\varsigma\sigma} = -\mathbb{J}_y^{x_\varsigma\sigma} = -\varsigma \mathbb{J}_\pi^{z_\varsigma\sigma} := J^{z\sigma} \\ \mathbb{J}_x^{x_\varsigma\sigma} = \mathbb{J}_y^{y_\varsigma\sigma} = \mathbb{J}_z^{z_\varsigma\sigma} := \varsigma J^{\pi\sigma} \\ D_\pi \Psi^\sigma(1, \varsigma) - \varsigma \nabla_d \times \Psi^\sigma(1, \varsigma) = i\vec{J}^\sigma \\ \nabla_d \cdot \Psi^\sigma(1, \varsigma) = -iJ^{\pi\sigma} \end{cases} \\ & \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1, \varsigma) = iJ^\sigma, \mathbb{J}_a^{\beta_\varsigma\sigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\sigma} \quad \square
\end{aligned}$$

另一种更解析更抽象的证法如下：

**定理2.8.3.**  $(D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \Leftrightarrow \mathbb{J}_a^{\beta_\varsigma\sigma} = \sigma_{\varsigma ab}^{\beta_\varsigma} \sigma_{\varsigma \gamma_\varsigma}^{bc} D_c \psi^{\gamma_\varsigma\sigma}$

**证明:**  $(D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \sigma_{\varsigma a}^{\beta_\varsigma c} \sigma_{\varsigma \gamma_\varsigma cb} D^b \psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}$$

$$\Leftrightarrow \mathbb{J}_a^{\beta_\varsigma\sigma} = \sigma_{\varsigma ab}^{\beta_\varsigma} \sigma_{\varsigma \alpha_\varsigma}^{bc} D_c \psi^{\alpha_\varsigma\sigma}$$

$$\begin{aligned}
& \Leftrightarrow \begin{cases} \mathbb{J}_y^{z_\varsigma\sigma} = -\mathbb{J}_z^{y_\varsigma\sigma} = -\varsigma \mathbb{J}_\pi^{x_\varsigma\sigma} = i\sigma_{\varsigma \alpha_\varsigma}^{xb} D_b \psi^{\alpha_\varsigma\sigma} \\ \mathbb{J}_z^{x_\varsigma\sigma} = -\mathbb{J}_x^{z_\varsigma\sigma} = -\varsigma \mathbb{J}_\pi^{y_\varsigma\sigma} = i\sigma_{\varsigma \alpha_\varsigma}^{yb} D_b \psi^{\alpha_\varsigma\sigma} \\ \mathbb{J}_x^{y_\varsigma\sigma} = -\mathbb{J}_y^{x_\varsigma\sigma} = -\varsigma \mathbb{J}_\pi^{z_\varsigma\sigma} = i\sigma_{\varsigma \alpha_\varsigma}^{zb} D_b \psi^{\alpha_\varsigma\sigma} \\ \mathbb{J}_x^{x_\varsigma\sigma} = \mathbb{J}_y^{y_\varsigma\sigma} = \mathbb{J}_z^{z_\varsigma\sigma} = i\varsigma \sigma_{\varsigma \alpha_\varsigma}^{\pi b} D_b \psi^{\alpha_\varsigma\sigma} \end{cases} \quad \square
\end{aligned}$$

这个定理表明此自旋方程源项受到一定限制，不是随意的，只有前一定理描述的源项情形才有解，其他情形全无解。

**推论2.8.1.**  $(D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma\sigma} = \mathbb{J}_a^{\beta_\varsigma\sigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma}$  有解  $\Leftrightarrow \mathbb{J}_a^{\beta_\varsigma\sigma} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\sigma}, \exists J^{b\sigma}$

## 2.9 Yang-Mills场方程的经典分离形式

**推论2.9.1.**  $(\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^\sigma(1, \varsigma) = iJ^\sigma \Leftrightarrow (\gamma, -i\varsigma)^a D_a \Psi^\sigma(1, \varsigma) = i\vec{J}^\sigma, \nabla_d \cdot \Psi^\sigma(1, \varsigma) = \varsigma J_\pi^\sigma$

# 第八章 引力场方程的重新表述

自我评述：本章提出了引力场方程的多种等价表述形式，并严格证明了它们之间的等价性。

## 1 引力场<sup>[11-14]</sup>物理量的各种描述

### 1.1 引力场<sup>[11-14]</sup>物理量的经典描述

#### 1.1.1 引力场的曲率张量

曲率张量的对称性质：

$$\text{反对称性： } R^{abcd} = -R^{bacd}, R^{abcd} = -R^{abdc} \quad (8.1)$$

$$\text{对称性： } R^{abcd} = R^{cdab} \quad (8.2)$$

$$\text{轮换对称性： } R^{abcd} + R^{adbc} + R^{acdb} = 0 \quad (8.3)$$

#### 1.1.2 引力场的里奇张量和标量曲率

定义1.1.1. 里奇张量 $R^{ab} := g_{cd}R^{cadb}$ ,  $R^{a*b} := g_{cd}R^{ca(*db)} \equiv 0$ , 标量曲率 $R := g_{ab}R^{ab} = R_{ab}{}^{ab}$

推论1.1.1.  $R^{abcd} + R^{adbc} + R^{acdb} = 0 \Rightarrow R^{a*b} = 0$

证明：  $R^{abcd} + R^{adbc} + R^{acdb} = 0$

$$\Rightarrow \varepsilon_{abcd}(R^{abcd} + R^{adbc} + R^{acdb}) = 0$$

$$\Rightarrow \varepsilon_{abcd}R^{abcd} + \varepsilon_{edbc}R^{adbc} + \varepsilon_{ecdb}R^{acdb} = 0$$

$$\Rightarrow 3\varepsilon_{abcd}R^{abcd} = 0$$

$$\Rightarrow g_{cd}R^{ca(*db)} = 0$$

$$\Rightarrow R^{a*b} = 0$$

□

#### 1.1.3 引力场的外尔张量

定义1.1.2.  $C^{abcd} := R^{abcd} + \frac{1}{2}g^{a[d}R^{c]b} + \frac{1}{2}g^{b[c}R^{d]a} + \frac{1}{6}g^{a[c}g^{d]b}R$

外尔张量的对称性质：

$$\text{反对称性： } C^{abcd} = -C^{bacd}, C^{abcd} = -C^{abdc} \quad (8.4)$$

$$\text{对称性： } C^{abcd} = C^{cdab} \quad (8.5)$$

$$\text{轮换对称性： } C^{abcd} + C^{adbc} + C^{acdb} = 0 \quad (8.6)$$

推论1.1.2.  $R^{abcd} = C^{abcd} - \frac{1}{2}g^{a[d}R^{c]b} - \frac{1}{2}g^{b[c}R^{d]a} - \frac{1}{6}g^{a[c}g^{d]b}R$

推论1.1.3.  $C^{ab} = g_{cd}C^{cadb} = 0, C^{a*b} = g_{cd}C^{ca(*db)} = 0$

## 1.2 引力场物理量的Yang-Mills描述<sup>[6]</sup>

### 1.2.1 引力场曲率张量的Yang-Mills描述

定义1.2.1. 引力场的YM曲率张量： $F^{ab\alpha\varsigma} := \frac{i}{2}\sigma_{\varsigma cd}^{\alpha\varsigma}R^{abcd}$

仿照电磁场情形的推理，有以下完全类似的结论。

$$\text{推论1.2.1. } \frac{1}{2}[R^{abcd} - \varsigma R^{ab(*cd)}] = \frac{i}{2}\sigma_{\varsigma\alpha\varsigma}^{cd}F^{ab\alpha\varsigma}$$

$$\text{推论1.2.2. } F^{ab\alpha\varsigma} = -\frac{i\varsigma}{2}\sigma_{\varsigma cd}^{\alpha\varsigma}R^{ab(*cd)}$$

$$\text{推论1.2.3. } \sigma_{\varsigma cd}^{\alpha\varsigma}[R^{abcd} + \varsigma R^{ab(*cd)}] = 0$$

$$\text{推论1.2.4. } F^{ab\alpha\zeta} = \frac{i}{2}\sigma_{\zeta cd}^{\alpha\zeta} \frac{1}{2}[R^{abcd} - \zeta R^{ab(*cd)}]$$

$$\text{推论1.2.5. } R^{abcd} - \zeta R^{ab(*cd)} = -\frac{1}{4}\sigma_{\zeta\alpha\zeta}^{cd} \sigma_{\zeta ef}^{\alpha\zeta} (R^{abef} - \zeta R^{ab(*ef)})$$

$$\text{推论1.2.6. } R^{abcd} = \frac{i}{2}(\sigma_{-\alpha'}^{cd} F^{ab\alpha'} + \sigma_{+\alpha}^{cd} F^{ab\alpha}), R^{ab(*cd)} = \frac{i}{2}(\sigma_{-\alpha'}^{cd} F^{ab\alpha'} - \sigma_{+\alpha}^{cd} F^{ab\alpha})$$

不同于电磁场情形，引力场还有以下不同的结论。

$$\text{推论1.2.7. } R^{ab} = -\frac{i}{2}(F^{\alpha'}\sigma_{-\alpha'} + F^{\alpha}\sigma_{+\alpha})^{ab}, 0 = R^{a*b} = -\frac{i}{2}(F^{\alpha'}\sigma_{-\alpha'} - F^{\alpha}\sigma_{+\alpha})^{ab}$$

$$\text{推论1.2.8. } R^{ab} = -i(F^{\alpha'}\sigma_{-\alpha'})^{ab} = -i(F^{\alpha}\sigma_{+\alpha})^{ab}, F^{\alpha'}\sigma_{-\alpha'} = F^{\alpha}\sigma_{+\alpha}$$

$$\text{推论1.2.9. } R^{ab} = -i(F^{\alpha\zeta}\sigma_{\zeta\alpha\zeta})^{ab}, F^{\alpha'\zeta}\sigma_{-\zeta\alpha'\zeta} = F^{\alpha\zeta}\sigma_{\zeta\alpha\zeta}$$

$$\text{推论1.2.10. } R = i\sigma_{\zeta\alpha\zeta}^{ab} F_{ab}^{\alpha\zeta}$$

### 1.2.2 引力场外尔张量的Yang-Mills描述

$$\text{定义1.2.2. 引力场的YM外尔张量: } C^{ab\alpha\zeta} := \frac{i}{2}\sigma_{\zeta cd}^{\alpha\zeta} C^{abcd}$$

仿照曲率张量情形的推理，有以下完全类似的结论。

$$\text{推论1.2.11. } \frac{1}{2}[C^{abcd} - \zeta C^{ab(*cd)}] = \frac{i}{2}\sigma_{\zeta\alpha\zeta}^{cd} C^{ab\alpha\zeta}$$

$$\text{推论1.2.12. } C^{ab\alpha\zeta} = -\frac{i\zeta}{2}\sigma_{\zeta cd}^{\alpha\zeta} C^{ab(*cd)}$$

$$\text{推论1.2.13. } \sigma_{\zeta cd}^{\alpha\zeta} [C^{abcd} + \zeta C^{ab(*cd)}] = 0$$

$$\text{推论1.2.14. } C^{ab\alpha\zeta} = \frac{i}{2}\sigma_{\zeta cd}^{\alpha\zeta} \frac{1}{2}[C^{abcd} - \zeta C^{ab(*cd)}]$$

$$\text{推论1.2.15. } C^{abcd} - \zeta C^{ab(*cd)} = -\frac{1}{4}\sigma_{\zeta\alpha\zeta}^{cd} \sigma_{\zeta ef}^{\alpha\zeta} (C^{abef} - \zeta C^{ab(*ef)})$$

$$\text{推论1.2.16. } C^{abcd} = \frac{i}{2}(\sigma_{-\alpha'}^{cd} C^{aba'} + \sigma_{+\alpha}^{cd} C^{aba}), C^{ab(*cd)} = \frac{i}{2}(\sigma_{-\alpha'}^{cd} C^{aba'} - \sigma_{+\alpha}^{cd} C^{aba})$$

不同于曲率张量情形，外尔张量还有以下不同的结论。

$$\text{推论1.2.17. } 0 = C^{ab} = -\frac{i}{2}(F^{\alpha'}\sigma_{-\alpha'} + F^{\alpha}\sigma_{+\alpha})^{ab}, 0 = C^{a*b} = -\frac{i}{2}(F^{\alpha'}\sigma_{-\alpha'} - F^{\alpha}\sigma_{+\alpha})^{ab}$$

$$\text{推论1.2.18. } C^{\alpha'}\sigma_{-\alpha'} = C^{\alpha}\sigma_{+\alpha} = 0, C^{\alpha'\zeta}\sigma_{-\zeta\alpha'\zeta} = C^{\alpha\zeta}\sigma_{\zeta\alpha\zeta} = 0$$

$$\text{推论1.2.19. } C = i\sigma_{\zeta\alpha\zeta}^{ab} C_{ab}^{\alpha\zeta} = 0$$

曲率张量和外尔张量之间的联系：

推论1.2.20.

$$R^{abcd} = C^{abcd} - \frac{1}{2}g^{a[d}R^{c]b} - \frac{1}{2}g^{b[c}R^{d]a} - \frac{1}{6}g^{a[c}g^{d]b}R \Rightarrow F^{ab\alpha\zeta} = C^{ab\alpha\zeta} + \frac{i}{2}\sigma_{\zeta}^{\alpha\zeta a} R^{cb} - \frac{i}{2}\sigma_{\zeta}^{\alpha\zeta b} R^{ca} - \frac{i}{6}\sigma_{\zeta}^{\alpha\zeta ab} R$$

$$\text{推论1.2.21. } C^{ab\alpha\zeta} = F^{ab\alpha\zeta} - \frac{1}{2}(\sigma_{\zeta}^{\alpha\zeta c[a} \sigma_{\zeta\beta\zeta}^{b]d} + \frac{1}{3}\sigma_{\zeta}^{\alpha\zeta ab} \sigma_{\zeta\beta\zeta}^{cd}) F_{cd}^{\beta\zeta}$$

## 1.3 引力场曲率张量的Ashtekar规范表述<sup>[36]</sup>

### 1.3.1 准备

$X, Y$  是正交标架下的实四维矢量或张量。

$$\text{引理1.3.1. } X_{a'\zeta}^* = \eta_{a'\zeta}^{a\zeta} X_{a\zeta}, X_{a'\zeta b'\zeta}^* = \eta_{a'\zeta}^{a\zeta} \eta_{b'\zeta}^{b\zeta} X_{a\zeta b\zeta}, X_{a'\zeta b'\zeta c'\zeta}^* = \eta_{a'\zeta}^{a\zeta} \eta_{b'\zeta}^{b\zeta} \eta_{c'\zeta}^{c\zeta} X_{a\zeta b\zeta c\zeta} \cdots$$

$$\text{引理1.3.2. } X_{a'\zeta}^* Y^{*a'\zeta} = X_{a\zeta} Y^{a\zeta}, X_{a'\zeta b'\zeta}^* Y^{*a'\zeta b'\zeta} = X_{a\zeta b\zeta} Y^{a\zeta b\zeta}, X_{a'\zeta b'\zeta c'\zeta}^* Y^{*a'\zeta b'\zeta c'\zeta} = X_{a\zeta b\zeta c\zeta} Y^{a\zeta b\zeta c\zeta}, \cdots$$

## 1.3.2 引力场曲率张量的Ashtekar规范表述 [36]

$$\text{引力场曲率张量 } R_{uv}{}^{cd} = \partial_u \omega_v{}^{cd} - \partial_v \omega_u{}^{cd} + \omega_{[u}{}^{ce} \omega_{v]e}{}^d \quad (8.7)$$

定义1.3.1. 引入Ashtekar变量 [36]  $A_u^{\alpha\zeta} := \frac{i}{2} \sigma_{\zeta cd}^{\alpha\zeta} \omega_u{}^{cd}$

推论1.3.1.  $[A_u^{\alpha\zeta}]^* = A_u^{*\alpha'\zeta} = \eta_u^u A_u^{\alpha'\zeta}$

证明:  $[A_u^{\alpha\zeta}]^* = \frac{i}{2} \sigma_{\zeta c'd'}^{\alpha'\zeta} \omega_u^{*c'd'} = \eta_u^u \frac{i}{2} \sigma_{-\zeta cd}^{\alpha'\zeta} \omega_u{}^{cd} = \eta_u^u A_u^{\alpha'\zeta}$  □

推论1.3.2.  $[F_{uv}^{\alpha\zeta}]^* = F_{u'v'}^{*\alpha'\zeta} = \eta_u^u \eta_{v'}^{v'} F_{uv}^{\alpha'\zeta}$

证明:  $[F_{uv}^{\alpha\zeta}]^* = F_{u'v'}^{*\alpha'\zeta} = \frac{i}{2} \sigma_{\zeta c'd'}^{\alpha'\zeta} R_{u'v'}^{*c'd'} = \frac{i}{2} \eta_u^u \eta_{v'}^{v'} \sigma_{-\zeta cd}^{\alpha'\zeta} R_{uv}{}^{cd} = \eta_u^u \eta_{v'}^{v'} F_{uv}^{\alpha'\zeta}$  □

推论1.3.3.  $\omega_{[u}{}^{ce} \omega_{v]e}{}^d = -\frac{i}{2} (\varepsilon^{\alpha\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} \sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'} + \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta})$

证明:  $\omega_{[u}{}^{ce} \omega_{v]e}{}^d = \omega_u^{ce} \omega_{ve}{}^d - \omega_v^{ce} \omega_{ue}{}^d$

$$\Leftrightarrow \omega_{[u}{}^{ce} \omega_{v]e}{}^d = \delta_{ef} \frac{i}{2} (\sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} A_u^{\alpha\zeta'} + \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} A_u^{\alpha\zeta}) \frac{i}{2} (\sigma_{-\zeta\beta\zeta'}^{\beta\zeta'} A_v^{\beta\zeta'} + \sigma_{\zeta\beta\zeta}^{\beta\zeta} A_v^{\beta\zeta})$$

$$- \delta_{ef} \frac{i}{2} (\sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} A_v^{\alpha\zeta'} + \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} A_v^{\alpha\zeta}) \frac{i}{2} (\sigma_{-\zeta\beta\zeta'}^{\beta\zeta'} A_u^{\beta\zeta'} + \sigma_{\zeta\beta\zeta}^{\beta\zeta} A_u^{\beta\zeta})$$

$$\Leftrightarrow \omega_{[u}{}^{ce} \omega_{v]e}{}^d = -\frac{1}{4} \delta_{ef} (\sigma_{-\zeta[\alpha\zeta'}^{\alpha\zeta} \sigma_{-\zeta\beta\zeta'}^{\beta\zeta'} A_u^{\alpha\zeta'} A_v^{\beta\zeta'} + \sigma_{\zeta[\alpha\zeta}^{\alpha\zeta} \sigma_{-\zeta\beta\zeta'}^{\beta\zeta'} A_u^{\alpha\zeta} A_v^{\beta\zeta'})$$

$$+ \sigma_{-\zeta[\alpha\zeta'}^{\alpha\zeta} \sigma_{\zeta\beta\zeta}^{\beta\zeta} A_u^{\alpha\zeta'} A_v^{\beta\zeta} + \sigma_{\zeta[\alpha\zeta}^{\alpha\zeta} \sigma_{-\zeta\beta\zeta'}^{\beta\zeta'} A_u^{\alpha\zeta} A_v^{\beta\zeta'})$$

$$\Leftrightarrow \omega_{[u}{}^{ce} \omega_{v]e}{}^d = -\frac{1}{4} (2i \varepsilon^{\alpha\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} \sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'} + 0 + 0 + 2i \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta})$$

$$\Leftrightarrow \omega_{[u}{}^{ce} \omega_{v]e}{}^d = -\frac{i}{2} (\varepsilon^{\alpha\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} \sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'} + \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta})$$

$$\Leftrightarrow \omega_{[u}{}^{ce} \omega_{v]e}{}^d = -\frac{i}{2} (\varepsilon^{\alpha\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} \sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'} + \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta}) \quad \square$$

推论1.3.4.  $R_{uv}{}^{cd} = \partial_u \omega_v{}^{cd} - \partial_v \omega_u{}^{cd} + \omega_{[u}{}^{ce} \omega_{v]e}{}^d \Leftrightarrow \begin{cases} F_{uv}^{\alpha\zeta} = \partial_u A_v^{\alpha\zeta} - \partial_v A_u^{\alpha\zeta} - \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta} \\ F_{uv}^{\alpha'\zeta} = \partial_u A_v^{\alpha'\zeta} - \partial_v A_u^{\alpha'\zeta} - \varepsilon^{\alpha'\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'} \end{cases}$

证明:  $R_{uv}{}^{cd} = \partial_u \omega_v{}^{cd} - \partial_v \omega_u{}^{cd} + \omega_{[u}{}^{ce} \omega_{v]e}{}^d$

$$\Leftrightarrow R_{uv}{}^{cd} = \frac{i}{2} (\sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} \partial_u A_v^{\alpha\zeta'} + \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} \partial_u A_v^{\alpha\zeta}) - \frac{i}{2} (\sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} \partial_v A_u^{\alpha\zeta'} + \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} \partial_v A_u^{\alpha\zeta})$$

$$- \frac{i}{2} (\varepsilon^{\alpha\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} \sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'} + \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta})$$

$$\Leftrightarrow \frac{i}{2} (\sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} F_{uv}^{\alpha\zeta'} + \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} F_{uv}^{\alpha\zeta}) = \frac{i}{2} \sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} (\partial_{[u} A_{v]}^{\alpha\zeta'} - \varepsilon^{\alpha\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'}) + \frac{i}{2} \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} (\partial_{[u} A_{v]}^{\alpha\zeta} - \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta})$$

$$\Leftrightarrow \begin{cases} F_{uv}^{\alpha\zeta} = \partial_u A_v^{\alpha\zeta} - \partial_v A_u^{\alpha\zeta} - \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta} \\ F_{uv}^{\alpha'\zeta} = \partial_u A_v^{\alpha'\zeta} - \partial_v A_u^{\alpha'\zeta} - \varepsilon^{\alpha'\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'} \end{cases} \quad \square$$

推论1.3.5.  $\frac{i}{2} \sigma_{\zeta cd}^{\alpha\zeta} \omega_{[u}{}^{ce} \omega_{v]e}{}^d = -\varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta}$

证明:  $\omega_{[u}{}^{ce} \omega_{v]e}{}^d = -\frac{i}{2} (\varepsilon^{\alpha\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} \sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'} + \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta})$

$$\Rightarrow \frac{i}{2} \sigma_{\zeta cd}^{\alpha\zeta} \omega_{[u}{}^{ce} \omega_{v]e}{}^d = \frac{1}{4} \sigma_{\zeta cd}^{\alpha\zeta} (\varepsilon^{\alpha\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} \sigma_{-\zeta\alpha\zeta'}^{\alpha\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'} + \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta})$$

$$\Leftrightarrow \frac{i}{2} \sigma_{\zeta cd}^{\alpha\zeta} \omega_{[u}{}^{ce} \omega_{v]e}{}^d = 0 - \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} \delta^{\rho\zeta}{}_{\zeta} A_u^{\alpha\zeta} A_v^{\beta\zeta}$$

$$\Leftrightarrow \frac{i}{2} \sigma_{\zeta cd}^{\alpha\zeta} \omega_{[u}{}^{ce} \omega_{v]e}{}^d = -\varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta} \quad \square$$

推论1.3.6.  $F_{uv}^{\alpha\zeta} = \partial_u A_v^{\alpha\zeta} - \partial_v A_u^{\alpha\zeta} - \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta} \Leftrightarrow F_{uv}^{\alpha'\zeta} = \partial_u A_v^{\alpha'\zeta} - \partial_v A_u^{\alpha'\zeta} - \varepsilon^{\alpha'\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'}$

证明:  $F_{uv}^{\alpha\zeta} = \partial_u A_v^{\alpha\zeta} - \partial_v A_u^{\alpha\zeta} - \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta}$

$$\Leftrightarrow [F_{uv}^{\alpha\zeta}]^* = F_{u'v'}^{*\alpha'\zeta} = \partial_{u'} A_{v'}^{*\alpha'\zeta} - \partial_{v'} A_{u'}^{*\alpha'\zeta} - \varepsilon^{\alpha'\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} A_{u'}^{\beta\zeta'} A_{v'}^{\gamma\zeta'}$$

$$\Leftrightarrow F_{u'v'}^{*\alpha'\zeta} = \eta_{u'}^u \eta_{v'}^{v'} (\partial_u A_v^{\alpha'\zeta} - \partial_v A_u^{\alpha'\zeta} - \varepsilon^{\alpha'\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'})$$

$$\Leftrightarrow \eta_{u'}^u \eta_{v'}^{v'} F_{u'v'}^{*\alpha'\zeta} = \eta_{u'}^u \eta_{v'}^{v'} (\partial_u A_v^{\alpha'\zeta} - \partial_v A_u^{\alpha'\zeta} - \varepsilon^{\alpha'\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'})$$

$$\Leftrightarrow F_{uv}^{\alpha'\zeta} = \partial_u A_v^{\alpha'\zeta} - \partial_v A_u^{\alpha'\zeta} - \varepsilon^{\alpha'\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'} \quad \square$$

推论1.3.7.  $R_{uv}{}^{cd} = \partial_u \omega_v{}^{cd} - \partial_v \omega_u{}^{cd} + \omega_{[u}{}^{ce} \omega_{v]e}{}^d \Leftrightarrow F_{uv}^{\alpha\zeta} = \partial_u A_v^{\alpha\zeta} - \partial_v A_u^{\alpha\zeta} - \varepsilon^{\alpha\zeta}{}_{\beta\zeta} \gamma_{\zeta}^{\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta}$

推论1.3.8.  $R_{uv}{}^{cd} = \partial_u \omega_v{}^{cd} - \partial_v \omega_u{}^{cd} + \omega_{[u}{}^{ce} \omega_{v]e}{}^d \Leftrightarrow F_{uv}^{\alpha'\zeta} = \partial_u A_v^{\alpha'\zeta} - \partial_v A_u^{\alpha'\zeta} - \varepsilon^{\alpha'\zeta}{}_{\beta\zeta'} \gamma_{\zeta'}^{\zeta} A_u^{\beta\zeta'} A_v^{\gamma\zeta'}$

## 1.4 引力场物理量的复张量描述

### 1.4.1 引力场的复张量

$$\text{定义1.4.1.} \quad \begin{cases} \text{曲率复张量 } \psi^{\alpha\epsilon\beta\kappa} := \frac{i}{2}\sigma_{\zeta ab}^{\alpha\epsilon} F^{ab\beta\kappa} = \frac{i}{2}\sigma_{\zeta ab}^{\alpha\epsilon} \frac{i}{2}\sigma_{\kappa cd}^{\beta\epsilon} R^{abcd} \\ \text{外尔复张量 } C^{\alpha\epsilon\beta\kappa} := \frac{i}{2}\sigma_{\zeta ab}^{\alpha\epsilon} C^{ab\beta\kappa} = \frac{i}{2}\sigma_{\zeta ab}^{\alpha\epsilon} \frac{i}{2}\sigma_{\kappa cd}^{\beta\epsilon} C^{abcd} \end{cases}$$

$$\text{推论1.4.1.} \quad \psi^{\alpha\epsilon\beta\kappa} = C^{\alpha\epsilon\beta\kappa} - \frac{1}{2}\zeta\kappa\sigma_{\zeta ac}^{\alpha\epsilon}\sigma_{\kappa}^{\beta\epsilon c} R^{ab} + \frac{1}{3}\zeta\kappa\delta_{\zeta\kappa}\delta^{\alpha\epsilon\beta\kappa} R$$

$$\text{证明: } \psi^{\alpha\epsilon\beta\kappa} := \frac{1}{2}\sigma_{\zeta ab}^{\alpha\epsilon} F^{ab\beta\kappa}$$

$$\Rightarrow \psi^{\alpha\epsilon\beta\kappa} = \frac{i}{2}\sigma_{\zeta ab}^{\alpha\epsilon} (C^{ab\beta\kappa} + \frac{i}{2}\sigma_{\kappa}^{\beta\epsilon a} R^{cb} - \frac{i}{2}\sigma_{\kappa}^{\beta\epsilon b} R^{ca} - \frac{i}{6}\sigma_{\kappa}^{\beta\epsilon ab} R)$$

$$\Rightarrow \psi^{\alpha\epsilon\beta\kappa} = C^{\alpha\epsilon\beta\kappa} + \frac{1}{2}\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\kappa}^{\beta\epsilon b} R^{ca} - \frac{1}{3}\delta_{\zeta\kappa}\delta^{\alpha\epsilon\beta\kappa} R$$

$$\Rightarrow \psi^{\alpha\epsilon\beta\kappa} = C^{\alpha\epsilon\beta\kappa} + \frac{1}{2}\sigma_{\zeta ac}^{\alpha\epsilon}\sigma_{\kappa}^{\beta\epsilon c} R^{ab} - \frac{1}{3}\delta_{\zeta\kappa}\delta^{\alpha\epsilon\beta\kappa} R \quad \square$$

$$\text{推论1.4.2.} \quad \psi^{\alpha\epsilon\beta\zeta} = C^{\alpha\epsilon\beta\zeta} + \frac{1}{2}\sigma_{\zeta ac}^{\alpha\epsilon}\sigma_{\zeta}^{\beta\epsilon c} R^{ab} - \frac{1}{3}\delta^{\alpha\epsilon\beta\zeta} R$$

$$\text{推论1.4.3.} \quad \psi^{\alpha\epsilon\beta\zeta} = C^{\alpha\epsilon\beta\zeta} + \frac{1}{6}\delta^{\alpha\epsilon\beta\zeta} R$$

$$\text{证明: } \psi^{\alpha\epsilon\beta\zeta} = C^{\alpha\epsilon\beta\zeta} + \frac{1}{2}\sigma_{\zeta ac}^{\alpha\epsilon}\sigma_{\zeta}^{\beta\epsilon c} R^{ab} - \frac{1}{3}\delta^{\alpha\epsilon\beta\zeta} R$$

$$\Leftrightarrow \psi^{\alpha\epsilon\beta\zeta} = C^{\alpha\epsilon\beta\zeta} + \frac{1}{2}(\delta^{\alpha\epsilon\beta\zeta}\delta_{ab} + i\varepsilon_{\alpha\epsilon\beta\zeta\gamma\delta}\sigma_{\zeta ab}^{\gamma\delta}) R^{ab} - \frac{1}{3}\delta^{\alpha\epsilon\beta\zeta} R$$

$$\Leftrightarrow \psi^{\alpha\epsilon\beta\zeta} = C^{\alpha\epsilon\beta\zeta} + \frac{1}{6}\delta^{\alpha\epsilon\beta\zeta} R \quad \square$$

$$\text{推论1.4.4.} \quad \frac{1}{2}(F^{ab\beta\kappa} - \zeta * F^{ab\beta\kappa}) = \frac{i}{2}\sigma_{\zeta\alpha\epsilon}^{ab} \psi^{\alpha\epsilon\beta\kappa}$$

$$\text{推论1.4.5.} \quad \psi^{\alpha\epsilon\beta\kappa} = \frac{i}{2}\sigma_{\zeta ab}^{\alpha\epsilon} \frac{1}{2}(F^{ab\beta\kappa} - \zeta * F^{ab\beta\kappa})$$

$$\text{推论1.4.6.} \quad \psi^{\alpha\epsilon\beta\kappa} = -\frac{i}{2}\zeta\sigma_{\zeta ab}^{\alpha\epsilon} * F^{ab\beta\kappa}$$

$$\text{推论1.4.7.} \quad \sigma_{\zeta ab}^{\alpha\epsilon}(F^{ab\beta\kappa} + \zeta * F^{ab\beta\kappa}) = 0$$

$$\text{推论1.4.8.} \quad F^{ab\beta\kappa} - \zeta * F^{ab\beta\kappa} = -\frac{1}{4}\sigma_{\zeta\alpha\epsilon}^{ab}\sigma_{\zeta cd}^{\alpha\epsilon}(F^{cd\beta\kappa} - \zeta * F^{cd\beta\kappa})$$

$$\text{推论1.4.9.} \quad F^{ab\beta\kappa} = \frac{i}{2}(\sigma_{-\alpha'}^{ab}\psi^{\alpha'\beta\kappa} + \sigma_{+\alpha}^{ab}\psi^{\alpha\beta\kappa}), *F^{ab\beta\kappa} = \frac{i}{2}(\sigma_{-\alpha'}^{ab}\psi^{\alpha'\beta\kappa} - \sigma_{+\alpha}^{ab}\psi^{\alpha\beta\kappa})$$

$$\text{推论1.4.10.} \quad \psi^{\alpha\epsilon\beta\kappa} = -\frac{1}{4}\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\kappa cd}^{\beta\epsilon} R^{abcd} = \frac{1}{4}\kappa\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\kappa cd}^{\beta\epsilon} R^{ab(*cd)} = \frac{1}{4}\zeta\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\kappa cd}^{\beta\epsilon} R^{(*ab)(*cd)} = -\frac{1}{4}\zeta\kappa\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\kappa cd}^{\beta\epsilon} R^{(*ab)cd}$$

### 1.4.2 引力场的复张量 $\psi^{\alpha\epsilon\beta\kappa}$ 的性质

$$\text{推论1.4.11.} \quad \psi^{\alpha\epsilon\beta\kappa} = \psi^{\beta\kappa\alpha\epsilon}$$

$$\text{证明: } R^{abcd} = R^{cdab}$$

$$\Rightarrow -\frac{1}{4}\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\kappa cd}^{\beta\epsilon} R^{abcd} = -\frac{1}{4}\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\kappa cd}^{\beta\epsilon} R^{cdab}$$

$$\Rightarrow \psi^{\alpha\epsilon\beta\kappa} = \psi^{\beta\kappa\alpha\epsilon} \quad \square$$

$$\text{推论1.4.12.} \quad \psi^{x_\zeta x_\zeta} + \psi^{y_\zeta y_\zeta} + \psi^{z_\zeta z_\zeta} = \frac{1}{2}R$$

$$\text{证明: } \sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\zeta\alpha\epsilon cd} = -(S_{abcd} - \zeta\varepsilon_{abcd})$$

$$\Rightarrow -\frac{1}{4}\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\zeta\alpha\epsilon cd} R^{abcd} = \frac{1}{4}(S_{abcd} - \zeta\varepsilon_{abcd}) R^{abcd}$$

$$\Rightarrow -\frac{1}{4}\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\zeta\alpha\epsilon cd} R^{abcd} = \frac{1}{2}(R_{ab}{}^{ab} - \zeta R_{*ab}{}^{ab})$$

$$\Rightarrow \psi^{x_\zeta x_\zeta} + \psi^{y_\zeta y_\zeta} + \psi^{z_\zeta z_\zeta} = \frac{1}{2}(R_{ab}{}^{ab} - \zeta R_{*ab}{}^{ab})$$

$$\Rightarrow \psi^{x_\zeta x_\zeta} + \psi^{y_\zeta y_\zeta} + \psi^{z_\zeta z_\zeta} = \frac{1}{2}R \quad \square$$

$$\text{推论1.4.13.} \quad C^{\alpha\epsilon\beta\zeta} = \psi^{\alpha\epsilon\beta\zeta} - \frac{1}{3}\delta^{\alpha\epsilon\beta\zeta}\psi^{\gamma\zeta}{}_{\gamma\zeta}$$

$$\text{推论1.4.14.} \quad \psi^{\alpha'\beta'\kappa} = (\psi^{\alpha\epsilon\beta\kappa})^*$$

$$\text{证明: } \psi^{\alpha\epsilon\beta\kappa} = -\frac{1}{4}\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\kappa cd}^{\beta\epsilon} R^{abcd}$$

$$\Leftrightarrow (\psi^{\alpha\epsilon\beta\kappa})^* = -\frac{1}{4}(\sigma_{\zeta ab}^{\alpha\epsilon}\sigma_{\kappa cd}^{\beta\epsilon} R^{abcd})^* = -\frac{1}{4}\sigma_{\zeta a'b'}^{\alpha'\epsilon'}\sigma_{\kappa c'd'}^{\beta'\epsilon'}\eta_a^{a'}\eta_b^{b'}\eta_c^{c'}\eta_d^{d'} R^{abcd}$$

$$\Leftrightarrow (\psi^{\alpha\epsilon\beta\kappa})^* = -\frac{1}{4}\sigma_{-\zeta ab}^{\alpha'\epsilon'}\sigma_{-\kappa cd}^{\beta'\epsilon'} R^{abcd}$$

$$\Leftrightarrow \psi^{\alpha'\beta'\kappa} = (\psi^{\alpha\epsilon\beta\kappa})^* \quad \square$$

### 1.4.3 引力场曲率张量的展开

$$\text{推论1.4.15. } R^{abcd} = -\frac{1}{4}(\sigma_{-\alpha'}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} + \sigma_{+\alpha}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} + \sigma_{-\alpha'}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha'\beta} + \sigma_{+\alpha}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha\beta})$$

$$\text{证明: } R^{abcd} = \frac{i}{2}(\sigma_{-\beta'}^{\alpha c}F^{ab\beta'} + \sigma_{+\beta}^{\alpha c}F^{ab\beta})$$

$$\Leftrightarrow R^{abcd} = -\frac{1}{4}[\sigma_{-\beta'}^{\alpha c}(\sigma_{-\alpha'}^{\alpha b}\psi^{\alpha'\beta'} + \sigma_{+\alpha}^{\alpha b}\psi^{\alpha\beta'}) + \sigma_{+\beta}^{\alpha c}(\sigma_{-\alpha'}^{\alpha b}\psi^{\alpha'\beta} + \sigma_{+\alpha}^{\alpha b}\psi^{\alpha\beta})]$$

$$\Leftrightarrow R^{abcd} = -\frac{1}{4}(\sigma_{-\alpha'}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} + \sigma_{+\alpha}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} + \sigma_{-\alpha'}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha'\beta} + \sigma_{+\alpha}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha\beta}) \quad \square$$

$$\text{推论1.4.16. } R^{ab(*cd)} = -\frac{1}{4}(\sigma_{-\alpha'}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} + \sigma_{+\alpha}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} - \sigma_{-\alpha'}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha'\beta} - \sigma_{+\alpha}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha\beta})$$

$$\text{推论1.4.17. } R^{(*ab)cd} = -\frac{1}{4}(\sigma_{-\alpha'}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} - \sigma_{+\alpha}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} + \sigma_{-\alpha'}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha'\beta} - \sigma_{+\alpha}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha\beta})$$

$$\text{推论1.4.18. } R^{(*ab)(*cd)} = -\frac{1}{4}(\sigma_{-\alpha'}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} - \sigma_{+\alpha}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} - \sigma_{-\alpha'}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha'\beta} + \sigma_{+\alpha}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha\beta})$$

$$\text{推论1.4.19. } R^{ab} = \frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'} = \frac{1}{4}\delta^{ab}R - \delta_{cd}S^{ac}{}_{AB}S^{db}{}_{C'D'}\psi^{ABC'D'}$$

$$\text{证明: } R^{ab} = \frac{1}{4}(\sigma_{-\alpha'}^{\alpha b}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'} + \sigma_{+\alpha}\sigma_{-\beta'}^{\alpha c}\psi^{\alpha\beta'} + \sigma_{-\alpha'}^{\alpha b}\sigma_{+\beta}^{\alpha c}\psi^{\alpha'\beta} + \sigma_{+\alpha}\sigma_{+\beta}^{\alpha c}\psi^{\alpha\beta})^{ab}$$

$$\Leftrightarrow R^{ab} = \frac{1}{8}(\{\sigma_{-\alpha'}, \sigma_{-\beta'}\}\psi^{\alpha'\beta'} + 2\{\sigma_{+\alpha}, \sigma_{-\beta'}\}\psi^{\alpha\beta'} + \{\sigma_{+\alpha}, \sigma_{+\beta}\}\psi^{\alpha\beta})^{ab}$$

$$\Leftrightarrow R^{ab} = \frac{1}{8}(R + 4\sigma_{+\alpha}\sigma_{-\beta'}\psi^{\alpha\beta'} + R)^{ab}$$

$$\Leftrightarrow R^{ab} = \frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'} = \frac{1}{4}\delta^{ab}R - \delta_{cd}S^{ac}{}_{AB}S^{db}{}_{C'D'}\psi^{ABC'D'} \quad \square$$

$$\text{推论1.4.20. } \frac{1}{4}(\sigma_{+}^{\alpha}\sigma_{-}^{\beta'})_{ab}(\sigma_{+\rho}\sigma_{-\sigma'})^{ab} = \delta^{\alpha\rho}\delta^{\beta'\sigma'}$$

$$\text{推论1.4.21. } \psi^{\alpha\beta\zeta} = \frac{1}{2}\sigma_{\zeta ac}^{\alpha\zeta}\sigma_{-\zeta b}^{\beta'\zeta}R^{ab} = \frac{1}{2}(\sigma_{\zeta}^{\alpha\zeta}\sigma_{-\zeta}^{\beta'\zeta})_{ab}R^{ab}$$

更一般的证明，不依赖各种量的定义

$$\text{推论1.4.22. } \psi^{\alpha\beta'} = \frac{1}{2}(\sigma_{+}^{\alpha}\sigma_{-}^{\beta'})_{ab}R^{ab} \Leftrightarrow R^{ab} = \frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'}$$

$$\text{证明: } \psi^{\alpha\beta'} = \frac{1}{2}(\sigma_{+}^{\alpha}\sigma_{-}^{\beta'})_{ab}R^{ab}$$

$$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'} = \frac{1}{4}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}(\sigma_{+}^{\alpha}\sigma_{-}^{\beta'})_{cd}R^{cd}$$

$$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'} = \frac{1}{4}(\sigma_{+\alpha}^{ae}\sigma_{-\beta'}^{e'b})(\sigma_{+c}^{\alpha}\sigma_{-d}^{\beta'})R^{cd}$$

$$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^a{}_b\psi^{\alpha\beta'} = \frac{1}{4}(\sigma_{+\alpha}^{ae}\sigma_{+}^{\alpha cf})(\sigma_{-\beta'}^{e'b}\sigma_{-}^{\beta' f d})R_c{}^d$$

$$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^a{}_b\psi^{\alpha\beta'} = \frac{1}{4}(S^{aecf} - \varepsilon^{aecf})(S_{ebfd} + \varepsilon_{ebfd})R_c{}^d$$

$$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^a{}_b\psi^{\alpha\beta'} = \frac{1}{4}(2\delta^{ac}\delta_{bd} + 2\delta^a{}_d\delta_b{}^c - \delta^a{}_b\delta^c{}_d)R_c{}^d$$

$$\Leftrightarrow \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^a{}_b\psi^{\alpha\beta'} = \frac{1}{4}(4R^a{}_b - \delta^a{}_b R)$$

$$\Leftrightarrow R^{ab} = \frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'} \quad \square$$

$$\text{定理1.4.1. } R^{ab}{}_{;b} \equiv \frac{1}{2}R^{;a} \Leftrightarrow (\sigma_{+\alpha}\sigma_{-\beta'})^{ab}D_b\psi^{\alpha\beta'} \equiv \frac{1}{2}R^{;a}$$

$$\text{证明: } R^{ab}{}_{;b} \equiv \frac{1}{2}R^{;a}$$

$$\Leftrightarrow [\frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}\psi^{\alpha\beta'}]_{;b} \equiv \frac{1}{2}R^{;a}$$

$$\Leftrightarrow (\sigma_{+\alpha}\sigma_{-\beta'})^{ab}D_b\psi^{\alpha\beta'} \equiv \frac{1}{2}R^{;a} \quad \square$$

### 1.4.4 引力场曲率张量的合成

$$\text{推论1.4.23. } R^{abcd} + R^{ab(*cd)} + R^{(*ab)cd} + R^{(*ab)(*cd)} = i\sigma_{-\alpha'}^{\alpha b}i\sigma_{-\beta'}^{\alpha c}\psi^{\alpha'\beta'}$$

$$\text{推论1.4.24. } R^{abcd} + R^{ab(*cd)} - R^{(*ab)cd} - R^{(*ab)(*cd)} = i\sigma_{+\alpha}^{\alpha b}i\sigma_{-\beta'}^{\alpha c}\psi^{\alpha\beta'}$$

$$\text{推论1.4.25. } R^{abcd} - R^{ab(*cd)} + R^{(*ab)cd} - R^{(*ab)(*cd)} = i\sigma_{-\alpha'}^{\alpha b}i\sigma_{+\beta}^{\alpha c}\psi^{\alpha'\beta}$$

$$\text{推论1.4.26. } R^{abcd} - R^{ab(*cd)} - R^{(*ab)cd} + R^{(*ab)(*cd)} = i\sigma_{+\alpha}^{\alpha b}i\sigma_{+\beta}^{\alpha c}\psi^{\alpha\beta}$$

统一描述：

$$\text{推论1.4.27. } R^{abcd} - \kappa R^{ab(*cd)} - \varsigma R^{(*ab)cd} + \zeta\kappa R^{(*ab)(*cd)} = i\sigma_{\zeta\alpha\varsigma}^{\alpha b}i\sigma_{\kappa\beta\zeta}^{\alpha c}\psi^{\alpha\beta\kappa}$$

### 1.4.5 引力场外尔张量的展开

$$\text{推论1.4.28. } 0 = C^{ab} = \frac{1}{4}\delta^{ab}C + \frac{1}{2}(\sigma_{+\alpha}\sigma_{-\beta'})^{ab}C^{\alpha\beta'} \Rightarrow C^{\alpha\beta'} = 0, C^{\alpha'\beta} = 0$$

$$\text{推论1.4.29. } C^{abcd} = -\frac{1}{4}(\sigma_{-\alpha'}\sigma_{-\beta'}^{cd}C^{\alpha'\beta'} + \sigma_{+\alpha}\sigma_{+\beta}^{cd}C^{\alpha\beta}), C^{\alpha'\beta'} = (C^{\alpha\beta})^*$$

$$\text{推论1.4.30. } C^{(*ab)cd} = C^{ab(*cd)}, C^{abcd} = C^{(*ab)(*cd)}$$

### 1.4.6 引力场源的复张量描述

$$\text{定义1.4.2. 引力场源旋量 } J_a^{\alpha\zeta} := \frac{i}{2}\sigma_{\zeta cd}^{\alpha\zeta}J_a^{cd}$$

仿照电磁场情形的推理，有以下完全类似的结论。

$$\text{推论1.4.31. } [J_a^{\alpha\zeta}]^* = J_{a'}^{*\alpha'\zeta} = \eta_{a'}^a J_a^{\alpha'\zeta}$$

$$\text{推论1.4.32. } \frac{1}{2}(J_a^{cd} - \zeta * J_a^{cd}) = \frac{i}{2}\sigma_{\zeta\alpha\zeta}^{cd}J_a^{\alpha\zeta}$$

$$\text{推论1.4.33. } J_a^{\alpha\zeta} = -\frac{i}{2}\zeta\sigma_{\zeta cd}^{\alpha\zeta} * J_a^{cd}$$

$$\text{推论1.4.34. } \sigma_{\zeta cd}^{\alpha\zeta}(J_a^{cd} + \zeta * J_a^{cd}) = 0$$

$$\text{推论1.4.35. } J_a^{\alpha\zeta} = \frac{i}{2}\zeta\sigma_{\zeta cd}^{\alpha\zeta}\frac{1}{2}(J_a^{cd} - \zeta * J_a^{cd})$$

$$\text{推论1.4.36. } J_a^{cd} - \zeta * J_a^{cd} = -\frac{1}{4}\sigma_{\zeta\alpha\zeta}^{cd}\sigma_{\zeta ef}^{\alpha\zeta}(J_a^{ef} - \zeta * J_a^{ef})$$

$$\text{推论1.4.37. } J_a^{cd} = \frac{i}{2}(\sigma_{-\alpha'}^{cd}J_a^{\alpha'} + \sigma_{+\alpha}^{cd}J_a^{\alpha}), *J_a^{cd} = \frac{i}{2}(\sigma_{-\alpha'}^{cd}J_a^{\alpha'} - \sigma_{+\alpha}^{cd}J_a^{\alpha})$$

$$\text{推论1.4.38. } J_a^{cd} = -J_a^{dc} \Leftrightarrow J_a^{cd} = \frac{i}{2}(\sigma_{-\alpha'}^{cd}J_a^{\alpha'} + \sigma_{+\alpha}^{cd}J_a^{\alpha\zeta})$$

## 1.5 引力场物理量的 $\frac{1}{2}$ -旋量描述 [1, 2]

### 1.5.1 引力场的曲率旋量 [1, 2]

定义1.5.1. 引力曲率旋量：

$$\psi^{A_\zeta B_\zeta C_\zeta D_\zeta} := \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha\zeta}^{A_\zeta B_\zeta}\frac{i\kappa}{\sqrt{2}}\sigma_{\beta\kappa}^{C_\zeta D_\zeta}\psi^{\alpha\zeta\beta\kappa} = \frac{i\zeta}{\sqrt{2}}S_{ab}^{A_\zeta B_\zeta}\frac{i\kappa}{\sqrt{2}}\sigma_{\beta\kappa}^{C_\zeta D_\zeta}F^{ab\beta\kappa} = \frac{i\zeta}{\sqrt{2}}S_{ab}^{A_\zeta B_\zeta}\frac{i\kappa}{\sqrt{2}}S_{cd}^{C_\zeta D_\zeta}R^{abcd}$$

$$\text{推论1.5.1. } \psi^{\alpha\zeta\beta\kappa} = \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\frac{i\kappa}{\sqrt{2}}\sigma_{C_\zeta D_\zeta}^{\beta\kappa}\psi^{A_\zeta B_\zeta C_\zeta D_\zeta}$$

$$\text{推论1.5.2. } \psi^{A_\zeta B_\zeta C_\zeta D_\zeta} = C^{A_\zeta B_\zeta C_\zeta D_\zeta} + \frac{1}{12}(\varepsilon^{A_\zeta C_\zeta}\varepsilon^{B_\zeta D_\zeta} - \varepsilon^{A_\zeta D_\zeta}\varepsilon^{C_\zeta B_\zeta})R$$

$$\text{证明: } \psi^{\alpha\zeta\beta\zeta} = C^{\alpha\zeta\beta\zeta} + \frac{1}{6}\delta^{\alpha\zeta\beta\zeta}R$$

$$\Leftrightarrow \psi^{A_\zeta B_\zeta C_\zeta D_\zeta} = C^{A_\zeta B_\zeta C_\zeta D_\zeta} - \frac{1}{12}\sigma_{\alpha\zeta}^{A_\zeta B_\zeta}\sigma_{\beta\zeta}^{C_\zeta D_\zeta}\delta^{\alpha\zeta\beta\zeta}R$$

$$\Leftrightarrow \psi^{A_\zeta B_\zeta C_\zeta D_\zeta} = C^{A_\zeta B_\zeta C_\zeta D_\zeta} - \frac{1}{12}\sigma_{\alpha\zeta}^{A_\zeta B_\zeta}\sigma^{\alpha\zeta C_\zeta D_\zeta}R$$

$$\Leftrightarrow \psi^{A_\zeta B_\zeta C_\zeta D_\zeta} = C^{A_\zeta B_\zeta C_\zeta D_\zeta} + \frac{1}{12}(\varepsilon^{A_\zeta C_\zeta}\varepsilon^{B_\zeta D_\zeta} - \varepsilon^{A_\zeta D_\zeta}\varepsilon^{C_\zeta B_\zeta})R \quad \square$$

$$\text{推论1.5.3. } \psi^{\alpha\zeta\beta\kappa} = \psi^{\beta\kappa\alpha\zeta} \Leftrightarrow \psi^{A_\zeta B_\zeta C_\zeta D_\zeta} = \psi^{B_\zeta A_\zeta C_\zeta D_\zeta}, \psi^{A_\zeta B_\zeta C_\zeta D_\zeta} = \psi^{A_\zeta B_\zeta D_\zeta C_\zeta}, \psi^{A_\zeta B_\zeta C_\zeta D_\zeta} = \psi^{C_\zeta D_\zeta A_\zeta B_\zeta}$$

$$\text{推论1.5.4. } \psi^{\alpha\zeta\beta\zeta} = \psi^{\beta\zeta\alpha\zeta} \Leftrightarrow \begin{cases} \psi^{1_\zeta 1_\zeta 1_\zeta 1_\zeta} \\ \psi^{1_\zeta 1_\zeta 1_\zeta 2_\zeta} = \psi^{1_\zeta 1_\zeta 2_\zeta 1_\zeta} = \psi^{1_\zeta 2_\zeta 1_\zeta 1_\zeta} = \psi^{2_\zeta 1_\zeta 1_\zeta 1_\zeta} \\ \psi^{1_\zeta 1_\zeta 2_\zeta 2_\zeta} = \psi^{2_\zeta 2_\zeta 1_\zeta 1_\zeta}, \psi^{1_\zeta 2_\zeta 1_\zeta 2_\zeta} = \psi^{1_\zeta 2_\zeta 2_\zeta 1_\zeta} = \psi^{2_\zeta 1_\zeta 1_\zeta 2_\zeta} = \psi^{2_\zeta 1_\zeta 2_\zeta 1_\zeta} \\ \psi^{1_\zeta 2_\zeta 2_\zeta 2_\zeta} = \psi^{2_\zeta 1_\zeta 2_\zeta 2_\zeta} = \psi^{2_\zeta 2_\zeta 1_\zeta 2_\zeta} = \psi^{2_\zeta 2_\zeta 2_\zeta 1_\zeta} \\ \psi^{2_\zeta 2_\zeta 2_\zeta 2_\zeta} \end{cases}$$

$$\text{推论1.5.5. } \psi^{A_\zeta B_\zeta}_{A_\zeta B_\zeta} = (-\zeta)\varepsilon_{A_\zeta C_\zeta}(-\zeta)\varepsilon_{B_\zeta D_\zeta}\psi^{A_\zeta B_\zeta C_\zeta D_\zeta} = 2(\psi^{1_\zeta 2_\zeta 1_\zeta 2_\zeta} - \psi^{1_\zeta 1_\zeta 2_\zeta 2_\zeta})$$

$$\text{推论1.5.6. } \psi^{A_\zeta B_\zeta}_{A_\zeta B_\zeta} = \delta_{\alpha\zeta\beta\zeta}\psi^{\alpha\zeta\beta\zeta} = \frac{1}{2}R$$

证明:  $\psi^{A_\zeta B_\zeta C_\zeta D_\zeta} := \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \frac{i\kappa}{\sqrt{2}} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \psi^{\alpha_\zeta \beta_\zeta}$

$$\Rightarrow \psi^{A_\zeta B_\zeta}_{A_\zeta B_\zeta} = -\frac{1}{2} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{A_\zeta B_\zeta} \psi^{\alpha_\zeta \beta_\zeta}$$

$$\Rightarrow \psi^{A_\zeta B_\zeta}_{A_\zeta B_\zeta} = \delta_{\alpha_\zeta \beta_\zeta} \psi^{\alpha_\zeta \beta_\zeta} \quad \square$$

推论1.5.7.  $C^{A_\zeta B_\zeta C_\zeta D_\zeta} = \psi^{A_\zeta B_\zeta C_\zeta D_\zeta} - \frac{1}{6} (\varepsilon^{A_\zeta C_\zeta} \varepsilon^{B_\zeta D_\zeta} - \varepsilon^{A_\zeta D_\zeta} \varepsilon^{B_\zeta C_\zeta}) \psi^{E_\zeta F_\zeta}_{E_\zeta F_\zeta}$

推论1.5.8.  $\psi^{1_\zeta 2_\zeta 1_\zeta 2_\zeta} - \psi^{1_\zeta 1_\zeta 2_\zeta 2_\zeta} = \frac{1}{4} R$

推论1.5.9.  $\psi^{A_\zeta B_\zeta C_\zeta D_\zeta}$  为全对称旋量  $\Leftrightarrow \psi^{\alpha_\zeta \beta_\zeta}$  为无迹对称张量, 即  $\psi^{\alpha_\zeta \beta_\zeta} = \psi^{\beta_\zeta \alpha_\zeta}$ ,  $\psi^{x_\zeta x_\zeta} + \psi^{y_\zeta y_\zeta} + \psi^{z_\zeta z_\zeta} = 0$

### 1.5.2 引力场的YM曲率张量 $F_{ab}^{\alpha_\zeta}$ 的约束条件

$$\text{定理1.5.1. } \begin{cases} \psi^{\alpha_\zeta \beta_\zeta} = \psi^{\beta_\zeta \alpha_\zeta} \\ \psi^{x_\zeta x_\zeta} + \psi^{y_\zeta y_\zeta} + \psi^{z_\zeta z_\zeta} = \frac{1}{2} R \end{cases} \Leftrightarrow \begin{cases} F_{yz}^{y_\zeta} - \zeta F_{x\pi}^{y_\zeta} = F_{zx}^{x_\zeta} - \zeta F_{y\pi}^{x_\zeta} \\ F_{zx}^{z_\zeta} - \zeta F_{y\pi}^{z_\zeta} = F_{xy}^{y_\zeta} - \zeta F_{z\pi}^{y_\zeta} \\ F_{xy}^{x_\zeta} - \zeta F_{z\pi}^{x_\zeta} = F_{yz}^{y_\zeta} - \zeta F_{x\pi}^{z_\zeta} \\ F_{yz}^{x_\zeta} - \zeta F_{x\pi}^{x_\zeta} + F_{zx}^{y_\zeta} - \zeta F_{y\pi}^{y_\zeta} + F_{xy}^{z_\zeta} - \zeta F_{z\pi}^{z_\zeta} = R \end{cases}$$

证明:  $\psi^{x_\zeta y_\zeta} = \psi^{y_\zeta x_\zeta}$

$$\Leftrightarrow \sigma_{\zeta cd}^{x_\zeta} F^{cdy_\zeta} = \sigma_{\zeta cd}^{y_\zeta} F^{cdx_\zeta}$$

$$\Leftrightarrow F_{yz}^{y_\zeta} - \zeta F_{x\pi}^{y_\zeta} = F_{zx}^{x_\zeta} - \zeta F_{y\pi}^{x_\zeta} \quad \square$$

证明:  $\psi^{y_\zeta z_\zeta} = \psi^{z_\zeta y_\zeta}$

$$\Leftrightarrow \sigma_{\zeta cd}^{y_\zeta} F^{cdz_\zeta} = \sigma_{\zeta cd}^{z_\zeta} F^{cdy_\zeta}$$

$$\Leftrightarrow F_{zx}^{z_\zeta} - \zeta F_{y\pi}^{z_\zeta} = F_{xy}^{y_\zeta} - \zeta F_{z\pi}^{y_\zeta} \quad \square$$

证明:  $\psi^{z_\zeta x_\zeta} = \psi^{x_\zeta z_\zeta}$

$$\Leftrightarrow \sigma_{\zeta cd}^{z_\zeta} F^{cdx_\zeta} = \sigma_{\zeta cd}^{x_\zeta} F^{cdz_\zeta}$$

$$\Leftrightarrow F_{xy}^{x_\zeta} - \zeta F_{z\pi}^{x_\zeta} = F_{yz}^{y_\zeta} - \zeta F_{x\pi}^{z_\zeta} \quad \square$$

证明:  $\psi^{x_\zeta x_\zeta} + \psi^{y_\zeta y_\zeta} + \psi^{z_\zeta z_\zeta} = \frac{1}{2} R$

$$\Leftrightarrow \frac{i}{2} [\sigma_{\zeta cd}^{x_\zeta} F^{cdx_\zeta} + \sigma_{\zeta cd}^{y_\zeta} F^{cdy_\zeta} + \sigma_{\zeta cd}^{z_\zeta} F^{cdz_\zeta}] = \frac{1}{2} R$$

$$\Leftrightarrow F_{yz}^{x_\zeta} - \zeta F_{x\pi}^{x_\zeta} + F_{zx}^{y_\zeta} - \zeta F_{y\pi}^{y_\zeta} + F_{xy}^{z_\zeta} - \zeta F_{z\pi}^{z_\zeta} = R \quad \square$$

### 1.5.3 引力场Ashtekar变量 $A_u^{\alpha_\zeta}$ 的约束条件

定理1.5.2.

$$\begin{cases} F_{yz}^{y_\zeta} - \zeta F_{x\pi}^{y_\zeta} = F_{zx}^{x_\zeta} - \zeta F_{y\pi}^{x_\zeta} \\ F_{zx}^{z_\zeta} - \zeta F_{y\pi}^{z_\zeta} = F_{xy}^{y_\zeta} - \zeta F_{z\pi}^{y_\zeta} \\ F_{xy}^{x_\zeta} - \zeta F_{z\pi}^{x_\zeta} = F_{yz}^{y_\zeta} - \zeta F_{x\pi}^{z_\zeta} \\ F_{yz}^{x_\zeta} - \zeta F_{x\pi}^{x_\zeta} + F_{zx}^{y_\zeta} - \zeta F_{y\pi}^{y_\zeta} \\ + F_{xy}^{z_\zeta} - \zeta F_{z\pi}^{z_\zeta} = R \end{cases} \Leftrightarrow \begin{cases} F_{uv}^{\alpha_\zeta} = \partial_u A_v^{\alpha_\zeta} - \partial_v A_u^{\alpha_\zeta} - \varepsilon^{\alpha_\zeta \beta_\zeta \gamma_\zeta} A_u^{\beta_\zeta} A_v^{\gamma_\zeta} \\ (\partial_y A_z^{y_\zeta} - \partial_z A_y^{y_\zeta} - A_y^{[z_\zeta} A_z^{x_\zeta]}) - \zeta (\partial_x A_\pi^{y_\zeta} - \partial_\pi A_x^{y_\zeta} - A_x^{[z_\zeta} A_\pi^{x_\zeta]}) \\ = (\partial_z A_x^{x_\zeta} - \partial_x A_z^{x_\zeta} - A_z^{[y_\zeta} A_x^{z_\zeta]}) - \zeta (\partial_y A_\pi^{x_\zeta} - \partial_\pi A_y^{x_\zeta} - A_y^{[z_\zeta} A_\pi^{x_\zeta]}) \\ (\partial_z A_x^{z_\zeta} - \partial_x A_z^{z_\zeta} - A_z^{[x_\zeta} A_x^{y_\zeta]}) - \zeta (\partial_y A_\pi^{z_\zeta} - \partial_\pi A_y^{z_\zeta} - A_y^{[x_\zeta} A_\pi^{y_\zeta]}) \\ = (\partial_x A_y^{y_\zeta} - \partial_y A_x^{y_\zeta} - A_x^{[z_\zeta} A_y^{x_\zeta]}) - \zeta (\partial_z A_\pi^{y_\zeta} - \partial_\pi A_z^{y_\zeta} - A_z^{[x_\zeta} A_\pi^{y_\zeta]}) \\ (\partial_x A_y^{x_\zeta} - \partial_y A_x^{x_\zeta} - A_x^{[y_\zeta} A_y^{z_\zeta]}) - \zeta (\partial_z A_\pi^{x_\zeta} - \partial_\pi A_z^{x_\zeta} - A_z^{[y_\zeta} A_\pi^{x_\zeta]}) \\ = (\partial_y A_z^{z_\zeta} - \partial_z A_y^{z_\zeta} - A_y^{[x_\zeta} A_z^{y_\zeta]}) - \zeta (\partial_x A_\pi^{z_\zeta} - \partial_\pi A_x^{z_\zeta} - A_x^{[y_\zeta} A_\pi^{z_\zeta]}) \\ (\partial_y A_z^{x_\zeta} - \partial_z A_y^{x_\zeta} - A_y^{[y_\zeta} A_z^{z_\zeta]}) - \zeta (\partial_x A_\pi^{x_\zeta} - \partial_\pi A_x^{x_\zeta} - A_x^{[y_\zeta} A_\pi^{z_\zeta]}) \\ + (\partial_z A_x^{y_\zeta} - \partial_x A_z^{y_\zeta} - A_z^{[z_\zeta} A_x^{x_\zeta]}) - \zeta (\partial_y A_\pi^{y_\zeta} - \partial_\pi A_y^{y_\zeta} - A_y^{[z_\zeta} A_\pi^{x_\zeta]}) \\ + (\partial_x A_y^{z_\zeta} - \partial_y A_x^{z_\zeta} - A_x^{[x_\zeta} A_y^{y_\zeta]}) - \zeta (\partial_z A_\pi^{z_\zeta} - \partial_\pi A_z^{z_\zeta} - A_z^{[x_\zeta} A_\pi^{y_\zeta]}) = R \end{cases}$$

引力场Ashtekar变量  $A_u^{\alpha_\zeta}$  的规范条件:  $\partial^u A_u^{\alpha_\zeta} = 0, A_\pi^{\alpha_\zeta} = 0$



## 1.5.4 引力场的外尔旋量 [1,2]

定义1.5.2. 引力外尔旋量:

$$C^{A_\zeta B_\zeta C_\kappa D_\kappa} := \frac{i_\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \frac{i_\kappa}{\sqrt{2}} \sigma_{\beta_\kappa}^{C_\kappa D_\kappa} C^{\alpha_\zeta \beta_\kappa} = \frac{i_\zeta}{\sqrt{2}} S_{ab}^{A_\zeta B_\zeta} \frac{i_\kappa}{\sqrt{2}} \sigma_{\beta_\kappa}^{C_\kappa D_\kappa} C^{ab \beta_\kappa} = \frac{i_\zeta}{\sqrt{2}} S_{ab}^{A_\zeta B_\zeta} \frac{i_\kappa}{\sqrt{2}} S_{cd}^{C_\kappa D_\kappa} C^{abcd}$$

$$\text{推论1.5.10. } C^{\alpha_\zeta \beta_\kappa} = \frac{i_\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \frac{i_\kappa}{\sqrt{2}} \sigma_{C_\kappa D_\kappa}^{\beta_\kappa} C^{A_\zeta B_\zeta C_\kappa D_\kappa}$$

$$\text{推论1.5.11. } C = 0$$

$$\text{推论1.5.12. } C^{\alpha \beta'} = 0, C^{\alpha' \beta} = 0 \Leftrightarrow C^{ABC'D'} = 0, C^{A'B'CD} = 0$$

$$\text{推论1.5.13. } C^{\alpha_\zeta \beta_\kappa} = C^{\beta_\kappa \alpha_\zeta} \Leftrightarrow C^{A_\zeta B_\zeta C_\kappa D_\kappa} = C^{B_\zeta A_\zeta C_\kappa D_\kappa}, C^{A_\zeta B_\zeta C_\kappa D_\kappa} = C^{A_\zeta B_\zeta D_\kappa C_\kappa}, C^{A_\zeta B_\zeta C_\kappa D_\kappa} = C^{C_\kappa D_\kappa A_\zeta B_\zeta}$$

$$\text{推论1.5.14. } C^{\alpha_\zeta \beta_\zeta} = C^{\beta_\zeta \alpha_\zeta} \Leftrightarrow \begin{cases} C^{1_\zeta 1_\zeta 1_\zeta 1_\zeta} \\ C^{1_\zeta 1_\zeta 1_\zeta 2_\zeta} = C^{1_\zeta 1_\zeta 2_\zeta 1_\zeta} = C^{1_\zeta 2_\zeta 1_\zeta 1_\zeta} = C^{2_\zeta 1_\zeta 1_\zeta 1_\zeta} \\ C^{1_\zeta 1_\zeta 2_\zeta 2_\zeta} = C^{2_\zeta 2_\zeta 1_\zeta 1_\zeta}, C^{1_\zeta 2_\zeta 1_\zeta 2_\zeta} = C^{1_\zeta 2_\zeta 2_\zeta 1_\zeta} = C^{2_\zeta 1_\zeta 1_\zeta 2_\zeta} = C^{2_\zeta 1_\zeta 2_\zeta 1_\zeta} \\ C^{1_\zeta 2_\zeta 2_\zeta 2_\zeta} = C^{2_\zeta 1_\zeta 2_\zeta 2_\zeta} = C^{2_\zeta 2_\zeta 1_\zeta 2_\zeta} = C^{2_\zeta 2_\zeta 2_\zeta 1_\zeta} \\ C^{2_\zeta 2_\zeta 2_\zeta 2_\zeta} \end{cases}$$

$$\text{推论1.5.15. } C^{A_\zeta B_\zeta}_{A_\zeta B_\zeta} = (-\zeta) \varepsilon_{A_\zeta C_\zeta} (-\zeta) \varepsilon_{B_\zeta D_\zeta} C^{A_\zeta B_\zeta C_\zeta D_\zeta} = 2(C^{1_\zeta 2_\zeta 1_\zeta 2_\zeta} - C^{1_\zeta 1_\zeta 2_\zeta 2_\zeta})$$

$$\text{推论1.5.16. } C^{A_\zeta B_\zeta}_{A_\zeta B_\zeta} = \delta_{\alpha_\zeta \beta_\zeta} C^{\alpha_\zeta \beta_\zeta} = 0$$

$$\text{推论1.5.17. } C^{1_\zeta 2_\zeta 1_\zeta 2_\zeta} - C^{1_\zeta 1_\zeta 2_\zeta 2_\zeta} = 0$$

$$\text{推论1.5.18. } \delta_{\alpha_\zeta \beta_\zeta} C^{\alpha_\zeta \beta_\zeta} = 0 \Leftrightarrow \sigma_{\alpha_\zeta} \sigma_{\beta_\zeta} C^{\alpha_\zeta \beta_\zeta} = 0 \Leftrightarrow (\sigma, -i_\zeta)_{\alpha_\zeta} \sigma_{\beta_\zeta} \tilde{C}^{\alpha_\zeta \beta_\zeta} = 0$$

$$\text{推论1.5.19. } C^{\alpha_\zeta \beta_\zeta} = C^{\beta_\zeta \alpha_\zeta}, C^{x_\zeta x_\zeta} + C^{y_\zeta y_\zeta} + C^{z_\zeta z_\zeta} = 0$$

推论1.5.20.  $C^{A_\zeta B_\zeta C_\zeta D_\zeta}$  是全对称旋量

## 1.5.5 引力场的外尔2-旋量

$$\text{定义1.5.3. } C^{k_\zeta} := \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta}^{k_\zeta}(2) C^{A_\zeta B_\zeta C_\zeta D_\zeta}, C_{k_\zeta} := \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta}(2) C_{A_\zeta B_\zeta C_\zeta D_\zeta}$$

$$\text{推论1.5.21. } C^{A_\zeta B_\zeta C_\zeta D_\zeta} = \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta}(2) C^{k_\zeta}, C_{A_\zeta B_\zeta C_\zeta D_\zeta} = \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta}^{k_\zeta}(2) C_{k_\zeta}$$

$$\text{推论1.5.22. } C^{\alpha_\zeta \beta_\zeta} = \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta}(2) C^{k_\zeta}, C^{k_\zeta} = \Gamma_{\alpha_\zeta \beta_\zeta}^{k_\zeta}(2) C^{\alpha_\zeta \beta_\zeta}$$

## 1.5.6 引力场源的矢量-旋量描述 [1,2]

$$\text{定义1.5.4. 引力场源旋量 } J_a^{A_\zeta B_\zeta} := \frac{i_\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} J_a^{\alpha_\zeta} = \frac{i_\zeta}{\sqrt{2}} S_{cd}^{A_\zeta B_\zeta} J_a^{cd}$$

仿照电磁场情形的推理，有以下完全类似的结论。

$$\text{推论1.5.23. } J_a^{A_\zeta B_\zeta} = J_a^{B_\zeta A_\zeta}$$

$$\text{推论1.5.24. } J_a^{A_\zeta B_\zeta} := \frac{i_\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} J_a^{\alpha_\zeta} \Leftrightarrow J_a^{\alpha_\zeta} = \frac{i_\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} J_a^{A_\zeta B_\zeta}$$

$$\text{推论1.5.25. } \frac{1}{2}(J_a^{cd} - \zeta * J_a^{cd}) = \frac{i_\zeta}{\sqrt{2}} S_{cd}^{A_\zeta B_\zeta} J_a^{A_\zeta B_\zeta} \Leftrightarrow J_a^{A_\zeta B_\zeta} = \frac{i_\zeta}{\sqrt{2}} S_{cd}^{A_\zeta B_\zeta} \frac{1}{2}(J_a^{cd} - \zeta * J_a^{cd})$$

$$\text{推论1.5.26. } J_a^{A_\zeta B_\zeta} = \frac{-i}{\sqrt{2}} S_{cd}^{A_\zeta B_\zeta} * J_a^{cd}$$

$$\text{推论1.5.27. } J_a^{cd} - \zeta * J_a^{cd} = -\frac{1}{2} S_{A_\zeta B_\zeta}^{cd} S_{ef}^{A_\zeta B_\zeta} (J_a^{ef} - \zeta * J_a^{ef})$$

$$\text{推论1.5.28. } J_a^{cd} = \frac{i}{\sqrt{2}} (S^{cdA'B'} J_{aA'B'} + S^{cdAB} J_a^{AB}), *J_a^{cd} = \frac{i}{\sqrt{2}} (S^{cdA'B'} J_{aA'B'} - S^{cdAB} J_a^{AB})$$

$$\text{推论1.5.29. } J_a^{cd} = -J_a^{dc} \Leftrightarrow J_a^{cd} = \frac{i}{\sqrt{2}} (S^{cdA'B'} J_{aA'B'} + S^{cdAB} J_a^{AB})$$

### 1.5.7 引力场源的 $\frac{1}{2}$ -旋量描述 [1, 2]

定义1.5.5. 引力场源旋量  $J_{A'_\zeta}{}^{B_\zeta C_\zeta D_\zeta} := \zeta \varepsilon^{B_\zeta A_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a J_a^{C_\zeta D_\zeta}$

推论1.5.30.  $J_{A'_\zeta}{}^{B_\zeta C_\zeta D_\zeta} = \zeta \varepsilon^{B_\zeta A_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \frac{i\kappa}{\sqrt{2}} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} J_a^{\alpha_\zeta} = \zeta \varepsilon^{B_\zeta A_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \frac{i\kappa}{\sqrt{2}} S_{cd}{}^{C_\zeta D_\zeta} J_a{}^{cd}$

推论1.5.31.  $J_a{}^{C_\zeta D_\zeta} = \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_{a A'_\zeta}^{A'_\zeta A_\zeta} (-\zeta) \varepsilon_{A_\zeta B_\zeta} J_{A'_\zeta}{}^{B_\zeta C_\zeta D_\zeta}$

推论1.5.32.  $J_{A'_\zeta}{}^{B_\zeta C_\zeta D_\zeta} = J_{A'_\zeta}{}^{B_\zeta D_\zeta C_\zeta}$

推论1.5.33.  $J_{A'_\zeta}{}^{B_\zeta C_\zeta D_\zeta} = \frac{\zeta}{2} \varepsilon^{B_\zeta A_\zeta} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} J_a^{\alpha_\zeta} = \frac{\zeta}{2} \varepsilon^{B_\zeta A_\zeta} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a S_{cd}{}^{C_\zeta D_\zeta} J_a{}^{cd}$

推论1.5.34.  $J_a^{\alpha_\zeta} = \frac{i\zeta}{\sqrt{2}} (\sigma, -i\zeta)_{a A'_\zeta}^{A'_\zeta A_\zeta} (-\zeta) \varepsilon_{A_\zeta B_\zeta} \frac{i\zeta}{\sqrt{2}} \sigma_{C_\zeta D_\zeta}^{\alpha_\zeta} J_{A'_\zeta}{}^{B_\zeta C_\zeta D_\zeta} = \frac{\zeta}{2} (\sigma, -i\zeta)_{a A'_\zeta}^{A'_\zeta A_\zeta} \varepsilon_{A_\zeta B_\zeta} \sigma_{C_\zeta D_\zeta}^{\alpha_\zeta} J_{A'_\zeta}{}^{B_\zeta C_\zeta D_\zeta}$

推论1.5.35.  $J_{A'_\zeta}{}^{B_\zeta C_\zeta D_\zeta} = J_{A'_\zeta}{}^{B_\zeta D_\zeta C_\zeta} \Leftrightarrow \begin{cases} J_{1'_\zeta}{}^{1_\zeta 1_\zeta 1_\zeta} \\ J_{1'_\zeta}{}^{1_\zeta 1_\zeta 2_\zeta} = J_{1'_\zeta}{}^{1_\zeta 2_\zeta 1_\zeta}, J_{1'_\zeta}{}^{2_\zeta 1_\zeta 2_\zeta} = J_{1'_\zeta}{}^{2_\zeta 2_\zeta 1_\zeta} \\ J_{2'_\zeta}{}^{1_\zeta 1_\zeta 2_\zeta} = J_{2'_\zeta}{}^{1_\zeta 2_\zeta 1_\zeta}, J_{2'_\zeta}{}^{2_\zeta 1_\zeta 2_\zeta} = J_{2'_\zeta}{}^{2_\zeta 2_\zeta 1_\zeta} \\ J_{2'_\zeta}{}^{2_\zeta 2_\zeta 2_\zeta} \end{cases}$

推论1.5.36.  $J_{A'_\zeta}{}^{1_\zeta 2_\zeta D_\zeta} = J_{A'_\zeta}{}^{2_\zeta 1_\zeta D_\zeta} \Leftrightarrow \begin{cases} \zeta J_\pi^{x_\zeta} = J_y^{z_\zeta} - J_z^{y_\zeta}, \zeta J_\pi^{y_\zeta} = J_z^{x_\zeta} - J_x^{z_\zeta}, \zeta J_\pi^{z_\zeta} = J_x^{y_\zeta} - J_y^{x_\zeta} \\ J_x^{x_\zeta} + J_y^{y_\zeta} + J_z^{z_\zeta} = 0 \end{cases}$

证明:  $J_{A'_\zeta}{}^{1_\zeta 2_\zeta D_\zeta} = J_{A'_\zeta}{}^{2_\zeta 1_\zeta D_\zeta}$

$\Leftrightarrow \frac{\zeta}{2} \varepsilon^{1_\zeta A_\zeta} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \sigma_{\alpha_\zeta}^{2_\zeta D_\zeta} J_a^{\alpha_\zeta} = \frac{\zeta}{2} \varepsilon^{2_\zeta A_\zeta} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \sigma_{\alpha_\zeta}^{1_\zeta D_\zeta} J_a^{\alpha_\zeta}$

$\Leftrightarrow \varepsilon^{1_\zeta 2_\zeta} (\sigma, i\zeta)_{2_\zeta A'_\zeta}^a \sigma_{\alpha_\zeta}^{2_\zeta D_\zeta} J_a^{\alpha_\zeta} = \varepsilon^{2_\zeta 1_\zeta} (\sigma, i\zeta)_{1_\zeta A'_\zeta}^a \sigma_{\alpha_\zeta}^{1_\zeta D_\zeta} J_a^{\alpha_\zeta}$

$\Leftrightarrow (\sigma, i\zeta)_{2_\zeta A'_\zeta}^a \sigma_{\alpha_\zeta}^{D_\zeta 2_\zeta} J_a^{\alpha_\zeta} = -(\sigma, i\zeta)_{1_\zeta A'_\zeta}^a \sigma_{\alpha_\zeta}^{D_\zeta 1_\zeta} J_a^{\alpha_\zeta}$

$\Leftrightarrow [\sigma_{\alpha_\zeta}^{D_\zeta 1_\zeta} (\sigma, i\zeta)_{1_\zeta A'_\zeta}^a + \sigma_{\alpha_\zeta}^{D_\zeta 2_\zeta} (\sigma, i\zeta)_{2_\zeta A'_\zeta}^a] J_a^{\alpha_\zeta} = 0$

$\Leftrightarrow \sigma_{\alpha_\zeta}^{D_\zeta A_\zeta} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a J_a^{\alpha_\zeta} = 0$

$\Leftrightarrow \sigma_{\alpha_\zeta D_\zeta}{}^{A_\zeta} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a J_a^{\alpha_\zeta} = 0$

$\Leftrightarrow \sigma_{\alpha_\zeta} (\sigma, i\zeta)^a J_a^{\alpha_\zeta} = 0$

$\Leftrightarrow (\sigma, i\zeta)^T \sigma_{\alpha_\zeta}^T J_a^{\alpha_\zeta} = 0$

$\Leftrightarrow \sigma_y (\sigma, i\zeta)^T \sigma_y \sigma_{\alpha_\zeta}^T \sigma_y J_a^{\alpha_\zeta} = 0$

$\Leftrightarrow (\sigma, -i\zeta)^a \sigma_{\alpha_\zeta} J_a^{\alpha_\zeta} = 0$

$\Leftrightarrow (J_x^{x_\zeta} + J_y^{y_\zeta} + J_z^{z_\zeta}) I + i(-\zeta J_\pi^{x_\zeta} + J_y^{y_\zeta} - J_z^{z_\zeta}) \sigma_x + i(-\zeta J_\pi^{y_\zeta} + J_z^{z_\zeta} - J_x^{x_\zeta}) \sigma_y + i(-\zeta J_\pi^{z_\zeta} + J_x^{x_\zeta} - J_y^{y_\zeta}) \sigma_z = 0$

$\Leftrightarrow \begin{cases} \zeta J_\pi^{x_\zeta} = J_y^{z_\zeta} - J_z^{y_\zeta}, \zeta J_\pi^{y_\zeta} = J_z^{x_\zeta} - J_x^{z_\zeta}, \zeta J_\pi^{z_\zeta} = J_x^{y_\zeta} - J_y^{x_\zeta} \\ J_x^{x_\zeta} + J_y^{y_\zeta} + J_z^{z_\zeta} = 0 \end{cases}$  □

推论1.5.37.  $[(\sigma, -i\zeta)^a \sigma_{\alpha_\zeta}] J_a^{\alpha_\zeta} = 0 \Leftrightarrow \begin{cases} \zeta J_\pi^{x_\zeta} = J_y^{z_\zeta} - J_z^{y_\zeta}, \zeta J_\pi^{y_\zeta} = J_z^{x_\zeta} - J_x^{z_\zeta}, \zeta J_\pi^{z_\zeta} = J_x^{y_\zeta} - J_y^{x_\zeta} \\ J_x^{x_\zeta} + J_y^{y_\zeta} + J_z^{z_\zeta} = 0 \end{cases}$

推论1.5.38.  $J_{A'_\zeta}{}^{B_\zeta C_\zeta D_\zeta}$  关于指标  $B_\zeta C_\zeta D_\zeta$  全对称  $\Leftrightarrow [(\sigma, -i\zeta)^a \sigma_{\alpha_\zeta}] J_a^{\alpha_\zeta} = 0$

推论1.5.39.  $J_{A'_\zeta}{}^{B_\zeta C_\zeta D_\zeta}$  关于指标  $B_\zeta C_\zeta D_\zeta$  全对称  $\Leftrightarrow [(\sigma, -i\zeta)^a S_{cd}(\frac{1}{2}, \zeta)] J_a{}^{cd} = 0$

## 2 引力场毕安奇恒等式的各种表述形式

### 2.1 引力场方程的经典表述形式

#### 2.1.1 无挠引力场的毕安奇恒等式 [11-14]

毕安奇恒等式:  $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0$

(8.8)

推论2.1.1.  $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \Rightarrow R^{(*ab)cd}{}_{;a} \equiv 0$

证明:  $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0$

$$\Rightarrow \varepsilon_{fcde}(R^{abcd;e} + R^{abde;c} + R^{abec;d}) \equiv 0$$

$$\Rightarrow 3\varepsilon_{fcde}R^{abcd;e} \equiv 0$$

$$\Rightarrow R^{ab(*cd)}{}_{;d} \equiv 0$$

$$\Rightarrow R^{(*ab)cd}{}_{;a} \equiv 0$$

□

推论2.1.2.  $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \Rightarrow R^{abcd}{}_{;a} \equiv -R^{b[c;d]}$

证明:  $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0$

$$\Rightarrow R^{abcd}{}_{;a} - R^{bd;c} + R^{bc;d} \equiv 0$$

$$\Rightarrow R^{abcd}{}_{;a} = R^{bd;c} - R^{bc;d}$$

$$\Rightarrow R^{cdba}{}_{;a} \equiv R^{b[c;d]}$$

$$\Rightarrow R^{abcd}{}_{;a} \equiv -R^{b[c;d]}$$

□

推论2.1.3.  $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \Rightarrow (R^{ab} - \frac{1}{2}g^{ab}R)_{;b} \equiv 0$

证明:  $R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0$

$$\Rightarrow R^{abcd}{}_{;a} \equiv -R^{b[c;d]}$$

$$\Rightarrow R^{ac}{}_{;a} \equiv R^{;c} - R^{ac}{}_{;a}$$

$$\Rightarrow R^{ac}{}_{;a} \equiv \frac{1}{2}R^{;c}$$

$$\Rightarrow (R^{ab} - \frac{1}{2}g^{ab}R)_{;b} \equiv 0$$

□

## 2.1.2 引力场方程经典形式

$$\begin{cases} R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \\ R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi GT^{ab} \end{cases} \Leftrightarrow \begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]}, R^{(*ab)cd}{}_{;a} \equiv 0, (R^{ab} - \frac{1}{2}g^{ab}R)_{;b} \equiv 0 \\ R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi GT^{ab} \end{cases} \quad (8.9)$$

## 2.2 引力场毕安奇恒等式的Yang-Mills形式

### 2.2.1 引力的Yang-Mills规范理论解释<sup>[48-51]</sup>

定义2.2.1.  $\theta^{\alpha\varsigma}(\varsigma) := \frac{i}{2}\sigma_{\varsigma ab}^{\alpha\varsigma}\vartheta^{ab} = -i(i\omega + \varsigma\epsilon)^{\alpha\varsigma}$

推论2.2.1.  $\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma) = i\theta^{\alpha\varsigma}\sigma_{\alpha\varsigma}(s) = (i\omega + \varsigma\epsilon) \cdot \sigma(s)$ ,  $\frac{i}{2}\omega_u{}^{ab}S_{ab}(s, \varsigma) = iA_u^{\alpha\varsigma}\sigma_{\alpha\varsigma}(s)$

线性无关性:

引理2.2.1.  $c^{cd}S_{cd} = 0 \Leftrightarrow c^{cd} = 0$

引理2.2.2.  $[\omega_u{}^{cd}(\frac{i}{2}S_{cd}), \omega_v{}^{ef}(\frac{i}{2}S_{ef})] = \omega_{[u}{}^{ce}\omega_{v]e}{}^d(\frac{i}{2}S_{cd})$

证明:  $[\omega_u{}^{cd}iS_{cd}, \omega_v{}^{ef}iS_{ef}] = \omega_u{}^{cd}\omega_v{}^{ef}[iS_{cd}, iS_{ef}]$

$$\Leftrightarrow [\omega_u{}^{cd}iS_{cd}, \omega_v{}^{ef}iS_{ef}] = \omega_u{}^{cd}\omega_v{}^{ef}[\delta_{cf}iS_{de} - \delta_{ce}iS_{df} + \delta_{de}iS_{cf} - \delta_{df}iS_{ce}]$$

$$\Leftrightarrow [\omega_u{}^{cd}iS_{cd}, \omega_v{}^{ef}iS_{ef}] = 4\omega_u{}^{ce}\omega_{ve}{}^diS_{cd}$$

$$\Leftrightarrow [\omega_u{}^{cd}iS_{cd}, \omega_v{}^{ef}iS_{ef}] = 2\omega_{[u}{}^{ce}\omega_{v]e}{}^diS_{cd}$$

$$\Leftrightarrow [\omega_u{}^{cd}(\frac{i}{2}S_{cd}), \omega_v{}^{ef}(\frac{i}{2}S_{ef})] = \omega_{[u}{}^{ce}\omega_{v]e}{}^d(\frac{i}{2}S_{cd})$$

□

推论2.2.2.  $R_{uv}{}^{cd} = \partial_u\omega_v{}^{cd} - \partial_v\omega_u{}^{cd} + \omega_{[u}{}^{ce}\omega_{v]e}{}^d$

$$\Leftrightarrow R_{uv}{}^{cd}(\frac{i}{2}S_{cd}) = \partial_u\omega_v{}^{cd}(\frac{i}{2}S_{cd}) - \partial_v\omega_u{}^{cd}(\frac{i}{2}S_{cd}) + [\omega_u{}^{cd}(\frac{i}{2}S_{cd}), \omega_v{}^{ef}(\frac{i}{2}S_{ef})]$$

推论2.2.3.  $R_{uv}{}^{cd} = \partial_u\omega_v{}^{cd} - \partial_v\omega_u{}^{cd} + \omega_{[u}{}^{ce}\omega_{v]e}{}^d$

$$\Leftrightarrow R_{uv}{}^{<cd>} = \partial_u\omega_v{}^{<cd>} - \partial_v\omega_u{}^{<cd>} + [\omega_u{}^{<cd>}, \omega_v{}^{<ef>}]$$

$$\begin{aligned} \text{推论2.2.4. } R_{uv}{}^{cd} &= \partial_u \omega_v{}^{cd} - \partial_v \omega_u{}^{cd} + \omega_{[u}{}^{ce} \omega_{v]}{}^d \\ &\Leftrightarrow R_{uv}{}^{cd} \frac{i}{2} S_{cd}(s, \varsigma) = \partial_u \omega_v{}^{cd} \frac{i}{2} S_{cd}(s, \varsigma) - \partial_v \omega_u{}^{cd} \frac{i}{2} S_{cd}(s, \varsigma) + [\omega_u{}^{cd} \frac{i}{2} S_{cd}(s, \varsigma), \omega_v{}^{ef} \frac{i}{2} S_{ef}(s, \varsigma)] \end{aligned}$$

$$\begin{aligned} \text{推论2.2.5. } R_{uv}{}^{cd} &= \partial_u \omega_v{}^{cd} - \partial_v \omega_u{}^{cd} + \omega_{[u}{}^{ce} \omega_{v]}{}^d \\ &\Leftrightarrow F_{uv}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s) = \partial_u A_v^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s) - \partial_v A_u^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s) + i[A_u^{\beta\varsigma} \sigma_{\beta\varsigma}(s), A_v^{\gamma\varsigma} \sigma_{\gamma\varsigma}(s)] \end{aligned}$$

$$\begin{aligned} \text{推论2.2.6. } \frac{i}{2} \omega_u{}^{ab} S_{ab}(s, \varsigma) &\rightarrow U(\theta) \frac{i}{2} \omega_u{}^{ab} S_{ab}(s, \varsigma) U^{-1}(\theta) + [\partial_u U(\theta)] U^{-1}(\theta) \\ &\Leftrightarrow A_u^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s) \rightarrow U(\theta) A_u^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s) U^{-1}(\theta) - i[\partial_u U(\theta)] U^{-1}(\theta) \end{aligned}$$

$$\begin{aligned} \text{推论2.2.7. } i[S_{ab}(s, \varsigma), S_{cd}(s, \varsigma)] &= \delta_{ad} S_{bc}(s, \varsigma) - \delta_{ac} S_{bd}(s, \varsigma) + \delta_{bc} S_{ad}(s, \varsigma) - \delta_{bd} S_{ac}(s, \varsigma) \\ &\Leftrightarrow [\sigma_{\alpha\varsigma}(s), \sigma_{\beta\varsigma}(s)] = i\varepsilon_{\alpha\varsigma\beta\varsigma} \gamma^\varsigma \sigma_{\gamma\varsigma}(s) \end{aligned}$$

$$\begin{aligned} \text{证明: } i[S_{ab}(s, \varsigma), S_{cd}(s, \varsigma)] &= \delta_{ad} S_{bc}(s, \varsigma) - \delta_{ac} S_{bd}(s, \varsigma) + \delta_{bc} S_{ad}(s, \varsigma) - \delta_{bd} S_{ac}(s, \varsigma) \\ &\Leftrightarrow \frac{1}{16} \sigma_{\varsigma\alpha\varsigma}^{ab} \sigma_{\varsigma\beta\varsigma}^{cd} [iS_{ab}(s, \varsigma), iS_{cd}(s, \varsigma)] = \frac{1}{16} \sigma_{\varsigma\alpha\varsigma}^{ab} \sigma_{\varsigma\beta\varsigma}^{cd} [\delta_{ad} iS_{bc}(s, \varsigma) - \delta_{ac} iS_{bd}(s, \varsigma) + \delta_{bc} iS_{ad}(s, \varsigma) - \delta_{bd} iS_{ac}(s, \varsigma)] \\ &\Leftrightarrow [\sigma_{\alpha\varsigma}(s), \sigma_{\beta\varsigma}(s)] = \frac{1}{16} \sigma_{\varsigma\alpha\varsigma}^{ab} \sigma_{\varsigma\beta\varsigma}^{cd} [\delta_{ad} iS_{bc}(s, \varsigma) - \delta_{ac} iS_{bd}(s, \varsigma) + \delta_{bc} iS_{ad}(s, \varsigma) - \delta_{bd} iS_{ac}(s, \varsigma)] \\ &\Leftrightarrow [\sigma_{\alpha\varsigma}(s), \sigma_{\beta\varsigma}(s)] = \frac{1}{4} \sigma_{\varsigma\alpha\varsigma}^{ab} \sigma_{\varsigma\beta\varsigma}^{cd} \delta_{ad} iS_{bc}(s, \varsigma) \\ &\Leftrightarrow [\sigma_{\alpha\varsigma}(s), \sigma_{\beta\varsigma}(s)] = \frac{1}{4} [\delta_{\alpha\varsigma\beta\varsigma} \delta^{bc} + i\varepsilon_{\alpha\varsigma\beta\varsigma} \gamma^\varsigma \sigma_{\varsigma\gamma\varsigma}{}^{bc}(s)] iS_{bc}(s, \varsigma) \\ &\Leftrightarrow [\sigma_{\alpha\varsigma}(s), \sigma_{\beta\varsigma}(s)] = i\varepsilon_{\alpha\varsigma\beta\varsigma} \gamma^\varsigma \sigma_{\gamma\varsigma}(s) \end{aligned} \quad \square$$

## 2.2.2 引力场毕安奇恒等式的Yang-Mills分量形式

$$\text{推论2.2.8. } R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \Rightarrow \begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases}$$

$$\text{引理2.2.3. } D^a F_{ab}^{\alpha\varsigma} = -J_b^{\alpha\varsigma} \Leftrightarrow D^a F_{ab}^{\alpha'\varsigma} = -J_b^{\alpha'\varsigma}$$

$$\begin{aligned} \text{证明: } D^a F_{ab}^{\alpha\varsigma} &= -J_b^{\alpha\varsigma} \\ &\Leftrightarrow (D^a F_{ab}^{\alpha\varsigma})^* = -(J_b^{\alpha\varsigma})^* \\ &\Leftrightarrow \eta_c^{\alpha'} D^c (\eta_a^a \eta_b^b F_{ab}^{\alpha'\varsigma}) = -\eta_b^b J_b^{\alpha'\varsigma} \\ &\Leftrightarrow \eta_b^b D^a (F_{ab}^{\alpha'\varsigma}) = -\eta_b^b J_b^{\alpha'\varsigma} \\ &\Leftrightarrow D^a (F_{ab}^{\alpha'\varsigma}) = -J_b^{\alpha'\varsigma} \end{aligned} \quad \square$$

$$\text{引理2.2.4. } D^a * F_{ab}^{\alpha\varsigma} \equiv 0 \Leftrightarrow D^a * F_{ab}^{\alpha'\varsigma} \equiv 0$$

$$\begin{aligned} \text{证明: } D^a * F_{ab}^{\alpha\varsigma} &\equiv 0 \\ &\Leftrightarrow (D^a * F_{ab}^{\alpha\varsigma})^* \equiv 0 \\ &\Leftrightarrow \eta_c^{\alpha'} D^c (\eta_a^a \eta_b^b * F_{ab}^{\alpha'\varsigma}) \equiv 0 \\ &\Leftrightarrow \eta_b^b D^a * F_{ab}^{\alpha'\varsigma} \equiv 0 \\ &\Leftrightarrow D^a * F_{ab}^{\alpha'\varsigma} \equiv 0 \end{aligned} \quad \square$$

$$\text{定理2.2.1. } \begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}^{\alpha\varsigma} \equiv -J_b^{\alpha\varsigma}, J^{b\alpha\varsigma} := \frac{i}{2} \sigma_{\varsigma cd} R^{b[c;d]} \\ D^a * F_{ab}^{\alpha\varsigma} \equiv 0 \end{cases}$$

$$\begin{aligned} \text{证明: } \begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} &\Leftrightarrow \begin{cases} R_{ab}{}^{cd;a} \equiv -R_b{}^{[c;d]} \\ R_{*ab}{}^{cd;a} \equiv 0 \end{cases} \\ &\Leftrightarrow \begin{cases} \frac{i}{2} (\sigma_{-\varsigma\alpha\varsigma}{}^{cd} F_{ab}^{\alpha'\varsigma} + \sigma_{\varsigma\alpha\varsigma}^{cd} F_{ab}^{\alpha\varsigma}){}_{;a} \equiv -\frac{i}{2} (\sigma_{-\varsigma\alpha\varsigma}{}^{cd} J_b^{\alpha'\varsigma} + \sigma_{\varsigma\alpha\varsigma}^{cd} J_b^{\alpha\varsigma}) \\ \frac{i}{2} (\sigma_{-\varsigma\alpha\varsigma}{}^{cd} * F_{ab}^{\alpha'\varsigma} + \sigma_{\varsigma\alpha\varsigma}^{cd} * F_{ab}^{\alpha\varsigma}){}_{;a} \equiv 0 \end{cases} \end{aligned}$$

$$\Leftrightarrow \begin{cases} D^a F_{ab}^{\alpha\zeta} \equiv -J_b^{\alpha\zeta}, D^a F_{ab}^{\alpha'\zeta} \equiv -J_b^{\alpha\zeta} \\ D^a * F_{ab}^{\alpha\zeta} \equiv 0, D^a * F_{ab}^{\alpha'\zeta} \equiv 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} D^a F_{ab}^{\alpha\zeta} \equiv -J_b^{\alpha\zeta}, J^{b\alpha\zeta} = \frac{i}{2} \sigma_{\zeta cd} R^{b[c;d]} \\ D^a * F_{ab}^{\alpha\zeta} \equiv 0 \end{cases} \quad \square$$

$$\text{定理2.2.2.} \quad \begin{cases} D^a F_{ab}^{\alpha\zeta} \equiv -J_b^{\alpha\zeta} \\ D^a * F_{ab}^{\alpha\zeta} \equiv 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \nabla_u F^{uv\alpha\zeta} - \varepsilon^{\alpha\zeta}{}_{\beta\gamma\zeta} A_u^{\beta\zeta} F^{uv\gamma\zeta} \equiv -J^{v\alpha\zeta} \\ \nabla_u F^{*uv\alpha\zeta} - \varepsilon^{\alpha\zeta}{}_{\beta\gamma\zeta} A_u^{\beta\zeta} F^{*uv\gamma\zeta} \equiv 0 \end{cases}$$

### 2.2.3 毕安奇恒等式Yang-Mills理论的矩阵描述

#### 1、毕安奇恒等式Yang-Mills理论的一般矩阵描述

$$\begin{cases} R_{uv}{}^{cd} \frac{i}{2} S_{cd} = \partial_u \omega_v{}^{cd} \frac{i}{2} S_{cd} - \partial_v \omega_u{}^{cd} \frac{i}{2} S_{cd} + [\omega_u{}^{cd} \frac{i}{2} S_{cd}, \omega_v{}^{ef} \frac{i}{2} S_{ef}] \\ i[S_{ab}, S_{cd}] = \delta_{ad} S_{bc} - \delta_{ac} S_{bd} + \delta_{bc} S_{ad} - \delta_{bd} S_{ac} \\ c^{ab} S_{ab} = 0, c^{ab} = -c^{ba} \Leftrightarrow c^{ab} = 0 \end{cases} \quad (8.10)$$

规范变换:

$$\begin{cases} \psi \rightarrow U(\theta)\psi, U(\theta) = e^{\frac{i}{2}\vartheta^{ab} S_{ab}} \\ \frac{i}{2} \omega_u{}^{ab} S_{ab} \rightarrow U(\theta) \frac{i}{2} \omega_u{}^{ab} S_{ab} U^{-1}(\theta) - [\partial_u U(\theta)] U^{-1}(\theta) \end{cases} \quad (8.11)$$

$$\text{推论2.2.9.} \quad D_u \psi \rightarrow U(\theta) D_u \psi, D_u = \partial_u + \frac{i}{2} \omega_u{}^{cd} S_{cd}$$

$$\text{推论2.2.10.} \quad R_{uv}{}^{cd} \frac{i}{2} S_{cd} \rightarrow U(\theta) R_{uv}{}^{cd} \frac{i}{2} S_{cd} U^{-1}(\theta)$$

$$\text{推论2.2.11.} \quad D_w R_{uv}{}^{cd} \frac{i}{2} S_{cd} \rightarrow U(\theta) D_w R_{uv}{}^{cd} \frac{i}{2} S_{cd} U^{-1}(\theta), D_w = \nabla_w + [\frac{i}{2} \omega_w{}^{cd} S_{cd}, \quad ]$$

毕安奇恒等式的规范方程形式:

$$\text{推论2.2.12.} \quad \begin{cases} \nabla_u R^{uvcd} \frac{i}{2} S_{cd} + [\frac{i}{2} \omega_u{}^{cd} S_{cd}, R^{uvcd} \frac{i}{2} S_{cd}] = 0 \\ \nabla_u R^{*(uv)cd} \frac{i}{2} S_{cd} + [\frac{i}{2} \omega_u{}^{cd} S_{cd}, R^{*(uv)cd} \frac{i}{2} S_{cd}] \equiv 0 \end{cases}$$

规范方程:

$$\text{推论2.2.13.} \quad \begin{cases} \nabla_u R^{uv\langle cd \rangle} + [\omega_u{}^{\langle cd \rangle}, R^{uv\langle cd \rangle}] \equiv -R_u{}^{\langle c;d \rangle} \\ \nabla_u R^{*(uv)\langle cd \rangle} + [\omega_u{}^{\langle cd \rangle}, R^{*(uv)\langle cd \rangle}] \equiv 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{*(ab)cd}{}_{;a} \equiv 0 \end{cases}$$

#### 2、毕安奇恒等式Yang-Mills理论的特殊矩阵描述

$$\begin{cases} R_{uv}{}^{cd} \frac{i}{2} S_{cd}(s, \zeta) = \partial_u \omega_v{}^{cd} \frac{i}{2} S_{cd}(s, \zeta) - \partial_v \omega_u{}^{cd} \frac{i}{2} S_{cd}(s, \zeta) + [\omega_u{}^{cd} \frac{i}{2} S_{cd}(s, \zeta), \omega_v{}^{ef} \frac{i}{2} S_{ef}(s, \zeta)] \\ i[S_{ab}(s, \zeta), S_{cd}(s, \zeta)] = \delta_{ad} S_{bc}(s, \zeta) - \delta_{ac} S_{bd}(s, \zeta) + \delta_{bc} S_{ad}(s, \zeta) - \delta_{bd} S_{ac}(s, \zeta) \\ c^{ab} i S_{ab}(s, \zeta) = 0, c^{ab} = -c^{ba} \Leftrightarrow c^{ab} = 0 \end{cases} \quad (8.12)$$

规范变换:

$$\begin{cases} \psi(s, \zeta) \rightarrow U(\theta)\psi(s, \zeta), U(\theta) = e^{\frac{i}{2}\vartheta^{ab} S_{ab}(s, \zeta)} \\ \frac{i}{2} \omega_u{}^{ab} S_{ab}(s, \zeta) \rightarrow U(\theta) \frac{i}{2} \omega_u{}^{ab} S_{ab}(s, \zeta) U^{-1}(\theta) - [\partial_u U(\theta)] U^{-1}(\theta) \end{cases} \quad (8.13)$$

$$\text{推论2.2.14.} \quad D_u \psi(s, \zeta) \rightarrow U(\theta) D_u \psi(s, \zeta), D_u = \partial_u + \frac{i}{2} \omega_u{}^{cd} S_{cd}(s, \zeta)$$

$$\text{推论2.2.15.} \quad R_{uv}{}^{cd} \frac{i}{2} S_{cd}(s, \zeta) \rightarrow U(\theta) R_{uv}{}^{cd} \frac{i}{2} S_{cd}(s, \zeta) U^{-1}(\theta)$$

$$\text{推论2.2.16.} \quad D_w R_{uv}{}^{cd} \frac{i}{2} S_{cd}(s, \zeta) \rightarrow U D_w R_{uv}{}^{cd} \frac{i}{2} S_{cd}(s, \zeta) U^{-1}, D_w = \nabla_w + [\frac{i}{2} \omega_w{}^{cd} S_{cd}(s, \zeta), \quad ]$$

毕安奇恒等式的规范方程形式:

$$\begin{aligned} \text{推论2.2.17. } & \begin{cases} \nabla_u R^{uvcd} \frac{i}{2} S_{cd}(s, \varsigma) + [\frac{i}{2} \omega_u{}^{cd} S_{cd}(s, \varsigma), R^{uvcd} \frac{i}{2} S_{cd}(s, \varsigma)] \equiv -R^{v[c;d]} \frac{i}{2} S_{cd}(s, \varsigma) \\ \nabla_u R^{(*uv)cd} \frac{i}{2} S_{cd}(s, \varsigma) + [\frac{i}{2} \omega_u{}^{cd} S_{cd}(s, \varsigma), R^{(*uv)cd} \frac{i}{2} S_{cd}(s, \varsigma)] \equiv 0 \end{cases} \\ \Leftrightarrow & \begin{cases} \nabla_u F^{uv\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma \gamma_\varsigma} A_u^{\beta_\varsigma} F^{uv\gamma_\varsigma} \equiv -J^{v\alpha_\varsigma} \\ \nabla_u F^{*uv\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma \gamma_\varsigma} A_u^{\beta_\varsigma} F^{*uv\gamma_\varsigma} \equiv 0 \end{cases} \end{aligned}$$

### 3、毕安奇恒等式Yang-Mills理论的标准矩阵描述

$$\begin{cases} F_{uv}^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) = \partial_u A_v^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) - \partial_v A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) + i[A_u^{\beta_\varsigma} \sigma_{\beta_\varsigma}(s), A_v^{\gamma_\varsigma} \sigma_{\gamma_\varsigma}(s)] \\ [\sigma_{\beta_\varsigma}(s), \sigma_{\gamma_\varsigma}(s)] = i\varepsilon_{\beta_\varsigma \gamma_\varsigma}{}^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s), c^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) = 0 \Leftrightarrow c^{\alpha_\varsigma} = 0 \end{cases} \quad (8.14)$$

规范变换:

$$\begin{cases} \psi(s, \varsigma) \rightarrow U(\theta)\psi(s, \varsigma), U(\theta) = e^{i\theta^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s)} = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma)} = e^{(i\omega + \varsigma \epsilon) \cdot \sigma(s)} \\ A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) \rightarrow U(\theta) A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) U^{-1}(\theta) + i[\partial_u U(\theta)] U^{-1}(\theta) \end{cases} \quad (8.15)$$

以上就是 $g=1$ 和 $T=\sigma(s)$ 的标准Yang-Mills理论, 所以有以下类似的结论。

$$\text{推论2.2.18. } D_u \psi(s, \varsigma) \rightarrow U(\theta) D_u \psi(s, \varsigma), D_u = \partial_u + i A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) = \partial_u + \frac{i}{2} \omega_u{}^{cd} S_{cd}(s, \varsigma)$$

$$\text{推论2.2.19. } F_{uv}^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) \rightarrow U(\theta) F_{uv}^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) U^{-1}(\theta)$$

$$\text{推论2.2.20. } D_w F_{uv}^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) \rightarrow U D_w F_{uv}^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) U^{-1}, D_w = \nabla_w + [i A_w^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s), \quad ]$$

毕安奇恒等式的规范方程形式:

推论2.2.21.

$$\begin{cases} \nabla_u F^{uv\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) + [i A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s), F^{uv\beta_\varsigma} \sigma_{\beta_\varsigma}(s)] \equiv -J^{v\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) \\ \nabla_u F^{*uv\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) + [i A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s), F^{*uv\beta_\varsigma} \sigma_{\beta_\varsigma}(s)] \equiv 0 \end{cases} \Leftrightarrow \begin{cases} \nabla_u F^{uv\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma \gamma_\varsigma} A_u^{\beta_\varsigma} F^{uv\gamma_\varsigma} \equiv -J^{v\alpha_\varsigma} \\ \nabla_u F^{*uv\alpha_\varsigma} - \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma \gamma_\varsigma} A_u^{\beta_\varsigma} F^{*uv\gamma_\varsigma} \equiv 0 \end{cases}$$

毕安奇恒等式的规范方程矩阵形式:

推论2.2.22.

$$\begin{cases} \nabla_u F^{uv} + i[A_u, F^{uv}] \equiv -J^v, A_u := A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) \\ \nabla_u F^{*uv} + i[A_u, F^{*uv}] \equiv 0, F^{uv} := F^{uv\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}^{\alpha_\varsigma} \equiv -J_b^{\alpha_\varsigma} \\ D^a * F_{ab}^{\alpha_\varsigma} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases}$$

#### 2.2.4 引力场Yang-Mills理论的分量描述

$$\text{定理2.2.3. } A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) \rightarrow U(\theta) A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) U^{-1}(\theta) + i[\partial_u U(\theta)] U^{-1}(\theta)$$

$$\Leftrightarrow \delta A_u^{\alpha_\varsigma} = i\theta^{\beta_\varsigma} (-i\varepsilon_{\beta_\varsigma}{}^{\alpha_\varsigma}{}_{\gamma_\varsigma}) A_u^{\gamma_\varsigma} - \partial_u \theta^{\alpha_\varsigma}$$

$$\Leftrightarrow \delta A_u = i\theta^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} A_u - \partial_u \theta$$

$$\text{证明: } A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) \rightarrow U(\theta) A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) U^{-1}(\theta) + i[\partial_u U(\theta)] U^{-1}(\theta)$$

$\Leftrightarrow$

$$\text{定理2.2.4. } \delta A_u^{\alpha_\varsigma} = \varepsilon^{\alpha_\varsigma}{}_{\beta_\varsigma \gamma_\varsigma} \theta^{\beta_\varsigma} A_u^{\gamma_\varsigma} - \partial_u \theta^{\alpha_\varsigma} \Leftrightarrow \delta \omega_u{}^{ab} = \vartheta^{ac} \omega_u{}^{cb} - \omega_u{}^{ac} \vartheta^{cb} - \partial_u \vartheta^{ab}$$

$$\text{证明: } \delta A_u^{\alpha_\varsigma} = i\theta^{\beta_\varsigma} (-i\varepsilon_{\beta_\varsigma}{}^{\alpha_\varsigma}{}_{\gamma_\varsigma}) A_u^{\gamma_\varsigma} - \partial_u \theta^{\alpha_\varsigma}$$

$$\Leftrightarrow A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) \rightarrow U(\theta) A_u^{\alpha_\varsigma} \sigma_{\alpha_\varsigma}(s) U^{-1}(\theta) + i[\partial_u U(\theta)] U^{-1}(\theta)$$

$$\Leftrightarrow [-\frac{1}{2} \omega_u{}^{ab} i S_{ab}(s, \varsigma)] \rightarrow U(\theta) [-\frac{1}{2} \omega_u{}^{ab} i S_{ab}(s, \varsigma)] U^{-1}(\theta) + [\partial_u U(\theta)] U^{-1}(\theta)$$

$$\Leftrightarrow [\frac{i}{2} \omega_u{}^{ab} S_{ab}(s, \varsigma)] \rightarrow U(\theta) [\frac{i}{2} \omega_u{}^{ab} S_{ab}(s, \varsigma)] U^{-1}(\theta) - [\partial_u U(\theta)] U^{-1}(\theta)$$

$$\Leftrightarrow [\frac{i}{2} \omega_u{}^{ab} S_{ab}(s, \varsigma)] \rightarrow \frac{1}{2} (\omega_u{}^{ab} - \partial_u \vartheta^{ab}) i S_{ab}(s, \varsigma) + \frac{1}{4} \vartheta^{ab} \omega_u{}^{cd} [i S_{ab}(s, \varsigma), i S_{cd}(s, \varsigma)]$$

$$\Leftrightarrow [\frac{i}{2} \omega_u{}^{ab} S_{ab}(s, \varsigma)] \rightarrow \frac{1}{2} (\omega_u{}^{ab} - \partial_u \vartheta^{ab}) i S_{ab}(s, \varsigma) + \frac{1}{2} (\vartheta^{ac} \omega_u{}^{cb} - \omega_u{}^{ac} \vartheta^{cb}) i S_{ab}(s, \varsigma)$$

$$\Leftrightarrow \omega_u{}^{ab} \rightarrow \omega_u{}^{ab} + \vartheta^{ac} \omega_u{}^{cb} - \omega_u{}^{ac} \vartheta^{cb} - \partial_u \vartheta^{ab}$$

$$\Leftrightarrow \delta \omega_u{}^{ab} = \vartheta^{ac} \omega_u{}^{cb} - \omega_u{}^{ac} \vartheta^{cb} - \partial_u \vartheta^{ab}$$

推论2.2.23. 规范变换: 
$$\begin{cases} \delta\psi(s, \varsigma) = i\theta^{\alpha\varsigma}\sigma_{\alpha\varsigma}(s)\psi(s, \varsigma) \\ \delta A_u^{\alpha\varsigma} = i\theta^{\beta\varsigma}(-i\varepsilon_{\beta\varsigma}^{\alpha\varsigma}\gamma_\varsigma)A_u^{\gamma\varsigma} - \partial_u\theta^{\alpha\varsigma} \end{cases}$$

推论2.2.24.  $\delta F_{uv}^{\alpha\varsigma} = i\theta^{\beta\varsigma}(-i\varepsilon_{\beta\varsigma}^{\alpha\varsigma}\gamma_\varsigma)F_{uv}^{\gamma\varsigma}, \delta F_{uv}^{[\alpha\varsigma]} = i\theta^{\beta\varsigma}\gamma_{\beta\varsigma}F_{uv}^{[\alpha\varsigma]} = (i\omega + \varsigma\epsilon) \cdot \gamma F_{uv}^{[\alpha\varsigma]}$

推论2.2.25.  $\delta\omega_u^{ab} = \frac{i}{2}(\sigma_{-\varsigma\alpha\varsigma'}^ab\delta A_u^{\alpha\varsigma'} + \sigma_{\varsigma\alpha\varsigma}^ab\delta A_u^{\alpha\varsigma})$

### 2.2.5 毕安奇恒等式的类电磁场方程形式

$$\begin{cases} \nabla_d \cdot \vec{E}^{\beta\kappa} \equiv \rho^{\beta\kappa}, \nabla_d \times \vec{E}^{\beta\kappa} \equiv -D_t \vec{B}^{\beta\kappa} \\ \nabla_d \cdot \vec{B}^{\beta\kappa} \equiv 0, \nabla_d \times \vec{B}^{\beta\kappa} \equiv \vec{J}^{\beta\kappa} + D_t \vec{E}^{\beta\kappa} \end{cases} \Leftrightarrow \begin{cases} D^u F_{uv}^{\beta\kappa} \equiv -J_v^{\beta\kappa} \\ D^u * F_{uv}^{\beta\kappa} \equiv 0 \end{cases} \quad (8.16)$$

推论2.2.26.  $F_{uv}^{\beta\kappa} = \partial_u A_v^{\beta\kappa} - \partial_v A_u^{\beta\kappa} - \varepsilon^{\beta\kappa\gamma\delta} A_u^{\gamma\kappa} A_v^{\delta\kappa} \Leftrightarrow D^a * F_{ab}^{\beta\kappa} \equiv 0; F_{ab}^{\beta\kappa} = e_a^u e_b^v F_{uv}^{\beta\kappa}$

### 2.3 毕安奇恒等式的复矢量表述形式

复矢量张量形式:

定理2.3.1.  $D^a F_{ab}^{\beta\kappa} \equiv -J_b^{\beta\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\varsigma} D_a \tilde{\Psi}^{\alpha\varsigma\beta\kappa} \equiv iJ_b^{\beta\kappa}; F_{ab}^{\beta\kappa} = e_a^u e_b^v F_{uv}^{\beta\kappa}, \tilde{\Psi}^{\alpha\varsigma\beta\kappa} = \left[ \psi^{\alpha\varsigma\beta\kappa} = \frac{i}{2}\sigma_{\varsigma ab}^{\alpha\varsigma} F^{ab\beta\kappa} \right]$

证明:  $D^a F_{ab}^{\beta\kappa} \equiv -J_b^{\beta\kappa}$

$\Leftrightarrow D^a F_{ab}^{\beta\kappa} \equiv -J_b^{\beta\kappa}, D^a * F_{ab}^{\beta\kappa} \equiv 0$

$\Leftrightarrow D^a (F_{ab}^{\beta\kappa} - \varsigma * F_{ab}^{\beta\kappa}) \equiv -J_b^{\beta\kappa}$

$\Leftrightarrow D^a (i\sigma_{\varsigma ab}^{\alpha\varsigma} \psi_{\alpha\varsigma}^{\beta\kappa}) \equiv -J_b^{\beta\kappa}, \alpha\varsigma = 1, 2, 3$

$\Leftrightarrow D^a [(\sigma_{\varsigma}, -i\varsigma)^{\alpha\varsigma} |_{ab} \tilde{\Psi}^{\alpha\varsigma\beta\kappa}] \equiv iJ_b^{\beta\kappa}, \alpha\varsigma = 1, 2, 3, 4$

$\Leftrightarrow D^a [(\sigma_{-\varsigma}, -i\varsigma)_a |_{b\alpha\varsigma} \tilde{\Psi}^{\alpha\varsigma\beta\kappa}] \equiv iJ_b^{\beta\kappa}, \alpha\varsigma = 1, 2, 3, 4$

$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\varsigma} D_a \tilde{\Psi}^{\alpha\varsigma\beta\kappa} \equiv iJ_b^{\beta\kappa}, \alpha\varsigma = 1, 2, 3, 4$  □

复矢量矩阵形式:

推论2.3.1.  $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\varsigma} D_a \tilde{\Psi}^{\alpha\varsigma\beta\kappa} \equiv iJ_b^{\beta\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^{\beta\kappa}(1, \varsigma) \equiv iJ^{\beta\kappa}$

复矢量方矩阵形式:

推论2.3.2.  $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\varsigma} D_a \tilde{\Psi}^{\alpha\varsigma\beta\kappa} \equiv iJ_b^{\beta\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a [\tilde{\Psi}(1, \varsigma)] \equiv i[J]$

表象变换:

推论2.3.3.  $(\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^{\beta\kappa}(1, \varsigma) \equiv iJ^{\beta\kappa} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{\psi}^{\beta\kappa}(1, \varsigma) \equiv i\tilde{J}^{\beta\kappa}(1, \varsigma)$

推论2.3.4.  $(\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^{\beta\kappa}(1, \varsigma) \equiv iJ^{\beta\kappa} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a [\tilde{\psi}(1, \varsigma)] \equiv i[\tilde{J}]$

### 2.4 毕安奇恒等式的 $\frac{1}{2}$ -旋量表述形式<sup>[1,2]</sup>

$\frac{1}{2}$ -旋量Penrose抽象指标形式:

定理2.4.1.  $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\varsigma} D_a \tilde{\Psi}^{\alpha\varsigma\beta\kappa} \equiv iJ_b^{\beta\kappa} \Leftrightarrow \nabla_d^{A'_\varsigma A_\varsigma} \psi_{A'_\varsigma B_\varsigma}^{\beta\kappa} \equiv \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma}^{\beta\kappa}, \nabla_d^{A'_\varsigma A_\varsigma} = \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D^a$

证明:  $(\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\varsigma} D_a \tilde{\Psi}^{\alpha\varsigma\beta\kappa} \equiv iJ_b^{\beta\kappa}$

$\Leftrightarrow D^a (i\sigma_{\varsigma ab}^{\alpha\varsigma} \psi_{\alpha\varsigma}^{\beta\kappa}) \equiv -J_b^{\beta\kappa}$

$\Leftrightarrow D^a (i\sigma_{\varsigma ab}^{\alpha\varsigma} \cdot \frac{i\varsigma}{\sqrt{2}} \sigma_{\alpha\varsigma}^{A'_\varsigma B_\varsigma} \psi_{A'_\varsigma B_\varsigma}^{\beta\kappa}) \equiv -J_b^{\beta\kappa}$

$\Leftrightarrow iS_{ab}^{A'_\varsigma B_\varsigma} D^a \psi_{A'_\varsigma B_\varsigma}^{\beta\kappa} \equiv \frac{-\varsigma}{\sqrt{2}} J_b^{\beta\kappa}$

$\Leftrightarrow (\frac{\delta}{2} \delta_{ab} \varepsilon^{A'_\varsigma B_\varsigma} + iS_{ab}^{A'_\varsigma B_\varsigma}) D^a \psi_{A'_\varsigma B_\varsigma}^{\beta\kappa} \equiv \frac{-\varsigma}{\sqrt{2}} J_b^{\beta\kappa}$

$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \varepsilon_{A'_\varsigma B'_\varsigma} \cdot \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_b^{B'_\varsigma B_\varsigma} D^a \psi_{A'_\varsigma B_\varsigma}^{\beta\kappa} \equiv \frac{-1}{\sqrt{2}} J_b^{\beta\kappa}$

$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \varepsilon_{A'_\varsigma B'_\varsigma} D^a \psi_{A'_\varsigma B_\varsigma}^{\beta\kappa} \equiv \frac{-1}{\sqrt{2}} J_b^{\beta\kappa} \cdot \frac{-i\varsigma}{\sqrt{2}} (\sigma, i\varsigma)_b^{B'_\varsigma B_\varsigma}$

$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D^a \psi_{A'_\varsigma B_\varsigma}^{\beta\kappa} \equiv \frac{-\varsigma}{\sqrt{2}} \varsigma \varepsilon^{A'_\varsigma B'_\varsigma} J_{B'_\varsigma B_\varsigma}$

$\Leftrightarrow \nabla_d^{A'_\varsigma A_\varsigma} \psi_{A'_\varsigma B_\varsigma}^{\beta\kappa} \equiv \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma}^{\beta\kappa}, \nabla_d^{A'_\varsigma A_\varsigma} = \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D^a$  □

$\frac{1}{2}$ -旋量张量形式：

$$\text{推论2.4.1. } \nabla_d^{A'_\zeta A_\zeta} \psi_{A_\zeta B_\zeta}^{\beta_\kappa} \equiv \frac{-\zeta}{\sqrt{2}} J_{A'_\zeta B_\zeta}^{\beta_\kappa} \Leftrightarrow (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} D^a \psi_{A_\zeta B_\zeta}^{\beta_\kappa} \equiv i J_{A'_\zeta B_\zeta}^{\beta_\kappa}$$

 $\frac{1}{2}$ -旋量矩阵形式：

$$\text{推论2.4.2. } (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} D^a \psi_{A_\zeta B_\zeta}^{\beta_\kappa} \equiv i J_{A'_\zeta B_\zeta}^{\beta_\kappa} \Leftrightarrow (\sigma \otimes I, -i\zeta)_a D^a \tilde{\psi}^{\beta_\kappa}(1, \zeta) \equiv i \tilde{J}^{\beta_\kappa}(1, \zeta)$$

 $\frac{1}{2}$ -旋量方矩阵形式：

$$\text{推论2.4.3. } (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} D^a \psi_{A_\zeta B_\zeta}^{\beta_\kappa} \equiv i J_{A'_\zeta B_\zeta}^{\beta_\kappa} \Leftrightarrow (\sigma, -i\zeta)_a D^a [\psi]^{\beta_\kappa} \equiv i [J]^{\beta_\kappa}$$

2.5 毕安奇恒等式的全 $\frac{1}{2}$ -旋量表述形式

$$\text{推论2.5.1. } \nabla_d^{A'_\zeta A_\zeta} \psi_{A_\zeta B_\zeta}^{\beta_\kappa} \equiv \frac{-\zeta}{\sqrt{2}} J_{A'_\zeta B_\zeta}^{\beta_\kappa} \Leftrightarrow \nabla_d^{A'_\zeta A_\zeta} \psi_{A_\zeta B_\zeta C_\zeta D_\zeta} \equiv \frac{-\zeta}{\sqrt{2}} J_{A'_\zeta B_\zeta C_\zeta D_\zeta}$$

$$\text{推论2.5.2. } \nabla_d^{A'_\zeta A_\zeta} \psi_{A_\zeta B_\zeta}^{\beta_\zeta} \equiv \frac{-\zeta}{\sqrt{2}} J_{A'_\zeta B_\zeta}^{\beta_\zeta} \Leftrightarrow \nabla_d^{A'_\zeta A_\zeta} \psi_{A_\zeta B_\zeta C_\zeta D_\zeta} \equiv \frac{-\zeta}{\sqrt{2}} J_{A'_\zeta B_\zeta C_\zeta D_\zeta}$$

$$\text{推论2.5.3. } \nabla_d^{A'_\zeta A_\zeta} \psi_{A_\zeta B_\zeta}^{\beta_\kappa} \equiv \frac{-\zeta}{\sqrt{2}} J_{A'_\zeta B_\zeta}^{\beta_\kappa} \Leftrightarrow (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} D^a \psi_{A_\zeta B_\zeta C_\zeta D_\zeta} \equiv i J_{A'_\zeta B_\zeta C_\zeta D_\zeta}$$

$$\text{推论2.5.4. } \nabla_d^{A'_\zeta A_\zeta} \psi_{A_\zeta B_\zeta}^{\beta_\zeta} \equiv \frac{-\zeta}{\sqrt{2}} J_{A'_\zeta B_\zeta}^{\beta_\zeta} \Leftrightarrow (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} D^a \psi_{A_\zeta B_\zeta C_\zeta D_\zeta} \equiv i J_{A'_\zeta B_\zeta C_\zeta D_\zeta}$$

以下三个推论的证明留待以后

$$\text{推论2.5.5. } (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} D^a \psi_{A_\zeta B_\zeta C_\zeta D_\zeta} \equiv i J_{A'_\zeta B_\zeta C_\zeta D_\zeta}, R = 0 \Leftrightarrow [2D_a + iS_{ab}(2, \zeta)D^b]_{k_\zeta}{}^{l_\zeta} \psi_{l_\zeta}(2, \zeta) = \mathbb{J}_{ak_\zeta}(2, \zeta)$$

$$\text{推论2.5.6. } \begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0, R = 0 \end{cases} \Leftrightarrow [2D_a + iS_{ab}(2, \zeta)D^b]_{k_\zeta}{}^{l_\zeta} \psi_{l_\zeta}(2, \zeta) = \mathbb{J}_{ak_\zeta}(2, \zeta)$$

$$\text{推论2.5.7. } \begin{cases} (\sigma_{-\zeta}, -i\zeta)^a{}_{b\alpha_\zeta} D_a \tilde{\Psi}^{\alpha_\zeta \beta_\zeta} \equiv i J_b^{\beta_\zeta} \\ \psi_{\alpha_\zeta \beta_\zeta} = \psi_{\beta_\zeta \alpha_\zeta}, \psi_{\alpha_\zeta}{}^{\alpha_\zeta} = 0, (\sigma, -i\zeta)^a \sigma_{\beta_\zeta} J_a^{\beta_\zeta} = 0 \end{cases} \Leftrightarrow (\sigma \otimes I_4, -i\zeta)^a D_a \tilde{\psi}(2, \zeta) = i \tilde{J}(2, \zeta)$$

## 2.6 猜测

$$\text{定理2.6.1. } D^a * F_{ab}^{\beta_\kappa} = 0 \Leftrightarrow F_{ab}^{\beta_\kappa} \Leftrightarrow D^a * F_{ab}^{\beta_\kappa} \equiv 0$$

$$\text{定理2.6.2. } D^a F_{ab}^{\beta_\kappa} = -J_b^{\beta_\kappa}, D^a * F_{ab}^{\beta_\kappa} = 0 \Leftrightarrow D^a F_{ab}^{\beta_\kappa} = -J_b^{\beta_\kappa}, F_{ab}^{\beta_\kappa}$$

2.7 毕安奇恒等式<sup>[7]</sup>的自旋张量表述形式

$$\text{引力场的自旋张量矩阵: } S_{ab} = i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta} \succ \begin{bmatrix} 0 & \gamma_z & -\gamma_y & -\zeta\gamma_x \\ -\gamma_z & 0 & \gamma_x & -\zeta\gamma_y \\ \gamma_y & -\gamma_x & 0 & -\zeta\gamma_z \\ \zeta\gamma_x & \zeta\gamma_y & \zeta\gamma_z & 0 \end{bmatrix} \quad (8.17)$$

$$\text{定理2.7.1. } (D_a + iS_{ab}D^b)^{\beta_\zeta}{}_{\gamma_\zeta} \psi^{\gamma_\zeta \delta_\kappa}(1, \zeta) \equiv -i\sigma_{\zeta ab}^{\beta_\zeta} J^{b\delta_\kappa}, S_{ab} = i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta} \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a D_a \tilde{\Psi}^{\delta_\kappa}(1, \zeta) \equiv i J^{\delta_\kappa}$$

一种直观证法如下：

$$\text{证明: } (D_a + iS_{ab}D^b)^{\beta_\zeta}{}_{\gamma_\zeta} \psi^{\gamma_\zeta \delta_\kappa} \equiv -i\sigma_{\zeta ab}^{\beta_\zeta} J^{b\delta_\kappa}, S_{ab} = i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta} \\ \Leftrightarrow \begin{cases} (D_x + i\gamma_z D_y - i\gamma_y D_z - i\zeta\gamma_x D_\pi)^{\beta_\zeta}{}_{\gamma_\zeta} \psi^{\gamma_\zeta \delta_\kappa} \equiv -i\sigma_{\zeta xb}^{\beta_\zeta} J^{b\delta_\kappa} \\ (D_y + i\gamma_x D_z - i\gamma_z D_x - i\zeta\gamma_y D_\pi)^{\beta_\zeta}{}_{\gamma_\zeta} \psi^{\gamma_\zeta \delta_\kappa} \equiv -i\sigma_{\zeta yb}^{\beta_\zeta} J^{b\delta_\kappa} \\ (D_z + i\gamma_y D_x - i\gamma_x D_y - i\zeta\gamma_z D_\pi)^{\beta_\zeta}{}_{\gamma_\zeta} \psi^{\gamma_\zeta \delta_\kappa} \equiv -i\sigma_{\zeta zb}^{\beta_\zeta} J^{b\delta_\kappa} \\ (D_\pi + i\zeta\gamma_x D_x + i\zeta\gamma_y D_y + i\zeta\gamma_z D_z)^{\beta_\zeta}{}_{\gamma_\zeta} \psi^{\gamma_\zeta \delta_\kappa} \equiv -i\sigma_{\zeta \pi b}^{\beta_\zeta} J^{b\delta_\kappa} \end{cases}$$



$$\begin{aligned}
& \Leftrightarrow \left\{ \begin{array}{l} \begin{bmatrix} D_x & D_y & D_z \\ -D_y & D_x & -\varsigma D_\pi \\ -D_z & \varsigma D_\pi & D_x \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma \delta_\kappa} \\ \psi^{y_\varsigma \delta_\kappa} \\ \psi^{z_\varsigma \delta_\kappa} \end{bmatrix} \equiv \begin{bmatrix} \varsigma J^{\pi \delta_\kappa} \\ J^{z \delta_\kappa} \\ -J^{y \delta_\kappa} \end{bmatrix}, \begin{bmatrix} D_y & -D_x & \varsigma D_\pi \\ D_x & D_y & D_z \\ -\varsigma D_\pi & -D_z & D_y \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma \delta_\kappa} \\ \psi^{y_\varsigma \delta_\kappa} \\ \psi^{z_\varsigma \delta_\kappa} \end{bmatrix} \equiv \begin{bmatrix} -J^{z \delta_\kappa} \\ \varsigma J^{\pi \delta_\kappa} \\ J^{x \delta_\kappa} \end{bmatrix} \\ \begin{bmatrix} D_z & -\varsigma D_\pi & -D_x \\ \varsigma D_\pi & D_z & -D_y \\ D_x & D_y & D_z \end{bmatrix} \begin{bmatrix} \psi^{x_\varsigma \delta_\kappa} \\ \psi^{y_\varsigma \delta_\kappa} \\ \psi^{z_\varsigma \delta_\kappa} \end{bmatrix} \equiv \begin{bmatrix} J^{y \delta_\kappa} \\ -J^{x \delta_\kappa} \\ \varsigma J^{\pi \delta_\kappa} \end{bmatrix}, iD_\pi \Psi^{\delta_\kappa}(1, \varsigma) \equiv \varsigma \gamma \cdot \nabla_d \Psi^{\delta_\kappa}(1, \varsigma) - i\varsigma \vec{J}^{\delta_\kappa} \\ \begin{bmatrix} iD_\pi \Psi^{\delta_\kappa}(1, \varsigma) \equiv i\varsigma \nabla_d \times \Psi^{\delta_\kappa}(1, \varsigma) - i\varsigma \vec{J}^{\delta_\kappa} \\ \nabla_d \cdot \Psi^{\delta_\kappa}(1, \varsigma) \equiv \varsigma J^{\pi \delta_\kappa} \end{bmatrix} \\ \begin{bmatrix} iD_\pi \Psi^{\delta_\kappa}(1, \varsigma) \equiv \varsigma \gamma \cdot \nabla_d \Psi^{\delta_\kappa}(1, \varsigma) - i\varsigma \vec{J}^{\delta_\kappa} \\ \nabla_d \cdot \Psi^{\delta_\kappa}(1, \varsigma) \equiv \varsigma J^{\pi \delta_\kappa} \end{bmatrix} \\ \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^{\delta_\kappa}(1, \varsigma) \equiv iJ \end{array} \right. \quad \square
\end{aligned}$$

另一种更解析更抽象的证法如下：

$$\begin{aligned}
& \text{证明: } (D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma \delta_\kappa} \equiv -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\delta_\kappa}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \\
& \Leftrightarrow \sigma_{\varsigma a}^{\beta_\varsigma} \sigma_{\varsigma \gamma_\varsigma cb} D^b \psi^{\gamma_\varsigma \delta_\kappa} \equiv -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\delta_\kappa} \\
& \Leftrightarrow \sigma_{\varsigma ac}^{\beta_\varsigma} \sigma_{\varsigma \gamma_\varsigma}^{cb} D_b \psi^{\gamma_\varsigma \delta_\kappa} \equiv -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\delta_\kappa} \\
& \Leftrightarrow \sigma_{\beta_\varsigma}^{\varsigma ad} \sigma_{\varsigma ac}^{\beta_\varsigma} \sigma_{\varsigma \gamma_\varsigma}^{cb} D_b \psi^{\gamma_\varsigma \delta_\kappa} \equiv -i\sigma_{\beta_\varsigma}^{\varsigma ad} \sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\delta_\kappa} \\
& \Leftrightarrow \sigma_{\varsigma \gamma_\varsigma}^{db} D_b \psi^{\gamma_\varsigma \delta_\kappa} \equiv -iJ^{d\delta_\kappa} \\
& \Leftrightarrow \sigma_{\varsigma \alpha_\varsigma}^{ab} D_a \psi^{\alpha_\varsigma \delta_\kappa} \equiv iJ^{b\delta_\kappa}, \alpha_\varsigma = 1, 2, 3 \\
& \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha_\varsigma} D_a \tilde{\Psi}^{\alpha_\varsigma \delta_\kappa} \equiv iJ_b^{\delta_\kappa}, \alpha_\varsigma = 1, 2, 3, 4 \quad \square
\end{aligned}$$

此方程(3.3.2)就是毕安奇恒等式的自旋张量表述形式。

$$\text{引理2.7.1. } \mathbb{J}_a^{\beta_\varsigma \delta_\kappa} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\delta_\kappa} \Leftrightarrow \begin{cases} \mathbb{J}_y^{z_\varsigma \delta_\kappa} = -\mathbb{J}_z^{y_\varsigma \delta_\kappa} = -\varsigma \mathbb{J}_\pi^{x_\varsigma \delta_\kappa} = J^{x\delta_\kappa} \\ \mathbb{J}_z^{x_\varsigma \delta_\kappa} = -\mathbb{J}_x^{z_\varsigma \delta_\kappa} = -\varsigma \mathbb{J}_\pi^{y_\varsigma \delta_\kappa} = J^{y\delta_\kappa} \\ \mathbb{J}_x^{y_\varsigma \delta_\kappa} = -\mathbb{J}_y^{x_\varsigma \delta_\kappa} = -\varsigma \mathbb{J}_\pi^{z_\varsigma \delta_\kappa} = J^{z\delta_\kappa} \\ \mathbb{J}_x^{z_\varsigma \delta_\kappa} = \mathbb{J}_y^{y_\varsigma \delta_\kappa} = \mathbb{J}_z^{z_\varsigma \delta_\kappa} = \varsigma J^{\pi \delta_\kappa} \end{cases}$$

展开即可证明。以上自旋方程是关于特殊的源项，那么对于一般的源项又会怎样呢？请看下面的定理。

定理2.7.2.

$$(D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma \delta_\kappa} = \mathbb{J}_a^{\beta_\varsigma \delta_\kappa}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^{\delta_\kappa}(1, \varsigma) = iJ^{\delta_\kappa}, \mathbb{J}_a^{\beta_\varsigma \delta_\kappa} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b\delta_\kappa}$$

$$\begin{aligned}
& \text{证明: } (D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma \delta_\kappa} = \mathbb{J}_a^{\beta_\varsigma \delta_\kappa}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \\
& \Leftrightarrow \begin{cases} (D_x + i\gamma_z D_y - i\gamma_y D_z - i\varsigma \gamma_x D_\pi)^{\beta_\varsigma} \psi^{\gamma_\varsigma \delta_\kappa} = \mathbb{J}_x^{\beta_\varsigma \delta_\kappa} \\ (D_y + i\gamma_x D_z - i\gamma_z D_x - i\varsigma \gamma_y D_\pi)^{\beta_\varsigma} \psi^{\gamma_\varsigma \delta_\kappa} = \mathbb{J}_y^{\beta_\varsigma \delta_\kappa} \\ (D_z + i\gamma_y D_x - i\gamma_x D_y - i\varsigma \gamma_z D_\pi)^{\beta_\varsigma} \psi^{\gamma_\varsigma \delta_\kappa} = \mathbb{J}_z^{\beta_\varsigma \delta_\kappa} \\ (D_\pi + i\varsigma \gamma_x D_x + i\varsigma \gamma_y D_y + i\varsigma \gamma_z D_z)^{\beta_\varsigma} \psi^{\gamma_\varsigma \delta_\kappa} = \mathbb{J}_\pi^{\beta_\varsigma \delta_\kappa} \end{cases}
\end{aligned}$$

$$\Leftrightarrow \left\{ \begin{array}{l} \begin{array}{l} \left[ \begin{array}{ccc} D_x & D_y & D_z \\ -D_y & D_x & -\varsigma D_\pi \\ -D_z & \varsigma D_\pi & D_x \end{array} \right] \begin{bmatrix} \psi^{x_\varsigma \delta_\kappa} \\ \psi^{y_\varsigma \delta_\kappa} \\ \psi^{z_\varsigma \delta_\kappa} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_{x_\varsigma \delta_\kappa} \\ \mathbb{J}_{y_\varsigma \delta_\kappa} \\ \mathbb{J}_{z_\varsigma \delta_\kappa} \end{bmatrix} \\ \left[ \begin{array}{ccc} D_y & -D_x & \varsigma D_\pi \\ D_x & D_y & D_z \\ -\varsigma D_\pi & -D_z & D_y \end{array} \right] \begin{bmatrix} \psi^{x_\varsigma \delta_\kappa} \\ \psi^{y_\varsigma \delta_\kappa} \\ \psi^{z_\varsigma \delta_\kappa} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_{y_\varsigma \delta_\kappa} \\ \mathbb{J}_{x_\varsigma \delta_\kappa} \\ \mathbb{J}_{y_\varsigma \delta_\kappa} \end{bmatrix} \\ \left[ \begin{array}{ccc} D_z & -\varsigma D_\pi & -D_x \\ \varsigma D_\pi & D_z & -D_y \\ D_x & D_y & D_z \end{array} \right] \begin{bmatrix} \psi^{x_\varsigma \delta_\kappa} \\ \psi^{y_\varsigma \delta_\kappa} \\ \psi^{z_\varsigma \delta_\kappa} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_{x_\varsigma \delta_\kappa} \\ \mathbb{J}_{z_\varsigma \delta_\kappa} \\ \mathbb{J}_{z_\varsigma \delta_\kappa} \end{bmatrix} \end{array} \Leftrightarrow \begin{cases} \nabla_d \cdot \Psi^{\delta_\kappa}(1, \varsigma) = \mathbb{J}_{x_\varsigma \delta_\kappa} \\ [\nabla_d \times \Psi^{\delta_\kappa}(1, \varsigma)]^{z_\varsigma \delta_\kappa} - \varsigma D_\pi \psi^{z_\varsigma \delta_\kappa}(1, \varsigma) = \mathbb{J}_{x_\varsigma \delta_\kappa} \\ -[\nabla_d \times \Psi^{\delta_\kappa}(1, \varsigma)]^{y_\varsigma \delta_\kappa} + \varsigma D_\pi \psi^{y_\varsigma \delta_\kappa}(1, \varsigma) = \mathbb{J}_{x_\varsigma \delta_\kappa} \\ -[\nabla_d \times \Psi^{\delta_\kappa}(1, \varsigma)]^{z_\varsigma \delta_\kappa} + \varsigma D_\pi \psi^{z_\varsigma \delta_\kappa}(1, \varsigma) = \mathbb{J}_{y_\varsigma \delta_\kappa} \\ \nabla_d \cdot \Psi^{\delta_\kappa}(1, \varsigma) = \mathbb{J}_{y_\varsigma \delta_\kappa} \\ [\nabla_d \times \Psi^{\delta_\kappa}(1, \varsigma)]^{x_\varsigma \delta_\kappa} - \varsigma D_\pi \psi^{x_\varsigma \delta_\kappa}(1, \varsigma) = \mathbb{J}_{y_\varsigma \delta_\kappa} \\ [\nabla_d \times \Psi^{\delta_\kappa}(1, \varsigma)]^{y_\varsigma \delta_\kappa} - \varsigma D_\pi \psi^{y_\varsigma \delta_\kappa}(1, \varsigma) = \mathbb{J}_{z_\varsigma \delta_\kappa} \\ -[\nabla_d \times \Psi^{\delta_\kappa}(1, \varsigma)]^{x_\varsigma \delta_\kappa} + \varsigma D_\pi \psi^{x_\varsigma \delta_\kappa}(1, \varsigma) = \mathbb{J}_{z_\varsigma \delta_\kappa} \\ \nabla_d \cdot \Psi^{\delta_\kappa}(1, \varsigma) = \mathbb{J}_{z_\varsigma \delta_\kappa} \end{cases} \\ D_\pi \Psi^{\delta_\kappa}(1, \varsigma) + i\varsigma \gamma \cdot \nabla_d \psi^{\delta_\kappa} = \mathbb{J}_\pi^{\delta_\kappa} \Leftrightarrow D_\pi \Psi^{\delta_\kappa}(1, \varsigma) - \varsigma \nabla_d \times \Psi^{\delta_\kappa}(1, \varsigma) = \mathbb{J}_\pi^{\delta_\kappa} \\ \begin{cases} \mathbb{J}_y^{z_\varsigma \delta_\kappa} = -\mathbb{J}_z^{y_\varsigma \delta_\kappa} = -\varsigma \mathbb{J}_\pi^{x_\varsigma \delta_\kappa} := J^{x \delta_\kappa} \\ \mathbb{J}_z^{x_\varsigma \delta_\kappa} = -\mathbb{J}_x^{z_\varsigma \delta_\kappa} = -\varsigma \mathbb{J}_\pi^{y_\varsigma \delta_\kappa} := J^{y \delta_\kappa} \\ \mathbb{J}_x^{y_\varsigma \delta_\kappa} = -\mathbb{J}_y^{x_\varsigma \delta_\kappa} = -\varsigma \mathbb{J}_\pi^{z_\varsigma \delta_\kappa} := J^{z \delta_\kappa} \\ \mathbb{J}_x^{x_\varsigma \delta_\kappa} = \mathbb{J}_y^{y_\varsigma \delta_\kappa} = \mathbb{J}_z^{z_\varsigma \delta_\kappa} := \varsigma J^{\pi \delta_\kappa} \\ D_\pi \Psi^{\delta_\kappa}(1, \varsigma) - \varsigma \nabla_d \times \Psi^{\delta_\kappa}(1, \varsigma) = i \vec{J}^{\delta_\kappa} \\ \nabla_d \cdot \Psi^{\delta_\kappa}(1, \varsigma) = -i J^{\pi \delta_\kappa} \end{cases} \\ \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^{\delta_\kappa}(1, \varsigma) = i J^{\delta_\kappa}, \mathbb{J}_a^{\beta_\varsigma \delta_\kappa} = -i \sigma_{\varsigma ab}^{\beta_\varsigma} J^{b \delta_\kappa} \end{array} \quad \square$$

另一种更解析更抽象的证法如下：

$$\text{定理2.7.3. } (D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma \delta_\kappa} = \mathbb{J}_a^{\beta_\varsigma \delta_\kappa}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \Leftrightarrow \mathbb{J}_a^{\beta_\varsigma \delta_\kappa} = \sigma_{\varsigma ab}^{\beta_\varsigma} (\sigma_{\varsigma \gamma_\varsigma}^{bc} D_c \psi^{\gamma_\varsigma \delta_\kappa})$$

$$\text{证明: } (D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma \delta_\kappa} = \mathbb{J}_a^{\beta_\varsigma \delta_\kappa}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma}$$

$$\Leftrightarrow \sigma_{\varsigma a}^{\beta_\varsigma c} \sigma_{\varsigma \gamma_\varsigma cb} D^b \psi^{\gamma_\varsigma \delta_\kappa} = \mathbb{J}_a^{\beta_\varsigma \delta_\kappa}$$

$$\Leftrightarrow \mathbb{J}_a^{\beta_\varsigma \delta_\kappa} = \sigma_{\varsigma ab}^{\beta_\varsigma} (\sigma_{\varsigma \alpha_\varsigma}^{bc} D_c \psi^{\alpha_\varsigma \delta_\kappa}) \quad \square$$

这个定理表明此自旋方程源项受到一定限制，不是随意的，只有前一定理描述的源项情形才有解，其他情形全无解。

$$\text{推论2.7.1. } (D_a + iS_{ab}D^b)^{\beta_\varsigma} \psi^{\gamma_\varsigma \delta_\kappa} = \mathbb{J}_a^{\beta_\varsigma \delta_\kappa}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma} \text{ 有解} \Leftrightarrow \mathbb{J}_a^{\beta_\varsigma \delta_\kappa} = -i\sigma_{\varsigma ab}^{\beta_\varsigma} J^{b \delta_\kappa}, \exists J^{b \delta_\kappa}$$

## 2.8 毕安奇恒等式的外尔表述形式

### 2.8.1 引力场外尔张量满足的经典毕安奇恒等式<sup>[14]</sup>

$$\text{定义2.8.1. } C^{abcd} \equiv R^{abcd} + \frac{1}{2}g^{a[d}R^{c]b} + \frac{1}{2}g^{b[c}R^{d]a} + \frac{1}{6}g^a[cg^{d]b}R$$

$$\text{推论2.8.1. } R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \Rightarrow C^{abcd}{}_{;a} \equiv -\frac{1}{2}R^{b[c;d]} + \frac{1}{12}g^{b[c}R^{d]}$$

$$\text{证明: } R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0$$

$$\Rightarrow R^{abcd}{}_{;a} \equiv -R^{b[c;d]}, R^{ba}{}_{;a} \equiv \frac{1}{2}R^{;b}$$

$$\Rightarrow C^{abcd}{}_{;a} \equiv R^{abcd}{}_{;a} + \frac{1}{2}g^{a[d}R^{c]b}{}_{;a} + \frac{1}{2}g^{b[c}R^{d]a}{}_{;a} + \frac{1}{6}g^a[cg^{d]b}R_{;a}$$

$$\Rightarrow C^{abcd}{}_{;a} \equiv -R^{b[c;d]} + \frac{1}{2}R^{b[c;d]} + \frac{1}{4}g^{b[c}R^{d]} - \frac{1}{6}g^{b[c}R^{d]}$$

$$\Leftrightarrow C^{abcd}{}_{;a} \equiv -\frac{1}{2}R^{b[c;d]} + \frac{1}{12}g^{b[c}R^{d]} \quad \square$$

$$\text{推论2.8.2. } C^{(*ab)cd} \equiv R^{(*ab)cd} + \frac{1}{2}\varepsilon^{abe[c}R^{d]}_e + \frac{1}{6}\varepsilon^{abcd}R$$

毕安奇恒等式的外尔张量形式：

$$\text{推论2.8.3. } \begin{cases} R^{abcd}{}_{;a} \equiv -R^{b[c;d]} \\ R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} C^{abcd}{}_{;a} \equiv -R^{b[c;d]} + \frac{1}{2}g^a[dR_{;a}^{c]b} + \frac{1}{2}g^{b[c}R_{;a}^{d]a} + \frac{1}{6}g^a[cg^{d]b}R_{;a} \\ C^{(*ab)cd}{}_{;a} \equiv \frac{1}{2}\varepsilon^{abe[c}R^d{}_{e;a} + \frac{1}{6}\varepsilon^{abcd}R_{;a} \end{cases}$$

$$\text{推论2.8.4. } \begin{cases} C^{abcd}{}_{;a} \equiv -\frac{1}{2}R^{b[c;d]} + \frac{1}{12}g^{b[c}R^{d]} \\ C^{(*ab)cd}{}_{;a} \equiv \frac{1}{2}\varepsilon^{abe[c}R^d{}_{e;a} + \frac{1}{6}\varepsilon^{abcd}R_{;a} \end{cases}$$

## 2.8.2 毕安奇恒等式的Weyl复矢量表述形式

定义2.8.2.  $\tilde{C}^{\alpha\kappa\beta\varsigma}(1, \varsigma) \equiv [C^{\alpha\kappa\beta\varsigma}, 0^{\beta\varsigma}]$

定理2.8.1.  $D^a F_{ab}^{\beta\varsigma} \equiv -J_b^{\beta\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\kappa} D_a \tilde{C}^{\alpha\kappa\beta\varsigma} \equiv i\frac{i}{2}\sigma_{\varsigma cd}^{\beta\varsigma}(R_b^{[c;d]} - \frac{1}{6}\delta_b^{[c}R^{d]})$

证明:  $D^a F_{ab}^{\beta\varsigma} \equiv -J_b^{\beta\varsigma}$

$$\Leftrightarrow D_a(i\sigma_{\varsigma\alpha\kappa}^{ab}\psi^{\alpha\kappa\beta\varsigma}) \equiv -J^{b\beta\varsigma}, \alpha\kappa = 1, 2, 3$$

$$\Leftrightarrow D_a[\sigma_{\varsigma\alpha\kappa}^{ab}(C^{\alpha\kappa\beta\varsigma} + \frac{1}{6}\delta^{\alpha\kappa\beta\varsigma}R)] \equiv iJ^{b\beta\varsigma}, \alpha\kappa = 1, 2, 3$$

$$\Leftrightarrow D_a(\sigma_{\varsigma\alpha\kappa}^{ab}C^{\alpha\kappa\beta\varsigma}) \equiv -\frac{1}{2}\sigma_{\varsigma cd}^{\beta\varsigma}R^{b[c;d]} - \frac{1}{6}\sigma_{\varsigma\alpha\kappa}^{ab}\delta^{\alpha\kappa\beta\varsigma}R_{;a}, \alpha\kappa = 1, 2, 3$$

$$\Leftrightarrow D_a(\sigma_{\varsigma\alpha\kappa}^{ab}C^{\alpha\kappa\beta\varsigma}) \equiv -\frac{1}{2}\sigma_{\varsigma cd}^{\beta\varsigma}R^{b[c;d]} + \frac{1}{6}\sigma_{\varsigma cd}^{\beta\varsigma}\delta^{b[c}R^{d]}, \alpha\kappa = 1, 2, 3$$

$$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\kappa} D_a \tilde{C}^{\alpha\kappa\beta\varsigma} \equiv i\frac{i}{2}\sigma_{\varsigma cd}^{\beta\varsigma}(R_b^{[c;d]} - \frac{1}{6}\delta_b^{[c}R^{d]}), \alpha\kappa = 1, 2, 3, 4$$

□

定义2.8.3.  $\bar{J}^{bcd} \equiv R^{b[c;d]} - \frac{1}{6}g^{b[c}R^{d]}, \bar{J}^{b\beta\varsigma} \equiv \frac{i}{2}\sigma_{\varsigma cd}^{\beta\varsigma}\bar{J}^{bcd}$

## 2.8.3 毕安奇恒等式的Weyl复矢量矩阵表述形式

复矢量矩阵形式:

$$\text{推论2.8.5. } (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\kappa} D_a \tilde{C}^{\alpha\kappa\beta\varsigma} \equiv i\bar{J}_b^{\beta\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\Psi}^{\beta\varsigma}(1, \varsigma) \equiv iJ^{\beta\varsigma}$$

复矢量方矩阵形式:

$$\text{推论2.8.6. } (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\kappa} D_a \tilde{C}^{\alpha\kappa\beta\varsigma} \equiv i\bar{J}_b^{\beta\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a [\tilde{C}(1, \varsigma)] \equiv i[\bar{J}]$$

表象变换:

$$\text{推论2.8.7. } (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{C}^{\beta\varsigma}(1, \varsigma) \equiv i\bar{J}^{\beta\varsigma} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a \tilde{c}^{\beta\varsigma}(1, \varsigma) \equiv i\tilde{J}^{\beta\varsigma}(1, \varsigma)$$

$$\text{推论2.8.8. } (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{C}^{\beta\varsigma}(1, \varsigma) \equiv i\bar{J}^{\beta\varsigma} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a D_a [\tilde{c}(1, \varsigma)] \equiv i[\tilde{J}]$$

## 2.8.4 毕安奇恒等式的 $\frac{1}{2}$ -外尔旋量表述形式<sup>[1, 2]</sup>

$\frac{1}{2}$ -旋量Penrose抽象指标形式:

$$\text{定理2.8.2. } (\sigma_{-\varsigma}, -i\varsigma)^a{}_{b\alpha\kappa} D_a \tilde{C}^{\alpha\kappa\beta\varsigma} \equiv i\bar{J}_b^{\beta\varsigma} \Leftrightarrow \nabla_d^{A'_\kappa A_\varsigma} C_{A_\varsigma B_\varsigma}^{\beta\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}\bar{J}_{A'_\kappa B_\varsigma}^{\beta\varsigma}$$

$\frac{1}{2}$ -旋量张量形式:

$$\text{推论2.8.9. } \nabla_d^{A'_\kappa A_\varsigma} C_{A_\varsigma B_\varsigma}^{\beta\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}\bar{J}_{A'_\kappa B_\varsigma}^{\beta\varsigma} \Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\kappa A_\varsigma} D^a C_{A_\varsigma B_\varsigma}^{\beta\varsigma} \equiv i\bar{J}_{A'_\kappa B_\varsigma}^{\beta\varsigma}$$

$\frac{1}{2}$ -旋量矩阵形式:

$$\text{推论2.8.10. } (\sigma, -i\varsigma)_a^{A'_\kappa A_\varsigma} D^a C_{A_\varsigma B_\varsigma}^{\beta\varsigma} \equiv i\bar{J}_{A'_\kappa B_\varsigma}^{\beta\varsigma} \Leftrightarrow (\sigma \otimes I, -i\varsigma)_a D^a \tilde{C}^{\beta\varsigma}(1, \varsigma) \equiv i\tilde{J}^{\beta\varsigma}(1, \varsigma)$$

$\frac{1}{2}$ -旋量方矩阵形式:

$$\text{推论2.8.11. } (\sigma, -i\varsigma)_a^{A'_\kappa A_\varsigma} D^a C_{A_\varsigma B_\varsigma}^{\beta\varsigma} \equiv i\bar{J}_{A'_\kappa B_\varsigma}^{\beta\varsigma} \Leftrightarrow (\sigma, -i\varsigma)_a D^a [C]^{\beta\varsigma} \equiv i[\bar{J}]^{\beta\varsigma}$$

## 2.8.5 毕安奇恒等式的全 $\frac{1}{2}$ -外尔旋量表述形式

$$\text{推论2.8.12. } \nabla_d^{A'_\kappa A_\varsigma} C_{A_\varsigma B_\varsigma}^{\beta\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}\bar{J}_{A'_\kappa B_\varsigma}^{\beta\varsigma} \Leftrightarrow \nabla_d^{A'_\kappa A_\varsigma} C_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}^{\beta\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}\bar{J}_{A'_\kappa B_\varsigma C_\varsigma D_\varsigma}^{\beta\varsigma}$$

$$\text{推论2.8.13. } \nabla_d^{A'_\kappa A_\varsigma} C_{A_\varsigma B_\varsigma}^{\beta\varsigma} \equiv \frac{-\varsigma}{\sqrt{2}}\bar{J}_{A'_\kappa B_\varsigma}^{\beta\varsigma} \Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\kappa A_\varsigma} D^a C_{A_\varsigma B_\varsigma C_\varsigma D_\varsigma}^{\beta\varsigma} \equiv i\bar{J}_{A'_\kappa B_\varsigma C_\varsigma D_\varsigma}^{\beta\varsigma}$$

### 2.8.6 毕安奇恒等式 [7] 的自旋张量外尔表述形式

定理2.8.3.  $(D_a + iS_{ab}D^b)^{\beta\zeta}_{\gamma\zeta} C^{\gamma\zeta\delta\zeta}(1, \zeta) \equiv -i\sigma_{\zeta ab}^{\beta\zeta} \bar{J}^{b\delta\zeta}, S_{ab} = i\sigma_{\zeta ab}^{\alpha\zeta} \gamma_{\alpha\zeta} \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a D_a \tilde{C}^{\delta\zeta}(1, \zeta) \equiv i\bar{J}^{\delta\zeta}$

此方程(2.8.3)就是毕安奇恒等式的自旋张量外尔表述形式。

定理2.8.4.

$(D_a + iS_{ab}D^b)^{\beta\zeta}_{\gamma\zeta} C^{\gamma\zeta\delta\zeta} = \bar{J}_a^{\beta\zeta\delta\zeta}, S_{ab} = i\sigma_{\zeta ab}^{\alpha\zeta} \gamma_{\alpha\zeta} \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a D_a \tilde{C}^{\delta\zeta}(1, \zeta) = i\bar{J}^{\delta\zeta}, \bar{J}_a^{\beta\zeta\delta\zeta} = -i\sigma_{\zeta ab}^{\beta\zeta} \bar{J}^{b\delta\zeta}$

这个定理表明此自旋方程源项受到一定限制，不是随意的，只有前一定理描述的源项情形才有解，其他情形全无解。

推论2.8.14.  $(D_a + iS_{ab}D^b)^{\beta\zeta}_{\gamma\zeta} C^{\gamma\zeta\delta\zeta} = \bar{J}_a^{\beta\zeta\delta\zeta}, S_{ab} = i\sigma_{\zeta ab}^{\alpha\zeta} \gamma_{\alpha\zeta} \Leftrightarrow \bar{J}_a^{\beta\zeta\delta\zeta} = \sigma_{\zeta ab}^{\beta\zeta} (\sigma_{\zeta\gamma\zeta}^{bc} D_c C^{\gamma\zeta\delta\zeta})$

推论2.8.15.  $(D_a + iS_{ab}D^b)^{\beta\zeta}_{\gamma\zeta} C^{\gamma\zeta\delta\zeta} = \bar{J}_a^{\beta\zeta\delta\zeta}, S_{ab} = i\sigma_{\zeta ab}^{\alpha\zeta} \gamma_{\alpha\zeta}$  有解  $\Leftrightarrow \bar{J}_a^{\beta\zeta\delta\zeta} = -i\sigma_{\zeta ab}^{\beta\zeta} \bar{J}^{b\delta\zeta}, \exists \bar{J}^{b\delta\zeta}$

### 2.8.7 毕安奇恒等式的全自旋张量外尔表述形式

推论2.8.16.  $(\sigma, -i\zeta)_{a^{A'}A_\zeta} D^a C_{A_\zeta B_\zeta C_\zeta D_\zeta} \equiv i\bar{J}^{A'}_{B_\zeta C_\zeta D_\zeta} \Leftrightarrow [2D_a + iS_{ab}(2, \zeta)D^b]_{k_\zeta}{}^{l_\zeta} c_{l_\zeta}(2, \zeta) \equiv \bar{J}_{ak_\zeta}(2, \zeta)$

推论2.8.17.  $(\sigma, -i\zeta)_{a^{A'}A_\zeta} D^a C_{A_\zeta B_\zeta C_\zeta D_\zeta} \equiv i\bar{J}^{A'}_{B_\zeta C_\zeta D_\zeta} \Leftrightarrow (\sigma \otimes I_4, -i\zeta)^a D_a \tilde{c}(2, \zeta) \equiv i\tilde{J}(2, \zeta)$

以上两个命题的证明以后章节会补上，在此略去。

推论2.8.18.  $(\sigma_{-\zeta}, -i\zeta)^a D_a \tilde{C}^{\beta\zeta}(1, \zeta) \equiv i\bar{J}^{\beta\zeta} \Leftrightarrow (\sigma \otimes I_4, -i\zeta)^a D_a \tilde{c}(2, \zeta) \equiv i\tilde{J}(2, \zeta)$

### 2.9 毕安奇恒等式的经典分离形式

推论2.9.1.  $(\sigma_{-\zeta}, -i\zeta)^a D_a \tilde{\Psi}^{\beta\zeta}(1, \zeta) = iJ^{\beta\zeta} \Leftrightarrow (\gamma, -i\zeta)^a D_a \Psi^{\beta\zeta}(1, \zeta) = i\vec{J}^{\beta\zeta}, i_\zeta \nabla_d \cdot \Psi^{\beta\zeta}(1, \zeta) = iJ_{\pi_\zeta}^{\beta\zeta}$

推论2.9.2.  $\begin{cases} (\sigma_{-\zeta}, -i\zeta)^a D_a \tilde{\Psi}^{\beta\zeta}(1, \zeta) = iJ^{\beta\zeta} \\ \psi_{\alpha\zeta\beta\zeta} = \psi_{\beta\zeta\alpha\zeta}, \psi_{\alpha\zeta}{}^{\alpha\zeta} = 0, (\sigma, -i\zeta)^a \sigma_{\beta\zeta} J_a^{\beta\zeta} = 0 \end{cases} \Leftrightarrow \begin{cases} (\frac{1}{2}G_m, -i\zeta)^a D_a \Psi(2, \zeta) = i\vec{J}(2, \zeta) \\ i_\zeta \nabla_d \cdot \Psi^{\beta\zeta}(1, \zeta) = iJ_{\pi_\zeta}^{\beta\zeta} \end{cases}$

推论2.9.3.  $\begin{cases} (\frac{1}{2}G_m, -i\zeta)^a D_a \Psi(2, \zeta) = i\vec{J}(2, \zeta) \\ i_\zeta \nabla_d \cdot \Psi^{\beta\zeta}(1, \zeta) = iJ_{\pi_\zeta}^{\beta\zeta} \end{cases} \Leftrightarrow \begin{cases} [\frac{1}{2}\sigma(2), -i\zeta]^a D_a \psi(2, \zeta) = i\vec{J}(2, \zeta) \\ i_\zeta \nabla_d \cdot \Psi^{\beta\zeta}(1, \zeta) = iJ_{\pi_\zeta}^{\beta\zeta} \end{cases}$

推论2.9.4.  $\begin{cases} [\sigma(s), -is\zeta]^a D_a \psi(s, \zeta) = is\vec{J}(s, \zeta) \\ i_\zeta \nabla_d \cdot \Psi^{l_\zeta}(1, \zeta) = iJ_{\pi_\zeta}^{l_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\frac{1}{s}\sigma(s), -i\zeta]^a D_a \psi(s, \zeta) = i\vec{J}(s, \zeta) \\ i_\zeta \nabla_d \cdot \Psi^{l_\zeta}(1, \zeta) = iJ_{\pi_\zeta}^{l_\zeta} \end{cases}$

推论2.9.5.

$$S = \frac{1}{\sqrt{4}} \begin{bmatrix} \sqrt{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & \sqrt{3/2i} & -\sqrt{1/2i} & 0 & 0 & -\sqrt{1/2i} & \sqrt{3/2i} & 0 \\ 0 & -\sqrt{3/2} & \sqrt{1/2} & 0 & 0 & -\sqrt{1/2} & \sqrt{3/2} & 0 \\ 0 & 0 & 0 & -\sqrt{2i} & \sqrt{2i} & 0 & 0 & 0 \end{bmatrix}, S^+ = \frac{1}{\sqrt{4}} \begin{bmatrix} \sqrt{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 & 0 & -\sqrt{3/2i} & -\sqrt{3/2} & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{1/2i} & \sqrt{1/2} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & \sqrt{2i} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & -\sqrt{2i} \\ 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{1/2i} & -\sqrt{1/2} & 0 \\ 0 & 0 & 0 & \sqrt{1} & 0 & -\sqrt{3/2i} & \sqrt{3/2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{4} & 0 & 0 & 0 \end{bmatrix}$$

推论2.9.6.

$(\sigma \otimes I_4, -i\zeta)^a \partial_a \tilde{\psi}(2, \zeta) = i\tilde{J}(2, \zeta) \Leftrightarrow \begin{cases} [\sigma(2), -i\zeta]^a \partial_a \psi(2, \zeta) = i\bar{N}(2)\vec{J}(2, \zeta) \\ i_\zeta \nabla \cdot S_m^{l_\zeta}(2) S_{im}(2, +)\psi(2, \zeta) = iJ_{\pi_\zeta}^{l_\zeta}, J_{\pi_\zeta}^{l_\zeta} \succ J_\pi \end{cases} \begin{cases} [\bar{N}(2)\vec{J}(2, \zeta)]_{J_\pi} = S\vec{J}(2, \zeta) \\ [\psi(2, \zeta)]_{0_3} = S\tilde{\psi}(2, \zeta) \end{cases}$

推论2.9.7.  $S_{im}^{l_\zeta}(2) = \left( \begin{bmatrix} 0 & 0 & -1 & 0 & \frac{1}{\sqrt{3}} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} \end{bmatrix} \right)$

推论2.9.8.

$\begin{cases} [\sigma(2), -i\zeta]^a \partial_a \psi(2, \zeta) = i\bar{N}(2)\vec{J}(2, \zeta) \\ i_\zeta \nabla \cdot S_m^{l_\zeta}(2) S_{im}(2, +)\psi(2, \zeta) = iJ_{\pi_\zeta}^{l_\zeta} \end{cases} \Leftrightarrow \begin{cases} (\frac{1}{2}G_{im}(+), -i\zeta)^a \partial_a \Psi(2, \zeta) = i\vec{J}(2), \vec{J}(2) = S_{im}(2, +)\bar{N}(2)\vec{J}(2, \zeta) \\ i_\zeta \nabla \cdot S_m^{l_\zeta}(2) \Psi(2, \zeta) = iJ_{\pi_\zeta}^{l_\zeta}, \Psi(2, \zeta) = S_{im}(2, +)\bar{N}(2)\tilde{\psi}(2, \zeta) \end{cases}$

推论2.9.9.

$$(\sigma \otimes I_{2s}, -i\zeta)^a \partial_a \tilde{\psi}(s, \zeta) = i\tilde{J}(s, \zeta) \Leftrightarrow \begin{cases} [\sigma(s), -i\zeta]^a \partial_a \psi(s, \zeta) = i\bar{N}(s)\tilde{J}(s, \zeta) \\ i\zeta \nabla \cdot S^{l\zeta}(s)\psi(s, \zeta) = iJ_\pi^{l\zeta}, J_\pi^{l\zeta} \succ J_\pi \end{cases} \quad \begin{cases} \left[ \begin{array}{c} \bar{N}(s)\tilde{J}(s, \zeta) \\ J_\pi \end{array} \right] = S\tilde{J}(s, \zeta) \\ \left[ \begin{array}{c} \psi(s, \zeta) \\ 0 \end{array} \right] = S\tilde{\psi}(s, \zeta) \end{cases}$$

## 2.10 毕安奇恒等式的特殊类电磁场表述形式

### 2.10.1 毕安奇恒等式的双电磁场表述形式???

$$\text{推论2.10.1. } (\sigma, -i\zeta)_{a'}^{A' A_\zeta} D_a C_{A_\zeta B_\zeta C_\zeta D_\zeta} \equiv i\bar{J}_{B_\zeta C_\zeta D_\zeta}^{A' A_\zeta} \Leftrightarrow (\sigma, -i\zeta)_{a'}^{A' A_\zeta} D_a C_{A_\zeta l_\zeta}(\frac{3}{2}) \equiv i\bar{J}_{l_\zeta}^{A' A_\zeta}(\frac{3}{2})$$

$$\text{推论2.10.2. } (\sigma, -i\zeta)_{a'}^{A' A_\zeta} D_a C_{A_\zeta l_\zeta}(\frac{3}{2}) \equiv i\bar{J}_{l_\zeta}^{A' A_\zeta}(\frac{3}{2}) \Leftrightarrow (\sigma_{-\zeta} \otimes I, -i\zeta)_a D^a \tilde{C}(2, \zeta) \equiv i\tilde{J}(2, \zeta)$$

以上后面那个方程形式上相当于两个同时有电荷和磁荷的电磁场方程，并且满足洛伦兹协变，表征无挠引力场，和Einstein方程成立与否无关。所以对电磁场的一些分析技术可以用在此，从而可以得到引力的一些性质。

$$\text{定义2.10.1. } \Omega(\zeta) = \left( \begin{bmatrix} 0 & 0 \\ -\sigma_{\zeta y} & \sigma_{\zeta x} \end{bmatrix}, \begin{bmatrix} \sigma_{\zeta y} & -\sigma_{\zeta x} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right)$$

$$\text{推论2.10.3. } \tilde{C}(2, \zeta) \sim e^{(i\omega + \zeta\epsilon) \cdot R \otimes I_4 + (i\omega + \zeta\epsilon) \cdot \Omega(\zeta)}$$

$$\text{证明: } \Lambda[\tilde{C}(2, \zeta)] = S_{em}(\zeta) \otimes S_{em}(\frac{1}{2}) e^{(i\omega + \zeta\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \zeta\epsilon) \cdot \sigma(\frac{3}{2})} S_{em}^+(\zeta) \otimes S_{em}^+(\frac{1}{2}) \\ = e^{(i\omega + \zeta\epsilon) \cdot [R \otimes I_4 + \Omega(\zeta)]} = e^{(i\omega + \zeta\epsilon) \cdot R \otimes I_4 + (i\omega + \zeta\epsilon) \cdot \Omega(\zeta)} \quad \square$$

$$\text{推论2.10.4. } \tilde{J}(2, \zeta) \sim e^{(i\omega \cdot R - \zeta\epsilon \cdot L) \otimes I_4 + (i\omega + \zeta\epsilon) \cdot \Omega(\zeta)}$$

$$\text{证明: } \Lambda[\tilde{J}(2, \zeta)] = S_{em}(\zeta) \otimes S_{em}(\frac{1}{2}) e^{(i\omega - \zeta\epsilon) \cdot \sigma(\frac{1}{2})} \otimes e^{(i\omega + \zeta\epsilon) \cdot \sigma(\frac{3}{2})} S_{em}^+(\zeta) \otimes S_{em}^+(\frac{1}{2}) \\ = e^{(i\omega \cdot R - \zeta\epsilon \cdot L) \otimes I_4 + (i\omega + \zeta\epsilon) \cdot \Omega(\zeta)} \quad \square$$

### 2.10.2 毕安奇恒等式的特殊类电磁场表述形式

$$\text{推论2.10.5. } J^{A'_{B_\zeta C_\zeta D_\zeta}} \text{ 关于指标 } B_\zeta C_\zeta D_\zeta \text{ 全对称} \Leftrightarrow [(\sigma, -i\zeta)^a \sigma_{\alpha\zeta}] J_a^{\alpha\zeta} = 0$$

$$\text{推论2.10.6. } X_l^{\alpha\zeta} = 0 \Leftrightarrow X_a^{\alpha\zeta} = 0; [(\sigma, -i\zeta)^a \sigma_{\alpha\zeta}] X_a^{\alpha\zeta} = 0, l = x, y, z$$

$$\text{推论2.10.7. } (\sigma_{-\zeta}, -i\zeta)_{b\alpha\zeta}^a D_a \tilde{C}^{\alpha\zeta\beta\zeta} \equiv i\bar{J}_b^{\beta\zeta} \Leftrightarrow (\gamma, -i\zeta)_{l\alpha\zeta}^a D_a C^{\alpha\zeta\beta\zeta} \equiv i\bar{J}_l^{\beta\zeta}$$

以上协变方程表明在某种特殊情况下 $(\gamma, -i\zeta)_a$ 表现出某种协变性。一般协变方程以Pauli矩阵为基础进行构造，此方程却用光子自旋矩阵构造了一个完整协变的方程，这是我第一次见到这样的情况。之所以会出现如此的情况，是场和源的全对称性的缘故。

## 3 引力场的物理Yang-Mills规范方程

### 3.1 Einstein方程<sup>[11]</sup>和引力场Yang-Mills规范方程

$$\text{Einstein方程: } R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \quad (8.18)$$

$$\text{推论3.1.1. } R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \Leftrightarrow T^{ab}{}_{;b} = 0$$

$$\text{证明: } R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \\ \Rightarrow (R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab})_{;b} = -8\pi G T^{ab}{}_{;b} \\ \Rightarrow 0 = -8\pi G T^{ab}{}_{;b} \\ \Rightarrow T^{ab}{}_{;b} = 0 \quad \square$$

$$\text{推论3.1.2. } R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \Leftrightarrow R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}$$

$$\text{证明: } R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi G T^{ab} \rightarrow R = 8\pi G T + 4\Lambda \\ \Leftrightarrow R^{ab} - \frac{1}{2}g^{ab}(8\pi G T + 4\Lambda) + \Lambda g^{ab} = -8\pi G T^{ab} \\ \Leftrightarrow R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \rightarrow R = 8\pi G T + 4\Lambda \quad \square$$

$$\text{推论3.1.3. } \begin{cases} R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi GT^{ab} \\ R^{abcd}{}_{;a} \equiv -R^{b[c;d]}, R^{(ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi GT^{ab} \\ R^{abcd}{}_{;a} = 8\pi G(T^{b[c;d]} - \frac{1}{2}g^{b[c}T^{;d]}), R^{(ab)cd}{}_{;a} \equiv 0 \end{cases}$$

$$\text{推论3.1.4. } \begin{cases} R^{abcd}{}_{;a} = -J^{bcd} \\ R^{(ab)cd}{}_{;a} \equiv 0 \\ J^{bcd} \equiv -8\pi G(T^{b[c;d]} - \frac{1}{2}g^{b[c}T^{;d]}) \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}^{\alpha\zeta} = -J^{b\alpha\zeta} \\ D^a * F_{ab}^{\alpha\zeta} \equiv 0 \\ J^{b\alpha\zeta} \equiv \frac{1}{2}\zeta\sigma_{\zeta cd}^{\alpha\zeta} J^{bcd}, J^{bcd} \equiv -8\pi G(T^{b[c;d]} - \frac{1}{2}g^{b[c}T^{;d]}) \end{cases}$$

$$\text{推论3.1.5. } \begin{cases} R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi GT^{ab} \\ R^{abcd;e} + R^{abde;c} + R^{abec;d} \equiv 0 \end{cases} \Rightarrow \begin{cases} R^{abcd}{}_{;a} = -J^{bcd} \\ R^{(ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}^{\alpha\zeta} = -J^{b\alpha\zeta} \\ D^a * F_{ab}^{\alpha\zeta} \equiv 0 \end{cases}$$

$$\text{推论3.1.6. } \begin{cases} D^a F_{ab}^{\alpha\zeta} = -J^{b\alpha\zeta} \\ D^a * F_{ab}^{\alpha\zeta} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}^{[\alpha\zeta]} = -J^{b[\alpha\zeta]} \\ D^a * F_{ab}^{[\alpha\zeta]} \equiv 0 \end{cases}$$

### 3.2 引力场Yang-Mills规范方程的旋量表述形式

只要将Einstein方程 $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}$ 代入毕安奇恒等式的各种表述形式的源项，并将相应的恒等号换为等号，即可得到引力场物理Yang-Mills规范方程的各种表述形式，形式上与毕安奇恒等式的各种表述形式完全一致，不再重复写出。从本质上讲，引力场Yang-Mills规范方程只是引力场的恒等式，与Einstein方程成立与否无关，但真正描述物理的是Einstein方程，只有将Einstein方程应用于引力场规范恒等式的源项后，引力场Yang-Mills规范方程才真正带有物理的引力源项，此时的引力场Yang-Mills规范方程才成为一个真正的物理方程。所以这一点完全不同于电磁场和Yang-Mills场情形，电磁场和Yang-Mills场的规范方程不只是恒等式，直接描述真正的物理。

### 3.3 自我评述

事实上，电磁场和引力场都可以归结到Yang-Mills场情形，当 $\sigma$ 为空时是电磁场；当 $\sigma = \beta_\kappa$ 时是引力场；当 $\sigma$ 为多个字母时，可以描述更一般情形。 $\sigma$ 既可以为内部指标也可以为外部指标，甚至是两类指标的混合，所以Yang-Mills场数学形式上已是很一般的情形了。

## 4 广义相对论Einstein方程<sup>[11-14]</sup>的等价矩阵形式

### 4.1 准备

$$\text{推论4.1.1. } R^{ab} = \zeta(F^{\alpha\zeta}\sigma_{\zeta\alpha\zeta})^{ab}, (F^{\alpha\zeta}\sigma_{-\zeta\alpha\zeta}')^{ab} = -(F^{\alpha\zeta}\sigma_{\zeta\alpha\zeta})^{ab}, F^{\alpha\zeta}{}_{ab} = F_{ab}^{\alpha\zeta}, R = -\zeta\sigma_{\zeta\alpha\zeta}{}^{ab}F_{ab}^{\alpha\zeta}$$

$$\text{推论4.1.2. } R^{ab} = -i(F^{\alpha\zeta}\sigma_{\zeta\alpha\zeta})^{ab}, F^{\alpha\zeta}\sigma_{-\zeta\alpha\zeta}' = F^{\alpha\zeta}\sigma_{\zeta\alpha\zeta}$$

$$\text{推论4.1.3. } R = i\sigma_{\zeta\alpha\zeta}{}^{ab}F_{ab}^{\alpha\zeta}$$

$$\text{推论4.1.4. } R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi GT^{ab} \Leftrightarrow R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}$$

$$\text{定义4.1.1. } \bar{T}^{ab} := 8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) - \Lambda g^{ab}, \bar{T}^b \equiv [\bar{T}_x{}^b, \bar{T}_y{}^b, \bar{T}_z{}^b, \bar{T}_\pi{}^b]^T$$

$$\text{定义4.1.2. } \mathcal{F}_{ab}(2, \zeta) \equiv [F_{ab}^{x\zeta}, F_{ab}^{y\zeta}, F_{ab}^{z\zeta}, 0_{ab}]^T, F_{ab}(2, \zeta) \equiv F_{ab}^{[\alpha\zeta]}, \mathcal{R} = [R, 0]$$

$$\text{定义4.1.3. } \mathcal{A}_u(\zeta) \equiv [A_u^{x\zeta}, A_u^{y\zeta}, A_u^{z\zeta}, 0_u]^T = A_u^{[\alpha\zeta]}(\zeta), \mathcal{J}_a(\zeta) \equiv [J_a^{x\zeta}, J_a^{y\zeta}, J_a^{z\zeta}, 0_a]^T = J_a^{[\alpha\zeta]}$$

$$\text{推论4.1.5. } F_{uv}^{\alpha\zeta} = \partial_u A_v^{\alpha\zeta} - \partial_v A_u^{\alpha\zeta} - \varepsilon^{\alpha\zeta}{}_{\beta\zeta\gamma\zeta} A_u^{\beta\zeta} A_v^{\gamma\zeta}$$

$$\Leftrightarrow \mathcal{F}_{uv}(\zeta) = \partial_u \mathcal{A}_v(\zeta) - \partial_v \mathcal{A}_u(\zeta) + i\mathcal{A}_u^T(\zeta)\mathcal{R}\mathcal{A}_v(\zeta) = [\partial_u + \frac{i}{2}\mathcal{A}_u^T(\zeta)\mathcal{R}]\mathcal{A}_v(\zeta) - [\partial_v + \frac{i}{2}\mathcal{A}_v^T(\zeta)\mathcal{R}]\mathcal{A}_u(\zeta)$$

### 4.2 Einstein方程的等价矩阵形式

$$\text{推论4.2.1. } R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)_a \mathcal{F}^{ab}(2, \zeta) = i\bar{T}^b$$

$$\text{证明: } R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab}$$

$$\Leftrightarrow (F^{\alpha\zeta}\sigma_{\zeta\alpha\zeta})^{ab} = -i\bar{T}^{ab}$$

$$\Leftrightarrow (\sigma_{\zeta\alpha\zeta}F^{\alpha\zeta})^{ab} = -i\bar{T}^{ab}$$

$$\Leftrightarrow [(\sigma_{\zeta}, -i\zeta)_{\alpha\zeta}F^{\alpha\zeta}]^{ab} = -i\bar{T}^{ab}$$

$$\Leftrightarrow (\sigma_{\zeta}, -i\zeta)_{ac}^{\alpha\zeta}F_{\alpha\zeta}^{cb} = -i\bar{T}_a^b$$

$$\Leftrightarrow (\sigma_{\zeta}, -i\zeta)_{ca}^{\alpha\zeta}F_{\alpha\zeta}^{cb} = i\bar{T}_a^b$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)_{ca}^{\alpha\zeta}F_{\alpha\zeta}^{cb} = i\bar{T}_a^b$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)_{ac}^{\alpha\zeta}F_{\alpha\zeta}^{ab} = i\bar{T}_c^b$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)_a\mathcal{F}^{ab}(2, \zeta) = i\bar{T}^b$$

□

$$\text{推论4.2.2. } R^{ab} = \frac{1}{4}\delta^{ab}R + \frac{1}{2}(\sigma_{\zeta\alpha\zeta}\sigma_{-\zeta\beta\zeta'})^{ab}\psi^{\alpha\zeta}\beta\zeta'$$

$$\text{推论4.2.3. } R_{ab} = -\bar{T}_{ab} \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)_a * \mathcal{F}^{ab}(2, \zeta) = i\zeta\bar{T}^b - \frac{i\zeta}{2}\delta^b\bar{T}$$

$$\text{证明: } R_{ab} = -\bar{T}_{ab}$$

$$\Leftrightarrow i\zeta(R_a^b - \frac{1}{2}\delta_a^b R) = -i\zeta(\bar{T}_a^b - \frac{1}{2}\delta_a^b\bar{T})$$

$$\Leftrightarrow \frac{i\zeta}{2}(2R_a^b - \frac{1}{2}\delta_a^b R - \frac{1}{2}\delta_a^b R) = -i\zeta(\bar{T}_a^b - \frac{1}{2}\delta_a^b\bar{T})$$

$$\Leftrightarrow \frac{i\zeta}{2}(\sigma_{\zeta\alpha\zeta}\sigma_{-\zeta}^{\beta\zeta'}\psi_{\beta\zeta'}^{\alpha\zeta} - \delta^{\alpha\zeta}\beta\zeta'\delta_a^b\psi_{\alpha\zeta}\beta\zeta') = -i\zeta(\bar{T}_a^b - \frac{1}{2}\delta_a^b\bar{T})$$

$$\Leftrightarrow \sigma_{\zeta\alpha\zeta}^{\alpha\zeta}\frac{i\zeta}{2}(\sigma_{-\zeta}^{\beta\zeta'}\psi_{\beta\zeta'}^{\alpha\zeta} - \sigma_{\zeta}^{\beta\zeta'}\psi_{\beta\zeta'}^{\alpha\zeta}) = -i\zeta(\bar{T}_a^b - \frac{1}{2}\delta_a^b\bar{T})$$

$$\Leftrightarrow \sigma_{\zeta\alpha\zeta}^{\alpha\zeta} * F_{\alpha\zeta}^{cb} = -i\zeta(\bar{T}_a^b - \frac{1}{2}\delta_a^b\bar{T})$$

$$\Leftrightarrow (\sigma_{\zeta}, -i\zeta)_{ac}^{\alpha\zeta} * F_{\alpha\zeta}^{cb} = -i\zeta(\bar{T}_a^b - \frac{1}{2}\delta_a^b\bar{T})$$

$$\Leftrightarrow (\sigma_{\zeta}, -i\zeta)_{ca}^{\alpha\zeta} * F_{\alpha\zeta}^{cb} = i\zeta(\bar{T}_a^b - \frac{1}{2}\delta_a^b\bar{T})$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)_{ca}^{\alpha\zeta} * F_{\alpha\zeta}^{cb} = i\zeta(\bar{T}_a^b - \frac{1}{2}\delta_a^b\bar{T})$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)_{ac}^{\alpha\zeta} * F_{\alpha\zeta}^{ab} = i\zeta(\bar{T}_c^b - \frac{1}{2}\delta_c^b\bar{T})$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)_a * \mathcal{F}^{ab}(2, \zeta) = i\zeta\bar{T}^b - \frac{i\zeta}{2}\delta^b\bar{T}$$

□

$$\text{推论4.2.4. } (\sigma_{-\zeta}, -i\zeta)_a\mathcal{F}^{ab}(2, \zeta) = i\bar{T}^b \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)_a * \mathcal{F}^{ab}(2, \zeta) = i\zeta\bar{T}^b - \frac{i\zeta}{2}\delta^b\bar{T}$$

自我评述：这里得到了一对美妙而简洁的都与Einstein方程等价的旋量方程，十分有意思。

$$\text{推论4.2.5. } \begin{cases} D^a F_{ab}^{\alpha\zeta} \equiv -J^{b\alpha\zeta} \\ D^a * F_{ab}^{\alpha\zeta} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} D_a \mathcal{F}^{ab}(2, \zeta) \equiv -\bar{\mathcal{J}}^b(\zeta) \\ D_a * \mathcal{F}^{ab}(2, \zeta) \equiv 0 \end{cases}$$

$$\text{推论4.2.6. } \begin{cases} R^{ab} - \frac{1}{2}g^{ab}R + \Lambda g^{ab} = -8\pi GT^{ab} \\ R^{abcd}{}_{;a} \equiv -R^{b[c;d]}, R^{(*ab)cd}{}_{;a} \equiv 0 \end{cases} \Leftrightarrow \begin{cases} \text{Einstein方程: } (\sigma_{-\zeta}, -i\zeta)_a\mathcal{F}^{ab}(2, \zeta) = -i\bar{T}^b \\ \text{毕安奇恒等式: } D_a \mathcal{F}^{ab}(2, \zeta) \equiv -\bar{\mathcal{J}}^b(\zeta), D_a * \mathcal{F}^{ab}(2, \zeta) \equiv 0 \end{cases}$$

$$\text{推论4.2.7. } (\sigma_{-\zeta}, -i\zeta)_a\mathcal{F}^{ab}(2, \zeta) = i\bar{T}^b, (\sigma_{-\zeta}, -i\zeta)^a\mathcal{A}_a(\zeta) = 0(\text{规范条件})$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a e_a^u e_b^v [\partial_u + i\mathcal{A}_u^T(\zeta)\mathcal{R}]\mathcal{A}_v(\zeta) = i\bar{\mathcal{T}}_b, (\sigma_{-\zeta}, -i\zeta)^a \mathcal{A}_a(\zeta) = 0$$

### 4.3 Einstein方程等价的低一阶导数自旋张量新形式

$$\text{推论4.3.1. } R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow [\delta_{ab} + iS_{ab}(\gamma, \zeta)]F^{bc}(2, \zeta) = -i\sigma_{\zeta ab}^{[\beta\zeta]}\bar{T}^{bc}$$

$$\text{推论4.3.2. } R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow [\delta_{ab} + iS_{ab}(\gamma, \zeta)] * F^{bc}(2, \zeta) = -i\sigma_{\zeta ab}^{[\beta\zeta]}(\bar{T}^{bc} - \frac{1}{2}\delta^{bc}\bar{T})$$

# 第九章 引力微子场方程的重新表述

## 1 应用常数张量定义引力微子场 [7] 的各种旋量

自我评述：本章提出了引力微子场方程的多种等价表述形式，并严格证明了它们之间的等价性。

### 1.1 引力微子理论的场强描述

定义1.1.1.  $F_{uv}(\frac{3}{2}, \varsigma) := D_u \psi_v(\varsigma) - D_v \psi_u(\varsigma)$

推论1.1.1.  $F_{uv}(\frac{3}{2}, \varsigma) = (\partial_u + \frac{i}{2} \sigma_{\alpha\varsigma} A_u^{\alpha\varsigma}) \psi_v(\varsigma) - (\partial_v + \frac{i}{2} \sigma_{\alpha\varsigma} A_v^{\alpha\varsigma}) \psi_u(\varsigma)$

证明:  $F_{uv}(\frac{3}{2}, \varsigma) := D_u \psi_v(\varsigma) - D_v \psi_u(\varsigma)$   
 $= [\partial_u \psi_v(\varsigma) + \Gamma_{uv}^\lambda \psi_\lambda(\varsigma) + \frac{i}{2} A_u^{\alpha\varsigma} \sigma_{\alpha\varsigma} \psi_v(\varsigma)] - [\partial_v \psi_u(\varsigma) + \Gamma_{vu}^\lambda \psi_\lambda(\varsigma) + \frac{i}{2} A_v^{\alpha\varsigma} \sigma_{\alpha\varsigma} \psi_u(\varsigma)]$   
 $= \partial_u \psi_v(\varsigma) - \partial_v \psi_u(\varsigma) + \frac{i}{2} \sigma_{\alpha\varsigma} [A_u^{\alpha\varsigma} \psi_v(\varsigma) - A_v^{\alpha\varsigma} \psi_u(\varsigma)]$   
 $= (\partial_u + \frac{i}{2} \sigma_{\alpha\varsigma} A_u^{\alpha\varsigma}) \psi_v(\varsigma) - (\partial_v + \frac{i}{2} \sigma_{\alpha\varsigma} A_v^{\alpha\varsigma}) \psi_u(\varsigma)$  □

推论1.1.2.  $\delta \psi_u(\varsigma) = i\theta^{\alpha\varsigma} \sigma_{\alpha\varsigma} (\frac{1}{2}) \psi_u(\varsigma), \delta F_{uv}(\frac{3}{2}, \varsigma) = i\theta^{\alpha\varsigma} \sigma_{\alpha\varsigma} (\frac{1}{2}) F_{uv}(\frac{3}{2}, \varsigma)$

与引力场对比:

推论1.1.3.  $F_{uv}(2, \varsigma) = (\partial_u + \frac{i}{2} \gamma_{\alpha\varsigma} A_u^{\alpha\varsigma}) \psi_v(\varsigma) - (\partial_v + \frac{i}{2} \gamma_{\alpha\varsigma} A_v^{\alpha\varsigma}) \psi_u(\varsigma)$

证明:  $F_{uv}(2, \varsigma) := \tilde{D}_u A_v(\varsigma) - \tilde{D}_v A_u(\varsigma)$   
 $= [\partial_u A_v(\varsigma) + \Gamma_{uv}^\lambda A_\lambda(\varsigma) + \frac{i}{2} A_u^{\alpha\varsigma} \gamma_{\alpha\varsigma} A_v(\varsigma)] - [\partial_v A_u(\varsigma) + \Gamma_{vu}^\lambda A_\lambda(\varsigma) + \frac{i}{2} A_v^{\alpha\varsigma} \gamma_{\alpha\varsigma} A_u(\varsigma)]$   
 $= \partial_u A_v(\varsigma) - \partial_v A_u(\varsigma) + \frac{i}{2} \gamma_{\alpha\varsigma} [A_u^{\alpha\varsigma} A_v(\varsigma) - A_v^{\alpha\varsigma} A_u(\varsigma)]$   
 $= (\partial_u + \frac{i}{2} \gamma_{\alpha\varsigma} A_u^{\alpha\varsigma}) A_v(\varsigma) - (\partial_v + \frac{i}{2} \gamma_{\alpha\varsigma} A_v^{\alpha\varsigma}) A_u(\varsigma)$  □

推论1.1.4.  $F_{uv}(2, \varsigma) = (\partial_u + \frac{i}{2} \gamma_{\alpha\varsigma} A_u^{\alpha\varsigma}) A_v(\varsigma) - (\partial_v + \frac{i}{2} \gamma_{\alpha\varsigma} A_v^{\alpha\varsigma}) A_u(\varsigma) \Leftrightarrow F_{uv}^{\alpha\varsigma} = \partial_u A_v^{\alpha\varsigma} - \partial_v A_u^{\alpha\varsigma} - \varepsilon^{\alpha\varsigma}{}_{\beta\zeta} \gamma_\zeta A_u^{\beta\varsigma} A_v^{\alpha\zeta}$

推论1.1.5.  $\delta A_u(\varsigma) = i\theta^{\alpha\varsigma} \gamma_{\alpha\varsigma} A_u - \partial_u \theta, \delta F_{uv}(2, \varsigma) = i\theta^{\alpha\varsigma} \gamma_{\alpha\varsigma} F_{uv}(2, \varsigma)$

### 1.2 引力微子场强的经典描述

$$F_{ab}^{Z\kappa} = \begin{bmatrix} 0 & B_z^{Z\kappa} & -B_y^{Z\kappa} & -iE_x^{Z\kappa} \\ -B_z^{Z\kappa} & 0 & B_x^{Z\kappa} & -iE_y^{Z\kappa} \\ B_y^{Z\kappa} & -B_x^{Z\kappa} & 0 & -iE_z^{Z\kappa} \\ iE_x^{Z\kappa} & iE_y^{Z\kappa} & iE_z^{Z\kappa} & 0 \end{bmatrix}, *F_{ab}^{Z\kappa} = \begin{bmatrix} 0 & -iE_z^{Z\kappa} & iE_y^{Z\kappa} & B_x^{Z\kappa} \\ iE_z^{Z\kappa} & 0 & -iE_x^{Z\kappa} & B_y^{Z\kappa} \\ -iE_y^{Z\kappa} & iE_x^{Z\kappa} & 0 & B_z^{Z\kappa} \\ -B_x^{Z\kappa} & -B_y^{Z\kappa} & -B_z^{Z\kappa} & 0 \end{bmatrix} \quad (9.1)$$

$$(\sigma, i\varsigma)_a (\sigma, -i\varsigma)_b F^{ab}(\frac{3}{2}, \varsigma) = 0, (\sigma, i\varsigma)_a (\sigma, -i\varsigma)_b *F^{ab}(\frac{3}{2}, \varsigma) = 0 \quad (9.2)$$

### 1.3 引力微子场强的复矢量描述

定义1.3.1. 引力微子场复矢量  $\psi_{\alpha\varsigma}^{Z\kappa} := \frac{i}{2} \sigma_{\varsigma\alpha\varsigma}^{ab} F_{ab}^{Z\kappa} = i\varsigma (E - i\varsigma B)_{\alpha\varsigma}^{Z\kappa} = (i\varsigma E + B)_{\alpha\varsigma}^{Z\kappa}$

推论1.3.1.  $\frac{1}{2} (F_{ab}^{Z\kappa} - \varsigma * F_{ab}^{Z\kappa}) = \frac{i}{2} \sigma_{\varsigma ab}^{\alpha\varsigma} \psi_{\alpha\varsigma}^{Z\kappa}$

证明:  $F_{ab}^{Z\kappa} = -F_{ba}^{Z\kappa}$   
 $\Leftrightarrow F_{ab}^{Z\kappa} = \frac{1}{2} S_{abcd} F^{cd}, *F_{ab}^{Z\kappa} := \frac{1}{2} \varepsilon_{abcd} F^{cd}$   
 $\Leftrightarrow F_{ab}^{Z\kappa} - \varsigma * F_{ab}^{Z\kappa} = \frac{1}{2} (S_{abcd} - \varsigma \varepsilon_{abcd}) F^{cd}$   
 $\Leftrightarrow F_{ab}^{Z\kappa} - \varsigma * F_{ab}^{Z\kappa} = -\frac{1}{2} \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\varsigma\alpha\varsigma cd} F^{cd}$   
 $\Leftrightarrow F_{ab}^{Z\kappa} - \varsigma * F_{ab}^{Z\kappa} = i\sigma_{\varsigma ab}^{\alpha\varsigma} \psi_{\alpha\varsigma}^{Z\kappa}$   
 $\Leftrightarrow \frac{1}{2} (F_{ab}^{Z\kappa} - \varsigma * F_{ab}^{Z\kappa}) = \frac{i}{2} \sigma_{\varsigma ab}^{\alpha\varsigma} \psi_{\alpha\varsigma}^{Z\kappa}$  □

推论1.3.2.  $\psi_{\alpha\varsigma}^{Z\kappa} = \frac{i}{2} \sigma_{\varsigma\alpha\varsigma}^{ab} \frac{1}{2} (F_{ab}^{Z\kappa} - \varsigma * F_{ab}^{Z\kappa})$



$$\text{推论1.3.3. } \psi_{\alpha\zeta}^{Z\kappa} = -\frac{i}{2}\zeta\sigma_{\zeta\alpha\zeta}^{ab} * F_{ab}^{Z\kappa}$$

$$\text{推论1.3.4. } \sigma_{\zeta\alpha\zeta}^{ab} (F_{ab}^{Z\kappa} + \zeta * F_{ab}^{Z\kappa}) = 0$$

$$\text{推论1.3.5. } F_{ab}^{Z\kappa} - \zeta * F_{ab}^{Z\kappa} = -\frac{1}{4}\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta}^{cd} (F_{cd}^{Z\kappa} - \zeta * F_{cd}^{Z\kappa})$$

$$\text{推论1.3.6. } F_{ab}^{Z\kappa} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^{Z\kappa} + \sigma_{+ab}^{\alpha}\psi_{\alpha}^{Z\kappa}), *F_{ab}^{Z\kappa} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^{Z\kappa} - \sigma_{+ab}^{\alpha}\psi_{\alpha}^{Z\kappa})$$

$$\text{证明: } F_{ab}^{Z\kappa} - \zeta * F_{ab}^{Z\kappa} = i\sigma_{\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}^{Z\kappa}$$

$$\Leftrightarrow F_{ab}^{Z\kappa} - *F_{ab}^{Z\kappa} = i\sigma_{+ab}^{\alpha}\psi_{\alpha}^{Z\kappa}, F_{ab}^{Z\kappa} + *F_{ab}^{Z\kappa} = i\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^{Z\kappa}$$

$$\Leftrightarrow F_{ab}^{Z\kappa} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^{Z\kappa} + \sigma_{+ab}^{\alpha}\psi_{\alpha}^{Z\kappa}), *F_{ab}^{Z\kappa} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^{Z\kappa} - \sigma_{+ab}^{\alpha}\psi_{\alpha}^{Z\kappa})$$

$$\Leftrightarrow F_{ab}^{Z\kappa} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^{Z\kappa} + \sigma_{+ab}^{\alpha}\psi_{\alpha}^{Z\kappa})$$

$$\Leftrightarrow *F_{ab}^{Z\kappa} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^{Z\kappa} - \sigma_{+ab}^{\alpha}\psi_{\alpha}^{Z\kappa})$$

□

$$\text{推论1.3.7. } F_{ab}^{Z\kappa} = -F_{ba}^{Z\kappa} \Leftrightarrow F_{ab}^{Z\kappa} = \frac{i}{2}(\sigma_{-ab}^{\alpha'}\psi_{\alpha'}^{Z\kappa} + \sigma_{+ab}^{\alpha}\psi_{\alpha}^{Z\kappa})$$

$$\text{推论1.3.8. } (\sigma, i\zeta)_a(\sigma, -i\zeta)_b F^{ab}(\zeta) = 0 \Leftrightarrow \sigma_{\alpha\zeta}\psi^{\alpha\zeta[Z\zeta]} = 0$$

## 1.4 引力微子场强的 $\frac{1}{2}$ -旋量描述 [1, 2]

$$\text{定义1.4.1. 引力微子场}\frac{1}{2}\text{-旋量张量: } \psi_{A_\zeta B_\zeta}^{Z\kappa} := \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\psi_{\alpha\zeta}^{Z\kappa} = \frac{i\zeta}{\sqrt{2}}S^{ab}{}_{A_\zeta B_\zeta} F_{ab}^{Z\kappa}$$

$$\text{推论1.4.1. } \psi_{A_\zeta B_\zeta}^{Z\kappa} = \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\psi_{\alpha\zeta}^{Z\kappa} \Leftrightarrow \psi_{\alpha\zeta}^{Z\kappa} = \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha\zeta}^{A_\zeta B_\zeta}\psi_{A_\zeta B_\zeta}^{Z\kappa}$$

$$\text{推论1.4.2. } \psi_{A_\zeta B_\zeta}^{Z\kappa} = \psi_{B_\zeta A_\zeta}^{Z\kappa}$$

$$\text{推论1.4.3. } \psi_{A_\zeta B_\zeta}^{Z\kappa} = \frac{-i}{\sqrt{2}}S^{ab}{}_{A_\zeta B_\zeta} * F_{ab}^{Z\kappa}$$

$$\text{推论1.4.4. } \frac{1}{2}(F_{ab}^{Z\kappa} - \zeta * F_{ab}^{Z\kappa}) = \frac{i\zeta}{\sqrt{2}}S_{ab}{}^{A_\zeta B_\zeta}\psi_{A_\zeta B_\zeta}^{Z\kappa} \Leftrightarrow \psi_{A_\zeta B_\zeta}^{Z\kappa} = \frac{i\zeta}{\sqrt{2}}S^{ab}{}_{A_\zeta B_\zeta} \frac{1}{2}(F_{ab}^{Z\kappa} - \zeta * F_{ab}^{Z\kappa})$$

$$\text{推论1.4.5. } F_{ab}^{Z\kappa} - \zeta * F_{ab}^{Z\kappa} = -\frac{1}{2}S_{ab}{}^{A_\zeta B_\zeta} S^{cd}{}_{A_\zeta B_\zeta} (F_{cd}^{Z\kappa} - \zeta * F_{cd}^{Z\kappa})$$

$$\text{推论1.4.6. } F_{ab}^{Z\kappa} = \frac{i\zeta}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'}^{Z\kappa} + S_{ab}{}^{AB}\psi_{AB}^{Z\kappa}), *F_{ab}^{Z\kappa} = \frac{i\zeta}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'}^{Z\kappa} - S_{ab}{}^{AB}\psi_{AB}^{Z\kappa})$$

$$\text{推论1.4.7. } F_{ab}^{Z\kappa} = -F_{ba}^{Z\kappa} \Leftrightarrow F_{ab}^{Z\kappa} = \frac{i\zeta}{\sqrt{2}}(S_{ab}{}^{A'B'}\psi_{A'B'}^{Z\kappa} + S_{ab}{}^{AB}\psi_{AB}^{Z\kappa})$$

结合推论1.3.6和(1.274), (1.275) 式可得Penrose对应记法 [1, 2]:

$$\text{推论1.4.8. } F_{ab}^{Z\kappa} \stackrel{P}{=} \frac{1}{\sqrt{2}}(\psi_{A'B'}^{Z\kappa}\varepsilon_{AB} + \psi_{AB}^{Z\kappa}\varepsilon_{A'B'}), *F_{ab}^{Z\kappa} \stackrel{P}{=} \frac{1}{\sqrt{2}}(\psi_{A'B'}^{Z\kappa}\varepsilon_{AB} - \psi_{AB}^{Z\kappa}\varepsilon_{A'B'})$$

$$\text{推论1.4.9. } (\sigma, i\zeta)_a(\sigma, -i\zeta)_b F^{ab}(\zeta) = 0 \Leftrightarrow \sigma_{\alpha\zeta}\psi^{\alpha\zeta[Z\zeta]} = 0 \Leftrightarrow \psi_{A_\zeta B_\zeta C_\zeta} = \frac{1}{3!}\psi(A_\zeta B_\zeta C_\zeta)$$

## 1.5 引力微子场强的1-旋量描述

$$\text{定义1.5.1. 引力微子场}1\text{-旋量}\psi_{k_\zeta}^{Z\kappa}(1) := \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1)\psi_{A_\zeta B_\zeta}^{Z\kappa} = \Gamma_{k_\zeta}^{\alpha\zeta}(1)\psi_{\alpha\zeta}^{Z\kappa}$$

$$\text{推论1.5.1. } \psi_{A_\zeta B_\zeta}^{Z\kappa} = \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\psi_{k_\zeta}^{Z\kappa}(1), \psi_{\alpha\zeta}^{Z\kappa} = \Gamma_{\alpha\zeta}^{k_\zeta}(1)\psi_{k_\zeta}^{Z\kappa}(1)$$

## 1.6 引力微子场源的 $\frac{1}{2}$ -旋量描述 [1, 2]

$$\text{定义1.6.1. 引力微子场源}\frac{1}{2}\text{-旋量张量}J^{A_\zeta A_\zeta}{}_{Z\kappa} := \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a^{A_\zeta A_\zeta} J^{aZ\kappa}, J_{A_\zeta A_\zeta}^{Z\kappa} := \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{A_\zeta A_\zeta}^a J_a^{Z\kappa}$$

$$\text{Penrose记法: } J^{aZ\kappa} \stackrel{P}{=} J^{A'AZ\kappa}, J_a^{Z\kappa} \stackrel{P}{=} J_{AA'}^{Z\kappa}$$

## 1.7 引力微子场的对称性条件证明

推论1.7.1.  $\psi^{A_\zeta B_\zeta C_\zeta} = \psi^{A_\zeta C_\zeta B_\zeta} \Leftrightarrow (\sigma, i\zeta)_a (\sigma, -i\zeta)_b F^{ab}(\zeta) = 0$

证明:  $\psi^{A_\zeta B_\zeta C_\zeta} = \psi^{A_\zeta C_\zeta B_\zeta}$

$$\Leftrightarrow \varepsilon_{B_\zeta C_\zeta} \psi^{A_\zeta B_\zeta C_\zeta} = 0$$

$$\Leftrightarrow -\frac{1}{\sqrt{2}} \zeta \varepsilon_{B_\zeta C_\zeta} i S_{ab}^{A_\zeta B_\zeta} F^{ab C_\zeta} = 0$$

$$\Leftrightarrow \varepsilon_{B_\zeta C_\zeta} i S_{ab}^{A_\zeta} \delta_{D_\zeta}^{B_\zeta} \bar{\varepsilon}^{D_\zeta B_\zeta} F^{ab C_\zeta} = 0$$

$$\Leftrightarrow i S_{ab}^{A_\zeta} \delta_{D_\zeta}^{B_\zeta} \delta^{D_\zeta C_\zeta} F^{ab C_\zeta} = 0$$

$$\Leftrightarrow i S_{ab}^{A_\zeta} F^{ab C_\zeta} = 0$$

$$\Leftrightarrow \frac{1}{4} (\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]} F^{ab [C_\zeta]} = 0$$

$$\Leftrightarrow (\sigma, i\zeta)_a (\sigma, -i\zeta)_b F^{ab [C_\zeta]} = 0$$

$$\Leftrightarrow (\sigma, i\zeta)_a (\sigma, -i\zeta)_b F^{ab}(\frac{3}{2}, \zeta) = 0 \quad \square$$

推论1.7.2.  $\psi_{A_\zeta B_\zeta C_\zeta} = \frac{1}{3!} \psi_{(A_\zeta B_\zeta C_\zeta)} \Leftrightarrow (\sigma, i\zeta)_a (\sigma, -i\zeta)_b F^{ab}(\frac{3}{2}, \zeta) = 0 \Leftrightarrow (\sigma, i\zeta)_a (\sigma, -i\zeta)_b * F^{ab}(\frac{3}{2}, \zeta) = 0$

推论1.7.3.  $J_{A'_\zeta}{}^{B_\zeta C_\zeta} = J_{A'_\zeta}{}^{C_\zeta B_\zeta} \Leftrightarrow (\sigma, -i\zeta)^a J_a(\zeta) = 0$

证明:  $J_{A'_\zeta}{}^{B_\zeta C_\zeta} = J_{A'_\zeta}{}^{C_\zeta B_\zeta}$

$$\Leftrightarrow \varepsilon_{B_\zeta C_\zeta} J_{A'_\zeta}{}^{B_\zeta C_\zeta} = 0$$

$$\Leftrightarrow \varepsilon_{B_\zeta C_\zeta} \frac{1}{\sqrt{2}} (\sigma, -i\zeta)^a \delta_{A'_\zeta A_\zeta} \bar{\varepsilon}^{A_\zeta B_\zeta} J_a{}^{C_\zeta} = 0$$

$$\Leftrightarrow (\sigma, -i\zeta)^a \delta_{A'_\zeta A_\zeta} \delta^{A_\zeta C_\zeta} J_a{}^{C_\zeta} = 0$$

$$\Leftrightarrow (\sigma, -i\zeta)^a J_a{}^{C_\zeta} = 0$$

$$\Leftrightarrow (\sigma, -i\zeta)^a J_a{}^{[C_\zeta]} = 0$$

$$\Leftrightarrow (\sigma, -i\zeta)^a J_a(\zeta) = 0 \quad \square$$

## 2 平坦时空中Penrose型引力微子场方程的多种等价表述形式

### 2.1 引力微子方程的标架描述

定义2.1.1.  $F_{ab}^{Z_\kappa} := e_a^u e_b^v F_{uv}^{Z_\kappa}, \psi_a^{Z_\kappa} := e_a^u \psi_u^{Z_\kappa}$

引力微子方程的标架描述

$$\partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa}, \partial^a * F_{ab}^{Z_\kappa} \equiv 0 \quad (9.3)$$

### 2.2 引力微子场方程的经典形式

$$\begin{cases} \nabla \cdot \vec{E}^{Z_\kappa} = \rho^{Z_\kappa}, \nabla \times \vec{E}^{Z_\kappa} = -\partial_t \vec{B}^{Z_\kappa} \\ \nabla \cdot \vec{B}^{Z_\kappa} = 0, \nabla \times \vec{B}^{Z_\kappa} = \vec{J}^{Z_\kappa} + \partial_t \vec{E}^{Z_\kappa} \end{cases} \Leftrightarrow \partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa}, \partial^a * F_{ab}^{Z_\kappa} \equiv 0 \quad (9.4)$$

### 2.3 引力微子场方程的复矢量表述形式

复矢量张量形式:

定理2.3.1.  $\partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa} \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a{}_{b\alpha_\zeta} \partial_a \tilde{\Psi}^{\alpha_\zeta \sigma} = iJ_b^{Z_\kappa}; F_{ab}^{Z_\kappa} = \partial_a A_b - \partial_b A_a, \tilde{\Psi}^{\alpha_\zeta \sigma} = \left[ \psi^{\alpha_\zeta \sigma} = \frac{i}{2} \sigma_{\zeta ab}^{\alpha_\zeta} F^{ab\sigma} \right]$

证明:  $\partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa}$

$$\Leftrightarrow \partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa}, \partial^a * F_{ab}^{Z_\kappa} \equiv 0$$

$$\Leftrightarrow \partial^a (F_{ab}^{Z_\kappa} - \zeta * F_{ab}^{Z_\kappa}) = -J_b^{Z_\kappa}$$

$$\Leftrightarrow \partial^a (i\sigma_{\zeta ab}^{\alpha_\zeta} \psi_{\alpha_\zeta}^{Z_\kappa}) = -J_b^{Z_\kappa}, \alpha_\zeta = 1, 2, 3$$

$$\Leftrightarrow \partial^a [(\sigma_\zeta, -i\zeta)^{\alpha_\zeta} |_{ab} \tilde{\Psi}^{\alpha_\zeta \sigma}] = iJ_b^{Z_\kappa}, \alpha_\zeta = 1, 2, 3, 4$$

$$\Leftrightarrow \partial^a [(\sigma_{-\zeta}, -i\zeta)_a |_{b\alpha_\zeta} \tilde{\Psi}^{\alpha_\zeta \sigma}] = iJ_b^{Z_\kappa}, \alpha_\zeta = 1, 2, 3, 4$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a{}_{b\alpha_\zeta} \partial_a \tilde{\Psi}^{\alpha_\zeta \sigma} = iJ_b^{Z_\kappa}, \alpha_\zeta = 1, 2, 3, 4 \quad \square$$

复矢量矩阵形式：

$$\text{推论2.3.1. } (\sigma_{-\varsigma}, -i\varsigma)_{b\alpha\varsigma}^a \partial_a \tilde{\Psi}^{\alpha\varsigma\sigma} = iJ_b^{Z\kappa} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a \partial_a \tilde{\Psi}^{Z\kappa}(1, \varsigma) = iJ^{Z\kappa}$$

表象变换：

$$\text{推论2.3.2. } (\sigma_{-\varsigma}, -i\varsigma)^a \partial_a \tilde{\Psi}^{Z\kappa}(1, \varsigma) = iJ^{Z\kappa} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a \partial_a \tilde{\Psi}^{Z\kappa}(1, \varsigma) = i\tilde{J}^{Z\kappa}(1, \varsigma)$$

## 2.4 引力微子场方程的 $\frac{1}{2}$ -旋量表述形式 [1, 2]

$\frac{1}{2}$ -旋量Penrose抽象指标形式：

$$\text{定理2.4.1. } (\sigma_{-\varsigma}, -i\varsigma)_{b\alpha\varsigma}^a \partial_a \tilde{\Psi}^{\alpha\varsigma\sigma} = iJ_b^{Z\kappa} \Leftrightarrow \nabla^{A'_\varsigma A_\varsigma} \psi_{A_\varsigma B_\varsigma}^{Z\kappa} = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma}, \nabla^{A'_\varsigma A_\varsigma} = \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial^a$$

$$\text{证明: } (\sigma_{-\varsigma}, -i\varsigma)_{b\alpha\varsigma}^a \partial_a \tilde{\Psi}^{\alpha\varsigma\sigma} = iJ_b^{Z\kappa}$$

$$\Leftrightarrow \partial^a (i\sigma_{\varsigma ab}^{\alpha\varsigma} \psi_{\alpha\varsigma}^{Z\kappa}) = -J_b^{Z\kappa}$$

$$\Leftrightarrow \partial^a (i\sigma_{\varsigma ab}^{\alpha\varsigma} \cdot \frac{i\varsigma}{\sqrt{2}} \sigma_{\alpha\varsigma}^{A'_\varsigma B'_\varsigma} \psi_{A'_\varsigma B'_\varsigma}^{Z\kappa}) = -J_b^{Z\kappa}$$

$$\Leftrightarrow iS_{ab}^{A'_\varsigma B'_\varsigma} \partial^a \psi_{A'_\varsigma B'_\varsigma}^{Z\kappa} = \frac{-\varsigma}{\sqrt{2}} J_b^{Z\kappa}$$

$$\Leftrightarrow (\frac{\varsigma}{2} \delta_{ab} \varepsilon^{A'_\varsigma B'_\varsigma} + iS_{ab}^{A'_\varsigma B'_\varsigma}) \partial^a \psi_{A'_\varsigma B'_\varsigma}^{Z\kappa} = \frac{-\varsigma}{\sqrt{2}} J_b^{Z\kappa}$$

$$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A'_\varsigma} \varepsilon_{A'_\varsigma B'_\varsigma} \cdot \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_b^{B'_\varsigma B'_\varsigma} \partial^a \psi_{A'_\varsigma B'_\varsigma}^{Z\kappa} = \frac{-1}{\sqrt{2}} J_b^{Z\kappa}$$

$$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A'_\varsigma} \varepsilon_{A'_\varsigma B'_\varsigma} \partial^a \psi_{A'_\varsigma B'_\varsigma}^{Z\kappa} = \frac{-1}{\sqrt{2}} J_b^{Z\kappa} \cdot \frac{-i\varsigma}{\sqrt{2}} (\sigma, i\varsigma)_b^{B'_\varsigma B'_\varsigma}$$

$$\Leftrightarrow \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A'_\varsigma} \partial^a \psi_{A'_\varsigma B'_\varsigma}^{Z\kappa} = \frac{-\varsigma}{\sqrt{2}} \varepsilon^{A'_\varsigma B'_\varsigma} J_{B'_\varsigma}^{Z\kappa}$$

$$\Leftrightarrow \nabla^{A'_\varsigma A_\varsigma} \psi_{A'_\varsigma B_\varsigma}^{Z\kappa} = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma}, \nabla^{A'_\varsigma A_\varsigma} = \frac{i\varsigma}{\sqrt{2}} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial^a \quad \square$$

$\frac{1}{2}$ -旋量张量形式：

$$\text{推论2.4.1. } \nabla^{A'_\varsigma A_\varsigma} \psi_{A'_\varsigma B_\varsigma}^{Z\kappa} = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma} \Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial_a \psi_{A'_\varsigma B_\varsigma}^{Z\kappa} = iJ_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma}$$

$\frac{1}{2}$ -旋量矩阵形式：

$$\text{推论2.4.2. } (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial_a \psi_{A'_\varsigma B_\varsigma}^{Z\kappa} = iJ_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma} \Leftrightarrow (\sigma \otimes I, -i\varsigma)^a \partial_a \tilde{\Psi}^{Z\kappa}(1, \varsigma) = i\tilde{J}^{Z\kappa}(1, \varsigma)$$

$\frac{1}{2}$ -旋量矩阵形式：

$$\text{推论2.4.3. } (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial_a \psi_{A'_\varsigma B_\varsigma}^{Z\kappa} = iJ_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma} \Leftrightarrow (\sigma, -i\varsigma)^a \partial_a [\psi]^{Z\kappa} = i[J]^{Z\kappa}$$

## 2.5 引力微子场方程的全 $\frac{1}{2}$ -旋量表述形式

$$\text{推论2.5.1. } \nabla^{A'_\varsigma A_\varsigma} \psi_{A'_\varsigma B_\varsigma}^{Z\kappa} = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma} \Leftrightarrow \nabla^{A'_\varsigma A_\varsigma} \partial_a \psi_{A'_\varsigma B_\varsigma C_\varsigma} = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma C_\varsigma}^{A'_\varsigma \sigma}$$

$$\text{推论2.5.2. } \nabla^{A'_\varsigma A_\varsigma} \psi_{A'_\varsigma B_\varsigma}^{Z\kappa} = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma B_\varsigma}^{A'_\varsigma \sigma} \Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial_a \psi_{A'_\varsigma B_\varsigma C_\varsigma} = iJ_{A'_\varsigma B_\varsigma C_\varsigma}^{A'_\varsigma \sigma}$$

## 2.6 全对称方程(广义协变推广)

推论2.6.1.

$$\begin{cases} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D_a \psi_{A'_\varsigma B_\varsigma C_\varsigma} = iJ_{A'_\varsigma B_\varsigma C_\varsigma}^{A'_\varsigma \sigma} \\ \psi_{A'_\varsigma B_\varsigma C_\varsigma} = \frac{1}{3!} \psi_{(A'_\varsigma B_\varsigma C_\varsigma)}, J_{A'_\varsigma B_\varsigma C_\varsigma}^{A'_\varsigma \sigma} = \frac{1}{2!} J_{A'_\varsigma (B_\varsigma C_\varsigma)}^{A'_\varsigma \sigma} \end{cases} \Leftrightarrow \begin{cases} D^a F_{ab}^{[C_\varsigma]} = -J_b^{[C_\varsigma]}, D^a * F_{ab}^{[C_\varsigma]} \equiv 0 \\ (\sigma, i\varsigma)_a (\sigma, -i\varsigma)_b F^{ab[C_\varsigma]} = 0, (\sigma, -i\varsigma)_a J^{a[C_\varsigma]} = 0 \end{cases}$$

以下两个推论的证明留待以后

推论2.6.2.

$$\begin{cases} (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D_a \psi_{A'_\varsigma B_\varsigma C_\varsigma} = iJ_{A'_\varsigma B_\varsigma C_\varsigma}^{A'_\varsigma \sigma} \\ \psi_{A'_\varsigma B_\varsigma C_\varsigma} = \frac{1}{3!} \psi_{(A'_\varsigma B_\varsigma C_\varsigma)}, J_{A'_\varsigma B_\varsigma C_\varsigma}^{A'_\varsigma \sigma} = \frac{1}{2!} J_{A'_\varsigma (B_\varsigma C_\varsigma)}^{A'_\varsigma \sigma} \end{cases} \Leftrightarrow [\frac{3}{2} D_a + iS_{ab}(\frac{3}{2}, \varsigma) D^b]_{k_\varsigma}{}^{l_\varsigma} (\frac{3}{2}, \varsigma) \psi_{l_\varsigma} = \mathbb{J}_{ak_\varsigma}(\frac{3}{2}, \varsigma)$$

推论2.6.3.

$$\begin{cases} D^a F_{ab}^{[C_\varsigma]} = -J_b^{[C_\varsigma]}, D^a * F_{ab}^{[C_\varsigma]} \equiv 0 \\ (\sigma, i\varsigma)_a (\sigma, -i\varsigma)_b F^{ab[C_\varsigma]} = 0, (\sigma, -i\varsigma)_a J^{a[C_\varsigma]} = 0 \end{cases} \Leftrightarrow [\frac{3}{2} D_a + iS_{ab}(\frac{3}{2}, \varsigma) D^b]_{k_\varsigma}{}^{l_\varsigma} (\frac{3}{2}, \varsigma) \psi_{l_\varsigma} = \mathbb{J}_{ak_\varsigma}(\frac{3}{2}, \varsigma)$$

## 2.7 猜测

定理2.7.1.  $\partial^a * F_{ab}^{Z_\kappa} = 0 \Leftrightarrow F_{ab}^{Z_\kappa} = \partial_a A_b^{Z_\kappa} - \partial_b A_a^{Z_\kappa} \Leftrightarrow \partial^a * F_{ab}^{Z_\kappa} \equiv 0$

定理2.7.2.  $\partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa}, \partial^a * F_{ab}^{Z_\kappa} = 0 \Leftrightarrow \partial^a F_{ab}^{Z_\kappa} = -J_b^{Z_\kappa}, F_{ab}^{Z_\kappa} = \partial_a A_b^{Z_\kappa} - \partial_b A_a^{Z_\kappa}$

## 2.8 引力微子场方程<sup>[7]</sup>的自旋张量表述形式

引力微子场的自旋张量矩阵:  $S_{ab} = i\sigma_{\varsigma ab}^{\alpha\varsigma} \gamma_{\alpha\varsigma} \succ \begin{bmatrix} 0 & \gamma_z & -\gamma_y & -\varsigma\gamma_x \\ -\gamma_z & 0 & \gamma_x & -\varsigma\gamma_y \\ \gamma_y & -\gamma_x & 0 & -\varsigma\gamma_z \\ \varsigma\gamma_x & \varsigma\gamma_y & \varsigma\gamma_z & 0 \end{bmatrix}$  (9.5)

定理2.8.1.  $(\partial_a + iS_{ab}\partial^b)^{\beta\varsigma} \gamma_\varsigma \psi^{\gamma\varsigma Z_\kappa}(1, \varsigma) = -i\sigma_{\varsigma ab}^{\beta\varsigma} J^{bZ_\kappa}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha\varsigma} \gamma_{\alpha\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a \partial_a \tilde{\Psi}^{Z_\kappa}(1, \varsigma) = iJ^{Z_\kappa}$

一种直观证法如下:

证明:  $(\partial_a + iS_{ab}\partial^b)^{\beta\varsigma} \gamma_\varsigma \psi^{\gamma\varsigma Z_\kappa} = -i\sigma_{\varsigma ab}^{\beta\varsigma} J^{bZ_\kappa}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha\varsigma} \gamma_{\alpha\varsigma}$

$$\Leftrightarrow \begin{cases} (\partial_x + i\gamma_z\partial_y - i\gamma_y\partial_z - i\varsigma\gamma_x\partial_\pi)^{\beta\varsigma} \gamma_\varsigma \psi^{\gamma\varsigma Z_\kappa} = -i\sigma_{\varsigma xb}^{\beta\varsigma} J^{bZ_\kappa} \\ (\partial_y + i\gamma_x\partial_z - i\gamma_z\partial_x - i\varsigma\gamma_y\partial_\pi)^{\beta\varsigma} \gamma_\varsigma \psi^{\gamma\varsigma Z_\kappa} = -i\sigma_{\varsigma yb}^{\beta\varsigma} J^{bZ_\kappa} \\ (\partial_z + i\gamma_y\partial_x - i\gamma_x\partial_y - i\varsigma\gamma_z\partial_\pi)^{\beta\varsigma} \gamma_\varsigma \psi^{\gamma\varsigma Z_\kappa} = -i\sigma_{\varsigma zb}^{\beta\varsigma} J^{bZ_\kappa} \\ (\partial_\pi + i\varsigma\gamma_x\partial_x + i\varsigma\gamma_y\partial_y + i\varsigma\gamma_z\partial_z)^{\beta\varsigma} \gamma_\varsigma \psi^{\gamma\varsigma Z_\kappa} = -i\sigma_{\varsigma \pi b}^{\beta\varsigma} J^{bZ_\kappa} \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} \partial_x & \partial_y & \partial_z \\ -\partial_y & \partial_x & -\varsigma\partial_\pi \\ -\partial_z & \varsigma\partial_\pi & \partial_x \end{bmatrix} \begin{bmatrix} \psi^{x\varsigma Z_\kappa} \\ \psi^{y\varsigma Z_\kappa} \\ \psi^{z\varsigma Z_\kappa} \end{bmatrix} = \begin{bmatrix} \varsigma J^{\pi Z_\kappa} \\ J^z Z_\kappa \\ -J^y Z_\kappa \end{bmatrix}, \begin{bmatrix} \partial_y & -\partial_x & \varsigma\partial_\pi \\ \partial_x & \partial_y & \partial_z \\ -\varsigma\partial_\pi & -\partial_z & \partial_y \end{bmatrix} \begin{bmatrix} \psi^{x\varsigma Z_\kappa} \\ \psi^{y\varsigma Z_\kappa} \\ \psi^{z\varsigma Z_\kappa} \end{bmatrix} = \begin{bmatrix} -J^z Z_\kappa \\ \varsigma J^{\pi Z_\kappa} \\ J^x Z_\kappa \end{bmatrix} \\ \begin{bmatrix} \partial_z & -\varsigma\partial_\pi & -\partial_x \\ \varsigma\partial_\pi & \partial_z & -\partial_y \\ \partial_x & \partial_y & \partial_z \end{bmatrix} \begin{bmatrix} \psi^{x\varsigma Z_\kappa} \\ \psi^{y\varsigma Z_\kappa} \\ \psi^{z\varsigma Z_\kappa} \end{bmatrix} = \begin{bmatrix} J^y Z_\kappa \\ -J^x Z_\kappa \\ \varsigma J^{\pi Z_\kappa} \end{bmatrix}, i\partial_\pi \Psi^{Z_\kappa}(1, \varsigma) = \varsigma\gamma \cdot \nabla \Psi^{Z_\kappa}(1, \varsigma) - i\varsigma \vec{J}^{Z_\kappa} \end{cases}$$

$$\Leftrightarrow \begin{cases} i\partial_\pi \Psi^{Z_\kappa}(1, \varsigma) = i\varsigma \nabla \times \Psi^{Z_\kappa}(1, \varsigma) - i\varsigma \vec{J}^{Z_\kappa} \\ \nabla \cdot \Psi^{Z_\kappa}(1, \varsigma) = \varsigma J^{\pi Z_\kappa} \end{cases}$$

$$\Leftrightarrow \begin{cases} i\partial_\pi \Psi^{Z_\kappa}(1, \varsigma) = \varsigma\gamma \cdot \nabla \Psi^{Z_\kappa}(1, \varsigma) - i\varsigma \vec{J}^{Z_\kappa} \\ \nabla \cdot \Psi^{Z_\kappa}(1, \varsigma) = \varsigma J^{\pi Z_\kappa} \end{cases}$$

$$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a \partial_a \tilde{\Psi}^{Z_\kappa}(1, \varsigma) = iJ^{Z_\kappa} \quad \square$$

另一种更解析更抽象的证法如下:

证明:  $(\partial_a + iS_{ab}\partial^b)^{\beta\varsigma} \gamma_\varsigma \psi^{\gamma\varsigma Z_\kappa} = -i\sigma_{\varsigma ab}^{\beta\varsigma} J^{bZ_\kappa}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha\varsigma} \gamma_{\alpha\varsigma}$

$$\Leftrightarrow \sigma_{\varsigma a}^{\beta\varsigma c} \sigma_{\varsigma\gamma\varsigma cb} \partial^b \psi^{\gamma\varsigma Z_\kappa} = -i\sigma_{\varsigma ab}^{\beta\varsigma} J^{bZ_\kappa}$$

$$\Leftrightarrow \sigma_{\varsigma ac}^{\beta\varsigma} \sigma_{\varsigma\gamma\varsigma}^{cb} \partial_b \psi^{\gamma\varsigma Z_\kappa} = -i\sigma_{\varsigma ab}^{\beta\varsigma} J^{bZ_\kappa}$$

$$\Leftrightarrow \sigma_{\beta\varsigma}^{\varsigma ad} \sigma_{\varsigma ac}^{\beta\varsigma} \sigma_{\varsigma\gamma\varsigma}^{cb} \partial_b \psi^{\gamma\varsigma Z_\kappa} = -i\sigma_{\beta\varsigma}^{\varsigma ad} \sigma_{\varsigma ab}^{\beta\varsigma} J^{bZ_\kappa}$$

$$\Leftrightarrow \sigma_{\varsigma\gamma\varsigma}^{db} \partial_b \psi^{\gamma\varsigma Z_\kappa} = -iJ^{dZ_\kappa}$$

$$\Leftrightarrow \sigma_{\varsigma\alpha\varsigma}^{ab} \partial_a \psi^{\alpha\varsigma Z_\kappa} = iJ^{bZ_\kappa}, \alpha_\varsigma = 1, 2, 3$$

$$\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_{b\alpha\varsigma}^a \partial_a \tilde{\Psi}^{\alpha\varsigma Z_\kappa} = iJ_b^{Z_\kappa}, \alpha_\varsigma = 1, 2, 3, 4 \quad \square$$

此方程(3.3.2)完全等价于引力微子场方程, 它就是引力微子场方程的自旋张量表述形式。

$$\text{引理2.8.1. } \mathbb{J}_a^{\beta\varsigma Z_\kappa} = -i\sigma_{\varsigma ab}^{\beta\varsigma} J^{bZ_\kappa} \Leftrightarrow \begin{cases} \mathbb{J}_y^{z\varsigma Z_\kappa} = -\mathbb{J}_z^{y\varsigma Z_\kappa} = -\varsigma \mathbb{J}_\pi^{x\varsigma Z_\kappa} = J^x Z_\kappa \\ \mathbb{J}_z^{x\varsigma Z_\kappa} = -\mathbb{J}_x^{z\varsigma Z_\kappa} = -\varsigma \mathbb{J}_\pi^{y\varsigma Z_\kappa} = J^y Z_\kappa \\ \mathbb{J}_x^{y\varsigma Z_\kappa} = -\mathbb{J}_y^{x\varsigma Z_\kappa} = -\varsigma \mathbb{J}_\pi^{z\varsigma Z_\kappa} = J^z Z_\kappa \\ \mathbb{J}_x^{x\varsigma Z_\kappa} = \mathbb{J}_y^{y\varsigma Z_\kappa} = \mathbb{J}_z^{z\varsigma Z_\kappa} = \varsigma J^{\pi Z_\kappa} \end{cases}$$

展开即可证明。以上自旋方程是关于特殊的源项，那么对于一般的源项又会怎样呢？请看下面的定理。

**定理2.8.2.**  $(\partial_a + iS_{ab}\partial^b)^{\beta_\zeta} \psi^{\gamma_\zeta Z_\kappa} = \mathbb{J}_a^{\beta_\zeta Z_\kappa}, S_{ab} = i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta} \Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a \partial_a \tilde{\Psi}^{Z_\kappa}(1, \zeta) = iJ^{Z_\kappa}, \mathbb{J}_a^{\beta_\zeta Z_\kappa} = -i\sigma_{\zeta ab}^{\beta_\zeta} J^b Z_\kappa$

**证明:**  $(\partial_a + iS_{ab}\partial^b)^{\beta_\zeta} \psi^{\gamma_\zeta Z_\kappa} = \mathbb{J}_a^{\beta_\zeta Z_\kappa}, S_{ab} = i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta}$

$$\Leftrightarrow \begin{cases} (\partial_x + i\gamma_z \partial_y - i\gamma_y \partial_z - i\zeta \gamma_x \partial_\pi)^{\beta_\zeta} \psi^{\gamma_\zeta Z_\kappa} = \mathbb{J}_x^{\beta_\zeta Z_\kappa} \\ (\partial_y + i\gamma_x \partial_z - i\gamma_z \partial_x - i\zeta \gamma_y \partial_\pi)^{\beta_\zeta} \psi^{\gamma_\zeta Z_\kappa} = \mathbb{J}_y^{\beta_\zeta Z_\kappa} \\ (\partial_z + i\gamma_y \partial_x - i\gamma_x \partial_y - i\zeta \gamma_z \partial_\pi)^{\beta_\zeta} \psi^{\gamma_\zeta Z_\kappa} = \mathbb{J}_z^{\beta_\zeta Z_\kappa} \\ (\partial_\pi + i\zeta \gamma_x \partial_x + i\zeta \gamma_y \partial_y + i\zeta \gamma_z \partial_z)^{\beta_\zeta} \psi^{\gamma_\zeta Z_\kappa} = \mathbb{J}_\pi^{\beta_\zeta Z_\kappa} \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} \partial_x & \partial_y & \partial_z \\ -\partial_y & \partial_x & -\zeta \partial_\pi \\ -\partial_z & \zeta \partial_\pi & \partial_x \end{bmatrix} \begin{bmatrix} \psi^{x_\zeta Z_\kappa} \\ \psi^{y_\zeta Z_\kappa} \\ \psi^{z_\zeta Z_\kappa} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_x^{\beta_\zeta Z_\kappa} \\ \mathbb{J}_y^{\beta_\zeta Z_\kappa} \\ \mathbb{J}_z^{\beta_\zeta Z_\kappa} \end{bmatrix} \Leftrightarrow \begin{cases} \nabla \cdot \Psi^{Z_\kappa}(1, \zeta) = \mathbb{J}_x^{\beta_\zeta Z_\kappa} \\ [\nabla \times \Psi^{Z_\kappa}(1, \zeta)]^{z_\zeta Z_\kappa} - \zeta \partial_\pi \psi^{z_\zeta Z_\kappa}(1, \zeta) = \mathbb{J}_x^{\beta_\zeta Z_\kappa} \\ -[\nabla \times \Psi^{Z_\kappa}(1, \zeta)]^{y_\zeta Z_\kappa} + \zeta \partial_\pi \psi^{y_\zeta Z_\kappa}(1, \zeta) = \mathbb{J}_x^{\beta_\zeta Z_\kappa} \end{cases} \\ \begin{bmatrix} \partial_y & -\partial_x & \zeta \partial_\pi \\ \partial_x & \partial_y & \partial_z \\ -\zeta \partial_\pi & -\partial_z & \partial_y \end{bmatrix} \begin{bmatrix} \psi^{x_\zeta Z_\kappa} \\ \psi^{y_\zeta Z_\kappa} \\ \psi^{z_\zeta Z_\kappa} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_y^{\beta_\zeta Z_\kappa} \\ \mathbb{J}_y^{\beta_\zeta Z_\kappa} \\ \mathbb{J}_y^{\beta_\zeta Z_\kappa} \end{bmatrix} \Leftrightarrow \begin{cases} -[\nabla \times \Psi^{Z_\kappa}(1, \zeta)]^{z_\zeta Z_\kappa} + \zeta \partial_\pi \psi^{z_\zeta Z_\kappa}(1, \zeta) = \mathbb{J}_y^{\beta_\zeta Z_\kappa} \\ \nabla \cdot \Psi^{Z_\kappa}(1, \zeta) = \mathbb{J}_y^{\beta_\zeta Z_\kappa} \\ [\nabla \times \Psi^{Z_\kappa}(1, \zeta)]^{x_\zeta Z_\kappa} - \zeta \partial_\pi \psi^{x_\zeta Z_\kappa}(1, \zeta) = \mathbb{J}_y^{\beta_\zeta Z_\kappa} \end{cases} \\ \begin{bmatrix} \partial_z & -\zeta \partial_\pi & -\partial_x \\ \zeta \partial_\pi & \partial_z & -\partial_y \\ \partial_x & \partial_y & \partial_z \end{bmatrix} \begin{bmatrix} \psi^{x_\zeta Z_\kappa} \\ \psi^{y_\zeta Z_\kappa} \\ \psi^{z_\zeta Z_\kappa} \end{bmatrix} = \begin{bmatrix} \mathbb{J}_z^{\beta_\zeta Z_\kappa} \\ \mathbb{J}_z^{\beta_\zeta Z_\kappa} \\ \mathbb{J}_z^{\beta_\zeta Z_\kappa} \end{bmatrix} \Leftrightarrow \begin{cases} [\nabla \times \Psi^{Z_\kappa}(1, \zeta)]^{y_\zeta Z_\kappa} - \zeta \partial_\pi \psi^{y_\zeta Z_\kappa}(1, \zeta) = \mathbb{J}_z^{\beta_\zeta Z_\kappa} \\ -[\nabla \times \Psi^{Z_\kappa}(1, \zeta)]^{x_\zeta Z_\kappa} + \zeta \partial_\pi \psi^{x_\zeta Z_\kappa}(1, \zeta) = \mathbb{J}_z^{\beta_\zeta Z_\kappa} \\ \nabla \cdot \Psi^{Z_\kappa}(1, \zeta) = \mathbb{J}_z^{\beta_\zeta Z_\kappa} \end{cases} \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial_\pi \Psi^{Z_\kappa}(1, \zeta) + i\zeta \gamma \cdot \nabla \psi^{Z_\kappa} = \mathbb{J}_\pi^{\beta_\zeta Z_\kappa} \Leftrightarrow \partial_\pi \Psi^{Z_\kappa}(1, \zeta) - \zeta \nabla \times \Psi^{Z_\kappa}(1, \zeta) = \mathbb{J}_\pi^{\beta_\zeta Z_\kappa} \\ \mathbb{J}_y^{z_\zeta Z_\kappa} = -\mathbb{J}_z^{y_\zeta Z_\kappa} = -\zeta \mathbb{J}_\pi^{x_\zeta Z_\kappa} := J^{x Z_\kappa} \\ \mathbb{J}_z^{x_\zeta Z_\kappa} = -\mathbb{J}_x^{z_\zeta Z_\kappa} = -\zeta \mathbb{J}_\pi^{y_\zeta Z_\kappa} := J^{y Z_\kappa} \\ \mathbb{J}_x^{y_\zeta Z_\kappa} = -\mathbb{J}_y^{x_\zeta Z_\kappa} = -\zeta \mathbb{J}_\pi^{z_\zeta Z_\kappa} := J^{z Z_\kappa} \\ \mathbb{J}_x^{x_\zeta Z_\kappa} = \mathbb{J}_y^{y_\zeta Z_\kappa} = \mathbb{J}_z^{z_\zeta Z_\kappa} := \zeta J^\pi Z_\kappa \\ \partial_\pi \Psi^{Z_\kappa}(1, \zeta) - \zeta \nabla \times \Psi^{Z_\kappa}(1, \zeta) = i\vec{J}^{Z_\kappa} \\ \nabla \cdot \Psi^{Z_\kappa}(1, \zeta) = -iJ^\pi Z_\kappa \end{cases}$$

$$\Leftrightarrow (\sigma_{-\zeta}, -i\zeta)^a \partial_a \tilde{\Psi}^{Z_\kappa}(1, \zeta) = iJ^{Z_\kappa}, \mathbb{J}_a^{\beta_\zeta Z_\kappa} = -i\sigma_{\zeta ab}^{\beta_\zeta} J^b Z_\kappa \quad \square$$

另一种更解析更抽象的证法如下：

**定理2.8.3.**  $(\partial_a + iS_{ab}\partial^b)^{\beta_\zeta} \psi^{\gamma_\zeta Z_\kappa} = \mathbb{J}_a^{\beta_\zeta Z_\kappa}, S_{ab} = i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta} \Leftrightarrow \mathbb{J}_a^{\beta_\zeta Z_\kappa} = \sigma_{\zeta ab}^{\beta_\zeta} \sigma_{\zeta \gamma_\zeta}^{bc} \partial_c \psi^{\gamma_\zeta Z_\kappa}$

**证明:**  $(\partial_a + iS_{ab}\partial^b)^{\beta_\zeta} \psi^{\gamma_\zeta Z_\kappa} = \mathbb{J}_a^{\beta_\zeta Z_\kappa}, S_{ab} = i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta}$

$$\Leftrightarrow \sigma_{\zeta a}^{\beta_\zeta c} \sigma_{\zeta \gamma_\zeta}^{cb} \partial^b \psi^{\gamma_\zeta Z_\kappa} = \mathbb{J}_a^{\beta_\zeta Z_\kappa}$$

$$\Leftrightarrow \mathbb{J}_a^{\beta_\zeta Z_\kappa} = \sigma_{\zeta ab}^{\beta_\zeta} \sigma_{\zeta \alpha_\zeta}^{bc} \partial_c \psi^{\alpha_\zeta Z_\kappa}$$

$$\Leftrightarrow \begin{cases} \mathbb{J}_y^{z_\zeta Z_\kappa} = -\mathbb{J}_z^{y_\zeta Z_\kappa} = -\zeta \mathbb{J}_\pi^{x_\zeta Z_\kappa} = i\sigma_{\zeta \alpha_\zeta}^{xb} \partial_b \psi^{\alpha_\zeta Z_\kappa} \\ \mathbb{J}_z^{x_\zeta Z_\kappa} = -\mathbb{J}_x^{z_\zeta Z_\kappa} = -\zeta \mathbb{J}_\pi^{y_\zeta Z_\kappa} = i\sigma_{\zeta \alpha_\zeta}^{yb} \partial_b \psi^{\alpha_\zeta Z_\kappa} \\ \mathbb{J}_x^{y_\zeta Z_\kappa} = -\mathbb{J}_y^{x_\zeta Z_\kappa} = -\zeta \mathbb{J}_\pi^{z_\zeta Z_\kappa} = i\sigma_{\zeta \alpha_\zeta}^{zb} \partial_b \psi^{\alpha_\zeta Z_\kappa} \\ \mathbb{J}_x^{x_\zeta Z_\kappa} = \mathbb{J}_y^{y_\zeta Z_\kappa} = \mathbb{J}_z^{z_\zeta Z_\kappa} = i\zeta \sigma_{\zeta \alpha_\zeta}^{\pi b} \partial_b \psi^{\alpha_\zeta Z_\kappa} \end{cases} \quad \square$$

这个定理表明此自旋方程源项受到一定限制，不是随意的，只有前一定理描述的源项情形才有解，其他情形全无解。

**推论2.8.1.**  $(\partial_a + iS_{ab}\partial^b)^{\beta_\zeta} \psi^{\gamma_\zeta Z_\kappa} = \mathbb{J}_a^{\beta_\zeta Z_\kappa}, S_{ab} = i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta}$  有解  $\Leftrightarrow \mathbb{J}_a^{\beta_\zeta Z_\kappa} = -i\sigma_{\zeta ab}^{\beta_\zeta} J^b Z_\kappa, \exists J^b Z_\kappa$

### 3 Rarita-Schwinger方程的分析 [20]

#### 3.1 准备

**Rarita-Schwinger拉氏量:**  $\mathcal{L}_{RS} = -\bar{\psi}^a \varepsilon_{abcd} \gamma_5(\varsigma) \gamma^d(\varsigma) [D^b + \frac{1}{2} m \gamma^b(\varsigma)] \psi^c(e, \varsigma)$

**引理3.1.1.**  $\varepsilon_{abcd} \gamma_5(\varsigma) \gamma^d(\varsigma) [D^b + \frac{1}{2} m \gamma^b(\varsigma)] \psi^c(e, \varsigma) = 0$

$$\Leftrightarrow \gamma_a(\varsigma) [\gamma_b(\varsigma) D^b - m] [\gamma_c(\varsigma) \psi^c(e, \varsigma)] + [\gamma_b(\varsigma) D^b + m] \psi_a(e, \varsigma) - \gamma_a(\varsigma) D_c \psi^c(e, \varsigma) - D_a [\gamma_c(\varsigma) \psi^c(e, \varsigma)] = 0$$

**证明:**  $\varepsilon_{abcd} \gamma_5(\varsigma) \gamma^d(\varsigma) [D^b + \frac{1}{2} m \gamma^b(\varsigma)] \psi^c(e, \varsigma) = 0$

$$\Leftrightarrow \varepsilon_{abcd} \gamma_5(\varsigma) \gamma^d(\varsigma) D^b \psi^c(e, \varsigma) + \frac{1}{2} m \varepsilon_{abcd} \gamma_5(\varsigma) \gamma^c(\varsigma) \gamma^d(\varsigma) \psi^b(e, \varsigma) = 0$$

**利用公式:**  $\varepsilon_{abcd} \gamma_5(\varsigma) \gamma^d(\varsigma) = 2i S_{ab}(e, \varsigma) \gamma_c(\varsigma) - \gamma_{[a}(\varsigma) \delta_{b]c}, \varepsilon_{abcd} S^{cd}(e, \varsigma) = -2\gamma_5(\varsigma) i S_{ab}(e, \varsigma)$

$$\Leftrightarrow [2i S_{ab}(e, \varsigma) \gamma_c(\varsigma) - \gamma_{[a}(\varsigma) \delta_{b]c}] D^b \psi^c(e, \varsigma) - m \gamma_5(\varsigma) i S_{ab}(e, \varsigma) \psi^b(e, \varsigma) = 0$$

$$\Leftrightarrow \gamma_a(\varsigma) [\gamma_b(\varsigma) D^b - m] [\gamma_c(\varsigma) \psi^c(e, \varsigma)] + [\gamma_b(\varsigma) D^b + m] \psi_a(e, \varsigma) - \gamma_a(\varsigma) D_c \psi^c(e, \varsigma) - D_a [\gamma_c(\varsigma) \psi^c(e, \varsigma)] = 0 \quad \square$$

**引理3.1.2.**  $\varepsilon_{abcd} \gamma_5(\varsigma) \gamma^d(\varsigma) [D^b + \frac{1}{2} m \gamma^b(\varsigma)] \psi^c(e, \varsigma) = 0$

$$\Rightarrow \begin{cases} m[\gamma_a(\varsigma) D^a] [\gamma_b(\varsigma) \psi^b(e, \varsigma)] - m[D^a \psi_a(e, \varsigma)] = 0 \\ 2[\gamma_a(\varsigma) D^a] [\gamma_b(\varsigma) \psi^b(e, \varsigma)] - 2[D^a \psi_a(e, \varsigma)] - 3m[\gamma_a(\varsigma) \psi^a(e, \varsigma)] = 0 \end{cases}$$

**证明:**  $\varepsilon_{abcd} \gamma_5(\varsigma) \gamma^d(\varsigma) [D^b + \frac{1}{2} m \gamma^b(\varsigma)] \psi^c(e, \varsigma) = 0$

$$\Leftrightarrow \gamma_a(\varsigma) [\gamma_b(\varsigma) D^b - m] [\gamma_c(\varsigma) \psi^c(e, \varsigma)] + [\gamma_b(\varsigma) D^b + m] \psi_a(e, \varsigma) - \gamma_a(\varsigma) D_c \psi^c(e, \varsigma) - D_a [\gamma_c(\varsigma) \psi^c(e, \varsigma)] = 0$$

$$\Rightarrow \begin{cases} [\gamma_a(\varsigma) D^a] [\gamma_b(\varsigma) D^b - m] [\gamma_c(\varsigma) \psi^c(e, \varsigma)] + [\gamma_b(\varsigma) D^b + m] [D^a \psi_a(e, \varsigma)] - [\gamma_a(\varsigma) D^a] D_c \psi^c(e, \varsigma) \\ - D^a D_a [\gamma_c(\varsigma) \psi^c(e, \varsigma)] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 4[\gamma_b(\varsigma) D^b - m] [\gamma_c(\varsigma) \psi^c(e, \varsigma)] + [\gamma_b(\varsigma) D^b + m] [\gamma^a(\varsigma) \psi_a(e, \varsigma)] - 4D_c \psi^c(e, \varsigma) - [\gamma^a(\varsigma) D_a] [\gamma_c(\varsigma) \psi^c(e, \varsigma)] = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} m[\gamma_a(\varsigma) D^a] [\gamma_b(\varsigma) \psi^b(e, \varsigma)] - m[D^a \psi_a(e, \varsigma)] = 0 \\ 2[\gamma_a(\varsigma) D^a] [\gamma_b(\varsigma) \psi^b(e, \varsigma)] - 2[D^a \psi_a(e, \varsigma)] - 3m[\gamma_a(\varsigma) \psi^a(e, \varsigma)] = 0 \end{cases}$$

$$\Leftrightarrow \gamma_a(\varsigma) \psi^a(e, \varsigma) = 0, D_a \psi^a(e, \varsigma) = 0 \quad \square$$

$$\text{引理3.1.3.} \begin{cases} m[\gamma_a(\varsigma) D^a] [\gamma_b(\varsigma) \psi^b(e, \varsigma)] - m[D^a \psi_a(e, \varsigma)] = 0 \\ 2[\gamma_a(\varsigma) D^a] [\gamma_b(\varsigma) \psi^b(e, \varsigma)] - 2[D^a \psi_a(e, \varsigma)] - 3m[\gamma_a(\varsigma) \psi^a(e, \varsigma)] = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \gamma_a(\varsigma) \psi^a(e, \varsigma) = 0, D_a \psi^a(e, \varsigma) = 0, m \neq 0 \\ D_a \psi^a(e, \varsigma) = [\gamma_a(\varsigma) D^a] [\gamma_b(\varsigma) \psi^b(e, \varsigma)] = 0, m = 0 \end{cases}$$

#### 3.2 有质量Rarita-Schwinger方程的等价形式

**推论3.2.1.**  $\varepsilon_{abcd} \gamma_5(\varsigma) \gamma^d(\varsigma) [D^b + \frac{1}{2} m \gamma^b(\varsigma)] \psi^c(e, \varsigma) = 0, m \neq 0$

$$\Leftrightarrow [\gamma_b(\varsigma) D^b + m] \psi^a(e, \varsigma) = 0, \gamma_a(\varsigma) \psi^a(e, \varsigma) = 0, D_a \psi^a(e, \varsigma) = 0, m \neq 0$$

**证明:**  $\varepsilon_{abcd} \gamma_5(\varsigma) \gamma^d(\varsigma) [D^b + \frac{1}{2} m \gamma^b(\varsigma)] \psi^c(e, \varsigma) = 0$

$$\Leftrightarrow \gamma_a(\varsigma) [\gamma_b(\varsigma) D^b - m] [\gamma_c(\varsigma) \psi^c(e, \varsigma)] + [\gamma_b(\varsigma) D^b + m] \psi_a(e, \varsigma) - \gamma_a(\varsigma) D_c \psi^c(e, \varsigma) - D_a [\gamma_c(\varsigma) \psi^c(e, \varsigma)] = 0$$

$$\Leftrightarrow [\gamma_b(\varsigma) D^b + m] \psi^a(e, \varsigma) = 0, \gamma_a(\varsigma) \psi^a(e, \varsigma) = 0, D_a \psi^a(e, \varsigma) = 0 \quad \square$$

**推论3.2.2.**  $[\gamma_b(\varsigma) D^b + m] \psi^a(e, \varsigma) = 0, \gamma_a(\varsigma) \psi^a(e, \varsigma) = 0, D_a \psi^a(e, \varsigma) = 0, m \neq 0$

$$\Leftrightarrow [\gamma_b(\varsigma) D^b + m] \psi^a(e, \varsigma) = 0, \gamma_a(\varsigma) \psi^a(e, \varsigma) = 0, m \neq 0$$

**重要结论:**

**定理3.2.1.**  $\varepsilon_{abcd} \gamma_5(\varsigma) \gamma^d(\varsigma) [D^b + \frac{1}{2} m \gamma^b(\varsigma)] \psi^c(e, \varsigma) = 0 \Leftrightarrow [\gamma_b(\varsigma) D^b + m] \psi^a(e, \varsigma) = 0, \gamma_a(\varsigma) \psi^a(e, \varsigma) = 0; m \neq 0$

### 3.3 无质量Rarita-Schwinger方程的等价形式

推论3.3.1.  $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e, \varsigma) = 0 \Leftrightarrow \gamma_b(\varsigma)[D^b\psi^a(e, \varsigma) - D^a\psi^b(e, \varsigma)] = 0, D_a\psi^a(e, \varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e, \varsigma)]$

证明:  $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e, \varsigma) = 0$

$$\Leftrightarrow \gamma_a(\varsigma)[\gamma_b(\varsigma)D^b][\gamma_c(\varsigma)\psi^c(e, \varsigma)] + [\gamma_b(\varsigma)D^b]\psi_a(\varsigma) - \gamma_a(\varsigma)D_c\psi^c(e, \varsigma) - D_a[\gamma_c(\varsigma)\psi^c(e, \varsigma)] = 0$$

$$\Leftrightarrow \gamma_b(\varsigma)[D^b\psi^a(e, \varsigma) - D^a\psi^b(e, \varsigma)] = 0, D_a\psi^a(e, \varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e, \varsigma)] \quad \square$$

推论3.3.2.  $\gamma_b(\varsigma)[D^b\psi^a(e, \varsigma) - D^a\psi^b(e, \varsigma)] = 0 \Rightarrow D_a\psi^a(e, \varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e, \varsigma)],$

推论3.3.3.  $\gamma_b(\varsigma)[D^b\psi^a(e, \varsigma) - D^a\psi^b(e, \varsigma)] = 0$

$$\Rightarrow [\gamma_b(\varsigma)D^b]\psi^a(e, \varsigma) = D^a[\gamma_b(\varsigma)\psi^b(e, \varsigma)]$$

$$\Rightarrow \gamma_a(\varsigma)\gamma_b(\varsigma)D^b\psi^a(e, \varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e, \varsigma)]$$

$$\Rightarrow [2\delta_{ab} - \gamma_b(\varsigma)\gamma_a(\varsigma)]D^b\psi^a(e, \varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e, \varsigma)]$$

$$\Rightarrow D_a\psi^a(e, \varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e, \varsigma)],$$

推论3.3.4.  $\gamma_b(\varsigma)[D^b\psi^a(e, \varsigma) - D^a\psi^b(e, \varsigma)] = 0, D_a\psi^a(e, \varsigma) = [\gamma_a(\varsigma)D^a][\gamma_b(\varsigma)\psi^b(e, \varsigma)]$

$$\Leftrightarrow \gamma_b(\varsigma)[D^b\psi^a(e, \varsigma) - D^a\psi^b(e, \varsigma)] = 0$$

推论3.3.5.  $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e, \varsigma) = 0 \Leftrightarrow \gamma_b(\varsigma)[D^b\psi^a(e, \varsigma) - D^a\psi^b(e, \varsigma)] = 0$

重要结论:

定理3.3.1.  $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e, \varsigma) = 0 \Leftrightarrow \gamma_a(\varsigma)F^{ab}(e, \varsigma) = 0, F^{ab}(e, \varsigma) \equiv D^a\psi^b(e, \varsigma) - D^b\psi^a(e, \varsigma)$

推论3.3.6.  $\varepsilon_{abcd}\gamma_5(\varsigma)\gamma^d(\varsigma)D^b\psi^c(e, \varsigma) = 0, \gamma_a(\varsigma)\psi^a(e, \varsigma) = 0$ (规范条件)

$$\Leftrightarrow \gamma_b(\varsigma)D^b\psi^a(e, \varsigma) = 0, \gamma_a(\varsigma)\psi^a(e, \varsigma) = 0$$

### 3.4 Weyl型R-S方程的等价形式

推论3.4.1.  $\varepsilon_{abcd}(\sigma, -i\varsigma)^d D^b\psi^c(\varsigma) = 0 \Leftrightarrow (\sigma, -i\varsigma)_b[D^b\psi^a(\varsigma) - D^a\psi^b(\varsigma)] = 0, D_a\psi^a(\varsigma) = [(\sigma, i\varsigma)_a D^a][(\sigma, -i\varsigma)_b \psi^b(\varsigma)]$

推论3.4.2.  $(\sigma, -i\varsigma)_b[D^b\psi^a(\varsigma) - D^a\psi^b(\varsigma)] = 0 \Rightarrow D_a\psi^a(\varsigma) = [(\sigma, i\varsigma)_a D^a][(\sigma, -i\varsigma)_b \psi^b(\varsigma)]$

推论3.4.3.  $\varepsilon_{abcd}(\sigma, -i\varsigma)^d D^b\psi^c(\varsigma) = 0 \Leftrightarrow (\sigma, -i\varsigma)_b[D^b\psi^a(\varsigma) - D^a\psi^b(\varsigma)] = 0$

重要结论:

定理3.4.1.  $\varepsilon_{abcd}(\sigma, -i\varsigma)^d D^b\psi^c(\varsigma) = 0 \Leftrightarrow (\sigma, -i\varsigma)_a F^{ab}(\frac{3}{2}, \varsigma) = 0, F^{ab}(\frac{3}{2}, \varsigma) := D^a\psi^b(\varsigma) - D^b\psi^a(\varsigma)$

推论3.4.4.  $F_{uv}(\frac{3}{2}, \varsigma) \equiv D_u\psi_v(\varsigma) - D_v\psi_u(\varsigma) \Leftrightarrow F_{uv}(\frac{3}{2}, \varsigma) = (\partial_u + \frac{i}{2}\sigma_{\alpha\varsigma}A_u^{\alpha\varsigma})\psi_v(\varsigma) - (\partial_v + \frac{i}{2}\sigma_{\alpha\varsigma}A_v^{\alpha\varsigma})\psi_u(\varsigma)$

推论3.4.5.  $\varepsilon_{abcd}(\sigma, -i\varsigma)^d D^b\psi^c(\varsigma) = 0, (\sigma, -i\varsigma)_a\psi^a(\varsigma) = 0 \Leftrightarrow (\sigma, -i\varsigma)_b D^b\psi^a(\varsigma) = 0, (\sigma, -i\varsigma)_a\psi^a(\varsigma) = 0$

### 3.5 Weyl型R-S方程等价的低一阶导数自旋张量新形式

推论3.5.1.  $\varepsilon_{abcd}(\sigma, -i\varsigma)^d D^b\psi^c(\varsigma) = 0 \Leftrightarrow [\frac{1}{2}\delta_{ab} + iS_{ab}(\varsigma)]F^{bc}(\frac{3}{2}, \varsigma) = 0, F^{bc}(\frac{3}{2}, \varsigma) \equiv D^b\psi^c(\varsigma) - D^c\psi^b(\varsigma)$

## 4 方程的对比

### 4.1 Weyl型引力微子方程和Penrose型引力微子方程的对比

Weyl型R-S方程:  $(\sigma, -i\varsigma)_a F^{ab}(\frac{3}{2}, \varsigma) = 0 \Leftrightarrow$  Penrose型R-S方程:  $\partial_a F^{ab}(\frac{3}{2}, \varsigma) = -J^b(\varsigma)$  (9.6)

$$F_{uv}(\frac{3}{2}, \varsigma) \equiv (\partial_u + \frac{i}{2}A_u^{\alpha\varsigma}\sigma_{\alpha\varsigma})\psi_v(\varsigma) - (\partial_v + \frac{i}{2}A_v^{\alpha\varsigma}\sigma_{\alpha\varsigma})\psi_u(\varsigma) \quad (9.7)$$

形式上相当于 $(\sigma, -i\varsigma)_a \leftrightarrow \partial_a$ , 引力场情形和引力微子情形两者形式上也十分相似。

### 4.2 引力场Einstein方程和引力场规范方程的对比

引力场Einstein方程:  $(\sigma_{-\varsigma}, -i\varsigma)_a \mathcal{F}^{ab}(2, \varsigma) = \varsigma \bar{T}^b \Leftrightarrow$  引力场规范方程:  $D_a \mathcal{F}^{ab}(2, \varsigma) = -\mathcal{J}^b(\varsigma)$  (9.8)

$$\mathcal{F}_{uv}(2, \varsigma) = (\partial_u + \frac{i}{2}A_u^{\alpha\varsigma}\mathcal{R}_{\alpha\varsigma})\mathcal{A}_v(\varsigma) - (\partial_v + \frac{i}{2}A_v^{\alpha\varsigma}\mathcal{R}_{\alpha\varsigma})\mathcal{A}_u(\varsigma) \quad (9.9)$$

形式上相当于 $(\sigma_{-\varsigma}, -i\varsigma)_a \leftrightarrow D_a$

# 第十章 各种粒子的自旋方程

自我评述：在此章中我独立创新地提出了一种全新的粒子方程表述形式：自旋方程，此方程直接用自旋和自旋张量矩阵构造，并注意到自旋张量同时也是场量相应表示的变换矩阵，所以此方程物理意义十分明确，可以根据粒子场量变换规律简单直接地写出相应的粒子方程，它正确描述了中微子、电磁场、Yang-Mills 场和电子等经典方程，并发现其无质量表示完全等价于全对称的Penrose旋量方程，当然它比Penrose旋量方程更广泛，可以描述更多物理方程。我继续利用自旋表述这个思想，进一步得到了正确描述Einstein引力场和引力微子的低一阶导数自旋方程。在以上这些自旋表述形式中可以很自然地引入一个标量场，从而推广得到了一个更有意思的方程：开关型自旋方程，在标量场为零时，自由粒子可以存在，在标量场不为零时，自由粒子不存在。此标量场的作用就象开关一样，控制粒子的产生与湮灭，这就为粒子的产生与湮灭提供了一种新的物理机制。

## 1 自旋矢量 $W_a$ 的描述

### 1.1 自旋矢量 $W_a, W_a(s, \varsigma)$ 定义

$$s\text{自旋张量矩阵: } S_{(ab)}(s, \varsigma) = \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -\varsigma\sigma_z(s) \\ \varsigma\sigma_x(s) & \varsigma\sigma_y(s) & \varsigma\sigma_z(s) & 0 \end{bmatrix} \quad (10.1)$$

$$\text{定义1.1.1. } W_a := -i * M_{ab} p^b = \frac{-i}{2} \varepsilon_{abcd} M^{bc} p^d$$

$$\text{定义1.1.2. } W_a(s, \varsigma) := -i * M_{ab}(s, \varsigma) p^b, M_{ab}(s, \varsigma) = L_{ab} + S_{ab}(s, \varsigma)$$

$$\text{性质1.1.1. } W_a p^a = 0, W_a(s, \varsigma) p^a = 0$$

说明自旋矢量与动量正交，且只有三个独立分量。

$$\text{性质1.1.2. } *L_{ab} p^b = 0$$

$$\text{证明: } *L_{ab} p^b = \frac{1}{2} \varepsilon_{abcd} (x^c p^d - x^d p^c) p^b = \varepsilon_{abcd} x^c p^d p^b = \varepsilon_{abcd} x^c p^b p^d = 0 \quad \square$$

说明轨道角动量对自旋矢量无贡献，因而有下面的结论。

$$\text{推论1.1.1. } \begin{cases} W_a = -i * S_{ab} p^b \\ W_a(s, \varsigma) = -i * S_{ab}(s, \varsigma) p^b = i\varsigma S_{ab}(s, \varsigma) p^b \end{cases}$$

### 1.2 自旋矢量 $W_a(s, \varsigma)$ 的性质

$$\text{性质1.2.1. } W_a(s, \varsigma) W^a(s, \varsigma) = m^2 s(s+1), m^2 = -p_a p^a$$

$$\text{证明: } W_a(s, \varsigma) W^a(s, \varsigma) = [i\varsigma S_{ab}(s, \varsigma) p^b] [i\varsigma S^{ac}(s, \varsigma) p_c]$$

$$\Leftrightarrow W_a(s, \varsigma) W^a(s, \varsigma) = -p^a S_{ca}(s, \varsigma) S^{cb}(s, \varsigma) p_b$$

$$\Leftrightarrow W_a(s, \varsigma) W^a(s, \varsigma) = p^a S_{ac}(s, \varsigma) S^c_b(s, \varsigma) p^b$$

$$\Leftrightarrow W_a(s, \varsigma) W^a(s, \varsigma) = -p^a s(s+1) \delta_{ab} p^b$$

$$\Leftrightarrow W_a(s, \varsigma) W^a(s, \varsigma) = -s(s+1) p_a p^a$$

$$\Leftrightarrow W_a(s, \varsigma) W^a(s, \varsigma) = m^2 s(s+1), m^2 = -p_a p^a \quad \square$$

利用 $S_{ab}$ 的常数张量性质可以证明以下一般结论：

$$\text{性质1.2.2. } W_a W^a = m^2 s(s+1), m^2 = -p_a p^a \neq 0$$

$$\text{证明: } W_a W^a = [-i * S_{ab} p^b] [-i * S^{ab} p_b]$$

$$= -[*S_{ab}(0, 0, 0, im)^b] [*S^{ab}(0, 0, 0, im)_b]$$

$$= -[*S_{a\pi} ip] [*S^{a\pi} ip]$$



$$\begin{aligned}
&= m^2 * S_{a\pi} * S^{a\pi} \\
&= -m^2(S_{xy}^2 + S_{yz}^2 + S_{zx}^2) \\
&= m^2 s(s+1)
\end{aligned}$$

□

$$\text{性质1.2.3. } W_a W^a = p^2 s(s+1) - p^2(S_{x\pi}^2 + S_{y\pi}^2 + S_{xy}^2), m^2 = -p_a p^a = 0$$

$$\begin{aligned}
\text{证明: } W_a W^a &= [-i * S_{ab} p^b] [-i * S^{ab} p_b] \\
&= -[*S_{ab}(0, 0, p, ip)^b] [*S^{ab}(0, 0, p, ip)_b] \\
&= -[*S_{az} p] [*S^{az} p] - [*S_{a\pi} ip] [*S^{a\pi} ip] \\
&= -p^2 * S_{az} * S^{az} - p^2 * S_{a\pi} * S^{a\pi} \\
&= p^2(S_{x\pi}^2 + S_{y\pi}^2 + S_{xy}^2) - p^2(S_{xy}^2 + S_{yz}^2 + S_{zx}^2) \\
&= p^2 s(s+1) - p^2(S_{x\pi}^2 + S_{y\pi}^2 + S_{xy}^2)
\end{aligned}$$

□

$$\text{性质1.2.4. } W_a(s, \varsigma) W^a(s, \varsigma) = 0, m^2 = -p_a p^a = 0$$

$$\text{性质1.2.5. } [W_a(s, \varsigma), W_b(s, \varsigma)] = \varsigma[W_a(s, \varsigma)p_b - W_b(s, \varsigma)p_a] - m^2 S_{ab}(s, \varsigma)$$

$$\text{性质1.2.6. } \vec{W}(s, \varsigma) = -i\varsigma\sigma(s) \times \vec{p} - i\sigma(s)p_\pi, W_\pi(s, \varsigma) = i\sigma(s) \cdot \vec{p}$$

$$\text{性质1.2.7. } \vec{W}(s, \varsigma) \times \vec{W}(s, \varsigma) = \varsigma\vec{W}(s, \varsigma) \times \vec{p} + im^2\sigma(s)$$

$$\text{性质1.2.8. } [\sigma(s), i\varsigma]_a W^a(s, \varsigma) = -is(s+1)p_\pi$$

### 1.3 自旋矢量 $W_a(s, \varsigma)$ 在特殊坐标系中的性质

有质量粒子  $W_a(s, \varsigma)$  自旋矢量在随动坐标系中的性质:

$$\text{性质1.3.1. } \vec{W}(s, \varsigma) = m\sigma(s), W_\pi(s, \varsigma) = 0 \text{ for } \vec{p} = 0$$

无质量粒子  $W_a(s, \varsigma)$  自旋矢量在运动方向坐标系中的性质:

$$\text{性质1.3.2. } \begin{cases} W_x(s, \varsigma) = [\sigma_x(s) - i\varsigma\sigma_y(s)]p, W_y(s, \varsigma) = [\sigma_y(s) + i\varsigma\sigma_x(s)]p \\ W_z(s, \varsigma) = \sigma_z(s)p, W_\pi(s, \varsigma) = i\sigma_z(s)p \end{cases} \text{ for } \begin{cases} m = 0, p_x = p_y = 0 \\ p_z = -ip_\pi = p > 0 \end{cases}$$

$$\text{推论1.3.1. } [M_{ab}, p_c p^c] = 0, [L_{ab}, p_c p^c] = 0, [S_{ab}, p_c p^c] = 0, [p_a, p_c p^c] = 0, [p_a, W_b] = 0$$

### 1.4 自旋矢量 $W_a(s, \varsigma)$ 和 $p_a, S_{ab}(s, \varsigma)$ 的对易关系

对易关系:

$$\begin{cases} i[S_{ab}(s, \varsigma), S_{cd}(s, \varsigma)] = g_{ad}S_{bc}(s, \varsigma) - g_{ac}S_{bd}(s, \varsigma) + g_{bc}S_{ad}(s, \varsigma) - g_{bd}S_{ac}(s, \varsigma) \\ [W_a(s, \varsigma), W_b(s, \varsigma)] = \varsigma[W_a(s, \varsigma)p_b + W_b(s, \varsigma)p_a + iS_{ab}(s, \varsigma)p_c p^c] \\ [W_a(s, \varsigma), S_{bc}(s, \varsigma)] = g_{ac}W_b(s, \varsigma) - g_{ab}W_c(s, \varsigma) - iS_{ac}(s, \varsigma)p_b + iS_{ab}(s, \varsigma)p_c \\ [p_a, W_b(s, \varsigma)] = 0, [p_a, S_{bc}(s, \varsigma)] = 0, [p_a, p_b] = 0 \end{cases} \quad (10.2)$$

### 1.5 有质量粒子彭加莱群のカ西米算子 [8]

$$\text{性质1.5.1. } W_a(s, \varsigma) W^a(s, \varsigma) = m^2 s(s+1), p_a p^a = -m^2, p_a W^a(s, \varsigma) = 0$$

### 1.6 无质量粒子彭加莱群のカ西米算子 [8]

$$\text{性质1.6.1. } W_a(s, \varsigma) W^a(s, \varsigma) = 0, p_a p^a = 0, p_a W^a(s, \varsigma) = 0$$

## 1.7 自旋张量 $S_{ab}(s, \varsigma)$ 的统一描述

$S_{ab}(s, \varsigma)$ 适用于任何一种分量形式。

$$\text{tr}[S_{ab}(s, \varsigma)S_{cd}(s, \varsigma)] = -\frac{2}{3}s(s + \frac{1}{2})(s + 1)\sigma_{\varsigma ab}^{\alpha\varsigma}\sigma_{\varsigma cd}^{\alpha\varsigma} \quad (10.3)$$

$$\text{tr}[S_{ab}(s, -\varsigma)S_{cd}(s, -\varsigma)] = -\frac{2}{3}s(s + \frac{1}{2})(s + 1)\sigma_{-\varsigma ab}^{\alpha\varsigma'}\sigma_{-\varsigma cd}^{\alpha\varsigma'} \quad (10.4)$$

$$S_{ac}(s, \varsigma)S_{cb}(s, \varsigma) = -s(s + 1)\delta_{ab}, S_{ac}(s, -\varsigma)S_{cb}(s, -\varsigma) = -s(s + 1)\delta_{ab} \quad (10.5)$$

$$\sigma^2(s) = \frac{1}{4}S_{ab}(s, \varsigma)S^{ab}(s, \varsigma) = \frac{1}{4}S_{ab}(s, -\varsigma)S^{ab}(s, -\varsigma) = s(s + 1) \quad (10.6)$$

## 2 自旋方程的构造

### 2.1 由自旋量直接构造的全新粒子方程

由自旋量直接构造如下粒子方程

$$[(s + \phi)D_a + iS_{ab}D^b]\psi = \mathbb{J}_a \quad (10.7)$$

$\psi$ 为粒子的态旋量, $s$ 为粒子自旋, $S_{ab}$ 为粒子自旋张量, $\phi$ 为标量场, $\mathbb{J}_a$ 为旋量源, $D_a$ 为协变导数。

### 2.2 新粒子方程的性质

$$\text{s-自旋矩阵: } S_{ab}(s, \varsigma) = i\sigma_{\varsigma ab}^{\alpha\varsigma}\sigma_{\alpha\varsigma}(s) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -\varsigma\sigma_z(s) \\ \varsigma\sigma_x(s) & \varsigma\sigma_y(s) & \varsigma\sigma_z(s) & 0 \end{bmatrix}$$

**定理2.2.1.**  $[(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi = 0 \Rightarrow \phi = 0$ 或 $\phi = -(2s + 1)$ 或 $\sigma(s) \cdot \nabla\psi = 0, \partial_\pi\psi = 0$

证明:  $[(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi = 0$

$$\Leftrightarrow \begin{cases} [(s + \phi)\partial_x + i\sigma_z(s)\partial_y - i\sigma_y(s)\partial_z - i\varsigma\sigma_x(s)\partial_\pi]\psi = 0 \\ [(s + \phi)\partial_y + i\sigma_x(s)\partial_z - i\sigma_z(s)\partial_x - i\varsigma\sigma_y(s)\partial_\pi]\psi = 0 \\ [(s + \phi)\partial_z + i\sigma_y(s)\partial_x - i\sigma_x(s)\partial_y - i\varsigma\sigma_z(s)\partial_\pi]\psi = 0 \\ [(s + \phi)\partial_\pi + i\varsigma\sigma_x(s)\partial_x + i\varsigma\sigma_y(s)\partial_y + i\varsigma\sigma_z(s)\partial_z]\psi = 0 \end{cases}$$

$$\Rightarrow \begin{cases} [(s + 1 + \phi)\sigma(s) \cdot \nabla - i\varsigma\sigma^2(s)\partial_\pi]\psi = 0 \\ [(s + \phi)\partial_\pi + i\varsigma\sigma \cdot \nabla]\psi = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \sigma(s) \cdot \nabla\psi = i\varsigma(s + \phi)\partial_\pi\psi \\ [(s + \phi)(s + 1 + \phi) - s(s + 1)]\partial_\pi\psi = 0 \end{cases}$$

$$\Leftrightarrow \phi = 0 \text{ 或 } \phi = -(2s + 1) \text{ 或 } \sigma(s) \cdot \nabla\psi = 0, \partial_\pi\psi = 0 \quad \square$$

**定理2.2.2.**  $[s\partial_a + iS_{ab}\partial^b]\psi = 0 \Rightarrow \partial_a\partial^a\psi = 0$

证明:  $[s\partial_a + iS_{ab}\partial^b]\psi = 0$

$$\Rightarrow \partial^a[s\partial_a + iS_{ab}\partial^b]\psi = 0$$

$$\Leftrightarrow [s\partial_a\partial^a + iS_{ab}\partial^a\partial^b]\psi = 0$$

$$\Leftrightarrow [s\partial_a\partial^a + 0]\psi = 0$$

$$\Leftrightarrow \partial_a\partial^a\psi = 0 \quad \square$$

即此方程描述的是无质量粒子。

**定理2.2.3.**  $\begin{cases} \text{当 } \phi \neq 0 \text{ 时, } [(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(s, \varsigma) = 0 \text{ 无平面波解} \\ \text{当 } \phi = 0 \text{ 时, } [(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(s, \varsigma) = 0 \text{ 有平面波解} \end{cases}$

证明: 因为此方程描述的是无质量粒子, 因此总可以选择粒子运动方向为 $z$ , 此时 $p_a = (0, 0, p, ip)$ , 则

$$[(s + \phi)p_a + iS_{ab}(s, \varsigma)p^b]\psi(s, \varsigma) = 0$$

$$\begin{aligned}
& \Leftrightarrow \begin{cases} [(s + \phi)p_x + i\sigma_z(s)p_y - i\sigma_y(s)p_z - i\zeta\sigma_x(s)p_\pi]\psi(s, \varsigma) = 0 \\ [(s + \phi)p_y + i\sigma_x(s)p_z - i\sigma_z(s)p_x - i\zeta\sigma_y(s)p_\pi]\psi(s, \varsigma) = 0 \\ [(s + \phi)p_z + i\sigma_y(s)p_x - i\sigma_x(s)p_y - i\zeta\sigma_z(s)p_\pi]\psi(s, \varsigma) = 0 \\ [(s + \phi)p_\pi + i\zeta\sigma_x(s)p_x + i\zeta\sigma_y(s)p_y + i\zeta\sigma_z(s)p_z]\psi(s, \varsigma) = 0 \end{cases} \\
& \Leftrightarrow \begin{cases} [-i\sigma_y(s)p_z - i\zeta\sigma_x(s)p_\pi]\psi(s, \varsigma) = 0 \\ [i\sigma_x(s)p_z - i\zeta\sigma_y(s)p_\pi]\psi(s, \varsigma) = 0 \\ [(s + \phi)p_z - i\zeta\sigma_z(s)p_\pi]\psi(s, \varsigma) = 0 \\ [(s + \phi)p_\pi + i\zeta\sigma_z(s)p_z]\psi(s, \varsigma) = 0 \end{cases} \\
& \Leftrightarrow \begin{cases} [\sigma_x(s) - i\zeta\sigma_y(s)]p\psi(s, \varsigma) = 0 \Leftrightarrow \psi_m(s, \varsigma) = 0, m = s - 1, \dots, -(s - 1), \zeta s \\ [(s + \phi) + \zeta\sigma_z(s)]p\psi(s, \varsigma) = 0 \end{cases} \\
& \Leftrightarrow \begin{cases} \psi_m(s, \varsigma) = 0, m = s - 1, \dots, -(s - 1), \zeta s \\ \phi\psi_{-\zeta s}(s, \varsigma) = 0 \end{cases} \\
& \Leftrightarrow \begin{cases} \text{当 } \phi \neq 0 \text{ 时, } \psi(s, \varsigma) = 0, \text{ 即所有自旋分量全为 } 0 \\ \text{当 } \phi = 0 \text{ 时, } \psi(s, \varsigma) = [\frac{1}{2}(\zeta - 1)\psi_s, 0, \dots, 0, \frac{1}{2}(\zeta + 1)\psi_{-s}]^T e^{ip \cdot x} \\ \text{即 } -\zeta s \text{ 自旋分量可以不为 } 0, \text{ 其余分量全为 } 0 \end{cases} \\
& \Leftrightarrow \begin{cases} \text{当 } \phi \neq 0 \text{ 时, } [(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(s, \varsigma) = 0 \text{ 无平面波解} \\ \text{当 } \phi = 0 \text{ 时, } [(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(s, \varsigma) = 0 \text{ 有平面波解} \end{cases} \quad \square
\end{aligned}$$

即此方程中的 $\phi$ 有类似开关的作用。

推论2.2.1.  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(s, \varsigma) = 0$ 有平面波解 $\psi(s, \varsigma) = [\frac{1}{2}(\zeta - 1)\psi_s, 0, \dots, 0, \frac{1}{2}(\zeta + 1)\psi_{-s}]^T e^{ip \cdot x}$

## 2.3 自旋方程的定义

定义2.3.1.  $[sD_a + iS_{ab}D^b]\psi = \mathbb{J}_a$ 称为自旋方程。

推论2.3.1.  $(s\delta_{ab} + iS_{ab})D^b\psi = \mathbb{J}_a$

## 2.4 自旋方程的一种等价表述

推论2.4.1.  $[s\hat{P}_a + \zeta\hat{W}_a(s, \varsigma)]\psi(s, \varsigma) = -i\mathbb{J}_a(s, \varsigma), \hat{P}_a := -i\partial_a, \hat{W}_a(s, \varsigma) := \zeta S_{ab}(s, \varsigma)\partial^b$

即自旋方程可以看作动量和自旋矢量关系决定的方程。

定理2.4.1.  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi = 0 \Leftrightarrow [s\hat{p}_a + \hat{W}_a(s, \varsigma)]\psi(s, \varsigma) = 0 \Leftrightarrow \hat{W}_a(s, \varsigma)\psi(s, \varsigma) = -s\zeta\hat{p}_a\psi(s, \varsigma)$

## 2.5 开关型自旋方程的定义

定义2.5.1.  $[(s + \phi)D_a + iS_{ab}D^b]\psi = \mathbb{J}_a$ 称为开关型自旋方程,  $\phi$ 称为开关型标量场。

推论2.5.1.  $[(s + \phi)\delta_{ab} + iS_{ab}]D^b\psi = \mathbb{J}_a$

## 2.6 开关型自旋方程的一种等价表述

推论2.6.1.  $[(s + \phi)\hat{p}_a + \zeta\hat{W}_a(s, \varsigma)]\psi(s, \varsigma) = -i\mathbb{J}_a(s, \varsigma)$

即开关型自旋方程可以看作动量和自旋矢量关系决定的方程。

### 3 各种粒子自旋方程

#### 3.1 中微子<sup>[5]</sup>自旋方程

$$\text{中微子自旋矩阵: } S_{ab}(\varsigma) = \frac{i}{2} \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma} \succ \frac{1}{2} \begin{bmatrix} 0 & \sigma_z & -\sigma_y & -\varsigma\sigma_x \\ -\sigma_z & 0 & \sigma_x & -\varsigma\sigma_y \\ \sigma_y & -\sigma_x & 0 & -\varsigma\sigma_z \\ \varsigma\sigma_x & \varsigma\sigma_y & \varsigma\sigma_z & 0 \end{bmatrix} \quad (10.8)$$

$$\text{定理3.1.1. } [\frac{1}{2}D_a + iS_{ab}(\varsigma)D^b]\psi(\frac{1}{2}, \varsigma) = 0 \Leftrightarrow (\sigma, -i\varsigma)^a D_a \psi(\frac{1}{2}, \varsigma) = 0$$

#### 3.2 任意N+1维时空中的电子<sup>[4]</sup>自旋方程

任意n=N+1维时空中的电子自旋方程:

$$\text{定理3.2.1. } [\frac{1}{2}(D_a + m\gamma_a) + iS_{ab}D^b]\psi = 0, S_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b] \Leftrightarrow (\gamma^a D_a + m)\psi = 0$$

$$\text{证明: } [\frac{1}{2}(D_a + m\gamma_a) + iS_{ab}D^b]\psi = 0, S_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b]$$

$$\Leftrightarrow [(2iS_{ab} + \delta_{ab})D^b + \gamma_a m]\psi = 0, S_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b]$$

$$\Leftrightarrow [\frac{1}{2}([\gamma_a, \gamma_b] + \{\gamma_a, \gamma_b\})D_b + \gamma_a m]\psi = 0$$

$$\Leftrightarrow \gamma_a(\gamma_b D^b + m)\psi = 0$$

$$\Leftrightarrow (\gamma_a D^a + m)\psi = 0$$

$$\Leftrightarrow (\gamma^a D_a + m)\psi = 0 \quad \square$$

4维时空中的电子自旋方程:

$$\text{推论3.2.1. } \{\frac{1}{2}[D_a + m\gamma_a(\varsigma)] + iS_{ab}(e, \varsigma)D^b\}\psi(e, \varsigma) = 0 \Leftrightarrow [\gamma^a(\varsigma)D_a + m]\psi(e, \varsigma) = 0$$

#### 3.3 Yang-Mills场<sup>[6]</sup>自旋方程

$$\text{定理3.3.1. } (D_a + iS_{ab}D^b)^{\beta\varsigma} \Psi^{\gamma\sigma}(1, \varsigma) = -i\sigma_{\varsigma ab}^{\beta\varsigma} J^{b\sigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha\varsigma} \gamma_{\alpha\varsigma} \\ \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\psi}^\sigma(1, \varsigma) = i\tilde{J}^\sigma(1, \varsigma)$$

$$\text{定理3.3.2. } (D_a + iS_{ab}D^b)^{\beta\varsigma} \psi^{\gamma\sigma}(1, \varsigma) = -i\sigma_{\varsigma ab}^{\beta\varsigma} J^{b\sigma}, S_{ab} = i\sigma_{\varsigma ab}^{\alpha\varsigma} \gamma_{\alpha\varsigma} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\psi}^\sigma(1, \varsigma) = iJ^\sigma$$

#### 3.4 s-自旋粒子的自旋方程: 全对称性Penrose方程<sup>[1,2]</sup>

##### 3.4.1 s-自旋方程

$$\text{s-自旋矩阵: } S_{ab}(s, \varsigma) = i\sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -\varsigma\sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -\varsigma\sigma_z(s) \\ \varsigma\sigma_x(s) & \varsigma\sigma_y(s) & \varsigma\sigma_z(s) & 0 \end{bmatrix} \quad (10.9)$$

$$\text{定理3.4.1. } \begin{cases} \nabla^{A'_\varsigma A_\varsigma} \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}} \\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}} = \frac{1}{(2s)!} \psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma \dots)_{2s}}_{2s}} \\ J^{A'_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}} = \frac{1}{(2s-1)!} J^{A'_\varsigma}_{\underbrace{(B_\varsigma C_\varsigma \dots)_{2s-1}}_{2s-1}} \end{cases} \Leftrightarrow [sD_a + iS_{ab}(s, \varsigma)D^b]\psi(s, \varsigma) = -\sqrt{2}\varsigma s \bar{Z}_a(s, \varsigma) \tilde{J}(s)$$

$$\text{证明: } \nabla^{A'_\varsigma A_\varsigma} \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}} = \frac{-\varsigma}{\sqrt{2}} J^{A'_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}} \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}} = \frac{1}{(2s)!} \psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma \dots)_{2s}}_{2s}}, J^{A'_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}} = \frac{1}{(2s-1)!} J^{A'_\varsigma}_{\underbrace{(B_\varsigma C_\varsigma \dots)_{2s-1}}_{2s-1}}$$

$$\Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D^a \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}} = i J^{A'_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}}$$

$$\Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \Gamma_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}}^{k_\varsigma}(s) D^a \psi_{k_\varsigma}(s) = i J^{A'_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}}$$

$$\Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s) \Gamma_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}}^{l_\varsigma}(s) (s - \frac{1}{2}) D^a \psi_{k_\varsigma}(s) = i J^{A'_\varsigma}_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}}$$

$$\Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s) D^a \psi_{k_\varsigma}(s) = i J^{A'_\varsigma}_{l_\varsigma}(s - \frac{1}{2})$$

$$\begin{aligned}
&\Leftrightarrow N_{j_\zeta}^{Z_\zeta l_\zeta}(s)(\sigma, i\zeta)_{a_{Z_\zeta A'_\zeta}}(\sigma, -i\zeta)_{b_{A'_\zeta A_\zeta}} N_{A_\zeta l_\zeta}^{k_\zeta}(s) D^b \psi_{k_\zeta}(s) = i N_{j_\zeta}^{Z_\zeta l_\zeta}(s)(\sigma, i\zeta)_{a_{Z_\zeta A'_\zeta}} J_{A'_\zeta l_\zeta}^{A'_\zeta}(s - \frac{1}{2}) \\
&\Leftrightarrow N_{j_\zeta}^{Z_\zeta l_\zeta}(s)[\delta_{ab} \delta_{Z_\zeta A_\zeta} + 2i S_{ab Z_\zeta A_\zeta}(\frac{1}{2}, \zeta)] N_{A_\zeta l_\zeta}^{k_\zeta}(s) D^b \psi_{k_\zeta}(s) = i N_{j_\zeta}^{Z_\zeta l_\zeta}(s)(\sigma, i\zeta)_{a_{Z_\zeta A'_\zeta}} J_{A'_\zeta l_\zeta}^{A'_\zeta}(s - \frac{1}{2}) \\
&\Leftrightarrow [s \delta_{ab} \delta_{j_\zeta}^{k_\zeta} + i S_{ab j_\zeta}^{k_\zeta}(s, \zeta)] D^b \psi_{k_\zeta}(s) = i s (\sigma, i\zeta)_{a_{A_\zeta A'_\zeta}} N_{j_\zeta}^{A_\zeta l_\zeta}(s) J_{A'_\zeta l_\zeta}^{A'_\zeta}(s - \frac{1}{2}) \\
&\Leftrightarrow [s \delta_{ab} \delta_{j_\zeta}^{k_\zeta} + i S_{ab j_\zeta}^{k_\zeta}(s, \zeta)] D^b \psi_{k_\zeta}(s) = i s \delta_{ab} (\sigma, i\zeta)_{A_\zeta A'_\zeta} \overbrace{\Gamma_{j_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}}^{2s}(s) J_{B_\zeta C_\zeta \dots}^{A'_\zeta} \\
&\Leftrightarrow [s \delta_{ab} \delta_{j_\zeta}^{k_\zeta} + i S_{ab j_\zeta}^{k_\zeta}(s, \zeta)] D^b \psi_{k_\zeta}(s) = -\sqrt{2} \zeta s Z_{A'_\zeta j_\zeta}^{a l_\zeta}(s, \zeta) J_{A'_\zeta l_\zeta}^{A'_\zeta}(s - \frac{1}{2}) \\
&\Leftrightarrow [s D_a + i S_{ab}(s, \zeta) D^b] \psi(s, \zeta) = -\sqrt{2} \zeta s \bar{Z}_a(s, \zeta) \tilde{J}(s), \psi(s, \zeta) \prec \psi_{k_\zeta}(s), \tilde{J}(s) \prec J_{A'_\zeta l_\zeta}^{A'_\zeta}(s - \frac{1}{2}) \quad \square
\end{aligned}$$

从上可知，s-自旋方程就是全对称性Penrose方程的自旋张量表述形式。

**推论3.4.1.**  $(\sigma \otimes I_{2^{2s-1}}, -i\zeta)^a D_a \hat{\psi}(s, \zeta) = i \hat{J}(s, \zeta) \Leftrightarrow (\sigma \otimes I_{2^s}, -i\zeta)^a D_a \tilde{\psi}(s, \zeta) = i \tilde{J}(s, \zeta)$

**证明:**  $(\sigma \otimes I_{2^{2s-1}}, -i\zeta)^a D_a \hat{\psi}(s, \zeta) = i \hat{J}(s, \zeta)$

$$\begin{aligned}
&\Leftrightarrow (\sigma \otimes I_{2^{2s-1}}, -i\zeta)^a \Gamma(s) D_a \psi(s, \zeta) = i [I \otimes \Gamma(s - \frac{1}{2})] \tilde{J}(s, \zeta) \\
&\Leftrightarrow (\sigma \otimes I_{2^{2s-1}}, -i\zeta)^a [I_{w+1} \otimes \Gamma(s - \frac{1}{2})] N(s) D_a \psi(s, \zeta) = i [I \otimes \Gamma(s - \frac{1}{2})] \tilde{J}(s, \zeta) \\
&\Leftrightarrow [I_{w+1} \otimes \Gamma(s - \frac{1}{2})] (\sigma \otimes I_{2^s}, -i\zeta)^a N(s) D_a \psi(s, \zeta) = i [I \otimes \Gamma(s - \frac{1}{2})] \tilde{J}(s, \zeta) \\
&\Leftrightarrow (\sigma \otimes I_{2^s}, -i\zeta)^a D_a \tilde{\psi}(s, \zeta) = i \tilde{J}(s, \zeta) \quad \square
\end{aligned}$$

**推论3.4.2.**  $[s D_a + i S_{ab}(s, \zeta) D^b] \psi(s, \zeta) = -\sqrt{2} \zeta s \bar{Z}_a(s, \zeta) \tilde{J}(s) \Leftrightarrow (\sigma \otimes I_{2^s}, -i\zeta)_a D^a \tilde{\psi}(s, \zeta) = i \tilde{J}(s, \zeta)$

**证明:**  $[s D_a + i S_{ab}(s, \zeta) D^b] \psi(s, \zeta) = -\sqrt{2} \zeta s \bar{Z}_a(s, \zeta) \tilde{J}(s, \zeta)$

$$\begin{aligned}
&\Leftrightarrow [s \delta_{ab} I_{2s+1} + i S_{ab}(s, \zeta)] D^b \psi(s, \zeta) = -\sqrt{2} \zeta s \bar{Z}_a(s, \zeta) \tilde{J}(s, \zeta) \\
&\Leftrightarrow 2s \bar{Z}_a(s, \zeta) Z_b(s, \zeta) D^b \psi(s, \zeta) = -\sqrt{2} \zeta s \bar{Z}_a(s, \zeta) \tilde{J}(s, \zeta) \\
&\Leftrightarrow Z_b(s, \zeta) D^b \psi(s, \zeta) = \frac{-\zeta}{\sqrt{2}} \tilde{J}(s, \zeta) \\
&\Leftrightarrow (\sigma \otimes I_{2^s}, -i\zeta)^a D_a \tilde{\psi}(s, \zeta) = i \tilde{J}(s, \zeta) \quad \square
\end{aligned}$$

**推论3.4.3.**

$$\begin{cases}
(\sigma, -i\zeta)_{a_{A'_\zeta A_\zeta}} D^a \underbrace{\psi_{A_\zeta B_\zeta C_\zeta \dots}}_{2s} = i J_{B_\zeta C_\zeta \dots}^{A'_\zeta} \\
\psi_{A_\zeta B_\zeta C_\zeta \dots} = \frac{1}{(2s)!} \underbrace{\psi_{(A_\zeta B_\zeta C_\zeta \dots)}}_{2s}, J_{B_\zeta C_\zeta \dots}^{A'_\zeta} = \frac{1}{(2s-1)!} \underbrace{J_{B_\zeta C_\zeta \dots}^{A'_\zeta}}_{2s-1}
\end{cases} \Leftrightarrow (\sigma \otimes I_{2^s}, -i\zeta)^a D_a \tilde{\psi}(s, \zeta) = i \tilde{J}(s, \zeta)$$

**推论3.4.4.**  $[s D_a + i S_{ab}(s, \zeta) D^b] \psi(s, \zeta) = \mathbb{J}_a(s, \zeta) \Rightarrow \mathbb{J}^a(s, \zeta) = \frac{2s}{2s+1} \bar{Z}_a(s, \zeta) Z_b(s, \zeta) \mathbb{J}^b(s, \zeta)$

**证明:**  $[s D_a + i S_{ab}(s, \zeta) D^b] \psi(s, \zeta) = \mathbb{J}_a(s, \zeta)$

$$\begin{aligned}
&\Leftrightarrow 2s \bar{Z}_a(s, \zeta) Z_b(s, \zeta) D^b \psi(s, \zeta) = \mathbb{J}_a(s, \zeta) \\
&\Rightarrow Z_b(s, \zeta) D^b \psi(s, \zeta) = \frac{1}{2s+1} Z^a(s, \zeta) \mathbb{J}_a(s, \zeta) \\
&\Rightarrow 2s \bar{Z}_a(s, \zeta) Z_b(s, \zeta) D^b \psi(s, \zeta) = \frac{1}{2s+1} 2s \bar{Z}_a(s, \zeta) Z_b(s, \zeta) \mathbb{J}^b(s, \zeta) \\
&\Rightarrow \mathbb{J}^a(s, \zeta) = \frac{2s}{2s+1} \bar{Z}_a(s, \zeta) Z_b(s, \zeta) \mathbb{J}^b(s, \zeta) \quad \square
\end{aligned}$$

**推论3.4.5.**  $[s D_a + i S_{ab}(s, \zeta) D^b] \psi(s, \zeta) = \mathbb{J}_a(s, \zeta)$  无解  $\Leftrightarrow \mathbb{J}^a(s, \zeta) \neq \frac{2s}{2s+1} \bar{Z}_a(s, \zeta) Z_b(s, \zeta) \mathbb{J}^b(s, \zeta)$

### 3.4.2 不同阶自旋方程之间的等价性(需完善)

**定理3.4.2.**  $[s D_a + i S_{ab}(s, \zeta) D^b] \psi(s, \zeta) = -\sqrt{2} \zeta s \bar{Z}_a(s, \zeta) \tilde{J}(s, \zeta)$

$$\Leftrightarrow [(s-l) D_a + i S_{ab}(s-l, \zeta) D^b] \underbrace{\psi_{A_\zeta B_\zeta \dots}}_{2l}(s-l, \zeta) = -\sqrt{2} \zeta s \bar{Z}_a(s-l, \zeta) \underbrace{\tilde{J}_{A_\zeta B_\zeta \dots}}_{2l}(s-l, \zeta)$$

$l = 0, \frac{1}{2}, 1, \dots, s +$  对称性条件

### 3.4.3 源 $\mathbb{J}_a(s, \varsigma)$ 的性质

推论3.4.6.  $\mathbb{J}^a(s, \varsigma) = \frac{2s}{2s+1} \bar{Z}_a(s, \varsigma) Z_b(s, \varsigma) \mathbb{J}^b(s, \varsigma) \Leftrightarrow \exists \tilde{J}(s, \varsigma), \mathbb{J}_a(s, \varsigma) = -\sqrt{2\varsigma s} \bar{Z}_a(s, \varsigma) \tilde{J}(s, \varsigma)$

推论3.4.7.  $\mathbb{J}^a(s, \varsigma) = \frac{2s}{2s+1} \bar{Z}_a(s, \varsigma) Z_b(s, \varsigma) \mathbb{J}^b(s, \varsigma) \Leftrightarrow \mathbb{J}_a(s, \varsigma) = \frac{1}{s+1} i S_{ab}(s, \varsigma) \mathbb{J}^b(s, \varsigma)$

推论3.4.8.  $\mathbb{J}_a(s, \varsigma) = \frac{1}{s+1} i S_{ab}(s, \varsigma) \mathbb{J}^b(s, \varsigma) \Leftrightarrow \begin{cases} (s+1)\mathbb{J} = -i\sigma(s) \times \mathbb{J} - i\varsigma\sigma(s)\mathbb{J}_\pi \\ \sigma(s) \cdot \mathbb{J} + i\varsigma(s+1)\mathbb{J}_\pi = 0 \end{cases}$

性质3.4.1.  $\sigma(s) \cdot [\sigma(s) \times \mathbb{J}] = i\sigma(s) \cdot \mathbb{J}$

推论3.4.9.  $(s+1)\mathbb{J} = -i\sigma(s) \times \mathbb{J} - i\varsigma\sigma(s)\mathbb{J}_\pi \Rightarrow \sigma(s) \cdot \mathbb{J} + i\varsigma(s+1)\mathbb{J}_\pi = 0$

推论3.4.10.  $\mathbb{J}_a(s, \varsigma) = \frac{1}{s+1} i S_{ab}(s, \varsigma) \mathbb{J}^b(s, \varsigma) \Leftrightarrow (s+1)\mathbb{J} = -i\sigma(s) \times \mathbb{J} - i\varsigma\sigma(s)\mathbb{J}_\pi$

### 3.4.4 无质量s-自旋粒子的螺旋度

定义3.4.1. s-自旋粒子的螺旋度： $\mathcal{P}(s) := \frac{\sigma(s) \cdot \vec{p}}{|\vec{p}|}$

推论3.4.11.  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(s, \varsigma) = 0 \Rightarrow [\sigma(s), -i\varsigma\sigma]^a \partial_a \varphi(s, \varsigma) = 0, \partial^a \partial_a \varphi(s, \varsigma) = 0$

推论3.4.12.  $\begin{cases} (\vec{p}^2 - E^2)\psi(s, \varsigma) = 0 \\ \sigma(s) \cdot \vec{p}\psi(s, \varsigma) = -s\varsigma E\psi(s, \varsigma) \end{cases} \Rightarrow \mathcal{P}(s)\psi(s, \varsigma) = \frac{\sigma(s) \cdot \vec{p}}{|\vec{p}|}\psi(s, \varsigma) = \begin{cases} -s\varsigma\psi(s, \varsigma), E = |\vec{p}| \\ s\varsigma\psi(s, \varsigma), E = -|\vec{p}| \end{cases}$

从上可知，无质量s-自旋粒子螺旋度本征值只能是 $\pm s$ ，没有其它值。

### 3.5 s-自旋方程的各种等价形式

推论3.5.1.  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x; s) = \mathbb{J}_a \Leftrightarrow [s\partial_a + i\Omega_{ab}(s)\partial^b]\Gamma(s)\psi(x; s) = \Gamma(s)\mathbb{J}_a$

$\Leftrightarrow \{s\partial_a + i[S_{ab}(\frac{1}{2}, \varsigma) \otimes I_{2s} + I \otimes S_{ab}(s - \frac{1}{2}, \varsigma)]\partial^b\}N(s)\psi(x; s) = N(s)\mathbb{J}_a$

证明： $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x; s) = \mathbb{J}_a$

$\Leftrightarrow \Gamma(s)[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x; s) = \Gamma(s)\mathbb{J}_a$

$\Leftrightarrow [s\partial_a + i\Omega_{ab}(s)\partial^b]\Gamma(s)\psi(x; s) = \Gamma(s)\mathbb{J}_a$  □

证明： $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x; s) = \mathbb{J}_a$

$\Leftrightarrow N(s)[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x; s) = N(s)\mathbb{J}_a$

$\Leftrightarrow \{s\partial_a + i[S_{ab}(\frac{1}{2}, \varsigma) \otimes I_{2s} + I \otimes S_{ab}(s - \frac{1}{2}, \varsigma)]\partial^b\}N(s)\psi(x; s) = N(s)\mathbb{J}_a$  □

### 3.6 偶数维时空中s-自旋粒子的自旋方程：偶数维时空全对称性Penrose方程<sup>[1,2]</sup>错??

#### 3.6.1 偶数维时空中的s-自旋方程

引理3.6.1.  $(\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} (\Gamma, i\varsigma)_{B_\varsigma B'_\varsigma}^a = 2\delta_{B_\varsigma}^{A'_\varsigma} \delta_{B'_\varsigma}^{A_\varsigma}$

引理3.6.2.  $(\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} N_{A'_\varsigma l'_\varsigma}^{k_\varsigma}(s; n) N_{k_\varsigma}^{B_\varsigma m_\varsigma}(s; n) (\Gamma, i\varsigma)_{B_\varsigma B'_\varsigma}^a = 2(1 + \frac{n}{2s}) \delta_{B'_\varsigma}^{A'_\varsigma} \delta_{l'_\varsigma}^{m_\varsigma}$

定理3.6.1.  $\begin{cases} \nabla_{A'_\varsigma A_\varsigma} \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}} = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma}^{A_\varsigma} \underbrace{B_\varsigma C_\varsigma \dots}_{2s-1} \\ \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}} = \frac{1}{(2s)!} \psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma \dots)_{2s}}_{2s}} \\ J_{A'_\varsigma}^{A_\varsigma} \underbrace{B_\varsigma C_\varsigma \dots}_{2s-1} = \frac{1}{(2s-1)!} J_{A'_\varsigma}^{A_\varsigma} \underbrace{(B_\varsigma C_\varsigma \dots)_{2s-1}}_{2s-1} \end{cases} \Leftrightarrow [sD_a + iS_{ab}(s, \varsigma; n)D^b]\psi(s, \varsigma; n) = -\sqrt{2\varsigma s} \bar{Z}_a(s, \varsigma) \tilde{J}(s; n)$

证明： $\nabla_{A'_\varsigma A_\varsigma} \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}} = \frac{-\varsigma}{\sqrt{2}} J_{A'_\varsigma}^{A_\varsigma} \underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}, \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}} = \frac{1}{(2s)!} \psi_{\underbrace{(A_\varsigma B_\varsigma C_\varsigma \dots)_{2s}}_{2s}}, J_{A'_\varsigma}^{A_\varsigma} \underbrace{B_\varsigma C_\varsigma \dots}_{2s-1} = \frac{1}{(2s-1)!} J_{A'_\varsigma}^{A_\varsigma} \underbrace{(B_\varsigma C_\varsigma \dots)_{2s-1}}_{2s-1}$

$\Leftrightarrow (\Gamma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} D^a \psi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}} = i J_{A'_\varsigma}^{A_\varsigma} \underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}$

$$\begin{aligned}
&\Leftrightarrow (\Gamma, -i\zeta)_a^{A'_\zeta A_\zeta} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; n) D^a \psi_{k_\zeta}(s; n) = i J_{B_\zeta C_\zeta \dots}^{A'_\zeta} \\
&\Leftrightarrow (\Gamma, -i\zeta)_a^{A'_\zeta A_\zeta} N_{A_\zeta l_\zeta}^{k_\zeta} (s; n) \Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta} (s - \frac{1}{2}; n) D^a \psi_{k_\zeta}(s; n) = i J_{B_\zeta C_\zeta \dots}^{A'_\zeta} \\
&\Leftrightarrow (\Gamma, -i\zeta)_a^{A'_\zeta A_\zeta} N_{A_\zeta l_\zeta}^{k_\zeta} (s; n) D^a \psi_{k_\zeta}(s; n) = i J_{A_\zeta l_\zeta}^{A'_\zeta} (s - \frac{1}{2}; n) \\
&\Leftrightarrow N_{j_\zeta}^{Z_\zeta l_\zeta} (s; n) (\Gamma, i\zeta)_{a Z_\zeta A'_\zeta} (\Gamma, -i\zeta)_b^{A'_\zeta A_\zeta} N_{A_\zeta l_\zeta}^{k_\zeta} (s; n) D^b \psi_{k_\zeta}(s; n) = i N_{j_\zeta}^{Z_\zeta l_\zeta} (s; n) (\Gamma, i\zeta)_{a Z_\zeta A'_\zeta} J_{A_\zeta l_\zeta}^{A'_\zeta} (s - \frac{1}{2}; n) \\
&\Leftrightarrow N_{j_\zeta}^{Z_\zeta l_\zeta} (s; n) [\delta_{ab} \delta_{Z_\zeta}^{A_\zeta} + 2i S_{ab Z_\zeta}^{A_\zeta} (\frac{1}{2}, \zeta)] N_{A_\zeta l_\zeta}^{k_\zeta} (s; n) D^b \psi_{k_\zeta}(s; n) = i N_{j_\zeta}^{Z_\zeta l_\zeta} (s; n) (\Gamma, i\zeta)_{a Z_\zeta A'_\zeta} J_{A_\zeta l_\zeta}^{A'_\zeta} (s - \frac{1}{2}; n) \\
&\Leftrightarrow [s \delta_{ab} \delta_{j_\zeta}^{k_\zeta} + i S_{ab j_\zeta}^{k_\zeta} (s, \zeta)] D^b \psi_{k_\zeta}(s; n) = i s (\Gamma, i\zeta)_{a A_\zeta A'_\zeta} N_{j_\zeta}^{A_\zeta l_\zeta} (s; n) J_{A_\zeta l_\zeta}^{A'_\zeta} (s - \frac{1}{2}; n) \\
&\Leftrightarrow [s \delta_{ab} \delta_{j_\zeta}^{k_\zeta} + i S_{ab j_\zeta}^{k_\zeta} (s, \zeta)] D^b \psi_{k_\zeta}(s; n) = i s \delta_{ab} (\Gamma, i\zeta)_{A_\zeta A'_\zeta}^b \Gamma_{j_\zeta}^{A_\zeta B_\zeta C_\zeta \dots} (s; n) J_{B_\zeta C_\zeta \dots}^{A'_\zeta} \\
&\Leftrightarrow [s \delta_{ab} \delta_{j_\zeta}^{k_\zeta} + i S_{ab j_\zeta}^{k_\zeta} (s, \zeta)] D^b \psi_{k_\zeta}(s; n) = -\sqrt{2} \zeta s Z_{A'_\zeta j_\zeta}^{a l_\zeta} (s, \zeta; n) J_{A_\zeta l_\zeta}^{A'_\zeta} (s - \frac{1}{2}; n) \\
&\Leftrightarrow \begin{cases} [s D_a + i S_{ab} (s, \zeta; n) D^b] \psi(s, \zeta; n) = -\sqrt{2} \zeta s \bar{Z}_a (s, \zeta; n) \tilde{J}(s; n) \\ \psi(s, \zeta) \prec \psi_{k_\zeta}(s; n), \tilde{J}(s; n) \prec J_{A_\zeta l_\zeta}^{A'_\zeta} (s - \frac{1}{2}; n) \end{cases} \quad \square
\end{aligned}$$

从上可知，s-自旋方程就是全对称性Penrose方程的自旋张量表述形式。

### 3.7 广义自旋方程

**定理3.7.1.**  $(\sigma \langle w \rangle \otimes I_{(w+1)^{2s-1}}, -i\zeta)^a D_a \hat{\psi}(s, \zeta; w) = i \hat{J}(s, \zeta; w)$

$$\Leftrightarrow (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)^a D_a \tilde{\psi}(s, \zeta; w) = i \tilde{J}(s, \zeta; w)$$

**证明:**  $(\sigma \langle w \rangle \otimes I_{(w+1)^{2s-1}}, -i\zeta)^a D_a \hat{\psi}(s, \zeta; w) = i \hat{J}(s, \zeta; w)$

$$\Leftrightarrow (\sigma \langle w \rangle \otimes I_{(w+1)^{2s-1}}, -i\zeta)^a \Gamma(s; w) D_a \psi(s, \zeta; w) = i [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{J}(s, \zeta; w)$$

$$\Leftrightarrow (\sigma \langle w \rangle \otimes I_{(w+1)^{2s-1}}, -i\zeta)^a [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] N(s; w) D_a \psi(s, \zeta; w) = i [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{J}(s, \zeta; w)$$

$$\Leftrightarrow [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)^a N(s; w) D_a \psi(s, \zeta; w) = i [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] \tilde{J}(s, \zeta; w)$$

$$\Leftrightarrow (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)^a D_a \tilde{\psi}(s, \zeta; w) = i \tilde{J}(s, \zeta; w) \quad \square$$

**定理3.7.2.**  $(\sigma \langle w \rangle \otimes I_{(w+1)^{2s-1}}, -i\zeta)^a D_a \hat{\psi}(s, \zeta; w) = i \hat{J}(s, \zeta; w)$

$$\Rightarrow \begin{cases} [s D_a + i S_{ab} (s, \zeta; w) D^b] \psi(s, \zeta; w) = i s \bar{N}(s; w) (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, i\zeta)_a \tilde{J}(s, \zeta; w) \\ [\sigma(s; w), -i s \zeta]_a D^a \psi(s, \zeta; w) = i s \bar{N}(s; w) \tilde{J}(s, \zeta; w) \end{cases}$$

**证明:**  $(\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)_b D^b \tilde{\psi}(s, \zeta; w) = i \tilde{J}(s, \zeta; w)$

$$\Rightarrow \bar{N}(s; w) (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, i\zeta)_a (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)_b N(s; w) D^b \psi(s, \zeta; w) = i \bar{N}(s; w) (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, i\zeta)_a \tilde{J}(s, \zeta; w)$$

$$\Leftrightarrow \bar{N}(s; w) [\delta_{ab} + 2i S_{ab} (\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-1+w}^{2s-1}}] N(s; w) D^b \psi(s, \zeta; w) = i \bar{N}(s; w) (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, i\zeta)_a \tilde{J}(s, \zeta; w)$$

$$\Leftrightarrow [\delta_{ab} + \frac{i}{s} S_{ab} (s, \zeta; w)] D^b \psi(s, \zeta; w) = i \bar{N}(s; w) (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, i\zeta)_a \tilde{J}(s, \zeta; w)$$

$$\Leftrightarrow [s D_a + i S_{ab} (s, \zeta; w) D^b] \psi(s, \zeta; w) = i s \bar{N}(s; w) (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, i\zeta)_a \tilde{J}(s, \zeta; w) \quad \square$$

**证明:**  $(\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)_a D^a \tilde{\psi}(s, \zeta; w) = i \tilde{J}(s, \zeta; w)$

$$\Rightarrow \bar{N}(s; w) (\sigma \langle w \rangle \otimes I_{C_{2s-1+w}^{2s-1}}, -i\zeta)_a N(s; w) D^a \psi(s, \zeta; w) = i \bar{N}(s; w) \tilde{J}(s, \zeta; w)$$

$$\Leftrightarrow [\frac{1}{s} \sigma(s; w), -i\zeta]_a D^a \psi(s, \zeta; w) = i \bar{N}(s; w) \tilde{J}(s, \zeta; w)$$

$$\Leftrightarrow [\sigma(s; w), -i s \zeta]_a D^a \psi(s, \zeta; w) = i s \bar{N}(s; w) \tilde{J}(s, \zeta; w) \quad \square$$

## 4 开关型自旋方程

### 4.1 开关型无源中微子自旋方程

**定理4.1.1.**  $[(\frac{1}{2} + \phi) D_a + i S_{ab}(\zeta) D^b] \psi(\frac{1}{2}, \zeta) = 0$

$$\Leftrightarrow \begin{cases} (\sigma, -i\zeta)^a D_a \psi(\frac{1}{2}, \zeta) = 0, \phi = 0 \\ \sigma_x D_x \psi(\frac{1}{2}, \zeta) = \sigma_y D_y \psi(\frac{1}{2}, \zeta) = \sigma_z D_z \psi(\frac{1}{2}, \zeta) = -i\zeta D_\pi \psi(\frac{1}{2}, \zeta), \phi = -2 \\ \psi(\frac{1}{2}, \zeta) = \text{常数解}, \phi \neq 0, -2 \end{cases}$$

证明:  $[(\frac{1}{2} + \phi)D_a + iS_{ab}(\varsigma)D^b]\psi(\frac{1}{2}, \varsigma) = 0$

$$\begin{aligned} &\Leftrightarrow [\frac{1}{2}D_a + iS_{ab}(\varsigma)D^b]\psi(\frac{1}{2}, \varsigma) = -\phi D_a\psi(\frac{1}{2}, \varsigma) \\ &\Leftrightarrow \sigma_a[\frac{1}{2}D_a + iS_{ab}(\varsigma)D^b]\psi(\frac{1}{2}, \varsigma) = -(\sigma, -i\varsigma)_a\phi D_a\psi(\frac{1}{2}, \varsigma) \\ &\Leftrightarrow (\sigma, -i\varsigma)^b D_b\psi(\frac{1}{2}, \varsigma) = -2\phi(\sigma, -i\varsigma)_a D_a\psi(\frac{1}{2}, \varsigma) \\ &\Leftrightarrow (\sigma, -i\varsigma)^a D_a\psi(\frac{1}{2}, \varsigma) = -2\phi\sigma_x D_x\psi(\frac{1}{2}, \varsigma) = -2\phi\sigma_y D_y\psi(\frac{1}{2}, \varsigma) = -2\phi\sigma_z D_z\psi(\frac{1}{2}, \varsigma) = -2\phi(-i\varsigma)D_\pi\psi(\frac{1}{2}, \varsigma) \\ &\Leftrightarrow \begin{cases} (\sigma, -i\varsigma)^a D_a\psi(\frac{1}{2}, \varsigma) = 0, \phi = 0 \\ \sigma_x D_x\psi(\frac{1}{2}, \varsigma) = \sigma_y D_y\psi(\frac{1}{2}, \varsigma) = \sigma_z D_z\psi(\frac{1}{2}, \varsigma) = -i\varsigma D_\pi\psi(\frac{1}{2}, \varsigma), \phi = -2 \\ D_a\psi(\frac{1}{2}, \varsigma) = 0, \phi \neq 0, -2 \end{cases} \quad \square \end{aligned}$$

推论4.1.1.  $[(\frac{1}{2} + \phi)\partial_a + iS_{ab}(\varsigma)\partial^b]\psi(\frac{1}{2}, \varsigma) = 0$

$$\Leftrightarrow \begin{cases} (\sigma, -i\varsigma)^a \partial_a\psi(\frac{1}{2}, \varsigma) = 0, \phi = 0 \\ \sigma_x \partial_x\psi(\frac{1}{2}, \varsigma) = \sigma_y \partial_y\psi(\frac{1}{2}, \varsigma) = \sigma_z \partial_z\psi(\frac{1}{2}, \varsigma) = -i\varsigma \partial_\pi\psi(\frac{1}{2}, \varsigma), \phi = -2 \\ \psi(\frac{1}{2}, \varsigma) = \text{常数解}, \phi \neq 0, -2 \end{cases}$$

推论4.1.2.  $\sigma_x \partial_x\psi(\frac{1}{2}, \varsigma) = \sigma_y \partial_y\psi(\frac{1}{2}, \varsigma) = \sigma_z \partial_z\psi(\frac{1}{2}, \varsigma) = -i\varsigma \partial_\pi\psi(\frac{1}{2}, \varsigma)$

$$\begin{aligned} &\Rightarrow \psi(\frac{1}{2}, \varsigma) = \omega_0 + (x\sigma_x + y\sigma_y + z\sigma_z + i\varsigma\pi)\pi_0 \Leftrightarrow \psi_{A_\varsigma}(\frac{1}{2}, \varsigma) = \omega_{A_\varsigma} + x_a(\sigma, i\varsigma)_{A_\varsigma A'_\varsigma}^a \pi^{A'_\varsigma} \\ &\Leftrightarrow (\sigma^*, i\varsigma)_{A'_\varsigma(A_\varsigma)}^a \partial_a \omega_{B_\varsigma} = 0 \end{aligned}$$

以上结论正是Penrose扭量<sup>[2,3]</sup>的投射关系。

## 4.2 开关型无源电磁场自旋方程

定理4.2.1.  $[(1 + \phi)D_a + iS_{ab}D^b]^{\beta_\varsigma} \Psi^{\gamma_\varsigma}(1, \varsigma) = 0, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\psi}(1, \varsigma) = 0, \phi = 0 \\ \begin{cases} -D_y \Psi_{z_\varsigma} = D_z \Psi_{y_\varsigma} = \varsigma D_\pi \Psi_{x_\varsigma}, -D_z \Psi_{x_\varsigma} = D_x \Psi_{z_\varsigma} = \varsigma D_\pi \Psi_{y_\varsigma} \\ -D_x \Psi_{y_\varsigma} = D_y \Psi_{x_\varsigma} = \varsigma D_\pi \Psi_{z_\varsigma}, D_x \Psi_{x_\varsigma} = D_y \Psi_{y_\varsigma} = D_z \Psi_{z_\varsigma} \end{cases}, \phi = -3 \\ D_a \Psi_{b_\varsigma} = 0, \phi \neq 0, -3 \end{cases}$$

证明:  $[(1 + \phi)D_a + iS_{ab}D^b]^{\beta_\varsigma} \Psi^{\gamma_\varsigma}(1, \varsigma) = 0, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma}$

$$\begin{aligned} &\Leftrightarrow (D_a + iS_{ab}D^b)^{\beta_\varsigma} \Psi^{\gamma_\varsigma}(1, \varsigma) = -\phi D_a \Psi^{\beta_\varsigma}(1, \varsigma) \\ &\Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\psi}(1, \varsigma) = i\tilde{\mathcal{J}}(1, \varsigma), -\phi D_a \Psi^{\beta_\varsigma}(1, \varsigma) = -i\sigma_{\varsigma ab}^{\beta_\varsigma} \mathcal{J}^b \\ &\Leftrightarrow \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)^a D_a \tilde{\psi}(1, \varsigma) = 0, \phi = 0 \\ \begin{cases} -D_y \Psi_{z_\varsigma} = D_z \Psi_{y_\varsigma} = \varsigma D_\pi \Psi_{x_\varsigma}, -D_z \Psi_{x_\varsigma} = D_x \Psi_{z_\varsigma} = \varsigma D_\pi \Psi_{y_\varsigma} \\ -D_x \Psi_{y_\varsigma} = D_y \Psi_{x_\varsigma} = \varsigma D_\pi \Psi_{z_\varsigma}, D_x \Psi_{x_\varsigma} = D_y \Psi_{y_\varsigma} = D_z \Psi_{z_\varsigma} \end{cases}, \phi = -3 \\ D_a \Psi_{b_\varsigma} = 0, \phi \neq 0, -3 \end{cases} \quad \square \end{aligned}$$

推论4.2.1.  $[(1 + \phi)\partial_a + iS_{ab}\partial^b]^{\beta_\varsigma} \Psi^{\gamma_\varsigma}(1, \varsigma) = 0, S_{ab} = i\sigma_{\varsigma ab}^{\alpha_\varsigma} \gamma_{\alpha_\varsigma}$

$$\Leftrightarrow \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)^a \partial_a \tilde{\psi}(1, \varsigma) = 0, \phi = 0 \\ \begin{cases} -\partial_y \Psi_{z_\varsigma} = \partial_z \Psi_{y_\varsigma} = \varsigma \partial_\pi \Psi_{x_\varsigma}, -\partial_z \Psi_{x_\varsigma} = \partial_x \Psi_{z_\varsigma} = \varsigma \partial_\pi \Psi_{y_\varsigma} \\ -\partial_x \Psi_{y_\varsigma} = \partial_y \Psi_{x_\varsigma} = \varsigma \partial_\pi \Psi_{z_\varsigma}, \partial_x \Psi_{x_\varsigma} = \partial_y \Psi_{y_\varsigma} = \partial_z \Psi_{z_\varsigma} \end{cases}, \phi = -3 \\ \Psi_{\alpha_\varsigma} = \text{常数解}, \phi \neq 0, -3 \end{cases}$$

推论4.2.2.  $\begin{cases} -\partial_y \Psi_{z_\varsigma} = \partial_z \Psi_{y_\varsigma} = \varsigma \partial_\pi \Psi_{x_\varsigma}, -\partial_z \Psi_{x_\varsigma} = \partial_x \Psi_{z_\varsigma} = \varsigma \partial_\pi \Psi_{y_\varsigma} \\ -\partial_x \Psi_{y_\varsigma} = \partial_y \Psi_{x_\varsigma} = \varsigma \partial_\pi \Psi_{z_\varsigma}, \partial_x \Psi_{x_\varsigma} = \partial_y \Psi_{y_\varsigma} = \partial_z \Psi_{z_\varsigma} \end{cases} \Rightarrow \Psi^{\alpha_\varsigma}(1, \varsigma) = x^a \sigma_{\varsigma ab}^{\alpha_\varsigma} C^b$



### 4.3 任意N+1维时空中的矢量场自旋方程及其无源开关型自旋方程

#### 任意N+1维时空中的矢量场自旋方程

定理4.3.1.  $(D_a\delta_{cd} + S_{abcd}D^b)A^d = X_{ac} \Leftrightarrow X_{ab} = D_aA_b - D_bA_a + \delta_{ab}D_cA^c$

证明:  $(D_a\delta_{cd} + S_{abcd}D^b)A^d = X_{ac}$   
 $\Leftrightarrow [D_a\delta_{cd} + (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})D^b]A^d = X_{ac}$   
 $\Leftrightarrow D_aA_c + \delta_{ac}D_bA^b - D_cA_a = X_{ac}$   
 $\Leftrightarrow D_aA_b - D_bA_a + \delta_{ab}D_cA^c = X_{ab}$   
 $\Leftrightarrow X_{ab} = D_aA_b - D_bA_a + \delta_{ab}D_cA^c$

□

推论4.3.1.  $(D_a\delta_{cd} + S_{abcd}D^b)A^d = 0 \Leftrightarrow D_aA_b - D_bA_a = 0, D_aA^a = 0$

推论4.3.2.  $(\partial_a\delta_{cd} + S_{abcd}\partial^b)A^d = 0 \Leftrightarrow \partial_aA_b - \partial_bA_a = 0, \partial_aA^a = 0 \Leftrightarrow \partial^a\partial_a\phi = 0, A_a = \partial_a\phi$

#### 任意N+1维时空中的无源矢量场开关型自旋方程

推论4.3.3.  $[(1 + \phi)D_a\delta_{cd} + S_{abcd}\partial^b]A^d = 0 \Leftrightarrow \begin{cases} D_aA_b - D_bA_a = 0, D_aA^a = 0, \phi = 0 \\ D_aA_b + D_bA_a = 0, \phi = -2 \\ D_aA_{b \neq a} = 0, D_xA_x = D_yA_y = D_zA_z = D_\pi A_\pi, \phi = -4 \\ D_aA_b = 0, \phi \neq 0, -2, -4 \end{cases}$

证明:  $[(1 + \phi)D_a\delta_{cd} + S_{abcd}\partial^b]A^d = 0$   
 $\Leftrightarrow (D_a\delta_{cd} + S_{abcd}D^b)A^d = -\phi D_aA_c$   
 $\Leftrightarrow -\phi D_aA_b = D_aA_b - D_bA_a + \delta_{ab}D_cA^c$   
 $\Leftrightarrow -\phi D_aA_a = D_cA^c, -\phi(D_aA_{b \neq a} + D_bA_{a \neq b}) = 0, (2 + \phi)(D_aA_b - D_bA_a) = 0$   
 $\Leftrightarrow \begin{cases} -\phi D_aA_a = D_cA^c, (4 + \phi)D_aA^a = 0 \\ -\phi(D_aA_{b \neq a} + D_bA_{a \neq b}) = 0, (2 + \phi)(D_aA_b - D_bA_a) = 0 \end{cases}$   
 $\Leftrightarrow \begin{cases} D_aA_b - D_bA_a = 0, D_aA^a = 0, \phi = 0 \\ D_aA_b + D_bA_a = 0, \phi = -2 \\ D_aA_{b \neq a} = 0, D_xA_x = D_yA_y = D_zA_z = D_\pi A_\pi, \phi = -4 \\ D_aA_b = 0, \phi \neq 0, -2, -4 \end{cases}$

□

推论4.3.4.  $[(1 + \phi)\partial_a\delta_{cd} + S_{abcd}\partial^b]A^d = 0 \Leftrightarrow \begin{cases} \partial_aA_b - \partial_bA_a = 0, \partial_aA^a = 0, \phi = 0 \\ \partial_aA_b + \partial_bA_a = 0, \phi = -2 \\ \partial_aA_{b \neq a} = 0, \partial_xA_x = \partial_yA_y = \partial_zA_z = \partial_\pi A_\pi, \phi = -4 \\ A_a = \text{常数解}, \phi \neq 0, -2, -4 \end{cases}$

推论4.3.5.  $\partial_aA_{b \neq a} = 0, \partial_xA_x = \partial_yA_y = \partial_zA_z = \partial_\pi A_\pi \Rightarrow A_a = kx_a$

### 4.4 任意N+1维时空中标量场的来源

#### 标量场的来源:

推论4.4.1.  $(\partial_a\delta_{cd} + S_{abcd}\partial^b)\partial^d\phi = m^2\phi\delta_{ac} \Leftrightarrow (\partial^a\partial_a - m^2)\phi = 0$

证明:  $(\partial_a\delta_{cd} + S_{abcd}\partial^b)\partial^d\phi = m^2\phi\delta_{ac}$   
 $\Leftrightarrow [\partial_a\delta_{cd} + (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\partial^b]\partial^d\phi = m^2\phi\delta_{ac}$   
 $\Leftrightarrow (\partial^b\partial_b - m^2)\delta_{ac}\phi = 0$   
 $\Leftrightarrow (\partial^a\partial_a - m^2)\phi = 0$

□

## 4.5 任意 $N+1$ 维时空中的无源开关型电子自旋方程

任意 $n=N+1$ 维时空中的无源开关型电子自旋方程:

**定理4.5.1.**  $[(\frac{1}{2} + \phi)(D_a + m\gamma_a) + iS_{ab}D^b]\psi = 0, S_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b] \Leftrightarrow (\gamma^a D_a + m)\psi = -2\phi\gamma_b D_b \psi$

**证明:**  $[(\frac{1}{2} + \phi)(D_a + m\gamma_a) + iS_{ab}D^b]\psi = 0, S_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b]$   
 $\Leftrightarrow [\frac{1}{2}(D_a + m\gamma_a) + iS_{ab}D^b]\psi = -\phi D_a \psi, S_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b]$   
 $\Leftrightarrow [(2iS_{ab} + \delta_{ab})D_b + \gamma_a m]\psi = -2\phi D_a \psi, S_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b]$   
 $\Leftrightarrow [\frac{1}{2}([\gamma_a, \gamma_b] + \{\gamma_a, \gamma_b\})D_b + \gamma_a m]\psi = -2\phi D_a \psi$   
 $\Leftrightarrow \gamma_a(\gamma_b D^b + m)\psi = -2\phi D_a \psi$   
 $\Leftrightarrow (\gamma_b D^b + m)\psi = -2\phi\gamma_a D_a \psi$   
 $\Leftrightarrow (\gamma^a D_a + m)\psi = -2\phi\gamma_b D_b \psi$  □

**推论4.5.1.**  $(\gamma^a D_a + m)\psi = -2\phi\gamma_b D_b \psi, \phi \neq 0$

$$\Leftrightarrow \begin{cases} \psi = 0, \phi = -\frac{n}{2}, m \neq 0 \\ \gamma_1 D_{x_1} \psi = \gamma_2 D_{x_2} \psi = \cdots = \gamma_n D_{x_n} \psi = -(n + 2\phi)^{-1} m \psi, \phi \neq -\frac{n}{2}, m \neq 0 \\ \gamma_1 D_{x_1} \psi = \gamma_2 D_{x_2} \psi = \cdots = \gamma_n D_{x_n} \psi, \phi = -\frac{n}{2}, m = 0 \\ \gamma_1 D_{x_1} \psi = \gamma_2 D_{x_2} \psi = \cdots = \gamma_n D_{x_n} \psi = 0, \phi \neq -\frac{n}{2}, m = 0 \end{cases}$$

$$\text{推论4.5.2. } (\gamma^a D_a + m)\psi = -2\phi\gamma_b D_b \psi, \phi \neq 0 \Rightarrow \begin{cases} \psi = 0, \phi = -\frac{n}{2}, m \neq 0 \\ \psi = 0, \phi \neq -\frac{n}{2}, m \neq 0 \\ \psi = x^a \gamma_a \lambda, \phi = -\frac{n}{2}, m = 0 \\ \psi = \text{常数解}, \phi \neq -\frac{n}{2}, m = 0 \end{cases}$$

## 4.6 s-自旋粒子的无源开关型自旋方程

**推论4.6.1.**  $[(s + \phi)D_a + iS_{ab}(s, \varsigma)D^b]\psi(s, \varsigma) = \mathbb{J}_a(s, \varsigma) \Rightarrow (2s + 1 + \phi)Z_a(s, \varsigma)D^a \psi(s, \varsigma) = Z_a(s, \varsigma)\mathbb{J}^a(s, \varsigma)$

**证明:**  $[(s + \phi)D_a + iS_{ab}(s, \varsigma)D^b]\psi(s, \varsigma) = \mathbb{J}_a(s, \varsigma)$   
 $\Leftrightarrow [sD_a + iS_{ab}(s, \varsigma)D^b]\psi(s, \varsigma) = \mathbb{J}_a(s, \varsigma) - \phi D_a \psi(s, \varsigma)$   
 $\Leftrightarrow 2s\bar{Z}_a(s, \varsigma)Z_b(s, \varsigma)D^b \psi(s, \varsigma) = \mathbb{J}_a(s, \varsigma) - \phi D_a \psi(s, \varsigma)$   
 $\Rightarrow Z_b(s, \varsigma)D^b \psi(s, \varsigma) = Z^a(s, \varsigma)\mathbb{J}_a(s, \varsigma) - \frac{\phi}{2s+1}Z^a(s, \varsigma)D_a \psi(s, \varsigma)$   
 $\Rightarrow (2s + 1 + \phi)Z_a(s, \varsigma)D^a \psi(s, \varsigma) = Z_a(s, \varsigma)\mathbb{J}^a(s, \varsigma)$  □

**推论4.6.2.**  $[(s + \phi)D_a + iS_{ab}(s, \varsigma)D^b]\psi(s, \varsigma) = 0 \Rightarrow (2s + 1 + \phi)\bar{Z}_a(s, \varsigma)D^a \psi(s, \varsigma) = 0$

**证明:**  $[(s + \phi)D_a + iS_{ab}(s, \varsigma)D^b]\psi(s, \varsigma) = 0$   
 $\Leftrightarrow [sD_a + iS_{ab}(s, \varsigma)D^b]\psi(s, \varsigma) = -\phi D_a \psi(s, \varsigma)$   
 $\Leftrightarrow 2s\bar{Z}_a(s, \varsigma)Z_b(s, \varsigma)D^b \psi(s, \varsigma) = -\phi D_a \psi(s, \varsigma)$   
 $\Rightarrow Z_b(s, \varsigma)D^b \psi(s, \varsigma) = \frac{-\phi}{2s+1}Z^a(s, \varsigma)D_a \psi(s, \varsigma)$   
 $\Rightarrow (2s + 1 + \phi)Z_a(s, \varsigma)D^a \psi(s, \varsigma) = 0$  □

$$\text{推论4.6.3. } [(s + \phi)D_a + iS_{ab}(s, \varsigma)D^b]\psi(s, \varsigma) = 0 \Leftrightarrow \begin{cases} (\sigma \otimes I_{2s}, -i\varsigma)^a D_a \tilde{\psi}(s, \varsigma) = 0, \phi = 0 \\ D_a \psi(s, \varsigma) = \bar{Z}_a(s, \varsigma)\tilde{J}(s, \varsigma), \phi = -(2s + 1) \\ D_a \psi(s, \varsigma) = 0, \phi \neq 0, -(2s + 1) \end{cases}$$

**推论4.6.4.**  $[-(s + 1)D_a + iS_{ab}(s, \varsigma)D^b]\psi(s, \varsigma) = 0 \Leftrightarrow D_a \psi(s, \varsigma) = \bar{Z}_a(s, \varsigma)\tilde{J}(s, \varsigma), \forall \tilde{J}(s, \varsigma)$

**推论4.6.5.**  $[-(s + 1)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(s, \varsigma) = 0 \Leftrightarrow \psi(s, \varsigma) = x^a \bar{Z}_a(s, \varsigma)\tilde{J}(s, \varsigma)$

推论4.6.6.  $[(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(s, \varsigma) = 0$

$$\Rightarrow \begin{cases} \text{当 } \phi = 0 \text{ 时, } (\sigma \otimes I_{2s}, -i\varsigma)^a \partial_a \tilde{\psi}(s, \varsigma) = 0, \text{ 有平面波解, 表征粒子的解} \\ \text{当 } \phi = -(2s + 1) \text{ 时, } \psi(s, \varsigma) = x^a \bar{Z}_a(s, \varsigma) \tilde{J}(s, \varsigma), \text{ 无平面波解, 退化为表征时空的解} \\ \text{当 } \phi \neq 0, -(2s + 1) \text{ 时, } \psi(s, \varsigma) = \text{常数解, 只有常数解, 退化为表征虚空的解} \end{cases}$$

#### 4.7 s-自旋粒子开关型自旋方程的各种等价形式

推论4.7.1.  $[(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x; s) = \mathbb{J}_a \Leftrightarrow [(s + \phi)\partial_a + i\Omega_{ab}(s)\partial^b]\Gamma(s)\psi(x; s) = \Gamma(s)\mathbb{J}_a$   
 $\Leftrightarrow \{(s + \phi)\partial_a + i[S_{ab}(\frac{1}{2}, \varsigma) \otimes I_{2s} + I \otimes S_{ab}(s - \frac{1}{2}, \varsigma)]\partial^b\}N(s)\psi(x; s) = N(s)\mathbb{J}_a$

证明:  $[(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x; s) = \mathbb{J}_a$

$$\Leftrightarrow \Gamma(s)[(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x; s) = \Gamma(s)\mathbb{J}_a$$

$$\Leftrightarrow [(s + \phi)\partial_a + i\Omega_{ab}(s)\partial^b]\Gamma(s)\psi(x; s) = \Gamma(s)\mathbb{J}_a \quad \square$$

证明:  $[(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x; s) = \mathbb{J}_a$

$$\Leftrightarrow N(s)[(s + \phi)\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x; s) = N(s)\mathbb{J}_a$$

$$\Leftrightarrow \{(s + \phi)\partial_a + i[S_{ab}(\frac{1}{2}, \varsigma) \otimes I_{2s} + I \otimes S_{ab}(s - \frac{1}{2}, \varsigma)]\partial^b\}N(s)\psi(x; s) = N(s)\mathbb{J}_a \quad \square$$

## 5 低一阶导数的自旋方程新形式

### 5.1 低一阶导数的s自旋方程

定义5.1.1. 低一阶导数的自旋方程:  $[s\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = -\sqrt{2}\varsigma s \bar{Z}_a(s, \varsigma) \tilde{J}^c(s, \varsigma)$

定理5.1.1.  $[s\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = -\sqrt{2}\varsigma s \bar{Z}_a(s, \varsigma) \tilde{J}^c(s, \varsigma) \Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_a \tilde{\psi}^{ab}(s, \varsigma) = i\tilde{J}^b(s, \varsigma)$

证明:  $[s\delta_{ab}I_{2s+1} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = -\sqrt{2}\varsigma s \bar{Z}_a(s, \varsigma) \tilde{J}^c(s, \varsigma)$

$$\Leftrightarrow 2s \bar{Z}_a(s, \varsigma) Z_b(s, \varsigma) \psi^{bc}(s, \varsigma) = -\sqrt{2}\varsigma s \bar{Z}_a(s, \varsigma) \tilde{J}^c(s, \varsigma)$$

$$\Leftrightarrow Z_b(s, \varsigma) \psi^{bc}(s, \varsigma) = \frac{-\varsigma}{\sqrt{2}} \tilde{J}^c(s, \varsigma)$$

$$\Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_a \tilde{\psi}^{ab}(s, \varsigma) = i\tilde{J}^b(s, \varsigma) \quad \square$$

### 5.2 低一阶导数的s = 1/2自旋方程: 引力微子方程

自旋s = 1/2情形, 即Weyl型引力微子方程的矩阵形式:

推论5.2.1.  $\varepsilon_{abcd}(\sigma, -i\varsigma)^d D^b \psi^c(\varsigma) = 0 \Leftrightarrow (\sigma, -i\varsigma)_a \psi^{ab}(\varsigma) = 0, \psi^{ab}(\varsigma) \equiv D^a \psi^b(\varsigma) - D^b \psi^a(\varsigma)$

推论5.2.2.  $\varepsilon_{abcd}(\sigma, -i\varsigma)^d D^b \psi^c(\varsigma) = 0 \Leftrightarrow [\frac{1}{2}\delta_{ab} + iS_{ab}(\varsigma)]\psi^{bc}(\varsigma) = 0, \psi^{bc}(\varsigma) \equiv D^b \psi^c(\varsigma) - D^c \psi^b(\varsigma)$

### 5.3 低一阶导数的s = 1自旋方程: Einstein方程

自旋s = 1情形, 即Einstein方程的矩阵形式:

推论5.3.1.  $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow (\sigma_{-\varsigma}, -i\varsigma)_a \mathcal{F}^{ab}(\varsigma) = i\bar{T}^b$

推论5.3.2.  $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow (\sigma \otimes I, -i\varsigma)_a \tilde{\psi}^{ab}(1, \varsigma) = i\tilde{J}^b$

$$\tilde{\psi}^{bc}(1, \varsigma) = S_{em}^+(\varsigma) \mathcal{F}^{bc}(\varsigma), \tilde{J}^c = S_{em}^+(\varsigma) \bar{T}^c$$

推论5.3.3.  $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow [\delta_{ab} + iS_{ab}(1, \varsigma)]\psi^{bc}(1, \varsigma) = -\sqrt{2}\varsigma \bar{Z}_a(1, \varsigma) \tilde{J}^c(1, \varsigma)$

$$\psi^{bc}(1, \varsigma) = \bar{N}(1) S_{em}^+(\varsigma) \mathcal{F}^{bc}(\varsigma), \tilde{J}^c = S_{em}^+(\varsigma) \bar{T}^c$$

从上面推论通过表象变换可以直接得到如下推论, 但也可以按如下方式证明。

推论5.3.4.  $R^{ab} = -8\pi G(T^{ab} - \frac{1}{2}g^{ab}T) + \Lambda g^{ab} \Leftrightarrow [\delta_{ab} + iS_{ab}(\gamma, \varsigma)]F^{bc}(2, \varsigma) = -i\sigma_{\varsigma ab}^{[\beta\varsigma]} \bar{T}^{bc}$

### 5.4 低一阶导数的 $s$ -自旋方程的各种等价形式

推论5.4.1.  $[s\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(x; s) = \mathbb{J}_a^c \Leftrightarrow [s\delta_{ab} + iS_{ab}(s, \varsigma)]\Gamma(s)\psi^{bc}(x; s) = \Gamma(s)\mathbb{J}_a^c$   
 $\Leftrightarrow \{s\delta_{ab} + i[S_{ab}(\frac{1}{2}, \varsigma) \otimes I_{2s} + I \otimes S_{ab}(s - \frac{1}{2}, \varsigma)]\}N(s)\psi^{bc}(x; s) = N(s)\mathbb{J}_a^c$

### 5.5 低一阶导数的开关型自旋方程新形式

定义5.5.1. 低一阶导数的开关型自旋方程:  $[(s + \phi)\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = \mathbb{J}_a^c(s, \varsigma)$

推论5.5.1.  $[(s + \phi)\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = 0 \Rightarrow (2s + 1 + \phi)Z_a(s, \varsigma)\psi^{ab}(s, \varsigma) = 0$

证明:  $[(s + \phi)\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = 0$   
 $\Leftrightarrow [s\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = -\phi\psi_a^c(s, \varsigma)$   
 $\Leftrightarrow 2s\bar{Z}_a(s, \varsigma)Z_b(s, \varsigma)\psi^{bc}(s, \varsigma) = -\phi\psi_a^c(s, \varsigma)$   
 $\Rightarrow Z_b(s, \varsigma)\psi^{bc}(s, \varsigma) = \frac{-\phi}{2s+1}Z^a(s, \varsigma)\psi_a^c(s, \varsigma)$   
 $\Rightarrow (2s + 1 + \phi)Z_a(s, \varsigma)\psi^{ab}(s, \varsigma) = 0$

□

推论5.5.2.  $[(s + \phi)\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = 0 \Leftrightarrow \begin{cases} Z_a(s, \varsigma)\psi^{ab}(s, \varsigma) = 0, \phi = 0 \\ \psi^{ab}(s, \varsigma) = \bar{Z}^a(s, \varsigma)\tilde{J}^b(s, \varsigma), \phi = -(2s + 1) \\ \psi^{ab}(s, \varsigma) = 0, \phi \neq 0, -(2s + 1) \end{cases}$

推论5.5.3.  $\begin{cases} [-(s + 1)\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = 0 \\ \psi^{ab}(s, \varsigma) + \psi^{ba}(s, \varsigma) = 0 \end{cases} \Leftrightarrow \begin{cases} \psi^{ab}(s, \varsigma) = \bar{Z}^a(s, \varsigma)\tilde{J}^b(s, \varsigma) \\ \bar{Z}^a(s, \varsigma)\tilde{J}^b(s, \varsigma) + \bar{Z}^b(s, \varsigma)\tilde{J}^a(s, \varsigma) = 0 \end{cases}$

### 5.6 低一阶导数的 $s$ -自旋开关型方程的各种等价形式

推论5.6.1.  $[(s + \phi)\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(x; s) = \mathbb{J}_a^c \Leftrightarrow [(s + \phi)\delta_{ab} + iS_{ab}(s, \varsigma)]\Gamma(s)\psi^{bc}(x; s) = \Gamma(s)\mathbb{J}_a^c$   
 $\Leftrightarrow \{(s + \phi)\delta_{ab} + i[S_{ab}(\frac{1}{2}, \varsigma) \otimes I_{2s} + I \otimes S_{ab}(s - \frac{1}{2}, \varsigma)]\}N(s)\psi^{bc}(x; s) = N(s)\mathbb{J}_a^c$

### 5.7 两种自旋方程的对比

低一阶导数的自旋方程:

$$[s\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = -\sqrt{2}\varsigma\bar{Z}_a(s, \varsigma)\tilde{J}^c(s, \varsigma) \Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_a\tilde{\psi}^{ab}(s, \varsigma) = i\tilde{J}^b(s, \varsigma)$$

自旋方程:

$$[s\delta_{ab} + iS_{ab}(s, \varsigma)]D^b\psi(s, \varsigma) = -\sqrt{2}\varsigma\bar{Z}_a(s, \varsigma)\tilde{J}(s, \varsigma) \Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_aD^a\tilde{\psi}(s, \varsigma) = i\tilde{J}(s, \varsigma)$$

### 5.8 两种开关型自旋方程的对比

低一阶导数的开关型自旋方程:

$$[(s + \phi)\delta_{ab} + iS_{ab}(s, \varsigma)]\psi^{bc}(s, \varsigma) = -\sqrt{2}\varsigma\bar{Z}_a(s, \varsigma)\tilde{J}^c(s, \varsigma)$$

开关型自旋方程:

$$[(s + \phi)\delta_{ab} + iS_{ab}(s, \varsigma)]D^b\psi(s, \varsigma) = -\sqrt{2}\varsigma\bar{Z}_a(s, \varsigma)\tilde{J}(s, \varsigma)$$

### 5.9 猜测: 一个新的物理方程

推论5.9.1.  $Z_a(s, \varsigma)D^a\psi(s, \varsigma) - m^2\tilde{A}(s, \varsigma) = \tilde{J}(s, \varsigma), \psi(s, \varsigma) = \bar{Z}_a(s, \varsigma)D^a\tilde{A}(s, \varsigma)$

引入规范条件后的方程:

推论5.9.2.  $Z_a(s, \varsigma)D^a\psi(s, \varsigma) - m^2\tilde{A}(s, \varsigma) = \tilde{J}(s, \varsigma), N(s)\psi(s, \varsigma) = (\sigma \otimes I_{2s}, i\varsigma)_aD^a\tilde{A}(s, \varsigma)$

# 第十一章 Penrose方程和扭量方程

自我评述：本章对Penrose方程进行了全面深入的分析，指出Penrose旋量方程等价于自旋方程，适合描写无质量粒子，并发现Penrose扭量方程就是开关型自旋方程的一个特殊解。

## 1 任意自旋全对称Penrose方程 [1,2]的重新表述

### 1.1 任意自旋全对称Penrose方程的整旋量等价形式

定理1.1.1.

$$\left\{ \begin{array}{l} (\sigma, -i\zeta)_a A'_\zeta A_\zeta D^a \psi_{A_\zeta B_\zeta C_\zeta \dots Z_\zeta} = i J_{A'_\zeta B_\zeta C_\zeta \dots Z_\zeta} \\ \psi_{A_\zeta B_\zeta C_\zeta \dots Z_\zeta} = \frac{1}{(2n)!} \psi_{(A_\zeta B_\zeta C_\zeta \dots)} Z_\zeta \\ J_{A'_\zeta B_\zeta C_\zeta \dots Z_\zeta} = \frac{1}{(2n-1)!} J_{A'_\zeta B_\zeta C_\zeta \dots} Z_\zeta \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \sigma_{\zeta ab} D^a \psi_{\alpha_\zeta \beta_\zeta \dots Z_\zeta} = i J_{b\beta_\zeta \gamma_\zeta \dots Z_\zeta} \\ \psi_{\alpha_\zeta \beta_\zeta \dots Z_\zeta} = \frac{1}{n!} \psi_{(\alpha_\zeta \beta_\zeta \dots)} Z_\zeta \\ J_{b\beta_\zeta \gamma_\zeta \dots Z_\zeta} = \frac{1}{(n-1)!} J_{b(\beta_\zeta \gamma_\zeta \dots)} Z_\zeta \\ \delta^{\alpha_\zeta \beta_\zeta} \psi_{\alpha_\zeta \beta_\zeta \dots Z_\zeta} = 0, (\sigma, -i\zeta)^a \sigma^{\alpha_\zeta} J_{a\alpha_\zeta \beta_\zeta \dots Z_\zeta} = 0 \\ \sigma^{\alpha_\zeta} \psi_{\alpha_\zeta \beta_\zeta \dots [Z_\zeta]} = 0, (\sigma, -i\zeta)^a J_{a\alpha_\zeta \beta_\zeta \dots [Z_\zeta]} = 0 \end{array} \right.$$

定理1.1.2.

$$\left\{ \begin{array}{l} (\sigma, -i\zeta)_a A'_\zeta A_\zeta D^a \psi_{A_\zeta B_\zeta C_\zeta \dots Z_\zeta} = i J_{A'_\zeta B_\zeta C_\zeta \dots Z_\zeta} \\ \psi_{A_\zeta B_\zeta C_\zeta \dots Z_\zeta} = \frac{1}{(2n)!} \psi_{(A_\zeta B_\zeta C_\zeta \dots)} Z_\zeta \\ J_{A'_\zeta B_\zeta C_\zeta \dots Z_\zeta} = \frac{1}{(2n-1)!} J_{A'_\zeta B_\zeta C_\zeta \dots} Z_\zeta \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (\sigma_{-\zeta}, -i\zeta)^a{}_{b\alpha_\zeta} D_a \Psi_{\alpha_\zeta \beta_\zeta \dots Z_\zeta} = i J_{b\beta_\zeta \gamma_\zeta \dots Z_\zeta} \\ \Psi_{\alpha_\zeta \beta_\zeta \dots Z_\zeta} = \frac{1}{n!} \Psi_{(\alpha_\zeta \beta_\zeta \dots)} Z_\zeta \\ J_{b\beta_\zeta \gamma_\zeta \dots Z_\zeta} = \frac{1}{(n-1)!} J_{b(\beta_\zeta \gamma_\zeta \dots)} Z_\zeta \\ \delta^{\alpha_\zeta \beta_\zeta} \Psi_{\alpha_\zeta \beta_\zeta \dots Z_\zeta} = 0, (\sigma, -i\zeta)^a (\sigma, i\zeta)^{\alpha_\zeta} J_{a\alpha_\zeta \beta_\zeta \dots Z_\zeta} = 0 \\ (\sigma, -i\zeta)^{\alpha_\zeta} \Psi_{\alpha_\zeta \beta_\zeta \dots [Z_\zeta]} = 0, (\sigma, -i\zeta)^a J_{a\alpha_\zeta \beta_\zeta \dots [Z_\zeta]} = 0 \end{array} \right.$$

定理1.1.3.

$$\left\{ \begin{array}{l} (\sigma, -i\zeta)_a A'_\zeta A_\zeta D^a \psi_{A_\zeta B_\zeta C_\zeta \dots} = i J_{A'_\zeta B_\zeta C_\zeta \dots} \\ \psi_{A_\zeta B_\zeta C_\zeta \dots} = \frac{1}{(2n)!} \psi_{(A_\zeta B_\zeta C_\zeta \dots)} \\ J_{A'_\zeta B_\zeta C_\zeta \dots} = \frac{1}{(2n-1)!} J_{A'_\zeta B_\zeta C_\zeta \dots} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \sigma_{\zeta ab} D^a \psi_{\alpha_\zeta \beta_\zeta \dots} = i J_{b\beta_\zeta \gamma_\zeta \dots} \\ \psi_{\alpha_\zeta \beta_\zeta \dots} = \frac{1}{n!} \psi_{(\alpha_\zeta \beta_\zeta \dots)}, J_{b\beta_\zeta \gamma_\zeta \dots} = \frac{1}{(n-1)!} J_{b(\beta_\zeta \gamma_\zeta \dots)} \\ \delta^{\alpha_\zeta \beta_\zeta} \psi_{\alpha_\zeta \beta_\zeta \dots} = 0, (\sigma, -i\zeta)^a \sigma^{\alpha_\zeta} J_{a\alpha_\zeta \beta_\zeta \dots} = 0 \end{array} \right.$$

定理1.1.4.

$$\left\{ \begin{array}{l} (\sigma, -i\zeta)_a A'_\zeta A_\zeta D^a \psi_{A_\zeta B_\zeta C_\zeta \dots} = i J_{A'_\zeta B_\zeta C_\zeta \dots} \\ \psi_{A_\zeta B_\zeta C_\zeta \dots} = \frac{1}{(2n)!} \psi_{(A_\zeta B_\zeta C_\zeta \dots)} \\ J_{A'_\zeta B_\zeta C_\zeta \dots} = \frac{1}{(2n-1)!} J_{A'_\zeta B_\zeta C_\zeta \dots} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (\sigma_{-\zeta}, -i\zeta)^a{}_{b\alpha_\zeta} D_a \Psi_{\alpha_\zeta \beta_\zeta \dots} = i J_{b\beta_\zeta \gamma_\zeta \dots} \\ \Psi_{\alpha_\zeta \beta_\zeta \dots} = \frac{1}{n!} \Psi_{(\alpha_\zeta \beta_\zeta \dots)}, J_{b\beta_\zeta \gamma_\zeta \dots} = \frac{1}{(n-1)!} J_{b(\beta_\zeta \gamma_\zeta \dots)} \\ \delta^{\alpha_\zeta \beta_\zeta} \Psi_{\alpha_\zeta \beta_\zeta \dots} = 0, (\sigma, -i\zeta)^a (\sigma, i\zeta)^{\alpha_\zeta} J_{a\alpha_\zeta \beta_\zeta \dots} = 0 \end{array} \right.$$

## 1.2 任意自旋全对称Penrose方程的矩阵等价形式

定理1.2.1.

$$\begin{cases} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} D^a \psi_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2n}} = i J_{\underbrace{B_\zeta C_\zeta \dots}_{2n-1}}^{A'_\zeta} \\ \psi_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2n}} = \frac{1}{(2n)!} \psi_{\underbrace{(A_\zeta B_\zeta C_\zeta \dots)_{2n}}_{2n}}, J_{\underbrace{B_\zeta C_\zeta \dots}_{2n-1}}^{A'_\zeta} = \frac{1}{(2n-1)!} J_{\underbrace{B_\zeta C_\zeta \dots}_{2n-1}}^{A'_\zeta} \end{cases} \Leftrightarrow (\sigma \otimes I_{2^{2s-1}}, -i\zeta)^a D_a \hat{\psi}(s, \zeta) = i \hat{J}(s, \zeta)$$

将分量改写成矩阵就可以得到以上定理。

$$\text{定理1.2.2. } (\sigma \otimes I_{2^{2s-1}}, -i\zeta)^a D_a \hat{\psi}(s, \zeta) = i \hat{J}(s, \zeta) \Leftrightarrow (\sigma \otimes I_{2^s}, -i\zeta)^a D_a \tilde{\psi}(s, \zeta) = i \tilde{J}(s, \zeta)$$

展开去掉冗余方程并整理就可以得到以上定理。

推论1.2.1.

$$\begin{cases} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} D^a \psi_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2n}} = i J_{\underbrace{B_\zeta C_\zeta \dots}_{2n-1}}^{A'_\zeta} \\ \psi_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2n}} = \frac{1}{(2n)!} \psi_{\underbrace{(A_\zeta B_\zeta C_\zeta \dots)_{2n}}_{2n}}, J_{\underbrace{B_\zeta C_\zeta \dots}_{2n-1}}^{A'_\zeta} = \frac{1}{(2n-1)!} J_{\underbrace{B_\zeta C_\zeta \dots}_{2n-1}}^{A'_\zeta} \end{cases} \Leftrightarrow (\sigma \otimes I_{2^s}, -i\zeta)^a D_a \tilde{\psi}(s, \zeta) = i \tilde{J}(s, \zeta)$$

$$\text{定理1.2.3. } (\sigma \otimes I_{2^{2n-1}}, -i\zeta)^a D_a \hat{\psi}(n, \zeta) = i \hat{J}(n, \zeta) \Leftrightarrow (\sigma_{-\zeta} \otimes I_{4^{n-1}}, -i\zeta)^a D_a \hat{\Psi}(n, \zeta) = i \hat{J}(n, \zeta)$$

作一个表象变换就可以得到以上定理。

$$\text{定理1.2.4. } (\sigma \otimes I_{2n}, -i\zeta)^a D_a \tilde{\psi}(n, \zeta) = i \tilde{J}(n, \zeta) \Leftrightarrow (\sigma_{-\zeta} \otimes I_n, -i\zeta)^a D_a \tilde{\Psi}(n, \zeta) = i \tilde{J}(n, \zeta)$$

对于  $n = 1, 2$  作一个表象变换就可以得到以上定理, 对于  $n > 2$  证明以后补上。

## 1.3 任意自旋全对称Penrose方程的自旋方程形式

$$\text{定理1.3.1. } \begin{cases} (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} D^a \psi_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}} = i J_{\underbrace{B_\zeta C_\zeta \dots}_{2s-1}}^{A'_\zeta} \\ \psi_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}} = \frac{1}{(2s)!} \psi_{\underbrace{(A_\zeta B_\zeta C_\zeta \dots)_{2s}}_{2s}} \\ J_{\underbrace{B_\zeta C_\zeta \dots}_{2s-1}}^{A'_\zeta} = \frac{1}{(2s-1)!} J_{\underbrace{(B_\zeta C_\zeta \dots)_{2s-1}}_{2s-1}}^{A'_\zeta} \end{cases} \Leftrightarrow [s D_a + i S_{ab}(s, \zeta) D^b] \psi(s, \zeta) = -\sqrt{2} \zeta_s \bar{Z}_a(s, \zeta) \tilde{J}(s)$$

## 2 扭量方程的重新表述

### 2.1 Penrose扭量方程 [2, 3]

$$\nabla_{A'_\zeta(A_\zeta \omega_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}})}(s) = 0, \nabla_{A'_\zeta(A_\zeta \omega_{B_\zeta})}(\frac{1}{2}) = 0 \quad (11.1)$$

$$\text{推论2.1.1. } \nabla_{A'_\zeta(A_\zeta \omega_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}})}(s) = 0 \Leftrightarrow (\sigma^*, i\zeta)_{A'_\zeta(A_\zeta}^a \partial_a \omega_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}})(s) = 0$$

$$\text{推论2.1.2. } \nabla_{A'_\zeta(A_\zeta \omega_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}})}(s) = 0 \Leftrightarrow (\sigma^*, i\zeta)_{A'_\zeta(A_\zeta}^a \Gamma_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}}^{k_\zeta}(s) \partial_a \omega_{k_\zeta}(s) = 0$$

### 2.2 类扭量方程的等价形式

$$\nabla_{A'_\zeta(A_\zeta \omega_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}})}(s) = 0, \omega_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}}(s) = \frac{1}{(2s)!} \omega_{\underbrace{(B_\zeta C_\zeta D_\zeta \dots)_{2s}}_{2s}}(s) \quad (11.2)$$

$$\text{推论2.2.1. } \nabla_{A'_\zeta(A_\zeta \omega_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}})}(s) = 0, \omega_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}}(s) = \frac{1}{(2s)!} \omega_{\underbrace{(B_\zeta C_\zeta D_\zeta \dots)_{2s}}_{2s}}(s) \Leftrightarrow \nabla_{A'_\zeta(A_\zeta} N_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}}^{k_\zeta}(s) \omega_{k_\zeta}(s) = 0$$

$$\text{推论2.2.2. } \nabla_{A'_\zeta(A_\zeta} N_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}}^{k_\zeta}(s) \omega_{k_\zeta}(s) = 0 \Leftrightarrow (\sigma^*, i\zeta)_{A'_\zeta(A_\zeta}^a N_{\underbrace{B_\zeta C_\zeta D_\zeta \dots}_{2s}}^{k_\zeta}(s) \partial^a \psi_{k_\zeta}(s) = 0$$

推论2.2.3.  $\nabla_{A'_\zeta(A_\zeta N_{B_\zeta}^{k_\zeta})l_\zeta}(s)\omega_{k_\zeta}(s) = 0 \Leftrightarrow [-(s+1)\partial_a + iS_{ab}(s, \zeta)\partial^b]\psi(s) = 0$

证明:  $\nabla_{A'_\zeta(A_\zeta N_{B_\zeta}^{k_\zeta})l_\zeta}(s)\omega_{k_\zeta}(s) = 0$

$$\Leftrightarrow \partial^a \omega_{k_\zeta}(s) = (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a N_{k_\zeta}^{A_\zeta l_\zeta}(s) \overset{\circ}{\pi}_{l_\zeta}^{A'_\zeta}(s), \forall \overset{\circ}{\pi}_{l_\zeta}^{A'_\zeta}(s)$$

$$\Leftrightarrow \partial^a \omega_{k_\zeta}(s) = (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a N_{k_\zeta}^{A_\zeta l_\zeta}(s) \overset{\circ}{\pi}_{l_\zeta}^{A'_\zeta}(s), \overset{\circ}{\pi}_{l_\zeta}^{A'_\zeta}(s) = \frac{s}{2s+1} N_{A_\zeta l_\zeta}^{k_\zeta}(s) (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \omega_{k_\zeta}(s)$$

$$\Leftrightarrow \partial^a \omega_{k_\zeta}(s) = (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a N_{k_\zeta}^{A_\zeta l_\zeta}(s) \frac{s}{2s+1} N_{B_\zeta l_\zeta}^{m_\zeta}(s) (\sigma, -i\zeta)_b^{A'_\zeta B_\zeta} \partial^b \omega_{m_\zeta}(s)$$

$$\Leftrightarrow (2s+1)\partial^a \omega_{k_\zeta}(s) = N_{k_\zeta}^{A_\zeta l_\zeta}(s) s (\delta_{ab} \delta_{A_\zeta}^{B_\zeta} + 2iS_{ab A_\zeta}^{B_\zeta}) N_{B_\zeta l_\zeta}^{m_\zeta}(s) \partial^b \omega_{m_\zeta}(s)$$

$$\Leftrightarrow (s+1)\partial^a \omega_{k_\zeta}(s) = iS_{ab k_\zeta}^{m_\zeta}(s) \partial^b \omega_{m_\zeta}(s)$$

$$\Leftrightarrow [-(s+1)\partial_a + iS_{ab}(s, \zeta)\partial^b]\omega(s, \zeta) = 0$$

□

推论2.2.4.  $\nabla_{A'_\zeta(A_\zeta \omega_{B_\zeta})}(\frac{1}{2}) = 0 \Leftrightarrow [-\frac{3}{2}\partial_a + iS_{ab}(\frac{1}{2}, \zeta)\partial^b]\omega(\frac{1}{2}) = 0$

### 2.3 类扭量方程的解

推论2.3.1.  $\nabla_{A'_\zeta(A_\zeta N_{B_\zeta}^{k_\zeta})l_\zeta}(s)\omega_{k_\zeta}(s) = 0 \Leftrightarrow \omega_{k_\zeta}(s) = \dot{\omega}_{k_\zeta}(s) + x_a(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a N_{k_\zeta}^{A_\zeta l_\zeta}(s) \overset{\circ}{\pi}_{l_\zeta}^{A'_\zeta}(s)$

推论2.3.2.  $\nabla_{A'_\zeta(A_\zeta \omega_{B_\zeta})}(\frac{1}{2}) = 0 \Leftrightarrow \omega_{A_\zeta}(\frac{1}{2}) = \dot{\omega}_{A_\zeta}(\frac{1}{2}) + x_a(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \overset{\circ}{\pi}_{l_\zeta}^{A'_\zeta}(\frac{1}{2})$

### 2.4 开关型自旋方程与类扭量方程的关系

推论2.4.1.

$$[(s+\phi)\partial_a + iS_{ab}(s, \zeta)\partial^b]\psi(s, \zeta) = 0 \Rightarrow \begin{cases} \text{粒子解: } (\sigma \otimes I_{2s}, -i\zeta)^a \partial_a \tilde{\psi}(s, \zeta) = 0, \phi = 0 \\ \text{类扭量解: } \psi(s, \zeta) = \dot{\psi}_0(s, \zeta) + x^a \bar{Z}_a(s, \zeta) \tilde{J}_0(s, \zeta), \phi = -(2s+1) \\ \text{真空解: } \psi(s, \zeta) = \text{常数}, \phi \neq 0, -(2s+1) \end{cases}$$

# 第十二章 Bargmann-Wigner方程的分析

自我评述：本章对Bargmann-Wigner方程进行了全面深入的分析。在平坦时空中严格证明了Bargmann-Wigner方程在半整数自旋情形等价于Rarita-Schwinger方程，在整数自旋情形等价于Klein-Gordon方程，揭示了Bargmann-Wigner方程的深刻物理内涵。通过对比研究，发现Bargmann-Wigner方程更适合描写有质量粒子，而Penrose旋量方程或自旋方程更适合描写无质量粒子。

## 1 Bargmann-Wigner方程

### 1.1 Bargmann-Wigner方程<sup>[17]</sup>

$$[\gamma^a(\varsigma)D_a + m]_{\kappa\zeta} \underbrace{\psi_{\lambda\mu\zeta\eta\zeta\xi\zeta\cdots\zeta\zeta}}_{2s} = \underbrace{J_{\kappa\zeta\mu\zeta\eta\zeta\xi\zeta\cdots\zeta\zeta}}_{2s} \underbrace{\psi_{\lambda\mu\zeta\eta\zeta\xi\zeta\cdots\zeta\zeta}}_{2s} \text{全对称}, \underbrace{J_{\kappa\zeta\mu\zeta\eta\zeta\xi\zeta\cdots\zeta\zeta}}_{2s} \text{除}\kappa\zeta\text{外全对称} \quad (12.1)$$

## 2 二阶矩阵完备展开

### 2.1 二阶矩阵完备泡利基展开

二阶矩阵完备泡利基： $\Gamma_a(\varsigma) = \{\sigma, i\varsigma\}$

性质2.1.1.  $x^a\Gamma_a(\varsigma) = 0 \Rightarrow x^a = 0$

证明： $x^a\Gamma_a(\varsigma) = 0$

$\Rightarrow x^a(\sigma, i\varsigma)_a = 0$

$\Rightarrow \{x^a(\sigma, i\varsigma)_a, (\sigma, -i\varsigma)_b\} = 0$

$\Rightarrow x^a(2\delta_{ab}) = 0$

$\Rightarrow x^a = 0$  □

推论2.1.1.  $x^a\Gamma_a(\varsigma) = 0 \Leftrightarrow x^a = 0$

性质2.1.2.  $X = \frac{1}{2}\text{tr}[\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma), \forall X \in \text{二阶矩阵}$

证明： $X = X^{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + X^{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + X^{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + X^{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \forall X \in \text{二阶矩阵}$

$\Leftrightarrow X = \frac{1}{2}[X^{11}(I + \sigma_z) + X^{12}(\sigma_x + i\sigma_y) + X^{21}(\sigma_x - i\sigma_y) + X^{22}(I - \sigma_z)], \forall X \in \text{二阶矩阵}$

$\Leftrightarrow X = \frac{1}{2}(X^{12} + X^{21})\sigma_x + \frac{i}{2}(X^{12} - X^{21})\sigma_y + \frac{1}{2}(X^{11} - X^{22})\sigma_z - i\varsigma\frac{1}{2}(X^{11} + X^{22})i\varsigma I, \forall X \in \text{二阶矩阵}$

$\Leftrightarrow X = \frac{1}{2}\text{tr}[\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma), \forall X \in \text{二阶矩阵}$  □

推论2.1.2.  $X = x^a\Gamma_a(\varsigma), x^a = \text{tr}[\Gamma^a(-\varsigma)X], \forall X \in \text{二阶矩阵}$

二阶矩阵完备基性质：

正交性： $\Gamma_a(-\varsigma)\Gamma_a(\varsigma) = I, \text{tr}[\Gamma_a(-\varsigma)\Gamma_b(\varsigma)] = 2\delta_{ab}$  (12.2)

线性无关性： $x^a\Gamma_a(\varsigma) = 0 \Leftrightarrow x^a = 0$  (12.3)

完备性： $X = x^a\Gamma_a, \forall X \in \text{二阶矩阵}$  (12.4)

展开唯一性： $X = x^a\Gamma_a \Leftrightarrow x^a = \frac{1}{2}\text{tr}[\Gamma_a(-\varsigma)X], \forall X \in \text{二阶矩阵}$  (12.5)

### 2.2 二阶矩阵对称和反对称基展开

二阶矩阵对称和反对称基： $\Gamma_a(\varsigma)\varepsilon = \{\sigma, i\varsigma\}\varepsilon, [\Gamma_a(\varsigma)\varepsilon]^T = \{\sigma, -i\varsigma\}\varepsilon, \sigma\varepsilon$ 是对称基, $i\varsigma\varepsilon$ 是反对称基。

性质2.2.1.  $x^a\Gamma_a(\varsigma)\varepsilon = 0 \Leftrightarrow x^a = 0$

性质2.2.2.  $X = \frac{1}{2}\text{tr}[\varepsilon\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma)\varepsilon, \forall X \in \text{二阶矩阵}$



证明:  $X\bar{\varepsilon} = \frac{1}{2}\text{tr}[\Gamma^a(-\varsigma)X\bar{\varepsilon}]\Gamma_a(\varsigma), \forall X \in$  二阶矩阵

$\Leftrightarrow X = \frac{1}{2}\text{tr}[\Gamma^a(-\varsigma)X\bar{\varepsilon}]\Gamma_a(\varsigma)\varepsilon, \forall X \in$  二阶矩阵

$\Leftrightarrow X = \frac{1}{2}\text{tr}[\varepsilon\bar{\varepsilon}\Gamma^a(-\varsigma)X\bar{\varepsilon}]\Gamma_a(\varsigma)\varepsilon, \forall X \in$  二阶矩阵

$\Leftrightarrow X = \frac{1}{2}\text{tr}[\bar{\varepsilon}\Gamma^a(-\varsigma)X]\Gamma_a(\varsigma)\varepsilon, \forall X \in$  二阶矩阵 □

### 3 四阶矩阵完备展开

#### 3.1 四阶矩阵双泡利基展开

性质3.1.1.  $X = \frac{1}{4}\text{tr}[\Gamma^a(-\varsigma) \otimes \Gamma^b(-\varsigma)X]\Gamma_a(\varsigma) \otimes \Gamma_b(\varsigma), \forall X$

证明:  $X = \frac{1}{2} \begin{bmatrix} \text{tr}[\Gamma^a(-\varsigma)X_{11}]\Gamma_a(\varsigma) & \text{tr}[\Gamma^a(-\varsigma)X_{12}]\Gamma_a(\varsigma) \\ \text{tr}[\Gamma^a(-\varsigma)X_{21}]\Gamma_a(\varsigma) & \text{tr}[\Gamma^a(-\varsigma)X_{22}]\Gamma_a(\varsigma) \end{bmatrix}, \forall X$

$\Leftrightarrow X = \frac{1}{2}\Gamma_a(\varsigma) \otimes \begin{bmatrix} \text{tr}[\Gamma^a(-\varsigma)X_{11}] & \text{tr}[\Gamma^a(-\varsigma)X_{12}] \\ \text{tr}[\Gamma^a(-\varsigma)X_{21}] & \text{tr}[\Gamma^a(-\varsigma)X_{22}] \end{bmatrix}, \forall X$

$\Leftrightarrow X = \frac{1}{4}\text{tr}\{\Gamma^b(-\varsigma) \begin{bmatrix} \text{tr}[\Gamma^a(-\varsigma)X_{11}] & \text{tr}[\Gamma^a(-\varsigma)X_{12}] \\ \text{tr}[\Gamma^a(-\varsigma)X_{21}] & \text{tr}[\Gamma^a(-\varsigma)X_{22}] \end{bmatrix}\}\Gamma_a(\varsigma) \otimes \Gamma_b(\varsigma), \forall X$

$\Leftrightarrow X = \frac{1}{4}\text{tr}[\Gamma^a(-\varsigma) \otimes \Gamma^b(-\varsigma)X]\Gamma_a(\varsigma) \otimes \Gamma_b(\varsigma), \forall X$  □

推论3.1.1.  $\text{tr}\{\Gamma^b(-\varsigma) \begin{bmatrix} \text{tr}[\Gamma^a(-\varsigma)X_{11}] & \text{tr}[\Gamma^a(-\varsigma)X_{12}] \\ \text{tr}[\Gamma^a(-\varsigma)X_{21}] & \text{tr}[\Gamma^a(-\varsigma)X_{22}] \end{bmatrix}\} = \text{tr}[\Gamma^a(-\varsigma) \otimes \Gamma^b(-\varsigma)X]$

#### 3.2 电荷共轭矩阵C<sup>[4,9]</sup>

定义3.2.1.  $\bar{C}\gamma_a(\varsigma)C = -\gamma_a^T(\varsigma), C^T = -C, C^+ = \bar{C}$

推论3.2.1.  $\gamma_a(\varsigma)C = [\gamma_a(\varsigma)C]^T$

证明:  $\gamma_a(\varsigma)C = C\bar{C}\gamma_a(\varsigma)C = -C\gamma_a^T(\varsigma) = C^T\gamma_a^T(\varsigma) = [\gamma_a(\varsigma)C]^T$  □

推论3.2.2.  $\bar{C}\gamma_a(\varsigma) = [\bar{C}\gamma_a(\varsigma)]^T$

证明:  $\bar{C}\gamma_a(\varsigma) = \bar{C}\gamma_a(\varsigma)C\bar{C} = -\gamma_a^T(\varsigma)\bar{C} = -[C^*\gamma_a(\varsigma)]^T = [\bar{C}\gamma_a(\varsigma)]^T$  □

推论3.2.3.  $S_{ab}(e, \varsigma)C = [S_{ab}(e, \varsigma)C]^T$

证明:  $S_{ab}(e, \varsigma)C = -\frac{i}{4}[\gamma_a(\varsigma)\gamma_b(\varsigma) - \gamma_b(\varsigma)\gamma_a(\varsigma)]C$   
 $= -\frac{i}{4}[C\bar{C}\gamma_a(\varsigma)C\bar{C}\gamma_b(\varsigma)C - C\bar{C}\gamma_b(\varsigma)C\bar{C}\gamma_a(\varsigma)C]$   
 $= -\frac{i}{4}C[\gamma_a^T(\varsigma)\gamma_b^T(\varsigma) - \gamma_b^T(\varsigma)\gamma_a^T(\varsigma)] = -\frac{i}{4}C^T[\gamma_b(\varsigma)\gamma_a(\varsigma) - \gamma_a(\varsigma)\gamma_b(\varsigma)]^T$   
 $= C^T S_{ab}^T(e, \varsigma) = [S_{ab}(e, \varsigma)C]^T$  □

推论3.2.4.  $\bar{C}S_{ab}(e, \varsigma) = [\bar{C}S_{ab}(e, \varsigma)]^T$

证明:  $\bar{C}S_{ab}(e, \varsigma) = -\frac{i}{4}\bar{C}[\gamma_a(\varsigma)\gamma_b(\varsigma) - \gamma_b(\varsigma)\gamma_a(\varsigma)]$   
 $= -\frac{i}{4}[\bar{C}\gamma_a(\varsigma)C\bar{C}\gamma_b(\varsigma)C\bar{C} - \bar{C}\gamma_b(\varsigma)C\bar{C}\gamma_a(\varsigma)C\bar{C}]$   
 $= -\frac{i}{4}[\gamma_a^T(\varsigma)\gamma_b^T(\varsigma) - \gamma_b^T(\varsigma)\gamma_a^T(\varsigma)]\bar{C} = \frac{i}{4}[\gamma_b(\varsigma)\gamma_a(\varsigma) - \gamma_a(\varsigma)\gamma_b(\varsigma)]^T\bar{C}^T$   
 $= S_{ab}^T(e, \varsigma)\bar{C}^T = [\bar{C}S_{ab}(e, \varsigma)]^T$  □

推论3.2.5.  $\bar{C}\gamma_5(\varsigma)C = \gamma_5^T(\varsigma)$

证明:  $\bar{C}\gamma_5(\varsigma)C = \bar{C}\gamma_x(\varsigma)\gamma_y(\varsigma)\gamma_z(\varsigma)\gamma_\pi(\varsigma)C$   
 $= \bar{C}\gamma_x(\varsigma)C\bar{C}\gamma_y(\varsigma)C\bar{C}\gamma_z(\varsigma)C\bar{C}\gamma_\pi(\varsigma)C$   
 $= \gamma_x^T(\varsigma)\gamma_y^T(\varsigma)\gamma_z^T(\varsigma)\gamma_\pi^T(\varsigma) = [\gamma_\pi(\varsigma)\gamma_z(\varsigma)\gamma_y(\varsigma)\gamma_x(\varsigma)]^T = \gamma_5^T(\varsigma)$  □

推论3.2.6.  $C = -C^T, \bar{C} = -\bar{C}^T,$

推论3.2.7.  $\gamma_5(\varsigma)C = -[\gamma_5(\varsigma)C]^T, \bar{C}\gamma_5(\varsigma) = -[\bar{C}\gamma_5(\varsigma)]^T$

推论3.2.8.  $\gamma_5(\varsigma)\gamma_a(\varsigma)C = -[\gamma_5(\varsigma)\gamma_a(\varsigma)C]^T$

证明:  $\gamma_5(\varsigma)\gamma_a(\varsigma)C = C\bar{C}\gamma_5(\varsigma)C\bar{C}\gamma_a(\varsigma)C = -C\gamma_5^T(\varsigma)\gamma_a^T(\varsigma)$   
 $= C^T\gamma_5^T(\varsigma)\gamma_a^T(\varsigma) = [\gamma_a(\varsigma)\gamma_5(\varsigma)C]^T = -[\gamma_5(\varsigma)\gamma_a(\varsigma)C]^T$  □

推论3.2.9.  $\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma) = -[\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma)]^T$

证明:  $\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma) = \bar{C}\gamma_5(\varsigma)C\bar{C}\gamma_a(\varsigma)C\bar{C} = -\gamma_5^T(\varsigma)\gamma_a^T(\varsigma)\bar{C}$   
 $= \gamma_5^T(\varsigma)\gamma_a^T(\varsigma)\bar{C}^T = [\bar{C}\gamma_a(\varsigma)\gamma_5(\varsigma)]^T = -[\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma)]^T$  □

总结:

对称基:  $\gamma_a(\varsigma)C = [\gamma_a(\varsigma)C]^T, \bar{C}\gamma_a(\varsigma) = [\bar{C}\gamma_a(\varsigma)]^T, S_{ab}(e, \varsigma)C = [S_{ab}(e, \varsigma)C]^T, \bar{C}S_{ab}(e, \varsigma) = [\bar{C}S_{ab}(e, \varsigma)]^T$

反对称基:  $C = -C^T, \bar{C} = -\bar{C}^T, \gamma_5(\varsigma)C = -[\gamma_5(\varsigma)C]^T, \bar{C}\gamma_5(\varsigma) = -[\bar{C}\gamma_5(\varsigma)]^T,$

$$\gamma_5(\varsigma)\gamma_a(\varsigma)C = -[\gamma_5(\varsigma)\gamma_a(\varsigma)C]^T, \bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma) = -[\bar{C}\gamma_5(\varsigma)\gamma_a(\varsigma)]^T$$

### 3.3 特殊表象下狄拉克矩阵<sup>[4,9]</sup>

取特殊表象的狄拉克矩阵:  $[\gamma_a(\varsigma), \gamma_b(\varsigma)] = [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_z]$

详细展开:

$$[\gamma_a(\varsigma), \gamma_b(\varsigma)] = [(\sigma_x \otimes \sigma_y, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_z]$$

$$[\gamma_a(\varsigma), \gamma_b(\varsigma)]\gamma_5(\varsigma) = -i\varsigma[(\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_x, \sigma_z \otimes \sigma_x, \varsigma I \otimes \sigma_y), i\varsigma I \otimes I]$$

$$S_{ab}(e, \varsigma) = -\frac{i}{4}[\gamma_a(\varsigma), \gamma_b(\varsigma)] = \frac{1}{2} \begin{bmatrix} 0 & \sigma_z \otimes I & -\sigma_y \otimes I & -\varsigma \sigma_x \otimes \sigma_z \\ -\sigma_z \otimes I & 0 & \sigma_x \otimes I & -\varsigma \sigma_y \otimes \sigma_z \\ \sigma_y \otimes I & -\sigma_x \otimes I & 0 & -\varsigma \sigma_z \otimes \sigma_z \\ \varsigma \sigma_x \otimes \sigma_z & \varsigma \sigma_y \otimes \sigma_z & \varsigma \sigma_z \otimes \sigma_z & 0 \end{bmatrix}$$

特殊表象下的电荷共轭矩阵  $C = \gamma_y(\varsigma)\gamma_\pi(\varsigma)$

### 3.4 四阶矩阵狄拉克基<sup>[4,9]</sup>展开

四阶矩阵狄拉克完备基:  $\Gamma_A(\varsigma) = [\gamma_a(\varsigma), 2S_{ab}(e, \varsigma), -I_4, -i\gamma_a(\varsigma)\gamma_5(\varsigma), -\gamma_5(\varsigma)]$

性质3.4.1.  $X = [im\gamma_a(\varsigma)A^a + S_{ab}(e, \varsigma)F^{ab}] - [I_4\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi], \forall X$

$$\begin{cases} \phi = -\frac{1}{4}trX \\ \Phi = -\frac{1}{4}tr[\gamma^5(\varsigma)X] \end{cases}, \begin{cases} imA^a = \frac{1}{4}tr[\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = \frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X] \end{cases}, F^{ab} = \frac{1}{2}tr[S^{ab}(e, \varsigma)X]$$

证明:  $X = \frac{1}{4}tr[\Gamma^a(-\varsigma) \otimes \Gamma^b(-\varsigma)X]\Gamma_a(\varsigma) \otimes \Gamma_b(\varsigma), \forall X$

$\Leftrightarrow X = [im\gamma_a(\varsigma)A^a - iS_{ab}(e, \varsigma)F^{ab}] - [I_4\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi], \forall X$

$$\begin{cases} imA^i = \frac{i\varsigma}{4}tr[\Gamma^i(-\varsigma) \otimes \Gamma^x(-\varsigma)X] = \frac{1}{4}tr[\gamma^i(\varsigma)\gamma^5(\varsigma)X] \\ imA^\pi = \frac{i}{4}tr[\Gamma^\pi(-\varsigma) \otimes \Gamma^x(-\varsigma)X] = \frac{1}{4}tr[\gamma^\pi(\varsigma)X] \end{cases}$$

$$\begin{cases} imA^i = \frac{1}{4}tr[\Gamma^i(-\varsigma) \otimes \Gamma^y(-\varsigma)X] = \frac{1}{4}tr[\gamma^i(\varsigma)X] \\ im\mathbf{A}^\pi = \frac{\varsigma}{4}tr[\Gamma^\pi(-\varsigma) \otimes \Gamma^y(-\varsigma)X] = \frac{1}{4}tr[\gamma^\pi(\varsigma)\gamma^5(\varsigma)X] \end{cases}$$

$$\begin{cases} F^{i\pi} = -F^{\pi i} = -\frac{\varsigma}{4}tr[\Gamma^i(-\varsigma) \otimes \Gamma^z(-\varsigma)X] = -\frac{i}{2}tr[S^{i\pi}(e, \varsigma)X] \\ \Phi = -\frac{i}{4}tr[\Gamma^\pi(-\varsigma) \otimes \Gamma^z(-\varsigma)X] = -\frac{1}{4}tr[\gamma^5(\varsigma)X] \end{cases}$$

$$\begin{cases} F^{yz} = -F^{zy} = \frac{i\varsigma}{4}tr[\Gamma^x(-\varsigma) \otimes \Gamma^\pi(-\varsigma)X] = -\frac{i}{2}tr[S^{yz}(e, \varsigma)X] \\ F^{zx} = -F^{xz} = \frac{i\varsigma}{4}tr[\Gamma^y(-\varsigma) \otimes \Gamma^\pi(-\varsigma)X] = -\frac{i}{2}tr[S^{zx}(e, \varsigma)X] \end{cases}$$

$$\begin{cases} F^{xy} = -F^{yx} = \frac{i\varsigma}{4}tr[\Gamma^z(-\varsigma) \otimes \Gamma^\pi(-\varsigma)X] = -\frac{i}{2}tr[S^{xy}(e, \varsigma)X] \\ \phi = \frac{1}{4}tr[\Gamma^\pi(-\varsigma) \otimes \Gamma^\pi(-\varsigma)X] = -\frac{1}{4}trX \end{cases}$$

$$\Leftrightarrow X = [im\gamma_a(\varsigma)A^a + S_{ab}(e, \varsigma)F^{ab}] - [I_4\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi], \forall X$$

$$\begin{cases} \phi = -\frac{1}{4}tr X \\ \Phi = -\frac{1}{4}tr[\gamma^5(\varsigma)X] \end{cases}, \begin{cases} imA^a = \frac{1}{4}tr[\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = \frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X] \end{cases}, F^{ab} = \frac{1}{2}tr[S^{ab}(e, \varsigma)X] \quad \square$$

推论3.4.1.  $X = [im\gamma_a(\varsigma)A^a + S_{ab}(e, \varsigma)F^{ab}] - [I_4\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi], \forall X$

$$\Leftrightarrow X = [im\gamma_a(\varsigma)A^a + S_{ab}(e, \varsigma)F^{ab}] - [I_4\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi], \forall X$$

$$\begin{cases} \phi = -\frac{1}{4}tr X \\ \Phi = -\frac{1}{4}tr[\gamma^5(\varsigma)X] \end{cases}, \begin{cases} imA^a = \frac{1}{4}tr[\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = \frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X] \end{cases}, F^{ab} = \frac{1}{2}tr[S^{ab}(e, \varsigma)X]$$

### 3.5 四阶矩阵对称和反对称基展开

四阶矩阵对称和反对称基:  $\Gamma_A(\varsigma) = [\gamma_a(\varsigma), 2S_{ab}(e, \varsigma), -I_4, -i\gamma_a(\varsigma)\gamma_5(\varsigma), -\gamma_5(\varsigma)]C$

性质3.5.1.  $X = [im\gamma_a(\varsigma)CA^a + S_{ab}(e, \varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi], \forall X$

$$F^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e, \varsigma)X], \begin{cases} imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = -\frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)X] \end{cases}, \begin{cases} \phi = -\frac{1}{4}tr[\bar{C}X] \\ \Phi = -\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)X] \end{cases}$$

证明:  $X\bar{C} = \frac{1}{4}tr[\Gamma^a(-\varsigma) \otimes \Gamma^b(-\varsigma)X\bar{C}]\Gamma_a(\varsigma) \otimes \Gamma_b(\varsigma), \forall X$

$$\Leftrightarrow X\bar{C} = [im\gamma_a(\varsigma)A^a + S_{ab}(e, \varsigma)F^{ab}] - [\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]$$

$$\begin{cases} \phi = -\frac{1}{4}tr[X\bar{C}] \\ \Phi = -\frac{1}{4}tr[\gamma^5(\varsigma)X\bar{C}] \end{cases}, \begin{cases} imA^a = \frac{1}{4}tr[\gamma^a(\varsigma)X\bar{C}] \\ im\mathbf{A}^a = \frac{1}{4}tr[\gamma^a(\varsigma)\gamma^5(\varsigma)X\bar{C}] \end{cases}, F^{ab} = \frac{1}{2}tr[S^{ab}(e, \varsigma)X\bar{C}]$$

$$\Leftrightarrow X = [im\gamma_a(\varsigma)CA^a + S_{ab}(e, \varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi]$$

$$F^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e, \varsigma)X], \begin{cases} imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)X] \end{cases}, \begin{cases} \phi = -\frac{1}{4}tr[\bar{C}X] \\ \Phi = -\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)X] \end{cases} \quad \square$$

推论3.5.1.  $X = [im\gamma_a(\varsigma)CA^a - iS_{ab}(e, \varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi], \forall X$

$$\Leftrightarrow X = [im\gamma_a(\varsigma)CA^a - iS_{ab}(e, \varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi], \forall X$$

$$iF^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e, \varsigma)X], \begin{cases} imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)X] \\ im\mathbf{A}^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)X] \end{cases}, \begin{cases} \phi = -\frac{1}{4}tr[\bar{C}X] \\ \Phi = -\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)X] \end{cases}$$

### 3.6 对称四阶矩阵展开

四阶矩阵对称基:  $\Gamma_A(\varsigma) = [\gamma_a(\varsigma), 2S_{ab}(e, \varsigma)]C, \bar{C}\gamma_a(\varsigma)C = -\gamma_a^T(\varsigma), C^T = \bar{C} = -C, C^+(\varsigma) = \bar{C}$

性质3.6.1.  $G = im\gamma_a(\varsigma)CA^a + S_{ab}(e, \varsigma)CF^{ab}, G = G^T, F^{ab} = tr[\bar{C}S^{ab}(e, \varsigma)G], imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)G]$

证明:  $G = [im\gamma_a(\varsigma)CA^a + S_{ab}(e, \varsigma)CF^{ab}] - [C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi]$

$$F^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e, \varsigma)G], G = G^T, \begin{cases} imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)G] \\ im\mathbf{A}^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)G] = 0 \end{cases}, \begin{cases} \phi = -\frac{1}{4}tr[\bar{C}G] = 0 \\ \Phi = -\frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)G] = 0 \end{cases}$$

$$\Leftrightarrow G = im\gamma_a(\varsigma)CA^a + S_{ab}(e, \varsigma)CF^{ab}, G = G^T, F^{ab} = \frac{1}{2}tr[\bar{C}S^{ab}(e, \varsigma)G], imA^a = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)G] \quad \square$$

## 4 自旋-1的Bargmann-Wigner方程 [17]

### 4.1 有质量自旋-1的Bargmann-Wigner方程的分析

引理4.1.1.  $[\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)CA_a^\sigma + S^{ab}(e, \varsigma)CF_{ab}^\sigma] = J_{[\kappa_c\mu_c]}^\sigma,$

$$\Leftrightarrow \begin{cases} i(D^b F_{ab}^\sigma + m^2 A_a^\sigma) = \frac{1}{4}tr[\bar{C}\gamma_a(\varsigma)J_{[\kappa_c\mu_c]}^\sigma], m[F_{ab}^\sigma - (D_a A_b^\sigma - D_b A_a^\sigma)] = \frac{1}{2}tr[\bar{C}S^{ab}(e, \varsigma)J_{[\kappa_c\mu_c]}^\sigma] \\ imD^a A_a^\sigma = \frac{1}{4}tr[\bar{C}J_{[\kappa_c\mu_c]}^\sigma], 0 = \frac{1}{4}tr[\bar{C}\gamma^5(\varsigma)J_{[\kappa_c\mu_c]}^\sigma], iD^b *F_{ab}^\sigma = \frac{1}{4}tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_c\mu_c]}^\sigma] \end{cases}$$

证明:  $[\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)CA_a^\sigma + S^{ab}(e, \varsigma)CF_{ab}^\sigma] = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma$ ,

$$\Leftrightarrow im\gamma^c(\varsigma)\gamma^a(\varsigma)D_cA_a^\sigma + \gamma^c(\varsigma)S^{ab}(e, \varsigma)D_cF_{ab}^\sigma + im^2\gamma^a(\varsigma)A_a^\sigma + mS^{ab}(e, \varsigma)F_{ab}^\sigma = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma \bar{C}$$

$$\Leftrightarrow im[\delta^{ca} + 2iS^{ca}(e, \varsigma)]D_cA_a^\sigma - \frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma) - \gamma^{[a}\delta^{b]c}]D_cF_{ab}^\sigma$$

$$+ im^2\gamma^a(\varsigma)A_a^\sigma + mS^{ab}(e, \varsigma)F_{ab}^\sigma = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma \bar{C},$$

$$\Leftrightarrow im[D^aA_a^\sigma + 2iS^{ab}(e, \varsigma)D_aA_b^\sigma] - \frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma) - \gamma^{[a}\delta^{b]c}]D_cF_{ab}^\sigma$$

$$+ im^2\gamma^a(\varsigma)A_a^\sigma + mS^{ab}(e, \varsigma)F_{ab}^\sigma = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma \bar{C}$$

$$\Leftrightarrow i(D^bF_{ab}^\sigma + m^2A_a^\sigma)\gamma^a(\varsigma)C + m[F_{ab}^\sigma - (D_aA_b^\sigma - D_bA_a^\sigma)]S^{ab}(e, \varsigma)C$$

$$+ imD^aA_a^\sigma C + iD^b*F_{ab}^\sigma\gamma_5(\varsigma)\gamma^a(\varsigma)C = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma$$

$$\Leftrightarrow \begin{cases} i(D^bF_{ab}^\sigma + m^2A_a^\sigma) = \frac{1}{4}\text{tr}[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], m[F_{ab}^\sigma - (D_aA_b^\sigma - D_bA_a^\sigma)] = \frac{1}{2}\text{tr}[\bar{C}S^{ab}(e, \varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] \\ imD^aA_a^\sigma = \frac{1}{4}\text{tr}[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], 0 = \frac{1}{4}\text{tr}[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], iD^b*F_{ab}^\sigma = \frac{1}{4}\text{tr}[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] \end{cases} \quad \square$$

引理4.1.2. 
$$\begin{cases} i(D^bF_{ab}^\sigma + m^2A_a^\sigma) = \frac{1}{4}\text{tr}[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] \\ im[F_{ab}^\sigma - (D_aA_b^\sigma - D_bA_a^\sigma)] = \frac{1}{2}\text{tr}[\bar{C}S^{ab}(e, \varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] = 0 \\ imD^aA_a^\sigma = \frac{1}{4}\text{tr}[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] = 0, 0 = \frac{1}{4}\text{tr}[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] \\ iD^b*F_{ab}^\sigma = \frac{1}{4}\text{tr}[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} D^bF_{ab}^\sigma + m^2A_a^\sigma = J_a^\sigma, D^b*F_{ab}^\sigma = 0, F_{ab}^\sigma = D_aA_b^\sigma - D_bA_a^\sigma, D^aA_a^\sigma = 0 \\ J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \end{cases}$$

证明: 
$$\begin{cases} i(D^bF_{ab}^\sigma + m^2A_a^\sigma) = \frac{1}{4}\text{tr}[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] \\ im[F_{ab}^\sigma - (D_aA_b^\sigma - D_bA_a^\sigma)] = \frac{1}{2}\text{tr}[\bar{C}S^{ab}(e, \varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] = 0 \\ imD^aA_a^\sigma = \frac{1}{4}\text{tr}[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] = 0, 0 = \frac{1}{4}\text{tr}[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] \\ iD^b*F_{ab}^\sigma = \frac{1}{4}\text{tr}[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} i(D^bF_{ab}^\sigma + m^2A_a^\sigma) = \frac{1}{4}\text{tr}[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], D^b*F_{ab}^\sigma = 0, F_{ab}^\sigma = D_aA_b^\sigma - D_bA_a^\sigma, D^aA_a^\sigma = 0 \\ J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma = \begin{bmatrix} 0 & J^{[1_\varsigma 2_\varsigma]\sigma} \\ J^{[2_\varsigma 1_\varsigma]\sigma} & 0 \end{bmatrix}, \frac{1}{4}\text{tr}[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} i(D^bF_{ab}^\sigma + m^2A_a^\sigma) = \frac{1}{4}\text{tr}[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma = \begin{bmatrix} 0 & J_a^\sigma\Gamma^a(\varsigma)\bar{\varepsilon} \\ J_a^\sigma\Gamma^a(-\varsigma)\bar{\varepsilon} & 0 \end{bmatrix} \\ D^b*F_{ab}^\sigma = 0, F_{ab}^\sigma = D_aA_b^\sigma - D_bA_a^\sigma, D^aA_a^\sigma = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} D^bF_{ab}^\sigma + m^2A_a^\sigma = J_a^\sigma, D^b*F_{ab}^\sigma = 0, F_{ab}^\sigma = D_aA_b^\sigma - D_bA_a^\sigma, D^aA_a^\sigma = 0 \\ J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \end{cases} \quad \square$$

推论4.1.1.  $[\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)CA_a^\sigma + S^{ab}(e, \varsigma)CF_{ab}^\sigma] = -iJ_a^\sigma\gamma^a(\varsigma)C$

$$\Leftrightarrow D^bF_{ab}^\sigma + m^2A_a^\sigma = -J_a^\sigma, D^b*F_{ab}^\sigma = 0, F_{ab}^\sigma = D_aA_b^\sigma - D_bA_a^\sigma, D^aA_a^\sigma = 0$$

推论4.1.2.  $[\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)CA_a^\sigma + S^{ab}(e, \varsigma)CF_{ab}^\sigma] = -iJ_a^\sigma\gamma^a(\varsigma)C$

$$\Leftrightarrow \begin{cases} [\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]A_a^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \\ F_{ab}^\sigma = D_aA_b^\sigma - D_bA_a^\sigma \end{cases}$$

推论4.1.3. 
$$\begin{cases} [\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]A_a^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \\ F_{ab}^\sigma = D_aA_b^\sigma - D_bA_a^\sigma \end{cases}$$

$$\Leftrightarrow D^bF_{ab}^\sigma + m^2A_a^\sigma = -J_a^\sigma, D^b*F_{ab}^\sigma = 0, F_{ab}^\sigma = D_aA_b^\sigma - D_bA_a^\sigma, D^aA_a^\sigma = 0$$

$$\text{推论4.1.4. } \begin{cases} [\gamma^c(\varsigma)\partial_c + m][im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]A_a^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \\ F_{ab}^\sigma = \partial_a A_b^\sigma - \partial_b A_a^\sigma \\ \Leftrightarrow \partial^b F_{ab}^\sigma + m^2 A_a^\sigma = -J_a^\sigma, F_{ab}^\sigma = \partial_a A_b^\sigma - \partial_b A_a^\sigma, \partial^a J_a^\sigma = 0 \end{cases}$$

$$\text{推论4.1.5. } \begin{cases} [\gamma^c(\varsigma)D_c + m]\psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \\ \psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = im\gamma^a(\varsigma)CA_a^\sigma + S^{ab}(e, \varsigma)CF_{ab}^\sigma \end{cases} \Leftrightarrow \begin{cases} [\gamma^c(\varsigma)D_c + m]\psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \\ \psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]A_a^\sigma \\ F_{ab}^\sigma = D_a A_b^\sigma - D_b A_a^\sigma \end{cases}$$

$$\text{推论4.1.6. } \begin{cases} [\gamma^c(\varsigma)D_c + m]\psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \\ \psi_{\lambda_\varsigma\mu_\varsigma}^\sigma = \psi_{\mu_\varsigma\lambda_\varsigma}^\sigma \end{cases} \Leftrightarrow \begin{cases} [\gamma^c(\varsigma)D_c + m]\psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \\ \psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]A_a^\sigma \end{cases}$$

$$\text{推论4.1.7. } \begin{cases} [\gamma^c(\varsigma)D_c + m]\psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \\ \psi_{\lambda_\varsigma\mu_\varsigma}^\sigma = \psi_{\mu_\varsigma\lambda_\varsigma}^\sigma \\ \Leftrightarrow \begin{cases} D^b F_{ab}^\sigma + m^2 A_a^\sigma = -J_a^\sigma, D^b *F_{ab}^\sigma = 0, F_{ab}^\sigma = D_a A_b^\sigma - D_b A_a^\sigma, D^a A_a^\sigma = 0 \\ \psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]A_a^\sigma \end{cases} \end{cases}$$

$$\text{推论4.1.8. } \begin{cases} [\gamma^c(\varsigma)\partial_c + m]\psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \\ \psi_{\lambda_\varsigma\mu_\varsigma}^\sigma = \psi_{\mu_\varsigma\lambda_\varsigma}^\sigma \\ \Leftrightarrow \begin{cases} \partial^b F_{ab}^\sigma + m^2 A_a^\sigma = -J_a^\sigma, \partial^a J_a^\sigma = 0, F_{ab}^\sigma = \partial_a A_b^\sigma - \partial_b A_a^\sigma \\ \psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]A_a^\sigma \end{cases} \end{cases}$$

## 4.2 有质量自旋-1的Bargmann-Wigner方程

$$\text{定理4.2.1. } \begin{cases} [\gamma^c(\varsigma)\partial_c + m]\psi_{[\lambda_\varsigma\mu_\varsigma]}^\sigma = -iJ_a^\sigma\gamma^a(\varsigma)C \\ \psi_{\lambda_\varsigma\mu_\varsigma}^\sigma = \psi_{\mu_\varsigma\lambda_\varsigma}^\sigma \end{cases} \Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2)A_a^\sigma = -J_a^\sigma \\ \partial^a A_a^\sigma = 0, \partial^a J_a^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_a^\sigma \end{cases}$$

## 4.3 无质量自旋-1的Bargmann-Wigner方程的分析

引理4.3.1.  $[\gamma^c(\varsigma)D_c + m][im\gamma^a(\varsigma)CA_a^\sigma + S^{ab}(e, \varsigma)CF_{ab}^\sigma] = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma$ ,

$$\Leftrightarrow \begin{cases} i(D^b F_{ab}^\sigma + m^2 A_a^\sigma) = \frac{1}{4}\text{tr}[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], m[F_{ab}^\sigma - (D_a A_b^\sigma - D_b A_a^\sigma)] = \frac{1}{2}\text{tr}[\bar{C}S^{ab}(e, \varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] \\ imD^a A_a^\sigma = \frac{1}{4}\text{tr}[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], 0 = \frac{1}{4}\text{tr}[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], iD^b *F_{ab}^\sigma = \frac{1}{4}\text{tr}[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] \end{cases}$$

证明:  $\gamma^c(\varsigma)D_c[im\gamma^a(\varsigma)CA_a^\sigma + S^{ab}(e, \varsigma)CF_{ab}^\sigma] = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma$

$$\Leftrightarrow im\gamma^c(\varsigma)\gamma^a(\varsigma)D_c A_a^\sigma + \gamma^c(\varsigma)S^{ab}(e, \varsigma)D_c F_{ab}^\sigma = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma \bar{C}$$

$$\Leftrightarrow im[\delta^{ca} + 2iS^{ca}(e, \varsigma)]D_c A_a^\sigma - \frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma) - \gamma^{[a}\delta^{b]c}]D_c F_{ab}^\sigma = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma \bar{C}$$

$$\Leftrightarrow im[D^a A_a^\sigma + 2iS^{ab}(e, \varsigma)D_a A_b^\sigma] - \frac{i}{2}[\varepsilon^{abcd}\gamma_5(\varsigma)\gamma_d(\varsigma) - \gamma^{[a}\delta^{b]c}]D_c F_{ab}^\sigma = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma \bar{C}$$

$$\Leftrightarrow iD^b F_{ab}^\sigma \gamma^a(\varsigma)C + im[(D_a A_b^\sigma - D_b A_a^\sigma)]S^{ab}(e, \varsigma)C + imD^a A_a^\sigma C + iD^b *F_{ab}^\sigma \gamma_5(\varsigma)\gamma^a(\varsigma)C = J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma$$

$$\Leftrightarrow \begin{cases} iD^b F_{ab}^\sigma = \frac{1}{4}\text{tr}[\bar{C}\gamma_a(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], -m(D_a A_b^\sigma - D_b A_a^\sigma) = \frac{1}{2}\text{tr}[\bar{C}S^{ab}(e, \varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] \\ imD^a A_a^\sigma = \frac{1}{4}\text{tr}[\bar{C}J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], 0 = \frac{1}{4}\text{tr}[\bar{C}\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma], iD^b *F_{ab}^\sigma = \frac{1}{4}\text{tr}[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)J_{[\kappa_\varsigma\mu_\varsigma]}^\sigma] \end{cases} \quad \square$$

## 4.4 无质量自旋-1的Bargmann-Wigner方程

推论4.4.1.  $\gamma^c(\varsigma)D_c[im\gamma^a(\varsigma)CA_a^\sigma + S^{ab}(e, \varsigma)CF_{ab}^\sigma] = -iJ_a^\sigma\gamma^a(\varsigma)C$

$$\Leftrightarrow D^b F_{ab}^\sigma = -J_a^\sigma, D^b *F_{ab}^\sigma = 0, D_a A_b^\sigma - D_b A_a^\sigma = 0, D^a A_a^\sigma = 0$$

$$\begin{aligned}
& \text{证明: } \gamma^c(\varsigma)D_c[im\gamma^a(\varsigma)CA_a^\sigma + S^{ab}(e, \varsigma)CF_{ab}^\sigma] = -iJ_a^\sigma\gamma^a(\varsigma)C \\
& \Leftrightarrow iD^bF_{ab}^\sigma\gamma^a(\varsigma)C - m(D_aA_b^\sigma - D_bA_a^\sigma)S^{ab}(e, \varsigma)C + imD^aA_a^\sigma C + iD^b*F_{ab}^\sigma\gamma_5(\varsigma)\gamma^a(\varsigma)C = -iJ_a^\sigma\gamma^a(\varsigma)C \\
& \Leftrightarrow D^bF_{ab}^\sigma = -J_a^\sigma, D^b*F_{ab}^\sigma = 0, D_aA_b^\sigma - D_bA_a^\sigma = 0, D^aA_a^\sigma = 0 \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{推论4.4.2. } \gamma^c(\varsigma)\partial_c[im\gamma^a(\varsigma)CA_a + S^{ab}(e, \varsigma)CF_{ab}] = -iJ_a\gamma^a(\varsigma)C \\
& \Leftrightarrow \partial^bF_{ab} = -J_a, \partial^b*F_{ab} = 0, \partial^a\partial_a\phi = 0, A_a = \partial_a\phi
\end{aligned}$$

在无质量情形，由于 $F_{ab}^\sigma, A_a^\sigma$ 完全相互独立，无法得到更简洁、更有意义的结论，且有多余的方程，显得拖泥带水不够简洁，也无法自然地高自旋情形推广。所以Bargmann-Wigner方程似乎不太适合描写无质量粒子，更适合描写无质量粒子的是Penrose旋量方程<sup>[1, 2]</sup>(自旋方程)。

## 5 自旋- $\frac{3}{2}$ , 2的Bargmann-Wigner方程<sup>[17]</sup>

### 5.1 有质量自旋- $\frac{3}{2}$ 的Bargmann-Wigner方程的分析

$$\begin{aligned}
& \text{推论5.1.1. } \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}^\sigma \quad tr[\bar{C}\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}^\sigma] = 0 \\
& \Rightarrow [im\gamma^a(\varsigma) - 2S^{ab}(e, \varsigma)D_b]A_{a[\eta_\varsigma]}^\sigma = 0
\end{aligned}$$

$$\begin{aligned}
& \text{推论5.1.2. } \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}^\sigma \quad tr[\bar{C}\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}^\sigma] = 0 \\
& \Rightarrow [im\gamma^a(\varsigma) + 2S^{ab}(e, \varsigma)D_b]A_{a[\eta_\varsigma]}^\sigma = 0
\end{aligned}$$

$$\begin{aligned}
& \text{推论5.1.3. } \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}^\sigma \quad tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}^\sigma] = 0, \\
& \Rightarrow [im\gamma^a(\varsigma)\gamma^c(\varsigma) + 2S^{ab}(e, \varsigma)\gamma^c(\varsigma)D_b]A_{a[\eta_\varsigma]}^\sigma = 0
\end{aligned}$$

$$\begin{aligned}
& \text{推论5.1.4. } \begin{cases} \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}^\sigma \\ tr[\bar{C}\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}^\sigma] = 0, tr[\bar{C}\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}^\sigma] = 0 \end{cases} \\
& \Leftrightarrow \begin{cases} \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}^\sigma \\ \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0, D^aA_{a[\eta_\varsigma]}^\sigma = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \text{推论5.1.5. } \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = \psi_{\lambda_\varsigma\eta_\varsigma\mu_\varsigma}^\sigma \\
& \Leftrightarrow tr[\bar{C}\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}^\sigma] = 0, tr[\bar{C}\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}^\sigma] = 0, tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma\eta_\varsigma]}^\sigma] = 0
\end{aligned}$$

$$\begin{aligned}
& \text{推论5.1.6. } \begin{cases} \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}^\sigma \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = \psi_{\lambda_\varsigma\eta_\varsigma\mu_\varsigma}^\sigma \end{cases} \\
& \Leftrightarrow \begin{cases} \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma\mu_\varsigma}A_{a\eta_\varsigma}^\sigma \\ [\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \text{推论5.1.7. } \begin{cases} A_{a\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^b(\varsigma)C - 2S^{bz}(e, \varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}^\sigma \\ \gamma^a(\varsigma)A_{a[\eta_\varsigma]\xi_\varsigma}^\sigma = 0 \end{cases} \\
& \Leftrightarrow \begin{cases} A_{a\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^b(\varsigma)C - 2S^{bz}(e, \varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}^\sigma \\ \delta^{ab}A_{ab}^\sigma = 0, A_{ab}^\sigma = A_{ba}^\sigma, \partial^aA_{ab}^\sigma = 0 \end{cases}
\end{aligned}$$

$$\text{证明: } A_{a\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^b(\varsigma)C - 2S^{bz}(e, \varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}^\sigma, \gamma^a(\varsigma)A_{a[\eta_\varsigma]\xi_\varsigma}^\sigma = 0$$

$$\begin{aligned}
& \Leftrightarrow \begin{cases} A_{a\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^b(\varsigma)C - 2S^{bz}(e, \varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}^\sigma \\ im[\delta^{ab} + 2iS^{ab}(e, \varsigma)]A_{ab}^\sigma - i\gamma^d(\varsigma)\gamma^5(\varsigma)\varepsilon^{abz}{}_d\partial_zA_{ab}^\sigma + i\gamma^z(\varsigma)(\delta^{ab}\partial_zA_{ab}^\sigma - \partial^aA_{az}^\sigma) = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \Leftrightarrow \begin{cases} A_{a\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^b(\varsigma)C - 2S^{bz}(e, \varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}^\sigma \\ \delta^{ab}A_{ab}^\sigma = 0, A_{ab}^\sigma - A_{ba}^\sigma = 0, (\varsigma)\varepsilon^{abz}{}_d\partial_zA_{ab}^\sigma = 0, (\delta^{ab}\partial_zA_{ab}^\sigma - \partial^aA_{az}^\sigma) = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
& \Leftrightarrow \begin{cases} A_{a\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^b(\varsigma)C - 2S^{bz}(e, \varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma}A_{ab}^\sigma \\ \delta^{ab}A_{ab}^\sigma = 0, A_{ab}^\sigma = A_{ba}^\sigma, \partial^aA_{ab}^\sigma = 0 \end{cases} \quad \square
\end{aligned}$$

## 5.2 弯曲时空中有质量自旋 $-\frac{3}{2}$ 的Bargmann-Wigner方程

引理5.2.1.  $[\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0 \Rightarrow D^a A_{a[\eta_\varsigma]}^\sigma = 0$

证明:  $[\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0$   
 $\Rightarrow \gamma^a(\varsigma)[\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0$   
 $\Rightarrow \gamma^a(\varsigma)\gamma^b(\varsigma)D_b A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0$   
 $\Rightarrow [\gamma^a(\varsigma)\gamma^b(\varsigma) + \gamma^b(\varsigma)\gamma^a(\varsigma) - \gamma^b(\varsigma)\gamma^a(\varsigma)]D_b A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0$   
 $\Rightarrow 2\delta^{ab}D_b A_{a[\eta_\varsigma]}^\sigma - \gamma^b(\varsigma)D_b[\gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma] = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0$   
 $\Rightarrow D^a A_{a[\eta_\varsigma]}^\sigma = 0$  □

引理5.2.2.  $[\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}^\sigma = 0 \Rightarrow (\partial_b\partial^b - m^2)A_{a[\eta_\varsigma]}^\sigma = 0$

证明:  $[\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}^\sigma = 0$   
 $\Rightarrow [\gamma^b(\varsigma)\partial_b - m][\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}^\sigma = 0$   
 $\Rightarrow [\gamma^b(\varsigma)\gamma^c(\varsigma)\partial_b\partial_c - m^2]A_{a[\eta_\varsigma]}^\sigma = 0$   
 $\Rightarrow \{[\delta^{bc} + 2iS^{ab}(e, \varsigma)]\partial_b\partial_c - m^2\}A_{a[\eta_\varsigma]}^\sigma = 0$   
 $\Rightarrow (\partial_b\partial^b - m^2)A_{a[\eta_\varsigma]}^\sigma = 0$  □

定理5.2.1.  $\begin{cases} [\gamma^a(\varsigma)D_a + m]_{\kappa_\varsigma}^{\lambda_\varsigma} \psi_{[\lambda_\varsigma \mu_\varsigma] \eta_\varsigma}^\sigma = -iJ_{a\eta_\varsigma}^\sigma \gamma^a(\varsigma)C \\ \psi_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma}^\sigma \text{除}\sigma\text{外全对称} \end{cases}$

$$\Leftrightarrow \begin{cases} \psi_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma \mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ D^b F_{ab\eta_\varsigma}^\sigma + m^2 A_{a\eta_\varsigma}^\sigma = -J_{a\eta_\varsigma}^\sigma, D^b *F_{ab\eta_\varsigma}^\sigma = 0, F_{ab\eta_\varsigma}^\sigma = D_a A_{b\eta_\varsigma}^\sigma - D_b A_{a\eta_\varsigma}^\sigma \\ [\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0 \end{cases}$$

证明:  $[\gamma^a(\varsigma)D_a + m]_{\kappa_\varsigma}^{\lambda_\varsigma} \psi_{[\lambda_\varsigma \mu_\varsigma] \eta_\varsigma}^\sigma = -iJ_{a\eta_\varsigma}^\sigma \gamma^a(\varsigma)C, \psi_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma}^\sigma \text{除}\sigma\text{外全对称}$

$$\Leftrightarrow \begin{cases} [\gamma^a(\varsigma)D_a + m]_{\kappa_\varsigma}^{\lambda_\varsigma} \psi_{[\lambda_\varsigma \mu_\varsigma] \eta_\varsigma}^\sigma = -iJ_{a\eta_\varsigma}^\sigma \gamma^a(\varsigma)C, \psi_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma}^\sigma = \psi_{\mu_\varsigma \lambda_\varsigma \eta_\varsigma}^\sigma \\ tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma \eta_\varsigma]}^\sigma] = 0, tr[\bar{C}\psi_{\lambda_\varsigma[\mu_\varsigma \eta_\varsigma]}^\sigma] = 0, tr[\bar{C}\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma \eta_\varsigma]}^\sigma] = 0 \\ [\gamma^a(\varsigma)D_a + m]_{\kappa_\varsigma}^{\lambda_\varsigma} \psi_{[\lambda_\varsigma \mu_\varsigma] \eta_\varsigma}^\sigma = -iJ_{a\eta_\varsigma}^\sigma \gamma^a(\varsigma)C, \psi_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma \mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ tr[\bar{C}\gamma^a(\varsigma)\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma \eta_\varsigma]}^\sigma] = 0, tr[\bar{C}\psi_{\lambda_\varsigma[\mu_\varsigma \eta_\varsigma]}^\sigma] = 0, tr[\bar{C}\gamma^5(\varsigma)\psi_{\lambda_\varsigma[\mu_\varsigma \eta_\varsigma]}^\sigma] = 0 \\ [\gamma^a(\varsigma)D_a + m]_{\kappa_\varsigma}^{\lambda_\varsigma} \psi_{[\lambda_\varsigma \mu_\varsigma] \eta_\varsigma}^\sigma = -iJ_{a\eta_\varsigma}^\sigma \gamma^a(\varsigma)C, \psi_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma \mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ [\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0, D^a A_{a[\eta_\varsigma]}^\sigma = 0 \\ \psi_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma \mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ D^b F_{ab\eta_\varsigma}^\sigma + m^2 A_{a\eta_\varsigma}^\sigma = -J_{a\eta_\varsigma}^\sigma, D^b *F_{ab\eta_\varsigma}^\sigma = 0, F_{ab\eta_\varsigma}^\sigma = D_a A_{b\eta_\varsigma}^\sigma - D_b A_{a\eta_\varsigma}^\sigma, D^a A_{a\eta_\varsigma}^\sigma = 0 \\ [\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0, D^a A_{a[\eta_\varsigma]}^\sigma = 0 \\ \psi_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)CD_b]_{\lambda_\varsigma \mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ D^b F_{ab\eta_\varsigma}^\sigma + m^2 A_{a\eta_\varsigma}^\sigma = -J_{a\eta_\varsigma}^\sigma, D^b *F_{ab\eta_\varsigma}^\sigma = 0, F_{ab\eta_\varsigma}^\sigma = D_a A_{b\eta_\varsigma}^\sigma - D_b A_{a\eta_\varsigma}^\sigma \\ [\gamma^b(\varsigma)D_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)J_{a[\eta_\varsigma]}^\sigma = 0 \end{cases}$$
 □

在弯曲时空中方程不能进一步简化，所以无法得到更简洁更有意义的结论。

5.3 平坦时空中有质量自旋 $-\frac{3}{2}$ 的Bargmann-Wigner方程的源项要求？

$$\text{定理5.3.1. } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]_{\kappa\varsigma} \lambda_\varsigma \psi_{[\lambda_\varsigma\mu_\varsigma]\eta_\varsigma}^\sigma = -iJ_{a\eta_\varsigma}^\sigma \gamma^a(\varsigma)C \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma \text{除}\sigma\text{外全对称} \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ \partial^b F_{ab\eta_\varsigma}^\sigma + m^2 A_{a\eta_\varsigma}^\sigma = 0, J_{a\eta_\varsigma}^\sigma = 0, F_{ab\eta_\varsigma}^\sigma = \partial_a A_{b\eta_\varsigma}^\sigma - \partial_b A_{a\eta_\varsigma}^\sigma \\ [\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0 \end{cases}$$

$$\text{证明: } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]_{\kappa\varsigma} \lambda_\varsigma \psi_{[\lambda_\varsigma\mu_\varsigma]\eta_\varsigma}^\sigma = -iJ_{a\eta_\varsigma}^\sigma \gamma^a(\varsigma)C \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma \text{除}\sigma\text{外全对称} \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ \partial^b F_{ab\eta_\varsigma}^\sigma + m^2 A_{a\eta_\varsigma}^\sigma = -J_{a\eta_\varsigma}^\sigma, \partial^b *F_{ab\eta_\varsigma}^\sigma = 0, F_{ab\eta_\varsigma}^\sigma = \partial_a A_{b\eta_\varsigma}^\sigma - \partial_b A_{a\eta_\varsigma}^\sigma \\ [\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ \partial^b F_{ab\eta_\varsigma}^\sigma + m^2 A_{a\eta_\varsigma}^\sigma = -J_{a\eta_\varsigma}^\sigma, F_{ab\eta_\varsigma}^\sigma = \partial_a A_{b\eta_\varsigma}^\sigma - \partial_b A_{a\eta_\varsigma}^\sigma \\ [\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0, \partial^a A_{a[\eta_\varsigma]}^\sigma = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ \partial^b F_{ab\eta_\varsigma}^\sigma + m^2 A_{a\eta_\varsigma}^\sigma = 0, J_{a\eta_\varsigma}^\sigma = 0, F_{ab\eta_\varsigma}^\sigma = \partial_a A_{b\eta_\varsigma}^\sigma - \partial_b A_{a\eta_\varsigma}^\sigma \\ [\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0 \end{cases} \quad \square$$

相对于弯曲时空情形，在平坦时空中方程进一步得到了简化，得到了更简洁更有意义的结论，方程本身自恰性也自动要求源项必须为零(???)。

5.4 平坦时空中有质量自旋 $-\frac{3}{2}$ 的Bargmann-Wigner方程<sup>[18]</sup>

$$\text{定理5.4.1. } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma \text{除}\sigma\text{外全对称} \end{cases} \Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma}^\sigma \end{cases}$$

$$\text{证明: } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma \text{除}\sigma\text{外全对称} \end{cases}$$

$$\Leftrightarrow \begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = \psi_{\mu_\varsigma\lambda_\varsigma\eta_\varsigma}^\sigma \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = \psi_{\lambda_\varsigma\eta_\varsigma\mu_\varsigma}^\sigma \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2)A_{a\eta_\varsigma}^\sigma = 0, \partial^a A_{a\eta_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = \psi_{\lambda_\varsigma\eta_\varsigma\mu_\varsigma}^\sigma \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2)A_{a\eta_\varsigma}^\sigma = 0, \partial^a A_{a\eta_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma}^\sigma \\ [\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma}^\sigma \end{cases} \quad \square$$



## 5.5 平坦时空中有质量自旋-2的Bargmann-Wigner方程

$$\text{定理5.5.1. } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma \text{除}^\sigma \text{外全对称} \end{cases} \\ \Leftrightarrow \begin{cases} (-\partial^d\partial_d + m^2)A_{ab}^\sigma = 0, \delta^{ab}A_{ab}^\sigma = 0, A_{ab}^\sigma = A_{ba}^\sigma, \partial^a A_{ab}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ac}(e, \varsigma)C\partial_c]_{\lambda_\varsigma\mu_\varsigma} [im\gamma^b(\varsigma)C - 2S^{bd}(e, \varsigma)C\partial_d]_{\eta_\varsigma\xi_\varsigma} A_{ab}^\sigma \end{cases}$$

$$\text{证明: } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma \text{除}^\sigma \text{外全对称} \end{cases}$$

$$\Leftrightarrow \begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma \text{除}^\sigma \text{外全对称} \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = \psi_{\lambda_\varsigma\mu_\varsigma\xi_\varsigma\eta_\varsigma}^\sigma \end{cases}$$

$$\Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]\xi_\varsigma}^\sigma = 0, \gamma^a(\varsigma)A_{a[\eta_\varsigma]\xi_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma\xi_\varsigma}^\sigma \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = \psi_{\lambda_\varsigma\mu_\varsigma\xi_\varsigma\eta_\varsigma}^\sigma \end{cases}$$

$$\Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m]A_{a[\eta_\varsigma]\xi_\varsigma}^\sigma = 0, A_{a\eta_\varsigma\xi_\varsigma}^\sigma = A_{a\xi_\varsigma\eta_\varsigma}^\sigma \\ \gamma^a(\varsigma)A_{a[\eta_\varsigma]\xi_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma\xi_\varsigma}^\sigma \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^d\partial_d + m^2)A_{ab}^\sigma = 0, \partial^b A_{ab}^\sigma = 0 \\ A_{a\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^b(\varsigma)C - 2S^{bz}(e, \varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma} A_{ab}^\sigma \\ \gamma^a(\varsigma)A_{a[\eta_\varsigma]\xi_\varsigma}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma\xi_\varsigma}^\sigma \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^d\partial_d + m^2)A_{ab}^\sigma = 0, \partial^b A_{ab}^\sigma = 0 \\ A_{a\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^b(\varsigma)C - 2S^{bz}(e, \varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma} A_{ab}^\sigma \\ \delta^{ab}A_{ab}^\sigma = 0, A_{ab}^\sigma = A_{ba}^\sigma, \partial^a A_{ab}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} A_{a\eta_\varsigma\xi_\varsigma}^\sigma \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^d\partial_d + m^2)A_{ab}^\sigma = 0, \delta^{ab}A_{ab}^\sigma = 0, A_{ab}^\sigma = A_{ba}^\sigma, \partial^a A_{ab}^\sigma = 0 \\ \psi_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma}^\sigma = [im\gamma^a(\varsigma)C - 2S^{ab}(e, \varsigma)C\partial_b]_{\lambda_\varsigma\mu_\varsigma} [im\gamma^b(\varsigma)C - 2S^{bz}(e, \varsigma)C\partial_z]_{\eta_\varsigma\xi_\varsigma} A_{ab}^\sigma \end{cases} \quad \square$$

$$\text{推论5.5.1. } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]_{\kappa_\varsigma} \lambda_\varsigma \mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a \mathbb{X}_{\eta_\varsigma\xi_\varsigma}^b A_{ab}^\sigma = 0 \\ \mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a \mathbb{X}_{\eta_\varsigma\xi_\varsigma}^b A_{ab}^\sigma = \mathbb{X}_{\lambda_\varsigma\eta_\varsigma}^a \mathbb{X}_{\mu_\varsigma\xi_\varsigma}^b A_{ab}^\sigma \end{cases} \Leftrightarrow \begin{cases} (-\partial^d\partial_d + m^2)A_{ab}^\sigma = 0 \\ \delta^{ab}A_{ab}^\sigma = 0, A_{ab}^\sigma = A_{ba}^\sigma, \partial^a A_{ab}^\sigma = 0 \end{cases}$$

$$\text{推论5.5.2. } \mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a \mathbb{X}_{\eta_\varsigma\xi_\varsigma}^b A_{ab} = \mathbb{X}_{\mu_\varsigma\eta_\varsigma}^a \mathbb{X}_{\lambda_\varsigma\xi_\varsigma}^b A_{ab} \Leftrightarrow ???$$

## 6 平坦时空中任意自旋粒子的Bargmann-Wigner方程

### 6.1 平坦时空中有质量自旋 $s = n$ 的Bargmann-Wigner方程 [17, 19, 22]

$$\text{定义6.1.1. } \mathbb{X}_a := [im\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)\partial^b]C, \mathbb{X}^a := [im\gamma^a(\varsigma) - 2S^{ab}(e, \varsigma)\partial_b]C$$

$$\text{定理6.1.1.} \quad \left\{ \begin{array}{l} [\gamma^a(\zeta)\partial_a + m]\psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\eta_\zeta\xi_\zeta\cdots\zeta_\zeta}_{2n}}^\sigma = 0 \\ \psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\eta_\zeta\xi_\zeta\cdots\zeta_\zeta}_{2n}}^\sigma \text{除}\sigma\text{外全对称} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (-\partial^d\partial_d + m^2)\underbrace{A_{abc\cdots}}_n^\sigma = 0, \underbrace{A_{abc\cdots}}_n^\sigma \text{除}\sigma\text{外全对称} \\ \delta^{ab}\underbrace{A_{abc\cdots}}_n^\sigma = 0, \partial^a\underbrace{A_{abc\cdots}}_n^\sigma = 0 \\ \psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\eta_\zeta\xi_\zeta\cdots\zeta_\zeta}_{2n}}^\sigma = \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a}_{n-1}\underbrace{\mathbb{X}_{\eta_\zeta\xi_\zeta}^b}_{n-1}\cdots\underbrace{A_{abc\cdots}}_n^\sigma \end{array} \right.$$

$$\text{推论6.1.1.} \quad \left\{ \begin{array}{l} [\gamma^a(\zeta)\partial_a + m]\underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a}_{n-1}\underbrace{\mathbb{X}_{\eta_\zeta\xi_\zeta}^b}_{n-1}\cdots\underbrace{A_{abc\cdots}}_n^\sigma = 0 \Leftrightarrow (-\partial^d\partial_d + m^2)\underbrace{A_{abc\cdots}}_n^\sigma = 0 \\ \underbrace{A_{abc\cdots}}_n^\sigma = \frac{1}{n!}\underbrace{A_{\{abc\cdots\}}}_n^\sigma, \delta^{ab}\underbrace{A_{abc\cdots}}_n^\sigma = 0, \partial^a\underbrace{A_{abc\cdots}}_n^\sigma = 0 \end{array} \right.$$

## 6.2 平坦时空中有质量自旋 $s = n + \frac{1}{2}$ 的Bargmann-Wigner方程 [17, 19, 20]

$$\text{定理6.2.1.} \quad \left\{ \begin{array}{l} [\gamma^a(\zeta)\partial_a + m]\psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\eta_\zeta\xi_\zeta\cdots\zeta_\zeta}_{2n+1}}^\sigma = 0 \\ \psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\eta_\zeta\xi_\zeta\cdots\zeta_\zeta}_{2n+1}}^\sigma \text{除}\sigma\text{外全对称} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} [\gamma^d(\zeta)\partial_d + m]\underbrace{A_{abc\cdots}[\zeta_\zeta]}_n^\sigma = 0, \underbrace{A_{abc\cdots}[\zeta_\zeta]}_n^\sigma \text{除}\zeta_\zeta\sigma\text{外全对称} \\ \delta^{ab}\underbrace{A_{abc\cdots}[\zeta_\zeta]}_n^\sigma = 0, \gamma^a(\zeta)\underbrace{A_{abc\cdots}[\zeta_\zeta]}_n^\sigma = 0 \\ \psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\eta_\zeta\xi_\zeta\cdots\zeta_\zeta}_{2n+1}}^\sigma = \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a}_{n-1}\underbrace{\mathbb{X}_{\eta_\zeta\xi_\zeta}^b}_{n-1}\cdots\underbrace{A_{abc\cdots}[\zeta_\zeta]}_n^\sigma \end{array} \right.$$

$$\text{推论6.2.1.} \quad \left\{ \begin{array}{l} [\gamma^a(\zeta)\partial_a + m]\underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a}_{n-1}\underbrace{\mathbb{X}_{\eta_\zeta\xi_\zeta}^b}_{n-1}\cdots\underbrace{A_{abc\cdots}[\zeta_\zeta]}_n^\sigma = 0 \Leftrightarrow [\gamma^d(\zeta)\partial_d + m]\underbrace{A_{abc\cdots}[\zeta_\zeta]}_n^\sigma = 0 \\ \underbrace{A_{abc\cdots}[\zeta_\zeta]}_n^\sigma = \frac{1}{n!}\underbrace{A_{\{abc\cdots\}[\zeta_\zeta]}}_n^\sigma, \delta^{ab}\underbrace{A_{abc\cdots}[\zeta_\zeta]}_n^\sigma = 0, \gamma^a(\zeta)\underbrace{A_{abc\cdots}[\zeta_\zeta]}_n^\sigma = 0 \end{array} \right.$$

采用数学归纳法并运用 $s = \frac{3}{2}$ 和 $s = 2$ 的推理技巧可容易并严格证明得到以上两个定理, 下面开始证明。

## 6.3 用数学归纳法严格证明以上两个定理

证明: 采用数学归纳法一起证明以上两个定理。

第一步:  $s = 1/2$ 时成立:

$$\left\{ \begin{array}{l} [\gamma^a(\zeta)\partial_a + m]\psi_{[\lambda_\zeta]}^\sigma = 0 \\ \psi_{[\lambda_\zeta]}^\sigma \text{除}\sigma\text{外全对称} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} [\gamma^d(\zeta)\partial_d + m]\underbrace{A_{ab\cdots}[\lambda_\zeta]}_0^\sigma = 0, \underbrace{A_{ab\cdots}[\lambda_\zeta]}_0^\sigma \text{除}\lambda_\zeta\sigma\text{外全对称} \\ \delta^{ab}\underbrace{A_{ab\cdots}[\lambda_\zeta]}_0^\sigma = 0, \gamma^a(\zeta)\underbrace{A_{ab\cdots}[\lambda_\zeta]}_0^\sigma = 0 \\ \psi_{[\lambda_\zeta]}^\sigma = \underbrace{A_{ab\cdots}[\lambda_\zeta]}_0^\sigma \end{array} \right.$$

第二步: 假设 $s = n - 1/2$ 时成立:

$$\left\{ \begin{array}{l} [\gamma^a(\zeta)\partial_a + m]\psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\cdots\eta_\zeta}_{2n-1}}^\sigma = 0 \\ \psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\cdots\eta_\zeta}_{2n-1}}^\sigma \text{除}\sigma\text{外全对称} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} [\gamma^d(\zeta)\partial_d + m]\underbrace{A_{ab\cdots}[\eta_\zeta]}_{n-1}^\sigma = 0, \underbrace{A_{ab\cdots}[\eta_\zeta]}_{n-1}^\sigma \text{除}\eta_\zeta\sigma\text{外全对称} \\ \delta^{ab}\underbrace{A_{ab\cdots}[\eta_\zeta]}_{n-1}^\sigma = 0, \gamma^a(\zeta)\underbrace{A_{ab\cdots}[\eta_\zeta]}_{n-1}^\sigma = 0 \\ \psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\cdots\eta_\zeta}_{2n-1}}^\sigma = \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a}_{n-2}\cdots\underbrace{A_{a\cdots}[\eta_\zeta]}_{n-1}^\sigma \end{array} \right.$$

第三步 $s = n$ 时,

$$1: \left\{ \begin{array}{l} [\gamma^a(\zeta)\partial_a + m]\psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\cdots\eta_\zeta\xi_\zeta}_{2n}}^\sigma = 0 \\ \psi_{\underbrace{[\lambda_\zeta]\mu_\zeta\cdots\eta_\zeta\xi_\zeta}_{2n}}^\sigma \text{除}\sigma\text{外全对称} \end{array} \right.$$

$$\begin{aligned}
& \Leftrightarrow \begin{cases} [\gamma^a(\zeta)\partial_a + m]\psi_{\underbrace{[\lambda_\zeta]\mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n}}^\sigma = 0, \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n}}^\sigma \text{除}_{\xi_\zeta}^\sigma \text{外全对称} \\ \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n}}^\sigma = \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \xi_\zeta \eta_\zeta}_{2n}}^\sigma \end{cases} \\
& \Leftrightarrow \begin{cases} [\gamma^d(\zeta)\partial_d + m]A_{\underbrace{ab \cdots [\eta_\zeta]\xi_\zeta}_{n-1}}^\sigma = 0, A_{\underbrace{ab \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma \text{除}_{\eta_\zeta \xi_\zeta}^\sigma \text{外全对称} \\ \delta^{ab}A_{\underbrace{ab \cdots [\eta_\zeta]\xi_\zeta}_{n-1}}^\sigma = 0, \gamma^a(\zeta)A_{\underbrace{ab \cdots [\eta_\zeta]\xi_\zeta}_{n-1}}^\sigma = 0 \\ \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n}}^\sigma = \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \cdots A_{\underbrace{a \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma, \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n}}^\sigma = \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \xi_\zeta \eta_\zeta}_{2n}}^\sigma \end{cases} \\
& \Leftrightarrow \begin{cases} [\gamma^d(\zeta)\partial_d + m]A_{\underbrace{ab \cdots [\eta_\zeta]\xi_\zeta}_{n-1}}^\sigma = 0, A_{\underbrace{ab \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma \text{除}_{\eta_\zeta \xi_\zeta}^\sigma \text{外全对称} \\ \delta^{ab}A_{\underbrace{ab \cdots [\eta_\zeta]\xi_\zeta}_{n-1}}^\sigma = 0, \gamma^a(\zeta)A_{\underbrace{ab \cdots [\eta_\zeta]\xi_\zeta}_{n-1}}^\sigma = 0 \\ A_{\underbrace{ab \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma = A_{\underbrace{ab \cdots \xi_\zeta \eta_\zeta}_{n-1}}^\sigma, \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n}}^\sigma = \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \cdots A_{\underbrace{a \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma \end{cases} \\
& \Leftrightarrow \begin{cases} (-\partial^d \partial_d + m^2)A_{\underbrace{ab \cdots c}_{n-1}}^\sigma = 0, A_{\underbrace{ab \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma \text{除}_{\eta_\zeta \xi_\zeta}^\sigma \text{外全对称} \\ \delta^{ab}A_{\underbrace{ab \cdots [\eta_\zeta]\xi_\zeta}_{n-1}}^\sigma = 0, \gamma^a(\zeta)A_{\underbrace{ab \cdots [\eta_\zeta]\xi_\zeta}_{n-1}}^\sigma = 0, \partial^c A_{\underbrace{ab \cdots c}_{n-1}}^\sigma = 0 \\ A_{\underbrace{ab \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma = \mathbb{X}_{\eta_\zeta \xi_\zeta}^c A_{\underbrace{ab \cdots c}_{n-1}}^\sigma, \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n}}^\sigma = \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \cdots A_{\underbrace{a \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma \end{cases} \\
& \Leftrightarrow \begin{cases} (-\partial^d \partial_d + m^2)A_{\underbrace{ab \cdots c}_{n-1}}^\sigma = 0, \partial^c A_{\underbrace{ab \cdots c}_{n-1}}^\sigma = 0, A_{\underbrace{ab \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma \text{除}_{\eta_\zeta \xi_\zeta}^\sigma \text{外全对称} \\ \delta^{ab}A_{\underbrace{ab \cdots c}_{n-1}}^\sigma = 0, A_{\underbrace{ab \cdots c}_{n-1}}^\sigma = A_{\underbrace{cb \cdots a}_{n-1}}^\sigma, \partial^a A_{\underbrace{ab \cdots c}_{n-1}}^\sigma = 0 \\ A_{\underbrace{ab \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma = \mathbb{X}_{\eta_\zeta \xi_\zeta}^c A_{\underbrace{ab \cdots c}_{n-1}}^\sigma, \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n}}^\sigma = \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \cdots A_{\underbrace{a \cdots \eta_\zeta \xi_\zeta}_{n-1}}^\sigma \end{cases} \\
& \Leftrightarrow \begin{cases} (-\partial^d \partial_d + m^2)A_{\underbrace{ab \cdots c}_{n-1}}^\sigma = 0, A_{\underbrace{ab \cdots c}_{n-1}}^\sigma \text{除}^\sigma \text{外全对称} \\ \delta^{ab}A_{\underbrace{ab \cdots c}_{n-1}}^\sigma = 0, \partial^a A_{\underbrace{ab \cdots c}_{n-1}}^\sigma = 0 \\ \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n}}^\sigma = \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \cdots \mathbb{X}_{\eta_\zeta \xi_\zeta}^c A_{\underbrace{a \cdots c}_{n-1}}^\sigma \end{cases}
\end{aligned}$$

第三步  $s = n + 1/2$  时,

$$\begin{aligned}
2: & \begin{cases} [\gamma^a(\zeta)\partial_a + m]\psi_{\underbrace{[\lambda_\zeta]\mu_\zeta \cdots \eta_\zeta \xi_\zeta \zeta_\zeta}_{2n+1}}^\sigma = 0 \\ \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta \zeta_\zeta}_{2n+1}}^\sigma \text{除}^\sigma \text{外全对称} \end{cases} \\
& \Leftrightarrow \begin{cases} [\gamma^a(\zeta)\partial_a + m]\psi_{\underbrace{[\lambda_\zeta]\mu_\zeta \cdots \eta_\zeta \xi_\zeta \zeta_\zeta}_{2n+1}}^\sigma = 0, \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta \zeta_\zeta}_{2n+1}}^\sigma \text{除}_{\zeta_\zeta}^\sigma \text{外全对称} \\ \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta \zeta_\zeta}_{2n+1}}^\sigma = \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \zeta_\zeta \xi_\zeta}_{2n+1}}^\sigma \end{cases}
\end{aligned}$$

$$\Leftrightarrow \begin{cases} (-\partial^d \partial_d + m^2) A_{ab \dots c \zeta_\zeta}^\sigma = 0, A_{ab \dots c \zeta_\zeta}^\sigma \text{除 } \zeta_\zeta^\sigma \text{外全对称} \\ \delta^{ab} A_{ab \dots c \zeta_\zeta}^\sigma = 0, \partial^a A_{ab \dots c \zeta_\zeta}^\sigma = 0, \psi_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta \zeta_\zeta}^\sigma = \overbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \dots \mathbb{X}_{\eta_\zeta \xi_\zeta}^c}^n A_{a \dots c \zeta_\zeta}^\sigma \\ \psi_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta \zeta_\zeta}^\sigma = \psi_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta \zeta_\zeta}^\sigma \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^d \partial_d + m^2) A_{ab \dots c \zeta_\zeta}^\sigma = 0, A_{ab \dots c \zeta_\zeta}^\sigma \text{除 } \zeta_\zeta^\sigma \text{外全对称} \\ \delta^{ab} A_{ab \dots c \zeta_\zeta}^\sigma = 0, \partial^a A_{ab \dots c \zeta_\zeta}^\sigma = 0, \psi_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta \zeta_\zeta}^\sigma = \overbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \dots \mathbb{X}_{\eta_\zeta \xi_\zeta}^c}^n A_{a \dots c \zeta_\zeta}^\sigma \\ [\gamma^d(\zeta) \partial_d + m] A_{ab \dots c [\zeta_\zeta]}^\sigma = 0, \gamma^a(\zeta) A_{ab \dots c [\zeta_\zeta]}^\sigma = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} [\gamma^d(\zeta) \partial_d + m] A_{ab \dots c [\zeta_\zeta]}^\sigma = 0, A_{ab \dots c \zeta_\zeta}^\sigma \text{除 } \zeta_\zeta^\sigma \text{外全对称} \\ \delta^{ab} A_{ab \dots c \zeta_\zeta}^\sigma = 0, \gamma^a(\zeta) A_{ab \dots c [\zeta_\zeta]}^\sigma = 0 \\ \psi_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta \zeta_\zeta}^\sigma = \overbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \dots \mathbb{X}_{\eta_\zeta \xi_\zeta}^c}^n A_{a \dots c \zeta_\zeta}^\sigma \end{cases}$$

此步证明了  $s = n$  和  $s = n + 1/2$  时命题都成立。

第四步：根据以上归纳法推理，命题成立，以上两个定理同时得证。  $\square$

## 6.4 对有质量自旋的Bargmann-Wigner方程的评述

从上可知，在平坦时空中Bargmann-Wigner方程在半整数自旋情形等价于Rarita-Schwinger方程<sup>[20]</sup>，在整数自旋情形等价于Klein-Gordon方程<sup>[22]</sup>，揭示了Bargmann-Wigner方程深刻丰富的物理内涵。但如果考虑一般的源项，并不能得到这个等价结果，只有满足一定条件的源项才能成立，并且只有自旋为  $s = \frac{1}{2}$  和  $s = 1$  才能带源项，对于自旋为  $s = \frac{3}{2}$  或以上，方程内在自恰性要求源项必须为零。另外在弯曲时空中，由于广义协变导数项的存在导致这个等价结论不再成立，这个情况不如Penrose旋量方程或自旋方程的性质好。总的来说，Penrose旋量方程或自旋方程更适合描写无质量粒子，而Bargmann-Wigner方程更适合描写有质量粒子。

## 6.5 Bargmann-Wigner自旋方程形式

缩减一对矢量指标：(右边是Penrose简记法，用 $\overset{P}{=}$ 表示。)

$$\frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \delta_b^a \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{B'_\zeta B'_\zeta}^b = \delta_{B'_\zeta}^{A'_\zeta} \delta_{B'_\zeta}^{A'_\zeta} \quad \delta_b^a \overset{P}{=} \delta_B^A \delta_{B'}^{A'} \quad (12.6)$$

$$\frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \delta^{ab} \frac{i\zeta}{\sqrt{2}}(\sigma, -i\zeta)_b^{B'_\zeta B'_\zeta} = \varepsilon^{AB} \varepsilon^{A'B'} \quad \delta^{ab} \overset{P}{=} \varepsilon^{AB} \varepsilon^{A'B'} \quad (12.7)$$

$$\frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{A'_\zeta A'_\zeta}^a \delta_{ab} \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{B'_\zeta B'_\zeta}^b = \varepsilon_{A_\zeta B_\zeta} \varepsilon_{A'_\zeta B'_\zeta} \quad \delta_{ab} \overset{P}{=} \varepsilon_{AB} \varepsilon_{A'B'} \quad (12.8)$$

引理6.5.1.  $\gamma^a_{\lambda_\zeta \mu_\zeta}$

$$= \begin{bmatrix} 0 & -i(\sigma, i\zeta)^a \\ i(\sigma, -i\zeta)_a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0_{A_\zeta B_\zeta}^a & -i(\sigma, i\zeta)_{A_\zeta B'_\zeta}^a \\ i(\sigma, -i\zeta)_{A'_\zeta B_\zeta} & 0_{A'_\zeta B'_\zeta} \end{bmatrix}$$

## 6.6 Bargmann-Wigner自旋方程形式

定义6.6.1.  $\begin{cases} S_{ab j_\zeta}^{k_\zeta}(e, s) := 2s N_{j_\zeta}^{\lambda_\zeta l_\zeta}(s, 3) S_{ab \lambda_\zeta \mu_\zeta}(e, \zeta) N_{\mu_\zeta l_\zeta}^{k_\zeta}(s, 3) \\ Z_{\rho_\zeta l_\zeta}^{a k_\zeta}(s, 3) := \gamma^a_{\rho_\zeta \lambda_\zeta} N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s, 3), \bar{Z}_{a j_\zeta}^{\rho_\zeta l_\zeta}(s, 3) := N_{j_\zeta}^{\lambda_\zeta l_\zeta}(s, 3) \gamma^a_{\lambda_\zeta \rho_\zeta} \end{cases}$

引理6.6.1.  $\bar{Z}_{a j_\zeta}^{\rho_\zeta l_\zeta}(s, 3) Z_{\rho_\zeta l_\zeta}^{b k_\zeta}(s, 3) = \frac{1}{s} [s \delta_a^b \delta_{j_\zeta}^{k_\zeta} + i S_a^b j_\zeta^{k_\zeta}(e, s)]$

$$\begin{aligned}
& \text{证明: } \bar{Z}_{aj_\zeta}^{\rho_\zeta l_\zeta}(s, 3) Z_{\rho_\zeta l_\zeta}^{bk_\zeta}(s, 3) \\
&= N_{j_\zeta}^{\lambda_\zeta l_\zeta}(s, 3) \gamma^a_{\lambda_\zeta} \rho_\zeta \gamma^b_{\rho_\zeta} \mu_\zeta N_{\mu_\zeta l_\zeta}^{k_\zeta}(s, 3) \\
&= N_{j_\zeta}^{\lambda_\zeta l_\zeta}(s, 3) (\gamma^a \gamma^b)_{\lambda_\zeta \mu_\zeta} N_{\mu_\zeta l_\zeta}^{k_\zeta}(s, 3) \\
&= N_{j_\zeta}^{\lambda_\zeta l_\zeta}(s, 3) [\delta_a^b + 2i S_a^b(e, \zeta)]_{\lambda_\zeta \mu_\zeta} N_{\mu_\zeta l_\zeta}^{k_\zeta}(s, 3) \\
&= \delta_a^b \delta_{j_\zeta}^{k_\zeta} + N_{j_\zeta}^{\lambda_\zeta l_\zeta}(s, 3) 2i S_a^b \lambda_\zeta \mu_\zeta(e, \zeta) N_{\mu_\zeta l_\zeta}^{k_\zeta}(s, 3) \\
&= \frac{1}{s} [s \delta_a^b \delta_{j_\zeta}^{k_\zeta} + i S_a^b \lambda_\zeta \mu_\zeta(e, s)] \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } Z_{\rho'_\zeta l'_\zeta}^{ak_\zeta}(s, 3) \bar{Z}_{ak_\zeta}^{\rho_\zeta l_\zeta}(s, 3) \\
&= \gamma^a_{\rho'_\zeta} \lambda'_\zeta N_{\lambda'_\zeta l'_\zeta}^{k_\zeta}(s, 3) N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s, 3) \gamma_{a\lambda_\zeta} \rho_\zeta \\
&= \gamma^a_{\rho'_\zeta} \lambda'_\zeta \gamma_{a\lambda_\zeta} \rho_\zeta N_{\lambda'_\zeta l'_\zeta}^{k_\zeta}(s, 3) N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s, 3) \quad \square
\end{aligned}$$

$$\text{定理6.6.1. } (\gamma^a \partial_a + m)_{\rho_\zeta} \lambda_\zeta \psi_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}_{2s} \dots} = 0 \Rightarrow \begin{cases} [\gamma^a(s) \partial_a + sm] \psi(e, s) = 0 \\ [s \partial_a + i S_{ab}(e, s) \partial^b] \psi(e, s) = -m \gamma_a(s) \psi(e, s) \end{cases}$$

$$\begin{aligned}
& \text{证明: } (\gamma^a \partial_a + m)_{\rho_\zeta} \lambda_\zeta \psi_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}_{2s} \dots} = 0 \\
&\Leftrightarrow (\gamma^a \partial_a + m)_{\rho_\zeta} \lambda_\zeta \Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s, 3) \psi_{k_\zeta}(e, s) = 0 \\
&\Leftrightarrow (\gamma^a \partial_a + m)_{\rho_\zeta} \lambda_\zeta N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s, 3) \Gamma_{\underbrace{\mu_\zeta \eta_\zeta \dots}_{2s-1}}^{l_\zeta}(s - \frac{1}{2}, 3) \psi_{k_\zeta}(e, s) = 0 \\
&\Leftrightarrow (\gamma^a \partial_a + m)_{\rho_\zeta} \lambda_\zeta N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s, 3) \psi_{k_\zeta}(e, s) = 0 \\
&\Rightarrow N_{j_\zeta}^{\rho_\zeta l_\zeta}(s, 3) (\gamma^a \partial_a + m)_{\rho_\zeta} \lambda_\zeta N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s, 3) \psi_{k_\zeta}(e, s) = 0 \\
&\Leftrightarrow [\gamma^a(s) \partial_a + sm]_{j_\zeta}^{k_\zeta} \psi_{k_\zeta}(e, s) = 0 \\
&\Leftrightarrow [\gamma^a(s) \partial_a + sm] \psi(e, s) = 0 \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } (\gamma^a \partial_a + m)_{\rho_\zeta} \lambda_\zeta \psi_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}_{2s} \dots} = 0 \\
&\Leftrightarrow (\gamma^a \partial_a + m)_{\rho_\zeta} \lambda_\zeta \Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s, 3) \psi_{k_\zeta}(e, s) = 0 \\
&\Leftrightarrow (\gamma^a \partial_a + m)_{\rho_\zeta} \lambda_\zeta N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s, 3) \Gamma_{\underbrace{\mu_\zeta \eta_\zeta \dots}_{2s-1}}^{l_\zeta}(s - \frac{1}{2}, 3) \psi_{k_\zeta}(e, s) = 0 \\
&\Leftrightarrow (\gamma^a \partial_a + m)_{\rho_\zeta} \lambda_\zeta N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s, 3) \psi_{k_\zeta}(e, s) = 0 \\
&\Leftrightarrow \gamma^a_{\rho_\zeta} \lambda_\zeta N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s, 3) \partial_a \psi_{k_\zeta}(e, s) = -m N_{\rho_\zeta l_\zeta}^{k_\zeta}(s, 3) \psi_{k_\zeta}(e, s) \\
&\Leftrightarrow Z_{\rho_\zeta l_\zeta}^{ak_\zeta}(s, 3) \partial_a \psi_{k_\zeta}(e, s) = -m N_{\rho_\zeta l_\zeta}^{k_\zeta}(s, 3) \psi_{k_\zeta}(e, s) \\
&\Rightarrow \bar{Z}_{aj_\zeta}^{\rho_\zeta l_\zeta}(s, 3) Z_{\rho_\zeta l_\zeta}^{bk_\zeta}(s, 3) \partial_b \psi_{k_\zeta}(e, s) = -m \bar{Z}_{aj_\zeta}^{\rho_\zeta l_\zeta}(s, 3) N_{\rho_\zeta l_\zeta}^{k_\zeta}(s, 3) \psi_{k_\zeta}(e, s) \\
&\Leftrightarrow [s \delta_{ab} \delta_{j_\zeta}^{k_\zeta} + i S_{abj_\zeta}^{k_\zeta}(e, s)] \partial^b \psi_{k_\zeta}(e, s) = -sm \bar{Z}_{aj_\zeta}^{\rho_\zeta l_\zeta}(s, 3) N_{\rho_\zeta l_\zeta}^{k_\zeta}(s, 3) \psi_{k_\zeta}(e, s) \\
&\Leftrightarrow [s \partial_a + i S_{ab}(e, s) \partial^b]_{j_\zeta}^{k_\zeta} \psi_{k_\zeta}(e, s) = -sm \bar{Z}_{aj_\zeta}^{\rho_\zeta l_\zeta}(s, 3) N_{\rho_\zeta l_\zeta}^{k_\zeta}(s, 3) \psi_{k_\zeta}(e, s) \\
&\Leftrightarrow [s \partial_a + i S_{ab}(e, s) \partial^b] \psi(e, s) = -m \gamma_a(s) \psi(e, s) \quad \square
\end{aligned}$$

$$\text{推论6.6.1. } \begin{cases} [\gamma^a(s) \partial_a + sm] \psi(e, s) = 0 \\ [s \partial_a + i S_{ab}(e, s) \partial^b] \psi(e, s) = -m \gamma_a(s) \psi(e, s) \end{cases} \Leftrightarrow \begin{cases} [\gamma^a(s) \partial_a + sm] \psi(e, s) = 0 \\ \frac{1}{s} \gamma_a(s) \gamma_b(s) \partial^b \psi(e, s) = [s \delta_{ab} + i S_{ab}(e, s)] \partial^b \psi(e, s) \end{cases}$$

## 7 反对称性的Dirac方程 [4]

### 7.1 有质量反对称性Dirac方程的分析

$$\text{定理7.1.1. } [\gamma^c(\zeta) \partial_c + m] F_{[\lambda_\zeta \mu_\zeta]} = J, F_{\lambda_\zeta \mu_\zeta} = -F_{\mu_\zeta \lambda_\zeta}$$

$$\begin{aligned}
& \Leftrightarrow \begin{cases} [-2m S_{ab}(e, \zeta) \partial^a \mathbf{A}^b - \gamma_a(\zeta) (im^2 \mathbf{A}^a + \partial^a \Phi)] C + [m(\Phi + i \partial_a \mathbf{A}^a) + m \gamma_5(\zeta) \phi - \gamma_a(\zeta) \gamma_5(\zeta) \partial^a \phi] C = -\gamma_5(\zeta) J \\ F = -[\phi + im \gamma_a(\zeta) \gamma_5(\zeta) \mathbf{A}^a + \gamma_5(\zeta) \Phi] C \end{cases}
\end{aligned}$$

证明:  $[\gamma^a(\varsigma)\partial_a + m]F_{[\lambda_\varsigma\mu_\varsigma]} = J, F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma}$

$$\begin{aligned} &\Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m]F_{[\lambda_\varsigma\mu_\varsigma]} = J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m][C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi] = -J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} [\gamma^b(\varsigma)\partial_b + m][\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi] = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} m\phi + \gamma_a(\varsigma)\partial^a\phi + im\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b + \gamma_a(\varsigma)\gamma_5(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi) + m\gamma_5(\varsigma)\Phi = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} m\phi + \gamma_a(\varsigma)\partial^a\phi - 2mS_{ab}(e, \varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b + \gamma_a(\varsigma)\gamma_5(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi) + m\gamma_5(\varsigma)(\Phi + i\partial_a\mathbf{A}^a) = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} m\gamma_5(\varsigma)\phi - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi - 2mS_{ab}(e, \varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi) + m(\Phi + i\partial_a\mathbf{A}^a) = -\gamma_5(\varsigma)J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} [-2mS_{ab}(e, \varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi)]C + [m(\Phi + i\partial_a\mathbf{A}^a) + m\gamma_5(\varsigma)\phi - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = -\gamma_5(\varsigma)J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \quad \square \end{aligned}$$

## 7.2 无质量反对称性Dirac方程的分析

定理7.2.1.  $\gamma^c(\varsigma)\partial_c F_{[\lambda_\varsigma\mu_\varsigma]} = J, F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma}$

$$\Leftrightarrow \begin{cases} [-2mS_{ab}(e, \varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)\partial^a\Phi]C + [im\partial_a\mathbf{A}^a - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = -\gamma_5(\varsigma)J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

证明:  $\gamma^a(\varsigma)\partial_a F_{[\lambda_\varsigma\mu_\varsigma]} = J, F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma}$

$$\begin{aligned} &\Leftrightarrow \begin{cases} \gamma^b(\varsigma)\partial_b F_{[\lambda_\varsigma\mu_\varsigma]} = J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} \gamma^b(\varsigma)\partial_b[C\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a + \gamma_5(\varsigma)C\Phi] = -J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} \gamma^b(\varsigma)\partial_b[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi] = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} \gamma_a(\varsigma)\partial^a\phi + im\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b + \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\Phi = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} \gamma_a(\varsigma)\partial^a\phi - 2mS_{ab}(e, \varsigma)\gamma_5(\varsigma)\partial^a\mathbf{A}^b + \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\Phi + im\gamma_5(\varsigma)\partial_a\mathbf{A}^a = -J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} -\gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi - 2mS_{ab}(e, \varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)\partial^a\Phi + im\partial_a\mathbf{A}^a = -\gamma_5(\varsigma)J\bar{C} \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \\ &\Leftrightarrow \begin{cases} [-2mS_{ab}(e, \varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)\partial^a\Phi]C + [im\partial_a\mathbf{A}^a - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = -\gamma_5(\varsigma)J \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases} \quad \square \end{aligned}$$

### 7.3 有质量膺标量场方程

$$\text{定理7.3.1. } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]F_{[\lambda_\varsigma\mu_\varsigma]} = \frac{j}{m}\gamma_5(\varsigma)C \\ F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma} \end{cases} \Leftrightarrow \begin{cases} (-\partial^a\partial_a + m^2)\Phi = -j \\ F = \frac{1}{m}[\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi \end{cases}$$

$$\text{证明: } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]F_{[\lambda_\varsigma\mu_\varsigma]} = \frac{j}{m}\gamma_5(\varsigma)C \\ F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} [-2mS_{ab}(e, \varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi)]C + [m(\Phi + i\partial_a\mathbf{A}^a) + m\gamma_5(\varsigma)\phi - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = -\frac{j}{m}C \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, (im^2\mathbf{A}^a + \partial^a\Phi) = 0, \phi = 0, \partial^a\phi = 0 \\ m^2(\Phi + i\partial_a\mathbf{A}^a) = -j \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^a\partial_a + m^2)\Phi = -j \\ \mathbf{A}^a = im^{-2}\partial^a\Phi, \phi = 0 \\ F = \frac{1}{m}[\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^a\partial_a + m^2)\Phi = -j \\ F = \frac{1}{m}[\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi \end{cases} \quad \square$$

$$\text{推论7.3.1. } [\gamma^a(\varsigma)\partial_a + m][\gamma^a(\varsigma)\partial_a - m]\gamma_5(\varsigma)C\Phi = j\gamma_5(\varsigma)C \Leftrightarrow (-\partial^a\partial_a + m^2)\Phi = -j$$

### 7.4 无质量膺标量场方程

$$\text{推论7.4.1. } \gamma^a(\varsigma)\partial_a[\gamma^b(\varsigma)\partial_b\gamma_5(\varsigma)C\Phi] = j\gamma_5(\varsigma)C \Leftrightarrow \gamma^a(\varsigma)\partial_a[\gamma^b(\varsigma)\partial_b C\Phi] = jC \Leftrightarrow \partial^a\partial_a\Phi = j$$

$$\text{推论7.4.2. } \gamma_a(\varsigma)\partial^a[\gamma_5(\varsigma)C\Phi] = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow \gamma_a(\varsigma)\partial^a[C\Phi] = \gamma_a(\varsigma)CJ^a \Leftrightarrow \partial^a\Phi = J^a$$

### 7.5 有质量膺矢量场方程

$$\text{定理7.5.1. } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]F_{[\lambda_\varsigma\mu_\varsigma]} = i\gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \\ F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma} \end{cases} \Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a \\ F = i[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a \end{cases}$$

$$\text{证明: } \begin{cases} [\gamma^a(\varsigma)\partial_a + m]F_{[\lambda_\varsigma\mu_\varsigma]} = i\gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \\ F_{\lambda_\varsigma\mu_\varsigma} = -F_{\mu_\varsigma\lambda_\varsigma} \end{cases}$$

$$\Leftrightarrow \begin{cases} [-2mS_{ab}(e, \varsigma)\partial^a\mathbf{A}^b - \gamma_a(\varsigma)(im^2\mathbf{A}^a + \partial^a\Phi)]C + [m(\Phi + i\partial_a\mathbf{A}^a) + m\gamma_5(\varsigma)\phi - \gamma_a(\varsigma)\gamma_5(\varsigma)\partial^a\phi]C = i\gamma_a(\varsigma)CJ^a \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \Phi = -i\partial_a\mathbf{A}^a, \phi = 0, \partial^a\phi = 0 \\ (im^2\mathbf{A}^a + \partial^a\Phi) = -iJ^a \\ F = -[\phi + im\gamma_a(\varsigma)\gamma_5(\varsigma)\mathbf{A}^a + \gamma_5(\varsigma)\Phi]C \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a \\ \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \Phi = -i\partial_a\mathbf{A}^a, \phi = 0 \\ F = i[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a \end{cases}$$

$$\Leftrightarrow \begin{cases} (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a \\ F = i[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a \end{cases} \quad \square$$

**推论7.5.1.**  $[\gamma_b(\varsigma)\partial^b + m][\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a$

**证明:**  $[\gamma_b(\varsigma)\partial^b + m][\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a$   
 $\Leftrightarrow -2mS_{ab}(e, \varsigma)\partial^a\mathbf{A}^b - i\gamma_a(\varsigma)(m^2\mathbf{A}^a - \partial^a\partial_b\mathbf{A}^b) = i\gamma_a(\varsigma)J^a$   
 $\Leftrightarrow -\partial^a\partial_b\mathbf{A}^b + m^2\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a$   
 $\Leftrightarrow (-\partial^b\partial_b + m^2)\mathbf{A}^a = -J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a$  □

## 7.6 无质量膺矢量场方程

**推论7.6.1.**  $\gamma_b(\varsigma)\partial^b[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow \partial^b\partial_b\mathbf{A}^a = J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \partial_a\mathbf{A}^a = 0$

**证明:**  $\gamma_b(\varsigma)\partial^b[\partial_a - m\gamma_a(\varsigma)]\gamma_5(\varsigma)C\mathbf{A}^a = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a$   
 $\Leftrightarrow [-2mS_{ab}(e, \varsigma)\partial^a\mathbf{A}^b + i\gamma_a(\varsigma)\partial^a\partial_b\mathbf{A}^b] + im\partial_a\mathbf{A}^a = i\gamma_a(\varsigma)J^a$   
 $\Leftrightarrow \partial^a\partial_b\mathbf{A}^b = J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \partial_a\mathbf{A}^a = 0$   
 $\Leftrightarrow \partial^b\partial_b\mathbf{A}^a = J^a, \partial^a\mathbf{A}^b = \partial^b\mathbf{A}^a, \partial_a\mathbf{A}^a = 0$  □

**推论7.6.2.**  $\gamma_a(\varsigma)\partial^a[\gamma_5(\varsigma)C\partial_b\mathbf{A}^b] = \gamma_a(\varsigma)\gamma_5(\varsigma)CJ^a \Leftrightarrow \gamma_a(\varsigma)\partial^a[C\partial_b\mathbf{A}^b] = \gamma_a(\varsigma)CJ^a \Leftrightarrow \partial^a\partial_b\mathbf{A}^b = J^a$

**推论7.6.3.**  $\gamma_b(\varsigma)\partial^b[\gamma_a(\varsigma)\gamma_5(\varsigma)C\mathbf{A}^a] = \gamma_5(\varsigma)[jC + J^{ab}S_{ab}(e, \varsigma)]$   
 $\Leftrightarrow \gamma_b(\varsigma)\partial^b[\gamma_a(\varsigma)C\mathbf{A}^a] = jC + J^{ab}S_{ab}(e, \varsigma)$   
 $\Leftrightarrow \partial^a\mathbf{A}^b - \partial^b\mathbf{A}^a = J^{ab}, \partial_a\mathbf{A}^a = j$



# 第十三章 高级表象变换技术

自我评述：本章是对前面章节内容的进一步充实和深化，我换成从表象变换的角度去研究同一个物理问题，为后续的各种自旋粒子洛伦兹变换多项式表示的证明提供了一个数学基础，并同时用表象变换技术提出了全新的粒子耦合理论。

## 1 高级表象变换技巧

### 1.1 表象变换与常数不变张量

定理1.1.1.  $\psi' = S\psi \Rightarrow \Lambda(\psi') = S\Lambda(\psi)S^{-1} \Leftrightarrow S = \Lambda(\psi')S\Lambda^{-1}(\psi) \Leftrightarrow S^{-1} = \Lambda(\psi)S^{-1}\Lambda^{-1}(\psi')$

所以表象变换就是一个二阶常数不变张量。写成分量形式如下：

推论1.1.1.  $\psi'^{\alpha'} = S^{\alpha'}_{\alpha}\psi^{\alpha}, \psi^{\alpha} = S^{-1\alpha}_{\alpha'}\psi'^{\alpha'}$

### 1.2 表象变换 $\tilde{S}(s)$ 和常数矩阵 $\Sigma(s)$ 的引入及其性质

#### 1.2.1 表象变换矩阵 $\tilde{S}(s)$ 的引入

定义1.2.1.  $\tilde{S}(s) := \begin{bmatrix} \bar{N}(s) \\ \bar{X}(s) \end{bmatrix}, \tilde{S}^+(s) = [N(s), X(s)]$

推论1.2.1.  $\begin{cases} \tilde{S}^+(s)\tilde{S}(s) = I_{4s} \Leftrightarrow N(s)\bar{N}(s) + X(s)\bar{X}(s) = I_{4s} \\ \tilde{S}(s)\tilde{S}^+(s) = I_{4s} \Leftrightarrow \bar{N}(s)N(s) = I_{2s+1}, \bar{X}(s)X(s) = I_{2s-1}, \bar{N}(s)X(s) = 0, \bar{X}(s)N(s) = 0 \end{cases}$

推论1.2.2.  $\begin{cases} \tilde{S}(s)[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})] = \begin{bmatrix} \sigma(s) & 0 \\ 0 & \sigma(s-1) \end{bmatrix} \tilde{S}(s) \\ [\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})]\tilde{S}^+(s) = \tilde{S}^+(s) \begin{bmatrix} \sigma(s) & 0 \\ 0 & \sigma(s-1) \end{bmatrix} \end{cases}$

推论1.2.3.  $\tilde{S}(s)[\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})]\tilde{S}^+(s) = \begin{bmatrix} \sigma(s) & 0 \\ 0 & \sigma(s-1) \end{bmatrix}$

推论1.2.4.  $\tilde{S}(s) = \frac{1}{\sqrt{2s}} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{2s-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{2s-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2s-1} & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2s} & 0 & 0 \\ 0 & -\sqrt{2s-1} & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2s-2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & \sqrt{2s-1} & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} N_{1\zeta}(s) \sqcup N_{2\zeta}(s) \\ \sqrt{1 - \frac{1}{2s}} \cdot \bar{N}_{2\zeta}(s - \frac{1}{2}) \sqcup [-\bar{N}_{1\zeta}(s - \frac{1}{2})] \end{bmatrix}$

推论1.2.5.  $\tilde{S}^+(s) = \frac{1}{\sqrt{2s}} \begin{bmatrix} \sqrt{2s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 & 0 & 0 & -\sqrt{2s-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2s-1} & 0 & 0 & 0 & 0 & 0 & \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & -\sqrt{2s-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2s-2} & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2s-1} & 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & \sqrt{2s-1} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2s} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

#### 1.2.2 表象变换矩阵 $\tilde{S}(s)$ 的几个具体表示

推论1.2.6.

$\tilde{S}(\frac{1}{2}, 1, \dots) = \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{1} \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{1} & \sqrt{1} & 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & -\sqrt{2} & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1} & \sqrt{2} & 0 \end{bmatrix}, \frac{1}{\sqrt{4}} \begin{bmatrix} \sqrt{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & \sqrt{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{4} \\ 0 & -\sqrt{3} & \sqrt{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & \sqrt{3} & 0 \end{bmatrix}, \dots$

推论1.2.7.

$$\tilde{S}^+(\frac{1}{2}, 1, \dots) = \begin{bmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{1} \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & -\sqrt{1} \\ 0 & \sqrt{1} & 0 & \sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{1} \\ 0 & 0 & \sqrt{1} & 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{4}} \begin{bmatrix} \sqrt{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & -\sqrt{1} \\ 0 & 0 & 0 & \sqrt{1} & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & \sqrt{4} & 0 & 0 & 0 \end{bmatrix}, \dots$$

### 1.2.3 常数矩阵 $O(s)$ 的引入及具体表示

$$\text{定义1.2.2. } \begin{cases} X_{m_\zeta}^{A_\zeta l_\zeta}(s) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta}(\frac{1}{2}) N_{B_\zeta l_\zeta}^{m_\zeta}(s) := \frac{1}{2s} O^{\alpha_\zeta l_\zeta n_\zeta}(s) \Leftrightarrow X^{A_\zeta}(s) \sigma^{\alpha_\zeta} A_\zeta^{B_\zeta}(\frac{1}{2}) \bar{N}_{B_\zeta}(s) = \frac{1}{2s} O(s) \\ \bar{X}(s) \sigma(\frac{1}{2}) \otimes I_{2s} N(s) = \frac{1}{2s} O(s) \Leftrightarrow \bar{N}(s) \sigma(\frac{1}{2}) \otimes I_{2s} X(s) = \frac{1}{2s} O^+(s) \end{cases}$$

$$\text{定理1.2.1. } \begin{cases} O^+(s) \cdot O(s) = s(2s-1) I_{2s+1}, O(s) \cdot O^+(s) = s(2s+1) I_{2s-1} \\ O(s) \cdot \sigma(s) = \sigma(s-1) \cdot O(s), \sigma(s) \cdot O^+(s) = O^+(s) \cdot \sigma(s-1) \end{cases}$$

$$\text{证明: } \tilde{S}(s) \sigma(\frac{1}{2}) \otimes I_{2s} \tilde{S}^+(s) \cdot \tilde{S}(s) \sigma(\frac{1}{2}) \otimes I_{2s} \tilde{S}^+(s) = \frac{1}{2s} \begin{bmatrix} \sigma(s) & O^+(s) \\ O(s) & -\sigma(s-1) \end{bmatrix} \cdot \frac{1}{2s} \begin{bmatrix} \sigma(s) & O^+(s) \\ O(s) & -\sigma(s-1) \end{bmatrix}$$

$$\Leftrightarrow \frac{3}{4} = \frac{1}{4s^2} \begin{bmatrix} \sigma^2(s) + O^+(s) \cdot O(s) & \sigma(s) \cdot O^+(s) - O^+(s) \cdot \sigma(s-1) \\ O(s) \cdot \sigma(s) - \sigma(s-1) \cdot O(s) & O(s) \cdot O^+(s) + \sigma^2(s-1) \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} O^+(s) \cdot O(s) = s(2s-1) I_{2s+1}, O(s) \cdot O^+(s) = s(2s+1) I_{2s-1} \\ O(s) \cdot \sigma(s) = \sigma(s-1) \cdot O(s), \sigma(s) \cdot O^+(s) = O^+(s) \cdot \sigma(s-1) \end{cases} \quad \square$$

$$\text{推论1.2.8. } O_x(s) = -\sqrt{s(s-\frac{1}{2})} [\bar{N}_{1_\zeta}(s-\frac{1}{2}) \bar{N}_{1_\zeta}(s) - \bar{N}_{2_\zeta}(s-\frac{1}{2}) \bar{N}_{2_\zeta}(s)]$$

$$= \frac{1}{2} \begin{bmatrix} -\sqrt{2s \cdot (2s-1)} & 0 & \sqrt{2 \cdot 1} & 0 & 0 & 0 \\ 0 & -\sqrt{(2s-1) \cdot (2s-2)} & 0 & \sqrt{3 \cdot 2} & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & -\sqrt{2 \cdot 1} & 0 & \sqrt{2s \cdot (2s-1)} \end{bmatrix}$$

$$\text{推论1.2.9. } O_y(s) = -i \sqrt{s(s-\frac{1}{2})} [\bar{N}_{1_\zeta}(s-\frac{1}{2}) \bar{N}_{1_\zeta}(s) + \bar{N}_{2_\zeta}(s-\frac{1}{2}) \bar{N}_{2_\zeta}(s)]$$

$$= \frac{i}{2} \begin{bmatrix} -\sqrt{2s \cdot (2s-1)} & 0 & -\sqrt{2 \cdot 1} & 0 & 0 & 0 \\ 0 & -\sqrt{(2s-1) \cdot (2s-2)} & 0 & -\sqrt{3 \cdot 2} & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & -\sqrt{2 \cdot 1} & 0 & -\sqrt{2s \cdot (2s-1)} \end{bmatrix}$$

$$\text{推论1.2.10. } O_z(s) = \sqrt{s(s-\frac{1}{2})} [\bar{N}_{1_\zeta}(s-\frac{1}{2}) \bar{N}_{2_\zeta}(s) + \bar{N}_{2_\zeta}(s-\frac{1}{2}) \bar{N}_{1_\zeta}(s)]$$

$$= \begin{bmatrix} 0 & \sqrt{1 \cdot (2s-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2 \cdot (2s-2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{(2s-1) \cdot 1} & 0 \end{bmatrix}, \bar{N}_{1_\zeta}(s-\frac{1}{2}) \bar{N}_{2_\zeta}(s) = \bar{N}_{2_\zeta}(s-\frac{1}{2}) \bar{N}_{1_\zeta}(s)$$

$$\text{推论1.2.11. } O(2) = \frac{1}{2} \begin{bmatrix} -\sqrt{4 \cdot 3} & 0 & \sqrt{2 \cdot 1} & 0 & 0 & 0 \\ 0 & -\sqrt{3 \cdot 2} & 0 & \sqrt{3 \cdot 2} & 0 & 0 \\ 0 & 0 & -\sqrt{2 \cdot 1} & 0 & \sqrt{4 \cdot 3} & 0 \end{bmatrix}, \frac{i}{2} \begin{bmatrix} -\sqrt{4 \cdot 3} & 0 & -\sqrt{2 \cdot 1} & 0 & 0 & 0 \\ 0 & -\sqrt{3 \cdot 2} & 0 & -\sqrt{3 \cdot 2} & 0 & 0 \\ 0 & 0 & -\sqrt{2 \cdot 1} & 0 & -\sqrt{4 \cdot 3} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{1 \cdot 3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2 \cdot 2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3 \cdot 1} & 0 & 0 \end{bmatrix}$$

### 1.2.4 常数矩阵 $\Sigma(s)$ 的引入及具体表示

$$\text{定义1.2.3. } \Sigma(s) := \tilde{S}(s) \sigma(\frac{1}{2}) \otimes I_{2s} \tilde{S}^+(s) = \frac{1}{2s} \begin{bmatrix} \sigma(s) & O^+(s) \\ O(s) & -\sigma(s-1) \end{bmatrix}, \tilde{S}(s) = \begin{bmatrix} \bar{N}(s) \\ \bar{X}(s) \end{bmatrix}$$

$$\text{推论1.2.12. } \Sigma(1) = \frac{1}{4} \left\{ \begin{bmatrix} 0 & \sqrt{1 \cdot 2} & 0 & -\sqrt{2 \cdot 1} \\ \sqrt{1 \cdot 2} & 0 & \sqrt{2 \cdot 1} & 0 \\ 0 & \sqrt{2 \cdot 1} & 0 & \sqrt{2 \cdot 1} \\ -\sqrt{2 \cdot 1} & 0 & \sqrt{2 \cdot 1} & 0 \end{bmatrix}, i \begin{bmatrix} 0 & -\sqrt{1 \cdot 2} & 0 & \sqrt{2 \cdot 1} \\ \sqrt{1 \cdot 2} & 0 & -\sqrt{2 \cdot 1} & 0 \\ 0 & \sqrt{2 \cdot 1} & 0 & \sqrt{2 \cdot 1} \\ -\sqrt{2 \cdot 1} & 0 & -\sqrt{2 \cdot 1} & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{1 \cdot 1} \\ 0 & 0 & -2 & 0 \\ 0 & 2\sqrt{1 \cdot 1} & 0 & 0 \end{bmatrix} \right\}$$

$$\text{推论1.2.13. } O(1) = \frac{1}{2} \left\{ \begin{bmatrix} -\sqrt{2 \cdot 1} & 0 & \sqrt{2 \cdot 1} \\ 0 & \sqrt{2 \cdot 1} & 0 \end{bmatrix}, i \begin{bmatrix} -\sqrt{2 \cdot 1} & 0 & -\sqrt{2 \cdot 1} \\ 0 & \sqrt{2 \cdot 1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2\sqrt{1 \cdot 1} & 0 \end{bmatrix} \right\}$$

$$\text{推论1.2.14. } \Sigma(\frac{3}{2})$$

$$= \frac{1}{6} \left\{ \begin{bmatrix} 0 & \sqrt{1 \cdot 3} & 0 & 0 & -\sqrt{3 \cdot 2} & 0 \\ \sqrt{1 \cdot 3} & 0 & \sqrt{2 \cdot 2} & 0 & 0 & -\sqrt{2 \cdot 1} \\ 0 & \sqrt{2 \cdot 2} & 0 & \sqrt{3 \cdot 1} & \sqrt{2 \cdot 1} & 0 \\ 0 & 0 & \sqrt{3 \cdot 1} & 0 & 0 & \sqrt{3 \cdot 2} \\ -\sqrt{3 \cdot 2} & 0 & \sqrt{2 \cdot 1} & 0 & 0 & -\sqrt{1 \cdot 1} \\ 0 & -\sqrt{2 \cdot 1} & 0 & \sqrt{3 \cdot 2} & -\sqrt{1 \cdot 1} & 0 \end{bmatrix}, i \begin{bmatrix} 0 & -\sqrt{1 \cdot 3} & 0 & 0 & \sqrt{3 \cdot 2} & 0 \\ \sqrt{1 \cdot 3} & 0 & -\sqrt{2 \cdot 2} & 0 & 0 & \sqrt{2 \cdot 1} \\ 0 & \sqrt{2 \cdot 2} & 0 & -\sqrt{3 \cdot 1} & \sqrt{2 \cdot 1} & 0 \\ 0 & 0 & \sqrt{3 \cdot 1} & 0 & 0 & \sqrt{3 \cdot 2} \\ -\sqrt{3 \cdot 2} & 0 & -\sqrt{2 \cdot 1} & 0 & 0 & \sqrt{1 \cdot 1} \\ 0 & -\sqrt{2 \cdot 1} & 0 & -\sqrt{3 \cdot 2} & -\sqrt{1 \cdot 1} & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2\sqrt{1 \cdot 2} & 0 \\ 0 & 0 & -1 & 0 & 0 & 2\sqrt{2 \cdot 1} \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 2\sqrt{1 \cdot 2} & 0 & 0 & -1 & 0 \\ 0 & 0 & 2\sqrt{2 \cdot 1} & 0 & 0 & 1 \end{bmatrix} \right\}$$

推论1.2.15.

$$\Sigma(2) = \frac{1}{8} \left\{ \begin{array}{l} \begin{bmatrix} 0 & \sqrt{1.4} & 0 & 0 & 0 & -\sqrt{4.3} & 0 & 0 \\ \sqrt{1.4} & 0 & \sqrt{2.3} & 0 & 0 & 0 & -\sqrt{3.2} & 0 \\ 0 & \sqrt{2.3} & 0 & \sqrt{3.2} & 0 & \sqrt{2.1} & 0 & -\sqrt{2.1} \\ 0 & 0 & \sqrt{3.2} & 0 & \sqrt{4.1} & 0 & \sqrt{3.2} & 0 \\ 0 & 0 & 0 & \sqrt{4.1} & 0 & 0 & 0 & \sqrt{4.3} \\ -\sqrt{4.3} & 0 & \sqrt{2.1} & 0 & 0 & 0 & -\sqrt{1.2} & 0 \\ 0 & -\sqrt{3.2} & 0 & \sqrt{3.2} & 0 & -\sqrt{1.2} & 0 & -\sqrt{2.1} \\ 0 & 0 & -\sqrt{2.1} & 0 & \sqrt{4.3} & 0 & -\sqrt{2.1} & 0 \end{bmatrix}, i \begin{bmatrix} 0 & -\sqrt{1.4} & 0 & 0 & 0 & \sqrt{4.3} & 0 & 0 \\ \sqrt{1.4} & 0 & -\sqrt{2.3} & 0 & 0 & 0 & \sqrt{3.2} & 0 \\ 0 & \sqrt{2.3} & 0 & -\sqrt{3.2} & 0 & \sqrt{2.1} & 0 & \sqrt{2.1} \\ 0 & 0 & \sqrt{3.2} & 0 & -\sqrt{4.1} & 0 & \sqrt{3.2} & 0 \\ 0 & 0 & 0 & \sqrt{4.1} & 0 & 0 & 0 & \sqrt{4.3} \\ -\sqrt{4.3} & 0 & -\sqrt{2.1} & 0 & 0 & 0 & 0 & \sqrt{1.2} \\ 0 & -\sqrt{3.2} & 0 & -\sqrt{3.2} & 0 & -\sqrt{1.2} & 0 & \sqrt{2.1} \\ 0 & 0 & -\sqrt{2.1} & 0 & -\sqrt{4.3} & 0 & -\sqrt{2.1} & 0 \end{bmatrix} \\ \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2\sqrt{1.3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2.2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{3.1} \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 2\sqrt{1.3} & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2\sqrt{2.2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{3.1} & 0 & 0 & 0 & 2 \end{bmatrix} \end{array} \right\}$$

## 1.2.5 无质量粒子的等价分离方程

$$\text{定理1.2.2. } (\sigma \otimes I_{2s}, -i\zeta)^a \partial_a \tilde{\psi}(s, \zeta) = i\tilde{J}(s, \zeta) \Leftrightarrow \begin{cases} [\frac{1}{s}\sigma(s), -i\zeta]^a \partial_a \psi(s, \zeta) = i\bar{N}(s)\tilde{J}(s, \zeta) \\ \frac{1}{s}O(s) \cdot \nabla \psi(s, \zeta) = i\bar{X}(s)\tilde{J}(s, \zeta) \end{cases}$$

$$\text{推论1.2.16. } \psi(s, \zeta) = \bar{N}(s)\tilde{\psi}(s, \zeta), 0_{2s-1} = \bar{X}(s)\tilde{\psi}(s, \zeta)$$

1.3 表象变换矩阵 $\hat{S}(s)$ 的引入

$$\text{定义1.3.1. } \hat{S}(s) = \begin{bmatrix} \tilde{S}(s) & 0 \\ 0 & I_{4^s-4s} \end{bmatrix} I \otimes \hat{S}(s - \frac{1}{2}), \tilde{S}(s) = \begin{bmatrix} \bar{N}(s) \\ \bar{X}(s) \end{bmatrix}$$

$$\text{推论1.3.1. } \hat{S}(s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots) = I, \begin{bmatrix} \bar{N}(1) \\ \bar{X}(1) \end{bmatrix}, \begin{bmatrix} \bar{N}(\frac{3}{2})[I \otimes \bar{N}(1)] \\ \bar{X}(\frac{3}{2})[I \otimes \bar{N}(1)] \\ I \otimes \bar{X}(1) \end{bmatrix}, \begin{bmatrix} \bar{N}(2)[I \otimes [\bar{N}(\frac{3}{2})[I \otimes \bar{N}(1)]]] \\ \bar{X}(2)[I \otimes [\bar{N}(\frac{3}{2})[I \otimes \bar{N}(1)]]] \\ I \otimes [\bar{X}(\frac{3}{2})[I \otimes \bar{N}(1)]] \\ I \otimes I \otimes \bar{X}(1) \end{bmatrix}, \dots$$

$$\text{推论1.3.2. } \hat{S}(s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots) = I, \begin{bmatrix} \bar{N}(1)[I \otimes \bar{\Gamma}(\frac{1}{2})] \\ \bar{X}(1)[I \otimes \bar{\Gamma}(\frac{1}{2})] \end{bmatrix}, \begin{bmatrix} \bar{N}(\frac{3}{2})[I \otimes \bar{\Gamma}(1)] \\ \bar{X}(\frac{3}{2})[I \otimes \bar{\Gamma}(1)] \\ I \otimes [\bar{X}(1)[I \otimes \bar{\Gamma}(\frac{1}{2})]] \end{bmatrix}, \begin{bmatrix} \bar{N}(2)[I \otimes \bar{\Gamma}(\frac{3}{2})] \\ \bar{X}(2)[I \otimes \bar{\Gamma}(\frac{3}{2})] \\ I \otimes [\bar{X}(\frac{3}{2})[I \otimes \bar{\Gamma}(1)]] \\ I \otimes I \otimes [\bar{X}(1)[I \otimes \bar{\Gamma}(\frac{1}{2})]] \end{bmatrix}, \dots$$

$$\text{推论1.3.3. } \hat{S}(s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots) = I, \begin{bmatrix} \bar{\Gamma}(1) \\ \bar{X}(1)[I \otimes \bar{\Gamma}(\frac{1}{2})] \end{bmatrix}, \begin{bmatrix} \bar{\Gamma}(\frac{3}{2}) \\ \bar{X}(\frac{3}{2})[I \otimes \bar{\Gamma}(1)] \\ I \otimes [\bar{X}(1)[I \otimes \bar{\Gamma}(\frac{1}{2})]] \end{bmatrix}, \begin{bmatrix} \bar{\Gamma}(2) \\ \bar{X}(2)[I \otimes \bar{\Gamma}(\frac{3}{2})] \\ I \otimes [\bar{X}(\frac{3}{2})[I \otimes \bar{\Gamma}(1)]] \\ I \otimes I \otimes [\bar{X}(1)[I \otimes \bar{\Gamma}(\frac{1}{2})]] \end{bmatrix}, \dots$$

$$\text{推论1.3.4. } \hat{S}(s) = \begin{bmatrix} \bar{\Gamma}(s) \\ \bar{X}(s)[I \otimes \bar{\Gamma}(s-\frac{1}{2})] \\ I \otimes [\bar{X}(s-\frac{1}{2})[I \otimes \bar{\Gamma}(s-\frac{3}{2})]] \\ \dots \\ (I \otimes)^{2s-3} [\bar{X}(\frac{3}{2})[I \otimes \bar{\Gamma}(1)]] \\ (I \otimes)^{2s-2} [\bar{X}(1)[I \otimes \bar{\Gamma}(\frac{1}{2})]] \end{bmatrix}, \hat{S}(s)\hat{S}^+(s) = \hat{S}^+(s)\hat{S}(s) = I_{4^s}$$

$$\text{推论1.3.5. } \hat{S}^+(s) = [\Gamma(s), [I \otimes \Gamma(s)]X(s - \frac{1}{2}), I \otimes [[I \otimes \Gamma(s - \frac{1}{2})]X(s - \frac{3}{2})], \dots, I \otimes \dots I \otimes [[I \otimes \Gamma(\frac{1}{2})]X(1)]]$$

$$\text{推论1.3.6. } \bar{\Gamma}(s)\Gamma(s) = I_{2s+1}, \bar{\Gamma}(s) \cdot I_{4^k} \otimes \{[I \otimes \Gamma(s - \frac{1}{2} - k)]X(s - k)\} = 0; k = 0, \frac{1}{2}, 1, \dots, s - 1$$

$$\text{推论1.3.7. } \hat{S}(s)[\sigma(\frac{1}{2}) \otimes I_{2^{2s-1}}]\hat{S}^+(s) = \frac{1}{2s} \begin{bmatrix} \sigma(s) & O^+(s) & 0 \\ O(s) & -\sigma(s-1) & 0 \\ 0 & 0 & 2s\sigma \otimes I_{(2^{2s-1}-2s)} \end{bmatrix}$$

1.3.1 表象变换矩阵 $\hat{S}(s)$ 的几个具体表示

$$\text{推论1.3.8. } \hat{S}(\frac{1}{2}) = I, \hat{S}^+(\frac{1}{2}) = I$$

$$\text{推论1.3.9. } \hat{S}(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{1} & \sqrt{1} & 0 \end{bmatrix}, \hat{S}^+(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & -\sqrt{1} \\ 0 & \sqrt{1} & 0 & \sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$\text{推论1.3.10. } \hat{S}(\frac{3}{2}) = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & -\sqrt{4} & \sqrt{1} & 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1} & 0 & -\sqrt{1} & \sqrt{4} & 0 \\ 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 \end{bmatrix}, \hat{S}^+(\frac{3}{2}) = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & -\sqrt{4} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{1} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{1} & 0 & -\sqrt{3} \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{1} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{1} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{4} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{推论1.3.11. } \hat{S}(2) = \frac{1}{\sqrt{12}} \begin{bmatrix} \sqrt{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{12} \\ 0 & -\sqrt{9} & \sqrt{1} & 0 & \sqrt{1} & 0 & 0 & 0 & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & 0 & 0 & 0 & -\sqrt{1} & 0 & \sqrt{9} \\ 0 & 0 & -\sqrt{8} & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{8} & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & \sqrt{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & \sqrt{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & \sqrt{8} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{6} & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 \end{bmatrix}$$

$$\text{推论1.3.12. } \hat{S}^+(2) = \frac{1}{\sqrt{12}} \begin{bmatrix} \sqrt{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & -\sqrt{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{1} & 0 & 0 & -\sqrt{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{1} & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{6} & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & -\sqrt{1} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{6} \\ 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{1} & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & \sqrt{6} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & -\sqrt{1} & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & \sqrt{6} \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & \sqrt{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & -\sqrt{1} & 0 & 0 & 0 & \sqrt{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{推论1.3.13. } \bar{X}(s=1, \frac{3}{2}, 2)\bar{\Gamma}(s=1, \frac{3}{2}, 2, \dots) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -\sqrt{1} & \sqrt{1} & 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & -2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 2 & 0 \end{bmatrix}, \frac{1}{\sqrt{12}} \begin{bmatrix} 0 & -3 & \sqrt{1} & 0 & \sqrt{1} & 0 & 0 & 0 & \sqrt{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{1} & 0 & 0 & 0 & -\sqrt{1} & 0 & -\sqrt{1} & 3 & 0 \end{bmatrix}$$

### 1.4 关于表象变换矩阵 \$\hat{S}(s)\$ 的一个重要定理及其证明

定义1.4.1. \$\pi(s, s') := \Omega(s') \otimes I\_{2(s-s')-1} + I\_{4s'} \otimes \sigma(s-s'-1); s' \ge 0, s-s' \ge 1\$

$$\text{引理1.4.1. } \hat{S}(s)\Omega(s=1, \frac{3}{2}, 2)\hat{S}^+(s) = \begin{bmatrix} \sigma(1) & 0 \\ 0 & \sigma(0) \end{bmatrix}, \begin{bmatrix} \sigma(\frac{3}{2}) & 0 & 0 \\ 0 & \sigma(\frac{1}{2}) & 0 \\ 0 & 0 & \sigma(\frac{1}{2}) \end{bmatrix}, \begin{bmatrix} \sigma(2) & 0 & 0 & 0 \\ 0 & \sigma(1) & 0 & 0 \\ 0 & 0 & \sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2}) & 0 \\ 0 & 0 & 0 & \sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2}) \end{bmatrix}$$

定理1.4.1.

$$\hat{S}(s)\Omega(s) = \begin{bmatrix} \sigma(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s,0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s, \frac{1}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s, s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s, s-1) \end{bmatrix} \hat{S}(s)[\Leftrightarrow]\Omega(s)\hat{S}^+(s) = \hat{S}^+(s) \begin{bmatrix} \sigma(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s,0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s, \frac{1}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s, s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s, s-1) \end{bmatrix}$$

证明: 采用数学归纳法证明此定理。

第一步: \$s' = 1\$ 时成立: \$\hat{S}(1)\Omega(1) = \begin{bmatrix} \sigma(1) & 0 \\ 0 & \sigma(0) \end{bmatrix} \hat{S}(1)\$

第二步: 假设 \$s' = s - \frac{1}{2}\$ 时成立:

$$\hat{S}(s-\frac{1}{2})\Omega(s-\frac{1}{2}) = \begin{bmatrix} \sigma(s-\frac{1}{2}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s-\frac{1}{2}, 0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s-\frac{1}{2}, \frac{1}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s-\frac{1}{2}, s-\frac{1}{2}-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s-\frac{1}{2}, s-\frac{1}{2}-1) \end{bmatrix} \hat{S}(s-\frac{1}{2})$$

第三步: \$s' = s\$ 时, \$\hat{S}(s)\Omega(s) = \begin{bmatrix} \tilde{\sigma}(s) & 0 \\ 0 & I\_{4s-4s} \end{bmatrix} [I \otimes \hat{S}(s-\frac{1}{2})][\sigma(\frac{1}{2}) \otimes I\_{2s-1} + I \otimes \Omega(s-\frac{1}{2})]\$

$$\begin{aligned}
&= \begin{bmatrix} \hat{S}(s) & 0 \\ 0 & I_{4^{s-4s}} \end{bmatrix} \left\{ \sigma\left(\frac{1}{2}\right) \otimes [I_{2^{2s-1}} \hat{S}(s)] + \{I \otimes [\hat{S}(s - \frac{1}{2}) \Omega(s - \frac{1}{2})]\} \right\} \\
&= \begin{bmatrix} \hat{S}(s) & 0 \\ 0 & I_{4^{s-4s}} \end{bmatrix} \left[ \sigma\left(\frac{1}{2}\right) \otimes I_{2^{2s-1}} + I \otimes \begin{bmatrix} \sigma(s-\frac{1}{2}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s-\frac{1}{2}, 0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s-\frac{1}{2}, \frac{1}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s-\frac{1}{2}, s-\frac{1}{2}-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s-\frac{1}{2}, s-\frac{1}{2}-1) \end{bmatrix} \right] [I \otimes \hat{S}(s - \frac{1}{2})] \\
&= \begin{bmatrix} \hat{S}(s) & 0 \\ 0 & I_{4^{s-4s}} \end{bmatrix} \left[ \begin{bmatrix} \sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s, \frac{1}{2}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s, \frac{3}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s, s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s, s-1) \end{bmatrix} \right] [I \otimes \hat{S}(s - \frac{1}{2})] \\
&= \begin{bmatrix} \hat{S}(s) [\sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s-\frac{1}{2})] & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s, \frac{1}{2}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s, \frac{3}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s, s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s, s-1) \end{bmatrix} [I \otimes \hat{S}(s - \frac{1}{2})] \\
&= \begin{bmatrix} \sigma(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s, 0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s, \frac{1}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s, s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s, s-1) \end{bmatrix} \begin{bmatrix} \hat{S}(s) & 0 \\ 0 & I_{4^{s-4s}} \end{bmatrix} [I \otimes \hat{S}(s - \frac{1}{2})] \\
&= \begin{bmatrix} \sigma(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s, 0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s, \frac{1}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s, s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s, s-1) \end{bmatrix} \hat{S}(s)
\end{aligned}$$

此步证明了  $s' = s$  时成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

$$\text{推论1.4.1. } \hat{S}(s) \Omega(s) \hat{S}^+(s) = \begin{bmatrix} \sigma(s) & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi(s, 0) & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi(s, \frac{1}{2}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi(s, s-\frac{3}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi(s, s-1) \end{bmatrix}$$

终于严格证明了以上的结论，之前的一些常数不变张量的复杂性质便可以轻松得到，如下：

推论1.4.2.

$$\begin{cases} \bar{\Gamma}(s) \Omega(s) = \sigma(s) \bar{\Gamma}(s), \Omega(s) \Gamma(s) = \Gamma(s) \sigma(s) \\ \bar{X}(s) [I \otimes \bar{\Gamma}(s - \frac{1}{2})] \Omega(s) = \sigma(s-1) \bar{X}(s) [I \otimes \bar{\Gamma}(s - \frac{1}{2})], \Omega(s) [I \otimes \Gamma(s - \frac{1}{2})] X(s) = [I \otimes \Gamma(s - \frac{1}{2})] X(s) \sigma(s-1) \end{cases}$$

推论1.4.3.

$$\begin{cases} I_{4^{s-1}} \otimes \{ \bar{X}(1) [I \otimes \bar{\Gamma}(\frac{1}{2})] \} \Omega(s) = \Omega(s-1) I_{4^{s-1}} \otimes \{ \bar{X}(1) [I \otimes \bar{\Gamma}(\frac{1}{2})] \} \\ \Omega(s) I_{4^{s-1}} \otimes \{ [I \otimes \Gamma(\frac{1}{2})] X(1) \} = I_{4^{s-1}} \otimes \{ [I \otimes \Gamma(\frac{1}{2})] X(1) \} \Omega(s-1) \end{cases}$$

推论1.4.4.

$$\begin{cases} I_{4^k} \otimes \{ \bar{X}(s-k) [I \otimes \bar{\Gamma}(s - \frac{1}{2} - k)] \} \Omega(s) = \pi(s, k) I_{4^k} \otimes \{ \bar{X}(s-k) [I \otimes \bar{\Gamma}(s - \frac{1}{2} - k)] \} \\ \Omega(s) I_{4^k} \otimes \{ [I \otimes \Gamma(s - \frac{1}{2} - k)] X(s-k) \} = I_{4^k} \otimes \{ [I \otimes \Gamma(s - \frac{1}{2} - k)] X(s-k) \} \pi(s, k) \end{cases}$$

推论1.4.5.

$$\begin{cases} \sigma(s) = \bar{\Gamma}(s) \Omega(s) \Gamma(s) \\ \sigma(s-1) = \bar{X}(s) [I \otimes \bar{\Gamma}(s - \frac{1}{2})] \Omega(s) [I \otimes \Gamma(s - \frac{1}{2})] X(s) \\ \Omega(s-1) = I_{4^{s-1}} \otimes \{ \bar{X}(1) [I \otimes \bar{\Gamma}(\frac{1}{2})] \} \Omega(s) I_{4^{s-1}} \otimes \{ [I \otimes \Gamma(\frac{1}{2})] X(1) \} \\ \pi(s, k) = I_{4^k} \otimes \{ \bar{X}(s-k) [I \otimes \bar{\Gamma}(s - \frac{1}{2} - k)] \} \Omega(s) I_{4^k} \otimes \{ [I \otimes \Gamma(s - \frac{1}{2} - k)] X(s-k) \} \end{cases}$$

### 1.5 常数矩阵 $\pi(s, s')$ 的表象变换

**推论1.5.1.**  $\pi(s, s') = \Omega(s' - \frac{1}{2}) \otimes I_{4(s-s')-2} + I_{2^{2s'-1}} \otimes [\sigma(\frac{1}{2}) \otimes I_{2(s-s')-1} + I \otimes \sigma(s - s' - 1)]$

**证明:**  $\pi(s, s') := \Omega(s') \otimes I_{2(s-s')-1} + I_{4s'} \otimes \sigma(s - s' - 1)$

$$= [\Omega(s' - \frac{1}{2}) \otimes I + I_{2^{2s'-1}} \otimes \sigma(\frac{1}{2})] \otimes I_{2(s-s')-1} + I_{4s'} \otimes \sigma(s - s' - 1)$$

$$= \Omega(s' - \frac{1}{2}) \otimes I_{4(s-s')-2} + I_{2^{2s'-1}} \otimes [\sigma(\frac{1}{2}) \otimes I_{2(s-s')-1} + I \otimes \sigma(s - s' - 1)] \quad \square$$

**推论1.5.2.**  $[I_{2^{2s'-1}} \otimes \tilde{S}(s - s' - \frac{1}{2})]\pi(s, s') = \begin{bmatrix} \pi(s, s' - \frac{1}{2}) & 0 \\ 0 & \pi(s-1, s' - \frac{1}{2}) \end{bmatrix} [I_{2^{2s'-1}} \otimes \tilde{S}(s - s' - \frac{1}{2})]; s' \geq \frac{1}{2}, s - s' \geq \frac{3}{2}$

**证明:**  $[I_{2^{2s'-1}} \otimes \tilde{S}(s - s' - \frac{1}{2})]\pi(s, s')$

$$= [I_{2^{2s'-1}} \otimes \tilde{S}(s - s' - \frac{1}{2})]\{\Omega(s' - \frac{1}{2}) \otimes I_{4(s-s')-2} + I_{2^{2s'-1}} \otimes [\sigma(\frac{1}{2}) \otimes I_{2(s-s')-1} + I \otimes \sigma(s - s' - 1)]\}$$

$$= \{\Omega(s' - \frac{1}{2}) \otimes I_{4(s-s')-2} + I_{2^{2s'-1}} \otimes \begin{bmatrix} \sigma(s-s'-\frac{1}{2}) & 0 \\ 0 & \sigma(s-s'-\frac{3}{2}) \end{bmatrix}\} [I_{2^{2s'-1}} \otimes \tilde{S}(s - s' - \frac{1}{2})]$$

$$= \begin{bmatrix} \Omega(s' - \frac{1}{2}) \otimes I_{2(s-s')-1} + I_{2^{2s'-1}} \otimes \sigma(s-s'-\frac{1}{2}) & 0 \\ 0 & \Omega(s' - \frac{1}{2}) \otimes I_{2(s-s')-2} + I_{2^{2s'-1}} \otimes \sigma(s-s'-\frac{3}{2}) \end{bmatrix} [I_{2^{2s'-1}} \otimes \tilde{S}(s - s' - \frac{1}{2})]$$

$$= \begin{bmatrix} \pi(s, s' - \frac{1}{2}) & 0 \\ 0 & \pi(s-1, s' - \frac{1}{2}) \end{bmatrix} [I_{2^{2s'-1}} \otimes \tilde{S}(s - s' - \frac{1}{2})] \quad \square$$

利用以上推理反复迭代可以得到以下推论:

**推论1.5.3.**  $I_{2^{2s'-2}} \otimes \left\{ \begin{bmatrix} \tilde{S}(s-s') & 0 \\ 0 & \tilde{S}(s-s'-1) \end{bmatrix} [I \otimes \tilde{S}(s - s' - \frac{1}{2})] \right\} \pi(s, s'); s' \geq 1, s - s' \geq 2$

$$= \begin{bmatrix} \pi(s, s'-1) & 0 & 0 & 0 \\ 0 & \pi(s-1, s'-1) & 0 & 0 \\ 0 & 0 & \pi(s-1, s'-1) & 0 \\ 0 & 0 & 0 & \pi(s-2, s'-1) \end{bmatrix} I_{2^{2s'-2}} \otimes \left\{ \begin{bmatrix} \tilde{S}(s-s') & 0 \\ 0 & \tilde{S}(s-s'-1) \end{bmatrix} [I \otimes \tilde{S}(s - s' - \frac{1}{2})] \right\}$$

**推论1.5.4.**

$$I_{2^{2s'-3}} \otimes \left\{ \begin{bmatrix} \tilde{S}(s-s'+\frac{1}{2}) & 0 \\ 0 & \tilde{S}(s-s'-\frac{1}{2}) \end{bmatrix} [I \otimes \tilde{S}(s-s')] \right. \\ \left. \begin{bmatrix} \tilde{S}(s-s'-\frac{1}{2}) & 0 \\ 0 & \tilde{S}(s-s'-\frac{3}{2}) \end{bmatrix} [I \otimes \tilde{S}(s-s'-1)] \right\} [I \otimes I \otimes \tilde{S}(s - s' - \frac{1}{2})] \pi(s, s')$$

$$= \begin{bmatrix} \begin{bmatrix} \pi(s, s' - \frac{3}{2}) & 0 \\ 0 & \pi(s-1, s' - \frac{3}{2}) \end{bmatrix} & 0 & 0 & 0 \\ 0 & \begin{bmatrix} \pi(s-1, s' - \frac{3}{2}) & 0 \\ 0 & \pi(s-2, s' - \frac{3}{2}) \end{bmatrix} & 0 & 0 \\ 0 & 0 & \begin{bmatrix} \pi(s-1, s' - \frac{3}{2}) & 0 \\ 0 & \pi(s-2, s' - \frac{3}{2}) \end{bmatrix} & 0 \\ 0 & 0 & 0 & \begin{bmatrix} \pi(s-2, s' - \frac{3}{2}) & 0 \\ 0 & \pi(s-3, s' - \frac{3}{2}) \end{bmatrix} \end{bmatrix}$$

$$I_{2^{2s'-3}} \otimes \left\{ \begin{bmatrix} \tilde{S}(s-s'+\frac{1}{2}) & 0 \\ 0 & \tilde{S}(s-s'-\frac{1}{2}) \end{bmatrix} [I \otimes \tilde{S}(s-s')] \right. \\ \left. \begin{bmatrix} \tilde{S}(s-s'-\frac{1}{2}) & 0 \\ 0 & \tilde{S}(s-s'-\frac{3}{2}) \end{bmatrix} [I \otimes \tilde{S}(s-s'-1)] \right\} [I \otimes I \otimes \tilde{S}(s - s' - \frac{1}{2})]$$

$$; s' \geq \frac{3}{2}, s - s' \geq \frac{5}{2}$$

### 1.6 常数矩阵 $\pi(s, s')$ 的表象变换一般形式

**推论1.6.1.**  $\{I_{2^{2s'-1}} \otimes \begin{bmatrix} [I_{20} \otimes \tilde{N}(s-s' - \frac{1}{2})] \\ [I_{20} \otimes \tilde{X}(s-s' - \frac{1}{2})] \end{bmatrix}\} \pi(s, s') = [\pi(s, s' - \frac{1}{2}) \oplus \pi(s - 1, s' - \frac{1}{2})] \{I_{2^{2s'-1}} \otimes \begin{bmatrix} [I_{20} \otimes \tilde{N}(s-s' - \frac{1}{2})] \\ [I_{20} \otimes \tilde{X}(s-s' - \frac{1}{2})] \end{bmatrix}\}$

$$; s' \geq \frac{1}{2}, s - s' \geq \frac{3}{2}$$

**推论1.6.2.**  $\{I_{2^{2s'-2}} \otimes \begin{bmatrix} [I_{20} \tilde{N}(s-s')] [I_{21} \otimes \tilde{N}(s-s' - \frac{1}{2})] \\ [I_{20} \tilde{X}(s-s')] [I_{21} \otimes \tilde{N}(s-s' - \frac{1}{2})] \\ [I_{20} \tilde{N}(s-s'-1)] [I_{21} \otimes \tilde{X}(s-s' - \frac{1}{2})] \\ [I_{20} \tilde{X}(s-s'-1)] [I_{21} \otimes \tilde{X}(s-s' - \frac{1}{2})] \end{bmatrix}\} \pi(s, s'); s' \geq 1, s - s' \geq 2$



推论1.7.3.  $S(s)\Omega(s)S^+(s) =$

$$\begin{bmatrix} \sigma(\frac{1}{2}) \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \Omega(\frac{1}{2}) \otimes I_{2s-2} + I \otimes \sigma(s - \frac{3}{2}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega(1) \otimes I_{2s-3} + [I \otimes]^2 \sigma(s-2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega(s - \frac{3}{2}) \otimes I_2 + [I \otimes]^{2s-1} \sigma(\frac{1}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega(s-1) + [I \otimes]^{2s-2} \sigma(0) \end{bmatrix}$$

定义1.7.1.  $\pi(s, s') := \Omega(s') \otimes I_{2(s-s')-1} + I_{4s'} \otimes \sigma(s - s' - 1); s' \geq 0, s - s' \geq 1$

推论1.7.4.  $S(s)(\sigma(\frac{1}{2}) \otimes I_{2s-1})S^+(s) = \sigma(\frac{1}{2}) \otimes I_{2s-1}$

推论1.7.5.  $(\sigma \otimes I_{2s-1}, -i\zeta)^a \partial_a \hat{\varphi}(s, \zeta) = i\hat{K}(s, \zeta) \Leftrightarrow (\sigma \otimes I_{2s-1}, -i\zeta)^a \partial_a S(s)\hat{\varphi}(s, \zeta) = iS(s)\hat{K}(s, \zeta)$

推论1.7.6.  $(\sigma \otimes I_{2s-1}, -i\zeta)^a \partial_a \hat{\psi}(s, \zeta) = i\hat{J}(s, \zeta) \Leftrightarrow \begin{cases} (\sigma \otimes I_{2s}, -i\zeta)^a \partial_a \tilde{\psi}(s, \zeta) = i\tilde{J}(s, \zeta) \\ (\sigma \otimes I_{2s-1-2s}, -i\zeta)^a \partial_a o(s, \zeta) = io(s, \zeta) \end{cases}$

推论1.7.7.  $(\sigma \otimes I_{2s-1}, -i\zeta)^a \partial_a \hat{\psi}(s, \zeta) = i\hat{J}(s, \zeta) \Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)^a \partial_a \tilde{\psi}(s, \zeta) = i\tilde{J}(s, \zeta)$

## 1.8 引力子的表象变换

$$\text{推论1.8.1. } \hat{S}_0(2) = \begin{bmatrix} \bar{N}(2)[I \otimes \bar{\Gamma}(\frac{3}{2})] \\ \bar{X}(2)[I \otimes \bar{\Gamma}(\frac{3}{2})] \\ \bar{N}(1)\{I \otimes [\bar{X}(\frac{3}{2})][I \otimes \bar{\Gamma}(1)]\} \\ \bar{X}(1)\{I \otimes [\bar{X}(\frac{3}{2})][I \otimes \bar{\Gamma}(1)]\} \\ \bar{N}(1)\{I \otimes I \otimes [\bar{X}(1)][I \otimes \bar{\Gamma}(\frac{1}{2})]\} \\ \bar{X}(1)\{I \otimes I \otimes [\bar{X}(1)][I \otimes \bar{\Gamma}(\frac{1}{2})]\} \end{bmatrix} = \begin{bmatrix} \tilde{S}(2)[I \otimes \bar{\Gamma}(\frac{3}{2})] \\ \tilde{S}(1)\{I \otimes [\bar{X}(\frac{3}{2})][I \otimes \bar{\Gamma}(1)]\} \\ \tilde{S}(1)\{I \otimes I \otimes [\bar{X}(1)][I \otimes \bar{\Gamma}(\frac{1}{2})]\} \end{bmatrix} = \begin{bmatrix} \tilde{S}(2) & 0 & 0 \\ 0 & \tilde{S}(1) & 0 \\ 0 & 0 & \tilde{S}(1) \end{bmatrix} [I \otimes \hat{S}(\frac{3}{2})]$$

$$\text{定理1.8.1. } \hat{S}_0(2)\Omega(2)\hat{S}_0^+(2) = \begin{bmatrix} \sigma(2) & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma(0) & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(1) & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma(0) \end{bmatrix}, \hat{S}_0(s)[\sigma(\frac{1}{2}) \otimes I_8]\hat{S}_0^+(s) = \begin{bmatrix} \Sigma(2) & 0 & 0 \\ 0 & \Sigma(1) & 0 \\ 0 & 0 & \Sigma(1) \end{bmatrix}$$

推论1.8.2.  $[\sigma \otimes I_8, -i\zeta]^a \partial_a \hat{\varphi}(2, \zeta) = i\hat{K}(2, \zeta) \Leftrightarrow \begin{cases} [2\Sigma(2), -i\zeta]^a \partial_a \tilde{\varphi}(2, \zeta) = i\tilde{K}(2, \zeta) \\ [2\Sigma(1), -i\zeta]^a \partial_a \tilde{\varphi}(1, \zeta) = i\tilde{K}(1, \zeta) \\ [2\Sigma(1'), -i\zeta]^a \partial_a \tilde{\varphi}(1', \zeta) = i\tilde{K}(1', \zeta) \end{cases}$

$$\text{定理1.8.2. } S_0(2) = \begin{bmatrix} \bar{N}(2)[I \otimes \bar{\Gamma}(\frac{3}{2})] \\ \bar{X}(2)[I \otimes \bar{\Gamma}(\frac{3}{2})] \\ \bar{N}(1)\{I \otimes [\bar{X}(\frac{3}{2})][I \otimes \bar{\Gamma}(1)]\} \\ \bar{N}(1)\{I \otimes I \otimes [\bar{X}(1)][I \otimes \bar{\Gamma}(\frac{1}{2})]\} \\ \bar{X}(1)\{I \otimes [\bar{X}(\frac{3}{2})][I \otimes \bar{\Gamma}(1)]\} \\ \bar{X}(1)\{I \otimes I \otimes [\bar{X}(1)][I \otimes \bar{\Gamma}(\frac{1}{2})]\} \end{bmatrix}, S_0(2)\Omega(2)S_0^+(2) = \begin{bmatrix} \sigma(2) & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma(1) & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma(0) & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma(0) \end{bmatrix}$$

## 1.9 常数矩阵 $\pi(s, s')$ 的深入分析

推论1.9.1.

$$\begin{cases} \pi(s, 0) := \sigma(s-1), s \geq 1; \pi(1, 0) = 0 \\ \pi(s, \frac{1}{2}) := \sigma(\frac{1}{2}) \otimes I_{2(s-1)} + I \otimes \sigma(s-1-\frac{1}{2}), s \geq \frac{3}{2}; \pi(\frac{3}{2}, \frac{1}{2}) = \sigma(\frac{1}{2}), \pi(2, \frac{1}{2}) = \pi(2, 1) = \Omega(1) \\ \pi(s, s-1) := \Omega(s-1), s \geq 1 \\ \pi(s, s') := \phi, s - s' \leq \frac{1}{2} \\ \Omega(s) = \pi(s+1, s) = \pi(s+1, s-\frac{1}{2}), s \geq \frac{1}{2} \end{cases}$$

推论1.9.2.  $\Omega(s) = \pi(s+1, s) = \pi(s+1, s-\frac{1}{2}), s \geq \frac{1}{2}$

$\rightarrow [\pi(s+1, s-1) \oplus \pi(s, s-1)], s \geq 1$



$$\begin{aligned}
&\rightarrow [\pi(s+1, s-\frac{3}{2}) \oplus \pi(s, s-\frac{3}{2})] \oplus [\pi(s, s-\frac{3}{2})], s \geq \frac{3}{2} \\
&\rightarrow [\pi(s+1, s-2) \oplus \pi(s, s-2)] \oplus [\pi(s, s-2) \oplus \pi(s-1, s-2)]^2, s \geq 2 \\
&\rightarrow [\pi(s+1, s-\frac{5}{2}) \oplus \pi(s, s-\frac{5}{2})] \oplus [\pi(s, s-\frac{5}{2}) \oplus \pi(s-1, s-\frac{5}{2})]^3 \oplus [\pi(s-1, s-\frac{5}{2})]^2, s \geq \frac{5}{2} \\
&\rightarrow [\pi(s+1, s-3) \oplus \pi(s, s-3)] \oplus [\pi(s, s-3) \oplus \pi(s-1, s-3)]^4 \oplus [\pi(s-1, s-3) \oplus \pi(s-2, s-3)]^5, s \geq 3 \\
&\rightarrow \dots
\end{aligned}$$

自我评述: 只要按以上方法, 任何一个 $\Omega(s)$ 都可以通过表象变换具体分解成多个单自旋态的直和, 这个问题原则上算是彻底解决了。实际运用还需作一些计算, 将具体的表象变换显式地写出来以便使用。同时也构造性地证明了 $\Omega(s)$ 确实完全由单自旋态组成, 没有多余成分。 $\Omega(s)$ 不会同时包含玻色和费米自旋态, 它表征了一个玻色或费米多重态。并且它遍历了高低玻色或费米自旋态, 除最高自旋态外一般情况下其它自旋态均有多个冗余态。

### 1.10 多重态自旋方程

定义1.10.1.  $[(S^+ \sqrt{[S\Omega(s)S^+]^2} + \frac{1}{4}S - \frac{1}{2})\partial_a + iS_{ab}(\Omega(s), \varsigma)\partial^b]\Psi(x) = 0, \partial^a\partial_a\Psi(x) = 0;$   
 $\Omega(s) \times \Omega(s) = i\Omega(s), S\Omega(s)S^+ = \sigma(s) \oplus \sigma(s-1) \oplus \dots \oplus \sigma(\frac{1}{2})|\sigma(0)$

推论1.10.1.

$$\begin{cases}
\{\tilde{S}^+(s)[sI_{2s+1} \oplus (s-1)I_{2s-1}]\tilde{S}(s)\partial_a + iS_{ab}(\pi(s+1, \frac{1}{2}), \varsigma)\partial^b\}\Psi(x) = 0, \partial^a\partial_a\Psi(x) = 0 \\
\{[s-1+N(s)\bar{N}(s)]\partial_a + iS_{ab}(\pi(s+1, \frac{1}{2}), \varsigma)\partial^b\}\Psi(x) = 0, \partial^a\partial_a\Psi(x) = 0 \\
\{[s-X(s)\bar{X}(s)]\partial_a + iS_{ab}(\pi(s+1, \frac{1}{2}), \varsigma)\partial^b\}\Psi(x) = 0, \partial^a\partial_a\Psi(x) = 0
\end{cases}$$

### 1.11 自旋 $s=2$ 的表象变换性质一

推论1.11.1.

$$S(1 \otimes 1) = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & \sqrt{4} & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 \end{bmatrix}, S^+(1 \otimes 1) = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 & 0 & -\sqrt{3} & 0 & \sqrt{2} \\ 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{4} & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \end{bmatrix}$$

推论1.11.2.  $\frac{1}{\sqrt{3}}[I_3 \otimes \begin{bmatrix} -\sqrt{2} & -1 \\ -1 & \sqrt{2} \end{bmatrix}][\sigma(1) \otimes I] \frac{1}{\sqrt{3}}[I_3 \otimes \begin{bmatrix} -\sqrt{2} & -1 \\ -1 & \sqrt{2} \end{bmatrix}] = \sigma(1) \otimes I$

定理1.11.1.  $S(1 \otimes 1)[\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]S^+(1 \otimes 1) = \begin{bmatrix} \sigma(2) & 0 & 0 \\ 0 & \sigma(1) & 0 \\ 0 & 0 & \sigma(0) \end{bmatrix}$

$$\text{推论1.11.3. } \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & \sqrt{4} & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \end{bmatrix} [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)] \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{4} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{6} \end{bmatrix} = \sigma(2)$$

$$\text{推论1.11.4. } \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 \end{bmatrix} [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)] \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 \\ 0 & -\sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{bmatrix} = \sigma(1)$$

$$\text{推论1.11.5. } \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 \end{bmatrix} [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)] \frac{1}{\sqrt{6}} \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \\ 0 \\ -\sqrt{2} \\ 0 \\ \sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \sigma(0)$$

## 1.12 自旋 $s = 2$ 的表象变换性质二

$$\text{定理1.12.1. } S(1 \otimes 1)[\sigma(1) \otimes I_3]S^+(1 \otimes 1) = \frac{1}{2} \begin{bmatrix} \sigma(2) & \frac{1}{\sqrt{3}}O^+(2) & 0 \\ \frac{1}{\sqrt{3}}O(2) & \sigma(1) & \frac{2}{\sqrt{3}}O^+(2) \\ 0 & \frac{2}{\sqrt{3}}O(2) & \sigma(0) \end{bmatrix}, 0(2) = \left\{ \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}, i \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2} & 0 \end{bmatrix} \right\}$$

## 1.13 更一般的表象变换性质(猜测)

$$\text{定义1.13.1. } S(s_1 \otimes s_2 \cdots \otimes s_n)[\sigma(s_1) \otimes I_* + I_{2s_1+1} \otimes \sigma(s_2) \otimes I_* + \cdots]S^+(s_1 \otimes s_2 \cdots \otimes s_n) = ?$$

$$\text{推论1.13.1. } \tilde{S}(s) := S[\frac{1}{2} \otimes (s - \frac{1}{2})], \hat{S}(s)? := S[(\frac{1}{2})_1 \otimes (\frac{1}{2})_2 \cdots \otimes (\frac{1}{2})_{2s}]$$

## 2 高级表象变换的物理应用

### 2.1 一般的新型耦合理论

#### 2.1.1 s-自旋粒子的新型耦合理论

推论2.1.1.

$$\begin{cases} (\sigma \otimes I_{2s}, -i\zeta)^a \partial_a \varphi(s, \zeta) = iJ(s, \zeta) \\ \psi(s, \zeta) = \bar{N}(s)\varphi(s, \zeta) \\ \psi(s-1, \zeta) = \bar{X}(s)\varphi(s, \zeta) \end{cases} \stackrel{S(s)}{\Leftrightarrow} \begin{cases} [\frac{1}{s}\sigma(s), -i\zeta]^a \partial_a \psi(s, \zeta) = -\frac{1}{s}O^+(s) \cdot \nabla \psi(s-1, \zeta) + i\bar{N}(s)J(s, \zeta) \\ [\frac{1}{s}\sigma(s-1), i\zeta]^a \partial_a \psi(s-1, \zeta) = \frac{1}{s}O(s) \cdot \nabla \psi(s, \zeta) - i\bar{X}(s)J(s, \zeta) \end{cases}$$

推论2.1.2.

$$\begin{cases} (\sigma \otimes I_{2s}, -i\zeta)^a \partial_a \varphi(s, \zeta) = iJ(s, \zeta) \\ \psi(s, \zeta) = \bar{N}(s)\varphi(s, \zeta) = 0 \\ \psi(s-1, \zeta) = \bar{X}(s)\varphi(s, \zeta) \end{cases} \stackrel{S(s)}{\Leftrightarrow} \begin{cases} \frac{1}{s}O^+(s) \cdot \nabla \psi(s-1, \zeta) = i\bar{N}(s)J(s, \zeta) \\ [\frac{1}{s}\sigma(s-1), i\zeta]^a \partial_a \psi(s-1, \zeta) = -i\bar{X}(s)J(s, \zeta) \end{cases}$$

推论2.1.3.

$$\begin{cases} (\sigma \otimes I_{2s}, -i\zeta)^a \partial_a \varphi(s, \zeta) = iJ(s, \zeta) \\ \psi(s, \zeta) = \bar{N}(s)\varphi(s, \zeta) \\ \psi(s-1, \zeta) = \bar{X}(s)\varphi(s, \zeta) = 0 \end{cases} \stackrel{S(s)}{\Leftrightarrow} \begin{cases} [\frac{1}{s}\sigma(s), -i\zeta]^a \partial_a \psi(s, \zeta) = i\bar{N}(s)J(s, \zeta) \\ \frac{1}{s}O(s) \cdot \nabla \psi(s, \zeta) = i\bar{X}(s)J(s, \zeta) \end{cases}$$

#### 2.1.2 低一阶导数s-自旋粒子的新型耦合理论

推论2.1.4.

$$\begin{cases} (\sigma \otimes I_4, -i\zeta)_a \varphi^{ab}(s, \zeta) = iJ(s, \zeta) \\ \psi(s, \zeta) = \bar{N}(s)\varphi(s, \zeta) \\ \psi(s-1, \zeta) = \bar{X}(s)\varphi(s, \zeta) \end{cases} \stackrel{S(s)}{\Leftrightarrow} \begin{cases} [\frac{1}{s}\sigma(s), -i\zeta]_a \psi^{ab}(s, \zeta) = -\frac{1}{s}O_i^+(s)\psi^{ib}(s-1, \zeta) + i\bar{N}(s)J(s, \zeta) \\ [\frac{1}{s}\sigma(s-1), i\zeta]_a \psi^{ab}(s-1, \zeta) = \frac{1}{s}O_i(s)\psi^{ib}(s, \zeta) - i\bar{X}(s)J(s, \zeta) \end{cases}$$

推论2.1.5.

$$\begin{cases} (\sigma \otimes I_4, -i\zeta)_a \varphi^{ab}(s, \zeta) = iJ(s, \zeta) \\ \psi(s, \zeta) = \bar{N}(s)\varphi(s, \zeta) = 0 \\ \psi(s-1, \zeta) = \bar{X}(s)\varphi(s, \zeta) \end{cases} \stackrel{S(s)}{\Leftrightarrow} \begin{cases} \frac{1}{s}O_i^+(s)\psi^{ib}(s-1, \zeta) = i\bar{N}(s)J(s, \zeta) \\ [\frac{1}{s}\sigma(s-1), i\zeta]_a \psi^{ab}(s-1, \zeta) = -i\bar{X}(s)J(s, \zeta) \end{cases}$$

推论2.1.6.

$$\begin{cases} (\sigma \otimes I_4, -i\zeta)_a \varphi^{ab}(s, \zeta) = iJ(s, \zeta) \\ \psi(s, \zeta) = \bar{N}(s)\varphi(s, \zeta) \\ \psi(s-1, \zeta) = \bar{X}(s)\varphi(s, \zeta) = 0 \end{cases} \stackrel{S(s)}{\Leftrightarrow} \begin{cases} [\frac{1}{s}\sigma(s), -i\zeta]_a \psi^{ab}(s, \zeta) = i\bar{N}(s)J(s, \zeta) \\ \frac{1}{s}O_i(s)\psi^{ib}(s, \zeta) = i\bar{X}(s)J(s, \zeta) \end{cases}$$

## 2.2 具体的新型耦合理论

### 2.2.1 引力微子和中微子的新型耦合理论

推论2.2.1.

$$\begin{cases} [\frac{2}{3}\sigma(\frac{3}{2}), -i\varsigma]^a \partial_a \psi(\frac{3}{2}, \varsigma) = -\frac{2}{3}O^+(\frac{3}{2}) \cdot \nabla \psi(\frac{1}{2}, \varsigma) \\ [-\frac{2}{3}\sigma(\frac{1}{2}), -i\varsigma]^a \partial_a \psi(\frac{1}{2}, \varsigma) = -\frac{2}{3}O(\frac{3}{2}) \cdot \nabla \psi(\frac{3}{2}, \varsigma) \\ [2\sigma(\frac{1}{2}), -i\varsigma]^a \partial_a \psi(\frac{1}{2}, \varsigma) = 0 \end{cases} \Leftrightarrow \begin{cases} i\varsigma \partial_\pi \psi(\frac{1}{2}, \varsigma) = \frac{1}{2}O(\frac{3}{2}) \cdot \nabla \psi(\frac{3}{2}, \varsigma) \\ i\varsigma \partial_\pi \psi(\frac{3}{2}, \varsigma) = \frac{2}{3}\sigma(\frac{3}{2}) \cdot \nabla \psi(\frac{3}{2}, \varsigma) + \frac{2}{3}O^+(\frac{3}{2}) \cdot \nabla \psi(\frac{1}{2}, \varsigma) \\ 4\sigma(\frac{1}{2}) \cdot \nabla \psi(\frac{1}{2}, \varsigma) - O(\frac{3}{2}) \cdot \nabla \psi(\frac{3}{2}, \varsigma) = 0 \end{cases}$$

### 2.2.2 引力子、光子和标量场新型耦合理论

推论2.2.2.

$$\begin{cases} [\frac{1}{2}\sigma(2), -i\varsigma]^a \partial_a \psi(2, \varsigma) = -\frac{1}{2}O^+(2) \cdot \nabla \psi(1, \varsigma) + i\bar{N}(2)J(2, \varsigma) \\ [-\frac{1}{2}\sigma(1), -i\varsigma]^a \partial_a \psi(1, \varsigma) = -\frac{1}{2}O(2) \cdot \nabla \psi(2, \varsigma) + i\bar{X}(2)J(2, \varsigma) \\ [\sigma(1), -i\varsigma]^a \partial_a \psi(1, \varsigma) = -O^+(1) \cdot \nabla \phi + i\bar{N}(1)J(1, \varsigma) \\ [-\sigma(0), -i\varsigma]^a \partial_a \phi = -O(1) \cdot \nabla \psi(1, \varsigma) + i\bar{X}(1)J(1, \varsigma) \end{cases}$$

推论2.2.3.

$$\begin{cases} [\frac{1}{2}\sigma(2), -i\varsigma]^a \partial_a \psi(2, \varsigma) = -\frac{1}{2}O^+(2) \cdot \nabla \psi(1, \varsigma) \\ [-\frac{1}{2}\sigma(1), -i\varsigma]^a \partial_a \psi(1, \varsigma) = -\frac{1}{2}O(2) \cdot \nabla \psi(2, \varsigma) \\ [\sigma(1), -i\varsigma]^a \partial_a \psi(1, \varsigma) = -O^+(1) \cdot \nabla \phi \\ [-\sigma(0), -i\varsigma]^a \partial_a \phi = -O(1) \cdot \nabla \psi(1, \varsigma) \end{cases} \Leftrightarrow \begin{cases} i\varsigma \partial_\pi \phi = O(1) \cdot \nabla \psi(1, \varsigma) \\ i\varsigma \partial_\pi \psi(1, \varsigma) = \frac{1}{3}O^+(1) \cdot \nabla \phi + \frac{1}{3}O(2) \cdot \nabla \psi(2, \varsigma) \\ i\varsigma \partial_\pi \psi(2, \varsigma) = \frac{1}{2}\sigma(2) \cdot \nabla \psi(2, \varsigma) + \frac{1}{2}O^+(2) \cdot \nabla \psi(1, \varsigma) \\ 2O^+(1) \cdot \nabla \phi + 3\sigma(1) \cdot \nabla \psi(1, \varsigma) - O(2) \cdot \nabla \psi(2, \varsigma) = 0 \end{cases}$$

### 2.2.3 引力子、光子和标量场新型耦合的自旋方程

推论2.2.4.  $\{\partial_a + iS_{ab}[\sigma(1), \varsigma]\partial^b\} \otimes I_3 \psi(1 \otimes 1, \varsigma) = 0$

$$\Leftrightarrow \begin{cases} \{2\partial_a + iS_{ab}[\sigma(2), \varsigma]\partial^b\} \psi(2, \varsigma) + iS_{ab}[\frac{1}{\sqrt{3}}O^+(2), \varsigma]\partial^b \psi(1, \varsigma) = 0 \\ iS_{ab}[\frac{1}{\sqrt{3}}O(2), \varsigma]\partial^b \psi(2, \varsigma) + \{2\partial_a + iS_{ab}[\sigma(1), \varsigma]\partial^b\} \psi(1, \varsigma) + iS_{ab}[\frac{2}{\sqrt{3}}O^+(2), \varsigma]\partial^b \psi(0, \varsigma) = 0 \\ iS_{ab}[\frac{2}{\sqrt{3}}O(2), \varsigma]\partial^b \psi(1, \varsigma) + \{2\partial_a + iS_{ab}[\sigma(0), \varsigma]\partial^b\} \psi(0, \varsigma) = 0 \end{cases}$$

### 2.2.4 一个束缚光子的理论

推论2.2.5. 无平面波解 (Z轴)

$$\begin{cases} (\sigma \otimes I_4, -i\varsigma)^a \partial_a \varphi(2, \varsigma) = 0 \\ \psi(2, \varsigma) = 0 \end{cases} \Leftrightarrow \begin{cases} O^+(2) \cdot \nabla \psi(1, \varsigma) = 0 \\ [\sigma(1), 2i\varsigma]^a \partial_a \psi(1, \varsigma) = 0 \end{cases} \Leftrightarrow \begin{cases} O^+(2)S_m^+(1) \cdot \nabla \Psi(1, \varsigma) = 0 \\ (\gamma, 2i\varsigma)^a \partial_a \Psi(1, \varsigma) = 0 \end{cases}$$

### 2.2.5 一个推广的新引力理论

推论2.2.6.

$$\begin{cases} (\sigma \otimes I_4, -i\varsigma)_a \varphi^{ab}(2, \varsigma) = iJ(2, \varsigma) \\ \psi(2, \varsigma) = \bar{N}(2)\varphi(2, \varsigma) \\ \psi(1, \varsigma) = \bar{X}(2)\varphi(2, \varsigma) \end{cases} \stackrel{S(2)}{\Leftrightarrow} \begin{cases} [\frac{1}{2}\sigma(2), -i\varsigma]_a \psi^{ab}(2, \varsigma) = -\frac{1}{2}O_i^+(2)\psi^{ib}(1, \varsigma) + i\bar{N}(2)J(2, \varsigma) \\ [\frac{1}{2}\sigma(1), i\varsigma]_a \psi^{ab}(1, \varsigma) = \frac{1}{2}O_i(2)\psi^{ib}(2, \varsigma) - i\bar{X}(2)J(2, \varsigma) \end{cases}$$

# 第十四章 洛伦兹变换的深入分析

自我评述：在本章我对洛伦兹变换作了细致深入的分析，揭示了基本粒子与质点系之间的某种等价关系。特别是得到了各种常见自旋粒子洛伦兹变换的多项式表示，为以后各种自旋粒子物理的研究提供了又一个十分有用的数学工具。

## 1 3+1维时空的洛伦兹群表示 [8, 12]

### 1.1 彭加莱群表示 [8]

彭加莱群生成元  $M_{ab}, p_a$  对易关系：

$$M_{ab} = L_{ab} + S_{ab}, L_{ab} = x_a p_b - x_b p_a, g_{ab} = \delta_{ab} \quad (14.1)$$

$$\begin{cases} [M_{ab}, M_{cd}] = -i(g_{ad}M_{bc} - g_{ac}M_{bd} + g_{bc}M_{ad} - g_{bd}M_{ac}) \\ [M_{ab}, p_c] = -i(g_{bc}p_a - g_{ac}p_b), [p_a, p_b] = 0 \end{cases} \quad (14.2)$$

彭加莱群生成元  $L_{ab}, S_{ab}, p_a$  对易关系：

$$\begin{cases} [L_{ab}, L_{cd}] = -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \\ [L_{ab}, p_c] = -i(g_{bc}p_a - g_{ac}p_b), [p_a, p_b] = 0 \end{cases} \quad (14.3)$$

$$[S_{ab}, S_{cd}] = -i(g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}) \quad (14.4)$$

$$[S_{ab}, L_{cd}] = 0, [S_{ab}, p_c] = 0 \quad (14.5)$$

### 1.2 从自旋张量中提取矢量 $\vec{X}, \vec{Y}, \vec{a}, \vec{b}$

定义1.2.1.  $X^i \equiv \frac{1}{2}\varepsilon^{ijk}S_{jk}, Y_i \equiv S_{\pi i}, a_i \equiv \frac{1}{2}(X_i + Y_i), b_i \equiv \frac{1}{2}(X_i - Y_i), g_{ab} := \delta_{ab}$

性质1.2.1.  $\varepsilon_{ijk}\varepsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl}$   
 $\varepsilon_{ijk}\varepsilon^k_{lm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \varepsilon_{ijk}\varepsilon^{jk}_l = 2\delta_{il}$

推论1.2.1.  $X^i = \frac{1}{2}\varepsilon^{ijk}S_{jk} \Leftrightarrow S_{ij} = \varepsilon_{ijk}X^k$

### 1.3 洛伦兹群表示关系命题的正向证明

定理1.3.1.  $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Rightarrow [X^i, X^l] = i\varepsilon^{il}_k X^k$

证明:  $[X^i, X^l] = [\frac{1}{2}\varepsilon^{ijk}S_{jk}, \frac{1}{2}\varepsilon^{lmn}S_{mn}]$

$$= \frac{1}{4}\varepsilon^{ijk}\varepsilon^{lmn}[S_{jk}, S_{mn}]$$

$$= -i\frac{1}{4}\varepsilon^{ijk}\varepsilon^{lmn}(g_{jn}S_{km} - g_{jm}S_{kn} + g_{km}S_{jn} - g_{kn}S_{jm})$$

$$= -i\frac{1}{4}\varepsilon^{ijk}\varepsilon^{lmn}(2g_{jn}S_{km} + 2g_{km}S_{jn})$$

$$= -\frac{i}{2}(\varepsilon^{ijk}\varepsilon^{lm}_j S_{km} - \varepsilon^{ijk}\varepsilon^{ln}_k S_{jn})$$

$$= -\frac{i}{2}(\varepsilon^{ikj}\varepsilon^{ln}_k S_{jn} - \varepsilon^{ijk}\varepsilon^{ln}_k S_{jn})$$

$$= i\varepsilon^{ijk}\varepsilon^{ln}_k S_{jn}$$

$$= (\delta^{il}\delta^{jn} - \delta^{in}\delta^{jl})S_{jn}$$

$$= iS^{il} = i\varepsilon^{il}_k X^k \quad \square$$

定理1.3.2.  $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Rightarrow [Y_i, Y_j] = i\varepsilon_{ij}^k X_k$

证明:  $[Y_i, Y_j] = [S_{\pi i}, S_{\pi j}]$

$$= [S_{i\pi}, S_{j\pi}]$$

$$= -i(g_{i\pi}S_{\pi j} - g_{ij}S_{\pi\pi} + g_{\pi j}S_{i\pi} - g_{\pi\pi}S_{ij})$$

$$= iS_{ij} = i\varepsilon_{ij}^k X_k \quad \square$$

**定理1.3.3.**  $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Rightarrow [X^i, Y_l] = i\varepsilon_l^i{}^k Y_k$

**证明:**  $[X^i, Y_l] = [\frac{1}{2}\varepsilon^{ijk}S_{jk}, S_{\pi l}]$

$$= -\frac{1}{2}\varepsilon^{ijk}[S_{jk}, S_{l\pi}]$$

$$= \frac{i}{2}\varepsilon^{ijk}(g_{j\pi}S_{kl} - g_{jl}S_{k\pi} + g_{kl}S_{j\pi} - g_{k\pi}S_{jl})$$

$$= \frac{i}{2}\varepsilon^{ijk}(-g_{jl}S_{k\pi} + g_{kl}S_{j\pi})$$

$$= i\varepsilon^{ijk}g_{kl}S_{j\pi} = i\varepsilon_l^i{}^k Y_k$$

□

**推论1.3.1.**  $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}$

$$\Rightarrow [X_i, X_j] = i\varepsilon_{ij}{}^k X^k, [Y_i, Y_j] = i\varepsilon_{ij}{}^k X^k, [X_i, Y_j] = i[Y_i, Y_j] = i\varepsilon_{ij}{}^k Y_k$$

## 1.4 洛伦兹群表示关系命题的反向证明

**定理1.4.1.**  $[X^i, X^l] = i\varepsilon^{il}{}^k X^k \Rightarrow i[S_{ij}, S_{lm}] = g_{im}S_{jl} - g_{il}S_{jm} + g_{jl}S_{im} - g_{jm}S_{il}$

**证明:**  $i[S_{ij}, S_{lm}] = i[\varepsilon_{ijk}X^k, \varepsilon_{lmn}X^n]$

$$= i\varepsilon_{ijk}\varepsilon_{lmn}[X^k, X^n] = -\varepsilon_{ijk}\varepsilon_{lmn}\varepsilon^{kn}{}_h X^h = -\varepsilon_{ijk}\varepsilon_{lmn}S^{kn}$$

$$= -(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl})S^{kn}$$

$$= -(\delta_{in}\delta_{jl}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl})S^{kn}$$

$$= -(\delta_{jl}S_{mi} + \delta_{im}S^{lj} - \delta_{il}S^{mj} - \delta_{jm}S^{li})$$

$$= \delta_{im}S_{jl} - \delta_{il}S_{jm} + \delta_{jl}S_{im} - \delta_{jm}S_{il}$$

$$= g_{im}S_{jl} - g_{il}S_{jm} + g_{jl}S_{im} - g_{jm}S_{il}$$

□

**定理1.4.2.**  $[X_i, Y_j] = i\varepsilon_{ij}{}^k Y_k \Rightarrow i[S_{ij}, S_{\pi l}] = g_{il}S_{j\pi} - g_{i\pi}S_{jl} + g_{j\pi}S_{il} - g_{jl}S_{i\pi}$

**证明:**  $i[S_{ij}, S_{\pi l}] = i[\varepsilon_{ijk}X^k, Y_l]$

$$= i\varepsilon_{ijk}[X^k, Y^l] = -\varepsilon_{ijk}\varepsilon^k{}_{lm}Y^m$$

$$= -(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})S^{\pi m} = \delta_{il}S_{j\pi} - \delta_{jl}S_{i\pi}$$

$$= \delta_{il}S_{j\pi} - \delta_{i\pi}S_{jl} + \delta_{j\pi}S_{il} - \delta_{jl}S_{i\pi}$$

$$= g_{il}S_{j\pi} - g_{i\pi}S_{jl} + g_{j\pi}S_{il} - g_{jl}S_{i\pi}$$

□

**定理1.4.3.**  $[Y_i, Y_j] = i\varepsilon_{ij}{}^k X^k \Rightarrow i[S_{\pi i}, S_{\pi j}] = g_{\pi j}S_{i\pi} - g_{\pi\pi}S_{ij} + g_{i\pi}S_{\pi j} - g_{ij}S_{\pi\pi}$

**证明:**  $i[S_{\pi i}, S_{\pi j}] = i[Y_i, Y_j]$

$$= -\varepsilon_{ijk}X^k = -S_{ij}$$

$$= \delta_{\pi j}S_{i\pi} - \delta_{\pi\pi}S_{ij} + \delta_{i\pi}S_{\pi j} - \delta_{ij}S_{\pi\pi}$$

$$= g_{\pi j}S_{i\pi} - g_{\pi\pi}S_{ij} + g_{i\pi}S_{\pi j} - g_{ij}S_{\pi\pi}$$

□

**推论1.4.1.**  $[X_i, X_j] = i\varepsilon_{ij}{}^k X^k, [Y_i, Y_j] = i\varepsilon_{ij}{}^k X^k, [X_i, Y_j] = i[Y_i, Y_j] = i\varepsilon_{ij}{}^k Y_k$

$$\Rightarrow i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}$$

## 1.5 洛伦兹群表示关系命题的综合结论

**推论1.5.1.**  $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}$

$$\Leftrightarrow [X_i, X_j] = i\varepsilon_{ij}{}^k X^k, [Y_i, Y_j] = i\varepsilon_{ij}{}^k X^k, [X_i, Y_j] = i[Y_i, Y_j] = i\varepsilon_{ij}{}^k Y_k$$

**推论1.5.2.**  $[X_i, X_j] = i\varepsilon_{ij}{}^k X^k, [Y_i, Y_j] = i\varepsilon_{ij}{}^k X^k, [X_i, Y_j] = i[Y_i, Y_j] = i\varepsilon_{ij}{}^k Y_k$

$$\Leftrightarrow \vec{X} \times \vec{X} = i\vec{X}, \vec{Y} \times \vec{Y} = i\vec{X}, \vec{X} \times \vec{Y} = i\vec{Y}, [X_i, Y_i] = 0$$

$$\Leftrightarrow \vec{a} \times \vec{a} = i\vec{a}, \vec{b} \times \vec{b} = i\vec{b}, [a_i, b_j] = 0$$

**推论1.5.3.**  $i[S_{ab}, S_{cd}] = g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac} \Leftrightarrow \vec{a} \times \vec{a} = i\vec{a}, \vec{b} \times \vec{b} = i\vec{b}, [a_i, b_j] = 0$

**洛伦兹态变换分解:**

**推论1.5.4.**  $e^{\frac{i}{2}\varepsilon^{ab}S_{ab}} = e^{i\omega \cdot \vec{X} + \epsilon \cdot \vec{Y}} = e^{(i\omega + \epsilon) \cdot \vec{a}} e^{(i\omega - \epsilon) \cdot \vec{b}}$

## 2 单粒子的相对论洛伦兹推动变换

### 2.1 坐标的洛伦兹变换 [23-25]

约定： $O$ 在 $O'$ 中的速度为 $\vec{v}$ ,  $v \neq 1$ ,  $O'$ 在 $O$ 中的速度为 $-\vec{v}$ , 此约定好处是可以直观描述运动粒子。

坐标及其坐标微分的相对论洛伦兹推动变换的一般形式：

$$\text{定义2.1.1.} \quad \begin{cases} \nabla' = \nabla - \gamma_v \vec{v} \partial_t + (\gamma_v - 1) \vec{v} / v^2 (\vec{v} \cdot \nabla) \\ \partial_{t'} = \gamma_v (\partial_t - \vec{v} \cdot \nabla), \gamma_v \equiv (1 - v^2)^{-\frac{1}{2}} \end{cases}$$

$$\text{定义2.1.2.} \quad \begin{cases} \vec{r}' = \vec{r} + \gamma_v \vec{v} t + (\gamma_v - 1) (\vec{v} \cdot \vec{r}) \vec{v} / v^2 \\ t' = \gamma_v (t + \vec{v} \cdot \vec{r}), \gamma_v \equiv (1 - v^2)^{-\frac{1}{2}} \end{cases} \quad \begin{cases} d\vec{r}' = d\vec{r} + \gamma_v \vec{v} dt + (\gamma_v - 1) (\vec{v} \cdot d\vec{r}) \vec{v} / v^2 \\ dt' = \gamma_v (dt + \vec{v} \cdot d\vec{r}) \end{cases}$$

以上变换是整个狭义相对论的重要基础, 另一个重要变换是矢量旋转变换。

推论2.1.1.  $\vec{r}'^2 - t'^2 = \vec{r}^2 - t^2 = \text{不变量}, d\vec{r}'^2 - dt'^2 = d\vec{r}^2 - dt^2 = \text{不变量}$

$$\text{定义2.1.3.} \quad L_{\vec{v}} := \begin{bmatrix} 1 & 0 & 0 & -i\gamma_v v_x \\ 0 & 1 & 0 & -i\gamma_v v_y \\ 0 & 0 & 1 & -i\gamma_v v_z \\ i\gamma_v v_x & i\gamma_v v_y & i\gamma_v v_z & \gamma_v \end{bmatrix} + \frac{\gamma_v - 1}{v^2} \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z & 0 \\ v_y v_x & v_y v_y & v_y v_z & 0 \\ v_z v_x & v_z v_y & v_z v_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, L_{\vec{v}} L_{-\vec{v}} = L_{-\vec{v}} L_{\vec{v}} = I$$

$$\text{推论2.1.2.} \quad L_{\vec{v}} \begin{bmatrix} \vec{0} \\ i \end{bmatrix} = e^{-\ln[\gamma_v(1+v)] \hat{v} \cdot L} \begin{bmatrix} \vec{0} \\ i \end{bmatrix} = \begin{bmatrix} \gamma_v \vec{v} \\ i\gamma_v \end{bmatrix}, L_{\vec{v}} \begin{bmatrix} \vec{0} \\ im \end{bmatrix} = e^{-\ln[\gamma_v(1+v)] \hat{v} \cdot L} \begin{bmatrix} \vec{0} \\ im \end{bmatrix} = \begin{bmatrix} \vec{p} \\ iE \end{bmatrix}$$

引理2.1.1.  $(\vec{v} \cdot R)^2 + (\vec{v} \cdot L)^2 = \vec{v}^2$

推论2.1.3.

$$L_{\vec{v}} = e^{-\ln[\gamma_v(1+v)] \hat{v} \cdot L} = 1 - \gamma_v (\vec{v} \cdot L) + \frac{\gamma_v - 1}{v^2} (\vec{v} \cdot L)^2 = \gamma_v (1 - \vec{v} \cdot L) - \frac{\gamma_v - 1}{v^2} (\vec{v} \cdot R)^2, L_{\vec{v}} L_{-\vec{v}} = L_{-\vec{v}} L_{\vec{v}} = I$$

$$\text{推论2.1.4.} \quad X' = L_{\vec{v}} dX, X \equiv \begin{bmatrix} \vec{r} \\ it \end{bmatrix}, X' \equiv \begin{bmatrix} \vec{r}' \\ it' \end{bmatrix}; dX' = L_{\vec{v}} dX, dX \equiv \begin{bmatrix} d\vec{r} \\ idt \end{bmatrix}, dX' \equiv \begin{bmatrix} d\vec{r}' \\ idt' \end{bmatrix}$$

### 2.2 速度合成公式

推论2.2.1.  $\vec{u}' = [\vec{u} + \gamma_v \vec{v} + (\gamma_v - 1) (\vec{v} \cdot \vec{u}) \vec{v} / v^2] / [\gamma_v (1 + \vec{v} \cdot \vec{u})]$

推论2.2.2.  $1 - \vec{u}'^2 = \frac{(1 - \vec{u}^2)(1 - \vec{v}^2)}{(1 + \vec{v} \cdot \vec{u})^2}$

$$\text{推论2.2.3.} \quad \begin{cases} \gamma_{u'} \vec{u}' = \gamma_u \vec{u} + \gamma_v \vec{v} \gamma_u + (\gamma_v - 1) [\vec{v} \cdot (\gamma_u \vec{u})] \vec{v} / v^2 \\ \gamma_{u'} = \gamma_v [\gamma_u + \vec{v} \cdot (\gamma_u \vec{u})] \end{cases} \Leftrightarrow \begin{bmatrix} \gamma_{u'} \vec{u}' \\ i\gamma_{u'} \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \gamma_u \vec{u} \\ i\gamma_u \end{bmatrix}$$

### 2.3 有质量粒子四动量的洛伦兹推动变换

有质量单粒子： $m_0 \neq 0, u \neq 1, u' \neq 1$

定义2.3.1.  $E \equiv m_0 (1 - u^2)^{-\frac{1}{2}}, E' \equiv m_0 (1 - u'^2)^{-\frac{1}{2}}, \vec{p} \equiv E \vec{u}, \vec{p}' \equiv E' \vec{u}'$

可以从坐标的洛伦兹推动变换出发, 推导得到下面的能量动量的洛伦兹推动变换。

$$\text{推论2.3.1.} \quad \begin{cases} \vec{p}' = \vec{p} + \gamma_v E \vec{v} + (\gamma_v - 1) (\vec{v} \cdot \vec{p}) \vec{v} / v^2 \\ E' = \gamma_v (E + \vec{v} \cdot \vec{p}) \end{cases} \Leftrightarrow \begin{bmatrix} \vec{p}' \\ iE' \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \vec{p} \\ iE \end{bmatrix}$$

推论2.3.2.  $\vec{p}'^2 - E'^2 = \vec{p}^2 - E^2 = -m_0^2 = \text{不变量}$

### 2.4 不同速度参考系间的温度洛伦兹变换猜想

猜想2.4.1.

$$\begin{cases} \text{运动系动能-质点系平动动能} = E'_k - E'_{k0} = \sum_i (\gamma_v E_i - m_0) - (\gamma_v - 1) \sum_i E_i = \frac{3}{2} N k_B T' \\ \text{静止系动能-质点系平动动能} = E_k - E_{k0} = \sum_i (E_i - m_0) - 0 = \frac{3}{2} N k_B T \end{cases} \Rightarrow T' = T$$

## 2.5 无质量粒子四动量的洛伦兹推动变换

无质量粒子：  $m_0 = 0, u = 1, u' = 1$

定义2.5.1.  $\vec{p} \equiv E\vec{u}, \vec{p}' \equiv E'\vec{u}'$

对于无质量粒子从坐标的洛伦兹推动变换出发，无法严格推导得到能量动量的洛伦兹推动变换，但可以使质量无限接近于零而得到。

$$\begin{cases} \vec{p}' = \vec{p} + \gamma_v E \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{p})\vec{v}/v^2 \\ E' = \gamma_v(E + \vec{v} \cdot \vec{p}) \end{cases}, \begin{bmatrix} \vec{p}' \\ iE' \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \vec{p} \\ iE \end{bmatrix} \quad (14.6)$$

推论2.5.1.  $\vec{p}'^2 - E'^2 = \vec{p}^2 - E^2 = 0 = \text{不变量}$

## 2.6 单粒子外力的洛伦兹推动变换

定义2.6.1.  $\vec{F} \equiv \frac{dp}{dt}, \vec{F}' \equiv \frac{dp'}{dt'}, \vec{f} \equiv \frac{\vec{F}}{\sqrt{1-u^2}}, \vec{f}' \equiv \frac{\vec{F}'}{\sqrt{1-u'^2}}$

推论2.6.1.  $\vec{a}' = [\vec{a} + (\gamma_v - 1)(\vec{v} \cdot \vec{a})\vec{v}/v^2]/[\gamma_v^2(1 + \vec{v} \cdot \vec{u})^2] - [\vec{u} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{u})\vec{v}/v^2](\vec{v} \cdot \vec{a})/[\gamma_v^2(1 + \vec{v} \cdot \vec{u})^3]$

推论2.6.2.  $\vec{F}' = [\vec{F} + \gamma_v(\vec{u} \cdot \vec{F})\vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{F})\vec{v}/v^2]/[\gamma_v(1 + \vec{v} \cdot \vec{u})]$

推论2.6.3.  $\vec{u}' \cdot \vec{F}' = \gamma_v(\vec{u} \cdot \vec{F} + \vec{v} \cdot \vec{F})/[\gamma_v(1 + \vec{v} \cdot \vec{u})] = \frac{\vec{v} + \vec{u}}{1 + \vec{v} \cdot \vec{u}} \cdot \vec{F}$

推论2.6.4.  $\begin{cases} \vec{f}' = \vec{f} + \gamma_v(\vec{u} \cdot \vec{f})\vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{f})\vec{v}/v^2 \\ \vec{u}' \cdot \vec{f}' = \gamma_v(\vec{u} \cdot \vec{f} + \vec{v} \cdot \vec{f}) = \gamma_v(\vec{u} + \vec{v}) \cdot \vec{f} \end{cases} \Leftrightarrow \begin{bmatrix} \vec{f}' \\ i\vec{u}' \cdot \vec{f}' \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \vec{f} \\ i\vec{u} \cdot \vec{f} \end{bmatrix}$

推论2.6.5.  $\vec{f}'^2 - (\vec{u}' \cdot \vec{f}')^2 = \vec{f}^2 - (\vec{u} \cdot \vec{f})^2 = \text{不变量}$

## 2.7 单粒子外力的广义相对论变换关系假设

定义2.7.1.  $\vec{a} \equiv \frac{d\vec{u}}{dt}, \vec{a}' \equiv \frac{d\vec{u}'}{dt'}, \vec{g} \equiv \frac{d\vec{v}}{dt}$

定义2.7.2.

$$\begin{cases} \vec{a}' = [\vec{a} + (\gamma_v - 1)(\vec{v} \cdot \vec{a})\vec{v}/v^2]/[\gamma_v^2(1 + \vec{v} \cdot \vec{u})^2] - [\vec{u} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{u})\vec{v}/v^2](\vec{v} \cdot \vec{a})/[\gamma_v^2(1 + \vec{v} \cdot \vec{u})^3] + \\ [\gamma_v \vec{g} + (\gamma_v - 1)(\vec{g} \cdot \vec{u})\vec{v}/v^2 + (\gamma_v - 1)(\vec{v} \cdot \vec{u})\vec{g}/v^2 - 2(\gamma_v - 1)(\vec{v} \cdot \vec{u})(\vec{v} \cdot \vec{g})\vec{v}/v^4 + \gamma_v^3(\vec{v} \cdot \vec{g})(\vec{v} \cdot \vec{u})\vec{v}/v^2]/[\gamma_v^2(1 + \vec{v} \cdot \vec{u})^2] \\ - [\vec{u} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{u})\vec{v}/v^2][(\vec{g} \cdot \vec{u}) + \gamma_v^2(1 + \vec{v} \cdot \vec{u})(\vec{v} \cdot \vec{g})]/[\gamma_v^2(1 + \vec{v} \cdot \vec{u})^3] \\ \vec{F}' = [\vec{F} + \gamma_v(\vec{u} \cdot \vec{F})\vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{F})\vec{v}/v^2 \\ + \gamma_v E \vec{g} + \gamma_v^3(\vec{v} \cdot \vec{g})E\vec{v} + (\gamma_v - 1)(\vec{g} \cdot \vec{p})\vec{v}/v^2 + (\gamma_v - 1)(\vec{v} \cdot \vec{p})\vec{g}/v^2 \\ + \gamma_v^3(\vec{v} \cdot \vec{g})(\vec{v} \cdot \vec{p})\vec{v}/v^2 - 2(\gamma_v - 1)(\vec{v} \cdot \vec{g})(\vec{v} \cdot \vec{p})\vec{v}/v^4]/[\gamma_v(1 + \vec{v} \cdot \vec{u})] \end{cases}$$

## 3 多粒子质点系的相对论洛伦兹变换

### 3.1 多粒子质点系的洛伦兹推动变换

$$\begin{cases} \vec{P}(v) = \sum_i [\vec{p}_i + \gamma_v E_i \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{p}_i)\vec{v}/v^2] \\ H(\vec{v}) = \sum_i \gamma_v(E_i + \vec{v} \cdot \vec{p}_i) \end{cases} \Leftrightarrow \begin{bmatrix} \vec{P}(v) \\ iH(\vec{v}) \end{bmatrix} = \sum_i L_{\vec{v}} \begin{bmatrix} \vec{p}_i \\ iE_i \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \sum_i \vec{p}_i \\ i \sum_i E_i \end{bmatrix} \quad (14.7)$$

### 3.2 不同速度参考系质点系间的洛伦兹推动变换

不同速度参考系质点系间的洛伦兹推动变换：

$$\begin{cases} \vec{u}' = [\vec{u} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{u})\vec{v}/v^2]/[\gamma_v(1 + \vec{v} \cdot \vec{u})] \\ \vec{P}(\vec{u}') = \vec{P}(\vec{u}) + \gamma_v H(\vec{u})\vec{v} + (\gamma_v - 1)[\vec{v} \cdot \vec{P}(\vec{u})]\vec{v}/v^2 \\ H(\vec{u}') = \gamma_v [H(\vec{u}) + \vec{v} \cdot \vec{P}(\vec{u})] \end{cases} \Leftrightarrow \begin{cases} \begin{bmatrix} \gamma_{u'} \vec{u}' \\ i\gamma_{u'} \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \gamma_u \vec{u} \\ i\gamma_u \end{bmatrix} \\ \begin{bmatrix} \vec{P}(\vec{u}') \\ iH(\vec{u}') \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \vec{P}(\vec{u}) \\ iH(\vec{u}) \end{bmatrix} \end{cases} \quad (14.8)$$

推论3.2.1.  $\vec{P}^2(\vec{u}') - H^2(\vec{u}') = \vec{P}^2(\vec{u}) - H^2(\vec{u}) = -M_0^2 = \text{不变量}$

不同质心速度质点系间外力的洛伦兹推动变换：

定义3.2.1.  $\vec{f}(\vec{u}) \equiv \frac{\vec{F}(\vec{u})}{\sqrt{1-u^2}}, \vec{f}'(\vec{u}') \equiv \frac{\vec{F}'(\vec{u}')}{\sqrt{1-u'^2}}, \vec{f}(\vec{u}) = \frac{d\vec{P}(\vec{u})}{d\tau}, \vec{f}'(\vec{u}') = \frac{d\vec{P}'(\vec{u}')}{d\tau'}$

引理3.2.1.  $\frac{dH(\vec{u})}{d\tau} \equiv \vec{u} \cdot \frac{d\vec{P}(\vec{u})}{d\tau}$

$$\text{推论3.2.2. } \begin{cases} \vec{f}'(\vec{u}') = \vec{f}(\vec{u}) + \gamma_v [\vec{u} \cdot \vec{f}(\vec{u})]\vec{v} + (\gamma_v - 1)[\vec{v} \cdot \vec{f}(\vec{u})]\vec{v}/v^2 \\ \vec{u}' \cdot \vec{f}'(\vec{u}') = \gamma_v [\vec{u} \cdot \vec{f}(\vec{u}) + \vec{v} \cdot \vec{f}(\vec{u})] = \gamma_v (\vec{u} + \vec{v}) \cdot \vec{f}(\vec{u}) \end{cases} \Leftrightarrow \begin{bmatrix} \vec{f}'(\vec{u}') \\ i\vec{u}' \cdot \vec{f}'(\vec{u}') \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \vec{f}(\vec{u}) \\ i\vec{u} \cdot \vec{f}(\vec{u}) \end{bmatrix}$$

推论3.2.3.  $\vec{f}'^2(\vec{u}') - [\vec{u}' \cdot \vec{f}'(\vec{u}')]^2 = \vec{f}^2(\vec{u}) - [\vec{u} \cdot \vec{f}(\vec{u})]^2 = \text{不变量}$

### 3.3 动质点系推动变换到静质点系

引理3.3.1.  $|\sum_i \vec{p}_i / \sum_i E_i| \leq 1$ , 等号有且仅当  $\vec{p}_i = E_i \vec{1}$  时成立。

定义3.3.1. 动质点系： $|\sum_i \vec{p}_i / \sum_i E_i| \neq 0$ , 静质点系： $|\sum_i \vec{p}_i / \sum_i E_i| = 0$

有质量动质点系到静质点系的洛伦兹推动变换：

$$\vec{v} = -\sum_i \vec{p}_i / \sum_i E_i \neq \vec{1} \Rightarrow \begin{cases} \vec{P}(\vec{v}) = \sum_i [\vec{p}_i + \gamma_v E_i \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{p}_i)\vec{v}/v^2] = 0 \\ M_0 = H(\vec{v}) = \sum_i \gamma_v (E_i + \vec{v} \cdot \vec{p}_i) = \sum_i E_i / \gamma_v \end{cases} \quad (14.9)$$

### 3.4 静质点系推动变换到动质点系

静质点系推动变换到动质点系：

$$\sum_i \vec{p}_i / \sum_i E_i = 0 \Rightarrow \begin{cases} H(\vec{v}) = \gamma_v \sum_i E_i = M, M \equiv \gamma_v M_0, M_0 \equiv \sum_i E_i \\ \vec{P}(\vec{v}) = \gamma_v \vec{v} \sum_i E_i = M\vec{v} \end{cases} \quad (14.10)$$

以上关系式的物理意义是：可以将质点系等效为一个粒子，当该质点系作运动时，可以等效为一个粒子的运动，并且与粒子一样符合相对论规律。并且该质点系质心静止时的总能就是该质点系的等效静止质量，质心运动时总能量就是等效的相对论动质量。因此质点系完全可以等效为一个粒子，反过来，也可以认为基本粒子就是一个质点系，难点是有相互作用的质点系是否还有如此的结论？是否可以按此线索对相互作用加一个约束，并得到新的物理？

### 3.5 单方向多光子系的洛伦兹推动变换

单方向多光子系的洛伦兹推动变换：

$$\vec{p}_i = E_i \vec{1}, \vec{1}' = [\vec{1} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{1})\vec{v}/v^2]/[\gamma_v(1 + \vec{v} \cdot \vec{1})] \quad (14.11)$$

$$\Rightarrow \begin{cases} \vec{P}(\vec{v}) = \sum_i [E_i \vec{1} + \gamma_v E_i \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{1})E_i \vec{v}/v^2] = \sum_i \gamma_v E_i (1 + \vec{v} \cdot \vec{1}) \vec{1}' \\ H(\vec{v}) = \sum_i \gamma_v E_i (1 + \vec{v} \cdot \vec{1}) \\ \vec{P}^2(\vec{v}) - H^2(\vec{v}) = -M_0^2, M_0 = \sum_i E_i \sqrt{1 - v^2} = 0 \end{cases} \quad (14.12)$$



### 3.6 质点系的普适静止质量公式

综合以上结论，可以得到以下普适的静止质量公式：

$$M_0 = \sum_i E_i \sqrt{1 - \left(\frac{\sum_i \vec{p}_i}{\sum_i E_i}\right)^2} = \sqrt{\left(\sum_i E_i\right)^2 - \left(\sum_i \vec{p}_i\right)^2} = \sqrt{\sum_{i,j} (E_i E_j - \vec{p}_i \cdot \vec{p}_j)} \quad (14.13)$$

简化记号的质量公式： $M_0 = \sqrt{\sum (E_i E_j - \vec{p}_i \cdot \vec{p}_j)}$

### 3.7 相互作用质点系势能的洛伦兹推动变换假设

相互作用动质点系势能状态假设：

$$\begin{cases} \vec{P}(\vec{0}) = \left(\frac{\sum_k \vec{p}_k}{\sum_k E_k}\right) \frac{1}{2} \sum_{i \neq j} V_{ij} \\ H(\vec{0}) = \frac{1}{2} \sum_{i \neq j} V_{ij} \end{cases} \quad (14.14)$$

相互作用动质点系势能的洛伦兹推动变换：

$$\begin{cases} \vec{P}(\vec{v}) = \vec{P}(\vec{0}) + \gamma_v H(\vec{0}) \vec{v} + (\gamma_v - 1) [\vec{v} \cdot \vec{P}(\vec{0})] \vec{v} / v^2 = \sum_{i \neq j} \frac{V_{ij}}{2E_k} \sum_i [\vec{p}_i + \gamma_v E_i \vec{v} + (\gamma_v - 1) (\vec{v} \cdot \vec{p}_i) \vec{v} / v^2] \\ H(\vec{v}) = \gamma_v [H(\vec{0}) + \vec{v} \cdot \vec{P}(\vec{0})] = \sum_{i \neq j} \frac{V_{ij}}{2E_k} \sum_i \gamma_v (E_i + \vec{v} \cdot \vec{p}_i) \\ \vec{P}^2(\vec{v}) - H^2(\vec{v}) = \vec{P}^2(\vec{0}) - H^2(\vec{0}) = -M_{V0}^2 \end{cases} \quad (14.15)$$

相互作用静质点系势能的洛伦兹推动变换：

$$\vec{v} = -\frac{\sum_i \vec{p}_i}{\sum_i E_i} \neq \vec{1} \Rightarrow \begin{cases} \vec{P}(\vec{v}) = \vec{0} \\ H(\vec{v}) = \sum_{i \neq j} V_{ij} \sqrt{1 - v^2} \equiv M_{V0} \end{cases} \quad (14.16)$$

相互作用质点系势能的质量公式： $M_{V0} = \left(\frac{1}{2} \sum_{i \neq j} V_{ij}\right) \sqrt{\sum_{i,j} (E_i E_j - \vec{p}_i \cdot \vec{p}_j) / \sum_{i,j} (E_i E_j)}$

### 3.8 多粒子相互作用质点系的洛伦兹推动变换

相互作用质点系的洛伦兹推动变换：

$$\text{推论 3.8.1.} \quad \begin{cases} \vec{P}(v) = \frac{1}{\sum_k 2E_k} \left(\sum_k 2E_k + \sum_{i \neq j} V_{ij}\right) \sum_i [\vec{p}_i + \gamma_v E_i \vec{v} + (\gamma_v - 1) (\vec{v} \cdot \vec{p}_i) \vec{v} / v^2] \\ H(\vec{v}) = \frac{1}{\sum_k 2E_k} \left(\sum_k 2E_k + \sum_{i \neq j} V_{ij}\right) \sum_i \gamma_v (E_i + \vec{v} \cdot \vec{p}_i) \end{cases}$$

证明：

$$\begin{aligned} \vec{P}(v) &= \sum_i [\vec{p}_i + \gamma_v E_i \vec{v} + (\gamma_v - 1) (\vec{v} \cdot \vec{p}_i) \vec{v} / v^2] + \frac{1}{2} \sum_{i,j} \{V_{ij} \left(\frac{\sum_k \vec{p}_k}{\sum_k E_k}\right) + \gamma_v V_{ij} \vec{v} + (\gamma_v - 1) [\vec{v} \cdot \left(\frac{\sum_k \vec{p}_k}{\sum_k E_k}\right)] \vec{v} / v^2\} \\ &= \frac{1}{\sum_k 2E_k} \left\{ \sum_k 2E_k \sum_i [\vec{p}_i + \gamma_v E_i \vec{v} + (\gamma_v - 1) (\vec{v} \cdot \vec{p}_i) \vec{v} / v^2] + \sum_{i \neq j} V_{ij} \sum_k \{ \vec{p}_k + \gamma_v E_k \vec{v} + (\gamma_v - 1) [\vec{v} \cdot (\vec{p}_k)] \vec{v} / v^2 \} \right\} \\ &= \frac{1}{\sum_k 2E_k} \left(\sum_k 2E_k + \sum_{i \neq j} V_{ij}\right) \sum_i [\vec{p}_i + \gamma_v E_i \vec{v} + (\gamma_v - 1) (\vec{v} \cdot \vec{p}_i) \vec{v} / v^2] \\ H(\vec{v}) &= \sum_i \gamma_v (E_i + \vec{v} \cdot \vec{p}_i) + \frac{1}{2} \sum_{i,j} \gamma_v [V_{ij} + \vec{v} \cdot \left(\frac{\sum_k \vec{p}_k}{\sum_k E_k}\right)] = \frac{1}{\sum_k 2E_k} \left(\sum_k 2E_k + \sum_{i \neq j} V_{ij}\right) \sum_i \gamma_v (E_i + \vec{v} \cdot \vec{p}_i) \quad \square \end{aligned}$$

相互作用动质点系到静质点系的洛伦兹推动变换：

$$\vec{v} = -\frac{\sum_i \vec{p}_i}{\sum_i E_i} \neq \vec{1} \Rightarrow \begin{cases} \vec{P}(\vec{v}) = \vec{0} \\ H(\vec{v}) = \left(\sum_k E_k + \frac{1}{2} \sum_{i \neq j} V_{ij}\right) \sqrt{1 - v^2} \equiv M_0 \end{cases} \quad (14.17)$$

相互作用质点系的质量公式： $M_0 = \left(\sum_i E_i + \frac{1}{2} \sum_{i \neq j} V_{ij}\right) \sqrt{\sum_{i,j} (E_i E_j - \vec{p}_i \cdot \vec{p}_j) / \sum_{i,j} (E_i E_j)}$

### 3.9 不同质心速度相互作用质点系间的洛伦兹推动变换

不同质心速度相互作用质点系间的洛伦兹推动变换：

$$\begin{cases} \vec{u}' = [\vec{u} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{u})\vec{v}/v^2]/[\gamma_v(1 + \vec{v} \cdot \vec{u})] \\ H(\vec{u}') = \gamma_v [H(\vec{u}) + \vec{v} \cdot \vec{P}(\vec{u})] \\ \vec{P}(\vec{u}') = \vec{P}(\vec{u}) + \gamma_v H(\vec{u})\vec{v} + (\gamma_v - 1)[\vec{v} \cdot \vec{P}(\vec{u})]\vec{v}/v^2 \\ \vec{P}^2(\vec{u}') - H^2(\vec{u}') = \vec{P}^2(\vec{u}) - H^2(\vec{u}) = -M_0^2 \end{cases} \quad (14.18)$$

不同质心速度相互作用质点系外力的洛伦兹推动变换：

$$\begin{cases} \vec{f}'(\vec{u}') = \vec{f}(\vec{u}) + \gamma_v [\vec{u} \cdot \vec{f}(\vec{u})]\vec{v} + (\gamma_v - 1)[\vec{v} \cdot \vec{f}(\vec{u})]\vec{v}/v^2 \\ \vec{u}' \cdot \vec{f}'(\vec{u}') = \gamma_v [\vec{u} \cdot \vec{f}(\vec{u}) + \vec{v} \cdot \vec{f}(\vec{u})] = \gamma_v (\vec{u} + \vec{v}) \cdot \vec{f}(\vec{u}) \end{cases} \quad (14.19)$$

### 3.10 相互作用质点系的普适静止质量公式

综合以上结论，可以得到以下普适的静止质量公式：

$$M_0 = \left( \sum_i E_i + \frac{1}{2} \sum_{i \neq j} V_{ij} \right) \sqrt{\frac{\sum_{i,j} (E_i E_j - \vec{p}_i \cdot \vec{p}_j)}{\sum_{i,j} (E_i E_j)}}$$

### 3.11 氢原子的普适静止质量公式(质心系)???

$$M_0 = \frac{[\sqrt{M^2 + \vec{p}_M^2} + \sqrt{m^2 + \vec{p}_m^2} + V(\vec{r}_M, \vec{r}_m)] \sqrt{(\sqrt{M^2 + \vec{p}_M^2} + \sqrt{m^2 + \vec{p}_m^2})^2 - (\vec{p}_M + \vec{p}_m)^2}}{\sqrt{M^2 + \vec{p}_M^2} + \sqrt{m^2 + \vec{p}_m^2}}$$

$$M_0 = [M + m + V(\vec{r}_{M0}, \vec{r}_{m0})]$$

$$M + m = [\sqrt{M^2 + \vec{p}^2} + \sqrt{m^2 + \vec{p}^2} + V(\vec{r}_M, \vec{r}_m)]$$

$$V(\vec{r}_M, \vec{r}_m) = M + m - (\sqrt{M^2 + \vec{p}^2} + \sqrt{m^2 + \vec{p}^2})$$

## 4 各种旋量的洛伦兹推动变换

### 4.1 反对称张量和电磁旋量的洛伦兹变换规律<sup>[23-25]</sup>

单粒子的角动量张量和电磁张量的洛伦兹变换规律：

定理4.1.1.

$$\begin{cases} F^{ab} = -F^{ba}, F' = L_{\vec{v}} F L_{\vec{v}}^T, F = \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix} \\ F' = \gamma_v \begin{bmatrix} 0 & (B_z + \vec{v} \times \vec{E})_z & -(B_y + \vec{v} \times \vec{E})_y & -i(\vec{E} - \vec{v} \times \vec{B})_x \\ -(B_z + \vec{v} \times \vec{E})_z & 0 & (B_x + \vec{v} \times \vec{E})_x & -i(\vec{E} - \vec{v} \times \vec{B})_y \\ (B_y + \vec{v} \times \vec{E})_y & -(B_x + \vec{v} \times \vec{E})_x & 0 & -i(\vec{E} - \vec{v} \times \vec{B})_z \\ i(\vec{E} - \vec{v} \times \vec{B})_x & i(\vec{E} - \vec{v} \times \vec{B})_y & i(\vec{E} - \vec{v} \times \vec{B})_z & 0 \end{bmatrix} - \frac{\gamma_v - 1}{v^2} \begin{bmatrix} 0 & (\vec{v} \cdot \vec{B})v_z & -(\vec{v} \cdot \vec{B})v_y & -i(\vec{v} \cdot \vec{E})v_x \\ -(\vec{v} \cdot \vec{B})v_z & 0 & (\vec{v} \cdot \vec{B})v_x & -i(\vec{v} \cdot \vec{E})v_y \\ (\vec{v} \cdot \vec{B})v_y & -(\vec{v} \cdot \vec{B})v_x & 0 & -i(\vec{v} \cdot \vec{E})v_z \\ i(\vec{v} \cdot \vec{E})v_x & i(\vec{v} \cdot \vec{E})v_y & i(\vec{v} \cdot \vec{E})v_z & 0 \end{bmatrix} \end{cases}$$

证明：

$$\begin{aligned} L_{\vec{v}} F &= \left( \begin{bmatrix} 1 & 0 & 0 & -i\gamma_v v_x \\ 0 & 1 & 0 & -i\gamma_v v_y \\ 0 & 0 & 1 & -i\gamma_v v_z \\ i\gamma_v v_x & i\gamma_v v_y & i\gamma_v v_z & \gamma_v \end{bmatrix} + \frac{\gamma_v - 1}{v^2} \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z & 0 \\ v_y v_x & v_y v_y & v_y v_z & 0 \\ v_z v_x & v_z v_y & v_z v_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix} \\ &= \begin{bmatrix} \gamma_v v_x E_x & B_z + \gamma_v v_x E_y & -B_y + \gamma_v v_x E_z & -iE_x \\ -B_z + \gamma_v v_y E_x & \gamma_v v_y E_y & B_x + \gamma_v v_y E_z & -iE_y \\ B_y + \gamma_v v_z E_x & -B_x + \gamma_v v_z E_y & \gamma_v v_z E_z & -iE_z \\ i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_x & i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_y & i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_z & \gamma_v \vec{v} \cdot \vec{E} \end{bmatrix} + \frac{\gamma_v - 1}{v^2} \begin{bmatrix} -v_x (\vec{v} \times \vec{B})_x & -v_x (\vec{v} \times \vec{B})_y & -v_x (\vec{v} \times \vec{B})_z & -iv_x \vec{v} \cdot \vec{E} \\ -v_y (\vec{v} \times \vec{B})_x & -v_y (\vec{v} \times \vec{B})_y & -v_y (\vec{v} \times \vec{B})_z & -iv_y \vec{v} \cdot \vec{E} \\ -v_z (\vec{v} \times \vec{B})_x & -v_z (\vec{v} \times \vec{B})_y & -v_z (\vec{v} \times \vec{B})_z & -iv_z \vec{v} \cdot \vec{E} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ F' &= L_{\vec{v}} F L_{\vec{v}}^T \\ &= \left( \begin{bmatrix} \gamma_v v_x E_x & B_z + \gamma_v v_x E_y & -B_y + \gamma_v v_x E_z & -iE_x \\ -B_z + \gamma_v v_y E_x & \gamma_v v_y E_y & B_x + \gamma_v v_y E_z & -iE_y \\ B_y + \gamma_v v_z E_x & -B_x + \gamma_v v_z E_y & \gamma_v v_z E_z & -iE_z \\ i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_x & i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_y & i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_z & \gamma_v \vec{v} \cdot \vec{E} \end{bmatrix} + \frac{\gamma_v - 1}{v^2} \begin{bmatrix} -v_x (\vec{v} \times \vec{B})_x & -v_x (\vec{v} \times \vec{B})_y & -v_x (\vec{v} \times \vec{B})_z & -iv_x \vec{v} \cdot \vec{E} \\ -v_y (\vec{v} \times \vec{B})_x & -v_y (\vec{v} \times \vec{B})_y & -v_y (\vec{v} \times \vec{B})_z & -iv_y \vec{v} \cdot \vec{E} \\ -v_z (\vec{v} \times \vec{B})_x & -v_z (\vec{v} \times \vec{B})_y & -v_z (\vec{v} \times \vec{B})_z & -iv_z \vec{v} \cdot \vec{E} \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \\ &= \left( \begin{bmatrix} 1 & 0 & 0 & i\gamma_v v_x \\ 0 & 1 & 0 & i\gamma_v v_y \\ 0 & 0 & 1 & i\gamma_v v_z \\ -i\gamma_v v_x & -i\gamma_v v_y & -i\gamma_v v_z & \gamma_v \end{bmatrix} + \frac{\gamma_v - 1}{v^2} \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z & 0 \\ v_y v_x & v_y v_y & v_y v_z & 0 \\ v_z v_x & v_z v_y & v_z v_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 & B_z + \gamma_v (\vec{v} \times \vec{E})_z & -(B_y + \gamma_v (\vec{v} \times \vec{E})_y) & -i(\gamma_v^2 v_x \vec{v} \cdot \vec{E} - i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_x) \\ -B_z - \gamma_v (\vec{v} \times \vec{E})_z & 0 & (B_x + \gamma_v (\vec{v} \times \vec{E})_x) & -i(\gamma_v^2 v_y \vec{v} \cdot \vec{E} - i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_y) \\ (B_y + \gamma_v (\vec{v} \times \vec{E})_y) & -(B_x + \gamma_v (\vec{v} \times \vec{E})_x) & 0 & i(\gamma_v^2 v_z \vec{v} \cdot \vec{E} - i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_z) \\ -i\gamma_v^2 v_x \vec{v} \cdot \vec{E} + i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_x & -i\gamma_v^2 v_y \vec{v} \cdot \vec{E} + i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_y & -i\gamma_v^2 v_z \vec{v} \cdot \vec{E} + i\gamma_v (\vec{E} - \vec{v} \times \vec{B})_z & \gamma_v^2 \vec{v} \cdot (\vec{v} \times \vec{B}) = 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 & + \frac{\gamma_v - 1}{v^2} \begin{bmatrix} v_x(\vec{v} \times \vec{B})_x + \gamma_v v_x v_x \vec{v} \cdot \vec{E} & v_y(\vec{v} \times \vec{B})_x + \gamma_v v_x v_y \vec{v} \cdot \vec{E} & v_z(\vec{v} \times \vec{B})_x + \gamma_v v_x v_z \vec{v} \cdot \vec{E} & 0 \\ v_x(\vec{v} \times \vec{B})_y + \gamma_v v_y v_x \vec{v} \cdot \vec{E} & v_y(\vec{v} \times \vec{B})_y + \gamma_v v_y v_y \vec{v} \cdot \vec{E} & v_z(\vec{v} \times \vec{B})_y + \gamma_v v_y v_z \vec{v} \cdot \vec{E} & 0 \\ v_x(\vec{v} \times \vec{B})_z + \gamma_v v_z v_x \vec{v} \cdot \vec{E} & v_y(\vec{v} \times \vec{B})_z + \gamma_v v_z v_y \vec{v} \cdot \vec{E} & v_z(\vec{v} \times \vec{B})_z + \gamma_v v_z v_z \vec{v} \cdot \vec{E} & 0 \\ i\gamma_v v_x \vec{v} \cdot \vec{E} & i\gamma_v v_y \vec{v} \cdot \vec{E} & i\gamma_v v_z \vec{v} \cdot \vec{E} & 0 \end{bmatrix} \\
 & + \frac{\gamma_v - 1}{v^2} \begin{bmatrix} -v_x(\vec{v} \times \vec{B})_x - \gamma_v v_x v_x \vec{v} \cdot \vec{E} & -v_x(\vec{v} \times \vec{B})_y - \gamma_v v_x v_y \vec{v} \cdot \vec{E} & -v_x(\vec{v} \times \vec{B})_z - \gamma_v v_x v_z \vec{v} \cdot \vec{E} & -i\gamma_v v_x \vec{v} \cdot \vec{E} \\ -v_y(\vec{v} \times \vec{B})_x - \gamma_v v_y v_x \vec{v} \cdot \vec{E} & -v_y(\vec{v} \times \vec{B})_y - \gamma_v v_y v_y \vec{v} \cdot \vec{E} & -v_y(\vec{v} \times \vec{B})_z - \gamma_v v_y v_z \vec{v} \cdot \vec{E} & -i\gamma_v v_y \vec{v} \cdot \vec{E} \\ -v_z(\vec{v} \times \vec{B})_x - \gamma_v v_z v_x \vec{v} \cdot \vec{E} & -v_z(\vec{v} \times \vec{B})_y - \gamma_v v_z v_y \vec{v} \cdot \vec{E} & -v_z(\vec{v} \times \vec{B})_z - \gamma_v v_z v_z \vec{v} \cdot \vec{E} & -i\gamma_v v_z \vec{v} \cdot \vec{E} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 & + \left(\frac{\gamma_v - 1}{v^2}\right)^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 & = \begin{bmatrix} 0 & B_z + \gamma_v(\vec{v} \times \vec{E})_z & -B_y - \gamma_v(\vec{v} \times \vec{E})_y & i\gamma_v^2 v_x \vec{v} \cdot \vec{E} - i\gamma_v(\vec{E} - \vec{v} \times \vec{B})_x \\ -B_z - \gamma_v(\vec{v} \times \vec{E})_z & 0 & B_x + \gamma_v(\vec{v} \times \vec{E})_x & i\gamma_v^2 v_y \vec{v} \cdot \vec{E} - i\gamma_v(\vec{E} - \vec{v} \times \vec{B})_y \\ B_y + \gamma_v(\vec{v} \times \vec{E})_y & -B_x - \gamma_v(\vec{v} \times \vec{E})_x & 0 & i\gamma_v^2 v_z \vec{v} \cdot \vec{E} - i\gamma_v(\vec{E} - \vec{v} \times \vec{B})_z \\ -i\gamma_v^2 v_x \vec{v} \cdot \vec{E} + i\gamma_v(\vec{E} - \vec{v} \times \vec{B})_x & -i\gamma_v^2 v_y \vec{v} \cdot \vec{E} + i\gamma_v(\vec{E} - \vec{v} \times \vec{B})_y & -i\gamma_v^2 v_z \vec{v} \cdot \vec{E} + i\gamma_v(\vec{E} - \vec{v} \times \vec{B})_z & \gamma_v^2 \vec{v} \cdot (\vec{v} \times \vec{B}) = 0 \end{bmatrix} \\
 & - \frac{\gamma_v - 1}{v^2} \begin{bmatrix} 0 & \vec{v} \times (\vec{v} \times \vec{B})_z & -\vec{v} \times (\vec{v} \times \vec{B})_y & i\gamma_v v_x \vec{v} \cdot \vec{E} \\ -\vec{v} \times (\vec{v} \times \vec{B})_z & 0 & \vec{v} \times (\vec{v} \times \vec{B})_x & i\gamma_v v_y \vec{v} \cdot \vec{E} \\ \vec{v} \times (\vec{v} \times \vec{B})_y & -\vec{v} \times (\vec{v} \times \vec{B})_x & 0 & i\gamma_v v_z \vec{v} \cdot \vec{E} \\ -i\gamma_v v_x \vec{v} \cdot \vec{E} & -i\gamma_v v_y \vec{v} \cdot \vec{E} & -i\gamma_v v_z \vec{v} \cdot \vec{E} & 0 \end{bmatrix} \\
 & = \gamma_v \begin{bmatrix} 0 & (B_z + \vec{v} \times \vec{E})_z & -(B_y + \vec{v} \times \vec{E})_y & -i(\vec{E} - \vec{v} \times \vec{B})_x \\ -(B_z + \vec{v} \times \vec{E})_z & 0 & (B_x + \vec{v} \times \vec{E})_x & -i(\vec{E} - \vec{v} \times \vec{B})_y \\ (B_y + \vec{v} \times \vec{E})_y & -(B_x + \vec{v} \times \vec{E})_x & 0 & -i(\vec{E} - \vec{v} \times \vec{B})_z \\ i(\vec{E} - \vec{v} \times \vec{B})_x & i(\vec{E} - \vec{v} \times \vec{B})_y & i(\vec{E} - \vec{v} \times \vec{B})_z & 0 \end{bmatrix} - \frac{\gamma_v - 1}{v^2} \begin{bmatrix} 0 & (\vec{v} \cdot \vec{B})_{v_z} & -(\vec{v} \cdot \vec{B})_{v_y} & -i(\vec{v} \cdot \vec{E})_{v_x} \\ -(\vec{v} \cdot \vec{B})_{v_z} & 0 & (\vec{v} \cdot \vec{B})_{v_x} & -i(\vec{v} \cdot \vec{E})_{v_y} \\ (\vec{v} \cdot \vec{B})_{v_y} & -(\vec{v} \cdot \vec{B})_{v_x} & 0 & -i(\vec{v} \cdot \vec{E})_{v_z} \\ i(\vec{v} \cdot \vec{E})_{v_x} & i(\vec{v} \cdot \vec{E})_{v_y} & i(\vec{v} \cdot \vec{E})_{v_z} & 0 \end{bmatrix} \quad \square
 \end{aligned}$$

推论4.1.1.  $\vec{E}' = \gamma_v(\vec{E} - \vec{v} \times \vec{B}) - (\gamma_v - 1)(\vec{v} \cdot \vec{E})\vec{v}/v^2$ ,  $\vec{B}' = \gamma_v(\vec{B} + \vec{v} \times \vec{E}) - (\gamma_v - 1)(\vec{v} \cdot \vec{B})\vec{v}/v^2$

推论4.1.2.  $\vec{\varphi}'_\zeta = \gamma_v(\vec{\varphi}_\zeta - i\zeta\vec{v} \times \vec{\varphi}_\zeta) - (\gamma_v - 1)(\vec{v} \cdot \vec{\varphi}_\zeta)\vec{v}/v^2$ ,  $\psi^{\alpha\zeta} := \frac{i}{2}\sigma_{\zeta ab}F^{ab} = -i\zeta(\vec{E} - i\zeta\vec{B}) := \vec{\varphi}_\zeta$

推论4.1.3.  $\vec{\varphi}'_\zeta = R_{\zeta\vec{v}}\vec{\varphi}_\zeta$ ,  $R_{\zeta\vec{v}} \equiv \gamma_v - \zeta\gamma_v \begin{bmatrix} 0 & -iv_z & iv_y \\ iv_z & 0 & -iv_x \\ -iv_y & iv_x & 0 \end{bmatrix} - \frac{\gamma_v - 1}{v^2} \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix}$

推论4.1.4.  $R_{\zeta\vec{v}} = 1 - \zeta\gamma_v\vec{v} \cdot \gamma + \frac{\gamma_v - 1}{v^2}(\vec{v} \cdot \gamma)^2$ ,  $R_{\zeta\vec{v}}R_{-\zeta\vec{v}} = R_{-\zeta\vec{v}}R_{\zeta\vec{v}} = I$

引理4.1.1.  $[1 + \frac{\gamma_v - 1}{v^2}(\vec{v} \cdot \gamma)^2]^2 = 1 + \gamma_v^2(\vec{v} \cdot \gamma)^2$

推论4.1.5.  $\begin{bmatrix} \vec{E}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 1 + \frac{\gamma_v - 1}{v^2}(\vec{v} \cdot \gamma)^2 & -i\zeta\gamma_v\vec{v} \cdot \gamma \\ i\zeta\gamma_v\vec{v} \cdot \gamma & 1 + \frac{\gamma_v - 1}{v^2}(\vec{v} \cdot \gamma)^2 \end{bmatrix} \begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix}, \begin{bmatrix} 1 + \frac{\gamma_v - 1}{v^2}(\vec{v} \cdot \gamma)^2 & -i\zeta\gamma_v\vec{v} \cdot \gamma \\ i\zeta\gamma_v\vec{v} \cdot \gamma & 1 + \frac{\gamma_v - 1}{v^2}(\vec{v} \cdot \gamma)^2 \end{bmatrix} \begin{bmatrix} 1 + \frac{\gamma_v - 1}{v^2}(\vec{v} \cdot \gamma)^2 & i\zeta\gamma_v\vec{v} \cdot \gamma \\ -i\zeta\gamma_v\vec{v} \cdot \gamma & 1 + \frac{\gamma_v - 1}{v^2}(\vec{v} \cdot \gamma)^2 \end{bmatrix} = I_6$

推论4.1.6.  $\begin{cases} \vec{E}' + i\vec{B}' = e^{(i\omega - \epsilon) \cdot \gamma}(\vec{E} + i\vec{B}) \\ \vec{E}' - i\vec{B}' = e^{(i\omega + \epsilon) \cdot \gamma}(\vec{E} - i\vec{B}) \end{cases} \Leftrightarrow \begin{cases} 2\vec{E}' = [e^{(i\omega - \epsilon) \cdot \gamma} + e^{(i\omega + \epsilon) \cdot \gamma}]\vec{E} + i[e^{(i\omega - \epsilon) \cdot \gamma} - e^{(i\omega + \epsilon) \cdot \gamma}]\vec{B} \\ 2\vec{B}' = [e^{(i\omega - \epsilon) \cdot \gamma} + e^{(i\omega + \epsilon) \cdot \gamma}]\vec{B} - i[e^{(i\omega - \epsilon) \cdot \gamma} - e^{(i\omega + \epsilon) \cdot \gamma}]\vec{E} \end{cases}$

推论4.1.7.  $\begin{bmatrix} \vec{E}' \\ \vec{B}' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} [e^{(i\omega - \epsilon) \cdot \gamma} + e^{(i\omega + \epsilon) \cdot \gamma}] & i[e^{(i\omega - \epsilon) \cdot \gamma} - e^{(i\omega + \epsilon) \cdot \gamma}] \\ -i[e^{(i\omega - \epsilon) \cdot \gamma} - e^{(i\omega + \epsilon) \cdot \gamma}] & [e^{(i\omega - \epsilon) \cdot \gamma} + e^{(i\omega + \epsilon) \cdot \gamma}] \end{bmatrix} \begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix}$

## 4.2 角动量变换规律

性质4.2.1.  $M^{ab} = x^a p^b - x^b p^a$ ,  $M' = L_{\vec{v}} M L_{\vec{v}}^T$ ,  $\vec{J} = \vec{B} = \vec{r} \times \vec{p}$ ,  $\vec{W} = \vec{E} = t\vec{p} - \vec{r}E$

性质4.2.2.  $M^{ab} = x^a p^b - x^b p^a$ ,  $M' = L_{\vec{v}} M L_{\vec{v}}^T$ ,  $\vec{J} = \vec{B} = \vec{r} \times \vec{p}$ ,  $\vec{W} = \vec{E} = r\vec{p} - \vec{r}E$

推论4.2.1.  $\begin{cases} \vec{W}' = \gamma_v(\vec{W} - \vec{v} \times \vec{J}) - (\gamma_v - 1)(\vec{v} \cdot \vec{W})\vec{v}/v^2 \\ \vec{J}' = \gamma_v(\vec{J} + \vec{v} \times \vec{W}) - (\gamma_v - 1)(\vec{v} \cdot \vec{J})\vec{v}/v^2 \end{cases}$

## 4.3 从光子旋量变换规律猜测得到旋量的变换规律

推论4.3.1.  $\begin{cases} \vec{\varphi}'_\zeta = \gamma_v(\vec{\varphi}_\zeta - i\zeta\vec{v} \times \vec{\varphi}_\zeta) - (\gamma_v - 1)(\vec{v} \cdot \vec{\varphi}_\zeta)\vec{v}/v^2 \\ (\vec{\varphi}'_\zeta, 0) = S_{em}(\kappa)\psi_\zeta \otimes \psi_\zeta \rightarrow \vec{\varphi}_\zeta = \frac{1}{\sqrt{2}}(\psi_{\zeta 1}^2 - \psi_{\zeta 2}^2, i\psi_{\zeta 1}^2 + i\psi_{\zeta 2}^2, -2\psi_{\zeta 1}\psi_{\zeta 2}) \end{cases}$

推论4.3.2.  $\Lambda(\psi_{\zeta 1}^2, \psi_{\zeta 2}^2, \psi_{\zeta 1}\psi_{\zeta 2}) = \frac{1}{2} \begin{bmatrix} 1 & -i & 0 \\ -1 & -i & 0 \\ 0 & 0 & -1 \end{bmatrix} (\gamma_v - \zeta\gamma_v \begin{bmatrix} 0 & -iv_z & iv_y \\ iv_z & 0 & -iv_x \\ -iv_y & iv_x & 0 \end{bmatrix} - \frac{\gamma_v - 1}{v^2} \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix}) \begin{bmatrix} 1 & -1 & 0 \\ i & i & 0 \\ 0 & 0 & -2 \end{bmatrix}$

猜测+推理:

推论4.3.3.  $\Lambda(\psi_{\zeta 1}^2, \psi_{\zeta 2}^2, \psi_{\zeta 1}\psi_{\zeta 2}) = \begin{bmatrix} (\gamma+1-\zeta\gamma v_z)^2/[2(\gamma+1)] & [\gamma(v_x - iv_y)]^2/[2(\gamma+1)] & -i\zeta v_y \\ [\gamma(v_x + iv_y)]^2/[2(\gamma+1)] & (\gamma+1+\zeta\gamma v_z)^2/[2(\gamma+1)] & i\zeta v_x \\ i\zeta v_y & -i\zeta v_x & 1 \end{bmatrix}$   
 $\Leftrightarrow \Lambda_{\zeta\vec{v}}(\psi_{\zeta 1}, \psi_{\zeta 2}) = \frac{1}{\sqrt{2(\gamma+1)}} \begin{bmatrix} \gamma+1-\zeta\gamma v_z & -\gamma\zeta(v_x - iv_y) \\ -\zeta\gamma(v_x + iv_y) & \gamma+1+\zeta\gamma v_z \end{bmatrix} = \frac{1}{\sqrt{2(\gamma+1)}}(1 + \gamma - \zeta\gamma\vec{v} \cdot \sigma)$

#### 4.4 从旋量变换规律推得光子旋量的变换规律

$$\text{推论4.4.1. } \Lambda_{\zeta\vec{v}} \otimes \Lambda_{\zeta\vec{v}} = \frac{\gamma+1}{2} - \frac{1}{2}\zeta\gamma\vec{v} \cdot (\sigma \otimes I + I \otimes \sigma) + \frac{\gamma-1}{2v^2}(\vec{v} \cdot \sigma) \otimes (\vec{v} \cdot \sigma)$$

$$\begin{aligned} \text{推论4.4.2. } R_{\zeta\vec{v}} &= S_{em}(\kappa)\Lambda_{\zeta\vec{v}} \otimes \Lambda_{\zeta\vec{v}}S_{em}^+(\kappa) = \frac{\gamma+1}{2} - \zeta\gamma\vec{v} \cdot R + \frac{\gamma-1}{2v^2}(\vec{v} \cdot \sigma_+)(\vec{v} \cdot \sigma_-) \\ &= 1 - \zeta\gamma\vec{v} \cdot R + \frac{\gamma-1}{v^2}(\vec{v} \cdot R)^2 = \begin{bmatrix} \gamma_v & 0 & 0 & 0 \\ 0 & \gamma_v & 0 & 0 \\ 0 & 0 & \gamma_v & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \zeta\gamma_v \begin{bmatrix} 0 & -iv_z & iv_y & 0 \\ iv_z & 0 & -iv_x & 0 \\ -iv_y & iv_x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \frac{\gamma-1}{v^2} \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z & 0 \\ v_y v_x & v_y v_y & v_y v_z & 0 \\ v_z v_x & v_z v_y & v_z v_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\text{推论4.4.3. } (\vec{v} \cdot \sigma_+)(\vec{v} \cdot \sigma_-) = -2 \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z & 0 \\ v_y v_x & v_y v_y & v_y v_z & 0 \\ v_z v_x & v_z v_y & v_z v_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + v^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\text{推论4.4.4. } (\vec{v} \cdot R)^2 = - \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z & 0 \\ v_y v_x & v_y v_y & v_y v_z & 0 \\ v_z v_x & v_z v_y & v_z v_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + v^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, (\vec{v} \cdot \gamma)^2 = - \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{bmatrix} + v^2$$

$$\text{推论4.4.5. } (\vec{v} \cdot \sigma_+)(\vec{v} \cdot \sigma_-) = 2(\vec{v} \cdot R)^2 - v^2 = v^2 - 2(\vec{v} \cdot L)^2$$

$$\text{推论4.4.6. } (\vec{v} \cdot R)(\vec{v} \cdot L) = (\vec{v} \cdot L)(\vec{v} \cdot R), (\vec{v} \cdot R)^2 + (\vec{v} \cdot L)^2 = v^2$$

#### 4.5 从旋量变换规律推得矢量的变换规律

$$\text{定义4.5.1. } L_{\vec{v}} \equiv \begin{bmatrix} 1 & 0 & 0 & -i\gamma_v v_x \\ 0 & 1 & 0 & -i\gamma_v v_y \\ 0 & 0 & 1 & -i\gamma_v v_z \\ i\gamma_v v_x & i\gamma_v v_y & i\gamma_v v_z & \gamma_v \end{bmatrix} + \frac{\gamma-1}{v^2} \begin{bmatrix} v_x v_x & v_x v_y & v_x v_z & 0 \\ v_y v_x & v_y v_y & v_y v_z & 0 \\ v_z v_x & v_z v_y & v_z v_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \gamma_v(1 - \vec{v} \cdot L) - \frac{\gamma-1}{v^2}(\vec{v} \cdot R)^2$$

推论4.5.1.

$$\Lambda_{\zeta\vec{v}} \otimes \Lambda_{-\zeta\vec{v}} = \frac{1}{2(\gamma_v+1)}(1 + \gamma_v - \zeta\gamma_v\vec{v} \cdot \sigma) \otimes (1 + \gamma_v + \zeta\gamma_v\vec{v} \cdot \sigma) = \frac{\gamma+1}{2} - \frac{1}{2}\zeta\gamma\vec{v} \cdot (\sigma \otimes I - I \otimes \sigma) - \frac{\gamma-1}{2v^2}(\vec{v} \cdot \sigma) \otimes (\vec{v} \cdot \sigma)$$

推论4.5.2.

$$L_{-\kappa\zeta\vec{v}} = S_{em}(\kappa)\Lambda_{\zeta\vec{v}} \otimes \Lambda_{-\zeta\vec{v}}S_{em}^+(\kappa) = \frac{\gamma+1}{2} + \kappa\zeta\gamma_v\vec{v} \cdot L - \frac{\gamma-1}{2v^2}(\vec{v} \cdot \sigma_+)(\vec{v} \cdot \sigma_-) = \gamma_v(1 + \kappa\zeta\vec{v} \cdot L) - \frac{\gamma-1}{v^2}(\vec{v} \cdot R)^2$$

#### 4.6 洛伦兹推动变换小结

旋量的洛伦兹推动变换:

$$\text{推论4.6.1. } \Lambda_{\zeta\vec{v}} = e^{-\frac{1}{2}\zeta\ln[\gamma_v(1+v)]\hat{v} \cdot \sigma} = \frac{1}{\sqrt{2(\gamma_v+1)}}(1 + \gamma_v - \zeta\gamma_v\vec{v} \cdot \sigma), \epsilon \sim -v, A_\zeta \sim e^{(i\omega+\zeta\epsilon) \cdot \sigma(s)}$$

狄拉克旋量的洛伦兹推动变换:

$$\text{推论4.6.2. } D_{\zeta\vec{v}} = e^{-\frac{1}{2}\zeta\ln[\gamma_v(1+v)]\hat{v} \cdot \sigma \otimes \sigma_z} = \frac{1}{\sqrt{2(\gamma_v+1)}}(1 + \gamma_v - \zeta\gamma_v\vec{v} \cdot \sigma \otimes \sigma_z), D_{\zeta\vec{v}} = \Lambda_{\zeta\vec{v}} \oplus \Lambda_{-\zeta\vec{v}}$$

矢量的洛伦兹推动变换:

$$\text{推论4.6.3. } L_{-\kappa\zeta\vec{v}} = \gamma_v(1 + \kappa\zeta\vec{v} \cdot L) - \frac{\gamma-1}{v^2}(\vec{v} \cdot R)^2, L_{-\kappa\zeta\vec{v}} = S_{em}(\kappa)\Lambda_{\zeta\vec{v}} \otimes \Lambda_{-\zeta\vec{v}}S_{em}^+(\kappa)$$

电磁旋量和角动量的洛伦兹推动变换:

$$\text{推论4.6.4. } R_{\zeta\vec{v}} = 1 - \zeta\gamma_v\vec{v} \cdot R + \frac{\gamma-1}{v^2}(\vec{v} \cdot R)^2, R_{\zeta\vec{v}} = S_{em}(\kappa)\Lambda_{\zeta\vec{v}} \otimes \Lambda_{\zeta\vec{v}}S_{em}^+(\kappa)$$

s-旋量的洛伦兹推动变换:

$$\text{推论4.6.5. } \Lambda_{\zeta\vec{v}}(s) = \bar{\mathcal{P}}(s + \frac{1}{2}) \overbrace{\Lambda_{\zeta\vec{v}} \otimes \cdots \otimes \Lambda_{\zeta\vec{v}}}^{2s} \mathcal{P}(s + \frac{1}{2}), \Lambda_{\zeta\vec{v}} = \frac{1}{\sqrt{2(\gamma_v+1)}}(1 + \gamma_v - \zeta\gamma_v\vec{v} \cdot \sigma)$$

### 5 各种自旋粒子洛伦兹变换的多项式表示

以上采用的是繁琐、直观、猜测的方法，下面将采用更解析、更严格、更有条理和更系统的分析与推导方法，得到更一般、更普适的结论。

## 5.1 数学准备

### 5.1.1 定义

$$\text{定义5.1.1. } e(s, n, \sigma) \equiv \left( \overbrace{I \otimes \cdots \otimes I}^{n-1} \sigma \overbrace{\otimes I \otimes \cdots \otimes I}^{2s-n} \right)$$

$$\text{定义5.1.2. } \hat{\Omega}(s) \equiv \hat{\Omega}(s, 1, \sigma) \equiv \sum_{n=1}^{2s} e(s, n, \sigma), \Omega(s) \equiv \frac{1}{2} \hat{\Omega}(s, 1, \sigma)$$

$$\hat{\Omega}(s, 1, \vec{\vartheta} \cdot \sigma) \equiv \sum_{n=1}^{2s} e(s, n, \vec{\vartheta} \cdot \sigma) = \vec{\vartheta} \cdot \hat{\Omega}(s, 1, \sigma)$$

$$\hat{\Omega}(s, 2, \vec{\vartheta} \cdot \sigma) \equiv \frac{1}{2!} \sum_{\substack{i,j \\ i \neq j}}^{1,2s} e(s, i, \vec{\vartheta} \cdot \sigma) e(s, j, \vec{\vartheta} \cdot \sigma)$$

$$\hat{\Omega}(s, n, \vec{\vartheta} \cdot \sigma) \equiv \frac{1}{n!} \sum_{i_1 \neq i_2 \cdots \neq i_n}^{1,2s} e(s, i_1, \vec{\vartheta} \cdot \sigma) e(s, i_2, \vec{\vartheta} \cdot \sigma) \cdots e(s, i_n, \vec{\vartheta} \cdot \sigma),$$

### 5.1.2 重要性质

$$\text{性质5.1.1. } \hat{\Omega}(s, 2, \vec{\vartheta} \cdot \sigma) = \frac{1}{2} \hat{\Omega}^2(s, 1, \vec{\vartheta} \cdot \sigma) - s \vec{\vartheta}^2$$

$$\begin{aligned} \text{性质5.1.2. } \hat{\Omega}(s < 2, n, \vec{\vartheta} \cdot \sigma) &= \frac{1}{n!} \hat{\Omega}^n(s, 1, \vec{\vartheta} \cdot \sigma) u(n-1) \\ &- \frac{1}{n!} \left( \frac{2s C_{2s-1}^{n-2}}{C_{2s}^{n-2}} \right) [C_n^2(n-2)!] \vec{\vartheta}^2 \hat{\Omega}(s, n-2, \vec{\vartheta} \cdot \sigma) u(n-2) - \frac{1}{n!} \left( \frac{2s C_{2s-1}^{n-3}}{C_{2s}^{n-2}} \right) [C_n^3(n-3)!] \vec{\vartheta}^2 \hat{\Omega}(s, n-2, \vec{\vartheta} \cdot \sigma) u(n-3) + \cdots \\ &= \frac{1}{n!} \hat{\Omega}^n(s, 1, \vec{\vartheta} \cdot \sigma) u(n-1) - [s - \frac{1}{2}(n-2)] \vec{\vartheta}^2 \hat{\Omega}(s, n-2, \vec{\vartheta} \cdot \sigma) u(n-2) - \frac{1}{6}(n-2) \vec{\vartheta}^2 \hat{\Omega}(s, n-2, \vec{\vartheta} \cdot \sigma) u(n-3) - \frac{5}{3} \vec{\vartheta}^4 \delta_{n,4} \end{aligned}$$

### 5.1.3 洛伦兹生成元矩阵性质<sup>[23]</sup>

$$\text{性质5.1.3. } \vec{\vartheta}^2 = 0 \Rightarrow (\vec{\vartheta} \cdot \sigma)^2 = 0, (\vec{\vartheta} \cdot \gamma)^2 = 0, (\vec{\vartheta} \cdot R)^2 = 0, (\vec{\vartheta} \cdot L)^2 = 0$$

$$\text{性质5.1.4. } \vec{\vartheta}^2 = 0 \Rightarrow [\vec{\vartheta} \cdot (\sigma \otimes I + I \otimes \sigma)]^2 = 0, (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) = 0$$

$$\text{性质5.1.5. } \vec{\vartheta}^2 = 0 \Rightarrow [\vec{\vartheta} \cdot (\sigma \otimes I \otimes I + I \otimes \sigma \otimes I + I \otimes I \otimes \sigma)]^2 = 0$$

$$\text{性质5.1.6. } \vec{\vartheta}^2 = 0 \Rightarrow [\vec{\vartheta} \cdot \hat{\Omega}(s)]^2 = 0, [\vec{\vartheta} \cdot \Omega(s)]^2 = 0, [\vec{\vartheta} \cdot \sigma(s)]^2 = 0$$

$$\text{性质5.1.7. } \vec{\vartheta}^2 = 1 \Rightarrow (\vec{\vartheta} \cdot \sigma)^3 = \vec{\vartheta} \cdot \sigma, (\vec{\vartheta} \cdot \gamma)^3 = \vec{\vartheta} \cdot \gamma, (\vec{\vartheta} \cdot R)^3 = \vec{\vartheta} \cdot R, (\vec{\vartheta} \cdot L)^3 = \vec{\vartheta} \cdot L$$

## 5.2 多项式展开的重要定理(终于严密化了2023.09.15)

$$\text{定理5.2.1. } (a_1 + a_2 + \cdots + a_m)^n = \sum_{\substack{i_k \\ \sum_k i_k = n}} \frac{n!}{i_1! i_2! \cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}$$

$$\text{定义5.2.1. } \langle k_1, k_2, \cdots, k_m \rangle = \frac{1}{m!} a_{\{1\}}^{k_1} a_2^{k_2} \cdots a_m^{k_m}; k_1 \geq k_2 \geq \cdots \geq k_m \geq 0, k_1 + k_2 + \cdots + k_m = n$$

$$\text{定义5.2.2. } \langle (n_1; l_1), (n_2; l_2), \cdots, (n_k; l_k) \rangle = \langle \overbrace{n_1, \cdots, n_1}^{l_1}, \overbrace{n_2, \cdots, n_2}^{l_2}, \cdots, \overbrace{n_k, \cdots, n_k}^{l_k} \rangle$$

$$n_1 > n_2 > \cdots > n_k \geq 0, l_1, l_2, \cdots, l_k \geq 1; l_1 + l_2 + \cdots + l_k = m, n_1 l_1 + n_2 l_2 + \cdots + n_k l_k = n$$

$$\begin{aligned} \text{推论5.2.1. } \langle (n_1; l_1), (n_2; l_2), \cdots, (n_k; l_k) \rangle &= \frac{1}{m!} (a_{\{1\}}^{n_1} \cdots a_{l_1}^{n_1}) (a_{l_1+1}^{n_2} \cdots a_{l_1+l_2}^{n_2}) \cdots (a_{l_1+\cdots+l_{k-1}+1}^{n_k} \cdots a_{l_1+\cdots+l_k}^{n_k}) \\ &= \frac{1}{m!} (l_1! a_{\{1\}}^{n_1} \cdots a_{l_1}^{n_1}) (l_2! a_{l_1+1}^{n_2} \cdots a_{l_1+l_2}^{n_2}) \cdots (l_k! a_{l_1+\cdots+l_{k-1}+1}^{n_k} \cdots a_{l_1+\cdots+l_k}^{n_k}) \\ &= \frac{l_1! l_2! \cdots l_k!}{m!} (a_{\{1\}}^{n_1} \cdots a_{l_1}^{n_1}) (a_{l_1+1}^{n_2} \cdots a_{l_1+l_2}^{n_2}) \cdots (a_{l_1+\cdots+l_{k-1}+1}^{n_k} \cdots a_{l_1+\cdots+l_k}^{n_k}) \end{aligned}$$

$$\text{定理5.2.2. } (a_1 + a_2 + \cdots + a_m)^n = \sum_{i_1+\cdots+i_m=n} \frac{n!}{i_1! i_2! \cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m}$$

$$= \sum \frac{n!}{(n_1!)^{l_1} (n_2!)^{l_2} \cdots (n_k!)^{l_k}} \frac{m!}{l_1! l_2! \cdots l_k!} \langle (n_1; l_1), (n_2; l_2), \cdots, (n_k; l_k) \rangle$$

$$n_1 > n_2 > \cdots > n_k \geq 0, l_1, l_2, \cdots, l_k \geq 1; l_1 + l_2 + \cdots + l_k = m, n_1 l_1 + n_2 l_2 + \cdots + n_k l_k = n$$

$$\begin{aligned} \text{证明: } (a_1 + a_2 + \cdots + a_m)^n &= \sum_{i_1+\cdots+i_m=n} \frac{n!}{i_1! i_2! \cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m} \\ &= \sum \frac{n!}{(n_1!)^{l_1} (n_2!)^{l_2} \cdots (n_k!)^{l_k}} (a_{\{1\}}^{n_1} \cdots a_{l_1}^{n_1}) (a_{l_1+1}^{n_2} \cdots a_{l_1+l_2}^{n_2}) \cdots (a_{l_1+\cdots+l_{k-1}+1}^{n_k} \cdots a_{l_1+\cdots+l_k}^{n_k}) \\ &= \sum \frac{n!}{(n_1!)^{l_1} (n_2!)^{l_2} \cdots (n_k!)^{l_k}} \frac{m!}{l_1! l_2! \cdots l_k!} \langle (n_1; l_1), (n_2; l_2), \cdots, (n_k; l_k) \rangle \end{aligned}$$

□

## 5.2.1 例子: 二项式展开

$$\begin{aligned}
\text{性质5.2.1. } (a_1 + a_2)^2 &= \sum \frac{2!}{i_1! i_2!} a_1^{i_1} a_2^{i_2} \\
&= \frac{2!}{2!0!} (a_1^2 a_2^0 + a_1^0 a_2^2) + \frac{2!}{1!1!} (a_1^1 a_2^1) \\
&= \frac{2!}{2!0!} \frac{2!}{1!1!} \langle 2, 0, 0 \rangle + \frac{2!}{1!1!} \frac{2!}{2!} \langle 1, 1, 0 \rangle, \langle 2, 0, 0 \rangle := \frac{1!1!}{2!} (a_1^2 a_2^0 + a_1^0 a_2^2), \langle 1, 1, 0 \rangle := \frac{2!}{2!} (a_1^1 a_2^1) \\
&= 2 \langle 2, 0, 0 \rangle + 2 \langle 1, 1, 0 \rangle
\end{aligned}$$

$$\begin{aligned}
\text{性质5.2.2. } (a_1 + a_2)^3 &= \sum \frac{3!}{i_1! i_2!} a_1^{i_1} a_2^{i_2} \\
&= \frac{3!}{3!0!} \frac{2!}{1!1!} \langle 3, 0, 0 \rangle + \frac{3!}{2!1!} \frac{2!}{1!1!} \langle 2, 1, 0 \rangle, \langle 3, 0, 0 \rangle := \frac{1!1!}{2!} (a_1^3 a_2^0 + a_1^0 a_2^3), \langle 2, 1, 0 \rangle := \frac{1!1!}{2!} (a_1^2 a_2^1 + a_1^1 a_2^2) \\
&= 2 \langle 3, 0, 0 \rangle + 6 \langle 2, 1, 0 \rangle
\end{aligned}$$

$$\begin{aligned}
\text{性质5.2.3. } (a_1 + a_2)^4 &= \sum \frac{4!}{i_1! i_2!} a_1^{i_1} a_2^{i_2} \\
&= \frac{4!}{4!0!} \frac{2!}{1!1!} \langle 4, 0, 0 \rangle + \frac{4!}{3!1!} \frac{2!}{1!1!} \langle 3, 1, 0 \rangle + \frac{4!}{2!2!} \frac{2!}{2!} \langle 2, 2, 0 \rangle \\
&= 2 \langle 4, 0, 0 \rangle + 8 \langle 3, 1, 0 \rangle + 6 \langle 2, 2, 0 \rangle
\end{aligned}$$

## 5.2.2 例子: 三项式展开

$$\begin{aligned}
\text{性质5.2.4. } (a_1 + a_2 + a_3)^2 &= \sum \frac{2!}{i_1! i_2! i_3!} a_1^{i_1} a_2^{i_2} a_3^{i_3} \\
&= \frac{2!}{2!0!0!} \frac{3!}{1!2!} \langle 2, 0, 0 \rangle + \frac{2!}{1!1!0!} \frac{3!}{2!1!} \langle 1, 1, 0 \rangle \\
&= 3 \langle 2, 0, 0 \rangle + 6 \langle 1, 1, 0 \rangle
\end{aligned}$$

$$\begin{aligned}
\text{性质5.2.5. } (a_1 + a_2 + a_3)^3 &= \sum \frac{3!}{i_1! i_2! i_3!} a_1^{i_1} a_2^{i_2} a_3^{i_3} \\
&= \frac{3!}{3!0!0!} \frac{3!}{1!2!} \langle 3, 0, 0 \rangle + \frac{3!}{2!1!0!} \frac{3!}{1!1!1!} \langle 2, 1, 0 \rangle + \frac{3!}{1!1!1!} \frac{3!}{3!} \langle 1, 1, 1 \rangle \\
&= 3 \langle 3, 0, 0 \rangle + 18 \langle 2, 1, 0 \rangle + 6 \langle 1, 1, 1 \rangle
\end{aligned}$$

$$\begin{aligned}
\text{性质5.2.6. } (a_1 + a_2 + a_3)^4 &= \sum \frac{4!}{i_1! i_2! i_3!} a_1^{i_1} a_2^{i_2} a_3^{i_3} \\
&= \frac{4!}{4!0!0!} \frac{3!}{1!2!} \langle 4, 0, 0 \rangle + \frac{4!}{3!1!0!} \frac{3!}{1!1!1!} \langle 3, 1, 0 \rangle + \frac{4!}{2!2!0!} \frac{3!}{2!1!} \langle 2, 2, 0 \rangle + \frac{4!}{2!1!1!} \frac{3!}{1!2!} \langle 2, 1, 1 \rangle \\
&= 3 \langle 4, 0, 0 \rangle + 24 \langle 3, 1, 0 \rangle + 18 \langle 2, 2, 0 \rangle + 36 \langle 2, 1, 1 \rangle
\end{aligned}$$

## 5.2.3 例子: 四项式展开

$$\begin{aligned}
\text{性质5.2.7. } (a_1 + a_2 + a_3 + a_4)^2 &= \sum \frac{2!}{i_1! i_2! i_3! i_4!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} \\
&= \frac{2!}{2!0!0!0!} \frac{4!}{1!3!} \langle 2, 0, 0, 0 \rangle + \frac{2!}{1!1!0!0!} \frac{4!}{2!2!} \langle 1, 1, 0, 0 \rangle \\
&= 4 \langle 2, 0, 0, 0 \rangle + 12 \langle 1, 1, 0, 0 \rangle
\end{aligned}$$

$$\begin{aligned}
\text{性质5.2.8. } (a_1 + a_2 + a_3 + a_4)^3 &= \sum \frac{3!}{i_1! i_2! i_3! i_4!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} \\
&= \frac{3!}{3!0!0!0!} \frac{4!}{1!3!} \langle 3, 0, 0, 0 \rangle + \frac{3!}{2!1!0!0!} \frac{4!}{1!1!2!} \langle 2, 1, 0, 0 \rangle + \frac{3!}{1!1!1!0!} \frac{4!}{3!1!} \langle 1, 1, 1, 0 \rangle \\
&= 4 \langle 3, 0, 0, 0 \rangle + 36 \langle 2, 1, 0, 0 \rangle + 24 \langle 1, 1, 1, 0 \rangle
\end{aligned}$$

$$\begin{aligned}
\text{性质5.2.9. } (a_1 + a_2 + a_3 + a_4)^4 &= \sum \frac{4!}{i_1! i_2! i_3! i_4!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} \\
&= \frac{4!}{4!0!0!0!} \frac{4!}{1!3!} \langle 4, 0, 0, 0 \rangle + \frac{4!}{3!1!0!0!} \frac{4!}{1!1!2!} \langle 3, 1, 0, 0 \rangle + \frac{4!}{2!2!0!0!} \frac{4!}{2!2!} \langle 2, 2, 0, 0 \rangle \\
&+ \frac{4!}{2!1!1!0!} \frac{4!}{1!2!1!} \langle 2, 1, 1, 0 \rangle + \frac{4!}{1!1!1!1!} \frac{4!}{4!} \langle 1, 1, 1, 1 \rangle \\
&= 4 \langle 4, 0, 0, 0 \rangle + 48 \langle 3, 1, 0, 0 \rangle + 36 \langle 2, 2, 0, 0 \rangle + 144 \langle 2, 1, 1, 0 \rangle + 24 \langle 1, 1, 1, 1 \rangle
\end{aligned}$$

$$\begin{aligned}
\text{性质5.2.10. } (a_1 + a_2 + a_3 + a_4)^5 &= \sum \frac{5!}{i_1! i_2! i_3! i_4!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} \\
&= \frac{5!}{5!0!0!0!} \frac{4!}{1!3!} \langle 5, 0, 0, 0 \rangle + \frac{5!}{4!1!0!0!} \frac{4!}{1!1!2!} \langle 4, 1, 0, 0 \rangle + \frac{5!}{3!2!0!0!} \frac{4!}{1!1!2!} \langle 3, 2, 0, 0 \rangle \\
&+ \frac{5!}{3!1!1!0!} \frac{4!}{1!2!1!} \langle 3, 1, 1, 0 \rangle + \frac{5!}{2!2!1!0!} \frac{4!}{2!1!1!} \langle 2, 2, 1, 0 \rangle + \frac{5!}{2!1!1!1!} \frac{4!}{1!3!} \langle 2, 1, 1, 1 \rangle \\
&= 4 \langle 5, 0, 0, 0 \rangle + 60 \langle 4, 1, 0, 0 \rangle + 120 \langle 3, 2, 0, 0 \rangle + 240 \langle 3, 1, 1, 0 \rangle + 360 \langle 2, 2, 1, 0 \rangle + 240 \langle 2, 1, 1, 1 \rangle
\end{aligned}$$

## 5.2.4 例子: 五项式展开

$$\begin{aligned}
\text{定理5.2.3. } (a_1 + a_2 + \cdots + a_m)^n &= \sum \frac{n!}{(n_1!)^{l_1} (n_2!)^{l_2} \cdots (n_k!)^{l_k}} \frac{m!}{l_1! l_2! \cdots l_k! (m-l_1-\cdots-l_k)!} \langle (n_1; l_1), (n_2; l_2), \cdots, (n_k; l_k) \rangle
\end{aligned}$$

$$\begin{aligned} \text{性质5.2.11. } (a_1 + a_2 + a_3 + a_4 + a_5)^2 &= \sum \frac{2!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5} \\ &= \frac{2!}{2!0!^4} \frac{5!}{1!4!} \langle 2, 0, 0, 0, 0 \rangle + \frac{2!}{1!^2 0!^3} \frac{5!}{2!3!} \langle 1, 1, 0, 0, 0 \rangle \\ &= 5 \langle 2, 0, 0, 0, 0 \rangle + 20 \langle 1, 1, 0, 0, 0 \rangle \end{aligned}$$

$$\begin{aligned} \text{性质5.2.12. } (a_1 + a_2 + a_3 + a_4 + a_5)^3 &= \sum \frac{3!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5} \\ &= \frac{3!}{3!0!^4} \frac{5!}{1!4!} \langle 3, 0, 0, 0, 0 \rangle + \frac{3!}{2!1!0!^3} \frac{5!}{1!1!3!} \langle 2, 1, 0, 0, 0 \rangle + \frac{3!}{1!^3 0!^2} \frac{5!}{3!2!} \langle 1, 1, 1, 0, 0 \rangle \\ &= 5 \langle 3, 0, 0, 0, 0 \rangle + 60 \langle 2, 1, 0, 0, 0 \rangle + 60 \langle 1, 1, 1, 0, 0 \rangle \end{aligned}$$

$$\begin{aligned} \text{性质5.2.13. } (a_1 + a_2 + a_3 + a_4 + a_5)^4 &= \sum \frac{4!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5} \\ &= \frac{4!}{4!0!^4} \frac{5!}{1!4!} \langle 4, 0, 0, 0, 0 \rangle + \frac{4!}{3!1!0!^3} \frac{5!}{1!1!3!} \langle 3, 1, 0, 0, 0 \rangle + \frac{4!}{2!^2 0!^3} \frac{5!}{2!3!} \langle 2, 2, 0, 0, 0 \rangle \\ &+ \frac{4!}{2!1!^2 0!^2} \frac{5!}{1!2!2!} \langle 2, 1, 1, 0, 0 \rangle + \frac{4!}{1!^4 0!} \frac{5!}{4!1!} \langle 1, 1, 1, 1, 0 \rangle \\ &= 5 \langle 4, 0, 0, 0, 0 \rangle + 80 \langle 3, 1, 0, 0, 0 \rangle + 60 \langle 2, 2, 0, 0, 0 \rangle + 360 \langle 2, 1, 1, 0, 0 \rangle + 120 \langle 1, 1, 1, 1, 0 \rangle \end{aligned}$$

$$\begin{aligned} \text{性质5.2.14. } (a_1 + a_2 + a_3 + a_4 + a_5)^5 &= \sum \frac{5!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5} \\ &= \frac{5!}{5!0!^4} \frac{5!}{1!4!} \langle 5, 0, 0, 0, 0 \rangle + \frac{5!}{4!1!0!^3} \frac{5!}{1!1!3!} \langle 4, 1, 0, 0, 0 \rangle + \frac{5!}{3!2!0!^3} \frac{5!}{1!1!3!} \langle 3, 2, 0, 0, 0 \rangle \\ &+ \frac{5!}{3!1!^2 0!^2} \frac{5!}{1!2!2!} \langle 3, 1, 1, 0, 0 \rangle + \frac{5!}{2!^2 1!0!^2} \frac{5!}{2!1!2!} \langle 2, 2, 1, 0, 0 \rangle + \frac{5!}{2!1!^3 0!} \frac{5!}{1!3!1!} \langle 2, 1, 1, 1, 0 \rangle + \frac{5!}{1!^5} \frac{5!}{5!} \langle 1, 1, 1, 1, 1 \rangle \\ &= 5 \langle 5, 0, 0, 0, 0 \rangle + 100 \langle 4, 1, 0, 0, 0 \rangle + 200 \langle 3, 2, 0, 0, 0 \rangle + 600 \langle 3, 1, 1, 0, 0 \rangle + 900 \langle 2, 2, 1, 0, 0 \rangle + 1200 \langle 2, 1, 1, 1, 0 \rangle + 120 \langle 1, 1, 1, 1, 1 \rangle \end{aligned}$$

### 5.3 归一化约束条件下的多项式展开

#### 5.3.1 归一化约束条件下的二项式展开

$$\text{定义5.3.1. } [a_i, a_j] = 0, a_i^2 = 1, \langle 1, 0 \rangle = \frac{1}{2}(a_1 + a_2)$$

$$\text{性质5.3.1. } (a_1 + a_2)^2 = 2 \langle 2, 0 \rangle + 2 \langle 1, 1 \rangle = 2 \langle 0, 0 \rangle + 2 \langle 1, 1 \rangle = 2 + 2 \langle 1, 1 \rangle$$

$$\text{推论5.3.1. } \langle 1, 1 \rangle = \frac{1}{2}(a_1 + a_2)^2 - 1$$

$$\text{性质5.3.2. } (a_1 + a_2)^3 = 2 \langle 3, 0 \rangle + 6 \langle 2, 1 \rangle = 2 \langle 1, 0 \rangle + 6 \langle 0, 1 \rangle = 8 \langle 1, 0 \rangle$$

$$\text{推论5.3.2. } (a_1 + a_2)^3 = 4(a_1 + a_2), 2^3 = 4 \cdot 2^1, 0^3 = 4 \cdot 0^1$$

$$\text{推论5.3.3. } \left[\frac{1}{2}(a_1 + a_2)\right]^3 = \left[\frac{1}{2}(a_1 + a_2)\right]$$

性质5.3.3.

$$(a_1 + a_2)^4 = 2 \langle 4, 0 \rangle + 8 \langle 3, 1 \rangle + 6 \langle 2, 2 \rangle = 2 \langle 0, 0 \rangle + 8 \langle 1, 1 \rangle + 6 \langle 0, 0 \rangle = 8 + 8 \langle 1, 1 \rangle$$

$$\text{推论5.3.4. } (a_1 + a_2)^4 = 4(a_1 + a_2)^2$$

#### 5.3.2 归一化约束条件下的三项式展开

$$\text{定义5.3.2. } [a_i, a_j] = 0, a_i^2 = 1, \langle 1, 0, 0 \rangle = \frac{1}{3}(a_1 + a_2 + a_3)$$

$$\text{推论5.3.5. } \frac{3!}{1!2!} \langle 1, 0, 0 \rangle = (a_1 + a_2 + a_3)$$

$$\text{性质5.3.4. } (a_1 + a_2 + a_3)^2 = 3 \langle 2, 0, 0 \rangle + 6 \langle 1, 1, 0 \rangle = 3 \langle 0, 0, 0 \rangle + 6 \langle 1, 1, 0 \rangle = 3 + 6 \langle 1, 1, 0 \rangle$$

$$\text{推论5.3.6. } \frac{3!}{2!1!} \langle 1, 1, 0 \rangle = \frac{1}{2}(a_1 + a_2 + a_3)^2 - \frac{3}{2}$$

$$\begin{aligned} \text{性质5.3.5. } (a_1 + a_2 + a_3)^3 &= 3 \langle 3, 0, 0 \rangle + 18 \langle 2, 1, 0 \rangle + 6 \langle 1, 1, 1 \rangle \\ &= 3 \langle 1, 0, 0 \rangle + 18 \langle 0, 1, 0 \rangle + 6 \langle 1, 1, 1 \rangle = 21 \langle 1, 0, 0 \rangle + 6 \langle 1, 1, 1 \rangle \end{aligned}$$

$$\text{推论5.3.7. } \frac{3!}{3!} \langle 1, 1, 1 \rangle = \frac{1}{6}(a_1 + a_2 + a_3)^3 - \frac{7}{6}(a_1 + a_2 + a_3)$$

$$\begin{aligned} \text{性质5.3.6. } (a_1 + a_2 + a_3)^4 &= 3 \langle 4, 0, 0 \rangle + 24 \langle 3, 1, 0 \rangle + 18 \langle 2, 2, 0, 0 \rangle + 36 \langle 2, 1, 1 \rangle \\ &= 3 \langle 0, 0, 0 \rangle + 24 \langle 1, 1, 0 \rangle + 18 \langle 0, 0, 0 \rangle + 36 \langle 0, 1, 1 \rangle = 21 + 60 \langle 1, 1, 0 \rangle \end{aligned}$$

$$\text{推论5.3.8. } (a_1 + a_2 + a_3)^4 = 10(a_1 + a_2 + a_3)^2 - 9, 3^4 = 10 \cdot 3^2 - 9, 1^4 = 10 \cdot 1^2 - 9$$

$$\text{推论5.3.9. } \left[\frac{1}{2}(a_1 + a_2 + a_3)\right]^4 = \frac{5}{2} \left[\frac{1}{2}(a_1 + a_2 + a_3)\right]^2 - \frac{9}{16}$$

### 5.3.3 归一化约束条件下的四项式展开

定义5.3.3.  $[a_i, a_j] = 0, a_i^2 = 1, \langle 1, 0, 0, 0 \rangle = \frac{1}{4}(a_1 + a_2 + a_3 + a_4)$

推论5.3.10.  $\frac{4!}{1!3!} \langle 1, 0, 0, 0 \rangle = (a_1 + a_2 + a_3 + a_4)$

性质5.3.7.  $(a_1 + a_2 + a_3 + a_4)^2 = 4 \langle 2, 0, 0, 0 \rangle + 12 \langle 1, 1, 0, 0 \rangle = 4 \langle 0, 0, 0, 0 \rangle + 12 \langle 1, 1, 0, 0 \rangle = 4 + 12 \langle 1, 1, 0, 0 \rangle$

推论5.3.11.  $\frac{4!}{2!2!} \langle 1, 1, 0, 0 \rangle = \frac{1}{2}(a_1 + a_2 + a_3 + a_4)^2 - 2$

性质5.3.8.  $(a_1 + a_2 + a_3 + a_4)^3 = 4 \langle 3, 0, 0, 0 \rangle + 36 \langle 2, 1, 0, 0 \rangle + 24 \langle 1, 1, 1, 0 \rangle = 4 \langle 1, 0, 0, 0 \rangle + 36 \langle 0, 1, 0, 0 \rangle + 24 \langle 1, 1, 1, 0 \rangle = 40 \langle 1, 0, 0, 0 \rangle + 24 \langle 1, 1, 1, 0 \rangle$

推论5.3.12.  $\frac{4!}{3!1!} \langle 1, 1, 1, 0 \rangle = \frac{1}{6}(a_1 + a_2 + a_3 + a_4)^3 - \frac{5}{3}(a_1 + a_2 + a_3 + a_4)$

性质5.3.9.  $(a_1 + a_2 + a_3 + a_4)^4 = 4 \langle 4, 0, 0, 0 \rangle + 48 \langle 3, 1, 0, 0 \rangle + 36 \langle 2, 2, 0, 0 \rangle + 144 \langle 2, 1, 1, 0 \rangle + 24 \langle 1, 1, 1, 1 \rangle = 4 \langle 0, 0, 0, 0 \rangle + 48 \langle 1, 1, 0, 0 \rangle + 36 \langle 0, 0, 0, 0 \rangle + 144 \langle 0, 1, 1, 0 \rangle + 24 \langle 1, 1, 1, 1 \rangle = 40 + 192 \langle 1, 1 \rangle + 24 \langle 1, 1, 1, 1 \rangle$

推论5.3.13.  $\frac{4!}{4!} \langle 1, 1, 1, 1 \rangle = \frac{1}{24}(a_1 + a_2 + a_3 + a_4)^4 - \frac{2}{3}(a_1 + a_2 + a_3 + a_4)^2 + 1$

性质5.3.10.  $(a_1 + a_2 + a_3 + a_4)^5 = 4 \langle 5, 0, 0, 0 \rangle + 60 \langle 4, 1, 0, 0 \rangle + 120 \langle 3, 2, 0, 0 \rangle + 240 \langle 3, 1, 1, 0 \rangle + 360 \langle 2, 2, 1, 0 \rangle + 240 \langle 2, 1, 1, 1 \rangle = 4 \langle 1, 0, 0, 0 \rangle + 60 \langle 0, 1, 0, 0 \rangle + 120 \langle 1, 0, 0, 0 \rangle + 240 \langle 1, 1, 1, 0 \rangle + 360 \langle 0, 0, 1, 0 \rangle + 240 \langle 0, 1, 1, 1 \rangle = 544 \langle 1, 0, 0, 0 \rangle + 480 \langle 1, 1, 1, 0 \rangle$

性质5.3.11.  $(a_1 + a_2 + a_3 + a_4)^5 = 20(a_1 + a_2 + a_3 + a_4)^3 - 64(a_1 + a_2 + a_3 + a_4)$   
 $4^5 = 20 \cdot 4^3 - 64 \cdot 4^1, 2^5 = 20 \cdot 2^3 - 64 \cdot 2^1, 0^5 = 20 \cdot 0^3 - 64 \cdot 0^1$

推论5.3.14.  $[\frac{1}{2}(a_1 + a_2 + a_3 + a_4)]^5 = 5[\frac{1}{2}(a_1 + a_2 + a_3 + a_4)]^3 - 4[\frac{1}{2}(a_1 + a_2 + a_3 + a_4)]$

### 5.3.4 符号简化例子：5项式展开

定义5.3.4.  $[a_i, a_j] = 0, a_i^2 = 1, \langle 1, 0_4 \rangle = \frac{1}{5}(a_1 + a_2 + a_3 + a_4 + a_5)$

推论5.3.15.  $\frac{5!}{1!4!} \langle 1, 0_4 \rangle = (a_1 + a_2 + a_3 + a_4 + a_5)$

性质5.3.12.  $(a_1 + a_2 + a_3 + \cdots + a_5)^2 = \sum \frac{2!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5}$   
 $= \frac{2!}{2!} \frac{5!}{1!4!} \langle 2, 0_4 \rangle + \frac{2!}{1!1!} \frac{5!}{2!3!} \langle 1_2, 0_3 \rangle$   
 $= 5 \langle 2, 0_4 \rangle + 5(5-1) \langle 1_2, 0_3 \rangle$   
 $\stackrel{1}{=} 5 + 2! \langle 1_2, 0_3 \rangle_+$

推论5.3.16.  $\langle 1_2, 0_3 \rangle_+ = \frac{1}{2!} [\langle 1, 0_4 \rangle_+^2 - 5]$

性质5.3.13.  $(a_1 + a_2 + a_3 + \cdots + a_5)^3 = \sum \frac{3!}{i_1!i_2!i_3!i_4!i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5}$   
 $= \frac{3!}{3!} \frac{5!}{1!4!} \langle 3, 0_4 \rangle + \frac{3!}{2!1!} \frac{5!}{1!1!3!} \langle 2, 1, 0_3 \rangle + \frac{3!}{1!1!1!1!} \frac{5!}{3!2!} \langle 1_3, 0_2 \rangle$   
 $= 5 \langle 3, 0_4 \rangle + 60 \langle 2, 1, 0_3 \rangle + 60 \langle 1_3, 0_2 \rangle$   
 $\stackrel{1}{=} 65 \langle 1, 0_4 \rangle + 60 \langle 1_3, 0_2 \rangle$   
 $\stackrel{1}{=} 13 \langle 1, 0_4 \rangle_+ + 3! \langle 1_3, 0_2 \rangle_+$

推论5.3.17.  $\langle 1_3, 0_2 \rangle_+ = \frac{1}{3!} [\langle 1, 0_4 \rangle_+^3 - 13 \langle 1, 0_4 \rangle_+]$



$$\begin{aligned}
\text{性质5.3.14. } (a_1 + a_2 + a_3 + \cdots + a_5)^4 &= \sum \frac{4!}{i_1! i_2! i_3! i_4! i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5} \\
&= \frac{4!}{4!} \frac{5!}{1!4!} \langle 4, 0_4 \rangle + \frac{4!}{3!1!} \frac{5!}{1!1!3!} \langle 3, 1, 0_3 \rangle + \frac{4!}{2!2!} \frac{5!}{2!1!3!} \langle 2, 2, 0_3 \rangle \\
&+ \frac{4!}{2!1!1!} \frac{5!}{1!2!2!} \langle 2, 1_2, 0_2 \rangle + \frac{4!}{1!1!1!1!} \frac{5!}{4!1!} \langle 1_4, 0 \rangle \\
&= 5 \langle 4, 0_4 \rangle + 80 \langle 3, 1, 0_3 \rangle + 60 \langle 2, 2, 0_3 \rangle + 360 \langle 2, 1_2, 0_2 \rangle + 120 \langle 1_4, 0 \rangle \\
&\stackrel{1}{=} 65 + 440 \langle 1_2, 0_3 \rangle + 120 \langle 1_4, 0 \rangle \\
&\stackrel{1}{=} 65 + 44 \langle 1_2, 0_3 \rangle + 4! \langle 1_4, 0 \rangle +
\end{aligned}$$

$$\text{推论5.3.18. } \langle 1_4, 0 \rangle + = \frac{1}{4!} [\langle 1, 0_4 \rangle +^4 - 22 \langle 1, 0_4 \rangle +^2 + 45]$$

$$\begin{aligned}
\text{性质5.3.15. } (a_1 + a_2 + a_3 + \cdots + a_5)^5 &= \sum \frac{5!}{i_1! i_2! i_3! i_4! i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5} \\
&= \frac{5!}{5!} \frac{5!}{1!4!} \langle 5, 0_4 \rangle + \frac{5!}{4!1!} \frac{5!}{1!1!3!} \langle 4, 1, 0_3 \rangle + \frac{5!}{3!2!} \frac{5!}{1!1!3!} \langle 3, 2, 0_3 \rangle \\
&+ \frac{5!}{3!1!1!} \frac{5!}{1!2!2!} \langle 3, 1_2, 0_2 \rangle + \frac{5!}{2!2!1!} \frac{5!}{2!1!2!} \langle 2, 2, 1, 0_2 \rangle \\
&+ \frac{5!}{2!1!1!1!} \frac{5!}{1!3!1!} \langle 2, 1_3, 0 \rangle + \frac{5!}{1!1!1!1!1!} \frac{5!}{5!} \langle 1_5 \rangle \\
&= 5 \langle 5, 0_4 \rangle + 100 \langle 4, 1, 0_3 \rangle + 200 \langle 3, 2, 0_3 \rangle + 600 \langle 3, 1_2, 0_2 \rangle + 900 \langle 2, 2, 1, 0_2 \rangle + 1200 \langle 2, 1_3, 0 \rangle \\
&+ 120 \langle 1_5 \rangle \\
&\stackrel{1}{=} 1205 \langle 1, 0_4 \rangle + 1800 \langle 1_3, 0_2 \rangle + 120 \langle 1_5 \rangle \\
&\stackrel{1}{=} 241 \langle 1, 0_4 \rangle + 180 \langle 1_3, 0_2 \rangle + 5! \langle 1_5 \rangle +
\end{aligned}$$

$$\text{推论5.3.19. } \langle 1_5 \rangle + = \frac{1}{5!} [\langle 1, 0_4 \rangle +^5 - 30 \langle 1, 0_4 \rangle +^3 + 149 \langle 1, 0_4 \rangle +]$$

$$\begin{aligned}
\text{性质5.3.16. } (a_1 + a_2 + a_3 + \cdots + a_5)^6 &= \sum \frac{6!}{i_1! i_2! i_3! i_4! i_5!} a_1^{i_1} a_2^{i_2} a_3^{i_3} a_4^{i_4} a_5^{i_5} \\
&= \frac{6!}{6!} \frac{5!}{1!4!} \langle 6, 0_4 \rangle + \frac{6!}{5!1!} \frac{5!}{1!1!3!} \langle 5, 1, 0_3 \rangle + \frac{6!}{4!2!} \frac{5!}{1!1!3!} \langle 4, 2, 0_3 \rangle + \frac{6!}{4!1!1!} \frac{5!}{1!2!2!} \langle 4, 1_2, 0_2 \rangle \\
&+ \frac{6!}{3!3!} \frac{5!}{2!3!} \langle 3, 2, 0_3 \rangle + \frac{6!}{3!2!1!} \frac{5!}{1!1!1!2!} \langle 3, 2, 1, 0_2 \rangle + \frac{6!}{3!1!1!1!} \frac{5!}{1!3!1!} \langle 3, 1_3, 0 \rangle \\
&+ \frac{6!}{2!2!2!} \frac{5!}{3!2!} \langle 2, 3, 0_2 \rangle + \frac{6!}{2!2!1!1!} \frac{5!}{2!2!1!} \langle 2, 2, 1_2, 0 \rangle + \frac{6!}{2!1!1!1!1!} \frac{5!}{1!4!0!} \langle 2, 1_4 \rangle \\
&= 5 + 120 \langle 1_2, 0_3 \rangle + 300 + 900 \langle 1_2, 0_3 \rangle \\
&+ 200 \langle 1_2, 0_3 \rangle + 3600 \langle 1_2, 0_3 \rangle + 2400 \langle 1_4, 0 \rangle \\
&+ 900 + 5400 \langle 1_2, 0_3 \rangle + 1800 \langle 1_4, 0 \rangle \\
&= 1205 + 10220 \langle 1_2, 0_3 \rangle + 4200 \langle 1_4, 0 \rangle \\
&= 1205 + 10220 \frac{1}{C_5^2} \langle 1_2, 0_3 \rangle + 4200 \frac{1}{C_5^4} \langle 1_4, 0 \rangle + \\
&= 1205 + 1022 \langle 1_2, 0_3 \rangle + 840 \langle 1_4, 0 \rangle + \\
&= 1205 + 1022 \frac{1}{2!} [\langle 1, 0_4 \rangle +^2 - 5] + 840 \frac{1}{4!} [\langle 1, 0_4 \rangle +^4 - 22 \langle 1, 0_4 \rangle +^2 + 45]
\end{aligned}$$

推论5.3.20.

$$\begin{aligned}
\langle 1, 0_4 \rangle +^6 &= 35 \langle 1, 0_4 \rangle +^4 - 259 \langle 1, 0_4 \rangle +^2 + 225 \\
5^6 &= 35 \cdot 5^4 - 259 \cdot 5^2 + 225 \\
3^6 &= 35 \cdot 3^4 - 259 \cdot 3^2 + 225 \\
1^6 &= 35 \cdot 1^4 - 259 \cdot 1^2 + 225 \\
\langle 1, 0_5 \rangle +^7 &= 56 \langle 1, 0_5 \rangle +^5 - 784 \langle 1, 0_5 \rangle +^3 + 2304 \langle 1, 0_5 \rangle +^1
\end{aligned}$$

### 5.3.5 例子：m项式展开(进一步简化符号)

$$\begin{aligned}
\text{性质5.3.17. } (a_1 + a_2 + a_3 + \cdots + a_m)^2 &= \sum \frac{2!}{i_1! i_2! \cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m} \\
&= \frac{2!}{2!} \frac{m!}{1!(m-1)!} \langle 2 \rangle + \frac{2!}{1!1!} \frac{m!}{2!(m-2)!} \langle 1, 1 \rangle \\
&= m \langle 2 \rangle + m(m-1) \langle 1, 1 \rangle \\
&\stackrel{1}{=} m + m(m-1) \langle 1, 1 \rangle + \stackrel{1}{=} m + 2! \langle 1, 1 \rangle +
\end{aligned}$$

$$\text{推论5.3.21. } \langle 1, 1 \rangle + = \frac{1}{2!} [\hat{\Omega}^2(m) - m]$$

$$\begin{aligned}
\text{性质5.3.18. } (a_1 + a_2 + a_3 + \cdots + a_m)^3 &= \sum \frac{3!}{i_1! i_2! \cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m} \\
&= \frac{3!}{3!} \frac{m!}{1!(m-1)!} \langle 3 \rangle + \frac{3!}{2!1!} \frac{m!}{1!1!(m-2)!} \langle 2, 1 \rangle + \frac{3!}{1!1!1!} \frac{m!}{3!(m-3)!} \langle 1, 1, 1 \rangle +
\end{aligned}$$

$$\begin{aligned}
&= m \langle 3 \rangle + 3m(m-1) \langle 2, 1 \rangle + m(m-1)(m-2) \langle 1, 1, 1 \rangle \\
&\stackrel{1}{=} [m + 3m(m-1)] \langle 1 \rangle + m(m-1)(m-2) \langle 1, 1, 1 \rangle \\
&\stackrel{1}{=} (3m-2) \langle 1 \rangle + 3! \langle 1, 1, 1 \rangle
\end{aligned}$$

$$\text{推论5.3.22. } \langle 1, 1, 1 \rangle_+ = \frac{1}{3!} [\hat{\Omega}^3(m) - (3m-2)\hat{\Omega}(m)]$$

$$\begin{aligned}
\text{性质5.3.19. } (a_1 + a_2 + a_3 + \cdots + a_m)^4 &= \sum \frac{4!}{i_1! i_2! \cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m} \\
&= \frac{4!}{4!} \frac{m!}{1!(m-1)!} \langle 4 \rangle + \frac{4!}{3!1!} \frac{m!}{1!1!(m-2)!} \langle 3, 1 \rangle + \frac{4!}{2!2!} \frac{m!}{2!(m-2)!} \langle 2, 2 \rangle \\
&+ \frac{4!}{2!1!1!} \frac{m!}{1!2!(m-3)!} \langle 2, 1, 1 \rangle + \frac{4!}{1!1!1!1!} \frac{m!}{4!(m-4)!} \langle 1, 1, 1, 1 \rangle \\
&= m \langle 4 \rangle + 4m(m-1) \langle 3, 1 \rangle + 3m(m-1) \langle 2, 2 \rangle \\
&+ 6m(m-1)(m-2) \langle 2, 1, 1 \rangle + m(m-1)(m-2)(m-3) \langle 1, 1, 1, 1 \rangle \\
&\stackrel{1}{=} [m + 3m(m-1)] + [4m(m-1) + 6m(m-1)(m-2)] \langle 1, 1 \rangle + m(m-1)(m-2)(m-3) \langle 1, 1, 1, 1 \rangle \\
&\stackrel{1}{=} m(3m-2) + 4(3m-4) \langle 1, 1 \rangle + 4! \langle 1, 1, 1, 1 \rangle
\end{aligned}$$

$$\text{推论5.3.23. } \langle 1, 1, 1, 1 \rangle_+ = \frac{1}{4!} [\hat{\Omega}^4(m) - 2(3m-4)\hat{\Omega}^2(m) + 3m(m-2)]$$

$$\begin{aligned}
\text{性质5.3.20. } (a_1 + a_2 + a_3 + \cdots + a_m)^5 &= \sum \frac{5!}{i_1! i_2! \cdots i_m!} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m} \\
&= \frac{5!}{5!} \frac{m!}{1!(m-1)!} \langle 5 \rangle + \frac{5!}{4!1!} \frac{m!}{1!1!(m-2)!} \langle 4, 1 \rangle + \frac{5!}{3!2!} \frac{m!}{1!1!(m-2)!} \langle 3, 2 \rangle \\
&+ \frac{5!}{3!1!1!} \frac{m!}{1!2!(m-3)!} \langle 3, 1, 1 \rangle + \frac{5!}{2!2!1!} \frac{m!}{2!1!(m-3)!} \langle 2, 2, 1 \rangle \\
&+ \frac{5!}{2!1!1!1!} \frac{m!}{1!3!(m-4)!} \langle 2, 1, 1, 1 \rangle + \frac{5!}{1!1!1!1!1!} \frac{m!}{5!(m-5)!} \langle 1, 1, 1, 1, 1 \rangle \\
&= m \langle 5 \rangle + 5m(m-1) \langle 4, 1 \rangle + 10m(m-1) \langle 3, 2 \rangle \\
&+ 10m(m-1)(m-2) \langle 3, 1, 1 \rangle + 15m(m-1)(m-2) \langle 2, 2, 1 \rangle \\
&+ 10m(m-1)(m-2)(m-3) \langle 2, 1, 1, 1 \rangle + m(m-1)(m-2)(m-3)(m-4) \langle 1, 1, 1, 1, 1 \rangle \\
&\stackrel{1}{=} m[1 + 15(m-1)^2] \langle 1 \rangle + 10m(m-1)(m-2)^2 \langle 1, 1, 1 \rangle \\
&+ m(m-1)(m-2)(m-3)(m-4) \langle 1, 1, 1, 1, 1 \rangle \\
&\stackrel{1}{=} [1 + 15(m-1)^2] \langle 1 \rangle + 3!10(m-2) \langle 1, 1, 1 \rangle + 5! \langle 1, 1, 1, 1, 1 \rangle
\end{aligned}$$

$$\text{推论5.3.24. } \langle 1, 1, 1, 1, 1 \rangle_+ = \frac{1}{5!} [\hat{\Omega}^5(m) - 10(m-2)\hat{\Omega}^3(m) + (15m^2 - 50m + 24)\hat{\Omega}(m)]$$

### 5.3.6 讨论

推论5.3.25.

$$\left\{ \begin{aligned}
\langle 1, 0_{m-1} \rangle_+ &= \frac{1}{1!} \langle 1, 0_{m-1} \rangle_+ \simeq C_m^1 \\
\langle 1_2, 0_{m-2} \rangle_+ &= \frac{1}{2!} [\langle 1, 0_{m-1} \rangle_+^2 - m] \simeq C_m^2 \\
\langle 1_3, 0_{m-3} \rangle_+ &= \frac{1}{3!} [\langle 1, 0_{m-1} \rangle_+^3 - (3m-2) \langle 1, 0_{m-1} \rangle_+] \simeq C_m^3 \\
\langle 1_4, 0_{m-4} \rangle_+ &= \frac{1}{4!} [\langle 1, 0_{m-1} \rangle_+^4 - 2(3m-4) \langle 1, 0_{m-1} \rangle_+^2 + 3m(m-2)] \simeq C_m^4 \\
\langle 1_5, 0_{m-5} \rangle_+ &= \frac{1}{5!} [\langle 1, 0_{m-1} \rangle_+^5 - 10(m-2) \langle 1, 0_{m-1} \rangle_+^3 + (15m^2 - 50m + 24) \langle 1, 0_{m-1} \rangle_+] \simeq C_m^5
\end{aligned} \right.$$

推论5.3.26.

$$\left\{ \begin{aligned}
\langle 1, 0_{m-1} \rangle_+ &= \frac{1}{1!} [C_1^0 \langle 1, 0_{m-1} \rangle_+ - (C_1^2 m - 2C_1^3)] \simeq C_m^1 \\
\langle 1_2, 0_{m-2} \rangle_+ &= \frac{1}{2!} [C_2^0 \langle 1, 0_{m-1} \rangle_+^2 - (C_2^2 m - 2C_2^3)] \simeq C_m^2 \\
\langle 1_3, 0_{m-3} \rangle_+ &= \frac{1}{3!} [C_3^0 \langle 1, 0_{m-1} \rangle_+^3 - (C_3^2 m - 2C_3^3) \langle 1, 0_{m-1} \rangle_+] \simeq C_m^3 \\
\langle 1_4, 0_{m-4} \rangle_+ &= \frac{1}{4!} [C_4^0 \langle 1, 0_{m-1} \rangle_+^4 - (C_4^2 m - 2C_4^3) \langle 1, 0_{m-1} \rangle_+^2 + 3m(m-2)] \simeq C_m^4 \\
\langle 1_5, 0_{m-5} \rangle_+ &= \frac{1}{5!} [C_5^0 \langle 1, 0_{m-1} \rangle_+^5 - (C_5^2 m - 2C_5^3) \langle 1, 0_{m-1} \rangle_+^3 + (15m^2 - 50m + 24) \langle 1, 0_{m-1} \rangle_+] \simeq C_m^5 \\
\text{一般通式???, 这是下一步需要攻克的难题. 先放一放, } &10.5
\end{aligned} \right.$$

性质5.3.21.

$$\langle \rangle_+^1 = 0^1$$

$$\begin{aligned}
\langle 1 \rangle_+^2 &= 1^2 \langle 1 \rangle_+ \\
\langle 1, 0 \rangle_+^3 &= 2^2 \langle 1, 0 \rangle_+^1 \\
\langle 1, 0_2 \rangle_+^4 &= C_5^2 \langle 1, 0_2 \rangle_+^2 - (1^2 3^2) \\
\langle 1, 0_3 \rangle_+^5 &= C_6^3 \langle 1, 0_3 \rangle_+^3 - (2^2 4^2) \langle 1, 0_3 \rangle_+^1 \\
\langle 1, 0_4 \rangle_+^6 &= C_7^4 \langle 1, 0_4 \rangle_+^4 - 259 \langle 1, 0_4 \rangle_+^2 + (1 \cdot 3 \cdot 5)^2 \\
\langle 1, 0_5 \rangle_+^7 &= C_8^5 \langle 1, 0_5 \rangle_+^5 - 784 \langle 1, 0_5 \rangle_+^3 + (2 \cdot 4 \cdot 6)^2 \langle 1, 0_5 \rangle_+^1 \\
\langle 1, 0_6 \rangle_+^8 &= C_9^6 \langle 1, 0_6 \rangle_+^6 - 1974 \langle 1, 0_6 \rangle_+^4 + 12916 \langle 1, 0_6 \rangle_+^2 - (1 \cdot 3 \cdot 5 \cdot 7)^2 \\
\langle 1, 0_7 \rangle_+^9 &= C_{10}^7 \langle 1, 0_7 \rangle_+^7 - \langle 1, 0_7 \rangle_+^5 + \langle 1, 0_7 \rangle_+^3 - (2 \cdot 4 \cdot 6 \cdot 8)^2 \langle 1, 0_7 \rangle_+^1 \\
1974 &= 1^2 3^2 + 3^2 5^2 + 5^2 7^2 + 7^2 1^2 + 1^2 5^2 + 3^2 7^2, 12916 = 3^2 5^2 7^2 + 5^2 7^2 1^2 + 1^2 3^2 5^2 + 1^2 3^2 7^2
\end{aligned}$$

性质5.3.22.

$$\begin{aligned}
\langle \rangle_+^1 &= 0^1 \\
\langle 1 \rangle_+^2 &= 1^2 \langle 1 \rangle_+ \\
\langle 1, 0 \rangle_+^3 &= 2^2 \langle 1, 0 \rangle_+^1 \\
\langle 1, 0_2 \rangle_+^4 &= (1^2 + 3^2) \langle 1, 0_2 \rangle_+^2 - (1^2 3^2) \\
\langle 1, 0_3 \rangle_+^5 &= (2^2 + 4^2) \langle 1, 0_3 \rangle_+^3 - (2^2 4^2) \langle 1, 0_3 \rangle_+^1 \\
\langle 1, 0_4 \rangle_+^6 &= (1^2 + 3^2 + 5^2) \langle 1, 0_4 \rangle_+^4 - (1^2 3^2 + 3^2 5^2 + 5^2 1^2) \langle 1, 0_4 \rangle_+^2 + (1^2 3^2 5^2) \\
\langle 1, 0_5 \rangle_+^7 &= (2^2 + 4^2 + 6^2) \langle 1, 0_5 \rangle_+^5 - (2^2 4^2 + 4^2 6^2 + 6^2 2^2) \langle 1, 0_5 \rangle_+^3 + (2^2 4^2 6^2) \langle 1, 0_5 \rangle_+^1 \\
\langle 1, 0_6 \rangle_+^8 &= C_{\{1^2, 3^2, 5^2, 7^2\}}^1 \langle 1, 0_6 \rangle_+^6 - C_{\{1^2, 3^2, 5^2, 7^2\}}^2 \langle 1, 0_6 \rangle_+^4 + C_{\{1^2, 3^2, 5^2, 7^2\}}^3 \langle 1, 0_6 \rangle_+^2 - C_{\{1^2, 3^2, 5^2, 7^2\}}^4 \\
\langle 1, 0_7 \rangle_+^9 &= C_{\{2^2, 4^2, 6^2, 8^2\}}^1 \langle 1, 0_7 \rangle_+^7 - C_{\{2^2, 4^2, 6^2, 8^2\}}^2 \langle 1, 0_7 \rangle_+^5 + C_{\{2^2, 4^2, 6^2, 8^2\}}^3 \langle 1, 0_7 \rangle_+^3 - C_{\{2^2, 4^2, 6^2, 8^2\}}^4 \langle 1, 0_7 \rangle_+^1
\end{aligned}$$

$$\langle 1, 0_{m-1} \rangle_+^{m+1} = \sum_{i=1}^{[(m+1)/2]} (-1)^{i-1} C_{\{m^2, (m-2)^2, \dots, (m\%2)^2\}}^i \langle 1, 0_{m-1} \rangle_+^{m+1-2i}$$

$$\text{定理5.3.1. } \sum_{i=0}^{[(m+1)/2]} (-1)^i C_{\{m^2, (m-2)^2, \dots, (m\%2)^2\}}^i \langle 1, 0_{m-1} \rangle_+^{m+1-2i} = 0$$

## 5.4 自然数拆分

定义5.4.1.  $\langle n-l, (l) \rangle := \langle n-l, \geq (l) \rangle$

推论5.4.1.  $(n) = \{ \langle n \rangle, \langle n-1, (1) \rangle, \langle n-2, (2) \rangle, \dots, \langle 2, (n-2) \rangle, \langle 1, (n-1) \rangle \}$  无限关联, 难得通式。

推论5.4.2.

$$\left. \begin{aligned}
 (1) &= \{ \langle 1 \rangle \} \\
 (2) &= \{ \langle 2 \rangle; \langle 1, (1) \rangle \} = \{ \langle 2 \rangle; \langle 1, 1 \rangle \} \\
 (3) &= \{ \langle 3 \rangle; \langle 2, (1) \rangle; \langle 1, (2) \rangle \} = \{ \langle 3 \rangle; \langle 2, 1 \rangle; \langle 1, 1, 1 \rangle \} \\
 (4) &= \{ \langle 4 \rangle; \langle 3, (1) \rangle; \langle 2, (2) \rangle; \langle 1, (3) \rangle \} = \{ \langle 4 \rangle, \langle 3, 1 \rangle; \langle 2, 2 \rangle, \langle 2, 1, 1 \rangle; \langle 1, 1, 1, 1 \rangle \} \\
 (5) &= \{ \langle 5 \rangle; \langle 4, (1) \rangle; \langle 3, (2) \rangle; \langle 2, (3) \rangle; \langle 1, (4) \rangle \} \\
 &= \{ \langle 5 \rangle; \langle 4, 1 \rangle; \langle 3, 2 \rangle, \langle 3, 1, 1 \rangle; \langle 2, 2, 1 \rangle, \langle 2, 1, 1, 1 \rangle; \langle 1, 1, 1, 1, 1 \rangle \} \\
 (6) &= \{ \langle 6 \rangle; \langle 5, (1) \rangle; \langle 4, (2) \rangle; \langle 3, (3) \rangle; \langle 2, (4) \rangle; \langle 1, (5) \rangle \} \\
 &= \{ \langle 6 \rangle; \langle 5, 1 \rangle; \langle 4, 2 \rangle, \langle 4, 1, 1 \rangle; \\
 &\langle 3, 3 \rangle, \langle 3, 2, 1 \rangle, \langle 3, 1, 1, 1 \rangle; \langle 2, 2, 2 \rangle, \langle 2, 2, 1, 1 \rangle, \langle 2, 1, 1, 1, 1 \rangle; \langle 1, 1, 1, 1, 1, 1 \rangle \} \\
 (7) &= \{ \langle 7 \rangle; \langle 6, (1) \rangle; \langle 5, (2) \rangle; \langle 4, (3) \rangle; \langle 3, (4) \rangle; \langle 2, (5) \rangle; \langle 1, (6) \rangle \} \\
 &= \{ \langle 7 \rangle; \langle 6, 1 \rangle; \langle 5, 2 \rangle, \langle 5, 1, 1 \rangle; \langle 4, 3 \rangle, \langle 4, 2, 1 \rangle, \langle 4, 1, 1, 1 \rangle \\
 &; \langle 3, 3, 1 \rangle, \langle 3, 2, 2 \rangle, \langle 3, 2, 1, 1 \rangle, \langle 3, 1, 1, 1, 1 \rangle \\
 &; \langle 2, 2, 2, 1 \rangle, \langle 2, 2, 1, 1, 1 \rangle, \langle 2, 1, 1, 1, 1, 1 \rangle; \langle 1, 1, 1, 1, 1, 1, 1 \rangle \} \\
 \dots
 \end{aligned} \right\}$$

## 5.5 平方零约束条件下的多项式展开

### 5.5.1 平方零约束条件下的二项式展开

定义5.5.1.  $[a_i, a_j] = 0, a_i^2 = 0, \langle 1, 0 \rangle = \frac{1}{2}(a_1 + a_2)$

性质5.5.1.  $(a_1 + a_2)^2 = 2 \langle 2, 0 \rangle + 2 \langle 1, 1 \rangle = 2 \langle 1, 1 \rangle$

推论5.5.1.  $\langle 1, 1 \rangle = \frac{1}{2}(a_1 + a_2)^2$

性质5.5.2.  $(a_1 + a_2)^3 = 2 \langle 3, 0 \rangle + 6 \langle 2, 1 \rangle = 0$

性质5.5.3.  $(a_1 + a_2)^4 = 2 \langle 4, 0 \rangle + 8 \langle 3, 1 \rangle + 6 \langle 2, 2 \rangle = 0$

### 5.5.2 平方零约束条件下的三项式展开

定义5.5.2.  $[a_i, a_j] = 0, a_i^2 = 0, \langle 1, 0, 0 \rangle = \frac{1}{3}(a_1 + a_2 + a_3)$

推论5.5.2.  $\frac{3!}{1!2!} \langle 1, 0, 0 \rangle = (a_1 + a_2 + a_3)$

性质5.5.4.  $(a_1 + a_2 + a_3)^2 = 3 \langle 2, 0, 0 \rangle + 6 \langle 1, 1, 0 \rangle = 6 \langle 1, 1, 0 \rangle$

推论5.5.3.  $\frac{3!}{2!1!} \langle 1, 1, 0 \rangle = \frac{1}{2}(a_1 + a_2 + a_3)^2$

性质5.5.5.  $(a_1 + a_2 + a_3)^3 = 3 \langle 3, 0, 0 \rangle + 18 \langle 2, 1, 0 \rangle + 6 \langle 1, 1, 1 \rangle = 6 \langle 1, 1, 1 \rangle$

推论5.5.4.  $\frac{3!}{3!} \langle 1, 1, 1 \rangle = \frac{1}{6}(a_1 + a_2 + a_3)^3$

性质5.5.6.  $(a_1 + a_2 + a_3)^4 = 3 \langle 4, 0, 0 \rangle + 24 \langle 3, 1, 0 \rangle + 18 \langle 2, 2, 0, 0 \rangle + 36 \langle 2, 1, 1 \rangle = 0$

### 5.5.3 平方零约束条件下的四项式展开

定义5.5.3.  $[a_i, a_j] = 0, a_i^2 = 0, \langle 1, 0, 0, 0 \rangle = \frac{1}{4}(a_1 + a_2 + a_3 + a_4)$

推论5.5.5.  $\frac{4!}{1!3!} \langle 1, 0, 0, 0 \rangle = (a_1 + a_2 + a_3 + a_4)$

性质5.5.7.  $(a_1 + a_2 + a_3 + a_4)^2 = 4 \langle 2, 0, 0, 0 \rangle + 12 \langle 1, 1, 0, 0 \rangle = 12 \langle 1, 1, 0, 0 \rangle$

推论5.5.6.  $\frac{4!}{2!2!} \langle 1, 1, 0, 0 \rangle = \frac{1}{2}(a_1 + a_2 + a_3 + a_4)^2$

性质5.5.8.  $(a_1 + a_2 + a_3 + a_4)^3 = 4 < 3, 0, 0, 0 > + 36 < 2, 1, 0, 0 > + 24 < 1, 1, 1, 0 > = 24 < 1, 1, 1, 0 >$

推论5.5.7.  $\frac{4!}{3!1!} < 1, 1, 1, 0 > = \frac{1}{6}(a_1 + a_2 + a_3 + a_4)^3$

性质5.5.9.  $(a_1 + a_2 + a_3 + a_4)^4 = 4 < 4, 0, 0, 0 > + 48 < 3, 1, 0, 0 > + 36 < 2, 2, 0, 0 > + 144 < 2, 1, 1, 0 > + 24 < 1, 1, 1, 1 > = 24 < 1, 1, 1, 1 >$

推论5.5.8.  $\frac{4!}{4!} < 1, 1, 1, 1 > = \frac{1}{24}(a_1 + a_2 + a_3 + a_4)^4$

性质5.5.10.  $(a_1 + a_2 + a_3 + a_4)^5 = 4 < 5, 0, 0, 0 > + 60 < 4, 1, 0, 0 > + 120 < 3, 2, 0, 0 > + 240 < 3, 1, 1, 0 > + 360 < 2, 2, 1, 0 > + 240 < 2, 1, 1, 1 > = 0$

## 5.6 更具体的直接求法

### 5.6.1 二次元性质

性质5.6.1.  $(\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) = \frac{1}{2}[\vec{\vartheta} \cdot (\sigma \otimes I + I \otimes \sigma)]^2 - \vec{\vartheta}^2$

性质5.6.2.  $(\vec{\vartheta} \cdot \sigma) \otimes (-\vec{\vartheta}^* \cdot \sigma) + (-\vec{\vartheta}^* \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) = [\vec{\vartheta} \cdot \sigma \otimes I + (-\vec{\vartheta}^*) \cdot I \otimes \sigma]^2 - \vec{\vartheta}^2 - (-\vec{\vartheta}^*) \cdot (-\vec{\vartheta}^*)$

性质5.6.3.  $(\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) \otimes I + (\vec{\vartheta} \cdot \sigma) \otimes I \otimes (\vec{\vartheta} \cdot \sigma) + I \otimes (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma)$   
 $= \frac{1}{2}[\vec{\vartheta} \cdot (\sigma \otimes I \otimes I + I \otimes \sigma \otimes I + I \otimes I \otimes \sigma)]^2 - \frac{3}{2}\vec{\vartheta}^2$

性质5.6.4.  $(\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) \otimes I \otimes I + (\vec{\vartheta} \cdot \sigma) \otimes I \otimes (\vec{\vartheta} \cdot \sigma) \otimes I + I \otimes (\vec{\vartheta} \cdot \sigma) \otimes I \otimes (\vec{\vartheta} \cdot \sigma)$   
 $+ I \otimes (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) \otimes I + I \otimes (\vec{\vartheta} \cdot \sigma) \otimes I \otimes (\vec{\vartheta} \cdot \sigma) + I \otimes I \otimes (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma)$   
 $= \frac{1}{2}[\vec{\vartheta} \cdot (\sigma \otimes I \otimes I \otimes I + I \otimes \sigma \otimes I \otimes I + I \otimes I \otimes \sigma \otimes I + I \otimes I \otimes I \otimes \sigma)]^2 - 2\vec{\vartheta}^2$

### 5.6.2 三次元性质

性质5.6.5.  $(\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma)$   
 $= \frac{1}{6}[\vec{\vartheta} \cdot (\sigma \otimes I \otimes I + I \otimes \sigma \otimes I + I \otimes I \otimes \sigma)]^3 - \frac{7}{6}\vec{\vartheta}^2[\vec{\vartheta} \cdot (\sigma \otimes I \otimes I + I \otimes \sigma \otimes I + I \otimes I \otimes \sigma)]$

### 5.6.3 四次元性质

性质5.6.6.  $(\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) \otimes I + (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) \otimes I \otimes (\vec{\vartheta} \cdot \sigma) + (\vec{\vartheta} \cdot \sigma) \otimes I \otimes (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) + I \otimes (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma)$   
 $= \frac{1}{6}[\vec{\vartheta} \cdot (\sigma \otimes I \otimes I \otimes I + I \otimes \sigma \otimes I \otimes I + I \otimes I \otimes \sigma \otimes I + I \otimes I \otimes I \otimes \sigma)]^3$   
 $- \frac{5}{3}\vec{\vartheta}^2[\vec{\vartheta} \cdot (\sigma \otimes I \otimes I \otimes I + I \otimes \sigma \otimes I \otimes I + I \otimes I \otimes \sigma \otimes I + I \otimes I \otimes I \otimes \sigma)]$

性质5.6.7.  $(\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma) \otimes (\vec{\vartheta} \cdot \sigma)$   
 $= \frac{1}{24}[\vec{\vartheta} \cdot (\sigma \otimes I \otimes I \otimes I + I \otimes \sigma \otimes I \otimes I + I \otimes I \otimes \sigma \otimes I + I \otimes I \otimes I \otimes \sigma)]^4$   
 $- \frac{2}{3}\vec{\vartheta}^2[\vec{\vartheta} \cdot (\sigma \otimes I \otimes I \otimes I + I \otimes \sigma \otimes I \otimes I + I \otimes I \otimes \sigma \otimes I + I \otimes I \otimes I \otimes \sigma)]^2 + \vec{\vartheta}^4$

## 5.7 中微子旋量的洛伦兹变换

### 5.7.1 数学准备

定义5.7.1.  $\cosh\theta := \frac{e^\theta + e^{-\theta}}{2} \sim \cos\theta, \sinh\theta := \frac{e^\theta - e^{-\theta}}{2} \sim i\sin\theta, \tanh\theta = \frac{\sinh\theta}{\cosh\theta} \sim i\tan\theta$

性质5.7.1.

$$\cosh^2 - \sinh^2\theta = 1$$

$$\cosh(-\theta) = \cosh\theta, \sinh(-\theta) = -\sinh\theta$$

性质5.7.2.

$$\cosh(\alpha + \beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta$$

$$\cosh(\alpha - \beta) = \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta$$

$$\sinh(\alpha + \beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta$$

$$\sinh(\alpha - \beta) = \sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta$$

性质5.7.3.

$$\begin{aligned} \cosh\alpha + \cosh\beta &= 2\cosh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2} \\ \cosh\alpha - \cosh\beta &= 2\sinh\frac{\alpha+\beta}{2}\sinh\frac{\alpha-\beta}{2} \\ \sinh\alpha + \sinh\beta &= 2\sinh\frac{\alpha+\beta}{2}\cosh\frac{\alpha-\beta}{2} \\ \sinh\alpha - \sinh\beta &= 2\cosh\frac{\alpha+\beta}{2}\sinh\frac{\alpha-\beta}{2} \end{aligned}$$

性质5.7.4.

$$\begin{aligned} \cosh(2\alpha) &= 2\cosh^2\alpha - 1, \sinh(2\alpha) = 2\sinh\alpha\cosh\alpha \\ \cosh^2\frac{\alpha}{2} &= \frac{\cosh\alpha+1}{2}, \sinh^2\frac{\alpha}{2} = \frac{\cosh\alpha-1}{2} \end{aligned}$$

### 5.7.2 中微子旋量的洛伦兹变换

$$\text{推论5.7.1. } \begin{cases} e^{\vec{v}\cdot\frac{\sigma}{2}} = \cosh\frac{1}{2}\sqrt{\vec{v}^2} + \frac{\sinh\frac{1}{2}\sqrt{\vec{v}^2}}{\sqrt{\vec{v}^2}}\vec{v}\cdot\sigma, \vec{v}^2 \neq 0 \\ e^{\vec{v}\cdot\frac{\sigma}{2}} = 1 + \vec{v}\cdot\frac{\sigma}{2}, \vec{v}^2 = 0, \vec{v} = i\vec{\omega} + \zeta\vec{e} \end{cases}$$

$$\text{定义5.7.2. } v := |\vec{v}|, c := \cosh\frac{1}{2}\sqrt{\vec{v}^2}, s := \frac{\sinh\frac{1}{2}\sqrt{\vec{v}^2}}{\sqrt{\vec{v}^2}}, c^2 - s^2\vec{v}^2 \equiv 1$$

$$\text{推论5.7.2. } e^{\vec{v}\cdot\sigma(\frac{1}{2})} = \cosh\frac{1}{2}\sqrt{\vec{v}^2} + \frac{\sinh\frac{1}{2}\sqrt{\vec{v}^2}}{\sqrt{\vec{v}^2}}\vec{v}\cdot\sigma \equiv c + s\vec{v}\cdot\sigma$$

$$\text{推论5.7.3. } \Lambda_{\zeta\vec{v}} = e^{-\zeta\ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(\frac{1}{2})} = \frac{1}{\sqrt{2(\gamma_v+1)}}(1 + \gamma_v - \zeta\gamma_v\vec{v}\cdot\sigma), c = \frac{(1+\gamma_v)}{\sqrt{2(\gamma_v+1)}}, s = -\frac{\zeta\gamma_v}{\sqrt{2(\gamma_v+1)}}$$

$$\text{推论5.7.4. } \Lambda_{\zeta\vec{v}} = e^{-\zeta\ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(\frac{1}{2})} = \frac{1}{\sqrt{2(\gamma_v+1)}}[1 + \gamma_v - 2\zeta\gamma_v v\hat{v}\cdot\sigma(\frac{1}{2})]$$

### 5.7.3 电子旋量的洛伦兹变换

$$\text{推论5.7.5. } D_{\zeta\vec{v}} = e^{-\zeta\ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_4)} = \frac{1}{\sqrt{2(\gamma_v+1)}}[1 + \gamma_v - i\zeta\gamma_v\vec{v}\cdot\vec{\gamma}\gamma_4]$$

## 5.8 光子旋量洛伦兹变换的多项式表示

### 5.8.1 光子旋量的一般洛伦兹变换的多项式表示

$$\text{定理5.8.1. } e^{\vec{v}\cdot\Omega(1)} = 1 + \frac{\sinh\sqrt{\vec{v}^2}}{\sqrt{\vec{v}^2}}[\vec{v}\cdot\Omega(1)] + \frac{\cosh\sqrt{\vec{v}^2}-1}{\vec{v}^2}[\vec{v}\cdot\Omega(1)]^2$$

$$\begin{aligned} \text{证明: } e^{\vec{v}\cdot\Omega(1)} &= (c + s\vec{v}\cdot\sigma) \otimes (c + s\vec{v}\cdot\sigma) \\ &= c^2 + cs[\vec{v}\cdot\hat{\Omega}(1)] + s^2[\hat{\Omega}(1, 2, \vec{v}\cdot\sigma)] \\ &= c^2 + cs[\vec{v}\cdot\hat{\Omega}(1)] + s^2\{\frac{1}{2}[\vec{v}\cdot\hat{\Omega}(1)]^2 - \vec{v}^2\} \\ &= (c^2 - s^2\vec{v}^2) + cs[\vec{v}\cdot\hat{\Omega}(1)] + \frac{1}{2}s^2[\vec{v}\cdot\hat{\Omega}(1)]^2 \\ &= 1 + 2cs[\vec{v}\cdot\Omega(1)] + 2s^2[\vec{v}\cdot\Omega(1)]^2 \\ &= 1 + \frac{\sinh\sqrt{\vec{v}^2}}{\sqrt{\vec{v}^2}}[\vec{v}\cdot\Omega(1)] + \frac{\cosh\sqrt{\vec{v}^2}-1}{\vec{v}^2}[\vec{v}\cdot\Omega(1)]^2 \end{aligned}$$

□

$$\text{推论5.8.1. } \begin{cases} e^{\vec{v}\cdot\Omega(1)} = 1 + \frac{\sinh\sqrt{\vec{v}^2}}{\sqrt{\vec{v}^2}}[\vec{v}\cdot\Omega(1)] + \frac{\cosh\sqrt{\vec{v}^2}-1}{\vec{v}^2}[\vec{v}\cdot\Omega(1)]^2 \\ e^{\vec{v}\cdot\sigma(1)} = 1 + \frac{\sinh\sqrt{\vec{v}^2}}{\sqrt{\vec{v}^2}}[\vec{v}\cdot\sigma(1)] + \frac{\cosh\sqrt{\vec{v}^2}-1}{\vec{v}^2}[\vec{v}\cdot\sigma(1)]^2 \\ e^{\vec{v}\cdot R} = 1 + \frac{\sinh\sqrt{\vec{v}^2}}{\sqrt{\vec{v}^2}}(\vec{v}\cdot R) + \frac{\cosh\sqrt{\vec{v}^2}-1}{\vec{v}^2}(\vec{v}\cdot R)^2 \\ e^{\vec{v}\cdot\gamma} = 1 + \frac{\sinh\sqrt{\vec{v}^2}}{\sqrt{\vec{v}^2}}(\vec{v}\cdot\gamma) + \frac{\cosh\sqrt{\vec{v}^2}-1}{\vec{v}^2}(\vec{v}\cdot\gamma)^2 \end{cases}$$

### 5.8.2 光子旋量的洛伦兹推动变换的多项式表示

推论5.8.2.  $\epsilon = \ln[\gamma_v(1+v)] \Leftrightarrow \sinh\epsilon = \gamma_v v \Leftrightarrow \cosh\epsilon = \gamma_v, \sinh\epsilon = \gamma_v v$

$$\text{推论5.8.3. } R_{\zeta\vec{v}} = \begin{cases} e^{-\zeta \ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(1)} = 1 - \zeta\gamma_v v[\hat{v}\cdot\Omega(1)] + (\gamma_v - 1)[\hat{v}\cdot\Omega(1)]^2 \\ e^{-\zeta \ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(1)} = 1 - \zeta\gamma_v v[\hat{v}\cdot\sigma(1)] + (\gamma_v - 1)[\hat{v}\cdot\sigma(1)]^2 \\ e^{-\zeta \ln[\gamma_v(1+v)]\hat{v}\cdot R} = 1 - \zeta\gamma_v v(\hat{v}\cdot R) + (\gamma_v - 1)(\hat{v}\cdot R)^2 \\ e^{-\zeta \ln[\gamma_v(1+v)]\hat{v}\cdot\gamma} = 1 - \zeta\gamma_v v(\hat{v}\cdot\gamma) + (\gamma_v - 1)(\hat{v}\cdot\gamma)^2 \end{cases}$$

## 5.9 引力微子旋量洛伦兹变换的多项式表示

### 5.9.1 引力微子旋量的一般洛伦兹变换的多项式表示

$$\text{定理5.9.1. } e^{\vec{\vartheta}\cdot\Omega(\frac{3}{2})} = \cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(1 - \frac{1}{2}\sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}) + 2\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[1 - \frac{1}{6}\sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}][\vec{\vartheta}\cdot\Omega(\frac{3}{2})] \\ + 2\cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^2 + \frac{4}{3}(\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^3[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^3$$

$$\text{证明: } e^{\vec{\vartheta}\cdot\Omega(\frac{3}{2})} = (c + s\vec{\vartheta}\cdot\sigma) \otimes (c + s\vec{\vartheta}\cdot\sigma) \otimes (c + s\vec{\vartheta}\cdot\sigma) \\ = c^3 + c^2s[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})] + cs^2[\hat{\Omega}(\frac{3}{2}, 2, \vec{\vartheta}\cdot\sigma)] + s^3[\hat{\Omega}(\frac{3}{2}, 3, \vec{\vartheta}\cdot\sigma)] \\ = c^3 + c^2s[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})] + cs^2\{\frac{1}{2}[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^2 - \frac{3}{2}\vec{\vartheta}^2\} + s^3\{\frac{1}{6}[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^3 - \frac{7}{6}\vec{\vartheta}^2[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]\} \\ = c(c^2 - \frac{3}{2}s^2\vec{\vartheta}^2) + s(c^2 - \frac{7}{6}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})] + \frac{1}{2}cs^2[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^2 + \frac{1}{6}cs^3[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^3 \\ = c(1 - \frac{1}{2}s^2\vec{\vartheta}^2) + s(1 - \frac{1}{6}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})] + \frac{1}{2}cs^2[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^2 + \frac{1}{6}cs^3[\vec{\vartheta}\cdot\hat{\Omega}(\frac{3}{2})]^3 \\ = c(1 - \frac{1}{2}s^2\vec{\vartheta}^2) + 2s(1 - \frac{1}{6}s^2\vec{\vartheta}^2)[\vec{\vartheta}\cdot\Omega(\frac{3}{2})] + 2cs^2[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^2 + \frac{4}{3}cs^3[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^3 \\ = \cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(1 - \frac{1}{2}\sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}) + 2\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[1 - \frac{1}{6}\sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}][\vec{\vartheta}\cdot\Omega(\frac{3}{2})] \\ + 2\cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^2 + \frac{4}{3}(\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^3[\vec{\vartheta}\cdot\Omega(\frac{3}{2})]^3 \quad \square$$

$$\text{推论5.9.1. } e^{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_3 + I\otimes\sigma(1)]} \\ = \cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(1 - \frac{1}{2}\sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}) + 2\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[1 - \frac{1}{6}\sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}]\{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_3 + I\otimes\sigma(1)]\} \\ + 2\cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2\{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_3 + I\otimes\sigma(1)]\}^2 + \frac{4}{3}(\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^3\{\vec{\vartheta}\cdot[\sigma(\frac{1}{2})\otimes I_3 + I\otimes\sigma(1)]\}^3$$

$$\text{推论5.9.2. } e^{\vec{\vartheta}\cdot\sigma(\frac{3}{2})} = \cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(1 - \frac{1}{2}\sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}) + 2\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}[1 - \frac{1}{6}\sinh^2\frac{1}{2}\sqrt{\vec{\vartheta}^2}][\vec{\vartheta}\cdot\sigma(\frac{3}{2})] \\ + 2\cosh\frac{1}{2}\sqrt{\vec{\vartheta}^2}(\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^2[\vec{\vartheta}\cdot\sigma(\frac{3}{2})]^2 + \frac{4}{3}(\frac{\sinh\frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})^3[\vec{\vartheta}\cdot\sigma(\frac{3}{2})]^3$$

### 5.9.2 引力微子旋量的洛伦兹推动变换的多项式表示

$$\text{推论5.9.3. } e^{-\zeta \ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(\frac{3}{2})} = \frac{(\gamma_v+1)}{\sqrt{2(\gamma_v+1)}}(1 - \frac{\gamma_v-1}{4}) - \frac{2\zeta\gamma_v v}{\sqrt{2(\gamma_v+1)}}(1 - \frac{\gamma_v-1}{12})[\hat{v}\cdot\Omega(\frac{3}{2})] \\ + \frac{\gamma_v^2-1}{\sqrt{2(\gamma_v+1)}}[\hat{v}\cdot\Omega(\frac{3}{2})]^2 - \frac{1}{3}\frac{2\zeta\gamma_v v(\gamma_v-1)}{\sqrt{2(\gamma_v+1)}}[\hat{v}\cdot\Omega(\frac{3}{2})]^3$$

推论5.9.4.

$$\Lambda_{\zeta\vec{v}}(\frac{3}{2}) = \begin{cases} e^{-\zeta \ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(\frac{3}{2})} = \frac{1}{\sqrt{2(\gamma_v+1)}}[1 + \gamma_v - 2\zeta\gamma_v\vec{v}\cdot\Omega(\frac{3}{2})] + \frac{\gamma_v-1}{\sqrt{2(\gamma_v+1)}}[1 + \gamma_v - \frac{2}{3}\zeta\gamma_v\vec{v}\cdot\Omega(\frac{3}{2})]\{[\hat{v}\cdot\Omega(\frac{3}{2})]^2 - \frac{1}{4}\} \\ e^{-\zeta \ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(\frac{3}{2})} = \frac{1}{\sqrt{2(\gamma_v+1)}}[1 + \gamma_v - 2\zeta\gamma_v\vec{v}\cdot\sigma(\frac{3}{2})] + \frac{\gamma_v-1}{\sqrt{2(\gamma_v+1)}}[1 + \gamma_v - \frac{2}{3}\zeta\gamma_v\vec{v}\cdot\sigma(\frac{3}{2})]\{[\hat{v}\cdot\sigma(\frac{3}{2})]^2 - \frac{1}{4}\} \end{cases}$$

## 5.10 引力子旋量洛伦兹变换的多项式表示

### 5.10.1 引力子旋量的一般洛伦兹变换的多项式表示

**定理5.10.1.** 
$$e^{\vec{\vartheta} \cdot \Omega(2)} = 1 + \left(\frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right) \left(1 - \frac{2}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2}\right) [\vec{\vartheta} \cdot \Omega(2)] + 2 \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^2 \left(1 - \frac{1}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2}\right) [\vec{\vartheta} \cdot \Omega(2)]^2$$

$$+ \frac{2}{3} \left(\frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right) \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^2 [\vec{\vartheta} \cdot \Omega(2)]^3 + \frac{2}{3} \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^4 [\vec{\vartheta} \cdot \Omega(2)]^4$$

**证明:** 
$$e^{\vec{\vartheta} \cdot \Omega(2)} = (c + s\vec{\vartheta} \cdot \sigma) \otimes (c + s\vec{\vartheta} \cdot \sigma) \otimes (c + s\vec{\vartheta} \cdot \sigma) \otimes (c + s\vec{\vartheta} \cdot \sigma)$$

$$= c^4 + c^3 s [\vec{\vartheta} \cdot \hat{\Omega}(2)] + c^2 s^2 [\hat{\Omega}(2, 2, \vec{\vartheta} \cdot \sigma)] + c s^3 [\hat{\Omega}(2, 3, \vec{\vartheta} \cdot \sigma)] + s^4 [\hat{\Omega}(2, 4, \vec{\vartheta} \cdot \sigma)]$$

$$= c^4 + c^3 s [\vec{\vartheta} \cdot \hat{\Omega}(2)] + c^2 s^2 \left\{ \frac{1}{2} [\vec{\vartheta} \cdot \hat{\Omega}(2)]^2 - 2\vec{\vartheta}^2 \right\}$$

$$+ c s^3 \left\{ \frac{1}{6} [\vec{\vartheta} \cdot \hat{\Omega}(2)]^3 - \frac{5}{3} \vec{\vartheta}^2 [\vec{\vartheta} \cdot \hat{\Omega}(2)] \right\} + s^4 \left\{ \frac{1}{24} [\vec{\vartheta} \cdot \hat{\Omega}(2)]^4 - \frac{2}{3} \vec{\vartheta}^2 [\vec{\vartheta} \cdot \hat{\Omega}(2)]^2 + \vec{\vartheta}^4 \right\}$$

$$= (c^2 - s^2 \vec{\vartheta}^2)^2 + c s (c^2 - \frac{5}{3} s^2 \vec{\vartheta}^2) [\vec{\vartheta} \cdot \hat{\Omega}(2)] + \frac{1}{2} s^2 (c^2 - \frac{4}{3} s^2 \vec{\vartheta}^2) [\vec{\vartheta} \cdot \hat{\Omega}(2)]^2 + \frac{1}{6} c s^3 [\vec{\vartheta} \cdot \hat{\Omega}(2)]^3 + \frac{1}{24} s^4 [\vec{\vartheta} \cdot \hat{\Omega}(2)]^4$$

$$= 1 + c s (1 - \frac{2}{3} s^2 \vec{\vartheta}^2) [\vec{\vartheta} \cdot \hat{\Omega}(2)] + \frac{1}{2} s^2 (1 - \frac{1}{3} s^2 \vec{\vartheta}^2) [\vec{\vartheta} \cdot \hat{\Omega}(2)]^2 + \frac{1}{6} c s^3 [\vec{\vartheta} \cdot \hat{\Omega}(2)]^3 + \frac{1}{24} s^4 [\vec{\vartheta} \cdot \hat{\Omega}(2)]^4$$

$$= 1 + 2c s (1 - \frac{2}{3} s^2 \vec{\vartheta}^2) [\vec{\vartheta} \cdot \Omega(2)] + 2s^2 (1 - \frac{1}{3} s^2 \vec{\vartheta}^2) [\vec{\vartheta} \cdot \Omega(2)]^2 + \frac{4}{3} c s^3 [\vec{\vartheta} \cdot \Omega(2)]^3 + \frac{2}{3} s^4 [\vec{\vartheta} \cdot \Omega(2)]^4$$

$$= 1 + \left(\frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right) \left(1 - \frac{2}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2}\right) [\vec{\vartheta} \cdot \Omega(2)] + 2 \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^2 \left(1 - \frac{1}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2}\right) [\vec{\vartheta} \cdot \Omega(2)]^2$$

$$+ \frac{2}{3} \left(\frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right) \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^2 [\vec{\vartheta} \cdot \Omega(2)]^3 + \frac{2}{3} \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^4 [\vec{\vartheta} \cdot \Omega(2)]^4 \quad \square$$

**推论5.10.1.** 
$$e^{\vec{\vartheta} \cdot [\sigma(\frac{1}{2}) \otimes I_4 + I \otimes \sigma(\frac{3}{2})]} = 1 + \left(\frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right) \left(1 - \frac{2}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2}\right) \{ \vec{\vartheta} \cdot [\sigma(\frac{1}{2}) \otimes I_4 + I \otimes \sigma(\frac{3}{2})] \}$$

$$+ 2 \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^2 \left(1 - \frac{1}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2}\right) \{ \vec{\vartheta} \cdot [\sigma(\frac{1}{2}) \otimes I_4 + I \otimes \sigma(\frac{3}{2})] \}^2$$

$$+ \frac{2}{3} \left(\frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right) \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^2 \{ \vec{\vartheta} \cdot [\sigma(\frac{1}{2}) \otimes I_4 + I \otimes \sigma(\frac{3}{2})] \}^3 + \frac{2}{3} \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^4 \{ \vec{\vartheta} \cdot [\sigma(\frac{1}{2}) \otimes I_4 + I \otimes \sigma(\frac{3}{2})] \}^4$$

**推论5.10.2.** 
$$e^{\vec{\vartheta} \cdot \sigma(2)} = 1 + \left(\frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right) \left(1 - \frac{2}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2}\right) [\vec{\vartheta} \cdot \sigma(2)] + 2 \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^2 \left(1 - \frac{1}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2}\right) [\vec{\vartheta} \cdot \sigma(2)]^2$$

$$+ \frac{2}{3} \left(\frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right) \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^2 [\vec{\vartheta} \cdot \sigma(2)]^3 + \frac{2}{3} \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^4 [\vec{\vartheta} \cdot \sigma(2)]^4$$

**推论5.10.3.** 
$$e^{\vec{\vartheta} \cdot G_m} = 1 + \left(\frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right) \left(1 - \frac{2}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2}\right) [\vec{\vartheta} \cdot G_m] + 2 \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^2 \left(1 - \frac{1}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2}\right) [\vec{\vartheta} \cdot G_m]^2$$

$$+ \frac{2}{3} \left(\frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right) \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^2 [\vec{\vartheta} \cdot G_m]^3 + \frac{2}{3} \left(\frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\right)^4 [\vec{\vartheta} \cdot G_m]^4$$

### 5.10.2 引力子旋量的洛伦兹推动变换的多项式表示

**推论5.10.4.** 
$$e^{-\zeta \ln[\gamma_v(1+v)] \hat{v} \cdot \Omega(2)} = 1 - \zeta \gamma_v (1 - \frac{\gamma_v - 1}{3}) [\vec{v} \cdot \Omega(2)] + \frac{\gamma_v - 1}{v^2} (1 - \frac{\gamma_v - 1}{6}) [\vec{v} \cdot \Omega(2)]^2$$

$$- \frac{1}{3} \frac{\zeta \gamma_v (\gamma_v - 1)}{v^2} [\vec{v} \cdot \Omega(2)]^3 + \frac{1}{6} \frac{(\gamma_v - 1)^2}{v^4} [\vec{v} \cdot \Omega(2)]^4$$

**推论5.10.5.**

$$\Lambda_{\zeta \vec{v}}(2) = \begin{cases} e^{-\zeta \ln[\gamma_v(1+v)] \hat{v} \cdot \Omega(2)} = 1 - \zeta \gamma_v v [\hat{v} \cdot \Omega(2)] + (\gamma_v - 1) [\hat{v} \cdot \Omega(2)]^2 \\ \quad + \frac{1}{3} (\gamma_v - 1) \{ -\zeta \gamma_v v [\hat{v} \cdot \Omega(2)] + \frac{1}{2} (\gamma_v - 1) [\hat{v} \cdot \Omega(2)]^2 \} \{ [\hat{v} \cdot \Omega(2)]^2 - 1 \} \\ e^{-\zeta \ln[\gamma_v(1+v)] \hat{v} \cdot \sigma(2)} = 1 - \zeta \gamma_v v [\hat{v} \cdot \sigma(2)] + (\gamma_v - 1) [\hat{v} \cdot \sigma(2)]^2 \\ \quad + \frac{1}{3} (\gamma_v - 1) \{ -\zeta \gamma_v v [\hat{v} \cdot \sigma(2)] + \frac{1}{2} (\gamma_v - 1) [\hat{v} \cdot \sigma(2)]^2 \} \{ [\hat{v} \cdot \sigma(2)]^2 - 1 \} \end{cases}$$

**推论5.10.6.** 
$$R_{\zeta \vec{v}}(2) = e^{-\zeta \ln[\gamma_v(1+v)] \hat{v} \cdot G_m} = 1 - \zeta \gamma_v v [\hat{v} \cdot G_m] + (\gamma_v - 1) [\hat{v} \cdot G_m]^2$$

$$+ \frac{1}{3} (\gamma_v - 1) \{ -\zeta \gamma_v v [\hat{v} \cdot G_m] + \frac{1}{2} (\gamma_v - 1) [\hat{v} \cdot G_m]^2 \} \{ [\hat{v} \cdot G_m]^2 - 1 \}$$

## 5.11 s旋量洛伦兹变换的多项式统一表示

**推论5.11.1.** 
$$e^{\vec{\vartheta} \cdot \Omega(s)} = \overbrace{e^{\vec{\vartheta} \cdot \sigma(\frac{1}{2})} \otimes \dots \otimes e^{\vec{\vartheta} \cdot \sigma(\frac{1}{2})}}^{2s}$$



推论5.11.2.

$$\begin{cases} e^{\vec{v} \cdot \sigma(s)} = \bar{\Gamma}(s) e^{\vec{v} \cdot \Omega(s)} \Gamma(s) \\ e^{\vec{v} \cdot \sigma(s-1)} = \bar{X}(s) [I \otimes \bar{\Gamma}(s - \frac{1}{2})] e^{\vec{v} \cdot \Omega(s)} [I \otimes \Gamma(s - \frac{1}{2})] X(s) \\ e^{\vec{v} \cdot [\sigma \frac{1}{2} \otimes I_{2s} + I \otimes \sigma(s - \frac{1}{2})]} = [I \otimes \bar{\Gamma}(s - \frac{1}{2})] e^{\vec{v} \cdot \Omega(s)} [I \otimes \Gamma(s - \frac{1}{2})] \\ e^{\vec{v} \cdot \Omega(s-1)} = I_{4^{s-1}} \otimes \{ \bar{X}(1) [I \otimes \bar{\Gamma}(\frac{1}{2})] \} e^{\vec{v} \cdot \Omega(s)} I_{4^{s-1}} \otimes \{ [I \otimes \Gamma(\frac{1}{2})] X(1) \} \\ e^{\vec{v} \cdot \pi(s,k)} = I_{4^k} \otimes \{ \bar{X}(s-k) [I \otimes \bar{\Gamma}(s-k - \frac{1}{2})] \} e^{\vec{v} \cdot \Omega(s)} I_{4^k} \otimes \{ [I \otimes \Gamma(s-k - \frac{1}{2})] X(s-k) \} \end{cases}$$

推论5.11.3.

$$\begin{cases} [\vec{v} \cdot \Omega(s)]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\vec{v} \cdot \Omega(s)]^{2s+1-2k}, [\vec{v} \cdot \sigma(s)]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\vec{v} \cdot \sigma(s)]^{2s+1-2k} \\ [\vec{v} \cdot \Omega(s-1)]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\vec{v} \cdot \Omega(s-1)]^{2s+1-2k}, [\vec{v} \cdot \sigma(s-1)]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\vec{v} \cdot \sigma(s-1)]^{2s+1-2k} \\ [\vec{v} \cdot \Omega(s-2)]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\vec{v} \cdot \Omega(s-2)]^{2s+1-2k}, [\vec{v} \cdot \sigma(s-2)]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\vec{v} \cdot \sigma(s-2)]^{2s+1-2k} \\ \dots \\ [\vec{v} \cdot \Omega(\frac{1}{2}|0)]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\vec{v} \cdot \Omega(\frac{1}{2}|0)]^{2s+1-2k}, [\vec{v} \cdot \sigma(\frac{1}{2}|0)]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\vec{v} \cdot \sigma(\frac{1}{2}|0)]^{2s+1-2k} \end{cases}$$

推论5.11.4.

$$\begin{cases} e^{\vec{v} \cdot \Omega(s)} = \sum_{k=0}^{2s} A_k(s) [\vec{v} \cdot \Omega(s)]^k, e^{\vec{v} \cdot \sigma(s)} = \sum_{k=0}^{2s} A_k(s) [\vec{v} \cdot \sigma(s)]^k \\ e^{\vec{v} \cdot \Omega(s-1)} = \sum_{k=0}^{2s} A_k(s) [\vec{v} \cdot \Omega(s-1)]^k, e^{\vec{v} \cdot \sigma(s-1)} = \sum_{k=0}^{2s} A_k(s) [\vec{v} \cdot \sigma(s-1)]^k \\ e^{\vec{v} \cdot \Omega(s-2)} = \sum_{k=0}^{2s} A_k(s) [\vec{v} \cdot \Omega(s-2)]^k, e^{\vec{v} \cdot \sigma(s-2)} = \sum_{k=0}^{2s} A_k(s) [\vec{v} \cdot \sigma(s-2)]^k \\ \dots \\ e^{\vec{v} \cdot \Omega(\frac{1}{2}|0)} = \sum_{k=0}^{2s} A_k(s) [\vec{v} \cdot \Omega(\frac{1}{2}|0)]^k, e^{\vec{v} \cdot \sigma(\frac{1}{2}|0)} = \sum_{k=0}^{2s} A_k(s) [\vec{v} \cdot \sigma(\frac{1}{2}|0)]^k \end{cases}$$

即粗略地讲有以下的说法，本质上是以上推论的意思，且本节结论是被严格证明的。

推论5.11.5.

$$\begin{cases} \left\{ e^{\vec{v} \cdot \Omega(n)}, e^{\vec{v} \cdot \Omega(n-1)}, \dots, e^{\vec{v} \cdot \Omega(1)}, e^{\vec{v} \cdot \Omega(0)} \right\} \text{的展开系数} = e^{\vec{v} \cdot \Omega(n)} \text{的展开系数} \\ \left\{ e^{\vec{v} \cdot \sigma(n)}, e^{\vec{v} \cdot \sigma(n-1)}, \dots, e^{\vec{v} \cdot \sigma(1)}, e^{\vec{v} \cdot \sigma(0)} \right\} \\ \left\{ e^{\vec{v} \cdot \Omega(n+\frac{1}{2})}, e^{\vec{v} \cdot \Omega(n-\frac{1}{2})}, \dots, e^{\vec{v} \cdot \Omega(\frac{3}{2})}, e^{\vec{v} \cdot \Omega(\frac{1}{2})} \right\} \text{的展开系数} = e^{\vec{v} \cdot \Omega(n+\frac{1}{2})} \text{的展开系数} \\ \left\{ e^{\vec{v} \cdot \sigma(n+\frac{1}{2})}, e^{\vec{v} \cdot \sigma(n-\frac{1}{2})}, \dots, e^{\vec{v} \cdot \sigma(\frac{3}{2})}, e^{\vec{v} \cdot \sigma(\frac{1}{2})} \right\} \end{cases}$$

推论5.11.6.  $e^{i2\pi\hat{\omega} \cdot \sigma(s)} = (-1)^{2s}$

推论5.11.7.  $\vec{v}^2 = 0$

$$\Rightarrow [\vec{v} \cdot \Omega(s)]^{2s+1} = 0, e^{\vec{v} \cdot \Omega(s)} = \sum_{n=0}^{2s} \frac{1}{n!} [\vec{v} \cdot \Omega(s)]^n \Rightarrow [\vec{v} \cdot \sigma(s)]^{2s+1} = 0, e^{\vec{v} \cdot \sigma(s)} = \sum_{n=0}^{2s} \frac{1}{n!} [\vec{v} \cdot \sigma(s)]^n$$

猜想5.11.1.  $e^{\vec{v} \cdot \sigma(s)}|_{\vec{v}^2=0} = \langle e^{\vec{v} \cdot \sigma(s)} \rangle_{\vec{v}^2 \rightarrow 0} ???$

## 5.12 s旋量的洛伦兹推动变换的多项式表示???

推论5.12.1.  $R_{\zeta\vec{v}}(n) = e^{-\zeta \ln[\gamma_v(1+v)] \hat{v} \cdot \sigma(n)} = \sum_{k=0}^{2n} f_k(v) [\hat{v} \cdot \sigma(n)]^k$

$$\Rightarrow \begin{cases} R_{\zeta\vec{v}}(l) = e^{-\zeta \ln[\gamma_v(1+v)] \hat{v} \cdot \sigma(l)} = \sum_{k=0}^{2n} f_k(v) [\hat{v} \cdot \sigma(l)]^k, f_0(v) = 1, 0 \leq l \leq n \\ R_{\zeta\vec{v}}(l + \frac{1}{2}) = e^{-\zeta \ln[\gamma_v(1+v)] \hat{v} \cdot \sigma(l)} = \sum_{k=0}^{2n} f_k(\frac{v}{2}) [2\hat{v} \cdot \sigma(l + \frac{1}{2})]^k = \sum_{k=0}^{2n} 2^k f_k(\frac{v}{2}) [\hat{v} \cdot \sigma(l + \frac{1}{2})]^k, 0 \leq l + \frac{1}{2} \leq n \end{cases}$$

$$\begin{aligned} \text{推论5.12.2. } R_{\zeta\bar{v}}(n + \frac{1}{2}) &= e^{-\zeta \ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(n+\frac{1}{2})} = \sum_{k=0}^{2n+1} g_k(v)[\hat{v}\cdot\sigma(n + \frac{1}{2})]^k \\ \Rightarrow \begin{cases} R_{\zeta\bar{v}}(l) = e^{-\zeta \ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(l)} = \sum_{k=0}^{2n+1} g_k(2v)[\frac{1}{2}\hat{v}\cdot\sigma(l)]^k = \sum_{k=0}^{2n+1} 2^{-k} g_k(2v)[\hat{v}\cdot\sigma(l)]^k, f_0(v) = 1, 0 \leq l \leq n + \frac{1}{2} \\ R_{\zeta\bar{v}}(l + \frac{1}{2}) = e^{-\zeta \ln[\gamma_v(1+v)]\hat{v}\cdot\sigma(l+\frac{1}{2})} = \sum_{k=0}^{2n+1} g_k(v)[\hat{v}\cdot\sigma(l + \frac{1}{2})]^k, 0 \leq l + \frac{1}{2} \leq n + \frac{1}{2} \end{cases} \end{aligned}$$

### 5.13 矢量洛伦兹变换的多项式表示

$$\begin{aligned} \text{推论5.13.1. } \Lambda(1, \epsilon) &= (c + s\epsilon \cdot \sigma) \otimes (c - s\epsilon \cdot \sigma) \\ &= c^2 + cs[\epsilon \cdot (\sigma \otimes I - I \otimes \sigma)] - s^2[\epsilon \cdot (\cdot)_2] \\ &= c^2 + cs[\epsilon \cdot (\sigma \otimes I - I \otimes \sigma)] - s^2\{\frac{1}{2}[\epsilon \cdot \hat{\Omega}(1)]^2 - \epsilon \cdot \epsilon\} \\ &= (c^2 + s^2\epsilon \cdot \epsilon) + cs[\epsilon \cdot (\sigma \otimes I - I \otimes \sigma)] - \frac{1}{2}s^2[\epsilon \cdot \hat{\Omega}(1)]^2 \\ &= (c^2 + s^2\epsilon \cdot \epsilon) + 2cs[\epsilon \cdot \frac{1}{2}(\sigma \otimes I - I \otimes \sigma)] - 2s^2[\epsilon \cdot \Omega(1)]^2 \\ &= \cosh\sqrt{\epsilon \cdot \epsilon} + \frac{\sinh\sqrt{\epsilon \cdot \epsilon}}{\sqrt{\epsilon \cdot \epsilon}}[\epsilon \cdot \frac{1}{2}(\sigma \otimes I - I \otimes \sigma)] - \frac{\cosh\sqrt{\epsilon \cdot \epsilon} - 1}{\epsilon \cdot \epsilon}[\epsilon \cdot \Omega(1)]^2 \end{aligned}$$

$$\text{推论5.13.2. } L(\epsilon) = e^{\epsilon \cdot L} = \cosh\sqrt{\epsilon \cdot \epsilon} + \frac{\sinh\sqrt{\epsilon \cdot \epsilon}}{\sqrt{\epsilon \cdot \epsilon}}\epsilon \cdot L - \frac{\cosh\sqrt{\epsilon \cdot \epsilon} - 1}{\epsilon \cdot \epsilon}(\epsilon \cdot R)^2 = 1 + \frac{\sinh\sqrt{\epsilon \cdot \epsilon}}{\sqrt{\epsilon \cdot \epsilon}}(\epsilon \cdot L) + \frac{\cosh\sqrt{\epsilon \cdot \epsilon} - 1}{\epsilon \cdot \epsilon}(\epsilon \cdot L)^2$$

$$\begin{aligned} \text{推论5.13.3. } \Lambda(1, i\omega) &= (c + is\omega \cdot \sigma) \otimes (c + is\omega \cdot \sigma) \\ &= c^2 + cs[i\omega \cdot (\sigma \otimes I + I \otimes \sigma)] + s^2[i\omega \cdot (\cdot)_2] \\ &= c^2 + cs[i\omega \cdot (\sigma \otimes I + I \otimes \sigma)] + s^2\{\frac{1}{2}[i\omega \cdot \hat{\Omega}(1)]^2 - i\omega \cdot i\omega\} \\ &= c^2 + cs[i\omega \cdot (\sigma \otimes I + I \otimes \sigma)] + s^2\{\frac{1}{2}[i\omega \cdot \hat{\Omega}(1)]^2 - i\omega \cdot i\omega\} \\ &= 1 + \frac{\sinh\sqrt{i\omega \cdot i\omega}}{\sqrt{i\omega \cdot i\omega}}[i\omega \cdot \frac{1}{2}(\sigma \otimes I + I \otimes \sigma)] + \frac{\cosh\sqrt{i\omega \cdot i\omega} - 1}{i\omega \cdot i\omega}[i\omega \cdot \Omega(1)]^2 \\ &= 1 + i\frac{\sinh\sqrt{\omega \cdot \omega}}{\sqrt{\omega \cdot \omega}}[\omega \cdot \frac{1}{2}(\sigma \otimes I + I \otimes \sigma)] + \frac{\cosh\sqrt{\omega \cdot \omega} - 1}{\omega \cdot \omega}[\omega \cdot \Omega(1)]^2 \end{aligned}$$

$$\text{推论5.13.4. } R(i\omega) = L(i\omega) = e^{i\omega \cdot R} = 1 + i\frac{\sinh\sqrt{\omega \cdot \omega}}{\sqrt{\omega \cdot \omega}}(\omega \cdot R) + \frac{\cosh\sqrt{\omega \cdot \omega} - 1}{\omega \cdot \omega}(\omega \cdot R)^2$$

$$\text{推论5.13.5. } R_3(i\omega) = L_3(i\omega) = e^{i\omega \cdot \gamma} = 1 + i\frac{\sinh\sqrt{\omega \cdot \omega}}{\sqrt{\omega \cdot \omega}}(\omega \cdot \gamma) + \frac{\cosh\sqrt{\omega \cdot \omega} - 1}{\omega \cdot \omega}(\omega \cdot \gamma)^2$$

$$\text{推论5.13.6. } e^{i\omega \cdot R + \zeta \epsilon \cdot L} = \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2}\frac{\cosh a - \cos b}{a^2 + b^2}(\vec{\vartheta} \cdot \sigma_+) (\vec{\vartheta}^* \cdot \sigma_-) - \frac{1}{2}\frac{\sinh a - i \sin b}{a - ib}(\vec{\vartheta}^* \cdot \sigma_-) + \frac{1}{2}\frac{\sinh a + i \sin b}{a + ib}(\vec{\vartheta} \cdot \sigma_+)$$

$$\begin{aligned} \text{证明: } e^{i\omega \cdot R + \zeta \epsilon \cdot L}, \sqrt{\vec{\vartheta}^2} &:= a + ib \\ &= e^{i\omega \cdot \frac{1}{2}(\sigma_+ + \sigma_-) + \zeta \epsilon \cdot \frac{1}{2}(\sigma_+ - \sigma_-)} = e^{(i\omega + \zeta \epsilon) \cdot \frac{1}{2}\sigma_+ + (i\omega - \zeta \epsilon) \cdot \frac{1}{2}\sigma_-} \\ &= e^{(i\omega + \zeta \epsilon) \cdot \frac{1}{2}\sigma_+} e^{(i\omega - \zeta \epsilon) \cdot \frac{1}{2}\sigma_-} = e^{\vec{\vartheta} \cdot \frac{1}{2}\sigma_+} e^{-\vec{\vartheta}^* \cdot \frac{1}{2}\sigma_-} \\ &= (\cosh \frac{1}{2}\sqrt{\vec{\vartheta}^2} + \frac{\sinh \frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}\vec{\vartheta} \cdot \sigma_+) (\cosh \frac{1}{2}\sqrt{\vec{\vartheta}^{*2}} - \frac{\sinh \frac{1}{2}\sqrt{\vec{\vartheta}^{*2}}}{\sqrt{\vec{\vartheta}^{*2}}}\vec{\vartheta}^* \cdot \sigma_-) \\ &= \cosh \frac{1}{2}\sqrt{\vec{\vartheta}^2} \cosh \frac{1}{2}\sqrt{\vec{\vartheta}^{*2}} - (\frac{\sinh \frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}})(\frac{\sinh \frac{1}{2}\sqrt{\vec{\vartheta}^{*2}}}{\sqrt{\vec{\vartheta}^{*2}}})(\vec{\vartheta} \cdot \sigma_+) (\vec{\vartheta}^* \cdot \sigma_-) \\ &\quad - \cosh \frac{1}{2}\sqrt{\vec{\vartheta}^2} \frac{\sinh \frac{1}{2}\sqrt{\vec{\vartheta}^{*2}}}{\sqrt{\vec{\vartheta}^{*2}}}(\vec{\vartheta}^* \cdot \sigma_-) + \cosh \frac{1}{2}\sqrt{\vec{\vartheta}^{*2}} \frac{\sinh \frac{1}{2}\sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}}(\vec{\vartheta} \cdot \sigma_+) \\ &= \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2}\frac{\cosh a - \cos b}{a^2 + b^2}(\vec{\vartheta} \cdot \sigma_+) (\vec{\vartheta}^* \cdot \sigma_-) - \frac{1}{2}\frac{\sinh a - i \sin b}{a - ib}(\vec{\vartheta}^* \cdot \sigma_-) + \frac{1}{2}\frac{\sinh a + i \sin b}{a + ib}(\vec{\vartheta} \cdot \sigma_+) \\ &= \begin{cases} \frac{1}{2}(\cosh a + 1) - \frac{1}{2}\frac{\cosh a - 1}{a^2}(\epsilon \cdot \sigma_+) (\epsilon \cdot \sigma_-) - \frac{1}{2}\frac{\sinh a}{a}(\epsilon \cdot \sigma_-) + \frac{1}{2}\frac{\sinh a}{a}(\epsilon \cdot \sigma_+); a = \sqrt{\epsilon \cdot \epsilon}, b = 0 \\ \frac{1}{2}(1 + \cos b) - \frac{1}{2}\frac{1 - \cos b}{b^2}(\omega \cdot \sigma_+) (\omega \cdot \sigma_-) + \frac{1}{2}\frac{\sin b}{b}(i\omega \cdot \sigma_-) + \frac{1}{2}\frac{\sin b}{b}(i\omega \cdot \sigma_+); a = 0, b = \sqrt{\omega \cdot \omega} \\ \text{???}; a \neq 0, b \neq 0 \end{cases} \\ &= \begin{cases} 1 + \frac{\sinh a}{a}(\epsilon \cdot L) + \frac{\cosh a - 1}{a^2}(\epsilon \cdot L)^2; a = \sqrt{\epsilon \cdot \epsilon}, b = 0 \\ 1 + i\frac{\sin b}{b}(\omega \cdot R) + \frac{\cos b - 1}{b^2}(\omega \cdot R)^2; a = 0, b = \sqrt{\omega \cdot \omega} \\ \text{???}; a \neq 0, b \neq 0 \end{cases} \quad \square \end{aligned}$$

### 5.14 矢量洛伦兹变换多项式表示的一般展开

$$\begin{aligned} \text{定理5.14.1. } e^{i\omega \cdot R + \zeta \epsilon \cdot L} &= \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2}\frac{\cosh a - \cos b}{a^2 + b^2}(\vec{\vartheta} \cdot \sigma_+) (\vec{\vartheta}^* \cdot \sigma_-) - \frac{1}{2}\frac{\sinh a - i \sin b}{a - ib}(\vec{\vartheta}^* \cdot \sigma_-) + \frac{1}{2}\frac{\sinh a + i \sin b}{a + ib}(\vec{\vartheta} \cdot \sigma_+) \\ &= \frac{1}{2}(\cosh a + \cos b) + \frac{1}{2}\frac{\cosh a - \cos b}{a^2 + b^2}(\omega^2 - \epsilon^2) + \frac{(a \sinh a + b \sin b)}{a^2 + b^2}(i\omega \cdot R + \zeta \epsilon \cdot L) + \frac{(b \sinh a - a \sin b)}{a^2 + b^2}(\omega \cdot L - i\zeta \epsilon \cdot R) \\ &\quad - \frac{\cosh a - \cos b}{a^2 + b^2}[(\omega \cdot R)^2 - (\epsilon \cdot L)^2 - i\zeta(\omega \cdot R)(\epsilon \cdot L) + i\zeta(\omega \cdot L)(\epsilon \cdot R)] \end{aligned}$$

$$\begin{aligned}
\text{证明: } e^{i\omega \cdot R + i\zeta \cdot L} &= \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2} \frac{\cosh a - \cos b}{a^2 + b^2} (\vec{v} \cdot \sigma_+) (\vec{v}^* \cdot \sigma_-) - \frac{1}{2} \frac{\sinh a - i \sin b}{a - ib} (\vec{v}^* \cdot \sigma_-) + \frac{1}{2} \frac{\sinh a + i \sin b}{a + ib} (\vec{v} \cdot \sigma_+) \\
&= \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2} \frac{\cosh a - \cos b}{a^2 + b^2} [(i\omega + \zeta\epsilon) \cdot \sigma_+] [(-i\omega + \zeta\epsilon) \cdot \sigma_-] \\
&\quad - \frac{1}{2} \frac{\sinh a - i \sin b}{a - ib} [(-i\omega + \zeta\epsilon) \cdot \sigma_-] + \frac{1}{2} \frac{\sinh a + i \sin b}{a + ib} [(i\omega + \zeta\epsilon) \cdot \sigma_+] \\
&= \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2} \frac{\cosh a - \cos b}{a^2 + b^2} [(\omega \cdot \sigma_+) (\omega \cdot \sigma_-) + (\epsilon \cdot \sigma_+) (\epsilon \cdot \sigma_-) + i\zeta(\omega \cdot \sigma_+) (\epsilon \cdot \sigma_-) - i\zeta(\epsilon \cdot \sigma_+) (\omega \cdot \sigma_-)] \\
&\quad + \frac{1}{2} \frac{\sinh a - i \sin b}{a - ib} [(i\omega - \zeta\epsilon) \cdot \sigma_-] + \frac{1}{2} \frac{\sinh a + i \sin b}{a + ib} [(i\omega + \zeta\epsilon) \cdot \sigma_+] \\
&= \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2} \frac{\cosh a - \cos b}{a^2 + b^2} [(\omega \cdot R)^2 - (\omega \cdot L)^2 + (\epsilon \cdot R)^2 - (\epsilon \cdot L)^2 + i\zeta(\omega \cdot \sigma_+) (\epsilon \cdot \sigma_-) - i\zeta(\epsilon \cdot \sigma_+) (\omega \cdot \sigma_-)] \\
&\quad + \frac{1}{2} \frac{(a+ib)(\sinh a - i \sin b)}{a^2 + b^2} [(i\omega - \zeta\epsilon) \cdot \sigma_-] + \frac{1}{2} \frac{(a-ib)(\sinh a + i \sin b)}{a^2 + b^2} [(i\omega + \zeta\epsilon) \cdot \sigma_+] \\
&= \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2} \frac{\cosh a - \cos b}{a^2 + b^2} [(\omega \cdot R)^2 - (\omega \cdot L)^2 + (\epsilon \cdot R)^2 - (\epsilon \cdot L)^2 + i\zeta(\omega \cdot \sigma_+) (\epsilon \cdot \sigma_-) - i\zeta(\epsilon \cdot \sigma_+) (\omega \cdot \sigma_-)] \\
&\quad + \frac{1}{2} \frac{(a \sinh a + b \sin b) + i(b \sinh a - a \sin b)}{a^2 + b^2} [(i\omega - \zeta\epsilon) \cdot \sigma_-] + \frac{1}{2} \frac{(a \sinh a + b \sin b) - i(b \sinh a - a \sin b)}{a^2 + b^2} [(i\omega + \zeta\epsilon) \cdot \sigma_+] \\
&= \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2} \frac{\cosh a - \cos b}{a^2 + b^2} [(\omega \cdot R)^2 - (\omega \cdot L)^2 + (\epsilon \cdot R)^2 - (\epsilon \cdot L)^2 + i\zeta(\omega \cdot \sigma_+) (\epsilon \cdot \sigma_-) - i\zeta(\epsilon \cdot \sigma_+) (\omega \cdot \sigma_-)] \\
&\quad + \frac{1}{2} \frac{(a \sinh a + b \sin b) + i(b \sinh a - a \sin b)}{a^2 + b^2} (i\omega \cdot \sigma_-) + \frac{1}{2} \frac{(a \sinh a + b \sin b) - i(b \sinh a - a \sin b)}{a^2 + b^2} (i\omega \cdot \sigma_+) \\
&\quad - \frac{1}{2} \frac{(a \sinh a + b \sin b) + i(b \sinh a - a \sin b)}{a^2 + b^2} (\zeta\epsilon \cdot \sigma_-) + \frac{1}{2} \frac{(a \sinh a + b \sin b) - i(b \sinh a - a \sin b)}{a^2 + b^2} (\zeta\epsilon \cdot \sigma_+) \\
&= \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2} \frac{\cosh a - \cos b}{a^2 + b^2} [(\omega \cdot R)^2 - (\omega \cdot L)^2 + (\epsilon \cdot R)^2 - (\epsilon \cdot L)^2 + i\zeta(\omega \cdot \sigma_+) (\epsilon \cdot \sigma_-) - i\zeta(\epsilon \cdot \sigma_+) (\omega \cdot \sigma_-)] \\
&\quad + \frac{(a \sinh a + b \sin b)}{a^2 + b^2} (i\omega \cdot R + \zeta\epsilon \cdot L) + \frac{(b \sinh a - a \sin b)}{a^2 + b^2} (\omega \cdot L - i\zeta\epsilon \cdot R) \\
&= \frac{1}{2}(\cosh a + \cos b) - \frac{1}{2} \frac{\cosh a - \cos b}{a^2 + b^2} [2(\omega \cdot R)^2 - \omega^2 + \epsilon^2 - 2(\epsilon \cdot L)^2 - 2i\zeta(\omega \cdot R) (\epsilon \cdot L) + 2i\zeta(\omega \cdot L) (\epsilon \cdot R)] \\
&\quad + \frac{(a \sinh a + b \sin b)}{a^2 + b^2} (i\omega \cdot R + \zeta\epsilon \cdot L) + \frac{(b \sinh a - a \sin b)}{a^2 + b^2} (\omega \cdot L - i\zeta\epsilon \cdot R) \\
&= \frac{1}{2}(\cosh a + \cos b) + \frac{1}{2} \frac{\cosh a - \cos b}{a^2 + b^2} (\omega^2 - \epsilon^2) - \frac{\cosh a - \cos b}{a^2 + b^2} [(\omega \cdot R)^2 - (\epsilon \cdot L)^2 - i\zeta(\omega \cdot R) (\epsilon \cdot L) + i\zeta(\omega \cdot L) (\epsilon \cdot R)] \\
&\quad + \frac{(a \sinh a + b \sin b)}{a^2 + b^2} (i\omega \cdot R + \zeta\epsilon \cdot L) + \frac{(b \sinh a - a \sin b)}{a^2 + b^2} (\omega \cdot L - i\zeta\epsilon \cdot R) \quad \square
\end{aligned}$$

## 5.15 任意运动电荷的电磁场 [23]

### 5.15.1 空间坐标和推迟位

$$\text{推论5.15.1. } \vec{r}' = \vec{r} + \gamma_v \vec{v} r + (\gamma_v - 1)(\vec{v} \cdot \vec{r}) \vec{v} / v^2 \Rightarrow r' = \gamma_v (r + \vec{v} \cdot \vec{r})$$

$$\text{推论5.15.2. } \vec{r}' = \vec{r} + \gamma_v \vec{v} r + (\gamma_v - 1)(\vec{v} \cdot \vec{r}) \vec{v} / v^2 \Rightarrow \hat{r}' = [\hat{r} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \hat{r}) \vec{v} / v^2] / [\gamma_v (1 + \vec{v} \cdot \hat{r})]$$

$$\text{推论5.15.3. } \vec{r}' = \vec{r} + \gamma_v \vec{v} r + (\gamma_v - 1)(\vec{v} \cdot \vec{r}) \vec{v} / v^2 \Leftrightarrow \begin{cases} \vec{r}' = \vec{r} + \gamma_v \vec{v} r + (\gamma_v - 1)(\vec{v} \cdot \vec{r}) \vec{v} / v^2 \\ r' = \gamma_v (r + \vec{v} \cdot \vec{r}), r'^2 - r'^2 = \vec{r}^2 - r^2 = 0 \end{cases}$$

$$\text{推论5.15.4. } \begin{cases} \vec{r} = \vec{r}_0 + \gamma_v \vec{v} r_0 + (\gamma_v - 1)(\vec{v} \cdot \vec{r}_0) \vec{v} / v^2 \\ r = \gamma_v (r_0 + \vec{v} \cdot \vec{r}_0) \end{cases} \quad \begin{cases} \vec{A} = \vec{A}_0 + \gamma_v \vec{v} \phi_0 + (\gamma_v - 1)(\vec{v} \cdot \vec{A}_0) \vec{v} / v^2 = \frac{e\gamma_v \vec{v}}{4\pi\epsilon_0 r_0} = \frac{e\vec{v}}{4\pi\epsilon_0 (r - \vec{v} \cdot \vec{r})} \\ \phi = \gamma_v (\phi_0 + \vec{v} \cdot \vec{A}_0) = \frac{e\gamma_v}{4\pi\epsilon_0 r_0} = \frac{e}{4\pi\epsilon_0 (r - \vec{v} \cdot \vec{r})} \end{cases}$$

### 5.15.2 偏导数分析

$$\text{推论5.15.5. } t = t' + R(t'), R(t') = \sqrt{[x'(t') - x]^2 + [y'(t') - y]^2 + [z'(t') - z]^2}$$

推论5.15.6.  $\partial_x$  含义:  $t, y, z$  固定不变,  $x$  变化, 且  $x$  与  $t'$  满足关系式  $t = t' + R(t')$

$$\begin{aligned}
\Rightarrow \begin{cases} \partial_x = \frac{\partial t'}{\partial x} \partial_{t'} \\ 0 = \frac{\partial t'}{\partial x} + \frac{\partial R(t')}{\partial x} \frac{\partial t'}{\partial x} \end{cases} &\Rightarrow \begin{cases} \frac{\partial R(t')}{\partial t'} = -\frac{\vec{v}(t') \cdot \vec{R}(t') - \frac{\partial x}{\partial t'} R_x(t')}{R(t')} = -1 \\ \frac{\partial t'}{\partial x} = -\frac{\hat{R}_x(t')}{1 - \vec{v}(t') \cdot \hat{R}(t')}, \partial_x = -\frac{\hat{R}_x(t')}{1 - \vec{v}(t') \cdot \hat{R}(t')} \partial_{t'} \Big|_x \end{cases} \\
\Rightarrow \begin{cases} \vec{v}(t') \cdot \partial_{t'} \vec{R}(t') = -\frac{1 - \vec{v}(t') \cdot \hat{R}(t')}{\hat{R}_x(t')} v_x - v^2(t') \\ \vec{v}(t') \cdot \partial_x \vec{R}(t') = v_x + \frac{v^2(t') \hat{R}_x(t')}{1 - \vec{v}(t') \cdot \hat{R}(t')} \end{cases}
\end{aligned}$$

推论5.15.7.  $\partial_y$  含义:  $x, t, z$  固定不变,  $y$  变化, 且  $y$  与  $t'$  满足关系式  $t = t' + R(t')$

$$\begin{aligned}
\Rightarrow \begin{cases} \partial_y = \frac{\partial t'}{\partial y} \partial_{t'} \\ 0 = \frac{\partial t'}{\partial y} + \frac{\partial R(t')}{\partial y} \frac{\partial t'}{\partial y} \end{cases} &\Rightarrow \begin{cases} \frac{\partial R(t')}{\partial t'} = -\frac{\vec{v}(t') \cdot \vec{R}(t') - \frac{\partial y}{\partial t'} R_y(t')}{R(t')} = -1 \\ \frac{\partial t'}{\partial y} = -\frac{\hat{R}_y(t')}{1 - \vec{v}(t') \cdot \hat{R}(t')}, \partial_y = -\frac{\hat{R}_y(t')}{1 - \vec{v}(t') \cdot \hat{R}(t')} \partial_{t'} \Big|_y \end{cases} \\
\Rightarrow \begin{cases} \vec{v}(t') \cdot \partial_{t'} \vec{R}(t') = -\frac{1 - \vec{v}(t') \cdot \hat{R}(t')}{\hat{R}_y(t')} v_y - v^2(t') \\ \vec{v}(t') \cdot \partial_y \vec{R}(t') = v_y + \frac{v^2(t') \hat{R}_y(t')}{1 - \vec{v}(t') \cdot \hat{R}(t')} \end{cases}
\end{aligned}$$

推论5.15.8.  $\partial_z$ 含义:  $x, y, t$ 固定不变,  $z$ 变化, 且 $z$ 与 $t'$ 满足关系式 $t = t' + R(t')$

$$\Rightarrow \begin{cases} \partial_z = \frac{\partial t'}{\partial z} \partial_{t'} \\ 0 = \frac{\partial t'}{\partial z} + \frac{\partial R(t')}{\partial t'} \frac{\partial t'}{\partial z} \end{cases} \Rightarrow \begin{cases} \frac{\partial R(t')}{\partial t'} = -\frac{\vec{v}(t') \cdot \vec{R}(t') - \frac{\partial z}{\partial t'} R_z(t')}{R(t')} = -1 \\ \frac{\partial t'}{\partial z} = -\frac{\hat{R}_z(t')}{1 - \vec{v}(t') \cdot \vec{R}(t')}, \partial_z = -\frac{\hat{R}_z(t')}{1 - \vec{v}(t') \cdot \vec{R}(t')} \partial_{t'} \Big|_z \end{cases}$$

$$\Rightarrow \begin{cases} \vec{v}(t') \cdot \partial_{t'} \vec{R}(t') = -\frac{1 - \vec{v}(t') \cdot \vec{R}(t')}{\hat{R}_z(t')} v_z - v^2(t') \\ \vec{v}(t') \cdot \partial_z \vec{R}(t') = v_z + \frac{v^2(t') \hat{R}_z(t')}{1 - \vec{v}(t') \cdot \vec{R}(t')} \end{cases}$$

推论5.15.9.  $\partial_t$ 含义:  $x, y, z$ 固定不变,  $t$ 变化, 且 $t$ 与 $t'$ 满足关系式 $t = t' + R(t')$

$$\Rightarrow \begin{cases} \partial_t = \frac{\partial t'}{\partial t} \partial_{t'} \\ 1 = \frac{\partial t'}{\partial t} + \frac{\partial R(t')}{\partial t'} \frac{\partial t'}{\partial t} \end{cases} \Rightarrow \begin{cases} \frac{\partial t'}{\partial t} = \frac{1}{1 - \vec{v}(t') \cdot \vec{R}(t')}, \partial_t = \frac{1}{1 - \vec{v}(t') \cdot \vec{R}(t')} \partial_{t'} \Big|_t \\ \frac{\partial R(t')}{\partial t'} = -\frac{\vec{v}(t') \cdot \vec{R}(t')}{R(t')} \end{cases}$$

$$\Rightarrow \begin{cases} \vec{v}(t') \cdot \partial_{t'} \vec{R}(t') = -v^2(t') \\ \vec{v}(t') \cdot \partial_t \vec{R}(t') = -\frac{v^2(t')}{1 - \vec{v}(t') \cdot \vec{R}(t')} \end{cases}$$

推论5.15.10. 
$$\begin{cases} \vec{A}(t, \vec{r}) = \frac{e\vec{v}(t')}{4\pi\epsilon_0[R(t') - \vec{v}(t') \cdot \vec{R}(t')]} \\ \phi(t, \vec{r}) = \frac{e}{4\pi\epsilon_0[R(t') - \vec{v}(t') \cdot \vec{R}(t')]} \end{cases}$$

推论5.15.11. 
$$\begin{cases} \vec{E}(t, \vec{r}) = -\nabla\phi(t, \vec{r}) - \partial_t \vec{A}(t, \vec{r}) = \frac{e[\vec{R}(t') - R(t')\vec{v}(t')]}{4\pi\epsilon_0\gamma_v^2[R(t') - \vec{v}(t') \cdot \vec{R}(t')]^3} + \frac{e\vec{R}(t') \times [\vec{R}(t') - R(t')\vec{v}(t')] \times \dot{\vec{v}}(t')}{4\pi\epsilon_0[R(t') - \vec{v}(t') \cdot \vec{R}(t')]^3} \\ \vec{B}(t, \vec{r}) = \nabla \times \vec{A}(t, \vec{r}) = \frac{\vec{R}(t') \times \vec{E}(t, \vec{r})}{R(t')} \end{cases}$$

推论5.15.12. 
$$\begin{cases} \vec{E}(t, \vec{r}) = \frac{e}{4\pi\epsilon_0\gamma_v^2[R(t') - \vec{v}(t') \cdot \vec{R}(t')]^3} \left\{ \frac{1}{\gamma_v^2} [\vec{R}(t') - R(t')\vec{v}(t')] + \vec{R}(t') \times [\vec{R}(t') - R(t')\vec{v}(t')] \times \dot{\vec{v}}(t') \right\} \\ \vec{B}(t, \vec{r}) = \frac{\vec{R}(t') \times \vec{E}(t, \vec{r})}{R(t')}, t = t' + R(t'), R(t') = \sqrt{[x'(t') - x]^2 + [y'(t') - y]^2 + [z'(t') - z]^2} \end{cases}$$

### 5.15.3 光子的能量动量与推迟矢量比较

推论5.15.13. 
$$\begin{cases} \vec{p}' = \vec{p} + \gamma_v \vec{v} p + (\gamma_v - 1)(\vec{v} \cdot \vec{p})\vec{v}/v^2 \\ p' = \gamma_v(p + \vec{v} \cdot \vec{p}), p'^2 - p^2 = p_0'^2 - p_0^2 = 0 \end{cases} \quad \begin{cases} \vec{r}' = \vec{r} + \gamma_v \vec{v} r + (\gamma_v - 1)(\vec{v} \cdot \vec{r})\vec{v}/v^2 \\ r' = \gamma_v(r + \vec{v} \cdot \vec{r}), r'^2 - r^2 = r^2 - r^2 = 0 \end{cases}$$

### 5.16 自旋矢量的变换规律 [23]

推论5.16.1. 
$$\begin{cases} \vec{S}(\vec{v}) = \vec{s} + (\gamma_v - 1)(\vec{v} \cdot \vec{s})\vec{v}/v^2 \\ S_0(\vec{v}) = \gamma_v(\vec{v} \cdot \vec{s}) \end{cases} \Rightarrow \vec{v} \cdot \vec{S}(\vec{v}) = \gamma_v \vec{v} \cdot \vec{s} = S_0(\vec{v})$$

推论5.16.2. 
$$\begin{cases} \vec{S}(\vec{v}) = \vec{s} + (\gamma_v - 1)(\vec{v} \cdot \vec{s})\vec{v}/v^2 \\ S_0(\vec{v}) = \gamma_v(\vec{v} \cdot \vec{s}) \end{cases} \Leftrightarrow \begin{cases} \vec{S}(\vec{v}) = \vec{s} + (\gamma_v - 1)(\vec{v} \cdot \vec{s})\vec{v}/v^2 \\ \vec{v} \cdot \vec{S}(\vec{v}) = \gamma_v(\vec{v} \cdot \vec{s}) \end{cases}$$

推论5.16.3. 
$$\begin{cases} \vec{u}' = [\vec{u} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{u})\vec{v}/v^2]/[\gamma_v(1 + \vec{v} \cdot \vec{u})] \\ \vec{S}(\vec{u}') = \vec{S}(\vec{u}) + \gamma_v \vec{v} [\vec{u} \cdot \vec{S}(\vec{u})] + (\gamma_v - 1)[\vec{v} \cdot \vec{S}(\vec{u})]\vec{v}/v^2 \\ \vec{u}' \cdot \vec{S}(\vec{u}') = \gamma_v [\vec{u} \cdot \vec{S}(\vec{u}) + \vec{v} \cdot \vec{S}(\vec{u})] \end{cases} \Leftrightarrow \begin{bmatrix} \vec{S}(\vec{u}') \\ i\vec{u}' \cdot \vec{S}(\vec{u}') \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \vec{S}(\vec{u}) \\ i\vec{u} \cdot \vec{S}(\vec{u}) \end{bmatrix}$$

自旋的变换规律与力的变换规律类似。

### 5.17 无质量粒子的角动量变换规律

推论5.17.1.  $\vec{r}' = \vec{r} + \gamma_v \vec{v} r + (\gamma_v - 1)(\vec{v} \cdot \vec{r})\vec{v}/v^2 \Rightarrow r' = \gamma_v(r + \vec{v} \cdot \vec{r})$

推论5.17.2.  $\vec{p}' = \vec{p} + \gamma_v \vec{v} p + (\gamma_v - 1)(\vec{v} \cdot \vec{p})\vec{v}/v^2 \Rightarrow p' = \gamma_v(p + \vec{v} \cdot \vec{p})$

推论5.17.3. 
$$M_{ab} = r_a p_b - r_b p_a = \begin{bmatrix} 0 & (\vec{r} \times \vec{p})_z & -(\vec{r} \times \vec{p})_y & -i(rp_x - xp) \\ -(\vec{r} \times \vec{p})_z & 0 & (\vec{r} \times \vec{p})_x & -i(rp_y - yp) \\ (\vec{r} \times \vec{p})_y & -(\vec{r} \times \vec{p})_x & 0 & -i(rp_z - zp) \\ i(xp - rp_x) & i(yp - rp_y) & i(zp - rp_z) & 0 \end{bmatrix}$$

推论5.17.4.  $\vec{J} = \vec{r} \times \vec{p}, \vec{W} = r\vec{p} - p\vec{r}$

## 5.18 质点系的角动量变换规律

$$\text{推论5.18.1. } M_{ab} = r_a p_b - r_b p_a = \begin{bmatrix} 0 & (\vec{r} \times \vec{p})_z & -(\vec{r} \times \vec{p})_y & -i(rp_x - xp) \\ -(\vec{r} \times \vec{p})_z & 0 & (\vec{r} \times \vec{p})_x & -i(rp_y - yp) \\ (\vec{r} \times \vec{p})_y & -(\vec{r} \times \vec{p})_x & 0 & -i(rp_z - zp) \\ i(xp - rp_x) & i(yp - rp_y) & i(zp - rp_z) & 0 \end{bmatrix}$$

$$\text{推论5.18.2. } \vec{J} = \sum_i (\vec{r}_i \times \vec{p}_i), \vec{W} = \sum_i (r_i \vec{p}_i - p_i \vec{r}_i)$$

$$\text{推论5.18.3. } \vec{u}' = [\vec{u} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \vec{u})\vec{v}/v^2]/[\gamma_v(1 + \vec{v} \cdot \vec{u})]$$

中微子的自旋

$$\text{推论5.18.4. } s(\nu) = \int \nu^+(\vec{0}) \sigma_y \sigma \nu(\vec{0}) dx^4$$

光子的自旋

$$\text{推论5.18.5. } s(\gamma) = \Psi(\vec{0})^T \gamma \Psi(\vec{0})$$

电子的自旋

$$\text{推论5.18.6. } s(e) = \bar{\psi}(\vec{0}) \gamma_e \psi(\vec{0})$$

## 5.19 魏格纳小群<sup>[37]</sup>

### 5.19.1 有质量粒子的小群

$$\text{推论5.19.1. } L_{\vec{v}} \Lambda[SO(3)] \begin{bmatrix} \vec{0} \\ im \end{bmatrix} = \begin{bmatrix} \gamma m \vec{v} \\ i\gamma m \end{bmatrix} = \begin{bmatrix} \vec{p} \\ iE \end{bmatrix}$$

$$\text{推论5.19.2. } L_p \equiv L_{\vec{v}} \Lambda[SO(3)]$$

$$\text{推论5.19.3. } L_p p_0 = p, L_{\Lambda p} p_0 = \Lambda p = \Lambda L_p p_0$$

$$\text{推论5.19.4. } p_0 = L_{\Lambda p}^{-1} \Lambda L_p p_0$$

$$\text{推论5.19.5. } W(\Lambda, p) \equiv L_{\Lambda p}^{-1} \Lambda L_p = \Lambda_2^{-1}[SO(3)] L_{\Lambda v}^{-1} \Lambda L_{\vec{v}} \Lambda_1[SO(3)] = \Lambda_3[SO(3)]$$

$$\text{推论5.19.6. } U(\Lambda, p) \equiv U_{\Lambda p}^{-1} U_{\Lambda} U_p = U^{-1}(\Lambda_2[SO(3)]) U_{\Lambda v}^{-1} U_{\Lambda} U_{\vec{v}} U(\Lambda_1[SO(3)])$$

### 5.19.2 无质量粒子的小群

$$\text{推论5.19.7. } L_p \Lambda[E(2)] \begin{bmatrix} 0 \\ p_0 \\ ip_0 \end{bmatrix} = p := \begin{bmatrix} \vec{p} \\ ip \end{bmatrix}$$

$$\text{推论5.19.8. } \tilde{L}_p \equiv L_p \Lambda[E(2)]$$

$$\text{推论5.19.9. } \tilde{L}_p \begin{bmatrix} 0 \\ p_0 \\ ip_0 \end{bmatrix} = \begin{bmatrix} \vec{p} \\ ip \end{bmatrix}, L_{\Lambda p} \begin{bmatrix} 0 \\ p_0 \\ ip_0 \end{bmatrix} = \Lambda \begin{bmatrix} \vec{p} \\ ip \end{bmatrix} = \Lambda \tilde{L}_p \begin{bmatrix} 0 \\ p_0 \\ ip_0 \end{bmatrix}$$

$$\text{推论5.19.10. } \begin{bmatrix} 0 \\ p_0 \\ ip_0 \end{bmatrix} = \tilde{L}_{\Lambda p}^{-1} \Lambda \tilde{L}_p \begin{bmatrix} 0 \\ p_0 \\ ip_0 \end{bmatrix}$$

$$\text{推论5.19.11. } W(\Lambda, p) \equiv \tilde{L}_{\Lambda p}^{-1} \Lambda \tilde{L}_p = \Lambda_2^{-1}[E(2)] L_{\Lambda p}^{-1} \Lambda L_p \Lambda_1[E(2)] = \Lambda_3[E(2)]$$

$$\text{推论5.19.12. } U(\Lambda, p) \equiv \tilde{U}_{\Lambda p}^{-1} U_{\Lambda} \tilde{U}_p = U^{-1}(\Lambda_2[SO(3)]) U_{\Lambda p}^{-1} U_{\Lambda} U_p U(\Lambda_1[SO(3)])$$

# 第十五章 螺旋度的数学分析

自我评述：为了进一步研究各种自旋粒子的物理，在本章中我发展了关于螺旋度的数学分析，为研究各种自旋粒子提供了一个有力的数学工具。

## 1 单位矢量的空间旋转变换

### 1.1 自旋-1旋量的空间旋转变换

$$\text{性质1.1.1. } e^{\vec{\nu} \cdot \bar{\Omega}(1)} = 1 + \frac{\sinh \sqrt{\vec{\nu}^2}}{\sqrt{\vec{\nu}^2}} [\vec{\nu} \cdot \bar{\Omega}(1)] + \frac{\cosh \sqrt{\vec{\nu}^2} - 1}{\vec{\nu}^2} [\vec{\nu} \cdot \bar{\Omega}(1)]^2$$

$$\Rightarrow e^{i\vec{\omega} \cdot \gamma} = 1 + i\hat{\omega} \cdot \gamma \sin \omega + (i\hat{\omega} \cdot \gamma)^2 (1 - \cos \omega), \omega := |\vec{\omega}|$$

$$\text{推论1.1.1. } e^{i\vec{\omega} \cdot \gamma} = 1 + \sin \omega \begin{bmatrix} 0 & \hat{\omega}_z & -\hat{\omega}_y \\ -\hat{\omega}_z & 0 & \hat{\omega}_x \\ \hat{\omega}_y & -\hat{\omega}_x & 0 \end{bmatrix} + (1 - \cos \omega) \begin{bmatrix} \hat{\omega}_x^2 - 1 & \hat{\omega}_x \hat{\omega}_y & \hat{\omega}_x \hat{\omega}_z \\ \hat{\omega}_y \hat{\omega}_x & \hat{\omega}_y^2 - 1 & \hat{\omega}_y \hat{\omega}_z \\ \hat{\omega}_z \hat{\omega}_x & \hat{\omega}_z \hat{\omega}_y & \hat{\omega}_z^2 - 1 \end{bmatrix}$$

$$\text{推论1.1.2. } e^{i\vec{\omega} \cdot \sigma(1)} = 1 + i\hat{\omega} \cdot \sigma(1) \sin \omega + [i\hat{\omega} \cdot \sigma(1)]^2 (1 - \cos \omega)$$

### 1.2 单位矢量 $\hat{p}$ 的空间旋转变换

$$\text{定义1.2.1. } \hat{\omega}_+ := \frac{1}{\sqrt{2}}(\hat{\omega}_x + i\hat{\omega}_y), \hat{\omega}_- := \frac{1}{\sqrt{2}}(\hat{\omega}_x - i\hat{\omega}_y), \hat{p}_+ := \frac{1}{\sqrt{2}}(\hat{p}_x + i\hat{p}_y), \hat{p}_- := \frac{1}{\sqrt{2}}(\hat{p}_x - i\hat{p}_y)$$

$$\text{定理1.2.1. } \hat{p} = \exp\left\{i \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\right\} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 0 \leq \arccos \hat{p}_z \leq \pi$$

$$\text{证明: } \hat{p} = e^{i\vec{\omega} \cdot \gamma} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \sin \omega \begin{bmatrix} -\hat{\omega}_y \\ \hat{\omega}_x \\ 0 \end{bmatrix} + (1 - \cos \omega) \begin{bmatrix} \hat{\omega}_x \hat{\omega}_z \\ \hat{\omega}_y \hat{\omega}_z \\ \hat{\omega}_z^2 - 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \hat{p}_x = -\hat{\omega}_y \sin \omega + \hat{\omega}_x \hat{\omega}_z (1 - \cos \omega) \\ \hat{p}_y = \hat{\omega}_x \sin \omega + \hat{\omega}_y \hat{\omega}_z (1 - \cos \omega) \\ \hat{p}_z = 1 + (\hat{\omega}_z^2 - 1)(1 - \cos \omega) \end{cases} \stackrel{\hat{\omega}_z=0}{\Leftrightarrow} \begin{cases} \hat{p}_x = -\hat{\omega}_y \sin \omega \\ \hat{p}_y = \hat{\omega}_x \sin \omega \\ \hat{p}_z = \cos \omega \end{cases} \Leftrightarrow \begin{cases} \hat{\omega}_x = \frac{\hat{p}_y}{\sqrt{1-\hat{p}_z^2}} \\ \hat{\omega}_y = \frac{-\hat{p}_x}{\sqrt{1-\hat{p}_z^2}} \\ \hat{\omega}_z = 0, 0 \leq \omega = \arccos \hat{p}_z \leq \pi \end{cases}$$

$$\Rightarrow \hat{p} = \exp\left\{i \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\right\} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 0 \leq \arccos \hat{p}_z \leq \pi \quad \square$$

$$\text{性质1.2.1. } \begin{cases} \hat{\omega} \cdot \gamma \stackrel{\hat{\omega}_z=0}{=} \gamma_x \hat{\omega}_x + \gamma_y \hat{\omega}_y = \frac{\gamma_x \hat{p}_y - \gamma_y \hat{p}_x}{\sqrt{1-\hat{p}_z^2}} = \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \\ \vec{\omega} \cdot \gamma \stackrel{\hat{\omega}_z=0}{=} \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z, 0 \leq \arccos \hat{p}_z \leq \pi \end{cases}$$

$$\text{性质1.2.2. } e^{i\vec{\omega} \cdot \gamma} \stackrel{\hat{\omega}_z=0}{=} \exp\left\{i \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\right\} = 1 + i(\gamma \times \hat{p})_z - (\gamma \times \hat{p})_z^2 / (1 + \hat{p}_z), 0 \leq \arccos \hat{p}_z \leq \pi$$

### 1.3 Wigner SO(2)小群

$$\text{性质1.3.1. } \hat{p} = e^{i\omega_z \gamma_z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

### 1.4 单位矢量 $\hat{p}$ 洛伦兹推动变换<sup>[25]</sup>与空间旋转变换的等价性

$$\text{推论1.4.1. } \hat{p}' = [\hat{p} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \hat{p})\vec{v}/v^2] / [\gamma_v(1 + \vec{v} \cdot \hat{p})]$$

$$\text{推论1.4.2. } \hat{p} = \left[ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})\vec{v}/v^2 \right] / [\gamma_v(1 + \vec{v} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})] = [1 + i(\gamma \times \hat{p})_z - (\gamma \times \hat{p})_z^2 / (1 + \hat{p}_z)] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{推论1.4.3. } e^{-s \ln[\gamma_v(1+v)] \hat{v} \cdot \sigma(s)}, \exp\left\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\right\}$$

## 2 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的分析

### 2.1 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的具体求法 [38]

定义2.1.1.  $\sigma(\frac{1}{2}) \cdot \hat{p}\lambda(\hat{p}, h) = h\lambda(\hat{p}, h), h = -\frac{1}{2}, \frac{1}{2}$

推论2.1.1.  $e^{\vec{\omega} \cdot \frac{\sigma}{2}} = \cosh \frac{1}{2} \sqrt{\vec{\omega}^2} + \frac{\sinh \frac{1}{2} \sqrt{\vec{\omega}^2}}{\sqrt{\vec{\omega}^2}} \vec{\omega} \cdot \sigma \Rightarrow e^{i\vec{\omega} \cdot \frac{\sigma}{2}} = \cos \frac{1}{2} \omega + i\hat{\omega} \cdot \sigma \sin \frac{1}{2} \omega = \frac{(1+\hat{p}_z)+i(\sigma \times \hat{p})_z}{\sqrt{2(1+\hat{p}_z)}}$

推论2.1.2.  $i\hat{\omega} \cdot \sigma = i\left\{ \begin{bmatrix} 0 & \hat{\omega}_x \\ \hat{\omega}_x & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i\hat{\omega}_y \\ i\hat{\omega}_y & 0 \end{bmatrix} + \begin{bmatrix} \hat{\omega}_z & 0 \\ 0 & -\hat{\omega}_z \end{bmatrix} \right\} = i \begin{bmatrix} \hat{\omega}_z & \sqrt{2}\hat{\omega}_- \\ \sqrt{2}\hat{\omega}_+ & -\hat{\omega}_z \end{bmatrix} \stackrel{\hat{\omega}_z=0}{=} i\sqrt{2} \begin{bmatrix} 0 & \hat{\omega}_- \\ \hat{\omega}_+ & 0 \end{bmatrix}$

推论2.1.3.  $e^{i\vec{\omega} \cdot \frac{\sigma}{2}} \stackrel{\hat{\omega}_z=0}{=} \cos \frac{1}{2} \omega + i\hat{\omega} \cdot \sigma \sin \frac{1}{2} \omega = \begin{bmatrix} \cos \frac{1}{2} \omega & i\sqrt{2}\hat{\omega}_- \sin \frac{1}{2} \omega \\ i\sqrt{2}\hat{\omega}_+ \sin \frac{1}{2} \omega & \cos \frac{1}{2} \omega \end{bmatrix}$

推论2.1.4.  $\begin{cases} \lambda(\hat{p}, \frac{1}{2}) = e^{i\vec{\omega} \cdot \sigma(\frac{1}{2})} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \frac{1}{2} \omega \\ i\sqrt{2}\hat{\omega}_+ \sin \frac{1}{2} \omega \end{bmatrix} = \frac{1}{\sqrt{1+\hat{p}_z}} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_+ \end{bmatrix}, \lambda(-\hat{p}, \frac{1}{2}) = -\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}} \lambda(\hat{p}, -\frac{1}{2}) \\ \lambda(\hat{p}, -\frac{1}{2}) = e^{i\vec{\omega} \cdot \sigma(\frac{1}{2})} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} i\sqrt{2}\hat{\omega}_- \sin \frac{1}{2} \omega \\ \cos \frac{1}{2} \omega \end{bmatrix} = \frac{1}{\sqrt{1+\hat{p}_z}} \begin{bmatrix} -\hat{p}_- \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z) \end{bmatrix}, \lambda(-\hat{p}, -\frac{1}{2}) = \frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}} \lambda(\hat{p}, \frac{1}{2}) \\ \lambda(\hat{p}, \frac{1}{2}) = i\sigma_y \lambda^*(\hat{p}, -\frac{1}{2}), \lambda(\hat{p}, -\frac{1}{2}) = -i\sigma_y \lambda^*(\hat{p}, \frac{1}{2}) \end{cases}$

推论2.1.5.  $\begin{cases} \lambda(\hat{p}, \frac{1}{2}) \lambda^+(\hat{p}, \frac{1}{2}) = \frac{1}{2}(\sigma \cdot \hat{p} + I) = \frac{1}{2}(\sigma, -i)^a \hat{p}_a, \hat{p}_a := (\hat{p}, i) \\ \lambda(\hat{p}, -\frac{1}{2}) \lambda^+(\hat{p}, -\frac{1}{2}) = -\frac{1}{2}(\sigma \cdot \hat{p} - I) = -\frac{1}{2}(\sigma, i)^a \hat{p}_a \\ \lambda(\hat{p}, \frac{1}{2}) \lambda^+(\hat{p}, -\frac{1}{2}) = \frac{1}{2}(\sigma \cdot \hat{p} + I) i\sigma_y = \frac{1}{2}(\sigma, i)^a \hat{p}_a i\sigma_y \\ \lambda(\hat{p}, -\frac{1}{2}) \lambda^+(\hat{p}, \frac{1}{2}) = -\frac{1}{2}(\sigma \cdot \hat{p} + I) i\sigma_y = -\frac{1}{2}(\sigma, i)^a \hat{p}_a i\sigma_y \end{cases}$

### 2.2 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的正交性与完备性

推论2.2.1.  $\lambda^+(\hat{p}, h)\lambda(\hat{p}, h') = \delta_{hh'}, \sum_{h=\frac{1}{2}}^{-\frac{1}{2}} \lambda(\hat{p}, h)\lambda^+(\hat{p}, h) = 1, \sum_{h=\frac{1}{2}}^{-\frac{1}{2}} h\lambda(\hat{p}, h)\lambda^+(\hat{p}, h) = \sigma(\frac{1}{2}) \cdot \hat{p}$

### 2.3 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的升降算符

定理2.3.1.

$$\begin{cases} e^{i\vec{\omega} \cdot \frac{\sigma}{2}} \sigma_x e^{-i\vec{\omega} \cdot \frac{\sigma}{2}} = \sigma_x - \hat{p}_x \frac{(\sigma \cdot \hat{p} + \sigma_z)}{(1+\hat{p}_z)} \\ e^{i\vec{\omega} \cdot \frac{\sigma}{2}} \sigma_y e^{-i\vec{\omega} \cdot \frac{\sigma}{2}} = \sigma_y - \hat{p}_y \frac{(\sigma \cdot \hat{p} + \sigma_z)}{(1+\hat{p}_z)} \\ e^{i\vec{\omega} \cdot \frac{\sigma}{2}} \sigma_z e^{-i\vec{\omega} \cdot \frac{\sigma}{2}} = \sigma \cdot \hat{p} \end{cases}$$

证明:  $e^{i\vec{\omega} \cdot \frac{\sigma}{2}} \sigma_x e^{-i\vec{\omega} \cdot \frac{\sigma}{2}} = (e^{-i\vec{\omega} \cdot \gamma})_x^k \sigma_k$   
 $= \frac{(1+\hat{p}_z)+i(\sigma_x \hat{p}_y - \sigma_y \hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}} \sigma_x \frac{(1+\hat{p}_z)-i(\sigma_x \hat{p}_y - \sigma_y \hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}$   
 $= \frac{(1+\hat{p}_z)+i(\sigma_x \hat{p}_y - \sigma_y \hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}} \frac{(1+\hat{p}_z)-i(\sigma_x \hat{p}_y + \sigma_y \hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}} \sigma_x$   
 $= \frac{(1+\hat{p}_z)^2 - 2i(1+\hat{p}_z)\sigma_y \hat{p}_x + (\sigma_x \hat{p}_y - \sigma_y \hat{p}_x)(\sigma_x \hat{p}_y + \sigma_y \hat{p}_x)}{2(1+\hat{p}_z)} \sigma_x$   
 $= \frac{(1+\hat{p}_z)^2 - 2i(1+\hat{p}_z)\sigma_y \hat{p}_x + \hat{p}_y^2 - \hat{p}_x^2 + 2i\sigma_z \hat{p}_x \hat{p}_y}{2(1+\hat{p}_z)} \sigma_x$   
 $= \frac{2(1+\hat{p}_z) - 2i(1+\hat{p}_z)\sigma_y \hat{p}_x - 2\hat{p}_x^2 + 2i\sigma_z \hat{p}_x \hat{p}_y}{2(1+\hat{p}_z)} \sigma_x$   
 $= \frac{(1+\hat{p}_z)(\sigma_x - \hat{p}_x \sigma_z) - \hat{p}_x(\hat{p}_x \sigma_x + \hat{p}_y \sigma_y)}{(1+\hat{p}_z)}$   
 $= \frac{\sigma_x + \hat{p}_z \sigma_x - \hat{p}_x \sigma_z - \hat{p}_x(\sigma \cdot \hat{p})}{(1+\hat{p}_z)}$   
 $= \sigma_x - \hat{p}_x \frac{(\sigma \cdot \hat{p} + \sigma_z)}{(1+\hat{p}_z)}$   
 $= \frac{\sigma_x - (\sigma \times \hat{p})_y - \hat{p}_x(\sigma \cdot \hat{p})}{(1+\hat{p}_z)}$

□

$$\begin{aligned}
& \text{证明: } e^{i\vec{\omega}\cdot\frac{\sigma}{2}}\sigma_y e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = (e^{-i\vec{\omega}\cdot\gamma})_y^k \sigma_k \\
& = \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\sigma_y \frac{(1+\hat{p}_z)-i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}} \\
& = \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}} \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y+\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\sigma_y \\
& = \frac{(1+\hat{p}_z)^2+2i(1+\hat{p}_z)(\sigma_x\hat{p}_y-(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)(\sigma_x\hat{p}_y+\sigma_y\hat{p}_x)}{2(1+\hat{p}_z)}\sigma_y \\
& = \frac{(1+\hat{p}_z)^2+2i(1+\hat{p}_z)\sigma_x\hat{p}_y+\hat{p}_x^2-\hat{p}_y^2-2i\sigma_x\hat{p}_x\hat{p}_y}{2(1+\hat{p}_z)}\sigma_y \\
& = \frac{2(1+\hat{p}_z)+2i(1+\hat{p}_z)\sigma_x\hat{p}_y-2\hat{p}_y^2-2i\sigma_x\hat{p}_x\hat{p}_y}{2(1+\hat{p}_z)}\sigma_y \\
& = \frac{(1+\hat{p}_z)(\sigma_y-\hat{p}_y\sigma_z)-\hat{p}_y(\hat{p}_x\sigma_x+\hat{p}_y\sigma_y)}{(1+\hat{p}_z)} \\
& = \frac{\sigma_y+\hat{p}_z\sigma_y-\hat{p}_y\sigma_z-\hat{p}_y(\sigma\cdot\hat{p})}{(1+\hat{p}_z)} \\
& = \sigma_y - \hat{p}_y \frac{(\sigma\cdot\hat{p}+\sigma_z)}{(1+\hat{p}_z)} \\
& = \frac{\sigma_y+(\sigma\times\hat{p})_x-\hat{p}_y(\sigma\cdot\hat{p})}{(1+\hat{p}_z)} \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } e^{i\vec{\omega}\cdot\frac{\sigma}{2}}\sigma_z e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = (e^{-i\vec{\omega}\cdot\gamma})_z^k \sigma_k \\
& = \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\sigma_z \frac{(1+\hat{p}_z)-i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}} \\
& = \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}} \frac{(1+\hat{p}_z)+i(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)}{\sqrt{2(1+\hat{p}_z)}}\sigma_z \\
& = \frac{(1+\hat{p}_z)^2+2i(1+\hat{p}_z)(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)-(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)^2}{2(1+\hat{p}_z)}\sigma_z \\
& = \frac{(1+\hat{p}_z)^2+2i(1+\hat{p}_z)(\sigma_x\hat{p}_y-\sigma_y\hat{p}_x)-(\hat{p}_x^2+\hat{p}_y^2)}{2(1+\hat{p}_z)}\sigma_z \\
& = [\hat{p}_z + i(\sigma_x\hat{p}_y - \sigma_y\hat{p}_x)]\sigma_z \\
& = \sigma\cdot\hat{p} \quad \square
\end{aligned}$$

推论2.3.1.

$$\begin{cases} e^{i\vec{\omega}\cdot\frac{\sigma}{2}}(\sigma_x + i\sigma_y)e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = (\sigma_x + i\sigma_y) - \frac{(\hat{p}_x+i\hat{p}_y)}{(1+\hat{p}_z)}(\sigma\cdot\hat{p} + \sigma_z) \\ e^{i\vec{\omega}\cdot\frac{\sigma}{2}}(\sigma_x - i\sigma_y)e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = (\sigma_x - i\sigma_y) - \frac{(\hat{p}_x-i\hat{p}_y)}{(1+\hat{p}_z)}(\sigma\cdot\hat{p} + \sigma_z) \\ e^{i\vec{\omega}\cdot\frac{\sigma}{2}}\sigma_z e^{-i\vec{\omega}\cdot\frac{\sigma}{2}} = \sigma\cdot\hat{p} \end{cases}$$

推论2.3.2.

$$\begin{cases} \{[\sigma_x(\frac{1}{2}) + i\sigma_y(\frac{1}{2})] - \frac{(\hat{p}_x+i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(\frac{1}{2})\cdot\hat{p} + \sigma_z(\frac{1}{2})]\}\lambda(\hat{p}, \frac{1}{2}) = 0 \\ \{[\sigma_x(\frac{1}{2}) + i\sigma_y(\frac{1}{2})] - \frac{(\hat{p}_x+i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(\frac{1}{2})\cdot\hat{p} + \sigma_z(\frac{1}{2})]\}\lambda(\hat{p}, -\frac{1}{2}) = \lambda(\hat{p}, \frac{1}{2}) \\ \{[\sigma_x(\frac{1}{2}) - i\sigma_y(\frac{1}{2})] - \frac{(\hat{p}_x-i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(\frac{1}{2})\cdot\hat{p} + \sigma_z(\frac{1}{2})]\}\lambda(\hat{p}, \frac{1}{2}) = \lambda(\hat{p}, -\frac{1}{2}) \\ \{[\sigma_x(\frac{1}{2}) - i\sigma_y(\frac{1}{2})] - \frac{(\hat{p}_x-i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(\frac{1}{2})\cdot\hat{p} + \sigma_z(\frac{1}{2})]\}\lambda(\hat{p}, -\frac{1}{2}) = 0 \end{cases}$$

## 2.4 螺旋度 $\sigma(\frac{1}{2})\cdot\hat{p}$ 本征函数的基本性质

性质2.4.1.  $\lambda^*(\hat{p}, -\frac{\zeta}{2}) \equiv -i\zeta\sigma_y\lambda(\hat{p}, \frac{\zeta}{2})$ ,  $\lambda^+(\hat{p}, -\frac{\zeta}{2}) \equiv i\zeta\lambda^T(\hat{p}, \frac{\zeta}{2})\sigma_y$

## 2.5 螺旋度 $\sigma(\frac{1}{2})\cdot\hat{p}$ 本征函数的复杂性质

性质2.5.1.  $\lambda^+(\hat{p}, -\frac{\zeta}{2})(\sigma, -i\zeta)_a\lambda(\hat{p}, -\frac{\zeta}{2}) = -\zeta\hat{p}_a$ ,  $\lambda^T(\hat{p}, \frac{\zeta}{2})\sigma_y(\sigma, -i\zeta)_a\lambda(\hat{p}, -\frac{\zeta}{2}) = i\hat{p}_a$

$$\text{性质2.5.2. } \lambda^+(\hat{p}, -\frac{\zeta}{2})(\sigma, -i\zeta)_a\lambda(\hat{p}, \frac{\zeta}{2}) = \begin{bmatrix} \hat{p}_x\hat{p}_z - i\zeta\hat{p}_y \\ \hat{p}_x - i\zeta\hat{p}_y \\ \hat{p}_y\hat{p}_z + i\zeta\hat{p}_x \\ \hat{p}_x - i\zeta\hat{p}_y \\ -\hat{p}_x - i\zeta\hat{p}_y \\ 0 \end{bmatrix}, \lambda^T(\hat{p}, \frac{\zeta}{2})(\sigma, 1)\lambda(\hat{p}, \frac{\zeta}{2}) = \begin{bmatrix} \zeta\hat{p}_x + i\hat{p}_y \\ 0 \\ \zeta\hat{p}_x\hat{p}_z - i\hat{p}_y \\ \hat{p}_x - i\zeta\hat{p}_y \\ \hat{p}_x - i\zeta\hat{p}_y\hat{p}_z \\ \hat{p}_x - i\zeta\hat{p}_y \end{bmatrix}$$

$$\text{性质2.5.3. } \lambda^+(\hat{p}, -\frac{\zeta}{2})\sigma_i\lambda(\hat{p}, \frac{\zeta}{2}) = \begin{bmatrix} \hat{p}_x\hat{p}_z - i\zeta\hat{p}_y \\ \hat{p}_x - i\zeta\hat{p}_y \\ \hat{p}_y\hat{p}_z + i\zeta\hat{p}_x \\ \hat{p}_x - i\zeta\hat{p}_y \\ -\hat{p}_x - i\zeta\hat{p}_y \end{bmatrix} = \begin{bmatrix} \hat{p}_x\hat{p}_z - i\zeta\hat{p}_y\delta_{xx} + i\zeta\hat{p}_x\delta_{xy} - \delta_{xz} \\ \hat{p}_x - i\zeta\hat{p}_y \\ \hat{p}_y\hat{p}_z - i\zeta\hat{p}_y\delta_{yx} + i\zeta\hat{p}_x\delta_{yy} - \delta_{yz} \\ \hat{p}_x - i\zeta\hat{p}_y \\ \hat{p}_z\hat{p}_z - i\zeta\hat{p}_y\delta_{yx} + i\zeta\hat{p}_x\delta_{yy} - \delta_{zz} \\ \hat{p}_x - i\zeta\hat{p}_y \end{bmatrix} = \frac{(\hat{p}_i\hat{p}_z - \delta_{iz}) - i\zeta(\hat{p}_y\delta_{ix} - \hat{p}_x\delta_{iy})}{\hat{p}_x - i\zeta\hat{p}_y}$$

$$\text{性质2.5.4. } \lambda^T(\hat{p}, \frac{\zeta}{2})(\sigma, 1)\lambda(\hat{p}, -\frac{\zeta}{2}) = \begin{bmatrix} \hat{p}_z \\ -i\zeta \\ -\hat{p}_x \\ i\hat{p}_y \end{bmatrix}$$



$$\text{性质2.5.5. } \lambda(\hat{p}, -\frac{\varsigma}{2})\lambda^T(\hat{p}, -\frac{\varsigma}{2}) = \begin{bmatrix} -\frac{1}{\sqrt{2}}\hat{p}_- & -\frac{1}{2}(\varsigma-\hat{p}_z) \\ \frac{1}{2}(\varsigma+\hat{p}_z) & \frac{1}{\sqrt{2}}\hat{p}_+ \end{bmatrix}$$

$$\text{性质2.5.6. } \lambda(\hat{p}, -\frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{\varsigma}{2}) = \begin{bmatrix} -\frac{1}{\sqrt{2}}\hat{p}_- & -\frac{1}{2}(\varsigma-\hat{p}_z) \\ \frac{1}{2}(\varsigma+\hat{p}_z) & \frac{1}{\sqrt{2}}\hat{p}_+ \end{bmatrix} = \frac{i}{2}(\sigma \cdot \hat{p} - \varsigma I)\sigma_y = \frac{i}{2}(\sigma, i\varsigma)^a \hat{p}_a \sigma_y$$

$$\text{性质2.5.7. } \lambda(\hat{p}, -\frac{\varsigma}{2})\lambda^+(\hat{p}, -\frac{\varsigma}{2}) = -\frac{\varsigma}{2}(\sigma, i\varsigma)^a \hat{p}_a, \lambda(\hat{p}, -\frac{\varsigma}{2})\lambda^T(\hat{p}, \frac{\varsigma}{2}) = \frac{i}{2}(\sigma, i\varsigma)^a \hat{p}_a \sigma_y$$

$$\text{性质2.5.8. } \begin{cases} \lambda(\hat{p}, \frac{1}{2})\lambda^+(\hat{p}, \frac{1}{2}) + \lambda(\hat{p}, -\frac{1}{2})\lambda^+(\hat{p}, -\frac{1}{2}) = I, \lambda(\hat{p}, \frac{1}{2})\lambda^+(\hat{p}, \frac{1}{2}) - \lambda(\hat{p}, -\frac{1}{2})\lambda^+(\hat{p}, -\frac{1}{2}) = \sigma \cdot \hat{p} \\ \lambda(\hat{p}, \frac{1}{2})\lambda^T(\hat{p}, -\frac{1}{2}) - \lambda(\hat{p}, -\frac{1}{2})\lambda^T(\hat{p}, \frac{1}{2}) = i\sigma_y, \lambda(\hat{p}, \frac{1}{2})\lambda^T(\hat{p}, -\frac{1}{2}) + \lambda(\hat{p}, -\frac{1}{2})\lambda^T(\hat{p}, \frac{1}{2}) = i\sigma \cdot \hat{p}\sigma_y \end{cases}$$

## 2.6 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数的导数性质

### 2.6.1 基本导数性质

推论2.6.1.

$$\begin{cases} \tilde{\partial}_i p = \hat{p}_i \\ \tilde{\partial}_i \hat{p}_j = \frac{p^2 \delta_{ij} - p_i p_j}{p^3} = \frac{\delta_{ij} - \hat{p}_i \hat{p}_j}{p} \\ \tilde{\partial}_i \hat{p}_+ = \frac{\frac{1}{\sqrt{2}}(\delta_{ix} + i\delta_{iy}) - \hat{p}_i \hat{p}_+}{p}, \tilde{\partial}_i \hat{p}_- = \frac{\frac{1}{\sqrt{2}}(\delta_{ix} - i\delta_{iy}) - \hat{p}_i \hat{p}_-}{p} \\ \tilde{\partial}_i \frac{\hat{p}_+}{\hat{p}_-} = \frac{i\hat{p}_x \delta_{iy} - i\hat{p}_y \delta_{ix}}{p\hat{p}_-^2}, \tilde{\partial}_i \frac{\hat{p}_-}{\hat{p}_+} = \frac{-i\hat{p}_x \delta_{iy} + i\hat{p}_y \delta_{ix}}{p\hat{p}_+^2} \\ \tilde{\partial}_i \frac{\hat{p}_+}{\sqrt{\hat{p}_+ \hat{p}_-}} = \frac{\hat{p}_-}{\sqrt{\hat{p}_+ \hat{p}_-}} \frac{i\hat{p}_x \delta_{iy} - i\hat{p}_y \delta_{ix}}{2p\hat{p}_-^2}, \tilde{\partial}_i \frac{\hat{p}_-}{\sqrt{\hat{p}_+ \hat{p}_-}} = \frac{\hat{p}_+}{\sqrt{\hat{p}_+ \hat{p}_-}} \frac{-i\hat{p}_x \delta_{iy} + i\hat{p}_y \delta_{ix}}{2p\hat{p}_+^2} \end{cases}$$

### 2.6.2 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数导数性质一

$$\text{推论2.6.2. } \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2}) = -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)} \lambda(\hat{p}, \frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}} \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix}$$

$$= \frac{1}{2p\sqrt{1+\hat{p}_z}^3} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ -\hat{p}_+(\delta_{iz} - \hat{p}_i \hat{p}_z) + \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} - \hat{p}_i \hat{p}_x) + i(\delta_{iy} - \hat{p}_i \hat{p}_y)] \end{bmatrix}$$

$$\text{证明: } \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2}) = \tilde{\partial}_i \frac{1}{\sqrt{1+\hat{p}_z}} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_+ \end{bmatrix} + \frac{1}{\sqrt{1+\hat{p}_z}} \tilde{\partial}_i \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_+ \end{bmatrix}$$

$$= \frac{-\delta_{iz} + \hat{p}_i \hat{p}_z}{2p\sqrt{1+\hat{p}_z}^3} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_+ \end{bmatrix} + \frac{1}{p\sqrt{1+\hat{p}_z}} \begin{bmatrix} \frac{1}{\sqrt{2}}(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ \frac{1}{\sqrt{2}}(\delta_{ix} + i\delta_{iy}) - \hat{p}_i \hat{p}_+ \end{bmatrix}$$

$$= \frac{1}{2p\sqrt{1+\hat{p}_z}^3} \left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ -\hat{p}_+(\delta_{iz} - \hat{p}_i \hat{p}_z) \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{2}}(\delta_{iz} - \hat{p}_i \hat{p}_z)2(1+\hat{p}_z) \\ [\frac{1}{\sqrt{2}}(\delta_{ix} + i\delta_{iy}) - \hat{p}_i \hat{p}_+]2(1+\hat{p}_z) \end{bmatrix} \right\}$$

$$= \frac{1}{2p\sqrt{1+\hat{p}_z}^3} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ -\hat{p}_+(\delta_{iz} - \hat{p}_i \hat{p}_z) + 2(1+\hat{p}_z)[\frac{1}{\sqrt{2}}(\delta_{ix} + i\delta_{iy}) - \hat{p}_i \hat{p}_+] \end{bmatrix}$$

$$= \frac{1}{2p\sqrt{1+\hat{p}_z}^3} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ -\hat{p}_+(\delta_{iz} - \hat{p}_i \hat{p}_z) + \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} - \hat{p}_i \hat{p}_x) + i(\delta_{iy} - \hat{p}_i \hat{p}_y)] \end{bmatrix}$$

$$= \frac{1}{2\sqrt{1+\hat{p}_z}^3} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z)\tilde{\partial}_i \hat{p}_z \\ -\hat{p}_+ \tilde{\partial}_i \hat{p}_z + 2(1+\hat{p}_z)\tilde{\partial}_i \hat{p}_+ \end{bmatrix}$$

$$= \frac{1}{2\sqrt{1+\hat{p}_z}} \begin{bmatrix} \frac{1}{\sqrt{2}}\tilde{\partial}_i \hat{p}_z \\ -\frac{\hat{p}_+}{1+\hat{p}_z}\tilde{\partial}_i \hat{p}_z + 2\tilde{\partial}_i \hat{p}_+ \end{bmatrix} \quad \square$$

$$\text{证明: } \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2}) = \frac{-\delta_{iz} + \hat{p}_i \hat{p}_z}{2p(1+\hat{p}_z)} \lambda(\hat{p}, \frac{1}{2}) + \frac{1}{p\sqrt{1+\hat{p}_z}} \begin{bmatrix} \frac{1}{\sqrt{2}}(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ \frac{1}{\sqrt{2}}(\delta_{ix} + i\delta_{iy}) - \hat{p}_i \hat{p}_+ \end{bmatrix}$$

$$= \frac{-\delta_{iz} + \hat{p}_i \hat{p}_z}{2p(1+\hat{p}_z)} \lambda(\hat{p}, \frac{1}{2}) + \frac{1}{\sqrt{2}p\sqrt{1+\hat{p}_z}} \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix} - \frac{\hat{p}_i}{p\sqrt{1+\hat{p}_z}} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_+ \end{bmatrix}$$

$$\begin{aligned}
&= \left[ \frac{-\delta_{iz} + \hat{p}_i \hat{p}_z}{2p(1+\hat{p}_z)} - \frac{\hat{p}_i}{p} \right] \lambda(\hat{p}, \frac{1}{2}) + \frac{1}{\sqrt{2p\sqrt{1+\hat{p}_z}}} \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix} \\
&= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)} \lambda(\hat{p}, \frac{1}{2}) + \frac{1}{\sqrt{2p\sqrt{1+\hat{p}_z}}} \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix} \quad \square
\end{aligned}$$

$$\begin{aligned}
\text{推论2.6.3. } \tilde{\partial}_i \lambda(\hat{p}, -\frac{1}{2}) &= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)} \lambda(\hat{p}, -\frac{1}{2}) + \frac{1}{\sqrt{2p\sqrt{1+\hat{p}_z}}} \begin{bmatrix} -\delta_{ix} + i\delta_{iy} \\ \delta_{iz} + \hat{p}_i \end{bmatrix} \\
&= \frac{1}{2p\sqrt{1+\hat{p}_z}^3} \begin{bmatrix} \hat{p}_- (\delta_{iz} - \hat{p}_i \hat{p}_z) - \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} - \hat{p}_i \hat{p}_x) - i(\delta_{iy} - \hat{p}_i \hat{p}_y)] \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \end{bmatrix}
\end{aligned}$$

$$\text{证明: } \tilde{\partial}_i \lambda(\hat{p}, -\frac{1}{2}) = -i\sigma_y \tilde{\partial}_i \lambda^*(\hat{p}, \frac{1}{2})$$

$$\begin{aligned}
&= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)} [-i\sigma_y \lambda^*(\hat{p}, \frac{1}{2})] + \frac{-i\sigma_y}{\sqrt{2p\sqrt{1+\hat{p}_z}}} \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} - i\delta_{iy} \end{bmatrix} \\
&= \frac{-i\sigma_y}{2p\sqrt{1+\hat{p}_z}^3} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ -\hat{p}_- (\delta_{iz} - \hat{p}_i \hat{p}_z) + \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} - \hat{p}_i \hat{p}_x) - i(\delta_{iy} - \hat{p}_i \hat{p}_y)] \end{bmatrix} \\
&= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)} \lambda(\hat{p}, -\frac{1}{2}) + \frac{1}{\sqrt{2p\sqrt{1+\hat{p}_z}}} \begin{bmatrix} -\delta_{ix} + i\delta_{iy} \\ \delta_{iz} + \hat{p}_i \end{bmatrix} \\
&= \frac{1}{2p\sqrt{1+\hat{p}_z}^3} \begin{bmatrix} \hat{p}_- (\delta_{iz} - \hat{p}_i \hat{p}_z) - \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} - \hat{p}_i \hat{p}_x) - i(\delta_{iy} - \hat{p}_i \hat{p}_y)] \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \end{bmatrix} \quad \square
\end{aligned}$$

### 2.6.3 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数导数性质二

$$\text{推论2.6.4. } \begin{cases} \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) = [\lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2})]^*, \lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) = [\lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2})]^* \\ \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) = -[\lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2})]^*, \lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) = -[\lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2})]^* \end{cases}$$

$$\text{性质2.6.1. } \begin{cases} \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) = \lambda^+(\hat{p}, \frac{1}{2}) \frac{[\sigma_k(\frac{1}{2}), \sigma_z(\frac{1}{2})]}{p(1+\hat{p}_z)} \lambda(\hat{p}, \frac{1}{2}) = \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{2p(1+\hat{p}_z)} \\ \lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) = \lambda^+(\hat{p}, -\frac{1}{2}) \frac{[\sigma_k(\frac{1}{2}), \sigma_z(\frac{1}{2})]}{p(1+\hat{p}_z)} \lambda(\hat{p}, -\frac{1}{2}) = -\frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{2p(1+\hat{p}_z)} \\ \lambda^+(-\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(-\hat{p}, \frac{1}{2}) = -\lambda^+(-\hat{p}, \frac{1}{2}) \frac{[\sigma_k(\frac{1}{2}), \sigma_z(\frac{1}{2})]}{p(1-\hat{p}_z)} \lambda(-\hat{p}, \frac{1}{2}) = \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{2p(1-\hat{p}_z)} \\ \lambda^+(-\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(-\hat{p}, -\frac{1}{2}) = -\lambda^+(-\hat{p}, -\frac{1}{2}) \frac{[\sigma_k(\frac{1}{2}), \sigma_z(\frac{1}{2})]}{p(1-\hat{p}_z)} \lambda(-\hat{p}, -\frac{1}{2}) = -\frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{2p(1-\hat{p}_z)} \\ \lambda^+(\hat{p}, h) \partial_z \lambda(\hat{p}, h) = 0 \end{cases}$$

$$\text{证明: } \lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, -\frac{1}{2})$$

$$\begin{aligned}
&= \frac{1}{2p(1+\hat{p}_z)^2} \begin{bmatrix} -\hat{p}_+ \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z) \end{bmatrix}^T \begin{bmatrix} \hat{p}_- (\delta_{iz} - \hat{p}_i \hat{p}_z) - \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} - \hat{p}_i \hat{p}_x) - i(\delta_{iy} - \hat{p}_i \hat{p}_y)] \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \end{bmatrix} \\
&= \lambda^+(\hat{p}, -\frac{1}{2}) \frac{[\sigma_i(\frac{1}{2}), \sigma_z(\frac{1}{2})]}{p(1+\hat{p}_z)} \lambda(\hat{p}, -\frac{1}{2}) = (\frac{i\hat{p}_y}{2p}, \frac{-i\hat{p}_x}{2p}, 0) = \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{2p(1+\hat{p}_z)} \quad \square
\end{aligned}$$

$$\text{证明: } \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2})$$

$$\begin{aligned}
&= \frac{1}{2p(1+\hat{p}_z)^2} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_- \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ -\hat{p}_+ (\delta_{iz} - \hat{p}_i \hat{p}_z) + \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} - \hat{p}_i \hat{p}_x) + i(\delta_{iy} - \hat{p}_i \hat{p}_y)] \end{bmatrix} \\
&= \lambda^+(\hat{p}, \frac{1}{2}) \frac{[\sigma_i(\frac{1}{2}), \sigma_z(\frac{1}{2})]}{p(1+\hat{p}_z)} \lambda(\hat{p}, \frac{1}{2}) = (\frac{-i\hat{p}_y}{2p}, \frac{i\hat{p}_x}{2p}, 0) = -\frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{2p(1+\hat{p}_z)} \quad \square
\end{aligned}$$

2.6.4 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数导数性质三

$$\text{性质2.6.2.} \quad \left\{ \begin{array}{l} \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, -\frac{1}{2}) = \frac{\sqrt{2}\hat{p}_- \hat{p}_i + \sqrt{2}\hat{p}_- \delta_{iz} - (1+\hat{p}_z)(\delta_{ix} - i\delta_{iy})}{2p(1+\hat{p}_z)} \\ \lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2}) = \frac{-\sqrt{2}\hat{p}_+ \hat{p}_i - \sqrt{2}\hat{p}_+ \delta_{iz} + (1+\hat{p}_z)(\delta_{ix} + i\delta_{iy})}{2p(1+\hat{p}_z)} \\ \lambda^+(-\hat{p}, -\frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, -\frac{1}{2}) = \frac{\hat{p}_+}{\sqrt{\hat{p}_+ \hat{p}_-}} \frac{\sqrt{2}\hat{p}_- \hat{p}_i + \sqrt{2}\hat{p}_- \delta_{iz} - (1+\hat{p}_z)(\delta_{ix} - i\delta_{iy})}{2p(1+\hat{p}_z)} \\ \lambda^+(-\hat{p}, \frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2}) = -\frac{\hat{p}_-}{\sqrt{\hat{p}_+ \hat{p}_-}} \frac{-\sqrt{2}\hat{p}_+ \hat{p}_i - \sqrt{2}\hat{p}_+ \delta_{iz} + (1+\hat{p}_z)(\delta_{ix} + i\delta_{iy})}{2p(1+\hat{p}_z)} \end{array} \right.$$

证明:  $\lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2})$

$$\begin{aligned} &= \frac{1}{2p(1+\hat{p}_z)^2} \begin{bmatrix} -\hat{p}_+ \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ -\hat{p}_+(\delta_{iz} - \hat{p}_i \hat{p}_z) + \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} - \hat{p}_i \hat{p}_x) + i(\delta_{iy} - \hat{p}_i \hat{p}_y)] \end{bmatrix} \\ &= \frac{1}{2p(1+\hat{p}_z)^2} \begin{bmatrix} -\hat{p}_+ \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ -\hat{p}_+(\delta_{iz} - \hat{p}_i \hat{p}_z) + \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} + i\delta_{iy}) - \sqrt{2}\hat{p}_i \hat{p}_+] \end{bmatrix} \\ &= \frac{1}{2p(1+\hat{p}_z)^2} \begin{bmatrix} -\hat{p}_+ \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z) \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \\ -\hat{p}_+ \delta_{iz} - 2\hat{p}_i \hat{p}_+ - \hat{p}_+ \hat{p}_i \hat{p}_z + \sqrt{2}(1+\hat{p}_z)(\delta_{ix} + i\delta_{iy}) \end{bmatrix} \\ &= \frac{1}{2p(1+\hat{p}_z)^2} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z) \end{bmatrix}^T \begin{bmatrix} -\hat{p}_+ \delta_{iz} + \hat{p}_+ \hat{p}_i \hat{p}_z \\ -\hat{p}_+ \delta_{iz} - 2\hat{p}_i \hat{p}_+ - \hat{p}_+ \hat{p}_i \hat{p}_z + \sqrt{2}(1+\hat{p}_z)(\delta_{ix} + i\delta_{iy}) \end{bmatrix} \\ &= \frac{-\sqrt{2}\hat{p}_+ \hat{p}_i - \sqrt{2}\hat{p}_+ \delta_{iz} + (1+\hat{p}_z)(\delta_{ix} + i\delta_{iy})}{2p(1+\hat{p}_z)} \quad \square \end{aligned}$$

证明:  $\lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, -\frac{1}{2})$

$$\begin{aligned} &= \frac{1}{2p(1+\hat{p}_z)^2} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_- \end{bmatrix}^T \begin{bmatrix} \hat{p}_-(\delta_{iz} - \hat{p}_i \hat{p}_z) - \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} - \hat{p}_i \hat{p}_x) - i(\delta_{iy} - \hat{p}_i \hat{p}_y)] \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \end{bmatrix} \\ &= \frac{1}{2p(1+\hat{p}_z)^2} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_- \end{bmatrix}^T \begin{bmatrix} \hat{p}_-(\delta_{iz} - \hat{p}_i \hat{p}_z) - \sqrt{2}(1+\hat{p}_z)[(\delta_{ix} - i\delta_{iy}) - \sqrt{2}\hat{p}_i \hat{p}_-] \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \end{bmatrix} \\ &= \frac{1}{2p(1+\hat{p}_z)^2} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_- \end{bmatrix}^T \begin{bmatrix} \hat{p}_- \delta_{iz} + 2\hat{p}_i \hat{p}_- + \hat{p}_i \hat{p}_- \hat{p}_z - \sqrt{2}(1+\hat{p}_z)(\delta_{ix} - i\delta_{iy}) \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z)(\delta_{iz} - \hat{p}_i \hat{p}_z) \end{bmatrix} \\ &= \frac{1}{2p(1+\hat{p}_z)^2} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z) \end{bmatrix}^T \begin{bmatrix} \hat{p}_- \delta_{iz} + 2\hat{p}_i \hat{p}_- + \hat{p}_i \hat{p}_- \hat{p}_z - \sqrt{2}(1+\hat{p}_z)(\delta_{ix} - i\delta_{iy}) \\ \hat{p}_- \delta_{iz} - \hat{p}_i \hat{p}_- \hat{p}_z \end{bmatrix} \\ &= \frac{\sqrt{2}\hat{p}_- \hat{p}_i + \sqrt{2}\hat{p}_- \delta_{iz} - (1+\hat{p}_z)(\delta_{ix} - i\delta_{iy})}{p(1+\hat{p}_z)} \quad \square \end{aligned}$$

2.6.5 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数导数性质四

$$\text{推论2.6.5.} \quad \lambda^+(\hat{p}, h) \lambda(\hat{p}, h') = \delta_{hh'}, \quad \sum_{h=\frac{1}{2}}^{-\frac{1}{2}} \lambda(\hat{p}, h) \lambda^+(\hat{p}, h) = 1, \quad \sum_{h=\frac{1}{2}}^{-\frac{1}{2}} \lambda^+(\hat{p}, h) \tilde{\partial}_k \lambda(\hat{p}, h) = 0$$

$$\text{推论2.6.6.} \quad (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a p_a = -2\zeta |\vec{p}| \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2})$$

证明:  $(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a p_a$

$$\begin{aligned} &= (\sigma \cdot \vec{p})_{A_\zeta} B_\zeta [\lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) + \lambda_{B_\zeta}(\hat{p}, \zeta) \lambda_{A'_\zeta}^+(\hat{p}, \zeta)] - \zeta |\vec{p}| [\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) + \lambda_{A_\zeta}(\hat{p}, \zeta) \lambda_{A'_\zeta}^+(\hat{p}, \zeta)] \\ &= [-\zeta |\vec{p}| \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) + \zeta |\vec{p}| \lambda_{A_\zeta}(\hat{p}, \zeta) \lambda_{A'_\zeta}^+(\hat{p}, \zeta)] - \zeta |\vec{p}| [\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) + \lambda_{A_\zeta}(\hat{p}, \zeta) \lambda_{A'_\zeta}^+(\hat{p}, \zeta)] \\ &= -2\zeta |\vec{p}| \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) \quad \square \end{aligned}$$

$$\text{推论2.6.7.} \quad \lambda_{A_\zeta}(\hat{p}, h) \lambda_{A'_\zeta}^+(\hat{p}, h) = h \sigma_{A_\zeta A'_\zeta}^k \hat{p}_k + \frac{1}{2} \delta_{A_\zeta A'_\zeta}$$

$$\text{推论2.6.8.} \quad e^{i\vec{\omega} \cdot \frac{\sigma}{2}} = \cos \frac{1}{2} \omega + i \hat{\omega} \cdot \sigma \sin \frac{1}{2} \omega$$

$$\begin{aligned} &= \frac{1}{\sqrt{2}} [p_x + p_z + i(\sigma_x p_y - \sigma_y p_x)] [p^2 + p p_z]^{-1/2} \\ &= \frac{1}{\sqrt{2}} [1 + \hat{p}_z + i(\sigma \times \hat{p})_z] [1 + \hat{p}_z]^{-1/2} \end{aligned}$$

推论2.6.9.  $\partial_z e^{i\vec{\omega} \cdot \frac{\sigma}{2}} = \frac{\sqrt{1+\hat{p}_z}}{\sqrt{2p}} [1 - \hat{p}_z - i(\sigma \times \hat{p})_z], e^{-i\vec{\omega} \cdot \frac{\sigma}{2}} \partial_z e^{i\vec{\omega} \cdot \frac{\sigma}{2}} = \frac{-i}{p} (\sigma \times \hat{p})_z$

推论2.6.10.  $\lambda^+(\hat{p}, h) \partial_z \lambda(\hat{p}, h) = 0$

### 2.6.6 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数导数性质的小结

性质2.6.3. 
$$\begin{cases} \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) = \frac{1}{2} \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{p(1+\hat{p}_z)} \\ \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, -\frac{1}{2}) = \frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x - i\hat{p}_y) - (1+\hat{p}_z)(\delta_{ix} - i\delta_{iy})}{2p(1+\hat{p}_z)} \\ \lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2}) = -\frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x + i\hat{p}_y) - (1+\hat{p}_z)(\delta_{ix} + i\delta_{iy})}{2p(1+\hat{p}_z)} \\ \lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) = -\frac{1}{2} \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{p(1+\hat{p}_z)} \end{cases}$$

### 2.7 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数导数性质的继续探索一?

推论2.7.1.  $\sigma_j(\frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2}) = -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)} \sigma_j \lambda(\hat{p}, \frac{1}{2}) + \frac{1}{\sqrt{2p}\sqrt{1+\hat{p}_z}} \sigma_j \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix}$

证明:  $\lambda^+(\hat{p}, \frac{1}{2}) \sigma_j(\frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2})$

$$= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)} \lambda^+(\hat{p}, \frac{1}{2}) \sigma_j(\frac{1}{2}) \lambda(\hat{p}, \frac{1}{2}) + \frac{1}{\sqrt{2p}\sqrt{1+\hat{p}_z}} \lambda^+(\hat{p}, \frac{1}{2}) \sigma_j(\frac{1}{2}) \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix}$$

$$= -\frac{\delta_{iz} \hat{p}_j + \hat{p}_i \hat{p}_j(2+\hat{p}_z)}{4p(1+\hat{p}_z)} + \frac{1}{4p(1+\hat{p}_z)} \begin{bmatrix} 1 + \hat{p}_z \\ \hat{p}_x - i\hat{p}_y \end{bmatrix}^T \sigma_j \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix} \quad \square$$

证明:  $\lambda^+(\hat{p}, \frac{1}{2}) [\sigma_i(\frac{1}{2}) \tilde{\partial}_j - \sigma_j(\frac{1}{2}) \tilde{\partial}_i] \lambda(\hat{p}, \frac{1}{2})$

$$= \frac{\delta_{iz} \hat{p}_j - \delta_{jz} \hat{p}_i}{4p(1+\hat{p}_z)} + \frac{1}{4p(1+\hat{p}_z)} \begin{bmatrix} 1 + \hat{p}_z \\ \hat{p}_x - i\hat{p}_y \end{bmatrix}^T (\sigma_i \begin{bmatrix} \delta_{jz} + \hat{p}_j \\ \delta_{jx} + i\delta_{jy} \end{bmatrix} - \sigma_j \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix}) \quad \square$$

### 2.8 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}$ 本征函数导数性质的继续探索二?

推论2.8.1.  $\sigma_j(\frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2}) = -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)} \sigma_j \lambda(\hat{p}, \frac{1}{2}) + \frac{1}{\sqrt{2p}\sqrt{1+\hat{p}_z}} \sigma_j \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix}$

证明:  $\lambda^+(\hat{p}, -\frac{1}{2}) \sigma_j(\frac{1}{2}) \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2})$

$$= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{2p(1+\hat{p}_z)} \lambda^+(\hat{p}, -\frac{1}{2}) \sigma_j(\frac{1}{2}) \lambda(\hat{p}, \frac{1}{2}) + \frac{1}{\sqrt{2p}\sqrt{1+\hat{p}_z}} \lambda^+(\hat{p}, -\frac{1}{2}) \sigma_j(\frac{1}{2}) \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix}$$

$$= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{4p(1+\hat{p}_z)} \frac{(\hat{p}_j \hat{p}_z - \delta_{jz}) - i(\hat{p}_y \delta_{jx} - \hat{p}_x \delta_{jy})}{\hat{p}_x - i\hat{p}_y} + \frac{1}{4p(1+\hat{p}_z)} \begin{bmatrix} -\hat{p}_x - i\hat{p}_y \\ 1 + \hat{p}_z \end{bmatrix}^T \sigma_j \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix} \quad \square$$

性质2.8.1.  $\lambda^+(\hat{p}, -\frac{\zeta}{2}) \sigma_i \lambda(\hat{p}, \frac{\zeta}{2}) = \begin{bmatrix} \hat{p}_x \hat{p}_z - i\zeta \hat{p}_y \\ \hat{p}_x - i\zeta \hat{p}_y \\ \hat{p}_y \hat{p}_z + i\zeta \hat{p}_x \\ \hat{p}_x - i\zeta \hat{p}_y \\ -\hat{p}_x - i\zeta \hat{p}_y \end{bmatrix} = \begin{bmatrix} \hat{p}_x \hat{p}_z - i\zeta \hat{p}_y \delta_{xx} + i\zeta \hat{p}_x \delta_{xy} - \delta_{xz} \\ \hat{p}_x - i\zeta \hat{p}_y \\ \hat{p}_y \hat{p}_z - i\zeta \hat{p}_y \delta_{yx} + i\zeta \hat{p}_x \delta_{yy} - \delta_{yz} \\ \hat{p}_x - i\zeta \hat{p}_y \\ \hat{p}_z \hat{p}_z - i\zeta \hat{p}_y \delta_{yx} + i\zeta \hat{p}_x \delta_{yy} - \delta_{zz} \\ \hat{p}_x - i\zeta \hat{p}_y \end{bmatrix} = \frac{(\hat{p}_i \hat{p}_z - \delta_{iz}) - i\zeta(\hat{p}_y \delta_{ix} - \hat{p}_x \delta_{iy})}{\hat{p}_x - i\zeta \hat{p}_y}$

证明:  $\lambda^+(\hat{p}, \frac{1}{2}) [\sigma_i(\frac{1}{2}) \tilde{\partial}_j - \sigma_j(\frac{1}{2}) \tilde{\partial}_i] \lambda(\hat{p}, \frac{1}{2})$

$$= \frac{\delta_{iz} \hat{p}_j - \delta_{jz} \hat{p}_i}{4p(1+\hat{p}_z)} + \frac{1}{4p(1+\hat{p}_z)} \begin{bmatrix} 1 + \hat{p}_z \\ \hat{p}_x - i\hat{p}_y \end{bmatrix}^T (\sigma_i \begin{bmatrix} \delta_{jz} + \hat{p}_j \\ \delta_{jx} + i\delta_{jy} \end{bmatrix} - \sigma_j \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix}) \quad \square$$

## 3 螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数的分析

### 3.1 螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数的具体求法I

推论3.1.1.  $\sigma(1) \cdot \hat{p} \lambda(\hat{p}, h; 1) = h \lambda(\hat{p}, h; 1), h = -1, 0, 1$

$$\text{推论3.1.2. } i\hat{\omega} \cdot \sigma(1) = i \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \hat{\omega}_x & 0 \\ \hat{\omega}_x & 0 & \hat{\omega}_x \\ 0 & \hat{\omega}_x & 0 \end{bmatrix} + \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & -\hat{\omega}_y & 0 \\ \hat{\omega}_y & 0 & -\hat{\omega}_y \\ 0 & \hat{\omega}_y & 0 \end{bmatrix} + \begin{bmatrix} \hat{\omega}_z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hat{\omega}_z \end{bmatrix} \right\} = i \begin{bmatrix} \hat{\omega}_z & \frac{1}{\sqrt{2}}(\hat{\omega}_x - i\hat{\omega}_y) & 0 \\ \frac{1}{\sqrt{2}}(\hat{\omega}_x + i\hat{\omega}_y) & 0 & \frac{1}{\sqrt{2}}(\hat{\omega}_x - i\hat{\omega}_y) \\ 0 & \frac{1}{\sqrt{2}}(\hat{\omega}_x + i\hat{\omega}_y) & -\hat{\omega}_z \end{bmatrix}$$

$$\text{推论3.1.3. } i\hat{\omega} \cdot \sigma(1) = i \begin{bmatrix} \hat{\omega}_z & \hat{\omega}_- & 0 \\ \hat{\omega}_+ & 0 & \hat{\omega}_- \\ 0 & \hat{\omega}_+ & -\hat{\omega}_z \end{bmatrix} \hat{\omega}_z \stackrel{=} {=} i \begin{bmatrix} 0 & \hat{\omega}_- & 0 \\ \hat{\omega}_+ & 0 & \hat{\omega}_- \\ 0 & \hat{\omega}_+ & 0 \end{bmatrix}$$

$$\text{推论3.1.4. } [i\hat{\omega} \cdot \sigma(1)]^2 = - \begin{bmatrix} \hat{\omega}_z & \hat{\omega}_- & 0 \\ \hat{\omega}_+ & 0 & \hat{\omega}_- \\ 0 & \hat{\omega}_+ & -\hat{\omega}_z \end{bmatrix}^2 = - \begin{bmatrix} \frac{1}{2}(\hat{\omega}_z^2 + 1) & \hat{\omega}_z \hat{\omega}_- & \hat{\omega}_-^2 \\ \hat{\omega}_z \hat{\omega}_+ & 2\hat{\omega}_+ \hat{\omega}_- & -\hat{\omega}_z \hat{\omega}_- \\ \hat{\omega}_+^2 & -\hat{\omega}_z \hat{\omega}_+ & \frac{1}{2}(\hat{\omega}_z^2 + 1) \end{bmatrix} \hat{\omega}_z \stackrel{=} {=} - \begin{bmatrix} \frac{1}{2} & 0 & \hat{\omega}_-^2 \\ 0 & 1 & 0 \\ \hat{\omega}_+^2 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\text{推论3.1.5. } e^{i\vec{\omega} \cdot \sigma(1)} \hat{\omega}_z \stackrel{=} {=} 1 + i\hat{\omega} \cdot \sigma(1) \sin \omega + [i\hat{\omega} \cdot \sigma(1)]^2 (1 - \cos \omega) = 1 + i \sin \omega \begin{bmatrix} 0 & \hat{\omega}_- & 0 \\ \hat{\omega}_+ & 0 & \hat{\omega}_- \\ 0 & \hat{\omega}_+ & 0 \end{bmatrix} - (1 - \cos \omega) \begin{bmatrix} \frac{1}{2} & 0 & \hat{\omega}_-^2 \\ 0 & 1 & 0 \\ \hat{\omega}_+^2 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\text{推论3.1.6. } e^{i\vec{\omega} \cdot \sigma(1)} \hat{\omega}_z \stackrel{=} {=} 1 + i \sin \omega \begin{bmatrix} 0 & \hat{\omega}_- & 0 \\ \hat{\omega}_+ & 0 & \hat{\omega}_- \\ 0 & \hat{\omega}_+ & 0 \end{bmatrix} - (1 - \cos \omega) \begin{bmatrix} \frac{1}{2} & 0 & \hat{\omega}_-^2 \\ 0 & 1 & 0 \\ \hat{\omega}_+^2 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 + \cos \omega) & i\hat{\omega}_- \sin \omega & -\hat{\omega}_-^2(1 - \cos \omega) \\ i\hat{\omega}_+ \sin \omega & \cos \omega & i\hat{\omega}_- \sin \omega \\ -\hat{\omega}_+^2(1 - \cos \omega) & i\hat{\omega}_+ \sin \omega & \frac{1}{2}(1 + \cos \omega) \end{bmatrix}$$

$$\text{推论3.1.7. } e^{i\vec{\omega} \cdot \sigma(1)} = \exp \left\{ i \frac{[\sigma(1) \times \hat{p}]_z}{\sqrt{1 - \hat{p}_z^2}} \arccos \hat{p}_z \right\} = 1 + i[\sigma(1) \times \hat{p}]_z - [\sigma(1) \times \hat{p}]_z^2 / (1 + \hat{p}_z), 0 \leq \arccos \hat{p}_z \leq \pi$$

$$\text{推论3.1.8. } \sigma(1) \cdot \hat{p} = e^{i\vec{\omega} \cdot \sigma(1)} \sigma_z(1) e^{-i\vec{\omega} \cdot \sigma(1)}$$

推论3.1.9.

$$\begin{cases} \lambda(\hat{p}, 1; 1) = e^{i\vec{\omega} \cdot \sigma(1)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 + \cos \omega) \\ i\hat{\omega}_+ \sin \omega \\ -\hat{\omega}_+^2(1 - \cos \omega) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 + \hat{p}_z) \\ \hat{p}_+ \\ \hat{p}_+^2 / (1 + \hat{p}_z) \end{bmatrix} = \frac{1}{\hat{p}_-} \begin{bmatrix} \frac{1}{2}\hat{p}_-(1 + \hat{p}_z) \\ \hat{p}_+ \hat{p}_- \\ \frac{1}{2}\hat{p}_+(1 - \hat{p}_z) \end{bmatrix}, \lambda(-\hat{p}, 1; 1) = \frac{\hat{p}_+}{\hat{p}_-} \lambda(\hat{p}, -1; 1) \\ \lambda(\hat{p}, 0; 1) = e^{i\vec{\omega} \cdot \sigma(1)} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i\hat{\omega}_- \sin \omega \\ \cos \omega \\ i\hat{\omega}_+ \sin \omega \end{bmatrix} = \begin{bmatrix} -\hat{p}_- \\ \hat{p}_z \\ \hat{p}_+ \end{bmatrix}, \lambda(-\hat{p}, 0; 1) = -\lambda(\hat{p}, 0; 1) \\ \lambda(\hat{p}, -1; 1) = e^{i\vec{\omega} \cdot \sigma(1)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\hat{\omega}_-^2(1 - \cos \omega) \\ i\hat{\omega}_- \sin \omega \\ \frac{1}{2}(1 + \cos \omega) \end{bmatrix} = \begin{bmatrix} \hat{p}_-^2 / (1 + \hat{p}_z) \\ -\hat{p}_- \\ \frac{1}{2}(1 + \hat{p}_z) \end{bmatrix} = \frac{1}{\hat{p}_+} \begin{bmatrix} \frac{1}{2}\hat{p}_-(1 - \hat{p}_z) \\ -\hat{p}_+ \hat{p}_- \\ \frac{1}{2}\hat{p}_+(1 + \hat{p}_z) \end{bmatrix}, \lambda(-\hat{p}, -1; 1) = \frac{\hat{p}_-}{\hat{p}_+} \lambda(\hat{p}, 1; 1) \end{cases}$$

### 3.2 螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数的具体求法II

$$\text{定理3.2.1. } \lambda(\hat{p}, h; 1) = \sqrt{C_2^{1-h} \Gamma(1)} \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{1+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{1-h}$$

### 3.3 螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数正交性与完备性的验证

$$\text{推论3.3.1. } \lambda^+(\hat{p}, h; 1) \lambda(\hat{p}, h'; 1) = \delta_{hh'}$$

$$\text{推论3.3.2. } \lambda(\hat{p}, 1; 1) \lambda^+(\hat{p}, 1; 1) = \frac{1}{2} \begin{bmatrix} \frac{1}{2}(1 + \cos \omega)^2 & -i\hat{\omega}_- \sin \omega (1 + \cos \omega) & -\hat{\omega}_-^2 \sin^2 \omega \\ i\hat{\omega}_+ \sin \omega (1 + \cos \omega) & \sin^2 \omega & -i\hat{\omega}_- \sin \omega (1 - \cos \omega) \\ -\hat{\omega}_+^2 \sin^2 \omega & i\hat{\omega}_+ \sin \omega (1 - \cos \omega) & \frac{1}{2}(1 - \cos \omega)^2 \end{bmatrix}$$

$$\text{推论3.3.3. } \lambda(\hat{p}, 0; 1) \lambda^+(\hat{p}, 0; 1) = \begin{bmatrix} \frac{1}{2} \sin^2 \omega & i\hat{\omega}_- \sin \omega \cos \omega & \hat{\omega}_-^2 \sin^2 \omega \\ -i\hat{\omega}_+ \sin \omega \cos \omega & \cos^2 \omega & -i\hat{\omega}_- \sin \omega \cos \omega \\ \hat{\omega}_+^2 \sin^2 \omega & i\hat{\omega}_+ \sin \omega \cos \omega & \frac{1}{2} \sin^2 \omega \end{bmatrix}$$

$$\text{推论3.3.4. } \lambda(\hat{p}, -1; 1) \lambda^+(\hat{p}, -1; 1) = \frac{1}{2} \begin{bmatrix} \frac{1}{2}(1 - \cos \omega)^2 & i\hat{\omega}_- \sin \omega (1 - \cos \omega) & -\hat{\omega}_-^2 \sin^2 \omega \\ -i\hat{\omega}_+ \sin \omega (1 - \cos \omega) & \sin^2 \omega & i\hat{\omega}_- \sin \omega (1 + \cos \omega) \\ -\hat{\omega}_+^2 \sin^2 \omega & -i\hat{\omega}_+ \sin \omega (1 + \cos \omega) & \frac{1}{2}(1 + \cos \omega)^2 \end{bmatrix}$$

$$\text{推论3.3.5. } \sum_{h=1}^{-1} \lambda(\hat{p}, h; 1) \lambda^+(\hat{p}, h; 1) = 1$$

$$\text{推论3.3.6. } \lambda_{\alpha_s}(\hat{p}, -s; 1) \lambda_{\alpha'_s}^+(\hat{p}, -s; 1) = \frac{1}{2} [(-1)^h (2 - |h|) [S_m^+(1) \hat{p} \hat{p}^T S_m(1)]_{A_s A'_s} + h \sigma^k(1)_{A_s A'_s} \hat{p}_k + |h| \delta_{A_s A'_s}]$$

### 3.4 螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数的正交性与完备性

$$\text{推论3.4.1. } \lambda^+(\hat{p}, h; 1) \lambda(\hat{p}, h'; 1) = \delta_{hh'}, \sum_{h=1}^{-1} \lambda(\hat{p}, h; 1) \lambda^+(\hat{p}, h; 1) = 1, \sum_{h=1}^{-1} h \lambda(\hat{p}, h; 1) \lambda^+(\hat{p}, h; 1) = \sigma(1) \cdot \hat{p}$$

### 3.5 螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数导数的性质

$$\text{推论3.5.1. } \lambda(\hat{p}, \frac{1}{2}) = \frac{1}{\sqrt{1+\hat{p}_z}} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_+ \end{bmatrix}, \lambda(-\hat{p}, \frac{1}{2}) = \frac{1}{\sqrt{1+\hat{p}_z}} \begin{bmatrix} -\hat{p}_- \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z) \end{bmatrix}$$

$$\text{引理3.5.1. } \lambda(\hat{p}, 1; 1) = \bar{\Gamma}(1)\lambda(\hat{p}, \frac{1}{2}) \otimes \lambda(\hat{p}, \frac{1}{2})$$

$$\text{定理3.5.1. } \tilde{\partial}_i \lambda(\hat{p}, 1; 1) = -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)} \lambda(\hat{p}, 1; 1) + \frac{1}{p(1+\hat{p}_z)} \begin{bmatrix} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ \frac{1}{\sqrt{2}}[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{bmatrix}$$

$$\begin{aligned} \text{证明: } \tilde{\partial}_i \lambda(\hat{p}, 1; 1) &= \bar{\Gamma}(1)[\tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2}) \otimes \lambda(\hat{p}, \frac{1}{2}) + \lambda(\hat{p}, \frac{1}{2}) \otimes \tilde{\partial}_i \lambda(\hat{p}, \frac{1}{2})] \\ &= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)} \lambda(\hat{p}, 1; 1) + \frac{1}{\sqrt{2p}\sqrt{1+\hat{p}_z}} \bar{\Gamma}(1) \left[ \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix} \otimes \lambda(\hat{p}, \frac{1}{2}) + \lambda(\hat{p}, \frac{1}{2}) \otimes \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix} \right] \\ &= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)} \lambda(\hat{p}, 1; 1) + \frac{\sqrt{2}}{p\sqrt{1+\hat{p}_z}} \bar{\Gamma}(1) \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix} \otimes \lambda(\hat{p}, \frac{1}{2}) \\ &= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)} \lambda(\hat{p}, 1; 1) + \frac{1}{p(1+\hat{p}_z)} \bar{\Gamma}(1) \begin{bmatrix} \delta_{iz} + \hat{p}_i \\ \delta_{ix} + i\delta_{iy} \end{bmatrix} \otimes \begin{bmatrix} 1 + \hat{p}_z \\ \hat{p}_x + i\hat{p}_y \end{bmatrix} \\ &= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)} \lambda(\hat{p}, 1; 1) + \frac{1}{p(1+\hat{p}_z)} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ (\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) \\ (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y) \\ (\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{bmatrix} \\ &= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)} \lambda(\hat{p}, 1; 1) + \frac{1}{p(1+\hat{p}_z)} \begin{bmatrix} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ \frac{1}{\sqrt{2}}[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{bmatrix} \quad \square \end{aligned}$$

$$\text{推论3.5.2. } \lambda^+(\hat{p}, -1; 1)\tilde{\partial}_i \lambda(\hat{p}, 1; 1) = 0$$

$$\begin{aligned} \text{证明: } \lambda^+(\hat{p}, -1; 1)\tilde{\partial}_i \lambda(\hat{p}, 1; 1) &= 0 + \frac{1}{\hat{p}_-} \begin{bmatrix} \frac{1}{2}\hat{p}_+(1-\hat{p}_z) \\ -\hat{p}_+\hat{p}_- \\ \frac{1}{2}\hat{p}_-(1+\hat{p}_z) \end{bmatrix}^T \frac{1}{p(1+\hat{p}_z)} \begin{bmatrix} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ \frac{1}{\sqrt{2}}[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{bmatrix} \\ &= \frac{1}{2p\hat{p}_-(1+\hat{p}_z)} \begin{bmatrix} \hat{p}_+(1-\hat{p}_z) \\ -2\hat{p}_+\hat{p}_- \\ \hat{p}_-(1+\hat{p}_z) \end{bmatrix}^T \begin{bmatrix} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ \frac{1}{\sqrt{2}}[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{bmatrix} \\ &= \frac{1}{2p\hat{p}_-(1+\hat{p}_z)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 2\hat{p}_+\hat{p}_-(\delta_{iz} + \hat{p}_i) \\ -\sqrt{2}\hat{p}_+\hat{p}_-[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ \sqrt{2}\hat{p}_+\hat{p}_-(1 + \hat{p}_z)(\delta_{ix} + i\delta_{iy}) \end{bmatrix} \\ &= \frac{\hat{p}_+}{\sqrt{2p}(1+\hat{p}_z)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} (\hat{p}_x + i\hat{p}_y)(\delta_{iz} + \hat{p}_i) \\ -[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (1 + \hat{p}_z)(\delta_{ix} + i\delta_{iy}) \end{bmatrix} \\ &= 0 \quad \square \end{aligned}$$

$$\text{推论3.5.3. } \lambda^+(\hat{p}, 1; 1)\tilde{\partial}_k \lambda(\hat{p}, 1; 1) = \frac{-i\hat{p}_y\delta_{kx} + i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)}$$

$$\begin{aligned} \text{证明: } \lambda^+(\hat{p}, 1; 1)\tilde{\partial}_i \lambda(\hat{p}, 1; 1) &= -\frac{\delta_{iz} + \hat{p}_i(2+\hat{p}_z)}{p(1+\hat{p}_z)} + \frac{1}{\hat{p}_+} \begin{bmatrix} \frac{1}{2}\hat{p}_+(1+\hat{p}_z) \\ \hat{p}_+\hat{p}_- \\ \frac{1}{2}\hat{p}_-(1-\hat{p}_z) \end{bmatrix}^T \frac{1}{p(1+\hat{p}_z)} \begin{bmatrix} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ \frac{1}{\sqrt{2}}[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= -\frac{\delta_{iz} + \hat{p}_i(2 + \hat{p}_z)}{p(1 + \hat{p}_z)} + \frac{1}{p\hat{p}_+(1 + \hat{p}_z)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2}\hat{p}_+(1 + \hat{p}_z)(\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ \frac{1}{\sqrt{2}}\hat{p}_+\hat{p}_-[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ \frac{1}{2}\hat{p}_-(1 - \hat{p}_z)(\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{bmatrix} \\
&= -\frac{\delta_{iz} + \hat{p}_i(2 + \hat{p}_z)}{p(1 + \hat{p}_z)} + \frac{1}{2p(1 + \hat{p}_z)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z)^2 \\ (\hat{p}_x - i\hat{p}_y)[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (\hat{p}_x - i\hat{p}_y)(1 - \hat{p}_z)(\delta_{ix} + i\delta_{iy}) \end{bmatrix} \\
&= -\frac{\delta_{iz} + \hat{p}_i(2 + \hat{p}_z)}{p(1 + \hat{p}_z)} + \frac{1}{2p(1 + \hat{p}_z)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z)^2 \\ (\hat{p}_x - i\hat{p}_y)(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(1 - \hat{p}_z^2) \\ (\hat{p}_x - i\hat{p}_y)(1 - \hat{p}_z)(\delta_{ix} + i\delta_{iy}) \end{bmatrix} \\
&= -\frac{\delta_{iz} + \hat{p}_i(2 + \hat{p}_z)}{p(1 + \hat{p}_z)} + \frac{1}{p(1 + \hat{p}_z)} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ 0 \\ (\hat{p}_x - i\hat{p}_y)(\delta_{ix} + i\delta_{iy}) \end{bmatrix} \\
&= \frac{(\hat{p}_x - i\hat{p}_y)(\delta_{ix} + i\delta_{iy}) + \delta_{iz}\hat{p}_z - \hat{p}_i}{p(1 + \hat{p}_z)} = \frac{-i\hat{p}_y\delta_{ix} + i\hat{p}_x\delta_{iy}}{p(1 + \hat{p}_z)} \quad \square
\end{aligned}$$

### 3.6 螺旋度 $\sigma(1) \cdot \hat{p}$ 本征函数导数性质的小结

推论3.6.1.

$$\left\{ \begin{aligned}
\lambda^+(\hat{p}, 1; 1)\tilde{\partial}_k\lambda(\hat{p}, 1; 1) &= \frac{-i\hat{p}_y\delta_{kx} + i\hat{p}_x\delta_{ky}}{p(1 + \hat{p}_z)} \\
\lambda^+(\hat{p}, 1; 1)\tilde{\partial}_k\lambda(\hat{p}, 0; 1) &= \frac{\sqrt{2}\hat{p}_-\hat{p}_k + \sqrt{2}\hat{p}_-\delta_{kz} - (1 + \hat{p}_z)(\delta_{kx} - i\delta_{ky})}{\sqrt{2}p(1 + \hat{p}_z)} \\
\lambda^+(\hat{p}, 1; 1)\tilde{\partial}_k\lambda(\hat{p}, -1; 1) &= 0 \\
\lambda^+(\hat{p}, 0; 1)\tilde{\partial}_k\lambda(\hat{p}, 0; 1) &= 0 \\
\lambda^+(\hat{p}, -1; 1)\tilde{\partial}_k\lambda(\hat{p}, 1; 1) &= 0 \\
\lambda^+(\hat{p}, -1; 1)\tilde{\partial}_k\lambda(\hat{p}, 0; 1) &= -\frac{\sqrt{2}\hat{p}_+\hat{p}_k + \sqrt{2}\hat{p}_+\delta_{kz} - (1 + \hat{p}_z)(\delta_{kx} + i\delta_{ky})}{\sqrt{2}p(1 + \hat{p}_z)} \\
\lambda^+(\hat{p}, -1; 1)\tilde{\partial}_k\lambda(\hat{p}, -1; 1) &= -\frac{-i\hat{p}_y\delta_{kx} + i\hat{p}_x\delta_{ky}}{p(1 + \hat{p}_z)}
\end{aligned} \right.$$

## 4 螺旋度 $\gamma \cdot \hat{p}$ 本征函数的分析

### 4.1 螺旋度 $\gamma \cdot \hat{p}$ 本征函数

推论4.1.1.  $S_m(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & -i \\ -1 & 0 & -1 \\ 0 & -i\sqrt{2} & 0 \end{bmatrix}$ ,  $S_m^+(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 \\ 0 & 0 & i\sqrt{2} \\ i & -1 & 0 \end{bmatrix}$ ,  $S_m(1)S_m^+(1) = S_m^+(1)S_m(1) = I_3$

推论4.1.2.  $\left\{ \begin{aligned}
\lambda(\hat{p}, 1; 1) &= e^{i\vec{\omega} \cdot \sigma(1)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 + \cos\omega) \\ i\hat{\omega}_+ \sin\omega \\ -\hat{\omega}_+^2(1 - \cos\omega) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 + \hat{p}_z) \\ \hat{p}_+ \\ \hat{p}_+^2/(1 + \hat{p}_z) \end{bmatrix} = \frac{1}{\hat{p}_-} \begin{bmatrix} \frac{1}{2}\hat{p}_-(1 + \hat{p}_z) \\ \hat{p}_+\hat{p}_- \\ \frac{1}{2}\hat{p}_+(1 - \hat{p}_z) \end{bmatrix}, \lambda(-\hat{p}, 1; 1) = \frac{\hat{p}_+}{\hat{p}_-} \lambda(\hat{p}, -1; 1) \\
\lambda(\hat{p}, 0; 1) &= e^{i\vec{\omega} \cdot \sigma(1)} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} i\hat{\omega}_- \sin\omega \\ \cos\omega \\ i\hat{\omega}_+ \sin\omega \end{bmatrix} = \begin{bmatrix} -\hat{p}_- \\ \hat{p}_z \\ \hat{p}_+ \end{bmatrix}, \lambda(-\hat{p}, 0; 1) = -\lambda(\hat{p}, 0; 1) \\
\lambda(\hat{p}, -1; 1) &= e^{i\vec{\omega} \cdot \sigma(1)} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\hat{\omega}_-^2(1 - \cos\omega) \\ i\hat{\omega}_- \sin\omega \\ \frac{1}{2}(1 + \cos\omega) \end{bmatrix} = \begin{bmatrix} \hat{p}_-^2/(1 + \hat{p}_z) \\ -\hat{p}_- \\ \frac{1}{2}(1 + \hat{p}_z) \end{bmatrix} = \frac{1}{\hat{p}_+} \begin{bmatrix} \frac{1}{2}\hat{p}_-(1 - \hat{p}_z) \\ -\hat{p}_+\hat{p}_- \\ \frac{1}{2}\hat{p}_+(1 + \hat{p}_z) \end{bmatrix}, \lambda(-\hat{p}, -1; 1) = \frac{\hat{p}_-}{\hat{p}_+} \lambda(\hat{p}, 1; 1)
\end{aligned} \right.$

推论4.1.3.  $\gamma \cdot \hat{p}\lambda_m(\hat{p}, h; 1) = h\lambda_m(\hat{p}, h; 1)$ ,  $\lambda_m(\hat{p}, h; 1) = S_m(1)\lambda(\hat{p}, h; 1)$ ,  $h = -1, 0, 1$

推论4.1.4.  $\left\{ \begin{aligned}
\lambda_m(\hat{p}, 1; 1) &= S_m(1)\lambda(\hat{p}, 1; 1) = e^{i\vec{\omega} \cdot \gamma} \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2\hat{p}_-} \begin{bmatrix} i(\hat{p}_x\hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ -2i(\hat{p}_+\hat{p}_-) \end{bmatrix}, \lambda_m(-\hat{p}, 1; 1) = \frac{\hat{p}_+}{\hat{p}_-} \lambda_m(\hat{p}, -1; 1) \\
\lambda_m(\hat{p}, 0; 1) &= S_m(1)\lambda(\hat{p}, 0; 1) = e^{i\vec{\omega} \cdot \gamma} \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = -i \begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{bmatrix} = -i\hat{p}, \lambda_m(-\hat{p}, 0; 1) = -\lambda_m(\hat{p}, 0; 1) \\
\lambda_m(\hat{p}, -1; 1) &= S_m(1)\lambda(\hat{p}, -1; 1) = e^{i\vec{\omega} \cdot \gamma} \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2\hat{p}_+} \begin{bmatrix} -i(\hat{p}_x\hat{p}_z + i\hat{p}_y) \\ -1(\hat{p}_x + i\hat{p}_y\hat{p}_z) \\ 2i(\hat{p}_+\hat{p}_-) \end{bmatrix}, \lambda_m(-\hat{p}, -1; 1) = \frac{\hat{p}_-}{\hat{p}_+} \lambda_m(\hat{p}, 1; 1)
\end{aligned} \right.$

推论4.1.5.  $\gamma \cdot \hat{p} = e^{i\vec{\omega} \cdot \gamma} \gamma_z e^{-i\vec{\omega} \cdot \gamma}$

引理4.1.1.  $\lambda_m^+(-\hat{p}, 1; 1) \begin{bmatrix} -\hat{p}_z \\ -i \\ \hat{p}_x \end{bmatrix} = 0$ ,  $\lambda_m^+(-\hat{p}, 1; 1) \begin{bmatrix} i \\ -\hat{p}_z \\ \hat{p}_y \end{bmatrix} = 0$ ,  $\lambda_m^+(-\hat{p}, -1; 1) \begin{bmatrix} -\hat{p}_z \\ i \\ \hat{p}_x \end{bmatrix} = 0$ ,  $\lambda_m^+(-\hat{p}, -1; 1) \begin{bmatrix} -i \\ -\hat{p}_z \\ \hat{p}_y \end{bmatrix} = 0$

$$\text{推论4.1.6.} \begin{cases} \lambda_m(\hat{p}, 1; 1) = \frac{1}{2\hat{p}_-} \left\{ -i\hat{p}_x \begin{bmatrix} -\hat{p}_z \\ -i \\ \hat{p}_x \end{bmatrix} - i\hat{p}_y \begin{bmatrix} i \\ -\hat{p}_z \\ \hat{p}_y \end{bmatrix} \right\} \\ \lambda_m(\hat{p}, 0; 1) = -i \begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{bmatrix} = -i\hat{p} \\ \lambda_m(\hat{p}, -1; 1) = \frac{1}{2\hat{p}_+} \left\{ i\hat{p}_x \begin{bmatrix} -\hat{p}_z \\ i \\ \hat{p}_x \end{bmatrix} + i\hat{p}_y \begin{bmatrix} -i \\ -\hat{p}_z \\ \hat{p}_y \end{bmatrix} \right\} \end{cases}$$

$$\text{推论4.1.7.} \begin{cases} \lambda_m \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 1; 1 \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ -1 \\ 0 \end{bmatrix} \\ \lambda_m \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 0; 1 \right) = \begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix} \\ \lambda_m \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, -1; 1 \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ -1 \\ 0 \end{bmatrix} \end{cases}$$

## 4.2 螺旋度 $\gamma \cdot \hat{p}$ 本征函数的基本性质

$$\text{推论4.2.1.} \lambda_m(\hat{p}, -1; 1) = \lambda_m^*(\hat{p}, 1; 1), \lambda_m(\hat{p}, 0; 1) = -\lambda_m^*(\hat{p}, 0; 1), \lambda_m(\hat{p}, 1; 1) = \lambda_m^*(\hat{p}, -1; 1)$$

推论4.2.2.

$$\begin{cases} \lambda_m(\hat{p}, -1; 1) \times \lambda_m(\hat{p}, 0; 1) = -\lambda_m(\hat{p}, -1; 1), \lambda_m(\hat{p}, 0; 1) \times \lambda_m(\hat{p}, 1; 1) = -\lambda_m(\hat{p}, 1; 1), \lambda_m(\hat{p}, 1; 1) \times \lambda_m(\hat{p}, -1; 1) = \lambda_m(\hat{p}, 0; 1) \\ \lambda_m(\hat{p}, \varsigma; 1) \cdot \lambda_m(\hat{p}, \varsigma; 1) = 0, \lambda_m(\hat{p}, 0; 1) \cdot \lambda_m(\hat{p}, \varsigma; 1) = 0, \lambda_m(\hat{p}, h; 1) \times \lambda_m(\hat{p}, h; 1) = 0 \\ \lambda_m(\hat{p}, 0; 1) \cdot \lambda_m(\hat{p}, 0; 1) = -1, \lambda_m(\hat{p}, \varsigma; 1) \cdot \lambda_m(\hat{p}, -\varsigma; 1) = 1 \end{cases}$$

## 4.3 螺旋度 $\gamma \cdot \hat{p}$ 本征函数的正交性与完备性

$$\text{推论4.3.1.} \lambda_m^+(\hat{p}, h)\lambda_m(\hat{p}, h') = \delta_{hh'}, \sum_{h=1}^{-1} \lambda_m(\hat{p}, h)\lambda_m^+(\hat{p}, h) = 1, \sum_{h=1}^{-1} h\lambda_m(\hat{p}, h)\lambda_m^+(\hat{p}, h) = \gamma \cdot \hat{p}$$

## 4.4 螺旋度 $\gamma \cdot \hat{p}$ 本征函数的复杂性质

$$\text{推论4.4.1.} \begin{cases} \gamma\lambda_m(\hat{p}, 1; 1) = \frac{1}{2\hat{p}_-} \gamma \begin{bmatrix} i(\hat{p}_x\hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ -2i(\hat{p}_+ \hat{p}_-) \end{bmatrix} = \frac{1}{2\hat{p}_-} \left\{ \begin{bmatrix} 0 \\ -2(\hat{p}_+ \hat{p}_-) \\ -i(\hat{p}_x - i\hat{p}_y\hat{p}_z) \end{bmatrix}, \begin{bmatrix} 2(\hat{p}_+ \hat{p}_-) \\ 0 \\ (\hat{p}_x\hat{p}_z - i\hat{p}_y) \end{bmatrix}, \begin{bmatrix} i(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ -1(\hat{p}_x\hat{p}_z - i\hat{p}_y) \\ 0 \end{bmatrix} \right\} \\ \gamma\lambda_m(\hat{p}, 0; 1) = -i\gamma \begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{bmatrix} = \left\{ \begin{bmatrix} 0 \\ \hat{p}_z \\ \hat{p}_y \end{bmatrix}, \begin{bmatrix} \hat{p}_z \\ 0 \\ -\hat{p}_x \end{bmatrix}, \begin{bmatrix} -\hat{p}_y \\ \hat{p}_x \\ 0 \end{bmatrix} \right\} \\ \gamma\lambda_m(\hat{p}, -1; 1) = \frac{1}{2\hat{p}_+} \gamma \begin{bmatrix} -i(\hat{p}_x\hat{p}_z + i\hat{p}_y) \\ -1(\hat{p}_x + i\hat{p}_y\hat{p}_z) \\ 2i(\hat{p}_+ \hat{p}_-) \end{bmatrix} = \frac{1}{2\hat{p}_+} \left\{ \begin{bmatrix} 0 \\ 2(\hat{p}_+ \hat{p}_-) \\ -i(\hat{p}_x + i\hat{p}_y\hat{p}_z) \end{bmatrix}, \begin{bmatrix} -2(\hat{p}_+ \hat{p}_-) \\ 0 \\ -(\hat{p}_x\hat{p}_z + i\hat{p}_y) \end{bmatrix}, \begin{bmatrix} i(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ 1(\hat{p}_x\hat{p}_z + i\hat{p}_y) \\ 0 \end{bmatrix} \right\} \end{cases}$$

$$\text{推论4.4.2.} \begin{cases} \lambda_m^+(\hat{p}, 1; 1)\gamma\lambda_m(\hat{p}, 1; 1) = \{\hat{p}_x, \hat{p}_y, \hat{p}_z\} = 1 \cdot \hat{p} & \begin{cases} \lambda_m^+(-\hat{p}, 1; 1)\gamma\lambda_m(\hat{p}, 1; 1) = 0 \\ \lambda_m^+(-\hat{p}, 0; 1)\gamma\lambda_m(\hat{p}, 0; 1) = 0 \\ \lambda_m^+(-\hat{p}, -1; 1)\gamma\lambda_m(\hat{p}, -1; 1) = 0 \end{cases} \\ \lambda_m^+(\hat{p}, 0; 1)\gamma\lambda_m(\hat{p}, 0; 1) = \{0, 0, 0\} = 0 \cdot \hat{p} \\ \lambda_m^+(\hat{p}, -1; 1)\gamma\lambda_m(\hat{p}, -1; 1) = \{-\hat{p}_x, -\hat{p}_y, -\hat{p}_z\} = -1 \cdot \hat{p} \end{cases}$$

$$\text{推论4.4.3.} \lambda_m^+(\hat{p}, h)\gamma\lambda_m(\hat{p}, h) = h\{\hat{p}_x, \hat{p}_y, \hat{p}_z\} = h\hat{p}, \lambda_m^+(-\hat{p}, h)\gamma\lambda_m(\hat{p}, h) = 0, \lambda_m^+(\hat{p}, -h)\gamma\lambda_m(\hat{p}, h) = 0$$

$$\text{推论4.4.4.} \sigma_{\alpha\alpha'}^{ab} p_a p_b = -2|\vec{p}|^2 \lambda_{m\alpha\alpha'}(\hat{p}, -\varsigma; 1) \lambda_{m\alpha'\alpha}^+(\hat{p}, -\varsigma; 1)$$

证明:  $\sigma_{\alpha\alpha'}^{ab} p_a p_b$

$$\begin{aligned} &= p_{\alpha\alpha'} p_{\alpha'} + \varsigma \gamma^k_{\alpha\alpha'} p_k |\vec{p}| - \delta_{\alpha\alpha'} |\vec{p}|^2 \\ &= p_{\alpha\alpha'} p_{\alpha'} + \varsigma |\vec{p}| \gamma^k_{\alpha\alpha'} p_k \delta_{\beta\alpha'} - \delta_{\alpha\alpha'} |\vec{p}|^2 \\ &= \lambda_{m\alpha\alpha'}(\hat{p}, 0; 1) \lambda_{m\alpha'\alpha}^+(\hat{p}, 0; 1) |\vec{p}|^2 + \varsigma |\vec{p}| \gamma^k_{\alpha\alpha'} p_k \sum_{h=1}^{-1} \lambda_{m\beta\alpha'}(\hat{p}, h) \lambda_{m\alpha'\alpha}^+(\hat{p}, h) - \delta_{\alpha\alpha'} |\vec{p}|^2 \\ &= \lambda_{m\alpha\alpha'}(\hat{p}, 0; 1) \lambda_{m\alpha'\alpha}^+(\hat{p}, 0; 1) |\vec{p}|^2 + \varsigma |\vec{p}| [\varsigma |\vec{p}| \lambda_{m\beta\alpha'}(\hat{p}, \varsigma; 1) \lambda_{m\alpha'\alpha}^+(\hat{p}, \varsigma; 1) - \varsigma |\vec{p}| \lambda_{m\beta\alpha'}(\hat{p}, -\varsigma; 1) \lambda_{m\alpha'\alpha}^+(\hat{p}, -\varsigma; 1)] - \delta_{\alpha\alpha'} |\vec{p}|^2 \\ &= |\vec{p}|^2 \sum_{h=1}^{-1} \lambda_{m\alpha\alpha'}(\hat{p}, h) \lambda_{m\alpha'\alpha}^+(\hat{p}, h) - \delta_{\alpha\alpha'} |\vec{p}|^2 - 2|\vec{p}|^2 \lambda_{m\alpha\alpha'}(\hat{p}, -\varsigma; 1) \lambda_{m\alpha'\alpha}^+(\hat{p}, -\varsigma; 1) \\ &= -2|\vec{p}|^2 \lambda_{m\alpha\alpha'}(\hat{p}, -\varsigma; 1) \lambda_{m\alpha'\alpha}^+(\hat{p}, -\varsigma; 1) \end{aligned}$$

□

$$\text{推论4.4.5.} \lambda_{m\alpha\alpha'}(\hat{p}, h) \lambda_{m\alpha'\alpha}^+(\hat{p}, h) = \frac{1}{2} [(-1)^h (2 - |h|) \hat{p}_{\alpha\alpha'} \hat{p}_{\alpha'} + h \gamma^k_{\alpha\alpha'} \hat{p}_k + |h| \delta_{\alpha\alpha'}]$$



$$\text{推论4.4.6.} \begin{cases} \lambda_{m\alpha_\zeta}(\hat{p}, 1; 1)\lambda_{m\alpha'_\zeta}^+(\hat{p}, 1; 1) = \frac{1}{2}(-\hat{p}_{\alpha_\zeta}\hat{p}_{\alpha'_\zeta} + \gamma^k \alpha_\zeta \alpha'_\zeta \hat{p}_k + \delta_{\alpha_\zeta \alpha'_\zeta}) \\ \lambda_{m\alpha_\zeta}(\hat{p}, 0; 1)\lambda_{m\alpha'_\zeta}^+(\hat{p}, 0; 1) = \hat{p}_{\alpha_\zeta}\hat{p}_{\alpha'_\zeta} \\ \lambda_{m\alpha_\zeta}(\hat{p}, -1; 1)\lambda_{m\alpha'_\zeta}^+(\hat{p}, -1; 1) = \frac{1}{2}(-\hat{p}_{\alpha_\zeta}\hat{p}_{\alpha'_\zeta} - \gamma^k \alpha_\zeta \alpha'_\zeta \hat{p}_k + \delta_{\alpha_\zeta \alpha'_\zeta}) \end{cases}$$

#### 4.5 螺旋度 $\gamma \cdot \hat{p}$ 本征函数的导数性质一

$$\text{引理4.5.1.} \lambda_m^+(-\hat{p}, 1; 1) \begin{bmatrix} -\hat{p}_z \\ -i \\ \hat{p}_x \end{bmatrix} = 0, \lambda_m^+(-\hat{p}, 1; 1) \begin{bmatrix} i \\ -\hat{p}_z \\ \hat{p}_y \end{bmatrix} = 0, \lambda_m^+(-\hat{p}, -1; 1) \begin{bmatrix} -\hat{p}_z \\ i \\ \hat{p}_x \end{bmatrix} = 0, \lambda_m^+(-\hat{p}, -1; 1) \begin{bmatrix} -i \\ -\hat{p}_z \\ \hat{p}_y \end{bmatrix} = 0$$

$$\text{引理4.5.2.} \begin{cases} \tilde{\partial}_k \lambda_m(\hat{p}, 1; 1) = \frac{1}{2p_-} \begin{bmatrix} -i\hat{p}_x(\hat{p}_k\hat{p}_z - \delta_{kz}) + i\delta_{kx}(\hat{p}_z - \frac{\hat{p}_x\hat{p}_z - i\hat{p}_y}{\hat{p}_x - i\hat{p}_y}) + \delta_{ky}(1 - \frac{\hat{p}_x\hat{p}_z - i\hat{p}_y}{\hat{p}_x - i\hat{p}_y}) \\ -i\hat{p}_y(\hat{p}_k\hat{p}_z - \delta_{kz}) - \delta_{ky}(1 - \frac{\hat{p}_x\hat{p}_z - i\hat{p}_y}{\hat{p}_x - i\hat{p}_y}) + i\delta_{ky}(\hat{p}_z - \frac{\hat{p}_x\hat{p}_z - i\hat{p}_y}{\hat{p}_x - i\hat{p}_y}) \\ i\hat{p}_k(\hat{p}_x^2 + \hat{p}_y^2) - i(\hat{p}_x - i\hat{p}_y)(\delta_{kx} + i\delta_{ky}) \end{bmatrix} \\ \tilde{\partial}_k \lambda_m(\hat{p}, -1; 1) = \frac{1}{2p_+} \begin{bmatrix} i\hat{p}_x(\hat{p}_k\hat{p}_z - \delta_{kz}) - i\delta_{kx}(\hat{p}_z - \frac{\hat{p}_x\hat{p}_z + i\hat{p}_y}{\hat{p}_x + i\hat{p}_y}) + \delta_{ky}(1 - \frac{\hat{p}_x\hat{p}_z + i\hat{p}_y}{\hat{p}_x + i\hat{p}_y}) \\ +i\hat{p}_y(\hat{p}_k\hat{p}_z - \delta_{kz}) - \delta_{ky}(1 - \frac{\hat{p}_x\hat{p}_z + i\hat{p}_y}{\hat{p}_x + i\hat{p}_y}) - i\delta_{ky}(\hat{p}_z - \frac{\hat{p}_x\hat{p}_z + i\hat{p}_y}{\hat{p}_x + i\hat{p}_y}) \\ -i\hat{p}_k(\hat{p}_x^2 + \hat{p}_y^2) + i(\hat{p}_x + i\hat{p}_y)(\delta_{kx} - i\delta_{ky}) \end{bmatrix} \end{cases}$$

$$\begin{aligned} \text{证明: } \tilde{\partial}_k \lambda_m(\hat{p}, 1; 1) &= \tilde{\partial}_k \frac{1}{2pp_-} \begin{bmatrix} i(p_x p_z - i p p_y) \\ -1(p p_x - i p_y p_z) \\ -2i(p_+ p_-) \end{bmatrix} \\ &= (2pp_- \tilde{\partial}_k \frac{1}{2pp_-}) \frac{1}{2pp_-} \begin{bmatrix} i(p_x p_z - i p p_y) \\ -1(p p_x - i p_y p_z) \\ -2i(p_+ p_-) \end{bmatrix} + \frac{1}{2pp_-} \tilde{\partial}_k \begin{bmatrix} i(p_x p_z - i p p_y) \\ -1(p p_x - i p_y p_z) \\ -2i(p_+ p_-) \end{bmatrix} \\ &= -\frac{1}{2p_-} [2\hat{p}_k \hat{p}_- + \sqrt{2}(\delta_{kx} - i\delta_{ky})] \lambda_m(\hat{p}, 1; 1) + \frac{1}{2p_-} \begin{bmatrix} i(\delta_{kx}\hat{p}_z + \hat{p}_x\delta_{kz} - i\delta_{ky} - i\hat{p}_k\hat{p}_y) \\ -1(\hat{p}_k\hat{p}_x + \delta_{kx} - i\delta_{ky}\hat{p}_z - i\hat{p}_y\delta_{kz}) \\ -2i(\hat{p}_x\delta_{kx} + \hat{p}_y\delta_{ky}) \end{bmatrix} \\ &= -\frac{1}{2p_-} [2\hat{p}_k \hat{p}_- + \sqrt{2}(\delta_{kx} - i\delta_{ky})] \frac{1}{2p_-} \begin{bmatrix} i(\hat{p}_x\hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ -2i(\hat{p}_+ \hat{p}_-) \end{bmatrix} + \frac{1}{2p_-} \begin{bmatrix} i(\delta_{kx}\hat{p}_z + \hat{p}_x\delta_{kz} - i\delta_{ky} - i\hat{p}_k\hat{p}_y) \\ -1(\hat{p}_k\hat{p}_x + \delta_{kx} - i\delta_{ky}\hat{p}_z - i\hat{p}_y\delta_{kz}) \\ -2i(\hat{p}_x\delta_{kx} + \hat{p}_y\delta_{ky}) \end{bmatrix} \\ &= -\frac{1}{2p_-} [\hat{p}_k + \frac{1}{\sqrt{2}p_-}(\delta_{kx} - i\delta_{ky})] \begin{bmatrix} i(\hat{p}_x\hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ -2i(\hat{p}_+ \hat{p}_-) \end{bmatrix} + \frac{1}{2p_-} \begin{bmatrix} i(\delta_{kx}\hat{p}_z + \hat{p}_x\delta_{kz} - i\delta_{ky} - i\hat{p}_k\hat{p}_y) \\ -1(\hat{p}_k\hat{p}_x + \delta_{kx} - i\delta_{ky}\hat{p}_z - i\hat{p}_y\delta_{kz}) \\ -2i(\hat{p}_x\delta_{kx} + \hat{p}_y\delta_{ky}) \end{bmatrix} \\ &= \frac{1}{2p_-} \begin{bmatrix} -i\hat{p}_x(\hat{p}_k\hat{p}_z - \delta_{kz}) + i\delta_{kx}(\hat{p}_z - \frac{\hat{p}_x\hat{p}_z - i\hat{p}_y}{\hat{p}_x - i\hat{p}_y}) + \delta_{ky}(1 - \frac{\hat{p}_x\hat{p}_z - i\hat{p}_y}{\hat{p}_x - i\hat{p}_y}) \\ -i\hat{p}_y(\hat{p}_k\hat{p}_z - \delta_{kz}) - \delta_{ky}(1 - \frac{\hat{p}_x\hat{p}_z - i\hat{p}_y}{\hat{p}_x - i\hat{p}_y}) + i\delta_{ky}(\hat{p}_z - \frac{\hat{p}_x\hat{p}_z - i\hat{p}_y}{\hat{p}_x - i\hat{p}_y}) \\ i\hat{p}_k(\hat{p}_x^2 + \hat{p}_y^2) - i(\hat{p}_x - i\hat{p}_y)(\delta_{kx} + i\delta_{ky}) \end{bmatrix} \quad \square \end{aligned}$$

$$\text{推论4.5.1.} \begin{cases} \tilde{\partial}_x \lambda_m(\hat{p}, -\varsigma; 1) = \frac{1}{2p_{+\varsigma}} \{ i\varsigma \hat{p}_x \hat{p}_z \begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{bmatrix} + i\varsigma \begin{bmatrix} -\hat{p}_z \\ i\varsigma \\ \hat{p}_x \end{bmatrix} - \sqrt{2} \lambda_m(\hat{p}, -\varsigma; 1) \} \\ \tilde{\partial}_y \lambda_m(\hat{p}, -\varsigma; 1) = \frac{1}{2p_{+\varsigma}} \{ i\varsigma \hat{p}_y \hat{p}_z \begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{bmatrix} + i\varsigma \begin{bmatrix} -i\varsigma \\ -\hat{p}_z \\ \hat{p}_y \end{bmatrix} - i\varsigma \sqrt{2} \lambda_m(\hat{p}, -\varsigma; 1) \} \\ \tilde{\partial}_z \lambda_m(\hat{p}, -\varsigma; 1) = -i\varsigma p_{-\varsigma} \begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{bmatrix} \end{cases}$$

$$\text{推论4.5.2.} \lambda_m^+(\hat{p}, -\varsigma; 1) = \frac{1}{2\hat{p}_{-\varsigma}} \begin{bmatrix} i\varsigma(\hat{p}_x\hat{p}_z - i\varsigma\hat{p}_y) \\ -1(\hat{p}_x - i\varsigma\hat{p}_y\hat{p}_z) \\ -2i\varsigma(\hat{p}_+ \hat{p}_-) \end{bmatrix}, \lambda_m(\hat{p}, -\varsigma; 1) = \frac{1}{2\hat{p}_{+\varsigma}} \begin{bmatrix} -i\varsigma(\hat{p}_x\hat{p}_z + i\varsigma\hat{p}_y) \\ -1(\hat{p}_x + i\varsigma\hat{p}_y\hat{p}_z) \\ 2i\varsigma(\hat{p}_+ \hat{p}_-) \end{bmatrix}$$

$$\text{推论4.5.3.} \lambda_m^+(\hat{p}, -\varsigma; 1) \tilde{\partial}_x \lambda_m(\hat{p}, -\varsigma; 1) = \frac{i\varsigma\hat{p}_y}{p(1+\hat{p}_z)}, \lambda_m^+(\hat{p}, -\varsigma; 1) \tilde{\partial}_y \lambda_m(\hat{p}, -\varsigma; 1) = \frac{-i\varsigma\hat{p}_x}{p(1+\hat{p}_z)}, \lambda_m^+(\hat{p}, -\varsigma; 1) \tilde{\partial}_z \lambda_m(\hat{p}, -\varsigma; 1) = 0$$

$$\lambda_m^+(\hat{p}, -\varsigma; 1) \hat{p} \cdot \tilde{\nabla} \lambda_m(\hat{p}, -\varsigma; 1) = 0$$

$$\text{推论4.5.4.} \begin{cases} \lambda_m^+(\hat{p}, 1; 1) \tilde{\partial}_k \lambda_m(\hat{p}, 1; 1) = \frac{-i\hat{p}_y\delta_{kx} + i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)} \\ \lambda_m^+(\hat{p}, -1; 1) \tilde{\partial}_k \lambda_m(\hat{p}, -1; 1) = -\frac{-i\hat{p}_y\delta_{kx} + i\hat{p}_x\delta_{ky}}{p(1+\hat{p}_z)} \\ \lambda_m^+(\hat{p}, 0; 1) \tilde{\partial}_k \lambda_m(\hat{p}, 0; 1) = 0 \\ \lambda_m^+(\hat{p}, -1; 1) \tilde{\partial}_k \lambda_m(\hat{p}, 1; 1) = 0 \\ \lambda_m^+(\hat{p}, 1; 1) \tilde{\partial}_k \lambda_m(\hat{p}, -1; 1) = 0 \end{cases}$$

## 4.6 螺旋度 $\gamma \cdot \hat{p}$ 本征函数的导数性质二?

$$\text{推论4.6.1. } \begin{cases} (\gamma_y \tilde{\partial}_z - \gamma_z \tilde{\partial}_y) \lambda_m(\hat{p}, -\varsigma; 1) = \frac{\varsigma}{2p+\varsigma} \left\{ \hat{p}_x \hat{p}_z \hat{p} - \begin{bmatrix} -\hat{p}_z \\ i\varsigma \\ \hat{p}_x \end{bmatrix} + i\sqrt{2} \gamma_z \lambda_m(\hat{p}, -\varsigma; 1) \right\} \\ (\gamma_z \tilde{\partial}_x - \gamma_x \tilde{\partial}_z) \lambda_m(\hat{p}, -\varsigma; 1) = \frac{\varsigma}{2p+\varsigma} \left\{ \hat{p}_y \hat{p}_z \hat{p} - \begin{bmatrix} -i\varsigma \\ -\hat{p}_z \\ \hat{p}_y \end{bmatrix} - \varsigma\sqrt{2} \gamma_z \lambda_m(\hat{p}, -\varsigma; 1) \right\} \\ (\gamma_x \tilde{\partial}_y - \gamma_y \tilde{\partial}_x) \lambda_m(\hat{p}, -\varsigma; 1) = \frac{\varsigma}{2p+\varsigma} \left\{ \hat{p}_z \hat{p}_x \hat{p} + \begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{bmatrix} - i\sqrt{2} (\gamma_x + i\varsigma \gamma_y) \lambda_m(\hat{p}, -\varsigma; 1) \right\} \end{cases}$$

$$\text{推论4.6.2. } \begin{cases} \lambda_m^+(\hat{p}, -\varsigma; 1) (\gamma_x \tilde{\partial}_y - \gamma_y \tilde{\partial}_x) \lambda_m(\hat{p}, -\varsigma; 1) = \frac{i\hat{p}_z}{p(1+\hat{p}_z)} + \frac{i}{p(1+\hat{p}_z)} \\ \lambda_m^+(\hat{p}, -\varsigma; 1) (\gamma_y \tilde{\partial}_z - \gamma_z \tilde{\partial}_y) \lambda_m(\hat{p}, -\varsigma; 1) = \frac{i\hat{p}_x}{p(1+\hat{p}_z)} \\ \lambda_m^+(\hat{p}, -\varsigma; 1) (\gamma_z \tilde{\partial}_x - \gamma_x \tilde{\partial}_z) \lambda_m(-\hat{p}, -\varsigma; 1) = \frac{i\hat{p}_y}{p(1+\hat{p}_z)} \end{cases}$$

$$\text{推论4.6.3. } \lambda_m^+(-\hat{p}, -\varsigma; 1) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma; 1) = 0, \lambda_m^+(\hat{p}, -\varsigma; 1) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(-\hat{p}, -\varsigma; 1) = 0$$

$$\text{推论4.6.4. } \left\{ \tilde{\partial}_k \lambda_m(\hat{p}, 1; 1) \right\}_{\hat{p}_z \rightarrow 1} = 0$$

## 5 螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数的分析

### 5.1 自旋-2洛伦兹变换 $e^{i\omega \cdot \sigma(2)}$

$$\text{推论5.1.1. } \sigma(2) = \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} & 0 \end{bmatrix}, \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & -\sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \right)$$

$$\text{推论5.1.2. } \sigma(2) \cdot \hat{p} \lambda(\hat{p}, h) = h \lambda(\hat{p}, h), h = -2, -1, 0, 1, 2$$

$$\text{推论5.1.3. } e^{\vec{\vartheta} \cdot \vec{\Omega}(2)} = 1 + \left( \frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}} \right) \left( 1 - \frac{2}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2} \right) [\vec{\vartheta} \cdot \vec{\Omega}(2)] + 2 \left( \frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}} \right)^2 \left( 1 - \frac{1}{3} \sinh^2 \frac{1}{2} \sqrt{\vec{\vartheta}^2} \right) [\vec{\vartheta} \cdot \vec{\Omega}(2)]^2 + \frac{2}{3} \left( \frac{\sinh \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}} \right) \left( \frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}} \right)^2 [\vec{\vartheta} \cdot \vec{\Omega}(2)]^3 + \frac{2}{3} \left( \frac{\sinh \frac{1}{2} \sqrt{\vec{\vartheta}^2}}{\sqrt{\vec{\vartheta}^2}} \right)^4 [\vec{\vartheta} \cdot \vec{\Omega}(2)]^4$$

$$\text{推论5.1.4. } e^{i\omega \cdot \sigma(2)} = 1 + i \sin \omega \left( 1 + \frac{2}{3} \sin^2 \frac{\omega}{2} \right) [\hat{\omega} \cdot \sigma(2)] - 2 \sin^2 \frac{\omega}{2} \left( 1 + \frac{1}{3} \sin^2 \frac{\omega}{2} \right) [\hat{\omega} \cdot \sigma(2)]^2 - \frac{2}{3} i \sin \omega \sin^2 \frac{\omega}{2} [\hat{\omega} \cdot \sigma(2)]^3 + \frac{2}{3} \sin^4 \frac{\omega}{2} [\hat{\omega} \cdot \sigma(2)]^4$$

推论5.1.5.

$$e^{i\omega \cdot \sigma(2)} = 1 + i \sin \omega [\hat{\omega} \cdot \sigma(2)] - 2 \sin^2 \frac{\omega}{2} [\hat{\omega} \cdot \sigma(2)]^2 + \frac{2}{3} \sin^2 \frac{\omega}{2} [i \sin \omega [\hat{\omega} \cdot \sigma(2)] - \sin^2 \frac{\omega}{2} [\hat{\omega} \cdot \sigma(2)]^2] [1 - [\hat{\omega} \cdot \sigma(2)]^2]$$

推论5.1.6.

$$e^{i\omega \cdot \sigma(1)} = 1 + i \sin \omega [\hat{\omega} \cdot \sigma(1)] - 2 \sin^2 \frac{\omega}{2} [\hat{\omega} \cdot \sigma(1)]^2 + \frac{2}{3} \sin^2 \frac{\omega}{2} [i \sin \omega [\hat{\omega} \cdot \sigma(1)] - \sin^2 \frac{\omega}{2} [\hat{\omega} \cdot \sigma(1)]^2] [1 - [\hat{\omega} \cdot \sigma(1)]^2] = 1 + i \hat{\omega} \cdot \sigma(1) \sin \omega + (1 - \cos \omega) [i \hat{\omega} \cdot \sigma(1)]^2$$

推论5.1.7.

$$e^{i\omega \cdot \sigma} = 1 + i \sin \omega (\hat{\omega} \cdot \sigma) - 2 \sin^2 \frac{\omega}{2} (\hat{\omega} \cdot \sigma)^2 + \frac{2}{3} \sin^2 \frac{\omega}{2} [i \sin \omega (\hat{\omega} \cdot \sigma) - \sin^2 \frac{\omega}{2} (\hat{\omega} \cdot \sigma)^2] [1 - (\hat{\omega} \cdot \sigma)^2] = \cos \omega + i \sin \omega (\hat{\omega} \cdot \sigma)$$

$$\text{推论5.1.8. } e^{i\omega \cdot \sigma(2)} = 1 + i \sin \omega [\hat{\omega} \cdot \sigma(2)] - (1 - \cos \omega) [\hat{\omega} \cdot \sigma(2)]^2 + \frac{1}{6} (1 - \cos \omega) [2 i \sin \omega [\hat{\omega} \cdot \sigma(2)] - (1 - \cos \omega) [\hat{\omega} \cdot \sigma(2)]^2] [1 - [\hat{\omega} \cdot \sigma(2)]^2]$$

$$\text{推论5.1.9. } e^{i\omega \cdot \sigma(2)} = 1 + [i\sigma(2) \times \hat{p}]_z + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z} + \frac{1}{6} [2[i\sigma(2) \times \hat{p}]_z + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z}] [(1-\hat{p}_z) + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z}]$$

$$\text{推论5.1.10. } \sigma(2) \cdot \hat{p} = e^{i\vec{\omega} \cdot \sigma(2)} \sigma_z(2) e^{-i\vec{\omega} \cdot \sigma(2)}$$

### 5.2 螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数的具体求法I

推论5.2.1.

$$[i\sigma(2) \times \hat{p}]_z = \begin{bmatrix} 0 & -\sqrt{2}\hat{p}_- & 0 & 0 & 0 \\ \sqrt{2}\hat{p}_+ & 0 & -\sqrt{3}\hat{p}_- & 0 & 0 \\ 0 & \sqrt{3}\hat{p}_+ & 0 & -\sqrt{3}\hat{p}_- & 0 \\ 0 & 0 & \sqrt{3}\hat{p}_+ & 0 & -\sqrt{2}\hat{p}_- \\ 0 & 0 & 0 & \sqrt{2}\hat{p}_+ & 0 \end{bmatrix}, [i\sigma(2) \times \hat{p}]_z^2 = \begin{bmatrix} -2\hat{p}_+\hat{p}_- & 0 & \sqrt{6}\hat{p}_-^2 & 0 & 0 \\ 0 & -5\hat{p}_+\hat{p}_- & 0 & 3\hat{p}_-^2 & 0 \\ \sqrt{6}\hat{p}_+^2 & 0 & -6\hat{p}_+\hat{p}_- & 0 & \sqrt{6}\hat{p}_-^2 \\ 0 & 3\hat{p}_+^2 & 0 & -5\hat{p}_+\hat{p}_- & 0 \\ 0 & 0 & \sqrt{6}\hat{p}_+^2 & 0 & -2\hat{p}_+\hat{p}_- \end{bmatrix}$$

$$\text{推论5.2.2. } [i\sigma(2) \times \hat{p}]_z^4 = \begin{bmatrix} 10\hat{p}_+^2\hat{p}_-^2 & 0 & -8\sqrt{6}\hat{p}_+\hat{p}_-^3 & 0 & 6\hat{p}_-^4 \\ 0 & 34\hat{p}_+^2\hat{p}_-^2 & 0 & -30\hat{p}_+\hat{p}_-^3 & 0 \\ -8\sqrt{6}\hat{p}_+^3\hat{p}_- & 0 & 48\hat{p}_+^2\hat{p}_-^2 & 0 & -8\sqrt{6}\hat{p}_+\hat{p}_-^3 \\ 0 & -30\hat{p}_+^3\hat{p}_- & 0 & 34\hat{p}_+^2\hat{p}_-^2 & 0 \\ 6\hat{p}_+^4 & 0 & -8\sqrt{6}\hat{p}_+^3\hat{p}_- & 0 & 10\hat{p}_+^2\hat{p}_-^2 \end{bmatrix}$$

$$\text{推论5.2.3. } [i\sigma(2) \times \hat{p}]_z + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z} = \frac{1}{1+\hat{p}_z} \begin{bmatrix} -2\hat{p}_+\hat{p}_- & -\sqrt{2}\hat{p}_-(1+\hat{p}_z) & \sqrt{6}\hat{p}_-^2 & 0 & 0 \\ \sqrt{2}\hat{p}_+(1+\hat{p}_z) & -5\hat{p}_+\hat{p}_- & -\sqrt{3}\hat{p}_-(1+\hat{p}_z) & 3\hat{p}_-^2 & 0 \\ \sqrt{6}\hat{p}_+^2 & \sqrt{3}\hat{p}_+(1+\hat{p}_z) & -6\hat{p}_+\hat{p}_- & -\sqrt{3}\hat{p}_-(1+\hat{p}_z) & \sqrt{6}\hat{p}_-^2 \\ 0 & -3\hat{p}_+^2 & \sqrt{3}\hat{p}_+(1+\hat{p}_z) & -5\hat{p}_+\hat{p}_- & -\sqrt{2}\hat{p}_-(1+\hat{p}_z) \\ 0 & 0 & \sqrt{6}\hat{p}_+^2 & \sqrt{2}\hat{p}_+(1+\hat{p}_z) & -2\hat{p}_+\hat{p}_- \end{bmatrix}$$

推论5.2.4.

$$\frac{1}{6}[2[i\sigma(2) \times \hat{p}]_z + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z}] = \frac{1}{6} \frac{1}{1+\hat{p}_z} \begin{bmatrix} -2\hat{p}_+\hat{p}_- & -2\sqrt{2}\hat{p}_-(1+\hat{p}_z) & \sqrt{6}\hat{p}_-^2 & 0 & 0 \\ 2\sqrt{2}\hat{p}_+(1+\hat{p}_z) & -5\hat{p}_+\hat{p}_- & -2\sqrt{3}\hat{p}_-(1+\hat{p}_z) & -3\hat{p}_-^2 & 0 \\ \sqrt{6}\hat{p}_+^2 & 2\sqrt{3}\hat{p}_+(1+\hat{p}_z) & -6\hat{p}_+\hat{p}_- & -2\sqrt{3}\hat{p}_-(1+\hat{p}_z) & \sqrt{6}\hat{p}_-^2 \\ 0 & 3\hat{p}_+^2 & 2\sqrt{3}\hat{p}_+(1+\hat{p}_z) & -5\hat{p}_+\hat{p}_- & -2\sqrt{2}\hat{p}_-(1+\hat{p}_z) \\ 0 & 0 & \sqrt{6}\hat{p}_+^2 & 2\sqrt{2}\hat{p}_+(1+\hat{p}_z) & -2\hat{p}_+\hat{p}_- \end{bmatrix}$$

$$\text{推论5.2.5. } [(1-\hat{p}_z) + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z}] = \frac{1}{1+\hat{p}_z} \begin{bmatrix} 0 & 0 & \sqrt{6}\hat{p}_-^2 & 0 & 0 \\ 0 & -3\hat{p}_+\hat{p}_- & 0 & 3\hat{p}_-^2 & 0 \\ \sqrt{6}\hat{p}_+^2 & 0 & -4\hat{p}_+\hat{p}_- & 0 & \sqrt{6}\hat{p}_-^2 \\ 0 & 3\hat{p}_+^2 & 0 & -3\hat{p}_+\hat{p}_- & 0 \\ 0 & 0 & \sqrt{6}\hat{p}_+^2 & 0 & 0 \end{bmatrix}$$

$$\text{推论5.2.6. } \frac{1}{6}[2[i\sigma(2) \times \hat{p}]_z + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z}][[(1-\hat{p}_z) + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z}]] \\ = \frac{1}{(1+\hat{p}_z)^2} \begin{bmatrix} \hat{p}_+^2\hat{p}_-^2 & \sqrt{2}\hat{p}_+\hat{p}_-^2(1+\hat{p}_z) & -\sqrt{6}\hat{p}_+\hat{p}_-^3 & -2\sqrt{2}\hat{p}_-^3(1+\hat{p}_z) & \hat{p}_-^4 \\ -\sqrt{2}\hat{p}_+^2\hat{p}_-(1+\hat{p}_z) & 4\hat{p}_+^2\hat{p}_-^2 & 2\sqrt{3}\hat{p}_+\hat{p}_-^2(1+\hat{p}_z) & -4\hat{p}_+\hat{p}_-^2 & -\sqrt{2}\hat{p}_-^3(1+\hat{p}_z) \\ -\sqrt{6}\hat{p}_+^3\hat{p}_- & -2\sqrt{3}\hat{p}_+^2\hat{p}_-(1+\hat{p}_z) & 6\hat{p}_+^2\hat{p}_-^2 & 2\sqrt{3}\hat{p}_+\hat{p}_-^2(1+\hat{p}_z) & -\sqrt{6}\hat{p}_+\hat{p}_-^3 \\ \sqrt{2}\hat{p}_+^3(1+\hat{p}_z) & -4\hat{p}_+^3\hat{p}_- & -2\sqrt{3}\hat{p}_+^2\hat{p}_-(1+\hat{p}_z) & 4\hat{p}_+^2\hat{p}_-^2 & \sqrt{2}\hat{p}_+\hat{p}_-^2(1+\hat{p}_z) \\ \hat{p}_+^4 & \sqrt{2}\hat{p}_+^3(1+\hat{p}_z) & -\sqrt{6}\hat{p}_+^3\hat{p}_- & -\sqrt{2}\hat{p}_+^2\hat{p}_-(1+\hat{p}_z) & \hat{p}_+^2\hat{p}_-^2 \end{bmatrix}$$

$$\text{推论5.2.7. } e^{i\omega \cdot \sigma(2)} \omega_z \stackrel{=} {=} 1 + [i\sigma(2) \times \hat{p}]_z + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z} + \frac{1}{6}[2[i\sigma(2) \times \hat{p}]_z + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z}][[(1-\hat{p}_z) + \frac{[i\sigma(2) \times \hat{p}]_z^2}{1+\hat{p}_z}]]$$

$$= \begin{bmatrix} \frac{1}{4}(1+\hat{p}_z)^2 & -\frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z) & \frac{\sqrt{6}}{2}\hat{p}_-^2 & -\sqrt{2}\hat{p}_-^3/(1+\hat{p}_z) & \hat{p}_-^4/(1+\hat{p}_z)^2 \\ \frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z) & \frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1) & -\sqrt{3}\hat{p}_-\hat{p}_z & \hat{p}_-^2(2\hat{p}_z+1)/(1+\hat{p}_z) & -\sqrt{2}\hat{p}_-^3/(1+\hat{p}_z) \\ \frac{\sqrt{6}}{2}\hat{p}_+^2 & \sqrt{3}\hat{p}_+\hat{p}_z & \frac{1}{2}(3\hat{p}_-^2-1) & -\sqrt{3}\hat{p}_-\hat{p}_z & \frac{\sqrt{6}}{2}\hat{p}_-^2 \\ \sqrt{2}\hat{p}_+^3/(1+\hat{p}_z) & \hat{p}_+^2(2\hat{p}_z+1)/(1+\hat{p}_z) & \sqrt{3}\hat{p}_+\hat{p}_z & \frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1) & -\frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z) \\ \hat{p}_+^4/(1+\hat{p}_z)^2 & \sqrt{2}\hat{p}_+^3/(1+\hat{p}_z) & \frac{\sqrt{6}}{2}\hat{p}_+^2 & \frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z) & \frac{1}{4}(1+\hat{p}_z)^2 \end{bmatrix}$$

$$\text{推论5.2.8. } e^{-i\omega \cdot \sigma(2)} \omega_z \stackrel{=} {=} \begin{bmatrix} \frac{1}{4}(1+\hat{p}_z)^2 & \frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z) & \frac{\sqrt{6}}{2}\hat{p}_-^2 & \sqrt{2}\hat{p}_-^3/(1+\hat{p}_z) & \hat{p}_-^4/(1+\hat{p}_z)^2 \\ -\frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z) & \frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1) & \sqrt{3}\hat{p}_-\hat{p}_z & \hat{p}_-^2(2\hat{p}_z+1)/(1+\hat{p}_z) & \sqrt{2}\hat{p}_-^3/(1+\hat{p}_z) \\ \frac{\sqrt{6}}{2}\hat{p}_+^2 & -\sqrt{3}\hat{p}_+\hat{p}_z & \frac{1}{2}(3\hat{p}_-^2-1) & \sqrt{3}\hat{p}_-\hat{p}_z & \frac{\sqrt{6}}{2}\hat{p}_-^2 \\ -\sqrt{2}\hat{p}_+^3/(1+\hat{p}_z) & \hat{p}_+^2(2\hat{p}_z+1)/(1+\hat{p}_z) & -\sqrt{3}\hat{p}_+\hat{p}_z & \frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1) & \frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z) \\ \hat{p}_+^4/(1+\hat{p}_z)^2 & -\sqrt{2}\hat{p}_+^3/(1+\hat{p}_z) & \frac{\sqrt{6}}{2}\hat{p}_+^2 & -\frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z) & \frac{1}{4}(1+\hat{p}_z)^2 \end{bmatrix}$$

### 5.3 螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数

推论5.3.1.

$$\lambda(\hat{p}, 2; 2) := \begin{bmatrix} \frac{1}{4}(1+\hat{p}_z)^2 \\ \frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z) \\ \frac{\sqrt{6}}{2}\hat{p}_+^2 \\ \sqrt{2}\hat{p}_+^3/(1+\hat{p}_z) \\ \hat{p}_+^4/(1+\hat{p}_z)^2 \end{bmatrix}, \lambda(\hat{p}, -2; 2) := \begin{bmatrix} \hat{p}_-^4/(1+\hat{p}_z)^2 \\ -\sqrt{2}\hat{p}_-^3/(1+\hat{p}_z) \\ \frac{\sqrt{6}}{2}\hat{p}_-^2 \\ -\frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z) \\ \frac{1}{4}(1+\hat{p}_z)^2 \end{bmatrix}, \begin{cases} \lambda(\hat{p}, 2; 2) = \frac{\hat{p}_+^2}{\hat{p}_-^2} \lambda(-\hat{p}, -2) \\ \lambda(-\hat{p}, 2; 2) = \frac{\hat{p}_+^2}{\hat{p}_-^2} \lambda(\hat{p}, -2) \\ \lambda(\hat{p}, -2; 2) = \frac{\hat{p}_-^2}{\hat{p}_+^2} \lambda(-\hat{p}, 2) \\ \lambda(-\hat{p}, -2; 2) = \frac{\hat{p}_-^2}{\hat{p}_+^2} \lambda(\hat{p}, 2) \end{cases}$$

推论5.3.2.

$$\lambda(\hat{p}, 1; 2) := \begin{bmatrix} -\frac{1}{\sqrt{2}}\hat{p}_-(1+\hat{p}_z) \\ \frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1) \\ \sqrt{3}\hat{p}_+\hat{p}_z \\ \hat{p}_+^2(2\hat{p}_z+1)/(1+\hat{p}_z) \\ \sqrt{2}\hat{p}_+^3/(1+\hat{p}_z) \end{bmatrix}, \lambda(\hat{p}, -1; 2) := \begin{bmatrix} -\sqrt{2}\hat{p}_-^3/(1+\hat{p}_z) \\ \hat{p}_-^2(2\hat{p}_z+1)/(1+\hat{p}_z) \\ -\sqrt{3}\hat{p}_-\hat{p}_z \\ \frac{1}{2}(1+\hat{p}_z)(2\hat{p}_z-1) \\ \frac{1}{\sqrt{2}}\hat{p}_+(1+\hat{p}_z) \end{bmatrix}, \begin{cases} \lambda(\hat{p}, 1; 2) = -\frac{\hat{p}_+}{\hat{p}_-} \lambda(-\hat{p}, -1; 2) \\ \lambda(-\hat{p}, 1; 2) = -\frac{\hat{p}_+}{\hat{p}_-} \lambda(\hat{p}, -1; 2) \\ \lambda(\hat{p}, -1; 2) = -\frac{\hat{p}_-}{\hat{p}_+} \lambda(-\hat{p}, 1; 2) \\ \lambda(-\hat{p}, -1; 2) = -\frac{\hat{p}_-}{\hat{p}_+} \lambda(\hat{p}, 1; 2) \end{cases}$$

推论5.3.3.

$$\lambda(\hat{p}, 0; 2) := \begin{bmatrix} \frac{\sqrt{6}}{2} \hat{p}_-^2 \\ -\sqrt{3} \hat{p}_- \hat{p}_z \\ \frac{1}{2} (3\hat{p}_z^2 - 1) \\ \sqrt{3} \hat{p}_+ \hat{p}_z \\ \frac{\sqrt{6}}{2} \hat{p}_+^2 \end{bmatrix}, \begin{cases} \lambda(\hat{p}, 0; 2) = \lambda(-\hat{p}, 0; 2) \\ \lambda(-\hat{p}, 0; 2) = \lambda(\hat{p}, 0; 2) \end{cases}$$

推论5.3.4.

$$\lambda(\hat{p}, 2; 2) \lambda^+(\hat{p}, 2; 2) = \begin{bmatrix} \frac{1}{4} (1 + \hat{p}_z)^2 \\ \frac{1}{\sqrt{2}} \hat{p}_+ (1 + \hat{p}_z) \\ \frac{\sqrt{6}}{2} \hat{p}_+^2 \\ \sqrt{2} \hat{p}_+^3 / (1 + \hat{p}_z) \\ \hat{p}_+^4 / (1 + \hat{p}_z)^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{16} (1 + \hat{p}_z)^4 & \frac{\sqrt{2}}{8} \hat{p}_- (1 + \hat{p}_z)^3 & \frac{\sqrt{6}}{8} \hat{p}_-^2 (1 + \hat{p}_z)^2 & \frac{\sqrt{2}}{4} \hat{p}_-^3 (1 + \hat{p}_z) & \frac{1}{4} \hat{p}_-^4 \\ 0 & \frac{1}{2} \hat{p}_+ \hat{p}_- (1 + \hat{p}_z)^2 & \frac{\sqrt{3}}{2} \hat{p}_+ \hat{p}_-^2 (1 + \hat{p}_z) & \hat{p}_+ \hat{p}_-^3 & \frac{\sqrt{2}}{2} \hat{p}_+ \hat{p}_-^4 / (1 + \hat{p}_z) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## 5.4 螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数的具体求法II

$$\text{定理5.4.1. } \lambda(\hat{p}, h; 2) = \sqrt{C_4^{2-h} \Gamma(2)} \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{2+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{2-h}$$

## 5.5 螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数的正交性与完备性

$$\text{推论5.5.1. } \lambda^+(\hat{p}, h; 2) \lambda(\hat{p}, h'; 2) = \delta_{hh'}, \sum_{h=2}^{-2} \lambda(\hat{p}, h; 2) \lambda^+(\hat{p}, h; 2) = 1, \sum_{h=2}^{-2} h \lambda(\hat{p}, h; 2) \lambda^+(\hat{p}, h; 2) = \sigma(2) \cdot \hat{p}$$

## 5.6 螺旋度 $\sigma(2) \cdot \hat{p}$ 本征函数导数性质的小结

推论5.6.1.

$$\begin{cases} \lambda^+(\hat{p}, 2; 2) \tilde{\partial}_k \lambda(\hat{p}, 2; 2) = 2 \frac{-i \hat{p}_y \delta_{kx} + i \hat{p}_x \delta_{ky}}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, 1; 2) \tilde{\partial}_k \lambda(\hat{p}, 1; 2) = \frac{-i \hat{p}_y \delta_{kx} + i \hat{p}_x \delta_{ky}}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, 0; 2) \tilde{\partial}_k \lambda(\hat{p}, 0; 2) = 0 \\ \lambda^+(\hat{p}, -1; 2) \tilde{\partial}_k \lambda(\hat{p}, -1; 2) = -\frac{-i \hat{p}_y \delta_{kx} + i \hat{p}_x \delta_{ky}}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, -2; 2) \tilde{\partial}_k \lambda(\hat{p}, -2; 2) = -2 \frac{-i \hat{p}_y \delta_{kx} + i \hat{p}_x \delta_{ky}}{p(1 + \hat{p}_z)} \end{cases} \begin{cases} \lambda^+(\hat{p}, -2; 2) \tilde{\partial}_k \lambda(\hat{p}, 2; 2) = 0 \\ \lambda^+(\hat{p}, -1; 2) \tilde{\partial}_k \lambda(\hat{p}, 1; 2) = 0 \\ \lambda^+(\hat{p}, 0; 2) \tilde{\partial}_k \lambda(\hat{p}, 0; 2) = 0 \\ \lambda^+(\hat{p}, 1; 2) \tilde{\partial}_k \lambda(\hat{p}, -1; 2) = 0 \\ \lambda^+(\hat{p}, 2; 2) \tilde{\partial}_k \lambda(\hat{p}, -2; 2) = 0 \end{cases}$$

推论5.6.2.

$$\begin{cases} \lambda^+(\hat{p}, 2; 2) \tilde{\partial}_k \lambda(\hat{p}, 2; 2) = 2 \frac{-i \hat{p}_y \delta_{kx} + i \hat{p}_x \delta_{ky}}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, 1; 2) \tilde{\partial}_k \lambda(\hat{p}, 2; 2) \\ = -\frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x + i \hat{p}_y) - (1 + \hat{p}_z)(\delta_{ix} + i \delta_{iy})}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, 0; 2) \tilde{\partial}_k \lambda(\hat{p}, 2; 2) = 0 \\ \lambda^+(\hat{p}, -1; 2) \tilde{\partial}_k \lambda(\hat{p}, 2; 2) = 0 \\ \lambda^+(\hat{p}, -2; 2) \tilde{\partial}_k \lambda(\hat{p}, 2; 2) = 0 \end{cases} \begin{cases} \lambda^+(\hat{p}, 2; 2) \tilde{\partial}_k \lambda(\hat{p}, 1; 2) = \frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x - i \hat{p}_y) - (1 + \hat{p}_z)(\delta_{ix} - i \delta_{iy})}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, 1; 2) \tilde{\partial}_k \lambda(\hat{p}, 1; 2) = \frac{-i \hat{p}_y \delta_{kx} + i \hat{p}_x \delta_{ky}}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, 0; 2) \tilde{\partial}_k \lambda(\hat{p}, 1; 2) = -\frac{\sqrt{6}}{2} \frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x + i \hat{p}_y) - (1 + \hat{p}_z)(\delta_{ix} + i \delta_{iy})}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, -1; 2) \tilde{\partial}_k \lambda(\hat{p}, 1; 2) = 0 \\ \lambda^+(\hat{p}, -2; 2) \tilde{\partial}_k \lambda(\hat{p}, 1; 2) = 0 \end{cases}$$

推论5.6.3.

$$\begin{cases} \lambda^+(\hat{p}, 2; 2) \tilde{\partial}_k \lambda(\hat{p}, 0; 2) = 0 \\ \lambda^+(\hat{p}, 1; 2) \tilde{\partial}_k \lambda(\hat{p}, 0; 2) = \frac{\sqrt{6}}{2} \frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x - i \hat{p}_y) - (1 + \hat{p}_z)(\delta_{ix} - i \delta_{iy})}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, 0; 2) \tilde{\partial}_k \lambda(\hat{p}, 0; 2) = 0 \\ \lambda^+(\hat{p}, -1; 2) \tilde{\partial}_k \lambda(\hat{p}, 0; 2) = -\frac{\sqrt{6}}{2} \frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x + i \hat{p}_y) - (1 + \hat{p}_z)(\delta_{ix} + i \delta_{iy})}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, -2; 2) \tilde{\partial}_k \lambda(\hat{p}, 0; 2) = 0 \end{cases}$$

推论5.6.4.

$$\left\{ \begin{array}{l} \lambda^+(\hat{p}, 2; 2) \tilde{\partial}_k \lambda(\hat{p}, -1; 2) = 0 \\ \lambda^+(\hat{p}, 1; 2) \tilde{\partial}_k \lambda(\hat{p}, -1; 2) = 0 \\ \lambda^+(\hat{p}, 0; 2) \tilde{\partial}_k \lambda(\hat{p}, -1; 2) = \frac{\sqrt{6}(\hat{p}_i + \delta_{iz})(\hat{p}_x - i\hat{p}_y) - (1 + \hat{p}_z)(\delta_{ix} - i\delta_{iy})}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, -1; 2) \tilde{\partial}_k \lambda(\hat{p}, -1; 2) = -\frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, -2; 2) \tilde{\partial}_k \lambda(\hat{p}, -1; 2) = -\frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x + i\hat{p}_y) - (1 + \hat{p}_z)(\delta_{ix} + i\delta_{iy})}{p(1 + \hat{p}_z)} \end{array} \right. \quad \left\{ \begin{array}{l} \lambda^+(\hat{p}, 2; 2) \tilde{\partial}_k \lambda(\hat{p}, -2; 2) = 0 \\ \lambda^+(\hat{p}, 1; 2) \tilde{\partial}_k \lambda(\hat{p}, -2; 2) = 0 \\ \lambda^+(\hat{p}, 0; 2) \tilde{\partial}_k \lambda(\hat{p}, -2; 2) = 0 \\ \lambda^+(\hat{p}, -1; 2) \tilde{\partial}_k \lambda(\hat{p}, -2; 2) \\ = \frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x - i\hat{p}_y) - (1 + \hat{p}_z)(\delta_{ix} - i\delta_{iy})}{p(1 + \hat{p}_z)} \\ \lambda^+(\hat{p}, -2; 2) \tilde{\partial}_k \lambda(\hat{p}, -2; 2) = -2 \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{p(1 + \hat{p}_z)} \end{array} \right.$$

## 6 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数的分析

### 6.1 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数的定义

定义6.1.1.  $\sigma(s) \cdot \hat{p} \lambda(\hat{p}, h; s) = h \lambda(\hat{p}, h; s)$ ,  $h = -s, \dots, s$

### 6.2 螺旋度 $\sigma(s) \cdot \hat{p}$ 的z-方向本征函数

定义6.2.1.  $\sigma(s) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s) = h \lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s)$

推论6.2.1.  $\lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s; s) = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$ ,  $\lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s-1; s) = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$ ,  $\dots$ ,  $\lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, -s+1; s) = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$ ,  $\lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, -s; s) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

推论6.2.2.

$\lambda(e^{i\omega_z \gamma_z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s; s) = e^{is\omega_z} \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$ ,  $\lambda(e^{i\omega_z \gamma_z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s-1; s) = e^{i(s-1)\omega_z} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$ ,  $\dots$ ,  $\lambda(e^{i\omega_z \gamma_z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, -s; s) = e^{-is\omega_z} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

推论6.2.3.

$\lambda(e^{\epsilon_z \cdot L_z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s; s) = e^{s\epsilon_z} \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$ ,  $\lambda(e^{\epsilon_z \cdot L_z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s-1; s) = e^{(s-1)\epsilon_z} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$ ,  $\dots$ ,  $\lambda(e^{\epsilon_z \cdot L_z} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, -s; s) = e^{-s\epsilon_z} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

### 6.3 螺旋度 $\sigma(s) \cdot \hat{p}$ 的一般本征函数

性质6.3.1.  $\left\{ \begin{array}{l} \hat{\omega} \cdot \sigma(s) \stackrel{\hat{\omega}_z=0}{=} \sigma_x(s) \hat{\omega}_x + \sigma_y(s) \hat{\omega}_y = \frac{\sigma_x(s) \hat{p}_y - \sigma_y(s) \hat{p}_x}{\sqrt{1-\hat{p}_z^2}} = \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \\ \vec{\omega} \cdot \sigma(s) \stackrel{\hat{\omega}_z=0}{=} \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z \end{array} \right.$

性质6.3.2.  $e^{i\vec{\omega} \cdot \sigma(s)} \stackrel{\hat{\omega}_z=0}{=} \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\}$

定理6.3.1.  $\lambda(\hat{p}, h; s) = \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} \lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s)$

证明:  $\sigma(s) \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s) = h \lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s)$

$\Leftrightarrow \exp\{-i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} \sigma(s) \cdot \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} \lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s) = h \lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s)$

$\Leftrightarrow \sigma(s) \cdot \hat{p} \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} \lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s) = h \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} \lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s)$

$\Rightarrow \lambda(\hat{p}, h; s) = \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} \lambda(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s)$  □

### 6.4 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数的正交性与完备性

推论6.4.1.  $\lambda^+(\hat{p}, h; s) \lambda(\hat{p}, h'; s) = \delta_{hh'}$ ,  $\sum_{h=s}^{-s} \lambda(\hat{p}, h; s) \lambda^+(\hat{p}, h; s) = 1$ ,  $\sum_{h=s}^{-s} h \lambda(\hat{p}, h; s) \lambda^+(\hat{p}, h; s) = \sigma(s) \cdot \hat{p}$

以上三个推论可以很容易证明得到。

## 6.5 自旋洛伦兹变换性质的合理性猜测(还需严格化)

$$\text{猜想6.5.1. } \begin{cases} \exp\{-i\pi \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}\} = (\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{2\sigma_z(s)} \varepsilon(s) = (-1)^{2s} \varepsilon^+(s) (\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{2\sigma_z(s)} \\ \exp\{i\pi \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}\} = \varepsilon^+(s) (\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{2\sigma_z(s)} = (-1)^{2s} (\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{2\sigma_z(s)} \varepsilon(s) \end{cases}, e^{i2\pi\hat{\omega} \cdot \sigma(s)} = (-1)^{2s}$$

$$\text{推论6.5.1. } \lambda(-\hat{p}, h; s) = (-1)^{s+h} (\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{2h} \lambda(\hat{p}, -h; s) = (-1)^{s+h} (\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}})^{-2h} \lambda(\hat{p}, -h; s)$$

$$\begin{aligned} \text{证明: } & \lambda(-\hat{p}, h; s) \\ &= \exp\{-i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos(-\hat{p}_z)\} \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \\ &= \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} \exp\{-i\pi \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}}\} \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \\ &= (-1)^{s+h} (\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}})^{2h} \lambda(\hat{p}, -h; s) \end{aligned}$$

□

6.6 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数的性质

$$\text{引理6.6.1. } \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \sigma(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) = h \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{定理6.6.1. } \lambda^+(\hat{p}, h; s) \sigma(s) \lambda(\hat{p}, h; s) = h\hat{p}, h = -s, \dots, s$$

$$\begin{aligned} \text{证明: } & \lambda^+(\hat{p}, h; s) \sigma_k(s) \lambda(\hat{p}, h; s) \\ &= \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) e^{-i\hat{\omega} \cdot \sigma(s)} \sigma_k(s) e^{i\hat{\omega} \cdot \sigma(s)} \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \\ &= \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) e^{-i\hat{\omega} \cdot \sigma(s)} [e^{i\hat{\omega} \cdot \gamma} |_k^l e^{i\hat{\omega} \cdot \sigma(s)} \sigma_l(s) e^{-i\hat{\omega} \cdot \sigma(s)}] e^{i\hat{\omega} \cdot \sigma(s)} \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \\ &= e^{i\hat{\omega} \cdot \gamma} |_k^l \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \sigma_l(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \\ &= h \{e^{i\hat{\omega} \cdot \gamma} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}_k = h\hat{p}_k \end{aligned}$$

□

$$\begin{aligned} \text{证明: } & \lambda^+(\hat{p}, h; s) \sigma_i(s) \sigma_j(s) \lambda(\hat{p}, h; s) \\ &= \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) e^{-i\hat{\omega} \cdot \sigma(s)} \sigma_i(s) \sigma_j(s) e^{i\hat{\omega} \cdot \sigma(s)} \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \\ &= \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) e^{-i\hat{\omega} \cdot \sigma(s)} [e^{i\hat{\omega} \cdot \gamma} |_i^k e^{i\hat{\omega} \cdot \sigma(s)} \sigma_k(s) e^{-i\hat{\omega} \cdot \sigma(s)}] [e^{i\hat{\omega} \cdot \gamma} |_j^l e^{i\hat{\omega} \cdot \sigma(s)} \sigma_l(s) e^{-i\hat{\omega} \cdot \sigma(s)}] e^{i\hat{\omega} \cdot \sigma(s)} \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \\ &= e^{i\hat{\omega} \cdot \gamma} |_i^k e^{i\hat{\omega} \cdot \gamma} |_j^l \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \sigma_k(s) \sigma_l(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \\ &= e^{i\hat{\omega} \cdot \gamma} |_i^k e^{i\hat{\omega} \cdot \gamma} |_j^l \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \sigma_k(s) \sum_{h'} [\lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h'; s\right) \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h'; s\right) \sigma_l(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right)] \\ &= e^{i\hat{\omega} \cdot \gamma} |_i^k e^{i\hat{\omega} \cdot \gamma} |_j^l \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \\ & \sigma_k(s) [\lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h-1; s\right) \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h-1; s\right) + \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) + \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h+1; s\right) \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h+1; s\right)] \sigma_l(s) \\ & \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) \\ & ? = h^2 \hat{p}_i \hat{p}_j \end{aligned}$$

□

$$\begin{aligned} \text{证明: } & \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s; s\right) \sigma_k(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s-1; s\right) \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^+ \sigma_k(s) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \sqrt{2s} \\ -i\sqrt{2s} \\ 0 \end{bmatrix}_k \end{aligned}$$

□

$$\begin{aligned} \text{证明: } & \lambda^+(\hat{p}, \varsigma s; s) \sigma_i(s) \sigma_j(s) \lambda(\hat{p}, \varsigma s; s) \\ &= e^{i\hat{\omega} \cdot \gamma} |_i^k e^{i\hat{\omega} \cdot \gamma} |_j^l \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \varsigma s; s\right) \sigma_k(s) [\lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \varsigma(s-1); s\right) \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \varsigma(s-1); s\right) \\ & + \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \varsigma s; s\right) \lambda^+\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \varsigma s; s\right)] \sigma_l(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \varsigma s; s\right) \\ &= s^2 \hat{p}_i \hat{p}_j + \frac{s}{2} (\delta_{ij} - \hat{p}_i \hat{p}_j + i\varsigma \varepsilon_{ij}^k \hat{p}_k) \\ &= s^2 \hat{p}_i \hat{p}_j - \frac{s}{2} \sigma_{ij}^{ab} \hat{p}_a \hat{p}_b \end{aligned}$$

□

证明:  $\lambda^+(\hat{p}, \zeta s; s) \sigma_i(s) \sigma_j(s) \lambda(\hat{p}, -\zeta s; s)$ ,  $s \geq \frac{3}{2}$

$$= e^{i\vec{\omega} \cdot \gamma} |i^k e^{i\vec{\omega} \cdot \gamma} |j^l \lambda^+\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \zeta s; s\right) \sigma_k(s) [\lambda\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \zeta(s-1); s\right) \lambda^+\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \zeta(s-1); s\right) \\ + \lambda\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \zeta s; s\right) \lambda^+\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \zeta s; s\right) \sigma_l(s) \lambda\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, -\zeta s; s\right) \\ = 0 \quad \square$$

证明:  $\lambda^+(\hat{p}, -\zeta s; s) \sigma_i(s) \sigma_j(s) [\sigma(s) \cdot \hat{p}]^n \lambda(\hat{p}, -\zeta s; s) = (-\zeta)^n s^n s^2 \hat{p}_i \hat{p}_j + (-\zeta)^n s^n \frac{s}{2} (\delta_{ij} - \hat{p}_i \hat{p}_j + i\zeta \varepsilon_{ij}^k \hat{p}_k)$   $\square$

推论6.6.1.  $\sigma_{\alpha\zeta\alpha'}^{ab} p_a p_b = p_{\alpha\zeta} p_{\alpha'} - \delta_{\alpha\zeta\alpha'} |\vec{p}|^2 - i\zeta \varepsilon_{\alpha\zeta\alpha'}^k p_k |\vec{p}|$

推论6.6.2.  $\lambda^+(\hat{p}, h; s) [\sigma(s), ih]_a \lambda(\hat{p}, h; s) = h(\hat{p}, i)_a = h \hat{p}_a, h = -s, \dots, s$

引理6.6.2.  $\lambda^+\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, h; s\right) \sigma(s) \lambda\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, h'; s\right) = 0, |h - h'| \geq 2$

定理6.6.2.  $\lambda^+(\hat{p}, h; s) \sigma(s) \lambda(\hat{p}, h'; s) = 0, h, h' = -s, \dots, s; |h - h'| \geq 2$

证明:  $\lambda^+(\hat{p}, h; s) \sigma_k(s) \lambda(\hat{p}, h'; s)$

$$= \lambda^+\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, h; s\right) e^{-i\vec{\omega} \cdot \sigma(s)} \sigma_k(s) e^{i\vec{\omega} \cdot \sigma(s)} \lambda\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, h'; s\right) \\ = \lambda^+\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, h; s\right) e^{-i\vec{\omega} \cdot \sigma(s)} [e^{i\vec{\omega} \cdot \gamma} |k^l e^{i\vec{\omega} \cdot \sigma(s)} \sigma_l(s) e^{-i\vec{\omega} \cdot \sigma(s)}] e^{i\vec{\omega} \cdot \sigma(s)} \lambda\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, h'; s\right) \\ = e^{i\vec{\omega} \cdot \gamma} |k^l \lambda^+\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, h; s\right) \sigma_l(s) \lambda\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, h'; s\right), |h - h'| \geq 2 \\ = e^{i\vec{\omega} \cdot \gamma} |k^l \cdot 0 \\ = 0 \quad \square$$

## 6.7 自旋矢量算符本征态 $\lambda(\hat{p}, -s\zeta)$

定义6.7.1.  $\lambda(\hat{p}, -s\zeta) := \lambda(\hat{p}, -s\zeta; s)$

定理6.7.1.  $[s \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}_a + i S_{ab}(s, \zeta) \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}_b] \lambda\left(\begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}, -s\zeta\right) \equiv 0 [\Leftrightarrow] [s \hat{p}_a + i S_{ab}(s, \zeta) \hat{p}^b] \lambda(\hat{p}, -s\zeta) \equiv 0$

定理6.7.2.  $[s \hat{p}_a + i S_{ab}(s, \zeta) \hat{p}^b] \lambda(\hat{p}, -s\zeta) = 0 [\Leftrightarrow] \begin{cases} W_a(\hat{p}, \zeta; s) \lambda(\hat{p}, -s\zeta) = -s\zeta \hat{p}_a \lambda(\hat{p}, -s\zeta) \\ W_a(\hat{p}, \zeta; s) := -i * S_{ab}(s, \zeta) \hat{p}^b = i\zeta S_{ab}(s, \zeta) \hat{p}^b \end{cases}$

性质6.7.1.  $W_a(\hat{p}, \zeta; s) = (\hat{W}(\hat{p}, \zeta; s), i\sigma(s) \cdot \hat{p}), \hat{W}(\hat{p}, \zeta; s) = \sigma(s) - i\zeta \sigma(s) \times \hat{p}$

推论6.7.1.  $\hat{W}(\hat{p}, \zeta; s) \lambda(\hat{p}, -s\zeta) = -s\zeta \hat{p} \lambda(\hat{p}, -s\zeta) [\Rightarrow] \sigma(s) \cdot \hat{p} \lambda(\hat{p}, -s\zeta) = -s\zeta \lambda(\hat{p}, -s\zeta)$

推论6.7.2.  $\hat{W}(\hat{p}, \zeta; s) \lambda(\hat{p}, -s\zeta) = -s\zeta \hat{p} \lambda(\hat{p}, -s\zeta) [\Leftrightarrow] W_a(\hat{p}, \zeta; s) \lambda(\hat{p}, -s\zeta) = -s\zeta \hat{p}_a \lambda(\hat{p}, -s\zeta)$

$\lambda(\hat{p}, -s\zeta)$  是螺旋度算符、四维自旋矢量算符和自旋矢量算符的共同本征态，本质上自旋矢量算符已完全包含了前两个算符。所以  $\lambda(\hat{p}, -s\zeta)$  本质上只是自旋矢量算符  $\hat{W}(\hat{p}, \zeta; s)$  的本征态，其他两个只是它的推论而已，而且  $\lambda(\hat{p}, -s\zeta)$  正是无质量粒子的本征态。

## 6.8 自旋矢量算符本征态 $\lambda(\hat{p}, -s\zeta)$ 的性质

性质6.8.1.  $\sigma(s) \times \hat{p} = [\sigma(s), i\sigma(s) \cdot \hat{p}]$

性质6.8.2.  $[s p_a + i S_{ab}(s, \zeta) p^b] \lambda(\hat{p}, -s\zeta) = 0 [\Leftrightarrow] \begin{cases} [\sigma(s) - i\zeta \sigma(s) \times \hat{p}] \lambda(\hat{p}, -s\zeta) = -s\zeta \hat{p} \lambda(\hat{p}, -s\zeta) \\ \sigma(s) \cdot \hat{p} \lambda(\hat{p}, -s\zeta) = -s\zeta \lambda(\hat{p}, -s\zeta) \end{cases}$

性质6.8.3.  $[s p_a + i S_{ab}(s, \zeta) p^b] \lambda(\hat{p}, -s\zeta) = 0 [\Leftrightarrow] \sigma(\frac{1}{2}) \otimes I_{2s} \cdot \hat{p} \begin{bmatrix} \lambda(\hat{p}, -s\zeta) \\ 0_{2s-1} \end{bmatrix} = -\frac{1}{2}\zeta \begin{bmatrix} \lambda(\hat{p}, -s\zeta) \\ 0_{2s-1} \end{bmatrix}$

性质6.8.4.  $\begin{cases} [s p_a + i S_{ab}(s, \zeta) p^b] \lambda(\hat{p}, -s\zeta) = 0 [\Leftrightarrow] -\zeta [\sigma(s) \cdot \hat{p} + \zeta(s-1)] \sigma(s) \lambda(\hat{p}, -s\zeta) = -s\zeta \hat{p} \lambda(\hat{p}, -s\zeta) \\ [\Updownarrow] \qquad \qquad \qquad [\Downarrow] \\ [\sigma(s) - i\zeta \sigma(s) \times \hat{p}] \lambda(\hat{p}, -s\zeta) = -s\zeta \hat{p} \lambda(\hat{p}, -s\zeta) [\Rightarrow] [\sigma(s) \cdot \hat{p}]^n \lambda(\hat{p}, -s\zeta) = (-s\zeta)^n \lambda(\hat{p}, -s\zeta) \end{cases}$

**推论6.8.1.**  $[sp_a + iS_{ab}(s, \varsigma)p^b]\lambda(\hat{p}, -s\varsigma) = 0[\Leftrightarrow][\sigma(s) \cdot \hat{p}]\sigma(s)\lambda(\hat{p}, -s\varsigma) = [s\hat{p} - \varsigma(s-1)\sigma(s)]\lambda(\hat{p}, -s\varsigma)$   
 $[\Leftrightarrow][\sigma(s) \cdot \hat{p}]^n\sigma(s)\lambda(\hat{p}, -s\varsigma) = \{(-\varsigma)^{n-1}s[s^n - (s-1)^n]\hat{p} + (-\varsigma)^n(s-1)^n\sigma(s)\}\lambda(\hat{p}, -s\varsigma)$

**证明:**

$$[\sigma(s) \cdot \hat{p}]\sigma(s)\lambda(\hat{p}, -s\varsigma) = [e_1\hat{p} + d_1\sigma(s)]\lambda(\hat{p}, -s\varsigma), e_1 = s, d_1 = -\varsigma(s-1)$$

..

$$[\sigma(s) \cdot \hat{p}]^{n-1}\sigma(s)\lambda(\hat{p}, -s\varsigma) = [e_{n-1}\hat{p} + d_{n-1}\sigma(s)]\lambda(\hat{p}, -s\varsigma)$$

$$[\sigma(s) \cdot \hat{p}]^n\sigma(s)\lambda(\hat{p}, -s\varsigma) = [e_n\hat{p} + d_n\sigma(s)]\lambda(\hat{p}, -s\varsigma)$$

..

$$[\sigma(s) \cdot \hat{p}]^n\sigma(s)\lambda(\hat{p}, -s\varsigma)$$

$$= [\sigma(s) \cdot \hat{p}][e_{n-1}\hat{p} + d_{n-1}\sigma(s)]\lambda(\hat{p}, -s\varsigma) = [(-s\varsigma e_{n-1} + d_{n-1}^{n-1}e_1)\hat{p} + d_{n-1}d_1\sigma(s)]\lambda(\hat{p}, -s\varsigma)$$

$$\begin{cases} e_n = -s\varsigma e_{n-1} + d_{n-1}^{n-1}e_1 \\ d_n = d_{n-1}d_1 \\ e_1 = s, d_1 = -\varsigma(s-1) \end{cases} \Leftrightarrow \begin{cases} e_n = (-\varsigma)^{n-1}s[s^n - (s-1)^n] \\ d_n = d_1^n = (-\varsigma)^n(s-1)^n \end{cases}$$

$$[\sigma(s) \cdot \hat{p}]^n\sigma(s)\lambda(\hat{p}, -s\varsigma) = \{(-\varsigma)^{n-1}s[s^n - (s-1)^n]\hat{p} + (-\varsigma)^n(s-1)^n\sigma(s)\}\lambda(\hat{p}, -s\varsigma) \quad \square$$

**推论6.8.2.**  $\lambda^+(\hat{p}, -s\varsigma)\sigma_i(s)[\sigma(s) \cdot \hat{p}]^n\sigma_j(s)\lambda(\hat{p}, -s\varsigma)$

$$= \lambda^+(\hat{p}, -s\varsigma)\sigma_i(s)\{(-\varsigma)^{n-1}s[s^n - (s-1)^n]\hat{p}_j + (-\varsigma)^n(s-1)^n\sigma_j(s)\}\lambda(\hat{p}, -s\varsigma)$$

$$= (-\varsigma)^n s^2 [s^n - (s-1)^n] \hat{p}_i \hat{p}_j + (-\varsigma)^n (s-1)^n \lambda^+(\hat{p}, -s\varsigma) \sigma_i(s) \sigma_j(s) \lambda(\hat{p}, -s\varsigma)$$

$$= (-\varsigma)^n s^2 [s^n - (s-1)^n] \hat{p}_i \hat{p}_j + (-\varsigma)^n (s-1)^n [s^2 \hat{p}_i \hat{p}_j + \frac{\varsigma}{2} (\delta_{ij} - \hat{p}_i \hat{p}_j - i\varsigma \varepsilon_{ij}^k \hat{p}_k)]$$

$$= (-\varsigma)^n s^2 s^n \hat{p}_i \hat{p}_j + (-\varsigma)^n (s-1)^n [\frac{\varsigma}{2} (\delta_{ij} - \hat{p}_i \hat{p}_j - i\varsigma \varepsilon_{ij}^k \hat{p}_k)]$$

**推论6.8.3.**  $\lambda^+(\hat{p}, -s\varsigma)[\sigma(s) \cdot \hat{p}]^n\sigma(s)\lambda(\hat{p}, -s\varsigma) = (-\varsigma s)^{n+1}\hat{p} = \lambda^+(\hat{p}, -s\varsigma)\sigma(s)[\sigma(s) \cdot \hat{p}]^n\lambda(\hat{p}, -s\varsigma)$

## 6.9 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征态 $\lambda(\hat{p}, h; s)$ 分解为 $\frac{1}{2}$ -自旋本征态

**定理6.9.1.**  $\lambda(\hat{p}, h; s) = \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h}$

**证明:**  $\lambda(\hat{p}, h; s) = e^{i\vec{\omega} \cdot \sigma(s)} \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right)$

$$= e^{i\vec{\omega} \cdot \bar{\Gamma}(s) \bar{\Omega}(s) \Gamma(s)} \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right)$$

$$= \bar{\Gamma}(s) e^{i\vec{\omega} \cdot \bar{\Omega}(s) \Gamma(s)} \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right)$$

$$= \bar{\Gamma}(s) e^{i\vec{\omega} \cdot \bar{\Omega}(s) \Gamma(s)} \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s) \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{s+h} \otimes \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{s-h}$$

$$= \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s) \Gamma(s) \bar{\Gamma}(s) e^{i\vec{\omega} \cdot \bar{\Omega}(s)} \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{s+h} \otimes \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{s-h}$$

$$= \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s) e^{i\vec{\omega} \cdot \bar{\Omega}(s)} \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{s+h} \otimes \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{s-h}$$

$$= \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s) e^{i\vec{\omega} \cdot \sigma(\frac{1}{2})} \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \cdots \otimes e^{i\vec{\omega} \cdot \sigma(\frac{1}{2})} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{s+h} \otimes \overbrace{e^{i\vec{\omega} \cdot \sigma(\frac{1}{2})} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \cdots \otimes e^{i\vec{\omega} \cdot \sigma(\frac{1}{2})} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{s-h}$$

$$= \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h} \quad \square$$

**定理6.9.2.**  $\lambda(-\hat{p}, h; s) = (-1)^{s+h} \left(\frac{\hat{p}_+}{\sqrt{\hat{p}_+ \hat{p}_-}}\right)^{2h} \lambda(\hat{p}, -h; s)$

**证明:**  $\lambda(-\hat{p}, h; s) = \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s) \overbrace{\lambda(-\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(-\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(-\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(-\hat{p}, -\frac{1}{2})}^{s-h}$

$$= \left(-\frac{\hat{p}_+}{\sqrt{\hat{p}_+ \hat{p}_-}}\right)^{s+h} \left(\frac{\hat{p}_-}{\sqrt{\hat{p}_+ \hat{p}_-}}\right)^{s-h} \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s-h}$$



$$\begin{aligned}
 &= \left(-\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}}\right)^{s+h} \left(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}}\right)^{s-h} \sqrt{C_{2s}^{s+h} \bar{\Gamma}(s)} \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s-h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s+h} \\
 &= \left(-\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}}\right)^{s+h} \left(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}}\right)^{s-h} \lambda(\hat{p}, -h; s) \\
 &= (-1)^{s+h} \left(\frac{\hat{p}_+}{\sqrt{\hat{p}_+\hat{p}_-}}\right)^{2h} \lambda(\hat{p}, -h; s)
 \end{aligned}$$

□

推论6.9.1.  $\lambda(-\hat{p}, -h; s) = (-1)^{s-h} \left(\frac{\hat{p}_-}{\sqrt{\hat{p}_+\hat{p}_-}}\right)^{2h} \lambda(\hat{p}, h; s)$

推论6.9.2.

$$\left\{ \begin{aligned}
 \lambda_{k_\zeta}(\hat{p}, h; s) &= \sqrt{C_{2s}^{s-h} \Gamma_{k_\zeta}^{A_\zeta B_\zeta \cdots C_\zeta D_\zeta \cdots}(s)} \overbrace{\lambda_{A_\zeta}(\hat{p}, \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) \cdots}^{s+h} \otimes \overbrace{\lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}) \lambda_{D_\zeta}(\hat{p}, -\frac{1}{2}) \cdots}^{s-h} \\
 \frac{1}{(2s)!} \overbrace{\lambda_{A_\zeta}(\hat{p}, \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) \cdots}^{s+h} \otimes \overbrace{\lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}) \lambda_{D_\zeta}(\hat{p}, -\frac{1}{2}) \cdots}^{s-h} &= \sqrt{C_{2s}^{h-s} \Gamma_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta \cdots}^{k_\zeta}(s)} \lambda_{k_\zeta}(\hat{p}, h; s)
 \end{aligned} \right.$$

推论6.9.3.

$$\left\{ \begin{aligned}
 \lambda_{k_\zeta}(\hat{p}, -s\zeta) &= \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}(s) \overbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}^{2s} \\
 \Gamma_{A_\zeta B_\zeta C_\zeta \cdots}^{k_\zeta}(s) \lambda_{k_\zeta}(\hat{p}, -s\zeta) &= \overbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}^{2s}
 \end{aligned} \right.$$

推论6.9.4.

$$\left\{ \begin{aligned}
 \lambda(\hat{p}, -s\zeta) &= \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\zeta}{2})}^{2s} \\
 \Gamma(s) \lambda(\hat{p}, -s\zeta) &= \overbrace{\lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\zeta}{2})}^{2s}
 \end{aligned} \right.$$

定理6.9.3.  $\lambda_{k_\zeta}(\hat{p}, -\zeta h; s) \lambda_{k'_\zeta}^+(\hat{p}, -\zeta h; s) = \left(-\frac{i}{2}\right)^{2h} 2^s C_{2s}^{s-h} \left(-\frac{i\zeta}{\sqrt{2}}\right)^{s+h} \left(\frac{i\zeta}{\sqrt{2}}\right)^{s-h}$

$$\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \cdots}(s) \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \cdots (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, -i\zeta)_{C_\zeta C'_\zeta}^c \cdots (\sigma, -i\zeta)_{D_\zeta D'_\zeta}^d}^{s+h} \hat{p}_a \cdots \hat{p}_b \hat{p}_c \cdots \hat{p}_d$$

证明:  $\lambda(\hat{p}, -\zeta h; s) \lambda^+(\hat{p}, -\zeta h; s) = C_{2s}^{s-h} \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\zeta}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{\zeta}{2})}^{s-h}$

$$\begin{aligned}
 &\overbrace{\lambda^+(\hat{p}, -\frac{\zeta}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{\zeta}{2})}^{s+h} \otimes \overbrace{\lambda^+(\hat{p}, \frac{\zeta}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{\zeta}{2})}^{s-h} \Gamma(s) \\
 &= C_{2s}^{s-h} \bar{\Gamma}(s)
 \end{aligned}$$

$$\overbrace{\lambda(\hat{p}, -\frac{\zeta}{2}) \lambda^+(\hat{p}, -\frac{\zeta}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\zeta}{2}) \lambda^+(\hat{p}, -\frac{\zeta}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, \frac{\zeta}{2}) \lambda^+(\hat{p}, \frac{\zeta}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{\zeta}{2}) \lambda^+(\hat{p}, \frac{\zeta}{2})}^{s-h} \Gamma(s)$$

$$= \left(-\frac{\zeta}{2}\right)^{2h} C_{2s}^{s-h} \bar{\Gamma}(s) \overbrace{(\sigma, i\zeta)^a \hat{p}_a \otimes \cdots \otimes (\sigma, i\zeta)^b \hat{p}_b}^{s+h} \otimes \overbrace{(\sigma, -i\zeta)^c \hat{p}_c \otimes \cdots \otimes (\sigma, -i\zeta)^d \hat{p}_d}^{s-h} \Gamma(s)$$

$$= \left(-\frac{\zeta}{2}\right)^{2h} C_{2s}^{s-h} \bar{\Gamma}(s) \overbrace{(\sigma, i\zeta)^a \otimes \cdots \otimes (\sigma, i\zeta)^b}^{s+h} \otimes \overbrace{(\sigma, -i\zeta)^c \otimes \cdots \otimes (\sigma, -i\zeta)^d}^{s-h} \Gamma(s) \hat{p}_a \cdots \hat{p}_b \hat{p}_c \cdots \hat{p}_d$$

$$= \left(-\frac{i}{2}\right)^{2h} 2^s C_{2s}^{s-h} \left(-\frac{i\zeta}{\sqrt{2}}\right)^{s+h} \left(\frac{i\zeta}{\sqrt{2}}\right)^{s-h}$$

$$\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \cdots}(s) \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \cdots (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, -i\zeta)_{C_\zeta C'_\zeta}^c \cdots (\sigma, -i\zeta)_{D_\zeta D'_\zeta}^d}^{s+h} \hat{p}_a \cdots \hat{p}_b \hat{p}_c \cdots \hat{p}_d$$

□

### 6.10 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数的性质

定理6.10.1.  $\lambda(\hat{p}, h; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) = C_{2s}^{s-h} h \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{p(1+\hat{p}_z)}$ ,  $\lambda(-\hat{p}, h; s) \tilde{\partial}_k \lambda(-\hat{p}, h; s) = C_{2s}^{s-h} h \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{p(1-\hat{p}_z)}$

推论6.10.1.

$$\begin{cases} \lambda^+(\hat{p}, h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) = \sqrt{C_{2s}^{s-h'} C_{2s}^{s-h}} [(h' + h) \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{2p(1+\hat{p}_z)} + (h' - h) \frac{\sqrt{2}\hat{p}_+ \hat{p}_i + \sqrt{2}\hat{p}_+ \delta_{iz} - (1+\hat{p}_z)(\delta_{ix} + i\delta_{iy})}{2p(1+\hat{p}_z)}], h' \leq h \\ \lambda^+(\hat{p}, h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) = \sqrt{C_{2s}^{s-h'} C_{2s}^{s-h}} [(h' + h) \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{2p(1+\hat{p}_z)} + (h' - h) \frac{\sqrt{2}\hat{p}_- \hat{p}_i + \sqrt{2}\hat{p}_- \delta_{iz} - (1+\hat{p}_z)(\delta_{ix} - i\delta_{iy})}{2p(1+\hat{p}_z)}], h' \geq h \\ \lambda^+(-\hat{p}, -h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) \\ = (-1)^{s-h'} \left(\frac{\hat{p}_-}{\sqrt{\hat{p}_+ \hat{p}_-}}\right)^{2h'} \sqrt{C_{2s}^{s-h'} C_{2s}^{s-h}} [(h' + h) \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{2p(1+\hat{p}_z)} + (h' - h) \frac{\sqrt{2}\hat{p}_+ \hat{p}_i + \sqrt{2}\hat{p}_+ \delta_{iz} - (1+\hat{p}_z)(\delta_{ix} + i\delta_{iy})}{2p(1+\hat{p}_z)}], h' \leq h \\ \lambda^+(-\hat{p}, -h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) \\ = (-1)^{s-h'} \left(\frac{\hat{p}_-}{\sqrt{\hat{p}_+ \hat{p}_-}}\right)^{2h'} \sqrt{C_{2s}^{s-h'} C_{2s}^{s-h}} [(h' + h) \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{2p(1+\hat{p}_z)} + (h' - h) \frac{\sqrt{2}\hat{p}_- \hat{p}_i + \sqrt{2}\hat{p}_- \delta_{iz} - (1+\hat{p}_z)(\delta_{ix} - i\delta_{iy})}{2p(1+\hat{p}_z)}], h' \geq h \end{cases}$$

### 6.11 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数的通用解法一

$$\text{引理6.11.1. } \lambda(\hat{p}, s; s) = \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{2s}, \lambda(\hat{p}, -s; s) = \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{2s}$$

$$\text{定理6.11.1. } \lambda^+(\hat{p}, -\varsigma s; s) \tilde{\partial}_k \lambda(\hat{p}, -\varsigma s; s) = 2s \lambda^+(\hat{p}, -\frac{\varsigma}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{\varsigma}{2})$$

$$\text{证明: } \lambda^+(\hat{p}, -\varsigma s; s) \tilde{\partial}_k \lambda(\hat{p}, -\varsigma s; s)$$

$$\begin{aligned} &= \overbrace{\lambda^+(\hat{p}, -\frac{\varsigma}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{\varsigma}{2})}^{2s} \Gamma(s) \bar{\Gamma}(s) \tilde{\partial}_k \overbrace{\lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\varsigma}{2})}^{2s} \\ &= 2s \overbrace{\lambda^+(\hat{p}, -\frac{\varsigma}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{\varsigma}{2})}^{2s} \Gamma(s) \bar{\Gamma}(s) \tilde{\partial}_k \overbrace{\lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\varsigma}{2})}^{2s} \\ &= 2s \overbrace{\lambda^+(\hat{p}, -\frac{\varsigma}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{\varsigma}{2})}^{2s} \tilde{\partial}_k \overbrace{\lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\varsigma}{2})}^{2s} \\ &= 2s \overbrace{\lambda^+(\hat{p}, -\frac{\varsigma}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{\varsigma}{2}) \lambda(\hat{p}, -\frac{\varsigma}{2})}^{2s} \\ &= 2s \lambda^+(\hat{p}, -\frac{\varsigma}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{\varsigma}{2}) \end{aligned}$$

□

$$\text{定理6.11.2. } \lambda^+(\hat{p}, -s; s) \tilde{\partial}_k \lambda(\hat{p}, s; s) = 0 [\Leftrightarrow] \lambda^+(\hat{p}, s; s) \tilde{\partial}_k \lambda(\hat{p}, -s; s) = 0; s \geq 1$$

$$\text{证明: } \lambda^+(\hat{p}, -\varsigma s; s) \tilde{\partial}_k \lambda(\hat{p}, \varsigma s; s); s \geq 1$$

$$\begin{aligned} &= \overbrace{\lambda^+(\hat{p}, -\frac{\varsigma}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{\varsigma}{2})}^{2s} \Gamma(s) \bar{\Gamma}(s) \tilde{\partial}_k \overbrace{\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{\varsigma}{2})}^{2s} \\ &= 2s \overbrace{\lambda^+(\hat{p}, -\frac{\varsigma}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{\varsigma}{2})}^{2s} \Gamma(s) \bar{\Gamma}(s) \tilde{\partial}_k \overbrace{\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{\varsigma}{2})}^{2s} \\ &= 2s \overbrace{\lambda^+(\hat{p}, -\frac{\varsigma}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{\varsigma}{2})}^{2s} \tilde{\partial}_k \overbrace{\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{\varsigma}{2})}^{2s} \\ &= 2s \overbrace{\lambda^+(\hat{p}, -\frac{\varsigma}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{\varsigma}{2}) \lambda(\hat{p}, \frac{\varsigma}{2})}^{2s} \\ &= 0 \end{aligned}$$

□

$$\text{证明: } \lambda^+(\hat{p}, -\varsigma s; s) \lambda(\hat{p}, \varsigma s; s) = 0, \lambda^+(\hat{p}, -\varsigma s; s) \tilde{\partial}_k \lambda(\hat{p}, \varsigma s; s) = 0$$

$$\Leftrightarrow \tilde{\partial}_k [\lambda^+(\hat{p}, -\varsigma s; s) \lambda(\hat{p}, \varsigma s; s)] = 0, \lambda^+(\hat{p}, -\varsigma s; s) \tilde{\partial}_k \lambda(\hat{p}, \varsigma s; s) = 0$$

$$\Leftrightarrow \tilde{\partial}_k \lambda^+(\hat{p}, -\varsigma s; s) \lambda(\hat{p}, \varsigma s; s) + \lambda^+(\hat{p}, -\varsigma s; s) \tilde{\partial}_k \lambda(\hat{p}, \varsigma s; s) = 0, \lambda^+(\hat{p}, -\varsigma s; s) \tilde{\partial}_k \lambda(\hat{p}, \varsigma s; s) = 0$$

$$\Rightarrow \tilde{\partial}_k \lambda^+(\hat{p}, -\varsigma s; s) \lambda(\hat{p}, \varsigma s; s) = 0$$

$$\Leftrightarrow [\tilde{\partial}_k \lambda^+(\hat{p}, -\varsigma s; s) \lambda(\hat{p}, \varsigma s; s)]^+ = 0$$

$$\Leftrightarrow \lambda^+(\hat{p}, \varsigma s; s) \tilde{\partial}_k \lambda(\hat{p}, -\varsigma s; s) = 0$$

□

### 6.12 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数的通用解法二

$$\text{引理6.12.1. } \lambda(\hat{p}, h; s) = \sqrt{C_{2s}^{s-h} \bar{\Gamma}(s)} \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h}$$

$$\text{定理6.12.1. } \lambda^+(\hat{p}, h; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) = 2h \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2})$$

证明:  $\lambda^+(\hat{p}, h; s)\tilde{\partial}_k\lambda(\hat{p}, h; s)$

$$\begin{aligned}
&= [\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^{s-h}]^+ \\
&\tilde{\partial}_k\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&= C_{2s}^{s-h}\overbrace{\lambda^+(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, \frac{1}{2})}^{s+h}\otimes\overbrace{\lambda^+(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, -\frac{1}{2})}^{s-h}\Gamma(s)\bar{\Gamma}(s) \\
&[(s+h)\overbrace{\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^{s-h} + (s-h)\overbrace{\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^{s+h}\otimes\overbrace{\tilde{\partial}_k\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^{s-h}] \\
&= (s+h)\overbrace{\lambda^+(\hat{p}, \frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, \frac{1}{2})\lambda(\hat{p}, \frac{1}{2})}^{s+h}\otimes\overbrace{\lambda^+(\hat{p}, -\frac{1}{2})\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, -\frac{1}{2})\lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&+ (s-h)\overbrace{\lambda^+(\hat{p}, \frac{1}{2})\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, \frac{1}{2})\lambda(\hat{p}, \frac{1}{2})}^{s+h}\otimes\overbrace{\lambda^+(\hat{p}, -\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, -\frac{1}{2})\lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&= (s+h)\lambda^+(\hat{p}, \frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2}) + (s-h)\lambda^+(\hat{p}, -\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, -\frac{1}{2}) \\
&= 2h\lambda^+(\hat{p}, \frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2}) \quad \square
\end{aligned}$$

定理6.12.2.  $\lambda^+(\hat{p}, -h; s)\tilde{\partial}_k\lambda(\hat{p}, h; s) = 0, |h| \geq 1$

证明:  $\lambda^+(\hat{p}, -h; s)\tilde{\partial}_k\lambda(\hat{p}, h; s)$

$$\begin{aligned}
&= [\sqrt{C_{2s}^{s+h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^{s-h}\otimes\overbrace{\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^{s+h}]^+ \\
&\tilde{\partial}_k\sqrt{C_{2s}^{s-h}}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^{s+h}\otimes\overbrace{\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&= C_{2s}^{s-h}\overbrace{\lambda^+(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, \frac{1}{2})}^{s-h}\otimes\overbrace{\lambda^+(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, -\frac{1}{2})}^{s+h}\Gamma(s)\bar{\Gamma}(s) \\
&[(s+h)\overbrace{\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^{s-h}\otimes\overbrace{\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^{s+h} + (s-h)\overbrace{\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^{s-h}\otimes\overbrace{\tilde{\partial}_k\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^{s+h}] \\
&= 0 + 0 = 0 \quad \square
\end{aligned}$$

定理6.12.3.  $\lambda^+(\hat{p}, 0; n)\tilde{\partial}_k\lambda(\hat{p}, 0; n) = 0$

证明:  $\lambda^+(\hat{p}, 0; n)\tilde{\partial}_k\lambda(\hat{p}, 0; n)$

$$\begin{aligned}
&= [\sqrt{C_{2n}^n}\bar{\Gamma}(n)\overbrace{\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^n\otimes\overbrace{\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^n]^+ \\
&\tilde{\partial}_k\sqrt{C_{2n}^n}\bar{\Gamma}(s)\overbrace{\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^n\otimes\overbrace{\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^n \\
&= C_{2n}^n\overbrace{\lambda^+(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, \frac{1}{2})}^n\otimes\overbrace{\lambda^+(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, -\frac{1}{2})}^n\Gamma(s)\bar{\Gamma}(s) \\
&[n\overbrace{\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^n\otimes\overbrace{\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^n + n\overbrace{\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^n\otimes\overbrace{\tilde{\partial}_k\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^n] \\
&= n[\overbrace{\lambda^+(\hat{p}, \frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, \frac{1}{2})\lambda(\hat{p}, \frac{1}{2})}^n\otimes\overbrace{\lambda^+(\hat{p}, -\frac{1}{2})\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, -\frac{1}{2})\lambda(\hat{p}, -\frac{1}{2})}^n \\
&+ \overbrace{\lambda^+(\hat{p}, \frac{1}{2})\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, \frac{1}{2})\lambda(\hat{p}, \frac{1}{2})}^n\otimes\overbrace{\lambda^+(\hat{p}, -\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda^+(\hat{p}, -\frac{1}{2})\lambda(\hat{p}, -\frac{1}{2})}^n] \\
&= n\lambda^+(\hat{p}, \frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2}) + n\lambda^+(\hat{p}, -\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, -\frac{1}{2}) \\
&= 0 \quad \square
\end{aligned}$$

定理6.12.4.  $\lambda^+(\hat{p}, -\frac{1}{2}; n + \frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2}; n + \frac{1}{2}) = (n + 1)\lambda^+(\hat{p}, -\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2})$

证明:  $\lambda^+(\hat{p}, -\frac{1}{2}; n + \frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2}; n + \frac{1}{2})$

$$= [\sqrt{C_{2n+1}^n}\bar{\Gamma}(n + \frac{1}{2})\overbrace{\lambda(\hat{p}, \frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, \frac{1}{2})}^n\otimes\overbrace{\lambda(\hat{p}, -\frac{1}{2})\otimes\cdots\otimes\lambda(\hat{p}, -\frac{1}{2})}^{n+1}]^+$$

$$\begin{aligned}
& \tilde{\partial}_k \sqrt{C_{2n+1}^n} \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{n+1} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^n \\
&= C_{2n+1}^n \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2})}^n \otimes \overbrace{\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2})}^{n+1} \Gamma(s) \bar{\Gamma}(s) \\
& [(n+1) \overbrace{\tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{n+1} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^n + n \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{n+1} \otimes \overbrace{\tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^n] \\
&= (n+1) \overbrace{[\lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2}) \lambda(\hat{p}, \frac{1}{2}) \otimes \lambda^+(\hat{p}, -\frac{1}{2}) \lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2}) \lambda(\hat{p}, -\frac{1}{2})]}^{n+1} + 0 \\
&= (n+1) \lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) \quad \square
\end{aligned}$$

**定理6.12.5.**  $\lambda^+(\hat{p}, \frac{1}{2}; n + \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}; n + \frac{1}{2}) = (n+1) \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2})$

$$\begin{aligned}
& \text{证明: } \lambda^+(\hat{p}, \frac{1}{2}; n + \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}; n + \frac{1}{2}) \\
&= [\sqrt{C_{2n+1}^n} \bar{\Gamma}(n + \frac{1}{2}) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{n+1} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^n]^+ \\
& \tilde{\partial}_k \sqrt{C_{2n+1}^n} \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^n \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{n+1} \\
&= C_{2n+1}^n \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2})}^{n+1} \otimes \overbrace{\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2})}^n \Gamma(s) \bar{\Gamma}(s) \\
& [n \overbrace{\tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^n \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{n+1} + (n+1) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^n \otimes \overbrace{\tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{n+1}] \\
&= 0 + (n+1) \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2}) \lambda(\hat{p}, \frac{1}{2})}^n \otimes \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2}) \lambda(\hat{p}, -\frac{1}{2})}^{n+1} \\
&= (n+1) \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) \quad \square
\end{aligned}$$

### 6.13 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数的通用解法三

**定理6.13.1.**  $\lambda^+(\hat{p}, h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) = 0, |h' - h| \geq 2$

$$\begin{aligned}
& \text{证明: } \lambda^+(\hat{p}, h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) \\
&= [\sqrt{C_{2s}^{s-h'}} \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h'} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h'}]^+ \\
& \tilde{\partial}_k \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&= \sqrt{C_{2s}^{s-h'}} \sqrt{C_{2s}^{s-h}} \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2})}^{s+h'} \otimes \overbrace{\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2})}^{s-h'} \Gamma(s) \bar{\Gamma}(s) \\
& [(s+h) \overbrace{\tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h} + (s-h) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h}] \\
&= 0 + 0 = 0 \quad \square
\end{aligned}$$

**定理6.13.2.**  $\lambda^+(\hat{p}, h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) = \sqrt{(s+h')(s-h)} \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}), h' - h = 1$

$$\begin{aligned}
& \text{证明: } \lambda^+(\hat{p}, h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) \\
&= [\sqrt{C_{2s}^{s-h'}} \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h'} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h'}]^+ \\
& \tilde{\partial}_k \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&= \sqrt{C_{2s}^{s-h'}} \sqrt{C_{2s}^{s-h}} \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2})}^{s+h'} \otimes \overbrace{\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2})}^{s-h'} \Gamma(s) \bar{\Gamma}(s) \\
& [(s+h) \overbrace{\tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h} + (s-h) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h}] \\
&= \sqrt{C_{2s}^{s-h'}} \sqrt{C_{2s}^{s-h}} \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2})}^{s+h'} \otimes \overbrace{\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2})}^{s-h'} \Gamma(s) \bar{\Gamma}(s)
\end{aligned}$$

$$\begin{aligned}
& (s-h) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&= \sqrt{C_{2s}^{s-h'}} \sqrt{C_{2s}^{s-h}} (C_{2s}^{s-h'})^{-1} (s-h) \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2}) \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2}) \lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&= \sqrt{(s+h')(s-h)} \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) \quad \square
\end{aligned}$$

**定理6.13.3.**  $\lambda^+(\hat{p}, h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) = \sqrt{(s-h')(s+h)} \lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}), h' - h = -1$

**证明:**  $\lambda^+(\hat{p}, h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s)$

$$\begin{aligned}
&= [\sqrt{C_{2s}^{s-h'}} \Gamma(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h'} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h'}]^+ \\
&\tilde{\partial}_k \sqrt{C_{2s}^{s-h}} \Gamma(s) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&= \sqrt{C_{2s}^{s-h'}} \sqrt{C_{2s}^{s-h}} \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2})}^{s+h'} \otimes \overbrace{\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2})}^{s-h'} \Gamma(s) \Gamma(s) \\
&[(s+h) \overbrace{\tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h'} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h} + (s-h) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\tilde{\partial}_k \lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h}] \\
&= \sqrt{C_{2s}^{s-h'}} \sqrt{C_{2s}^{s-h}} \overbrace{\lambda^+(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2})}^{s-h} \Gamma(s) \Gamma(s) \\
&(s+h) \overbrace{\tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h'} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&= \sqrt{C_{2s}^{s-h'}} \sqrt{C_{2s}^{s-h}} (C_{2s}^{s-h'})^{-1} (s+h) \\
&\overbrace{\lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{1}{2}) \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda^+(\hat{p}, -\frac{1}{2}) \lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, -\frac{1}{2}) \lambda(\hat{p}, -\frac{1}{2})}^{s-h} \\
&= \sqrt{(s-h')(s+h)} \lambda^+(\hat{p}, -\frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) \quad \square
\end{aligned}$$

## 6.14 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数性质的小结

**定理6.14.1.**

$$\begin{cases}
\lambda^+(\hat{p}, h; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) = 2h \lambda^+(\hat{p}, \frac{1}{2}) \tilde{\partial}_k \lambda(\hat{p}, \frac{1}{2}) = h \frac{-i\hat{p}_y \delta_{kx} + i\hat{p}_x \delta_{ky}}{p(1+\hat{p}_z)} \\
\lambda^+(\hat{p}, h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) = \sqrt{(s+\zeta h')(s-\zeta h)} \lambda^+(\hat{p}, \frac{\zeta}{2}) \tilde{\partial}_k \lambda(\hat{p}, -\frac{\zeta}{2}) \\
= \zeta \sqrt{(s+\zeta h')(s-\zeta h)} \frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x - i\zeta \hat{p}_y) - (1+\hat{p}_z)(\delta_{ix} - i\zeta \delta_{iy})}{2p(1+\hat{p}_z)}, h' - h = \zeta \\
\lambda^+(\hat{p}, h'; s) \tilde{\partial}_k \lambda(\hat{p}, h; s) = 0, |h' - h| \geq 2
\end{cases}$$

## 6.15 螺旋度 $\sigma(s) \cdot \hat{p}$ 本征函数导数的通用解法四

**引理6.15.1.**  $\Gamma(s; w) \bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w) = \Omega(s; w) \Gamma(s; w)$

**定理6.15.1.**  $\lambda^+(\hat{p}, \zeta s; s) \sigma(s) \tilde{\partial}_k \lambda(\hat{p}, -\zeta s; s) = 0, s \geq \frac{3}{2}$

**证明:**  $\lambda^+(\hat{p}, \zeta s; s) \sigma(s) \tilde{\partial}_k \lambda(\hat{p}, -\zeta s; s)$

$$\begin{aligned}
&= \overbrace{\lambda^+(\hat{p}, \frac{\zeta}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{\zeta}{2})}^{2s} \Gamma(s) [\bar{\Gamma}(s) \Omega(s) \Gamma(s)] \bar{\Gamma}(s) \overbrace{\tilde{\partial}_k \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\zeta}{2})}^{2s} \\
&= 2s \overbrace{\lambda^+(\hat{p}, \frac{\zeta}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{\zeta}{2})}^{2s} \Gamma(s) \bar{\Gamma}(s) \Omega(s) \Gamma(s) \bar{\Gamma}(s) \overbrace{\tilde{\partial}_k \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\zeta}{2})}^{2s} \\
&= 2s \overbrace{\lambda^+(\hat{p}, \frac{\zeta}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{\zeta}{2})}^{2s} \Gamma(s) \bar{\Gamma}(s) \Omega(s) \overbrace{\tilde{\partial}_k \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\zeta}{2})}^{2s} \\
&= 2s \overbrace{\lambda^+(\hat{p}, \frac{\zeta}{2}) \otimes \cdots \otimes \lambda^+(\hat{p}, \frac{\zeta}{2})}^{2s} \Omega(s) \overbrace{\tilde{\partial}_k \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{\zeta}{2})}^{2s} \\
&= 0 \quad \square
\end{aligned}$$

**推论6.15.1.**  $\lambda^+(\hat{p}, \zeta s; s) [\sigma_i(s) \tilde{\partial}_j - \sigma_j(s) \tilde{\partial}_i] \lambda(\hat{p}, -\zeta s; s) = 0, s \geq \frac{3}{2}$

定理6.15.2.  $\lambda^+(\hat{p}, -1; 1)\sigma(s)\tilde{\partial}_k\lambda(\hat{p}, 1; 1) = 0$

证明:  $\lambda^+(\hat{p}, -1; s)\sigma(s)\tilde{\partial}_k\lambda(\hat{p}, 1; 1)$   
 $= \lambda^+(\hat{p}, -\frac{1}{2}) \otimes \lambda^+(\hat{p}, -\frac{1}{2})\Gamma(1)\bar{\Gamma}(1)\Omega(1)\Gamma(1)\bar{\Gamma}(1)\tilde{\partial}_k[\lambda(\hat{p}, \frac{1}{2}) \otimes \lambda(\hat{p}, \frac{1}{2})]$   
 $= 2s\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \lambda^+(\hat{p}, -\frac{1}{2})\Gamma(1)\bar{\Gamma}(1)\Omega(1)\Gamma(1)\bar{\Gamma}(1)\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2}) \otimes \lambda(\hat{p}, \frac{1}{2})$   
 $= 2\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \lambda^+(\hat{p}, -\frac{1}{2})\Gamma(1)\bar{\Gamma}(1)\Omega(1)\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2}) \otimes \lambda(\hat{p}, \frac{1}{2})$   
 $= 2\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \lambda^+(\hat{p}, -\frac{1}{2})\Omega(1)\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2}) \otimes \lambda(\hat{p}, \frac{1}{2})$   
 $= \lambda^+(\hat{p}, -\frac{1}{2}) \otimes \lambda^+(\hat{p}, -\frac{1}{2})(\sigma \otimes I + I \otimes \sigma)\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2}) \otimes \lambda(\hat{p}, \frac{1}{2})$   
 $= [\lambda^+(\hat{p}, -\frac{1}{2})\tilde{\partial}_k\lambda(\hat{p}, \frac{1}{2})][\lambda^+(\hat{p}, -\frac{1}{2})\sigma\lambda(\hat{p}, \frac{1}{2})]$

□

性质6.15.1.

$$\lambda^+(\hat{p}, -\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p}, \frac{1}{2}) = -\frac{(\hat{p}_i + \delta_{iz})(\hat{p}_x + i\hat{p}_y) - (1 + \hat{p}_z)(\delta_{ix} + i\delta_{iy})}{2p(1 + \hat{p}_z)}$$

$$= -\left[ \begin{array}{c} \frac{\hat{p}_x(\hat{p}_x + i\hat{p}_y) - (1 + \hat{p}_z)}{2p(1 + \hat{p}_z)} \\ \frac{\hat{p}_y(\hat{p}_x + i\hat{p}_y) - i(1 + \hat{p}_z)}{2p(1 + \hat{p}_z)} \\ \frac{(1 + \hat{p}_z)(\hat{p}_x + i\hat{p}_y)}{2p(1 + \hat{p}_z)} \end{array} \right] = -\frac{1}{2p(1 + \hat{p}_z)} [(\hat{p}_x + i\hat{p}_y)\hat{p} + \left[ \begin{array}{c} -(1 + \hat{p}_z) \\ -i(1 + \hat{p}_z) \\ (\hat{p}_x + i\hat{p}_y) \end{array} \right]]$$

$$\lambda^+(\hat{p}, -\frac{1}{2})(\sigma, -i\varsigma)_a\lambda(\hat{p}, \frac{1}{2}) = \left[ \begin{array}{c} \frac{\hat{p}_x\hat{p}_z - i\hat{p}_y}{\hat{p}_x - i\hat{p}_y} \\ \frac{\hat{p}_y\hat{p}_z + i\hat{p}_x}{\hat{p}_x - i\hat{p}_y} \\ \frac{\hat{p}_z\hat{p}_z - 1}{\hat{p}_x - i\hat{p}_y} \\ 0 \end{array} \right] = \frac{1}{\hat{p}_x - i\hat{p}_y} (\hat{p}_z\hat{p} + \left[ \begin{array}{c} -i\hat{p}_y \\ i\hat{p}_x \\ -1 \\ 0 \end{array} \right])$$

推论6.15.2.  $\lambda^+(\hat{p}, -1; 1)[\sigma_i(1)\tilde{\partial}_j - \sigma_j(1)\tilde{\partial}_i]\lambda(\hat{p}, 1; 1)$   
 $= [\lambda^+(\hat{p}, -\frac{1}{2})\tilde{\partial}_j\lambda(\hat{p}, \frac{1}{2})][\lambda^+(\hat{p}, -\frac{1}{2})\sigma_i\lambda(\hat{p}, \frac{1}{2})] - [\lambda^+(\hat{p}, -\frac{1}{2})\tilde{\partial}_i\lambda(\hat{p}, \frac{1}{2})][\lambda^+(\hat{p}, -\frac{1}{2})\sigma_j\lambda(\hat{p}, \frac{1}{2})]$   
 $= 0$

推论6.15.3.  $\lambda^+(\hat{p}, \varsigma s; s)[\sigma_i(s)\tilde{\partial}_j - \sigma_j(s)\tilde{\partial}_i]\lambda(\hat{p}, -\varsigma s; s) = 0, s \geq 1$

6.16 特例:1-自旋本征态 $\lambda(\hat{p}, h; 1)$ 分解为 $\frac{1}{2}$ -自旋本征态

引理6.16.1.  $\left\{ \begin{array}{l} \Gamma_{\alpha_\varsigma}^{k_\varsigma}(1) \succ S_m(1) = \frac{1}{\sqrt{2}} \left[ \begin{array}{ccc} i & 0 & -i \\ -1 & 0 & -1 \\ 0 & -i\sqrt{2} & 0 \end{array} \right] \\ \Gamma_{k_\varsigma}^{\alpha_\varsigma}(1) \succ S_m^+(1) = \frac{1}{\sqrt{2}} \left[ \begin{array}{ccc} -i & -1 & 0 \\ 0 & 0 & i\sqrt{2} \\ i & -1 & 0 \end{array} \right] \end{array} \right\} \left\{ \begin{array}{l} \Gamma^{\alpha_\varsigma k_\varsigma}(1) \succ S_m^*(1) \\ \Gamma^{k_\varsigma \alpha_\varsigma}(1) \succ S_m^T(1) \end{array} \right.$

引理6.16.2.  $[S_m(1)\bar{\Gamma}(1)]_{\alpha_\varsigma}^{A_\varsigma B_\varsigma} = \frac{1}{\sqrt{2}} \left[ \begin{array}{ccc} i & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -i & -i \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \\ -1 \end{array} \right]$ ,  $[S_m(1)\bar{\Gamma}(1)]_{\alpha_\varsigma}^{A_\varsigma B_\varsigma} = -\frac{1}{\sqrt{2}}\sigma_y\sigma = \frac{i\varsigma}{\sqrt{2}}\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}$

定理6.16.1.  $\lambda(\hat{p}, h; 1) = \sqrt{C_2^{1-h}}\bar{\Gamma}(1) \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, \frac{1}{2})}^{1+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \cdots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{1-h}$

推论6.16.1.  $\left\{ \begin{array}{l} \lambda(\hat{p}, -\varsigma; 1) = \bar{\Gamma}(1)\lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \lambda(\hat{p}, -\frac{\varsigma}{2}) = \lambda^T(\hat{p}, -\frac{\varsigma}{2})\Gamma\lambda(\hat{p}, -\frac{\varsigma}{2}) \\ \lambda(\hat{p}, 0; 1) = \sqrt{C_2^1}\bar{\Gamma}(1)\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \lambda(\hat{p}, -\frac{\varsigma}{2}) = \sqrt{C_2^1}\lambda^T(\hat{p}, \frac{\varsigma}{2})\Gamma\lambda(\hat{p}, -\frac{\varsigma}{2}) \end{array} \right.$

推论6.16.2.  $\left\{ \begin{array}{l} \lambda_m(\hat{p}, -\varsigma; 1) = S_m(1)\bar{\Gamma}(1)\lambda(\hat{p}, -\frac{\varsigma}{2}) \otimes \lambda(\hat{p}, -\frac{\varsigma}{2}) = -\frac{1}{\sqrt{2}}\lambda^T(\hat{p}, -\frac{\varsigma}{2})\sigma_y\sigma\lambda(\hat{p}, -\frac{\varsigma}{2}) \\ \lambda_m(\hat{p}, 0; 1) = \sqrt{C_2^1}S_m(1)\bar{\Gamma}(1)\lambda(\hat{p}, \frac{\varsigma}{2}) \otimes \lambda(\hat{p}, -\frac{\varsigma}{2}) = -\lambda^T(\hat{p}, \frac{\varsigma}{2})\sigma_y\sigma\lambda(\hat{p}, -\frac{\varsigma}{2}) \end{array} \right.$

推论6.16.3.  $\left\{ \begin{array}{l} \lambda_{k_\varsigma}(\hat{p}, -\varsigma; 1) = \Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \\ \lambda_{k_\varsigma}(\hat{p}, 0; 1) = \sqrt{C_2^1}\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p}, \frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \end{array} \right.$

推论6.16.4.  $\left\{ \begin{array}{l} \lambda_{m\alpha_\varsigma}(\hat{p}, -\varsigma; 1) = \frac{i\varsigma}{\sqrt{2}}\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) = -\frac{1}{\sqrt{2}}(\sigma_y\sigma)_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \\ \lambda_{m\alpha_\varsigma}(\hat{p}, 0; 1) = \sqrt{C_2^1}\frac{i\varsigma}{\sqrt{2}}\sigma_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p}, \frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) = -(\sigma_y\sigma)_{\alpha_\varsigma}^{A_\varsigma B_\varsigma}\lambda_{A_\varsigma}(\hat{p}, \frac{\varsigma}{2})\lambda_{B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \end{array} \right.$

## 6.16.1 螺旋度的上升和下降

定义6.16.1.

$$\begin{cases} \hat{Q}(\hat{p}, s) := \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} \hat{Q} \exp\{-i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\}, \hat{Q}(s) := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{(2s+1) \times (2s+1)} \\ \hat{Q}^+(\hat{p}, s) := \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} \hat{Q}^+(s) \exp\{-i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\}, \hat{Q}^+(s) := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}_{(2s+1) \times (2s+1)} \end{cases}$$

$$\text{推论6.16.5. } \hat{Q}(\hat{p}, s) \hat{Q}^+(\hat{p}, s) = \hat{Q}^+(\hat{p}, s) \hat{Q}(\hat{p}, s) = \hat{Q}(s) \hat{Q}^+(s) = \hat{Q}^+(s) \hat{Q}(s) = 1$$

推论6.16.6.

$$\begin{cases} \hat{Q}(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) = \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h-1; s\right), \hat{Q}(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, -s; s\right) = \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s; s\right), h = s, s-1, \dots, -(s-1) \\ \hat{Q}^+(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) = \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h+1; s\right), \hat{Q}^+(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, s; s\right) = \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, -s; s\right), h = -s, -(s-1), \dots, s-1 \end{cases}$$

推论6.16.7.

$$\begin{cases} \hat{Q}(\hat{p}, s) \lambda(\hat{p}, h; s) = \lambda(\hat{p}, h-1; s), \hat{Q}(\hat{p}, s) \lambda(\hat{p}, -s; s) = \lambda(\hat{p}, s; s), h = s, s-1, \dots, -(s-1) \\ \hat{Q}^+(\hat{p}, s) \lambda(\hat{p}, h; s) = \lambda(\hat{p}, h+1; s), \hat{Q}^+(\hat{p}, s) \lambda(\hat{p}, s; s) = \lambda(\hat{p}, -s; s), h = -s, -(s-1), \dots, s-1 \end{cases}$$

推论6.16.8.

$$\begin{cases} \sigma(s) \cdot \hat{p} \hat{Q}(\hat{p}, s) \lambda(\hat{p}, h; s) = (h-1) \lambda(\hat{p}, h-1; s), \hat{Q}(\hat{p}, s) \sigma(s) \cdot \hat{p} \lambda(\hat{p}, h; s) = h \lambda(\hat{p}, h-1; s) \\ \sigma(s) \cdot \hat{p} \hat{Q}(\hat{p}, s) \lambda(\hat{p}, -s; s) = s \lambda(\hat{p}, s; s), \hat{Q}(\hat{p}, s) \sigma(s) \cdot \hat{p} \lambda(\hat{p}, -s; s) = -s \lambda(\hat{p}, s; s) \\ h = -(s-1), \dots, s-1, s \end{cases}$$

推论6.16.9.

$$\begin{cases} \sigma(s) \cdot \hat{p} \hat{Q}^+(\hat{p}, s) \lambda(\hat{p}, h; s) = (h+1) \lambda(\hat{p}, h+1; s), \hat{Q}^+(\hat{p}, s) \sigma(s) \cdot \hat{p} \lambda(\hat{p}, h; s) = h \lambda(\hat{p}, h+1; s) \\ \sigma(s) \cdot \hat{p} \hat{Q}^+(\hat{p}, s) \lambda(\hat{p}, s; s) = -s \lambda(\hat{p}, -s; s), \hat{Q}^+(\hat{p}, s) \sigma(s) \cdot \hat{p} \lambda(\hat{p}, s; s) = s \lambda(\hat{p}, -s; s) \\ h = -s, -(s-1), \dots, s-1 \end{cases}$$

推论6.16.10.

$$\begin{cases} [\sigma(s) \cdot \hat{p}, \hat{Q}(\hat{p}, s)] \lambda(\hat{p}, h; s) = -\hat{Q}(\hat{p}, s) \lambda(\hat{p}, h; s), [\sigma(s) \cdot \hat{p}, \hat{Q}(\hat{p}, s)] \lambda(\hat{p}, -s; s) = 2s \hat{Q}(\hat{p}, s) \lambda(\hat{p}, -s; s) \\ \{\sigma(s) \cdot \hat{p}, \hat{Q}(\hat{p}, s)\} \lambda(\hat{p}, h; s) = (2h-1) \hat{Q}(\hat{p}, s) \lambda(\hat{p}, h; s), \{\sigma(s) \cdot \hat{p}, \hat{Q}(\hat{p}, s)\} \lambda(\hat{p}, -s; s) = 0 \\ h = -(s-1), \dots, s-1, s \end{cases}$$

推论6.16.11.

$$\begin{cases} [\sigma(s) \cdot \hat{p}, \hat{Q}^+(\hat{p}, s)] \lambda(\hat{p}, h; s) = \hat{Q}^+(\hat{p}, s) \lambda(\hat{p}, h; s), [\sigma(s) \cdot \hat{p}, \hat{Q}^+(\hat{p}, s)] \lambda(\hat{p}, s; s) = -2\hat{Q}^+(\hat{p}, s) \lambda(\hat{p}, s; s) \\ \{\sigma(s) \cdot \hat{p}, \hat{Q}^+(\hat{p}, s)\} \lambda(\hat{p}, h; s) = (2h+1) \hat{Q}^+(\hat{p}, s) \lambda(\hat{p}, h; s), \{\sigma(s) \cdot \hat{p}, \hat{Q}^+(\hat{p}, s)\} \lambda(\hat{p}, s; s) = 0 \\ h = -s, -(s-1), \dots, s-1 \end{cases}$$

## 6.17 螺旋度本征函数的算符化—新数学工具

$$\text{定义6.17.1. } \lambda(\hat{\nabla}, h; s) := \lambda(\hat{p}, h; s)|_{\hat{p} \rightarrow \hat{\nabla}}, \hat{\nabla} := \frac{-i\nabla}{\sqrt{-\nabla^2}}$$

$$\text{推论6.17.1. } \lambda(\hat{\nabla}, h; s) = \exp\{i \frac{[\sigma(s) \times \hat{\nabla}]_z}{\sqrt{1-\hat{\nabla}_z^2}} \arccos \hat{\nabla}_z\} \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right)$$

$$\text{推论6.17.2. } \sigma(s) \cdot \hat{\nabla} \lambda(\hat{\nabla}, h; s) = h \lambda(\hat{\nabla}, s; s), h = -s, \dots, s$$

$$\text{推论6.17.3. } \lambda^+(\hat{\nabla}, h; s) \lambda(\hat{\nabla}, h'; s) = \delta_{hh'}, \sum_{h=s}^{-s} \lambda(\hat{\nabla}, h; s) \lambda^+(\hat{\nabla}, h; s) = 1$$

$$\text{推论6.17.4. } \lambda(-\hat{\nabla}, h; s) = (-1)^{s+|h|} \left(\frac{\hat{\nabla}_+}{\hat{\nabla}_-}\right)^h \lambda(\hat{\nabla}, -h; s)$$

$$\text{推论6.17.5. } \lambda^+(\hat{\nabla}, h; s) \sigma(s) \lambda(\hat{\nabla}, h; s) = h \hat{\nabla}, h = -s, \dots, s$$

推论6.17.6.  $\lambda^+(-\hat{\nabla}, h; s)\sigma(s)\lambda(\hat{\nabla}, h; s) = 0, \lambda^+(\hat{\nabla}, -h; s)\sigma(s)\lambda(\hat{\nabla}, h; s) = 0, h = -s, \dots, s$

$$\text{推论6.17.7. } \begin{cases} \Gamma_{k_\zeta k'_\zeta}^{abc \cdot \cdot} (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdot \cdot}^{2s} = (i\sqrt{2})^{2s} \lambda_{k_\zeta}(\hat{p}, -s) \lambda_{k'_\zeta}^+(\hat{p}, -s) \\ \Gamma_{k_\zeta k'_\zeta}^{abc \cdot \cdot} (s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \cdot \cdot}^{2s} = (i\sqrt{2})^{2s} \lambda_{k_\zeta}(\hat{\nabla}, -s) \lambda_{k'_\zeta}^+(\hat{\nabla}, -s) \end{cases}$$

$$\text{推论6.17.8. } \tilde{\partial}_k \left\{ \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z \right\} \\ = - \left\{ \frac{[\sigma(s) \times \hat{p}]_z}{1-\hat{p}_z^2} \right\} \tilde{\partial}_k \hat{p}_z + \left\{ \frac{[\sigma(s) \times \hat{p}]_z \hat{p}_z}{(1-\hat{p}_z^2)^{3/2}} \arccos \hat{p}_z \right\} \tilde{\partial}_k \hat{p}_z + \left\{ \frac{\arccos \hat{p}_z}{\sqrt{1-\hat{p}_z^2}} \right\} \tilde{\partial}_k [\sigma(s) \times \hat{p}]_z$$

## 7 螺旋度 $\sigma(s) \cdot \hat{p}$ 的解析延拓(还需严格化)

### 7.1 螺旋度 $\sigma(\frac{1}{2}) \cdot \hat{p}, \hat{p} \in C$ 本征函数的分析

定义7.1.1.  $\tilde{\lambda}^T(\hat{p}, \frac{1}{2}) := -i\lambda^T(\hat{p}, -\frac{1}{2})\sigma_y, \tilde{\lambda}^T(\hat{p}, -\frac{1}{2}) := i\lambda^T(\hat{p}, \frac{1}{2})\sigma_y, \hat{p} = \frac{\vec{p}}{\sqrt{\vec{p} \cdot \vec{p}}} \in C$

$$\text{推论7.1.1. } \lambda(\hat{p}, \frac{1}{2}) = \frac{1}{\sqrt{1+\hat{p}_z}} \begin{bmatrix} \frac{1}{\sqrt{2}}(1+\hat{p}_z) \\ \hat{p}_+ \end{bmatrix}, \lambda(\hat{p}, -\frac{1}{2}) = \frac{1}{\sqrt{1+\hat{p}_z}} \begin{bmatrix} -\hat{p}_- \\ \frac{1}{\sqrt{2}}(1+\hat{p}_z) \end{bmatrix}, \hat{p}^2 = 1, \hat{p} \in C$$

推论7.1.2.  $[\sigma(\frac{1}{2}) \cdot \hat{p}]\lambda(\hat{p}, \frac{1}{2}) = \frac{1}{2}\lambda(\hat{p}, \frac{1}{2}), [\sigma(\frac{1}{2}) \cdot \hat{p}]\lambda(\hat{p}, -\frac{1}{2}) = -\frac{1}{2}\lambda(\hat{p}, -\frac{1}{2}), \hat{p}^2 = 1, \hat{p} \in C$

性质7.1.1.  $\tilde{\lambda}^T(\hat{p}, \frac{1}{2}) = \lambda^+(\hat{p}, \frac{1}{2}), \tilde{\lambda}^T(\hat{p}, -\frac{1}{2}) = \lambda^+(\hat{p}, -\frac{1}{2}), \hat{p} \in R$

$$\text{推论7.1.3. } \tilde{\lambda}^T(\hat{p}, h)\lambda(\hat{p}, h') = \delta_{hh'}, \sum_{h=\frac{1}{2}}^{-\frac{1}{2}} \lambda(\hat{p}, h)\tilde{\lambda}^T(\hat{p}, h) = 1, \sum_{h=\frac{1}{2}}^{-\frac{1}{2}} h\lambda(\hat{p}, h)\tilde{\lambda}^T(\hat{p}, h) = \sigma(\frac{1}{2}) \cdot \hat{p}, \hat{p} \in C$$

### 7.2 螺旋度 $\sigma(s) \cdot \hat{p}, \hat{p} \in C$ 本征函数的分析

$$\text{定义7.2.1. } \lambda(\hat{p}, h; s) := \sqrt{C_{2s}^{s-h}} \overline{\Gamma(s)} \overbrace{\lambda(\hat{p}, \frac{1}{2}) \otimes \dots \otimes \lambda(\hat{p}, \frac{1}{2})}^{s+h} \otimes \overbrace{\lambda(\hat{p}, -\frac{1}{2}) \otimes \dots \otimes \lambda(\hat{p}, -\frac{1}{2})}^{s-h}, \hat{p} \in C$$

定义7.2.2.

$$\tilde{\lambda}^T(\hat{p}, h; s) := (-1)^h \sqrt{C_{2s}^{s-h}} \overbrace{\lambda^T(\hat{p}, \frac{1}{2})\sigma_y \otimes \dots \otimes \lambda^T(\hat{p}, \frac{1}{2})\sigma_y}^{s+h} \otimes \overbrace{\lambda^T(\hat{p}, -\frac{1}{2})\sigma_y \otimes \dots \otimes \lambda^T(\hat{p}, -\frac{1}{2})\sigma_y}^{s-h} \Gamma(s), \hat{p} \in C$$

推论7.2.1.  $[\sigma(s) \cdot \hat{p}]\lambda(\hat{p}, h; s) = h\lambda(\hat{p}, h; s), \hat{p}^2 = 1, \hat{p} \in C$

性质7.2.1.  $\tilde{\lambda}^T(\hat{p}, h; s) = \lambda^+(\hat{p}, h; s), \hat{p} \in R$

$$\text{推论7.2.2. } \tilde{\lambda}^T(\hat{p}, h; s)\lambda(\hat{p}, h'; s) = \delta_{hh'}, \sum_{h=s}^{-s} \lambda(\hat{p}, h; s)\tilde{\lambda}^T(\hat{p}, h; s) = 1, \sum_{h=s}^{-s} h\lambda(\hat{p}, h; s)\tilde{\lambda}^T(\hat{p}, h; s) = \sigma(s) \cdot \hat{p}, \hat{p} \in C$$

### 7.3 螺旋度 $\sigma(s) \cdot \hat{p}, \hat{p} \in C$ 杂分析

$$\text{定理7.3.1. } \tilde{\partial}_i \lambda(\hat{p}, 1; 1) = \frac{1}{p(1+\hat{p}_z)} \{ -[\delta_{iz} + \hat{p}_i(2 + \hat{p}_z)]\lambda(\hat{p}, 1; 1) + [(\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) + (\hat{p}_x - i\hat{p}_y)(\delta_{ix} + i\delta_{iy})]\lambda(\hat{p}, 1; 1) \\ + \left[ \begin{array}{c} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ \frac{1}{\sqrt{2}}[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{array} \right] - [(\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) + (\hat{p}_x - i\hat{p}_y)(\delta_{ix} + i\delta_{iy})]\lambda(\hat{p}, 1; 1) \} \\ = \frac{1}{p(1+\hat{p}_z)} \{ (-i\hat{p}_y\delta_{ix} + i\hat{p}_x\delta_{iy})\lambda(\hat{p}, 1; 1) \\ + \left[ \begin{array}{c} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ \frac{1}{\sqrt{2}}[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{array} \right] \\ - [(\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) + (\hat{p}_x - i\hat{p}_y)(\delta_{ix} + i\delta_{iy})] \frac{1}{\hat{p}_-} \left[ \begin{array}{c} \frac{1}{2}\hat{p}_-(1+\hat{p}_z) \\ \hat{p}_+ \hat{p}_- \\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z) \end{array} \right] \}$$



$$\begin{aligned}
&= \frac{1}{p(1+\hat{p}_z)} \{ (-i\hat{p}_y\delta_{ix} + i\hat{p}_x\delta_{iy})\lambda(\hat{p}, 1; 1) \\
&+ \left[ \begin{array}{c} (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \\ \frac{1}{\sqrt{2}}[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{array} \right] \\
&- (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \frac{1}{\hat{p}_-} \left[ \begin{array}{c} \frac{1}{2}\hat{p}_-(1+\hat{p}_z) \\ \hat{p}_+\hat{p}_- \\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z) \end{array} \right] - \sqrt{2}(\delta_{ix} + i\delta_{iy}) \left[ \begin{array}{c} \frac{1}{2}\hat{p}_-(1+\hat{p}_z) \\ \hat{p}_+\hat{p}_- \\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z) \end{array} \right] \} \\
&= \frac{1}{p(1+\hat{p}_z)} \{ (-i\hat{p}_y\delta_{ix} + i\hat{p}_x\delta_{iy})\lambda(\hat{p}, 1; 1) \\
&+ \left[ \begin{array}{c} \frac{1}{2}(\delta_{iz} + \hat{p}_i)(1 - \hat{p}_z)^2 \\ \frac{1}{\sqrt{2}}[(\delta_{ix} + i\delta_{iy})(1 + \hat{p}_z) + (\delta_{iz} + \hat{p}_i)(\hat{p}_x + i\hat{p}_y)] \\ (\delta_{ix} + i\delta_{iy})(\hat{p}_x + i\hat{p}_y) \end{array} \right] \\
&- (\delta_{iz} + \hat{p}_i)(1 + \hat{p}_z) \frac{1}{\hat{p}_-} \left[ \begin{array}{c} \frac{1}{2}\hat{p}_-(1+\hat{p}_z) \\ \hat{p}_+\hat{p}_- \\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z) \end{array} \right] - \sqrt{2}(\delta_{ix} + i\delta_{iy}) \left[ \begin{array}{c} \frac{1}{2}\hat{p}_-(1+\hat{p}_z) \\ \hat{p}_+\hat{p}_- \\ \frac{1}{2}\hat{p}_+(1-\hat{p}_z) \end{array} \right] \}
\end{aligned}$$

# 第十六章 自旋代数的数学分析

自我评述：为了进一步研究一般自旋粒子的物理内容，在本章节中我独自发展了以下关于自旋代数的数学分析，为研究各种自旋粒子提供了又一个新的数学工具。

## 1 自旋-单位矢量算符的基本代数性质

### 1.1 $\sigma(s), \vec{p}$ 的基本运算规则

$$\text{性质1.1.1.} \quad \begin{cases} \sigma(s) \times \vec{p} = [\sigma(s), i\sigma(s) \cdot \vec{p}] = i\{\sigma(s)[\sigma(s) \cdot \vec{p}] - [\sigma(s) \cdot \vec{p}]\sigma(s)\} \\ \vec{p} \times \sigma(s) = -[\sigma(s), i\sigma(s) \cdot \vec{p}] = i\{[\sigma(s) \cdot \vec{p}]\sigma(s) - \sigma(s)[\sigma(s) \cdot \vec{p}]\} \end{cases}$$

$$\text{性质1.1.2.} \quad \begin{cases} \sigma(s) \times \vec{p} = -\vec{p} \times \sigma(s) & \begin{cases} \sigma(s) \cdot \vec{p} = \vec{p} \cdot \sigma(s) \\ \sigma^2(s) = s(s+1) \end{cases} & \begin{cases} [\sigma(s) \times \vec{p}] \cdot \vec{p} = 0 \\ \sigma(s) \cdot [\sigma(s) \times \vec{p}] = i\sigma(s) \cdot \vec{p} \\ [\vec{p} \times \sigma(s)] \cdot \sigma(s) = i\sigma(s) \cdot \vec{p} \end{cases} \end{cases}$$

$$\text{性质1.1.3.} \quad \begin{cases} [\sigma(s) \times \vec{p}] \times \vec{p} = [\sigma(s) \cdot \vec{p}]\vec{p} - \sigma(s)(\vec{p} \cdot \vec{p}) \\ \sigma(s) \times [\sigma(s) \times \vec{p}] = [\sigma(s) \cdot \vec{p}]\sigma(s) - \sigma^2(s)\vec{p} \\ [\vec{p} \times \sigma(s)] \times \sigma(s) = \sigma(s)[\sigma(s) \cdot \vec{p}] - \sigma^2(s)\vec{p} \\ \sigma(s) \times \{\sigma(s) \times [\sigma(s) \times \vec{p}]\} = \sigma(s) \times \{[\sigma(s) \cdot \vec{p}]\sigma(s)\} - \sigma(s) \times \vec{p}\sigma^2(s) \\ \sigma(s) \times \{\sigma(s) \times [\sigma(s) \times \vec{p}]\} = \sigma(s) \times \{[\sigma(s) \cdot \vec{p}]\sigma(s)\} - \sigma(s) \times \vec{p}\sigma^2(s) \end{cases}$$

$$\text{性质1.1.4.} \quad \begin{cases} i^{2k-1}\hat{p} \cdot \{\sigma(s) \times \{\sigma(s)[\times\hat{p}]^{2k-1}\}\} = i\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} \\ i^{2k}\hat{p} \cdot \{\sigma(s) \times \{\sigma(s)[\times\hat{p}]^{2k}\}\} = i\sigma(s) \cdot \hat{p} \end{cases}$$

$$\text{性质1.1.5.} \quad \begin{cases} \{[\sigma(s) \times \vec{p}] \times \vec{p}\} \cdot \vec{p} = 0 & \begin{cases} \vec{p} \cdot \{[\sigma(s) \times \vec{p}] \times \vec{p}\} = 0 \\ \sigma(s) \cdot \{\sigma(s) \times [\sigma(s) \times \vec{p}]\} = -\sigma(s) \cdot \vec{p} \\ \{[\vec{p} \times \sigma(s)] \times \sigma(s)\} \cdot \sigma(s) = -\sigma(s) \cdot \vec{p} \end{cases} \\ \{\sigma(s) \times [\sigma(s) \times \vec{p}]\} \cdot \sigma(s) = 0 \\ \sigma(s) \cdot \{[\vec{p} \times \sigma(s)] \times \sigma(s)\} = 0 \end{cases}$$

$$\text{性质1.1.6.} \quad \begin{cases} \{[\sigma(s) \times \vec{p}] \times \vec{p}\} \cdot \sigma(s) = [\sigma(s) \cdot \vec{p}]^2 - \sigma^2(s)\vec{p}^2 \\ \sigma(s) \cdot \{[\sigma(s) \times \vec{p}] \times \vec{p}\} = [\sigma(s) \cdot \vec{p}]^2 - \sigma^2(s)\vec{p}^2 \\ \{\sigma(s) \times [\sigma(s) \times \vec{p}]\} \cdot \vec{p} = [\sigma(s) \cdot \vec{p}]^2 - \sigma^2(s)\vec{p}^2 \\ \vec{p} \cdot \{[\vec{p} \times \sigma(s)] \times \sigma(s)\} = [\sigma(s) \cdot \vec{p}]^2 - \sigma^2(s)\vec{p}^2 \\ -[\sigma(s) \times \vec{p}] \cdot [\sigma(s) \times \vec{p}] = [\sigma(s) \cdot \vec{p}]^2 - \sigma^2(s)\vec{p}^2 \end{cases}$$

$$\text{性质1.1.7.} \quad \begin{cases} [\sigma(s) \times \vec{p}] \times [\sigma(s) \times \vec{p}] = i[\sigma(s) \cdot \vec{p}]\vec{p} = i\{[\sigma(s) \times \vec{p}] \times \vec{p} + \sigma(s)\} \\ \{[\sigma(s) \times \vec{p}] \times [\sigma(s) \times \vec{p}]\} \cdot \vec{p} = i[\sigma(s) \cdot \vec{p}]\vec{p}^2 \\ \sigma(s) \cdot \{[\sigma(s) \times \vec{p}] \times [\sigma(s) \times \vec{p}]\} = i[\sigma(s) \cdot \vec{p}]^2 \\ \{[\sigma(s) \times \vec{p}] \times [\sigma(s) \times \vec{p}]\} \cdot \sigma(s) = i[\sigma(s) \cdot \vec{p}]^2 \end{cases}$$

$$\text{性质1.1.8.} \quad \begin{cases} [\sigma(s) \times \vec{p}] \times \{[\sigma(s) \times \vec{p}] \times [\sigma(s) \times \vec{p}]\} = i\{[\sigma(s) \times \vec{p}] \times \vec{p}\}[\sigma(s) \cdot \vec{p}] \\ \{[\sigma(s) \times \vec{p}] \times [\sigma(s) \times \vec{p}]\} \times [\sigma(s) \times \vec{p}] = -i[\sigma(s) \cdot \vec{p}]\{[\sigma(s) \times \vec{p}] \times \vec{p}\} \\ [\sigma(s) \times \vec{p}] \cdot \{[\sigma(s) \times \vec{p}] \times [\sigma(s) \times \vec{p}]\} = 0 \\ \{[\sigma(s) \times \vec{p}] \times [\sigma(s) \times \vec{p}]\} \cdot [\sigma(s) \times \vec{p}] = 0 \end{cases}$$

$$\text{性质1.1.9.} \begin{cases} [\sigma(s) \times \vec{p}] \times [\sigma(s) \times \vec{p}] = i[\sigma(s) \cdot \vec{p}]\vec{p} = i\{[\sigma(s) \times \vec{p}] \times \vec{p} + \sigma(s)\} \\ [\sigma(s) \times \vec{p}] \times \{[\sigma(s) \times \vec{p}] \times [\sigma(s) \times \vec{p}]\} = i\{[\sigma(s) \times \vec{p}] \times \vec{p}\}[\sigma(s) \cdot \vec{p}] \end{cases}$$

$$\text{性质1.1.10.} \begin{cases} \{[\sigma(s) \times \hat{p}] \times \hat{p}\} \times \vec{p} = -\sigma(s) \times \hat{p} \\ \sigma(s) \times [\sigma(s) \times \vec{p}] = [\sigma(s) \cdot \vec{p}]\sigma(s) - \sigma^2(s)\vec{p} \\ [\vec{p} \times \sigma(s)] \times \sigma(s) = \sigma(s)[\sigma(s) \cdot \vec{p}] - \sigma^2(s)\vec{p} \end{cases}$$

$$\text{性质1.1.11.} \begin{cases} \{\hat{p}\} \times \hat{p} = \{\vec{p}; \hat{0}\} \times \hat{p} = \{\vec{p}; \hat{0}\} \\ \{\sigma(s)\} \times \hat{p} = \{\sigma(s); \sigma(s) \times \hat{p}\} \times \hat{p} = \dots = \{\sigma(s); \pm\sigma(s) \times \hat{p}, \pm[\sigma(s) \times \hat{p}] \times \hat{p}\} \end{cases}$$

$$\text{性质1.1.12.} \begin{cases} \{\sigma(s)\} \times \sigma(s) = \dots = \{\pm\sigma(s), \pm i\sigma(s)\} \\ \{\hat{p}\} \times \sigma(s) = \{\hat{p}; \hat{p} \times \sigma(s)\} \times \sigma(s) = \dots \end{cases}$$

$$\text{性质1.1.13.} \{\sigma[\sigma \cdot \hat{p}]\} \times \sigma = -2i\hat{p}$$

## 1.2 自旋-单位矢量算符叉乘代数的分类

性质1.2.1.

$$\begin{cases} [\hat{p} \times ]^n \hat{p} = \hat{p} [ \times \hat{p}]^n = \vec{0} \\ \langle \hat{p}; \times \hat{p} \rangle = \langle \hat{p}; \hat{p} \times \rangle \prec (\{\hat{p}, \vec{0}\}, \times) \\ \text{满足封闭性; 满足交换律和结合律; 有零元但无单位元。} \end{cases}$$

性质1.2.2.

$$\begin{cases} [\sigma(s) \times ]^n \sigma(s) = \sigma(s) [ \times \sigma(s)]^n = i^n \sigma(s) \\ \langle \sigma(s); \times \sigma(s) \rangle = \langle \sigma(s); \sigma(s) \times \rangle \prec (\{i^0 \sigma(s), i^1 \sigma(s), i^2 \sigma(s), i^3 \sigma(s)\}, \times) \\ \text{满足封闭性; 满足交换律和结合律; 无零元和单位元。} \end{cases}$$

性质1.2.3.

$$\begin{cases} \sigma(s) [ \times \hat{p}]^n = i^{-n} \{1: i\sigma(s) \times \hat{p}, 2: -\{[\sigma(s) \cdot \hat{p}]\hat{p} - \sigma(s)\}\} \\ \langle \sigma(s); \times \hat{p} \rangle \prec (\{\sigma(s), \pm\sigma(s) \times \hat{p}, \pm\{[\sigma(s) \cdot \hat{p}]\hat{p} - \sigma(s)\}\}, \times \hat{p}) \\ \text{满足封闭性; 不满足交换律和结合律; 无零元和单位元。} \end{cases}$$

性质1.2.4.

$$\begin{cases} [\hat{p} \times ]^n \sigma(s) = i^n \{1: -i\hat{p} \times \sigma(s), 2: -\{[\sigma(s) \cdot \hat{p}]\hat{p} - \sigma(s)\}\} \\ \langle \sigma(s); \hat{p} \times \rangle \prec (\{\sigma(s), \pm\hat{p} \times \sigma(s), \pm\{[\sigma(s) \cdot \hat{p}]\hat{p} - \sigma(s)\}\}, \hat{p} \times) \\ \text{满足封闭性; 不满足交换律和结合律; 无零元和单位元。} \end{cases}$$

性质1.2.5.

$$\begin{cases} \hat{p} [ \times \sigma(s)]^n = i^n \{a_n [\sigma(s) \cdot \hat{p}]\sigma(s) + b_n \sigma(s) [\sigma(s) \cdot \hat{p}] - c_n \sigma^2(s) \hat{p}\} \\ \langle \hat{p}; \times \sigma(s) \rangle \prec (i^n \{a_n [\sigma(s) \cdot \hat{p}]\sigma(s) + b_n \sigma(s) [\sigma(s) \cdot \hat{p}] - c_n \sigma^2(s) \hat{p} | n \geq 0\}, \times \sigma(s)) \\ \text{满足封闭性; 不满足交换律和结合律; 无零元和单位元。} \end{cases}$$

性质1.2.6.

$$\begin{cases} [\sigma(s) \times ]^n \hat{p} = i^n \{a_n \sigma(s) [\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}]\sigma(s) - c_n \sigma^2(s) \hat{p}\} \\ \langle \hat{p}; \sigma(s) \times \rangle \prec (i^n \{a_n \sigma(s) [\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}]\sigma(s) - c_n \sigma^2(s) \hat{p} | n \geq 0\}, \sigma(s) \times) \\ \text{满足封闭性; 不满足交换律和结合律; 无零元和单位元。} \end{cases}$$

性质1.2.7.

$$\langle \sigma(s) \times \hat{p}; \sigma(s) \times \hat{p} \times \rangle, \text{满足封闭性; 不满足交换律和结合律; 无零元和单位元。}$$

## 自旋-单位矢量的最大代数：无限离散代数

性质1.2.8.

$$\left\{ \begin{array}{l} \langle \sigma(s), \hat{p}; \times \rangle, \text{ 满足封闭性; 不满足交换律和结合律; 有零元但无单位元.} \\ \langle \sigma(s), \hat{p}; \times \rangle = \sum_{i,j,k=0}^{2s} a_{ij} [\sigma(s) \cdot \hat{p}]^i \sigma(s) [\sigma(s) \cdot \hat{p}]^j + b_k \hat{p} [\sigma(s) \cdot \hat{p}]^k \\ \text{线性无关性: } \sum_{i,j,k=0}^{2s} a_{ij} h^{i+j+1} + b_k h^k = 0, h = s, s-1, \dots, -(s-1), -s \end{array} \right.$$

## 自旋-单位矢量标积代数：有限离散代数

性质1.2.9.

$$\left\{ \begin{array}{l} \langle \sigma(s) \cdot \hat{p}; \cdot \rangle, \text{ 满足封闭性; 满足交换律和结合律; 无零元和单位元.} \\ \langle \sigma(s) \cdot \hat{p}; \cdot \rangle = \sum_{k=0}^{2s} c_k [\sigma(s) \cdot \hat{p}]^k \\ \text{线性无关性: } \sum_{k=0}^{2s} c_k h^k = 0, h = s, s-1, \dots, -(s-1), -s \end{array} \right.$$

用 $2s+1$ 个自旋螺旋度函数可望解决以上两个代数的线性无关性难题，而完备性已可以被证明。

## 2 基本展开式

2.1  $\sigma(s)[|\times\hat{p}]^n$ 和 $i^n[\hat{p}\times]{}^n\sigma(s)$ 的两种展开式2.1.1  $i^n\sigma(s)[|\times\hat{p}]^n$ 和 $i^n[\hat{p}\times]{}^n\sigma(s)$ 的简单展开式和通项公式

$$\text{性质2.1.1. } \sigma(s)[|\times\hat{p}]^1 = \sigma(s) \times \hat{p} \quad \sigma(s)[|\times\hat{p}]^2 = [\sigma(s) \cdot \hat{p}]\hat{p} - \sigma(s) \quad \sigma(s)[|\times\hat{p}]^3 = -\sigma(s) \times \hat{p}$$

定理2.1.1.

$$i^n\sigma(s)[|\times\hat{p}]^n = \begin{cases} i\sigma(s) \times \hat{p}, n = 2k - 1 \\ \sigma(s) - [\sigma(s) \cdot \hat{p}]\hat{p}, n = 2k \end{cases}, i^n[\hat{p}\times]{}^n\sigma(s) = \begin{cases} i\hat{p} \times \sigma(s), n = 2k - 1 \\ \sigma(s) - [\sigma(s) \cdot \hat{p}]\hat{p}, n = 2k \end{cases}, k \geq 1$$

$$\text{推论2.1.1. } \sigma(s)[|\times\hat{p}]^n = (-1)^n[\hat{p}\times]{}^n\sigma(s)$$

$$\text{推论2.1.2. } i^n\sigma(s) \cdot \{\sigma(s)[|\times\hat{p}]^n\} = i^n\{[\hat{p}\times]{}^n\sigma(s)\} \cdot \sigma(s) = \begin{cases} -\sigma(s) \cdot \hat{p}, n = 2k - 1 \\ -\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\}, n = 2k \end{cases}, k \geq 1$$

$$\text{推论2.1.3. } i^n\sigma(s) \cdot \{\sigma(s)[|\times\hat{p}]^n\} = i^n\{[\hat{p}\times]{}^n\sigma(s)\} \cdot \sigma(s) = -\{[\sigma(s) \cdot \hat{p}]^{2-n\%2} - (1-n\%2)\sigma^2(s)\}, n \geq 1$$

$$\text{推论2.1.4. } i^n\sigma(s)[|\times\hat{p}]^n \cdot \hat{p} = i^n\hat{p} \cdot [|\times\hat{p}]{}^n\sigma(s) = 0, n \geq 1$$

推论2.1.5.

$$\begin{cases} \sigma(s)[|\times\hat{p}]^{2k-1} = (-1)^{k+1}\sigma(s) \times \hat{p} \\ \sigma(s)[|\times\hat{p}]^{2k} = (-1)^{k+1}\{[\sigma(s) \cdot \hat{p}]\hat{p} - \sigma(s)\} \end{cases} \Rightarrow \begin{cases} i^{2k-1}i\hat{p} \cdot \{\sigma(s) \times \{\sigma(s)[|\times\hat{p}]^{2k-1}\}\} = -\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} \\ i^{2k}i\hat{p} \cdot \{\sigma(s) \times \{\sigma(s)[|\times\hat{p}]^{2k}\}\} = -\sigma(s) \cdot \hat{p} \end{cases}$$

2.1.2  $\sigma(s)[|\times\hat{p}]^n$ 和 $i^n[\hat{p}\times]{}^n\sigma(s)$ 的类二项式展开

$$\text{性质2.1.2. } \sigma(s) \times \hat{p} = [\sigma(s), i\sigma(s) \cdot \hat{p}] = i\{\sigma(s)[\sigma(s) \cdot \hat{p}] - [\sigma(s) \cdot \hat{p}]\sigma(s)\} = i\{\sigma(s)A + B\sigma(s)\}$$

$$\text{性质2.1.3. } \sigma(s) \times \hat{p} \times \hat{p} = i^2\{\sigma(s)[\sigma(s) \cdot \hat{p}]^2 - 2[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] + [\sigma(s) \cdot \hat{p}]^2\sigma(s)\} \\ = i^2[\sigma(s)A^2 + 2B\sigma(s)A + B^2\sigma(s)] \simeq i^2[\sigma^{\frac{1}{2}}(s)A + B\sigma^{\frac{1}{2}}(s)]_{B\parallel A}^2$$

$$\text{定理2.1.2. } \sigma(s)[|\times\hat{p}]^n = i^n \sum_{k=0}^n C_n^k [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{n-k}, n \geq 1$$

证明：采用数学归纳法证明：

$$\text{第一步: } i = 1 \text{ 时成立, 即 } \sigma(s)[|\times\hat{p}]^1 = i^1 \sum_{k=0}^1 C_1^k [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{1-k}$$

第二步: 假设  $i = n$  时成立, 即  $\sigma(s)[|\times \hat{p}]^n = i^n \sum_{k=0}^n C_n^k [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{n-k}$

第三步:  $i = n + 1$  时

$$\begin{aligned} \sigma(s)[|\times \hat{p}]^{n+1} &= \sigma(s)[|\times \hat{p}]^n \times \hat{p} \\ &= i^n \sum_{k=0}^n C_n^k [-\sigma(s) \cdot \hat{p}]^k [\sigma(s) \times \hat{p}] [\sigma(s) \cdot \hat{p}]^{n-k} \\ &= i^{n+1} \sum_{k=0}^n C_n^k [-\sigma(s) \cdot \hat{p}]^k \{ \sigma(s) [\sigma(s) \cdot \hat{p}] - [\sigma(s) \cdot \hat{p}] \sigma(s) \} [\sigma(s) \cdot \hat{p}]^{n-k} \\ &= i^{n+1} \sum_{k=0}^n C_n^k [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{n+1-k} + i^{n+1} \sum_{k=0}^n C_n^k [-\sigma(s) \cdot \hat{p}]^{k+1} \sigma(s) [\sigma(s) \cdot \hat{p}]^{n-k} \\ &= i^{n+1} \sum_{k=0}^n C_n^k [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{n+1-k} + i^{n+1} \sum_{k=1}^{n+1} C_n^{k-1} [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{n+1-k} \\ &= i^{n+1} \sum_{k=0}^{n+1} C_n^k [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{n+1-k} + i^{n+1} \sum_{k=0}^{n+1} C_n^{k-1} [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{n+1-k} \\ &= i^{n+1} \sum_{k=0}^{n+1} (C_n^k + C_n^{k-1}) [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{n+1-k} \\ &= i^{n+1} \sum_{k=0}^{n+1} C_{n+1}^k [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{n+1-k} \end{aligned}$$

此步证明了  $i = n + 1$  时成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

$$\text{推论2.1.6.} \quad \begin{cases} i^n \sigma(s)[|\times \hat{p}]^n = \sum_{k=0}^n C_n^k [\sigma(s) \cdot \hat{p}]^k \sigma(s) [-\sigma(s) \cdot \hat{p}]^{n-k}, n \geq 0 \\ i^n [\hat{p} \times |]^n \sigma(s) = \sum_{k=0}^n C_n^k [-\sigma(s) \cdot \hat{p}]^{n-k} \sigma(s) [\sigma(s) \cdot \hat{p}]^k, n \geq 0 \end{cases}$$

## 2.2 $[\sigma(s) \cdot \hat{p}]^n \sigma(s)$ 和 $\sigma(s)[\sigma(s) \cdot \hat{p}]^n$ 的递推公式

定理2.2.1.

$$\begin{cases} [\sigma(s) \cdot \hat{p}]^n \sigma(s) = i^n \sigma(s)[|\times \hat{p}]^n - \sum_{k=0}^{n-1} C_n^k [\sigma(s) \cdot \hat{p}]^k \sigma(s) [-\sigma(s) \cdot \hat{p}]^{n-k}, n \geq 0 \\ \sigma(s)[\sigma(s) \cdot \hat{p}]^n = i^n [\hat{p} \times |]^n \sigma(s) - \sum_{k=0}^{n-1} C_n^k [-\sigma(s) \cdot \hat{p}]^{n-k} \sigma(s) [\sigma(s) \cdot \hat{p}]^k, n \geq 0 \end{cases}$$

推论2.2.1.

$$\begin{cases} \sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s) = i^n \sigma(s) \cdot \{ \sigma(s)[|\times \hat{p}]^n \} - \sum_{k=0}^{n-1} C_n^k \sigma^\alpha(s) [\sigma(s) \cdot \hat{p}]^k \sigma_\alpha(s) [-\sigma(s) \cdot \hat{p}]^{n-k}, n \geq 0 \\ \sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s) = i^n \{ [\hat{p} \times |]^n \sigma(s) \} \cdot \sigma(s) - \sum_{k=0}^{n-1} C_n^k [-\sigma(s) \cdot \hat{p}]^{n-k} \sigma^\alpha(s) [\sigma(s) \cdot \hat{p}]^k \sigma_\alpha(s), n \geq 0 \end{cases}$$

推论2.2.2.

$$\begin{cases} \sigma(s) \times \{ [\sigma(s) \cdot \hat{p}]^n \sigma(s) \} = i^n \sigma(s) \times \{ \sigma(s)[|\times \hat{p}]^n \} - \sum_{k=0}^{n-1} C_n^k \sigma(s) \times \{ [\sigma(s) \cdot \hat{p}]^k \sigma(s) \} [-\sigma(s) \cdot \hat{p}]^{n-k}, n \geq 0 \\ \{ \sigma(s)[\sigma(s) \cdot \hat{p}]^n \} \times \sigma(s) = i^n \{ [\hat{p} \times |]^n \sigma(s) \} \times \sigma(s) - \sum_{k=0}^{n-1} C_n^k [-\sigma(s) \cdot \hat{p}]^{n-k} \{ \sigma(s)[\sigma(s) \cdot \hat{p}]^k \} \times \sigma(s), n \geq 0 \end{cases}$$

$$\text{推论2.2.3.} \quad X(n) = O(n) - \sum_{k=0}^{n-1} C_n^k X(k) [-\sigma(s) \cdot \hat{p}]^{n-k}, n \geq 0$$

## 2.3 $i^{-n}[\sigma(s) \times ]^n \hat{p}$ 和 $i^{-n} \hat{p} [ \times \sigma(s) ]^n$ 的通项公式

### 2.3.1 $i^{-n}[\sigma(s) \times ]^n \hat{p}$ 的通项公式

性质2.3.1.

$$\begin{cases} i^{-0}[\sigma(s) \times ]^0 \hat{p} = \hat{p} \\ i^{-1}[\sigma(s) \times ]^1 \hat{p} = \{\sigma(s)[\sigma(s) \cdot \hat{p}] - [\sigma(s) \cdot \hat{p}]\sigma(s)\} \\ i^{-2}[\sigma(s) \times ]^2 \hat{p} = -[\sigma(s) \cdot \hat{p}]\sigma(s) + \sigma^2(s)\hat{p} \\ i^{-3}[\sigma(s) \times ]^3 \hat{p} = -[1 - \sigma^2(s)]\sigma(s)[\sigma(s) \cdot \hat{p}] - [1 + \sigma^2(s)][\sigma(s) \cdot \hat{p}]\sigma(s) + \sigma^2(s)\hat{p} \\ i^{-4}[\sigma(s) \times ]^4 \hat{p} = -[2 - \sigma^2(s)]\sigma(s)[\sigma(s) \cdot \hat{p}] - [1 + 2\sigma^2(s)][\sigma(s) \cdot \hat{p}]\sigma(s) + [1 + \sigma^2(s)]\sigma^2(s)\hat{p} \\ i^{-5}[\sigma(s) \times ]^5 \hat{p} = -[3 - \sigma^4(s)]\sigma(s)[\sigma(s) \cdot \hat{p}] - [1 + 3\sigma^2(s) + \sigma^4(s)][\sigma(s) \cdot \hat{p}]\sigma(s) + [1 + 2\sigma^2(s)]\sigma^2(s)\hat{p} \end{cases}$$

性质2.3.2.

$$\begin{cases} \hat{p} \cdot [ \sigma(s) \times ]^2 \hat{p} = [\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s) \\ \hat{p} \cdot [ \sigma(s) \times ]^3 \hat{p} = 2i[\sigma(s) \cdot \hat{p}]^2 - i\sigma^2(s) \\ \hat{p} \cdot [ \sigma(s) \times ]^4 \hat{p} = i^2[3 + \sigma^2(s)][\sigma(s) \cdot \hat{p}]^2 - i^2[1 + \sigma^2(s)]\sigma^2(s) \end{cases}$$

引理2.3.1.  $-(2s+1) = [(s+1)^4 - (-s)^4] - 2[(s+1)^3 - (-s)^3]$

定理2.3.1.

$$\begin{cases} i^{-n}[\sigma(s) \times ]^n \hat{p} = a_n \sigma(s)[\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}]\sigma(s) - c_n \sigma^2(s)\hat{p} \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)} \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, c_n = -\frac{(s+1)^{n-1} - (-s)^{n-1}}{2s+1}, n \geq 0 \end{cases}$$

证明: 采用数学归纳法证明:

第一步:  $i = 0$  时成立, 即

$$\begin{cases} i^{-0}[\sigma(s) \times ]^0 \hat{p} = a_0 \sigma(s)[\sigma(s) \cdot \hat{p}] + b_0 [\sigma(s) \cdot \hat{p}]\sigma(s) - c_0 \sigma^2(s)\hat{p} \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s) \end{cases}$$

第二步: 假设  $i = n$  时成立, 即

$$i^{-n}[\sigma(s) \times ]^n \hat{p} = a_n \sigma(s)[\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}]\sigma(s) - c_n \sigma^2(s)\hat{p}$$

第三步:  $i = n + 1$  时

$$\begin{aligned} & i^{-(n+1)}[\sigma(s) \times ]^{n+1} \hat{p} \\ &= a_n \sigma(s)[\sigma(s) \cdot \hat{p}] - i b_n \sigma(s) \times [ \sigma(s) \cdot \hat{p} ] \sigma(s) + i c_n \sigma(s) \times \sigma^2(s)\hat{p} \\ &= a_n \sigma(s)[\sigma(s) \cdot \hat{p}] + b_n \{ \sigma(s)[\sigma(s) \cdot \hat{p}] + [\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma^2(s)\hat{p} \} - c_n \{ \sigma(s)[\sigma(s) \cdot \hat{p}] - [\sigma(s) \cdot \hat{p}]\sigma(s) \} \sigma^2(s) \\ &= (a_n + b_n - c_n \sigma^2(s))\sigma(s)[\sigma(s) \cdot \hat{p}] + (b_n + c_n \sigma^2(s))[\sigma(s) \cdot \hat{p}]\sigma(s) - b_n \sigma^2(s)\hat{p} \\ &= a_{n+1} \sigma(s)[\sigma(s) \cdot \hat{p}] + b_{n+1} [\sigma(s) \cdot \hat{p}]\sigma(s) - c_{n+1} \sigma^2(s)\hat{p} \end{aligned}$$

此步证明了  $i = n + 1$  时成立。

第四步: 得到如下递推关系

$$\begin{cases} a_{n+1} = a_n + b_n - c_n \sigma^2(s), b_{n+1} = b_n + c_n \sigma^2(s), c_{n+1} = b_n \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s); a_1 = 1, b_1 = -1, c_1 = 0 \\ \Rightarrow a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)} \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, c_n = -\frac{(s+1)^{n-1} - (-s)^{n-1}}{2s+1}, n \geq 0 \end{cases}$$

第五步: 根据以上归纳法推理, 命题成立, 定理得证。 □

推论2.3.1.

$$\begin{cases} i^{-n}[\sigma(s) \times ]^n \hat{p} = a_n \sigma(s)[\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}]\sigma(s) - b_{n-1} \sigma^2(s)\hat{p}, n \geq 0 \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1} \end{cases}$$

推论2.3.2.

$$\begin{cases} i^{-n}\hat{p} \cdot \{[\sigma(s) \times \hat{p}]^n\} = -k_n[\sigma(s) \cdot \hat{p}]^2 + b_{n-1}\sigma^2(s), n \geq 0 \\ k_n = -(a_n + b_n) = \frac{(s+1)^{n+1} - (-s)^{n+1} - (2s+1)}{s(2s+1)(s+1)}, b_{n-1} = -\frac{(s+1)^{n-1} - (-s)^{n-1}}{2s+1} \end{cases}$$

2.3.2  $i^{-n}\hat{p}[\times \sigma(s)]^n$ 的通项公式

$$\text{定理2.3.2.} \begin{cases} i^{-n}\hat{p}[\times \sigma(s)]^n = a_n[\sigma(s) \cdot \hat{p}]\sigma(s) + b_n\sigma(s)[\sigma(s) \cdot \hat{p}] - c_n\sigma^2(s)\hat{p} \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)} \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, c_n = -\frac{(s+1)^{n-1} - (-s)^{n-1}}{2s+1}, n \geq 0 \end{cases}$$

证明: 采用数学归纳法证明:

第一步:  $i = 0$ 时成立, 即

$$\begin{cases} i^{-0}\hat{p}[\times \sigma(s)]^0 = a_0[\sigma(s) \cdot \hat{p}]\sigma(s) + b_0\sigma(s)[\sigma(s) \cdot \hat{p}] - c_0\sigma^2(s)\hat{p} \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s) \end{cases}$$

第二步: 假设 $i = n$ 时成立, 即

$$i^{-n}\hat{p}[\times \sigma(s)]^n = a_n[\sigma(s) \cdot \hat{p}]\sigma(s) + b_n\sigma(s)[\sigma(s) \cdot \hat{p}] - c_n\sigma^2(s)\hat{p}$$

第三步:  $i = n + 1$ 时

$$\begin{aligned} & i^{-(n+1)}\hat{p}[\times \sigma(s)]^{n+1} \\ &= a_n[\sigma(s) \cdot \hat{p}]\sigma(s) - ib_n\sigma(s)[\sigma(s) \cdot \hat{p}][\times \sigma(s)] + ic_n\hat{p} \times \sigma(s)\sigma^2(s) \\ &= a_n[\sigma(s) \cdot \hat{p}]\sigma(s) + b_n\{[\sigma(s) \cdot \hat{p}]\sigma(s) + \sigma(s)[\sigma(s) \cdot \hat{p}] - \sigma^2(s)\hat{p}\} - c_n\{[\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma(s)[\sigma(s) \cdot \hat{p}]\}\sigma^2(s) \\ &= (a_n + b_n - c_n\sigma^2(s))[\sigma(s) \cdot \hat{p}]\sigma(s) + (b_n + c_n\sigma^2(s))\sigma(s)[\sigma(s) \cdot \hat{p}] - b_n\sigma^2(s)\hat{p} \\ &= a_{n+1}[\sigma(s) \cdot \hat{p}]\sigma(s) + b_{n+1}\sigma(s)[\sigma(s) \cdot \hat{p}] - c_{n+1}\sigma^2(s)\hat{p} \end{aligned}$$

此步证明了 $i = n + 1$ 时成立。

第四步: 得到如下递推关系

$$\begin{cases} a_{n+1} = a_n + b_n - c_n\sigma^2(s), b_{n+1} = b_n + c_n\sigma^2(s), c_{n+1} = b_n \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s); a_1 = 1, b_1 = -1, c_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)} \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, c_n = -\frac{(s+1)^{n-1} - (-s)^{n-1}}{2s+1}, n \geq 0 \end{cases}$$

第五步: 根据以上归纳法推理, 命题成立, 定理得证。 □

$$\text{推论2.3.3.} \begin{cases} i^{-n}\hat{p}[\times \sigma(s)]^n = a_n[\sigma(s) \cdot \hat{p}]\sigma(s) + b_n\sigma(s)[\sigma(s) \cdot \hat{p}] - b_{n-1}\sigma^2(s)\hat{p}, n \geq 0 \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1} \end{cases}$$

推论2.3.4.

$$\begin{cases} i^{-n}\{\hat{p}[\times \sigma(s)]^n\} \cdot \hat{p} = -k_n[\sigma(s) \cdot \hat{p}]^2 + b_{n-1}\sigma^2(s), n \geq 0 \\ k_n = -(a_n + b_n) = \frac{(s+1)^{n+1} - (-s)^{n+1} - (2s+1)}{s(2s+1)(s+1)}, b_{n-1} = -\frac{(s+1)^{n-1} - (-s)^{n-1}}{2s+1} \end{cases}$$

推论2.3.5.  $\hat{p} \cdot \{[\sigma(s) \times \hat{p}]^n\} = \{\hat{p}[\times \sigma(s)]^n\} \cdot \hat{p}, n \geq 0$

2.3.3 参数汇总

$$\text{推论2.3.6.} \begin{cases} a_{n+1} = a_n + b_n - c_n\sigma^2(s), b_{n+1} = b_n + c_n\sigma^2(s), c_{n+1} = b_n \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s); a_1 = 1, b_1 = -1, c_1 = 0; a_2 = 0, b_2 = -1, c_2 = -1 \end{cases}$$

$$\Rightarrow \begin{cases} a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, c_n = -\frac{(s+1)^{n-1} - (-s)^{n-1}}{2s+1} \\ k_n = -(a_n + b_n) = \frac{(s+1)^{n+1} - (-s)^{n+1} - (2s+1)}{s(2s+1)(s+1)}, c_n - (a_n + b_n) = \frac{(s+1)^n - (-s)^n - (2s+1)}{s(2s+1)(s+1)}, n \geq 0 \end{cases}$$

$$\Rightarrow \begin{cases} \sigma^2(s)a_n = -b_{n+2} + 2b_{n+1} + 1, \sigma^2(s)k_n = -b_{n+1} - 1, c_n = b_{n-1}, k_{n+1} = k_n - b_n \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, \sigma^2(s)(c_n - a_n - b_n) = -b_n - 1, n \geq 0 \end{cases}$$

## 2.3.4 参数一般化

定理2.3.3.  $b_{n+1} = b_n + b_{n-1}\sigma^2(s), b_0 = 0, b_{-1} = -\sigma^{-2}(s) \Leftrightarrow b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}; n \in Z$

证明:  $b_{n+1} = b_n + b_{n-1}\sigma^2(s), b_0 = 0, b_{-1} = -\sigma^{-2}(s); n \in Z$

$$\Leftrightarrow b_{n+1} + sb_n = (s+1)(b_n + sb_{n-1}), b_0 = 0, b_{-1} = -\sigma^{-2}(s); n \in Z$$

$$\Leftrightarrow b_n + sb_{n-1} = (s+1)^n(b_0 + sb_{-1}), b_0 = 0, b_{-1} = -\sigma^{-2}(s); n \in Z$$

$$\Leftrightarrow b_n + sb_{n-1} = -(s+1)^{n-1}, b_0 = 0, b_{-1} = -\sigma^{-2}(s); n \in Z$$

$$\Leftrightarrow \begin{cases} b_n - (-s)b_{n-1} = -(s+1)^{n-1}(-s)^0 \\ (-s)b_{n-1} - (-s)^2b_{n-2} = -(s+1)^{n-2}(-s) \\ (-s)^2b_{n-2} - (-s)^3b_{n-3} = -(s+1)^{n-3}(-s)^2 \\ \dots\dots\dots \\ (-s)^{n-1}b_1 - (-s)^nb_0 = -(s+1)^0(-s)^{n-1} \\ (-s)^nb_0 - (-s)^{n+1}b_{-1} = -(s+1)^{-1}(-s)^n \\ (-s)^{n+1}b_{-1} - (-s)^{n+2}b_{-2} = -(s+1)^{-2}(-s)^{n+1} \\ \dots\dots\dots \\ (-s)^{n-l-2}b_{l+2} - (-s)^{n-l-1}b_{l+1} = -(s+1)^{l+1}(-s)^{n-l-2} \\ (-s)^{n-l-1}b_{l+1} - (-s)^{n-l}b_l = -(s+1)^l(-s)^{n-l-1} \end{cases}, b_0 = 0, b_{-1} = -\sigma^{-2}(s); n \geq 0, l \leq -1$$

$$\Leftrightarrow \begin{cases} b_n - (-s)^nb_0 = -(s+1)^{n-1} \sum_{i=0}^{n-1} \left(\frac{-s}{s+1}\right)^i \\ (-s)^nb_0 - (-s)^{n-l}b_l = -(s+1)^{n-1} \left(\frac{-s}{s+1}\right)^n \sum_{i=0}^{-l-1} \left(\frac{-s}{s+1}\right)^i \end{cases}, b_0 = 0; n \geq 1, l \leq -1$$

$$\Leftrightarrow \begin{cases} b_n - (-s)^nb_0 = -(s+1)^{n-1} \sum_{i=0}^{n-1} \left(\frac{-s}{s+1}\right)^i \\ (-s)^lb_0 - b_l = -\frac{(-s)^l}{s+1} \sum_{i=0}^{-l-1} \left(\frac{-s}{s+1}\right)^i \end{cases}, b_0 = 0; n \geq 1, l \leq -1$$

$$\Leftrightarrow b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, b_l = -\frac{(s+1)^l - (-s)^l}{2s+1}, b_0 = 0; n \geq 1, l \leq -1$$

$$\Leftrightarrow b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}; n \in Z \quad \square$$

定理2.3.4.

$$\begin{cases} a_{n+1} = a_n + b_n - b_{n-1}\sigma^2(s), a_0 = 0 \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1} \end{cases} \Leftrightarrow \begin{cases} a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)} \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1} \end{cases}; n \in Z$$

证明:  $\begin{cases} a_{n+1} = a_n + b_n - b_{n-1}\sigma^2(s), a_0 = 0 \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1} \end{cases}; n \in Z$

$$\Leftrightarrow \begin{cases} a_{n+1} = a_n + b_n - b_{n-1}\sigma^2(s), b_{n+1} = b_n + b_{n-1}\sigma^2(s) \\ a_0 = 0, b_0 = 0, b_{-1} = -\sigma^{-2}(s) \end{cases}; n \in Z$$

$$\Leftrightarrow \begin{cases} (a_{n+1} + b_{n+1}) - (a_n + b_n) = b_n, b_{n+1} = b_n + b_{n-1}\sigma^2(s) \\ a_0 + b_0 = 0, b_0 = 0, b_{-1} = -\sigma^{-2}(s) \end{cases}; n \in Z$$

$$\Leftrightarrow \begin{cases} a_n = \sum_{i=0}^{n-1} b_i - b_n, a_l = \sum_{i=l}^0 b_i - b_l; n \geq 1, l \leq 0 \\ b_{n+1} = b_n + b_{n-1}\sigma^2(s), b_0 = 0, b_{-1} = -\sigma^{-2}(s); n \in Z \end{cases}$$

$$\Leftrightarrow \begin{cases} a_n = \sum_{i=0}^{n-1} b_i - b_n, a_l = -\sum_{i=l}^0 b_i - b_l; n \geq 1, l \leq 0 \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}; n \in Z \end{cases}$$



$$\Leftrightarrow \begin{cases} a_n = -\sum_{i=0}^{n-1} \frac{(s+1)^i - (-s)^i}{2s+1} + \frac{(s+1)^n - (-s)^n}{2s+1}; n \geq 1 \\ a_l = \sum_{i=l}^0 \frac{(s+1)^i - (-s)^i}{2s+1} + \frac{(s+1)^l - (-s)^l}{2s+1}; l \leq 0 \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}; n \in Z \end{cases}$$

$$\Leftrightarrow \begin{cases} a_n = -\sum_{i=0}^{n-1} \frac{(s+1)^i}{2s+1} + \sum_{i=0}^{n-1} \frac{(-s)^i}{2s+1} + \frac{(s+1)^n - (-s)^n}{2s+1} = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}; n \geq 1 \\ a_l = \sum_{i=l}^0 \frac{(s+1)^i}{2s+1} - \sum_{i=l}^0 \frac{(-s)^i}{2s+1} + \frac{(s+1)^l - (-s)^l}{2s+1} = \frac{1}{2s+1} \frac{1 - (\frac{1}{s+1})^{-l+1}}{1 - \frac{1}{s+1}} - \frac{1}{2s+1} \frac{1 - (\frac{1}{-s})^{-l+1}}{1 - \frac{1}{-s}} + \frac{(s+1)^l - (-s)^l}{2s+1} \\ = \frac{s+1 - (s+1)^l}{s(2s+1)} - \frac{s + (-s)^l}{(2s+1)(s+1)} + \frac{(s+1)^l - (-s)^l}{2s+1} = \frac{(s+1)^2 - (s+1)^{l+1}}{s(2s+1)(s+1)} - \frac{s^2 - (-s)^{l+1}}{s(2s+1)(s+1)} + \frac{s(s+1)^{l+1} + (s+1)(-s)^{l+1}}{s(2s+1)(s+1)} \\ = \frac{[(s+1)^{l+2} - (-s)^{l+2}] - 2[(s+1)^{l+1} - (-s)^{l+1}] + (2s+1)}{s(2s+1)(s+1)}; l \leq 0 \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}; n \in Z \end{cases}$$

$$\Leftrightarrow \begin{cases} a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)} \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1} \end{cases}; n \in Z$$

$$\Leftrightarrow \begin{cases} \sigma^2(s)a_n = -b_{n+2} + 2b_{n+1} + 1 \\ b_n = -\frac{(s+1)^n - (-s)^n}{2s+1} \end{cases}; n \in Z$$

$$\Leftrightarrow \begin{cases} \sigma^2(s)a_n = -b_{n+2} + 2b_{n+1} + 1 \\ b_{n+1} = b_n + b_{n-1}\sigma^2(s), b_0 = 0, b_{-1} = -\sigma^{-2}(s) \end{cases}; n \in Z \quad \square$$

推论2.3.7.

$$\begin{cases} a_{n+1} = a_n + b_n - b_{n-1}\sigma^2(s) \\ b_{n+1} = b_n + b_{n-1}\sigma^2(s) \\ a_0 = 0, b_0 = 0, b_{-1} = -\sigma^{-2}(s) \end{cases} \Leftrightarrow \begin{cases} \sigma^2(s)a_n = -b_{n+2} + 2b_{n+1} + 1 \\ b_{n+1} = b_n + b_{n-1}\sigma^2(s) \\ b_0 = 0, b_{-1} = -\sigma^{-2}(s) \end{cases}; n \in Z$$

### 2.3.5 $i^{l-n}\{\sigma(s) \times ||^n \hat{p}\} || \times \hat{p}^l$ 的通项公式

$$\text{推论2.3.8.} \begin{cases} i^{l-n}\{\sigma(s) \times ||^n \hat{p}\} || \times \hat{p}^l = a_n i^l \sigma(s) || \times \hat{p}^l [\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}] i^l \sigma(s) || \times \hat{p}^l \\ = \begin{cases} -a_n \sigma(s) [\sigma(s) \cdot \hat{p}]^2 + (a_n - b_n) [\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}]^2 \sigma(s), l = 2k - 1 \\ -(a_n + b_n) [\sigma(s) \cdot \hat{p}]^2 \hat{p} + a_n \sigma(s) [\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}] \sigma(s), l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

$$\text{推论2.3.9.} \begin{cases} i^{l-n}\{\sigma(s) \times ||^n \hat{p}\} || \times \hat{p}^l \cdot \sigma(s) = \begin{cases} (2a_n + b_n) [\sigma(s) \cdot \hat{p}]^2 - a_n \sigma^2(s), l = 2k - 1 \\ -(a_n + b_n) [\sigma(s) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s) - a_n] [\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

$$\text{推论2.3.10.} \begin{cases} i^{l-n} \sigma(s) \cdot \{[\sigma(s) \times ||^n \hat{p}\} || \times \hat{p}^l = \begin{cases} -(a_n + 2b_n) [\sigma(s) \cdot \hat{p}]^2 + b_n \sigma^2(s), l = 2k - 1 \\ -(a_n + b_n) [\sigma(s) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s) - b_n] [\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

推论2.3.11.  $i^{l-n} \hat{p} \cdot \{[\sigma(s) \times ||^n \hat{p}\} || \times \hat{p}^l = 0, n \geq 0, l \geq 1$

### 2.3.6 $i^{l-n} [\hat{p} \times ||^l \{\hat{p} || \times \sigma(s)\}^n]$ 的通项公式

$$\text{推论2.3.12.} \begin{cases} i^{l-n} [\hat{p} \times ||^l \{\hat{p} || \times \sigma(s)\}^n] = a_n [\sigma(s) \cdot \hat{p}] i^l [\hat{p} \times ||^l \sigma(s) + b_n i^l [\hat{p} \times ||^l \sigma(s) [\sigma(s) \cdot \hat{p}] - c_n \sigma^2(s) \hat{p}] \\ = \begin{cases} -a_n [\sigma(s) \cdot \hat{p}]^2 \sigma(s) + (a_n - b_n) [\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}] + b_n \sigma(s) [\sigma(s) \cdot \hat{p}]^2, l = 2k - 1 \\ -(a_n + b_n) [\sigma(s) \cdot \hat{p}]^2 \hat{p} + a_n [\sigma(s) \cdot \hat{p}] \sigma(s) + b_n \sigma(s) [\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

$$\text{推论2.3.13. } \begin{cases} i^{l-n}\sigma(s) \cdot \|\hat{p} \times \|\|^l \{\hat{p} \|\times \sigma(s)\|^n\} = \begin{cases} (2a_n + b_n)[\sigma(s) \cdot \hat{p}]^2 - a_n\sigma^2(s), l = 2k - 1 \\ -(a_n + b_n)[\sigma(s) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s) - a_n][\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

$$\text{推论2.3.14. } \begin{cases} i^{l-n}[\hat{p} \times \|\|^l \{\hat{p} \|\times \sigma(s)\|^n\}] \cdot \sigma(s) = \begin{cases} -(a_n + 2b_n)[\sigma(s) \cdot \hat{p}]^2 + b_n\sigma^2(s), l = 2k - 1 \\ -(a_n + b_n)[\sigma(s) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s) - b_n][\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0, l \geq 1 \end{cases}$$

$$\text{推论2.3.15. } i^{l-n}\hat{p} \cdot \|\hat{p} \times \|\|^l \{\hat{p} \|\times \sigma(s)\|^n\} = 0, n \geq 0, l \geq 1$$

$$\text{推论2.3.16. } \begin{cases} i^{l-n}\{\sigma(s) \times \|\|^n \hat{p}\} \|\times \hat{p}\|^l \cdot \sigma(s) = i^{l-n}\sigma(s) \cdot \|\hat{p} \times \|\|^l \{\hat{p} \|\times \sigma(s)\|^n\} \\ i^{l-n}\sigma(s) \cdot \|\{\sigma(s) \times \|\|^n \hat{p}\} \|\times \hat{p}\|^l = i^{l-n}[\hat{p} \times \|\|^l \{\hat{p} \|\times \sigma(s)\|^n\}] \cdot \sigma(s) \end{cases}$$

### 3 两类基本自旋复合算符的通项公式

#### 3.1 $\sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s)$ 的通项公式

##### 3.1.1 $\sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s)$ 通项公式的试探和猜想

推论3.1.1.

$$\sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\} = -\{[\sigma(s) \cdot \hat{p}]^{2-n\%2} - (1 - n\%2)\sigma^2(s)\} - \sum_{k=0}^{n-1} C_n^k \{\sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^k \sigma(s)\}\} [-\sigma(s) \cdot \hat{p}]^{n-k}$$

性质3.1.1.

$$\begin{cases} \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^0 \sigma(s)\} = \sigma^2(s) \\ \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^1 \sigma(s)\} = [\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}] \\ \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^2 \sigma(s)\} = [\sigma^2(s) - 3][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \\ \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^3 \sigma(s)\} = [\sigma^2(s) - 6][\sigma(s) \cdot \hat{p}]^3 + [3\sigma^2(s) - 1]\sigma(s) \cdot \hat{p} \\ \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^4 \sigma(s)\} = [\sigma^2(s) - 10][\sigma(s) \cdot \hat{p}]^4 + [6\sigma^2(s) - 5][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \\ \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^5 \sigma(s)\} = [\sigma^2(s) - 15][\sigma(s) \cdot \hat{p}]^5 + [10\sigma^2(s) - 15][\sigma(s) \cdot \hat{p}]^3 + [5\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}] \\ \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^6 \sigma(s)\} = [\sigma^2(s) - 21][\sigma(s) \cdot \hat{p}]^6 + [15\sigma^2(s) - 35][\sigma(s) \cdot \hat{p}]^4 + [15\sigma^2(s) - 7][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \\ \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^7 \sigma(s)\} = [\sigma^2(s) - 28][\sigma(s) \cdot \hat{p}]^7 + [21\sigma^2(s) - 70][\sigma(s) \cdot \hat{p}]^5 + [35\sigma^2(s) - 28][\sigma(s) \cdot \hat{p}]^3 + [7\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}] \end{cases}$$

证明:  $\sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^4 \sigma(s)\}$

$$\begin{aligned} &= -\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} - \sum_{k=0}^3 C_4^k \{\sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^k \sigma(s)\}\} [-\sigma(s) \cdot \hat{p}]^{4-k} \\ &= -\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} - \sigma^2(s)[\sigma(s) \cdot \hat{p}]^4 - \sum_{k=1}^3 C_4^k \{\sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^k \sigma(s)\}\} [-\sigma(s) \cdot \hat{p}]^{4-k} \\ &= -\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} - C_4^0 \sigma^2(s)[\sigma(s) \cdot \hat{p}]^4 + C_4^1 [\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]^4 - \sum_{k=2}^3 C_4^k \{\sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^k \sigma(s)\}\} [-\sigma(s) \cdot \hat{p}]^{4-k} \\ &= -\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} - C_4^0 \sigma^2(s)[\sigma(s) \cdot \hat{p}]^4 + C_4^1 [\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]^4 - C_4^2 \{[\sigma^2(s) - 3][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s)\} [\sigma(s) \cdot \hat{p}]^2 \\ &\quad - \sum_{k=3}^3 C_4^k \{\sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^k \sigma(s)\}\} [-\sigma(s) \cdot \hat{p}]^{4-k} \\ &= -\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} - C_4^0 \sigma^2(s)[\sigma(s) \cdot \hat{p}]^4 + C_4^1 [\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]^4 - C_4^2 \{[\sigma^2(s) - 3][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s)\} [\sigma(s) \cdot \hat{p}]^2 \\ &\quad + C_4^3 \{[\sigma^2(s) - 6][\sigma(s) \cdot \hat{p}]^3 + [3\sigma^2(s) - 1]\sigma(s) \cdot \hat{p}\} [\sigma(s) \cdot \hat{p}] \\ &= [\sigma^2(s) - 10][\sigma(s) \cdot \hat{p}]^4 + [6\sigma^2(s) - 5][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \quad \square \end{aligned}$$

证明:  $\sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^5 \sigma(s)\}$

$$\begin{aligned} &= -[\sigma(s) \cdot \hat{p}] + C_5^0 \sigma^2(s)[\sigma(s) \cdot \hat{p}]^5 - C_5^1 [\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]^5 + C_5^2 \{[\sigma^2(s) - 3][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s)\} [\sigma(s) \cdot \hat{p}]^3 \\ &\quad - C_5^3 \{[\sigma^2(s) - 6][\sigma(s) \cdot \hat{p}]^3 + [3\sigma^2(s) - 1]\sigma(s) \cdot \hat{p}\} [\sigma(s) \cdot \hat{p}]^2 \end{aligned}$$

$$\begin{aligned}
& + C_5^4 \{ [\sigma^2(s) - 10][\sigma(s) \cdot \hat{p}]^4 + [6\sigma^2(s) - 5][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \} [\sigma(s) \cdot \hat{p}] \\
& = [\sigma^2(s) - 15][\sigma(s) \cdot \hat{p}]^5 + [10\sigma^2(s) - 15][\sigma(s) \cdot \hat{p}]^3 + [5\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]
\end{aligned} \quad \square$$

证明:  $\sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^6 \sigma(s) \}$

$$\begin{aligned}
& = -\{ [\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s) \} - C_6^0 \sigma^2(s) [\sigma(s) \cdot \hat{p}]^6 + C_6^1 [\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]^6 \\
& - C_6^2 \{ [\sigma^2(s) - 3][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \} [\sigma(s) \cdot \hat{p}]^4 \\
& + C_6^3 \{ [\sigma^2(s) - 6][\sigma(s) \cdot \hat{p}]^3 + [3\sigma^2(s) - 1]\sigma(s) \cdot \hat{p} \} [\sigma(s) \cdot \hat{p}]^3 \\
& - C_6^4 \{ [\sigma^2(s) - 10][\sigma(s) \cdot \hat{p}]^4 + [6\sigma^2(s) - 5][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \} [\sigma(s) \cdot \hat{p}]^2 \\
& + C_6^5 \{ [\sigma^2(s) - 15][\sigma(s) \cdot \hat{p}]^5 + [10\sigma^2(s) - 15][\sigma(s) \cdot \hat{p}]^3 + [5\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}] \} [\sigma(s) \cdot \hat{p}] \\
& = [\sigma^2(s) - 21][\sigma(s) \cdot \hat{p}]^6 + [15\sigma^2(s) - 35][\sigma(s) \cdot \hat{p}]^4 + [15\sigma^2(s) - 7][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s)
\end{aligned} \quad \square$$

证明:  $\sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^7 \sigma(s) \}$

$$\begin{aligned}
& = -[\sigma(s) \cdot \hat{p}] + C_7^0 \sigma^2(s) [\sigma(s) \cdot \hat{p}]^7 - C_7^1 [\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]^7 \\
& + C_7^2 \{ [\sigma^2(s) - 3][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \} [\sigma(s) \cdot \hat{p}]^5 \\
& - C_7^3 \{ [\sigma^2(s) - 6][\sigma(s) \cdot \hat{p}]^3 + [3\sigma^2(s) - 1]\sigma(s) \cdot \hat{p} \} [\sigma(s) \cdot \hat{p}]^4 \\
& + C_7^4 \{ [\sigma^2(s) - 10][\sigma(s) \cdot \hat{p}]^4 + [6\sigma^2(s) - 5][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \} [\sigma(s) \cdot \hat{p}]^3 \\
& - C_7^5 \{ [\sigma^2(s) - 15][\sigma(s) \cdot \hat{p}]^5 + [10\sigma^2(s) - 15][\sigma(s) \cdot \hat{p}]^3 + [5\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}] \} [\sigma(s) \cdot \hat{p}]^2 \\
& + C_7^6 \{ [\sigma^2(s) - 21][\sigma(s) \cdot \hat{p}]^6 + [15\sigma^2(s) - 35][\sigma(s) \cdot \hat{p}]^4 + [15\sigma^2(s) - 7][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \} [\sigma(s) \cdot \hat{p}] \\
& = [\sigma^2(s) - 28][\sigma(s) \cdot \hat{p}]^7 + [21\sigma^2(s) - 70][\sigma(s) \cdot \hat{p}]^5 + [35\sigma^2(s) - 28][\sigma(s) \cdot \hat{p}]^3 + [7\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]
\end{aligned} \quad \square$$

重新整理得到:

性质3.1.2.

$$\left\{ \begin{aligned}
& \sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^0 \sigma(s) \} = \sigma^2(s) \\
& \sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^1 \sigma(s) \} = [\sigma^2(s) - C_2^2][\sigma(s) \cdot \hat{p}] \\
& \sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^2 \sigma(s) \} = [\sigma^2(s) - C_3^2][\sigma(s) \cdot \hat{p}]^2 + C_2^2 \sigma^2(s) \\
& \sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^3 \sigma(s) \} = [\sigma^2(s) - C_4^2][\sigma(s) \cdot \hat{p}]^3 + [C_3^2 \sigma^2(s) - C_4^4] \sigma(s) \cdot \hat{p} \\
& \sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^4 \sigma(s) \} = [\sigma^2(s) - C_5^2][\sigma(s) \cdot \hat{p}]^4 + [C_4^2 \sigma^2(s) - C_5^4][\sigma(s) \cdot \hat{p}]^2 + C_4^4 \sigma^2(s) \\
& \sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^5 \sigma(s) \} = [\sigma^2(s) - C_6^2][\sigma(s) \cdot \hat{p}]^5 + [C_5^2 \sigma^2(s) - C_6^4][\sigma(s) \cdot \hat{p}]^3 + [C_5^4 \sigma^2(s) - C_6^6][\sigma(s) \cdot \hat{p}] \\
& \sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^6 \sigma(s) \} = [\sigma^2(s) - C_7^2][\sigma(s) \cdot \hat{p}]^6 + [C_6^2 \sigma^2(s) - C_7^4][\sigma(s) \cdot \hat{p}]^4 + [C_6^4 \sigma^2(s) - C_7^6][\sigma(s) \cdot \hat{p}]^2 + C_6^6 \sigma^2(s) \\
& \sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^7 \sigma(s) \} = [\sigma^2(s) - C_8^2][\sigma(s) \cdot \hat{p}]^7 + [C_7^2 \sigma^2(s) - C_8^4][\sigma(s) \cdot \hat{p}]^5 + [C_7^4 \sigma^2(s) - C_8^6][\sigma(s) \cdot \hat{p}]^3 + [C_7^6 \sigma^2(s) - C_8^8][\sigma(s) \cdot \hat{p}]
\end{aligned} \right.$$

$$\begin{aligned}
& \text{猜想3.1.1. } \sigma(s) \cdot \{ [\sigma(s) \cdot \hat{p}]^n \sigma(s) \} = [C_n^0 \sigma^2(s) - C_{n+1}^2][\sigma(s) \cdot \hat{p}]^n + [C_n^2 \sigma^2(s) - C_{n+1}^4][\sigma(s) \cdot \hat{p}]^{n-2} \\
& + [C_n^4 \sigma^2(s) - C_{n+1}^6][\sigma(s) \cdot \hat{p}]^{n-4} + [C_n^6 \sigma^2(s) - C_{n+1}^8][\sigma(s) \cdot \hat{p}]^{n-6} + [C_n^8 \sigma^2(s) - C_{n+1}^{10}][\sigma(s) \cdot \hat{p}]^{n-8} + \dots
\end{aligned}$$

$$\text{猜想3.1.2. } \sigma^\alpha(s) [\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s) = \sum_{k=0}^{[n/2]} [C_n^{2k} \sigma^2(s) - C_{n+1}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{n-2k}, n \geq 0$$

3.1.2  $\sigma^\alpha(s) [\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s)$  通项公式的有关引理及其证明

$$\text{引理3.1.1. } \sum_{l=0}^n \sum_{k=0}^{[l/2]} A(k, l) = \sum_{k=0}^{[n/2]} \sum_{l=2k}^n A(k, l)$$

$$\text{引理3.1.2. } \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l C_l^{2k} = C_{n+1}^{2k}$$

$$\text{证明: } \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l C_l^{2k} = C_{n+1}^{2k}$$

$$\Leftrightarrow \sum_{l=0}^{n-2k} (-1)^l C_{n+1}^{n-l} C_{n-l}^{2k} = C_{n+1}^{2k}$$

$$\begin{aligned}
&\Leftrightarrow \sum_{l=0}^{n-2k} (-1)^l \frac{1}{(l+1)!(n-l-2k)!} = \frac{1}{(n+1-2k)!} \\
&\Leftrightarrow \sum_{l=0}^{n-2k} (-1)^l \frac{(n+1-2k)!}{(l+1)!(n-l-2k)!} = 1 \\
&\Leftrightarrow \sum_{l=0}^{n-2k} (-1)^l C_{n+1-2k}^{l+1} = 1 \\
&\Leftrightarrow \sum_{l=0}^{n+1-2k} (-1)^l C_{n+1-2k}^l = 0 \\
&\Leftrightarrow [1 + (-1)]^{n+1-2k} = 0
\end{aligned}$$

□

**引理3.1.3.**  $\sum_{l=2k+1}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{2(k+1)} = C_{n+2}^{2(k+1)}, n > 2k + 1$

**证明:**

$$\begin{aligned}
&\sum_{l=2k+1}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{2(k+1)} = C_{n+2}^{2(k+1)} \\
&\Leftrightarrow \sum_{l=0}^{n-1-2k} (-1)^l C_{n+1}^{n-l} C_{n+1-l}^{2(k+1)} = C_{n+2}^{2(k+1)} \\
&\Leftrightarrow \sum_{l=0}^{n-1-2k} (-1)^l \frac{n+1-l}{(l+1)!(n-l-1-2k)!} = \frac{n+2}{(n-2k)!} \\
&\Leftrightarrow \sum_{l=0}^{n-1-2k} (-1)^l \frac{(n+1-l)(n-2k)!}{(l+1)!(n-l-1-2k)!} = n+2 \\
&\Leftrightarrow \sum_{l=0}^{n-1-2k} (-1)^l (n+1-l) C_{n-2k}^{l+1} = n+2 \\
&\Leftrightarrow \sum_{l=0}^{n-1-2k} (-1)^{l+1} (l+1) C_{n-2k}^{l+1} = -(n+2) \sum_{l=0}^{n-2k} (-1)^l C_{n-2k}^l \\
&\Leftrightarrow \sum_{l=0}^{n-1-2k} (-1)^{l+1} (l+1) C_{n-2k}^{l+1} = 0 \\
&\Leftrightarrow \sum_{l=1}^{n-2k} (-1)^l l C_{n-2k}^l = 0 \\
&\Leftrightarrow (n-2k) \sum_{l=1}^{n-2k} (-1)^l C_{n-1-2k}^{l-1} = 0 \\
&\Leftrightarrow \sum_{l=0}^{n-1-2k} (-1)^l C_{n-1-2k}^l = 0, n > 2k + 1 \\
&\Leftrightarrow [1 + (-1)]^{n-1-2k} = 0, n > 2k + 1
\end{aligned}$$

□

**引理3.1.4.**  $\sum_{l=2k+1}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{2(k+1)} = C_{n+2}^{2(k+1)} - 1, n = 2k + 1$

**证明:**

$$\begin{aligned}
&\sum_{l=2k+1}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{2(k+1)} = C_{2k+2}^{2(k+1)}, n = 2k + 1 \\
&\Leftrightarrow \sum_{l=2k+1}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{2(k+1)} = C_{n+2}^{2(k+1)} - 1, n = 2k + 1
\end{aligned}$$

□

**定理3.1.1.**  $\sum_{l=2k+1}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{2(k+1)} = C_{n+2}^{2(k+1)} - \delta_{n,2k+1}$

**定理3.1.2.**  $\sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k} \sigma^2(s) - C_{n+2}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{n+1-2k}$

$$= -[\sigma(s) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2]\sigma^2(s) + \sum_{k=0}^{[n/2]} \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l [C_l^{2k} \sigma^2(s) - C_{l+1}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{n+1-2k}$$

**证明:**

$$\begin{aligned}
&\sum_{k=0}^{[n/2]} \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l [C_l^{2k} \sigma^2(s) - C_{l+1}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{n+1-2k} \\
&= \sum_{k=0}^{[n/2]} [C_{n+1}^{2k} \sigma^2(s) - C_{n+2}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{n+1-2k} + [\sigma(s) \cdot \hat{p}]^{2-(n+1)\%2} \\
&= \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k} \sigma^2(s) - C_{n+2}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{n+1-2k} + [\sigma(s) \cdot \hat{p}]^{2-(n+1)\%2} + [1 - (n+1)\%2]\sigma^2(s)
\end{aligned}$$

□

### 3.1.3 $\sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s)$ 通项公式的数学归纳法证明

**定理3.1.3.**  $\sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s) = \sum_{k=0}^{[n/2]} [C_n^{2k} \sigma^2(s) - C_{n+1}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{n-2k}, n \geq 0$

证明:

采用数学归纳法证明此定理。

第一步:  $i = 0$  时成立:  $\sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^0 \sigma_\alpha(s) = \sum_{k=0}^{[0/2]} [C_0^{2k} \sigma^2(s) - C_{0+1}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{0-2k}$

第二步: 假设  $i \leq n$  时成立:  $\sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^i \sigma_\alpha(s) = \sum_{k=0}^{[i/2]} [C_i^{2k} \sigma^2(s) - C_{i+1}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{i-2k}, 0 \leq i \leq n$

第三步:  $i = n + 1$  时,  $\sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^{n+1} \sigma(s)\}$

$$= -\{[\sigma(s) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2] \sigma^2(s)\} - \sum_{l=0}^n C_{n+1}^l \{\sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}\} [-\sigma(s) \cdot \hat{p}]^{n+1-l}$$

$$= -\{[\sigma(s) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2] \sigma^2(s)\} - \sum_{l=0}^n C_{n+1}^l \sum_{k=0}^{[l/2]} [C_l^{2k} \sigma^2(s) - C_{l+1}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{l-2k} [-\sigma(s) \cdot \hat{p}]^{n+1-l}$$

$$= -\{[\sigma(s) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2] \sigma^2(s)\} - \sum_{l=0}^n \sum_{k=0}^{[l/2]} (-1)^{n+1-l} C_{n+1}^l [C_l^{2k} \sigma^2(s) - C_{l+1}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{n+1-2k}$$

$$= -\{[\sigma(s) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2] \sigma^2(s)\} + \sum_{k=0}^{[n/2]} \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l [C_l^{2k} \sigma^2(s) - C_{l+1}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{n+1-2k}$$

$$= \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k} \sigma^2(s) - C_{n+2}^{2(k+1)}][\sigma(s) \cdot \hat{p}]^{n+1-2k}$$

此步证明了  $i = n + 1$  时成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

### 3.1.4 $\sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s)$ 的通项公式的推论

**引理3.1.5.**  $\sum_{k=0}^{[n/2]} s^{n-2k} [C_n^{2k} \sigma^2(s) - C_{n+1}^{2(k+1)}] = s[s^{n+1} + (s-1)^n]$

**推论3.1.2.**  $\sigma^\alpha(s)[\sigma(s) \cdot \hat{p}]^n \sigma_\alpha(s) \lambda(\hat{p}, -s\zeta) = s[s^{n+1} + (s-1)^n] [-\zeta]^n \lambda(\hat{p}, -s\zeta)$

**推论3.1.3.**  $[\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^2 \psi = \psi \Rightarrow \sigma^\alpha(s)[\sigma(s) \cdot \hat{\nabla}]^n \sigma_\alpha(s) \psi = s[s^{n+1} + (s-1)^n] [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^n \psi$

## 3.2 $i^{-1} \sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\}$ 的通项公式

### 3.2.1 $i^{-1} \sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\}$ 的通项公式试探

**性质3.2.1.** 
$$\begin{cases} \sigma(s) \times \hat{p} = [\sigma(s), i\sigma(s) \cdot \hat{p}] = i\{\sigma(s)[\sigma(s) \cdot \hat{p}] - [\sigma(s) \cdot \hat{p}]\sigma(s)\} \\ \hat{p} \times \sigma(s) = -[\sigma(s), i\sigma(s) \cdot \hat{p}] = i\{[\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma(s)[\sigma(s) \cdot \hat{p}]\} \end{cases}$$

**性质3.2.2.** 
$$\begin{cases} i^n \sigma(s) [|\times \hat{p}]^n = \begin{cases} i\sigma(s) \times \hat{p}, n = 2k - 1 \\ -\{[\sigma(s) \cdot \hat{p}]\hat{p} - \sigma(s)\}, n = 2k \end{cases}, k \geq 1 \\ i^n [\hat{p} \times |]^n \sigma(s) = \begin{cases} i\hat{p} \times \sigma(s), n = 2k - 1 \\ -\{[\sigma(s) \cdot \hat{p}]\hat{p} - \sigma(s)\}, n = 2k \end{cases}, k \geq 1 \end{cases}$$

**性质3.2.3.** 
$$\begin{cases} i^n \sigma(s) \times \{\sigma(s) [|\times \hat{p}]^n\} = \begin{cases} i\{[\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma^2(s)\hat{p}\}, n = 2k - 1 \\ -[\sigma(s) \times \hat{p}][\sigma(s) \cdot \hat{p}] + i\sigma(s), n = 2k \end{cases}, k \geq 1 \\ i^n i^{-1} \sigma(s) \times \{\sigma(s) [|\times \hat{p}]^n\} = \begin{cases} [\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma^2(s)\hat{p}, n = 2k - 1 \\ -\sigma(s)[\sigma(s) \cdot \hat{p}]^2 + [\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] + \sigma(s), n = 2k \end{cases}, k \geq 1 \end{cases}$$

**推论3.2.1.**

$$i^{-1} \sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\} = i^n i^{-1} \sigma(s) \times \{\sigma(s) [|\times \hat{p}]^n\} - \sum_{k=0}^{n-1} C_n^k \{i^{-1} \sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^k \sigma(s)\}\} [-\sigma(s) \cdot \hat{p}]^{n-k}$$

$$\Rightarrow \begin{cases} i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^0\sigma(s)\} = \sigma(s) \\ i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^1\sigma(s)\} = \begin{cases} [\sigma(s) \cdot \hat{p}]\sigma(s) \\ \sigma(s)[\sigma(s) \cdot \hat{p}] - \sigma^2(s)\hat{p} \end{cases} \\ i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^2\sigma(s)\} = \begin{cases} 3[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] \\ -2\sigma^2(s)\hat{p}[\sigma(s) \cdot \hat{p}] + \sigma(s) \end{cases} \\ i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^3\sigma(s)\} = \begin{cases} 6[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}]^2 + [\sigma(s) \cdot \hat{p}]\sigma(s) \\ -2\sigma(s)[\sigma(s) \cdot \hat{p}]^3 - 3\sigma^2(s)\hat{p}[\sigma(s) \cdot \hat{p}]^2 + 3\sigma(s)[\sigma(s) \cdot \hat{p}] - \sigma^2(s)\hat{p} \end{cases} \\ i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^4\sigma(s)\} = \begin{cases} [\sigma(s) \cdot \hat{p}]\sigma(s)\{10[\sigma(s) \cdot \hat{p}]^2 + 5\}[\sigma(s) \cdot \hat{p}] \\ +\sigma(s)\{-5[\sigma(s) \cdot \hat{p}]^4 + 5[\sigma(s) \cdot \hat{p}]^2 + 1\} - \sigma^2(s)\hat{p}\{4[\sigma(s) \cdot \hat{p}]^2 + 4\}[\sigma(s) \cdot \hat{p}]^1 \end{cases} \end{cases}$$

证明:

$$\begin{aligned} & i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^1\sigma(s)\} \\ &= \{[\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma^2(s)\hat{p}\} - i\{\sigma(s) \times \sigma(s)\}[\sigma(s) \cdot \hat{p}] \\ &= \{\sigma(s) \cdot \hat{p}, \sigma(s)\} - \sigma^2(s)\hat{p} \end{aligned} \quad \square$$

证明:

$$\begin{aligned} & i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^2\sigma(s)\} \\ &= -[\sigma(s), \sigma(s) \cdot \hat{p}][\sigma(s) \cdot \hat{p}] + \sigma(s) + \sum_{k=0}^1 C_2^k \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^k\sigma(s)\}\}[-\sigma(s) \cdot \hat{p}]^{2-k} \\ &= -[\sigma(s), \sigma(s) \cdot \hat{p}][\sigma(s) \cdot \hat{p}] + \sigma(s) - C_2^0\sigma(s)[\sigma(s) \cdot \hat{p}]^2 + \sum_{k=1}^1 C_2^k \{i\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^k\sigma(s)\}\}[-\sigma(s) \cdot \hat{p}]^{2-k} \\ &= -[\sigma(s), \sigma(s) \cdot \hat{p}][\sigma(s) \cdot \hat{p}] - \sigma(s) - C_2^0\sigma(s)[\sigma(s) \cdot \hat{p}]^2 + C_2^1\{[\sigma(s), \sigma(s) \cdot \hat{p}] - \sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}] \\ &= 3[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] - 2\sigma^2(s)\hat{p}[\sigma(s) \cdot \hat{p}] + \sigma(s) \end{aligned} \quad \square$$

证明:

$$\begin{aligned} & i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^3\sigma(s)\} \\ &= -i^3 i\sigma(s) \times \{\sigma(s)[\times \hat{p}]^3\} + \sum_{k=0}^2 C_3^k \{i\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^k\sigma(s)\}\}[-\sigma(s) \cdot \hat{p}]^{3-k} \\ &= \{[\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma^2(s)\hat{p}\} + C_3^0\sigma(s)[\sigma(s) \cdot \hat{p}]^3 - \sum_{k=1}^2 C_3^k \{i\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^k\sigma(s)\}\}[-\sigma(s) \cdot \hat{p}]^{3-k} \\ &= -\{[\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma^2(s)\hat{p}\} - C_3^0\sigma(s)[\sigma(s) \cdot \hat{p}]^3 + C_3^1\{-[\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma(s)[\sigma(s) \cdot \hat{p}] + \sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}]^2 + \sum_{k=2}^2 C_3^k \{i\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^k\sigma(s)\}\}[-\sigma(s) \cdot \hat{p}]^{3-k} \\ &= \{[\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma^2(s)\hat{p}\} + C_3^0\sigma(s)[\sigma(s) \cdot \hat{p}]^3 + C_3^1\{-[\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma(s)[\sigma(s) \cdot \hat{p}] + \sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}]^2 \\ &\quad - C_3^2\{-3[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] + 2\sigma^2(s)\hat{p}[\sigma(s) \cdot \hat{p}] - \sigma(s)\}[\sigma(s) \cdot \hat{p}] \\ &= 6[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}]^2 - 2\sigma(s)[\sigma(s) \cdot \hat{p}]^3 - 3\sigma^2(s)\hat{p}[\sigma(s) \cdot \hat{p}]^2 + [\sigma(s) \cdot \hat{p}]\sigma(s) + 3\sigma(s)[\sigma(s) \cdot \hat{p}] - \sigma^2(s)\hat{p} \end{aligned} \quad \square$$

证明:  $i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^4\sigma(s)\}$ 

$$\begin{aligned} &= i^4 i^{-1}\sigma(s) \times \{\sigma(s)[\times \hat{p}]^4\} - \sum_{k=0}^3 C_4^k \{i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^k\sigma(s)\}\}[-\sigma(s) \cdot \hat{p}]^{4-k} \\ &= -\sigma(s)[\sigma(s) \cdot \hat{p}]^2 + [\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] + \sigma(s) \\ &\quad - C_4^0\sigma(s)[\sigma(s) \cdot \hat{p}]^4 \\ &\quad + C_4^1\{[\sigma(s) \cdot \hat{p}]\sigma(s) + \sigma(s)[\sigma(s) \cdot \hat{p}] - \sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}]^3 \\ &\quad - C_4^2\{3[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] - 2\sigma^2(s)\hat{p}[\sigma(s) \cdot \hat{p}] + \sigma(s)\}[\sigma(s) \cdot \hat{p}]^2 \\ &\quad + C_4^3\{6[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}]^2 + [\sigma(s) \cdot \hat{p}]\sigma(s) - 2\sigma(s)[\sigma(s) \cdot \hat{p}]^3 - 3\sigma^2(s)\hat{p}[\sigma(s) \cdot \hat{p}]^2 + 3\sigma(s)[\sigma(s) \cdot \hat{p}] - \sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}] \\ &= [\sigma(s) \cdot \hat{p}]\sigma(s)\{(1 + C_4^3)[\sigma(s) \cdot \hat{p}] + (C_4^1 - 3C_4^2 + 6C_4^3)[\sigma(s) \cdot \hat{p}]^3\} \\ &\quad + \{(-C_4^0 + C_4^1 - 2C_4^3)\sigma(s)\}[\sigma(s) \cdot \hat{p}]^4 + \{-C_4^1 + 2C_4^2 - 3C_4^3\}\sigma^2(s)\hat{p}[\sigma(s) \cdot \hat{p}]^3 \\ &\quad + \{-1 - C_4^2 + 3C_4^3\}\sigma(s)[\sigma(s) \cdot \hat{p}]^2 \end{aligned}$$

$$\begin{aligned}
& + \{-C_4^3\}\sigma^2(s)[\sigma(s) \cdot \hat{p}]^1 \\
& + \sigma(s) \\
& = [\sigma(s) \cdot \hat{p}]\sigma(s)\{(C_4^1 + C_4^2)[\sigma(s) \cdot \hat{p}]^2 + (C_4^3 + C_4^4)\}[\sigma(s) \cdot \hat{p}] + \sigma(s)\{(C_4^0 - C_4^2)[\sigma(s) \cdot \hat{p}]^4 + (C_4^2 - C_4^4)[\sigma(s) \cdot \hat{p}]^2 + (C_4^4 - C_4^6)\} - \sigma^2(s)\hat{p}\{C_4^1[\sigma(s) \cdot \hat{p}]^2 + C_4^3\}[\sigma(s) \cdot \hat{p}]^1 \\
& = [\sigma(s) \cdot \hat{p}]\sigma(s)\{10[\sigma(s) \cdot \hat{p}]^2 + 5\}[\sigma(s) \cdot \hat{p}] + \sigma(s)\{-5[\sigma(s) \cdot \hat{p}]^4 + 5\}[\sigma(s) \cdot \hat{p}]^2 + 1\} - \sigma^2(s)\hat{p}\{4[\sigma(s) \cdot \hat{p}]^2 + 4\}[\sigma(s) \cdot \hat{p}]^1 \quad \square
\end{aligned}$$

推论3.2.2.

$$\left\{ \begin{aligned}
i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^0\sigma(s)\} &= \sigma(s) \\
i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^1\sigma(s)\} &= \begin{cases} (C_1^1 + C_1^2)[\sigma(s) \cdot \hat{p}]\sigma(s) \\ +(C_1^0 - C_1^2)\sigma(s)[\sigma(s) \cdot \hat{p}] \\ -C_1^1\sigma^2(s)\hat{p} \end{cases} \\
i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^2\sigma(s)\} &= \begin{cases} (C_2^1 + C_2^2)[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] \\ +(C_2^0 - C_2^2)\sigma(s)[\sigma(s) \cdot \hat{p}]^2 + (C_2^2 - C_2^4)\sigma(s) \\ -C_2^1\sigma^2(s)\hat{p}[\sigma(s) \cdot \hat{p}] \end{cases} \\
i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^3\sigma(s)\} &= \begin{cases} [\sigma(s) \cdot \hat{p}]\sigma(s)\{(C_3^1 + C_3^2)[\sigma(s) \cdot \hat{p}]^2 + (C_3^3 + C_3^4)\} \\ +\sigma(s)\{(C_3^0 - C_3^2)[\sigma(s) \cdot \hat{p}]^2 + (C_3^2 - C_3^4)\}[\sigma(s) \cdot \hat{p}] \\ -\sigma^2(s)\hat{p}\{C_3^1[\sigma(s) \cdot \hat{p}]^2 + C_3^3\} \end{cases} \\
i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^4\sigma(s)\} &= \begin{cases} [\sigma(s) \cdot \hat{p}]\sigma(s)\{(C_4^1 + C_4^2)[\sigma(s) \cdot \hat{p}]^2 + (C_4^3 + C_4^4)\}[\sigma(s) \cdot \hat{p}] \\ +\sigma(s)\{(C_4^0 - C_4^2)[\sigma(s) \cdot \hat{p}]^4 + (C_4^2 - C_4^4)[\sigma(s) \cdot \hat{p}]^2 + (C_4^4 - C_4^6)\} \\ -\sigma^2(s)\hat{p}\{C_4^1[\sigma(s) \cdot \hat{p}]^2 + C_4^3\}[\sigma(s) \cdot \hat{p}]^1 \end{cases}
\end{aligned} \right.$$

猜想3.2.1.  $i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n\sigma(s)\}, n \geq 0$

$$\begin{aligned}
& = [\sigma(s) \cdot \hat{p}]\sigma(s) \sum_{k=0}^{[(n-1)/2]} (C_n^{2k+1} + C_n^{2k+2})[\sigma(s) \cdot \hat{p}]^{n-2k-1} + \sigma(s) \sum_{k=0}^{[n/2]} (C_n^{2k} - C_n^{2k+2})[\sigma(s) \cdot \hat{p}]^{n-2k} \\
& - \sigma^2(s)\hat{p} \sum_{k=0}^{[(n-1)/2]} C_n^{2k+1}[\sigma(s) \cdot \hat{p}]^{n-2k-1}
\end{aligned}$$

3.2.2  $i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n\sigma(s)\}$ 通项公式的数学归纳法证明

定理3.2.1.  $i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n\sigma(s)\}, n \geq 0$

$$= \sum_{k=0}^{[n/2]} \{(C_n^{2k+1} + C_n^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) + (C_n^{2k} - C_n^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] - C_n^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}]^{n-2k-1}$$

证明:

采用数学归纳法证明此定理。

第一步:  $i = 0$ 时成立:

$$i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^0\sigma(s)\} = \sum_{k=0}^{[0/2]} \{(C_0^{2k+1} + C_0^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) + (C_0^{2k} - C_0^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] - C_0^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}]^{0-2k-1}$$

第二步: 假设  $0 \leq l \leq n$ 时成立:

$$i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^l\sigma(s)\} = \sum_{k=0}^{[l/2]} \{(C_i^{2k+1} + C_i^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) + (C_i^{2k} - C_i^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] - C_i^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}]^{i-2k-1}$$

第三步:  $i = n + 1$ 时,  $i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^{n+1}\sigma(s)\}$

$$\begin{aligned}
& = i^{n+1}i^{-1}\sigma(s) \times \{\sigma(s)[\times \hat{p}]^{n+1}\} - \sum_{l=0}^n C_{n+1}^l \{i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^l\sigma(s)\}\}[-\sigma(s) \cdot \hat{p}]^{n+1-l} \\
& = i^{n+1}i^{-1}\sigma(s) \times \{\sigma(s)[\times \hat{p}]^{n+1}\} \\
& - \sum_{l=0}^n C_{n+1}^l \sum_{k=0}^{[l/2]} \{(C_l^{2k+1} + C_l^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) + (C_l^{2k} - C_l^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] - C_l^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}]^{l-2k-1}[-\sigma(s) \cdot \hat{p}]^{n+1-l}
\end{aligned}$$

$$\begin{aligned}
&= i^{n+1}i^{-1}\sigma(s) \times \{\sigma(s)[\cdot \hat{p}]^{n+1}\} \\
&+ \sum_{l=0}^n \sum_{k=0}^{[l/2]} (-1)^{n-l} C_{n+1}^l \{(C_l^{2k+1} + C_l^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) + (C_l^{2k} - C_l^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] - C_l^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}]^{n-2k} \\
&= i^{n+1}i^{-1}\sigma(s) \times \{\sigma(s)[\cdot \hat{p}]^{n+1}\} \\
&+ \sum_{k=0}^{[n/2]} \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l \{(C_l^{2k+1} + C_l^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) + (C_l^{2k} - C_l^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] - C_l^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}]^{n-2k} \\
&= \sum_{k=0}^{[(n+1)/2]} \{(C_{n+1}^{2k+1} + C_{n+1}^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) + (C_{n+1}^{2k} - C_{n+1}^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] - C_{n+1}^{2k+1}\sigma^2(s)\hat{p}\}[\sigma(s) \cdot \hat{p}]^{n-2k}
\end{aligned}$$

此步证明了  $i = n + 1$  时成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。  $\square$

### 3.2.3 $i^{-1}\{\sigma(s)[\sigma(s) \cdot \hat{p}]^n\} \times \sigma(s)$ 通项公式的归纳法证明

**定理3.2.2.**  $i^{-1}\{\sigma(s)[\sigma(s) \cdot \hat{p}]^n\} \times \sigma(s), n \geq 0$

$$\begin{aligned}
&= \left\{ \sum_{k=0}^{[(n-1)/2]} (C_n^{2k+1} + C_n^{2k+2})[\sigma(s) \cdot \hat{p}]^{n-2k-1} \right\} \sigma(s)[\sigma(s) \cdot \hat{p}] + \left\{ \sum_{k=0}^{[n/2]} (C_n^{2k} - C_n^{2k+2})[\sigma(s) \cdot \hat{p}]^{n-2k} \right\} \sigma(s) \\
&- \left\{ \sum_{k=0}^{[(n-1)/2]} C_n^{2k+1}[\sigma(s) \cdot \hat{p}]^{n-2k-1} \right\} \sigma^2(s)\hat{p}
\end{aligned}$$

**定理3.2.3.**  $i^{-1}\{\sigma(s)[\sigma(s) \cdot \hat{p}]^n\} \times \sigma(s), n \geq 0$

$$= \sum_{k=0}^{[n/2]} [\sigma(s) \cdot \hat{p}]^{n-2k-1} \{(C_n^{2k+1} + C_n^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] + (C_n^{2k} - C_n^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) - C_n^{2k+1}\sigma^2(s)\hat{p}\}$$

**证明:**

采用数学归纳法证明此定理。

第一步:  $i = 0$  时成立:

$$i^{-1}\{\sigma(s)[\sigma(s) \cdot \hat{p}]^0\} \times \sigma(s) = \sum_{k=0}^{[0/2]} [\sigma(s) \cdot \hat{p}]^{0-2k-1} \{(C_0^{2k+1} + C_0^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] + (C_0^{2k} - C_0^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) - C_0^{2k+1}\sigma^2(s)\hat{p}\}$$

第二步: 假设  $0 \leq l \leq n$  时成立:

$$i^{-1}\{\sigma(s)[\sigma(s) \cdot \hat{p}]^l\} \times \sigma(s) = \sum_{k=0}^{[l/2]} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \{(C_i^{2k+1} + C_i^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] + (C_i^{2k} - C_i^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) - C_i^{2k+1}\sigma^2(s)\hat{p}\}$$

第三步:  $i = n + 1$  时,  $i^{-1}\{\sigma(s)[\sigma(s) \cdot \hat{p}]^{n+1}\} \times \sigma(s) = i^{n+1}i^{-1}\{[\hat{p} \times ]^{n+1}\sigma(s)\} \times \sigma(s) - \sum_{l=0}^n C_{n+1}^l [-\sigma(s) \cdot \hat{p}]^{n+1-l} \{i^{-1}\sigma(s)[\sigma(s) \cdot \hat{p}]^l\} \times \sigma(s)$

$$\begin{aligned}
&= i^{n+1}i^{-1}\{[\hat{p} \times ]^{n+1}\sigma(s)\} \times \sigma(s) \\
&- \sum_{l=0}^n C_{n+1}^l \sum_{k=0}^{[l/2]} [-\sigma(s) \cdot \hat{p}]^{n+1-l} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \{(C_l^{2k+1} + C_l^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] + (C_l^{2k} - C_l^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) - C_l^{2k+1}\sigma^2(s)\hat{p}\} \\
&= i^{n+1}i^{-1}\{[\hat{p} \times ]^{n+1}\sigma(s)\} \times \sigma(s) \\
&+ \sum_{l=0}^n \sum_{k=0}^{[l/2]} [\sigma(s) \cdot \hat{p}]^{n-2k} (-1)^{n-l} C_{n+1}^l \{(C_l^{2k+1} + C_l^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] + (C_l^{2k} - C_l^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) - C_l^{2k+1}\sigma^2(s)\hat{p}\} \\
&= i^{n+1}i^{-1}\{[\hat{p} \times ]^{n+1}\sigma(s)\} \times \sigma(s) \\
&+ \sum_{k=0}^{[n/2]} \sum_{l=2k}^n [\sigma(s) \cdot \hat{p}]^{n-2k} (-1)^{n-l} C_{n+1}^l \{(C_l^{2k+1} + C_l^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] + (C_l^{2k} - C_l^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) - C_l^{2k+1}\sigma^2(s)\hat{p}\} \\
&= \sum_{k=0}^{[(n+1)/2]} [\sigma(s) \cdot \hat{p}]^{n-2k} \{(C_{n+1}^{2k+1} + C_{n+1}^{2k+2})\sigma(s)[\sigma(s) \cdot \hat{p}] + (C_{n+1}^{2k} - C_{n+1}^{2k+2})[\sigma(s) \cdot \hat{p}]\sigma(s) - C_{n+1}^{2k+1}\sigma^2(s)\hat{p}\}
\end{aligned}$$

此步证明了  $i = n + 1$  时成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。  $\square$

### 3.2.4 $i^{-1}\hat{p} \cdot |\sigma(s) \times \{\sigma(s) \cdot \hat{p}\}^n \sigma(s)\}$ 的通项公式

$$\text{推论3.2.3. } i^{-1}\hat{p} \cdot |\sigma(s) \times \{\sigma(s) \cdot \hat{p}\}^n \sigma(s)\} = \sum_{k=0}^{(n+1)/2} [C_{n+1}^{2k+1} - C_n^{2k-1}\sigma^2(s)][\sigma(s) \cdot \hat{p}]^{n+1-2k}, n \geq 0$$



$$\begin{aligned}
& \text{证明: } i^{-1}\hat{p} \cdot |\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\} = i^{-1}\{\sigma(s)[\sigma(s) \cdot \hat{p}]^n\} \times \sigma(s) \cdot \hat{p} \\
& = \sum_{k=0}^{[n/2]} \{(C_n^{2k+1} + C_n^{2k+2})[\sigma(s) \cdot \hat{p}]^2 + (C_n^{2k} - C_n^{2k+2})[\sigma(s) \cdot \hat{p}]^2 - C_n^{2k+1}\sigma^2(s)\}[\sigma(s) \cdot \hat{p}]^{n-2k-1} \\
& = \sum_{k=0}^{[n/2]} \{(C_n^{2k+1} + C_n^{2k+2}) + (C_n^{2k} - C_n^{2k+2})\}[\sigma(s) \cdot \hat{p}]^{n+1-2k} - \sum_{k=0}^{[n/2]} C_n^{2k+1}\sigma^2(s)[\sigma(s) \cdot \hat{p}]^{n-1-2k} \\
& = \sum_{k=0}^{[n/2]} C_{n+1}^{2k+1}[\sigma(s) \cdot \hat{p}]^{n+1-2k} - \sum_{k=1}^{[n/2]+1} C_n^{2k-1}\sigma^2(s)[\sigma(s) \cdot \hat{p}]^{n+1-2k} \\
& = \sum_{k=0}^{[(n+1)/2]} C_{n+1}^{2k+1}[\sigma(s) \cdot \hat{p}]^{n+1-2k} - \sum_{k=0}^{[(n-1)/2]+1} C_n^{2k-1}\sigma^2(s)[\sigma(s) \cdot \hat{p}]^{n+1-2k} \\
& = \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k+1} - C_n^{2k-1}\sigma^2(s)][\sigma(s) \cdot \hat{p}]^{n+1-2k}
\end{aligned}$$

□

## 4 复杂自旋复合算符的通项公式

### 4.1 $i^{-n}[\sigma(s) \times ||^n[\sigma(s) \cdot \hat{p}]\sigma(s)$ 和 $i^{-n}\sigma(s)[\sigma(s) \cdot \hat{p}]|| \times \sigma(s)^n$ 的通项公式

#### 4.1.1 $i^{-n}[\sigma(s) \times ||^n[\sigma(s) \cdot \hat{p}]\sigma(s)$ 的通项公式

性质4.1.1.

$$\begin{cases}
i^{-0}[\sigma(s) \times ||^0\{\sigma(s) \cdot \hat{p}\}\sigma(s)] = [\sigma(s) \cdot \hat{p}]\sigma(s) \\
i^{-1}[\sigma(s) \times ||^1\{\sigma(s) \cdot \hat{p}\}\sigma(s)] = \sigma(s)[\sigma(s) \cdot \hat{p}] + [\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma^2(s)\hat{p} \\
i^{-2}[\sigma(s) \times ||^2\{\sigma(s) \cdot \hat{p}\}\sigma(s)] = [2 - \sigma^2(s)]\sigma(s)[\sigma(s) \cdot \hat{p}] + [1 + \sigma^2(s)][\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma^2(s)\hat{p} \\
i^{-3}[\sigma(s) \times ||^3\{\sigma(s) \cdot \hat{p}\}\sigma(s)] = [3 - \sigma^2(s)]\sigma(s)[\sigma(s) \cdot \hat{p}] + [1 + 2\sigma^2(s)][\sigma(s) \cdot \hat{p}]\sigma(s) - [1 + \sigma^2(s)]\sigma^2(s)\hat{p} \\
i^{-4}[\sigma(s) \times ||^4\{\sigma(s) \cdot \hat{p}\}\sigma(s)] = [4 - \sigma^4(s)]\sigma(s)[\sigma(s) \cdot \hat{p}] + [1 + 3\sigma^2(s) + \sigma^4(s)][\sigma(s) \cdot \hat{p}]\sigma(s) - [1 + 2\sigma^2(s)]\sigma^2(s)\hat{p}
\end{cases}$$

性质4.1.2.

$$\begin{cases}
i^{-n}[\sigma(s) \times ||^n[\sigma(s) \cdot \hat{p}]\sigma(s)] = \sigma(s)[\sigma(s) \cdot \hat{p}] - i^{-(n+1)}[\sigma(s) \times ||^{n+1}\hat{p} \\
i^{-n}\hat{p} \cdot ||[\sigma(s) \times ||^n[\sigma(s) \cdot \hat{p}]\sigma(s)] = [\sigma(s) \cdot \hat{p}]^2 - i^{-(n+1)}\hat{p} \cdot ||[\sigma(s) \times ||^{n+1}\hat{p}
\end{cases}$$

证明:  $i^{-(n+1)}[\sigma(s) \times ||^{n+1}\hat{p}$

$$\begin{aligned}
& = i^{-n}[\sigma(s) \times ||^n i^{-1}\sigma(s) \times \hat{p} \\
& = i^{-n}[\sigma(s) \times ||^n \{\sigma(s)[\sigma(s) \cdot \hat{p}] - [\sigma(s) \cdot \hat{p}]\sigma(s)\} \\
& = \sigma(s)[\sigma(s) \cdot \hat{p}] - i^{-n}[\sigma(s) \times ||^n[\sigma(s) \cdot \hat{p}]\sigma(s)] \\
& \Rightarrow i^{-n}[\sigma(s) \times ||^n[\sigma(s) \cdot \hat{p}]\sigma(s)] = \sigma(s)[\sigma(s) \cdot \hat{p}] - i^{-(n+1)}[\sigma(s) \times ||^{n+1}\hat{p}
\end{aligned}$$

□

推论4.1.1.

$$\begin{cases}
i^{-n}[\sigma(s) \times ||^n[\sigma(s) \cdot \hat{p}]\sigma(s)] = (1 - a_{n+1})\sigma(s)[\sigma(s) \cdot \hat{p}] - b_{n+1}[\sigma(s) \cdot \hat{p}]\sigma(s) + c_{n+1}\sigma^2(s)\hat{p} \\
a_{n+1} = \frac{[(s+1)^{n+3} - (-s)^{n+3}] - 2[(s+1)^{n+2} - (-s)^{n+2}] + (2s+1)}{s(2s+1)(s+1)} \\
b_{n+1} = -\frac{(s+1)^{n+1} - (-s)^{n+1}}{2s+1}, c_{n+1} = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0
\end{cases}$$

推论4.1.2.

$$\begin{cases}
i^{-n}[\sigma(s) \times ||^n[\sigma(s) \cdot \hat{p}]\sigma(s)] = (1 - a_{n+1})\sigma(s)[\sigma(s) \cdot \hat{p}] - b_{n+1}[\sigma(s) \cdot \hat{p}]\sigma(s) + b_n\sigma^2(s)\hat{p}, n \geq 0 \\
a_{n+1} = \frac{[(s+1)^{n+3} - (-s)^{n+3}] - 2[(s+1)^{n+2} - (-s)^{n+2}] + (2s+1)}{s(2s+1)(s+1)}, b_{n+1} = -\frac{(s+1)^{n+1} - (-s)^{n+1}}{2s+1}
\end{cases}$$

推论4.1.3.

$$\begin{cases}
i^{-n}\hat{p} \cdot ||[\sigma(s) \times ||^n[\sigma(s) \cdot \hat{p}]\sigma(s)] = (1 - a_{n+1} - b_{n+1})[\sigma(s) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s)\hat{p} \\
a_{n+1} = \frac{[(s+1)^{n+3} - (-s)^{n+3}] - 2[(s+1)^{n+2} - (-s)^{n+2}] + (2s+1)}{s(2s+1)(s+1)} \\
b_{n+1} = -\frac{(s+1)^{n+1} - (-s)^{n+1}}{2s+1}, c_{n+1} = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0
\end{cases}$$

推论4.1.4.

$$\begin{cases} i^{-n}\hat{p} \cdot [|\sigma(s) \times \rangle^n [\sigma(s) \cdot \hat{p}] \sigma(s) = (1 - a_n - 2b_n)[\sigma(s) \cdot \hat{p}]^2 + b_n \sigma^2(s) \hat{p} \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0 \end{cases}$$

推论4.1.5.  $i^{-n}\hat{p} \cdot [|\sigma(s) \times \rangle^n [\sigma(s) \cdot \hat{p}] \sigma(s) = [1 + \frac{(s+1)^{n+2} - (-s)^{n+2} - (2s+1)}{s(2s+1)(s+1)}][\sigma(s) \cdot \hat{p}]^2 - \frac{(s+1)^n - (-s)^n}{2s+1} \sigma^2(s), n \geq 0$

#### 4.1.2 $i^{-n}\sigma(s)[\sigma(s) \cdot \hat{p}][|\times \sigma(s)\rangle^n$ 的通项公式

性质4.1.3.

$$\begin{cases} i^{-n}\sigma(s)[\sigma(s) \cdot \hat{p}][|\times \sigma(s)\rangle^n = [\sigma(s) \cdot \hat{p}]\sigma(s) - i^{-(n+1)}\hat{p}[|\times \sigma(s)\rangle^{n+1} \\ i^{-n}\hat{p} \cdot |\sigma(s)[\sigma(s) \cdot \hat{p}][|\times \sigma(s)\rangle^n = [\sigma(s) \cdot \hat{p}]^2 - i^{-(n+1)}\hat{p}[|\times \sigma(s)\rangle^{n+1}] \cdot \hat{p} \end{cases}$$

证明:  $i^{-(n+1)}\hat{p}[|\times \sigma(s)\rangle^{n+1}$

$$= i^{-1}\hat{p} \times \sigma(s)[|\times \sigma(s)\rangle^n i^{-n}$$

$$= \{[\sigma(s) \cdot \hat{p}]\sigma(s) - \sigma(s)[\sigma(s) \cdot \hat{p}]\}[|\times \sigma(s)\rangle^n i^{-n}$$

$$= [\sigma(s) \cdot \hat{p}]\sigma(s) - i^{-n}\sigma(s)[\sigma(s) \cdot \hat{p}][|\times \sigma(s)\rangle^n$$

$$\Rightarrow i^{-n}\sigma(s)[\sigma(s) \cdot \hat{p}][|\times \sigma(s)\rangle^n = [\sigma(s) \cdot \hat{p}]\sigma(s) - i^{-(n+1)}\hat{p}[|\times \sigma(s)\rangle^{n+1} \quad \square$$

推论4.1.6.

$$\begin{cases} i^{-n}[\sigma(s) \times \rangle^n [\sigma(s) \cdot \hat{p}]\sigma(s) = (1 - a_{n+1})[\sigma(s) \cdot \hat{p}]\sigma(s) - b_{n+1}\sigma(s)[\sigma(s) \cdot \hat{p}] + c_{n+1}\sigma^2(s)\hat{p} \\ a_{n+1} = \frac{[(s+1)^{n+3} - (-s)^{n+3}] - 2[(s+1)^{n+2} - (-s)^{n+2}] + (2s+1)}{s(2s+1)(s+1)} \\ b_{n+1} = -\frac{(s+1)^{n+1} - (-s)^{n+1}}{2s+1}, c_{n+1} = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0 \end{cases}$$

推论4.1.7.

$$\begin{cases} i^{-n}[\sigma(s) \times \rangle^n [\sigma(s) \cdot \hat{p}]\sigma(s) = (1 - a_{n+1})[\sigma(s) \cdot \hat{p}]\sigma(s) - b_{n+1}\sigma(s)[\sigma(s) \cdot \hat{p}] + b_n \sigma^2(s)\hat{p}, n \geq 0 \\ a_{n+1} = \frac{[(s+1)^{n+3} - (-s)^{n+3}] - 2[(s+1)^{n+2} - (-s)^{n+2}] + (2s+1)}{s(2s+1)(s+1)}, b_{n+1} = -\frac{(s+1)^{n+1} - (-s)^{n+1}}{2s+1} \end{cases}$$

推论4.1.8.

$$\begin{cases} i^{-n}\hat{p} \cdot |\sigma(s)[\sigma(s) \cdot \hat{p}][|\times \sigma(s)\rangle^n = (1 - a_{n+1} - b_{n+1})[\sigma(s) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s)\hat{p} \\ a_{n+1} = \frac{[(s+1)^{n+3} - (-s)^{n+3}] - 2[(s+1)^{n+2} - (-s)^{n+2}] + (2s+1)}{s(2s+1)(s+1)} \\ b_{n+1} = -\frac{(s+1)^{n+1} - (-s)^{n+1}}{2s+1}, c_{n+1} = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0 \end{cases}$$

推论4.1.9.

$$\begin{cases} i^{-n}\hat{p} \cdot |\sigma(s)[\sigma(s) \cdot \hat{p}][|\times \sigma(s)\rangle^n = (1 - a_n - 2b_n)[\sigma(s) \cdot \hat{p}]^2 + b_n \sigma^2(s)\hat{p} \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0 \end{cases}$$

推论4.1.10.  $i^{-n}\hat{p} \cdot |\sigma(s)[\sigma(s) \cdot \hat{p}][|\times \sigma(s)\rangle^n = [1 + \frac{(s+1)^{n+2} - (-s)^{n+2} - (2s+1)}{s(2s+1)(s+1)}][\sigma(s) \cdot \hat{p}]^2 - \frac{(s+1)^n - (-s)^n}{2s+1} \sigma^2(s), n \geq 0$

推论4.1.11.  $i^{-n}\hat{p} \cdot [|\sigma(s) \times \rangle^n [\sigma(s) \cdot \hat{p}]\sigma(s) = i^{-n}\hat{p} \cdot |\sigma(s)[\sigma(s) \cdot \hat{p}][|\times \sigma(s)\rangle^n, n \geq 0$

## 4.2 $i^{-n}[\sigma(s) \times \rangle^n [\sigma(s) \cdot \hat{p}]^l \sigma(s)$ 和 $i^{-n}\sigma(s)[\sigma(s) \cdot \hat{p}]^l[|\times \sigma(s)\rangle^n$ 的通项公式

### 4.2.1 $i^{-n}[\sigma(s) \times \rangle^n [\sigma(s) \cdot \hat{p}]^l \sigma(s)$ 的通项公式

定理4.2.1.  $i^{-n}[\sigma(s) \times \rangle^n \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}, n \geq 1, l \geq 0$

$$\begin{aligned} &= \sum_{k=0}^{\lfloor l/2 \rfloor} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1}] \sigma(s) [\sigma(s) \cdot \hat{p}] - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s) b_{n-1}] [\sigma(s) \cdot \hat{p}] \sigma(s) \\ &+ [C_{l+1}^{2k+2} \sigma^2(s) b_{n-1} + C_l^{2k+1} \sigma^4(s) b_{n-2}] \hat{p} \} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \end{aligned}$$

证明:  $i^{-n}[\sigma(s) \times \rangle^n \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}$

$$= i^{-(n-1)}[\sigma(s) \times \rangle^{n-1} i^{-1}\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}$$

$$= i^{-(n-1)}[\sigma(s) \times \rangle^{n-1} \sum_{k=0}^{\lfloor l/2 \rfloor} \{ (C_l^{2k+1} + C_l^{2k+2}) [\sigma(s) \cdot \hat{p}] \sigma(s) + (C_l^{2k} - C_l^{2k+2}) \sigma(s) [\sigma(s) \cdot \hat{p}] - C_l^{2k+1} \sigma^2(s) \hat{p} \}$$

$$\begin{aligned}
& [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{ (C_l^{2k+1} + C_l^{2k+2}) \{ \sigma(s) [\sigma(s) \cdot \hat{p}] - i^{-n} [\sigma(s) \times ||^n \hat{p}] \} + (C_l^{2k} - C_l^{2k+2}) \sigma(s) [\sigma(s) \cdot \hat{p}] \\
&\quad - C_l^{2k+1} \sigma^2(s) i^{-(n-1)} [\sigma(s) \times ||^{n-1} \hat{p}] \} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{ -C_{l+1}^{2k+2} i^{-n} [\sigma(s) \times ||^n \hat{p} + C_{l+1}^{2k+1} \sigma(s) [\sigma(s) \cdot \hat{p}] - C_l^{2k+1} \sigma^2(s) i^{-(n-1)} [\sigma(s) \times ||^{n-1} \hat{p}] \} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{ -C_{l+1}^{2k+2} \{ a_n \sigma(s) [\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}] \sigma(s) - c_n \sigma^2(s) \hat{p} \} + C_{l+1}^{2k+1} \sigma(s) [\sigma(s) \cdot \hat{p}] \\
&\quad - C_l^{2k+1} \sigma^2(s) \{ a_{n-1} \sigma(s) [\sigma(s) \cdot \hat{p}] + b_{n-1} [\sigma(s) \cdot \hat{p}] \sigma(s) - c_{n-1} \sigma^2(s) \hat{p} \} \} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{ [-C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1} + C_{l+1}^{2k+1}] \sigma(s) [\sigma(s) \cdot \hat{p}] + [-C_{l+1}^{2k+2} b_n - C_l^{2k+1} \sigma^2(s) b_{n-1}] [\sigma(s) \cdot \hat{p}] \sigma(s) \\
&\quad + [C_{l+1}^{2k+2} c_n + C_l^{2k+1} \sigma^2(s) c_{n-1}] \sigma^2(s) \hat{p} \} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1}] \sigma(s) [\sigma(s) \cdot \hat{p}] - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s) b_{n-1}] [\sigma(s) \cdot \hat{p}] \sigma(s) \\
&\quad + [C_{l+1}^{2k+2} \sigma^2(s) c_n + C_l^{2k+1} \sigma^4(s) c_{n-1}] \hat{p} \} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1}] \sigma(s) [\sigma(s) \cdot \hat{p}] - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s) b_{n-1}] [\sigma(s) \cdot \hat{p}] \sigma(s) \\
&\quad + [C_{l+1}^{2k+2} \sigma^2(s) b_{n-1} + C_l^{2k+1} \sigma^4(s) b_{n-2}] \hat{p} \} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \quad \square
\end{aligned}$$

**推论4.2.1.**  $i^{-n} \hat{p} \cdot ||[\sigma(s) \times ||^n \{ [\sigma(s) \cdot \hat{p}]^l \sigma(s) \}, n \geq 1, l \geq 0$

$$\begin{aligned}
&= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} \sigma^2(s) (a_{n-1} + b_{n-1})] [\sigma(s) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2} \sigma^2(s) c_n + C_l^{2k+1} \sigma^4(s) c_{n-1}] \} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} (1 + b_n)] [\sigma(s) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2} \sigma^2(s) c_n + C_l^{2k+1} \sigma^4(s) c_{n-1}] \} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{ [C_l^{2k} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} b_n] [\sigma(s) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2} \sigma^2(s) b_{n-1} + C_l^{2k+1} \sigma^4(s) b_{n-2}] \} [\sigma(s) \cdot \hat{p}]^{l-2k-1}
\end{aligned}$$

**推论4.2.2.**  $i^{-n} \hat{p} \cdot ||[\sigma(s) \times ||^n \{ [\sigma(s) \cdot \hat{p}]^l \sigma(s) \}, n \geq 1, l \geq 0$

$$= \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s) + C_l^{2k}] [\sigma(s) \cdot \hat{p}]^{l+1-2k} - b_{n-1} \sigma^2(s) [\sigma(s) \cdot \hat{p}]^{l+1}$$

**证明:**  $i^{-n} \hat{p} \cdot ||[\sigma(s) \times ||^n \{ [\sigma(s) \cdot \hat{p}]^l \sigma(s) \}, n \geq 1, l \geq 0$

$$\begin{aligned}
&= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} \sigma^2(s) (a_{n-1} + b_{n-1})] [\sigma(s) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2} \sigma^2(s) c_n + C_l^{2k+1} \sigma^4(s) c_{n-1}] \} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} \sigma^2(s) (a_{n-1} + b_{n-1})] [\sigma(s) \cdot \hat{p}]^2 [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&\quad + \sum_{k=0}^{[(l-1)/2]} [C_{l+1}^{2k+2} \sigma^2(s) c_n + C_l^{2k+1} \sigma^4(s) c_{n-1}] [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} \sigma^2(s) (a_{n-1} + b_{n-1})] [\sigma(s) \cdot \hat{p}]^{l+1-2k} \\
&\quad + \sum_{k=1}^{[(l+1)/2]} [C_{l+1}^{2k} \sigma^2(s) c_n + C_l^{2k-1} \sigma^4(s) c_{n-1}] [\sigma(s) \cdot \hat{p}]^{l+1-2k} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} \sigma^2(s) (a_{n-1} + b_{n-1})] [\sigma(s) \cdot \hat{p}]^{l+1-2k} \\
&\quad + \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k} \sigma^2(s) c_n + C_l^{2k-1} \sigma^4(s) c_{n-1}] [\sigma(s) \cdot \hat{p}]^{l+1-2k} - \sigma^2(s) c_n [\sigma(s) \cdot \hat{p}]^{l+1} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} \sigma^2(s) (a_{n-1} + b_{n-1}) + C_{l+1}^{2k} \sigma^2(s) c_n + C_l^{2k-1} \sigma^4(s) c_{n-1}] [\sigma(s) \cdot \hat{p}]^{l+1-2k} \\
&\quad - \sigma^2(s) c_n [\sigma(s) \cdot \hat{p}]^{l+1} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} c_{n+1} + C_l^{2k} c_n) \sigma^2(s) + C_l^{2k}] [\sigma(s) \cdot \hat{p}]^{l+1-2k} - c_n \sigma^2(s) [\sigma(s) \cdot \hat{p}]^{l+1} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s) + C_l^{2k}] [\sigma(s) \cdot \hat{p}]^{l+1-2k} - b_{n-1} \sigma^2(s) [\sigma(s) \cdot \hat{p}]^{l+1} \quad \square
\end{aligned}$$

**推论4.2.3.**  $i^{-n}\hat{p} \cdot [|\sigma(s) \times |]^n \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}, n \geq 0, l \geq 0$

$$= \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s) + C_l^{2k}][\sigma(s) \cdot \hat{p}]^{l+1-2k} - b_{n-1} \sigma^2(s) [\sigma(s) \cdot \hat{p}]^{l+1}$$

#### 4.2.2 $i^{-n}\sigma(s)[\sigma(s) \cdot \hat{p}]^l [|\times \sigma(s)]^n$ 的通项公式

**定理4.2.2.**  $i^{-n}\sigma(s)[\sigma(s) \cdot \hat{p}]^l [|\times \sigma(s)]^n, n \geq 1, l \geq 0$

$$= \sum_{k=0}^{[l/2]} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1}] [\sigma(s) \cdot \hat{p}] \sigma(s) - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s) b_{n-1}] \sigma(s) [\sigma(s) \cdot \hat{p}] + [C_{l+1}^{2k+2} \sigma^2(s) b_{n-1} + C_l^{2k+1} \sigma^4(s) b_{n-2}] \hat{p} \}$$

**证明:**  $i^{-n}\sigma(s)[\sigma(s) \cdot \hat{p}]^l [|\times \sigma(s)]^n, n \geq 1, l \geq 0$

$$= i^{-(n-1)} i^{-1} \sigma(s) [\sigma(s) \cdot \hat{p}]^l \times \sigma(s) [|\times \sigma(s)]^{n-1}$$

$$= i^{-(n-1)} \sum_{k=0}^{[l/2]} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \{ (C_l^{2k+1} + C_l^{2k+2}) \sigma(s) [\sigma(s) \cdot \hat{p}] + (C_l^{2k} - C_l^{2k+2}) [\sigma(s) \cdot \hat{p}] \sigma(s) - C_l^{2k+1} \sigma^2(s) \hat{p} \} [|\times \sigma(s)]^{n-1}$$

$$= \sum_{k=0}^{[l/2]} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \{ (C_l^{2k+1} + C_l^{2k+2}) \{ [\sigma(s) \cdot \hat{p}] \sigma(s) - i^{-n} \hat{p} [|\times \sigma(s)]^n \} + (C_l^{2k} - C_l^{2k+2}) [\sigma(s) \cdot \hat{p}] \sigma(s) - C_l^{2k+1} \sigma^2(s) i^{-(n-1)} \hat{p} [|\times \sigma(s)]^{n-1} \}$$

$$- C_l^{2k+1} \sigma^2(s) i^{-(n-1)} \hat{p} [|\times \sigma(s)]^{n-1} \}$$

$$= \sum_{k=0}^{[l/2]} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \{ -C_{l+1}^{2k+2} i^{-n} \hat{p} [|\times \sigma(s)]^n + C_{l+1}^{2k+1} [\sigma(s) \cdot \hat{p}] \sigma(s) - C_l^{2k+1} \sigma^2(s) i^{-(n-1)} \hat{p} [|\times \sigma(s)]^{n-1} \}$$

$$- C_l^{2k+1} \sigma^2(s) \{ a_n [\sigma(s) \cdot \hat{p}] \sigma(s) + b_n \sigma(s) [\sigma(s) \cdot \hat{p}] - c_n \sigma^2(s) \hat{p} \} + C_{l+1}^{2k+1} [\sigma(s) \cdot \hat{p}] \sigma(s) - C_l^{2k+1} \sigma^2(s) \{ a_{n-1} [\sigma(s) \cdot \hat{p}] \sigma(s) + b_{n-1} \sigma(s) [\sigma(s) \cdot \hat{p}] - c_{n-1} \sigma^2(s) \hat{p} \} \}$$

$$= \sum_{k=0}^{[l/2]} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1}] [\sigma(s) \cdot \hat{p}] \sigma(s) - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s) b_{n-1}] \sigma(s) [\sigma(s) \cdot \hat{p}] + [C_{l+1}^{2k+2} \sigma^2(s) c_n + C_l^{2k+1} \sigma^4(s) c_{n-1}] \hat{p} \}$$

$$= \sum_{k=0}^{[l/2]} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1}] [\sigma(s) \cdot \hat{p}] \sigma(s) - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s) b_{n-1}] \sigma(s) [\sigma(s) \cdot \hat{p}] + [C_{l+1}^{2k+2} \sigma^2(s) b_{n-1} + C_l^{2k+1} \sigma^4(s) b_{n-2}] \hat{p} \}$$

$$\square$$

**推论4.2.4.**  $i^{-n}\hat{p} \cdot |\sigma(s)[\sigma(s) \cdot \hat{p}]^l [|\times \sigma(s)]^n, n \geq 1, l \geq 0$

$$= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} \sigma^2(s) (a_{n-1} + b_{n-1})] [\sigma(s) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2} \sigma^2(s) c_n + C_l^{2k+1} \sigma^4(s) c_{n-1}] \} [\sigma(s) \cdot \hat{p}]^{l-2k-1}$$

$$= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} (1 + b_n)] [\sigma(s) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2} \sigma^2(s) c_n + C_l^{2k+1} \sigma^4(s) c_{n-1}] \} [\sigma(s) \cdot \hat{p}]^{l-2k-1}$$

$$= \sum_{k=0}^{[l/2]} \{ [C_l^{2k} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} b_n] [\sigma(s) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2} \sigma^2(s) b_{n-1} + C_l^{2k+1} \sigma^4(s) b_{n-2}] \} [\sigma(s) \cdot \hat{p}]^{l-2k-1}$$

**推论4.2.5.**  $i^{-n}\hat{p} \cdot |\sigma(s)[\sigma(s) \cdot \hat{p}]^l [|\times \sigma(s)]^n, n \geq 0, l \geq 0$

$$= \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s) + C_l^{2k}][\sigma(s) \cdot \hat{p}]^{l+1-2k} - b_{n-1} \sigma^2(s) [\sigma(s) \cdot \hat{p}]^{l+1}$$

**推论4.2.6.**  $i^{-n}\hat{p} \cdot [|\sigma(s) \times |]^n \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\} = i^{-n}\hat{p} \cdot |\sigma(s)[\sigma(s) \cdot \hat{p}]^l [|\times \sigma(s)]^n, n \geq 0, l \geq 0$

#### 4.2.3 $i^{-n}\sigma(s) \cdot [|\sigma(s) \times |]^n \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}, i^{-n}\{[\sigma(s)[\sigma(s) \cdot \hat{p}]^l [|\times \sigma(s)]^n \cdot \sigma(s)\}$ 的通项公式

**定理4.2.3.**  $i^{-n}\sigma(s) \cdot [|\sigma(s) \times |]^n \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\} = i^{-n}\{[\sigma(s)[\sigma(s) \cdot \hat{p}]^l [|\times \sigma(s)]^n \cdot \sigma(s)\}$

$$= \sum_{k=0}^{[l/2]} [C_l^{2k} \sigma^2(s) - C_{l+1}^{2k+2}] [\sigma(s) \cdot \hat{p}]^{l-2k}, n \geq 1, l \geq 0$$

**证明:**  $i^{-n}\sigma(s) \cdot [|\sigma(s) \times |]^n \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\} = i^{-n}\{[\sigma(s)[\sigma(s) \cdot \hat{p}]^l [|\times \sigma(s)]^n \cdot \sigma(s)\}, n \geq 1, l \geq 0$

$$= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1}] \sigma^2(s) [\sigma(s) \cdot \hat{p}] - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s) b_{n-1}] \sigma(s) \cdot [\sigma(s) \cdot \hat{p}] \sigma(s) + [C_{l+1}^{2k+2} \sigma^2(s) b_{n-1} + C_l^{2k+1} \sigma^4(s) b_{n-2}] [\sigma(s) \cdot \hat{p}] \} [\sigma(s) \cdot \hat{p}]^{l-2k-1}$$

$$= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1}] \sigma^2(s) [\sigma(s) \cdot \hat{p}] - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s) b_{n-1}] [\sigma^2(s) - 1] [\sigma(s) \cdot \hat{p}] \}$$

$$\begin{aligned}
& + [C_{l+1}^{2k+2}\sigma^2(s)b_{n-1} + C_l^{2k+1}\sigma^4(s)b_{n-2}][\sigma(s) \cdot \hat{p}][\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
& = \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s)a_{n-1} - C_{l+1}^{2k+2}b_n - C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma^2(s) \\
& + [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s)b_{n-1}] + [C_{l+1}^{2k+2}\sigma^2(s)b_{n-1} + C_l^{2k+1}\sigma^4(s)b_{n-2}]\}[\sigma(s) \cdot \hat{p}]^{l-2k} \\
& = \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} + C_{l+1}^{2k+2}k_n + C_l^{2k+1}\sigma^2(s)k_{n-1}]\sigma^2(s) + [C_{l+1}^{2k+2}b_{n+1} + C_l^{2k+1}\sigma^2(s)b_n]\}[\sigma(s) \cdot \hat{p}]^{l-2k} \\
& = \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}\sigma^{-2}(s)(b_{n+1} + 1) - C_l^{2k+1}(b_n + 1)]\sigma^2(s) + [C_{l+1}^{2k+2}b_{n+1} + C_l^{2k+1}\sigma^2(s)b_n]\}[\sigma(s) \cdot \hat{p}]^{l-2k} \\
& = \sum_{k=0}^{[l/2]} [C_l^{2k}\sigma^2(s) - C_{l+1}^{2k+2}][\sigma(s) \cdot \hat{p}]^{l-2k}
\end{aligned}$$

□

**推论4.2.7.**  $i^{-n}\sigma(s) \cdot |[\sigma(s) \times |^n\{\sigma(s) \cdot \hat{p}\}^l\sigma(s)\rangle| = i^{-n}\{\sigma(s)[\sigma(s) \cdot \hat{p}]^l[|\times\sigma(s)]^n \cdot \sigma(s) = \sigma(s) \cdot [\sigma(s) \cdot \hat{p}]^l\sigma(s)$

$$= \sum_{k=0}^{[l/2]} [C_l^{2k}\sigma^2(s) - C_{l+1}^{2k+2}][\sigma(s) \cdot \hat{p}]^{l-2k}, n \geq 0, l \geq 0$$

#### 4.2.4 $i^{-n}[\sigma(s) \times |^n\{\sigma(s) \cdot \hat{p}\}^l\sigma(s)\rangle \cdot \sigma(s), i^{-n}\sigma(s) \cdot \{[\sigma(s)[\sigma(s) \cdot \hat{p}]^l[|\times\sigma(s)]^n\}$ 的通项公式

**定理4.2.4.**  $i^{-n}[\sigma(s) \times |^n\{\sigma(s) \cdot \hat{p}\}^l\sigma(s)\rangle \cdot \sigma(s) = i^{-n}\sigma(s) \cdot \{[\sigma(s)[\sigma(s) \cdot \hat{p}]^l[|\times\sigma(s)]^n\}$

$$= \sum_{k=0}^{[l/2]} [C_l^{2k}\sigma^2(s) - C_{l+2}^{2k+2} + C_{l+1}^{2k+2}(a_n - b_n) + C_l^{2k+1}\sigma^2(s)(a_{n-1} - b_{n-1})][\sigma(s) \cdot \hat{p}]^{l-2k}, n \geq 1, l \geq 0$$

**证明:**  $i^{-n}[\sigma(s) \times |^n\{\sigma(s) \cdot \hat{p}\}^l\sigma(s)\rangle \cdot \sigma(s) = i^{-n}\sigma(s) \cdot \{[\sigma(s)[\sigma(s) \cdot \hat{p}]^l[|\times\sigma(s)]^n, n \geq 1, l \geq 0$

$$\begin{aligned}
& = \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s)a_{n-1}]\sigma(s)[\sigma(s) \cdot \hat{p}] \cdot \sigma(s) - [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s)b_{n-1}][\sigma(s) \cdot \hat{p}]\sigma^2(s) \\
& + [C_{l+1}^{2k+2}\sigma^2(s)b_{n-1} + C_l^{2k+1}\sigma^4(s)b_{n-2}][\sigma(s) \cdot \hat{p}][\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
& = \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s)a_{n-1}][\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}] - [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma^2(s)[\sigma(s) \cdot \hat{p}] \\
& + [C_{l+1}^{2k+2}\sigma^2(s)b_{n-1} + C_l^{2k+1}\sigma^4(s)b_{n-2}][\sigma(s) \cdot \hat{p}][\sigma(s) \cdot \hat{p}]^{l-2k-1} \\
& = \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s)a_{n-1} - C_{l+1}^{2k+2}b_n - C_l^{2k+1}\sigma^2(s)b_{n-1}]\sigma^2(s) \\
& + [-C_{l+1}^{2k+1} + C_{l+1}^{2k+2}a_n + C_l^{2k+1}\sigma^2(s)a_{n-1}] + [C_{l+1}^{2k+2}\sigma^2(s)b_{n-1} + C_l^{2k+1}\sigma^4(s)b_{n-2}]\}[\sigma(s) \cdot \hat{p}]^{l-2k} \\
& = \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} + C_{l+1}^{2k+2}k_n + C_l^{2k+1}\sigma^2(s)k_{n-1}]\sigma^2(s) + [C_{l+1}^{2k+2}b_{n+1} + C_l^{2k+1}\sigma^2(s)b_n] \\
& + [-C_{l+1}^{2k+1} + C_{l+1}^{2k+2}(a_n - b_n) + C_l^{2k+1}\sigma^2(s)(a_{n-1} - b_{n-1})]\}[\sigma(s) \cdot \hat{p}]^{l-2k} \\
& = \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}\sigma^{-2}(s)(b_{n+1} + 1) - C_l^{2k+1}(b_n + 1)]\sigma^2(s) + [C_{l+1}^{2k+2}b_{n+1} + C_l^{2k+1}\sigma^2(s)b_n] \\
& + [-C_{l+1}^{2k+1} + C_{l+1}^{2k+2}(a_n - b_n) + C_l^{2k+1}\sigma^2(s)(a_{n-1} - b_{n-1})]\}[\sigma(s) \cdot \hat{p}]^{l-2k} \\
& = \sum_{k=0}^{[l/2]} \{C_l^{2k}\sigma^2(s) - C_{l+1}^{2k+2} + [-C_{l+1}^{2k+1} + C_{l+1}^{2k+2}(a_n - b_n) + C_l^{2k+1}\sigma^2(s)(a_{n-1} - b_{n-1})]\}[\sigma(s) \cdot \hat{p}]^{l-2k} \\
& = \sum_{k=0}^{[l/2]} [C_l^{2k}\sigma^2(s) - C_{l+2}^{2k+2} + C_{l+1}^{2k+2}(a_n - b_n) + C_l^{2k+1}\sigma^2(s)(a_{n-1} - b_{n-1})][\sigma(s) \cdot \hat{p}]^{l-2k}
\end{aligned}$$

□

## 5 复杂自旋复合算符通项公式的另一种独立求法

### 5.1 $i^{-1}\hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n\sigma(s)\}\}$ 通项公式的另一种求法

#### 5.1.1 $i^{-1}\hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n\sigma(s)\}\}$ 的通项公式的试探和猜想

**定义5.1.1.**  $A(1, n) := i^{-1}\hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n\sigma(s)\}\}, A(1, 0) = \sigma(s) \cdot \hat{p}$

**推论5.1.1.**

$$\begin{cases} \sigma(s)[\times\hat{p}]^{2k-1} = (-1)^{k+1}\sigma(s) \times \hat{p} \\ \sigma(s)[\times\hat{p}]^{2k} = (-1)^{k+1}[\sigma(s) \cdot \hat{p}]\hat{p} - \sigma(s) \end{cases} \Rightarrow \begin{cases} i^{2k-1}i^{-1}\hat{p} \cdot \{\sigma(s) \times \{\sigma(s)[\times\hat{p}]^{2k-1}\}\} = \{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} \\ i^{2k}i^{-1}\hat{p} \cdot \{\sigma(s) \times \{\sigma(s)[\times\hat{p}]^{2k}\}\} = \sigma(s) \cdot \hat{p} \end{cases}$$

**推论5.1.2.**  $i^{-1}\hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n\sigma(s)\}\}$

$$= i^{-1}i^n\hat{p} \cdot \{\sigma(s) \times \{\sigma(s)[|\times\hat{p}]^n\}\} - \sum_{k=0}^{n-1} C_n^k i^{-1}\hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^k\sigma(s)\}\}[-\sigma(s) \cdot \hat{p}]^{n-k}$$

推论5.1.3.  $A(1, n) = \{[\sigma(s) \cdot \hat{p}]^{1+n\%2} - (n\%2)\sigma^2(s)\} - \sum_{k=0}^{n-1} C_n^k A(1, k)[- \sigma(s) \cdot \hat{p}]^{n-k}$ ,  $A(1, 0) = -\sigma(s) \cdot \hat{p}$

$$\text{性质5.1.1.} \quad \begin{cases} A(1, 0) = 1[\sigma(s) \cdot \hat{p}] \\ A(1, 1) = 2[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s) \\ A(1, 2) = 3[\sigma(s) \cdot \hat{p}]^3 - [2\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}] \\ A(1, 3) = 4[\sigma(s) \cdot \hat{p}]^4 - [3\sigma^2(s) - 4][\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s) \\ A(1, 4) = 5[\sigma(s) \cdot \hat{p}]^5 - [4\sigma^2(s) - 10][\sigma(s) \cdot \hat{p}]^3 - [4\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]^1 \\ A(1, 5) = 6[\sigma(s) \cdot \hat{p}]^6 - [5\sigma^2(s) - 20][\sigma(s) \cdot \hat{p}]^4 - [10\sigma^2(s) - 6][\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s) \end{cases}$$

证明:  $-A(1, 1)$

$$\begin{aligned} &= -\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} + C_1^0 A(1, 0)[\sigma(s) \cdot \hat{p}] \\ &= -2[\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \end{aligned} \quad \square$$

证明:  $-A(1, 2)$

$$\begin{aligned} &= -[\sigma(s) \cdot \hat{p}] - \sum_{k=0}^1 C_2^k A(1, k)[- \sigma(s) \cdot \hat{p}]^{2-k} \\ &= -[\sigma(s) \cdot \hat{p}] - C_2^0 A(1, 0)[\sigma(s) \cdot \hat{p}]^2 - \sum_{k=1}^1 C_2^k A(1, k)[- \sigma(s) \cdot \hat{p}]^{2-k} \\ &= -[\sigma(s) \cdot \hat{p}] + C_2^0 [\sigma(s) \cdot \hat{p}]^3 + C_2^1 A_1(1)[\sigma(s) \cdot \hat{p}] \\ &= -[\sigma(s) \cdot \hat{p}] + C_2^0 [\sigma(s) \cdot \hat{p}]^3 + C_2^1 \{-2[\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s)\}[\sigma(s) \cdot \hat{p}] \\ &= -3[\sigma(s) \cdot \hat{p}]^3 + [2\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}] \end{aligned} \quad \square$$

证明:  $-A(1, 3)$

$$\begin{aligned} &= -\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} + C_3^0 [-\sigma(s) \cdot \hat{p}][\sigma(s) \cdot \hat{p}]^3 \\ &\quad - C_3^1 \{-2[\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s)\}[\sigma(s) \cdot \hat{p}]^2 \\ &\quad + C_3^2 \{-3[\sigma(s) \cdot \hat{p}]^3 + [2\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]\}[\sigma(s) \cdot \hat{p}]^1 \\ &= -4[\sigma(s) \cdot \hat{p}]^4 + [3\sigma^2(s) - 4][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \end{aligned} \quad \square$$

证明:  $-A(1, 4)$

$$\begin{aligned} &= -[\sigma(s) \cdot \hat{p}] - C_4^0 [-\sigma(s) \cdot \hat{p}][\sigma(s) \cdot \hat{p}]^4 \\ &\quad + C_4^1 \{-2[\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s)\}[\sigma(s) \cdot \hat{p}]^3 \\ &\quad - C_4^2 \{-3[\sigma(s) \cdot \hat{p}]^3 + [2\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]\}[\sigma(s) \cdot \hat{p}]^2 \\ &\quad + C_4^3 \{-4[\sigma(s) \cdot \hat{p}]^4 + [3\sigma^2(s) - 4][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s)\}[\sigma(s) \cdot \hat{p}]^1 \\ &= -5[\sigma(s) \cdot \hat{p}]^5 + [4\sigma^2(s) - 10][\sigma(s) \cdot \hat{p}]^3 + [4\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]^1 \end{aligned} \quad \square$$

证明:  $-A(1, 5)$

$$\begin{aligned} &= -\{[\sigma(s) \cdot \hat{p}]^2 - \sigma^2(s)\} + C_5^0 [-\sigma(s) \cdot \hat{p}][\sigma(s) \cdot \hat{p}]^5 \\ &\quad - C_5^1 \{-2[\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s)\}[\sigma(s) \cdot \hat{p}]^4 \\ &\quad + C_5^2 \{-3[\sigma(s) \cdot \hat{p}]^3 + [2\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]\}[\sigma(s) \cdot \hat{p}]^3 \\ &\quad - C_5^3 \{-4[\sigma(s) \cdot \hat{p}]^4 + [3\sigma^2(s) - 4][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s)\}[\sigma(s) \cdot \hat{p}]^2 \\ &\quad + C_5^4 \{-5[\sigma(s) \cdot \hat{p}]^5 + [4\sigma^2(s) - 10][\sigma(s) \cdot \hat{p}]^3 + [4\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}]\}[\sigma(s) \cdot \hat{p}]^1 \\ &= -6[\sigma(s) \cdot \hat{p}]^6 + [5\sigma^2(s) - 20][\sigma(s) \cdot \hat{p}]^4 + [10\sigma^2(s) - 6][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \end{aligned} \quad \square$$

重新整理得到:

$$\text{性质5.1.2.} \quad \begin{cases} A(1, 0) = C_1^1[\sigma(s) \cdot \hat{p}] \\ A(1, 1) = C_2^1[\sigma(s) \cdot \hat{p}]^2 - C_1^1\sigma^2(s) \\ A(1, 2) = C_3^1[\sigma(s) \cdot \hat{p}]^3 - [C_2^1\sigma^2(s) - C_3^3][\sigma(s) \cdot \hat{p}] \\ A(1, 3) = C_4^1[\sigma(s) \cdot \hat{p}]^4 - [C_3^1\sigma^2(s) - C_4^3][\sigma(s) \cdot \hat{p}]^2 - C_3^3\sigma^2(s) \\ A(1, 4) = C_5^1[\sigma(s) \cdot \hat{p}]^5 - [C_4^1\sigma^2(s) - C_5^3][\sigma(s) \cdot \hat{p}]^3 - [C_4^3\sigma^2(s) - C_5^5][\sigma(s) \cdot \hat{p}]^1 \\ A(1, 5) = C_6^1[\sigma(s) \cdot \hat{p}]^6 - [C_5^1\sigma^2(s) - C_6^3][\sigma(s) \cdot \hat{p}]^4 - [C_5^3\sigma^2(s) - C_6^5][\sigma(s) \cdot \hat{p}]^2 - C_5^5\sigma^2(s) \end{cases}$$

$$\text{猜想5.1.1.} \quad A(1, n) = C_{n+1}^1[\sigma(s) \cdot \hat{p}]^{n+1} - [C_n^1\sigma^2(s) - C_{n+1}^3][\sigma(s) \cdot \hat{p}]^{n-1} - [C_n^3\sigma^2(s) - C_{n+1}^5][\sigma(s) \cdot \hat{p}]^{n-3} + \dots$$

$$\text{猜想5.1.2.} \quad i^{-1}\hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\}\} = \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k+1} - C_n^{2k-1}\sigma^2(s)][\sigma(s) \cdot \hat{p}]^{n+1-2k}, n \geq 0$$

下面用数学归纳法严格证明以上猜想。

**5.1.2**  $i^{-1}\hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\}\}$  的通项公式的有关引理及其证明

$$\text{推论5.1.4.} \quad A(1, n) = \{[\sigma(s) \cdot \hat{p}]^{1+n\%2} - (n\%2)\sigma^2(s)\} - \sum_{l=0}^{n-1} C_n^l A(1, l)[- \sigma(s) \cdot \hat{p}]^{n-l}, A(1, 0) = -\sigma(s) \cdot \hat{p}$$

$$\text{引理5.1.1.} \quad \sum_{l=0}^n \sum_{k=0}^{[(l+1)/2]} A(k, l) = \sum_{k=0}^{[(n+1)/2]} \sum_{l=2k-1|0}^n A(k, l)$$

$$\text{引理5.1.2.} \quad \sum_{l=2k-1|0}^n (-1)^{n-l} C_{n+1}^l C_l^{2k-1} = C_{n+1}^{2k-1}$$

$$\text{引理5.1.3.} \quad \sum_{l=r}^n (-1)^{n-l} C_{n+1}^l C_l^r = C_{n+1}^r$$

$$\text{证明:} \quad \sum_{l=r}^n (-1)^{n-l} C_{n+1}^l C_l^r = C_{n+1}^r$$

$$\Leftrightarrow \sum_{l=0}^{n-r} (-1)^l C_{n+1}^{n-l} C_{n-l}^r = C_{n+1}^r$$

$$\Leftrightarrow \sum_{l=0}^{n-r} (-1)^l \frac{1}{(l+1)!(n-l-r)!} = \frac{1}{(n+1-r)!}$$

$$\Leftrightarrow \sum_{l=0}^{n-r} (-1)^l \frac{(n+1-r)!}{(l+1)!(n-l-r)!} = 1$$

$$\Leftrightarrow \sum_{l=0}^{n-r} (-1)^l C_{n+1-r}^{l+1} = 1$$

$$\Leftrightarrow \sum_{l=0}^{n+1-r} (-1)^l C_{n+1-r}^l = 0$$

$$\Leftrightarrow [1 + (-1)]^{n+1-r} = 0 \quad \square$$

$$\text{推论5.1.5.} \quad \sum_{l=2k-1}^n (-1)^{n-l} C_{n+1}^l C_l^{2k-1} = C_{n+1}^{2k-1}, k \geq 0$$

$$\text{引理5.1.4.} \quad \sum_{l=r}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{r+1} = C_{n+2}^{r+1} - \delta_{nr}$$

$$\text{证明:} \quad \sum_{l=r}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{r+1} = C_{n+2}^{r+1}, r < n$$

$$\Leftrightarrow \sum_{l=0}^{n-r} (-1)^l C_{n+1}^{n-l} C_{n+1-l}^{r+1} = C_{n+2}^{r+1}$$

$$\Leftrightarrow \sum_{l=0}^{n-r} (-1)^l \frac{n+1-l}{(l+1)!(n-l-r)!} = \frac{n+2}{(n+1-r)!}$$

$$\Leftrightarrow \sum_{l=0}^{n-r} (-1)^l \frac{(n+1-l)(n+1-r)!}{(l+1)!(n-l-r)!} = n+2$$

$$\Leftrightarrow \sum_{l=0}^{n-r} (-1)^l (n+1-l) C_{n+1-r}^{l+1} = n+2$$

$$\begin{aligned}
&\Leftrightarrow \sum_{l=0}^{n-r} (-1)^{l+1} (l+1) C_{n+1-r}^{l+1} = -(n+2) \sum_{l=0}^{n+1-r} (-1)^l C_{n+1-r}^l \\
&\Leftrightarrow \sum_{l=0}^{n-r} (-1)^{l+1} (l+1) C_{n+1-r}^{l+1} = 0 \\
&\Leftrightarrow \sum_{l=1}^{n+1-r} (-1)^l l C_{n+1-r}^l = 0 \\
&\Leftrightarrow (n+1-r) \sum_{l=1}^{n+1-r} (-1)^l C_{n-r}^{l-1} = 0 \\
&\Leftrightarrow \sum_{l=0}^{n-r} (-1)^l C_{n-r}^l = 0 \\
&\Leftrightarrow [1 + (-1)]^{n-r} = 0
\end{aligned}$$

□

$$\text{证明: } \sum_{l=r}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{r+1} = C_{n+1}^n, r = n \Leftrightarrow \sum_{l=r}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{r+1} = C_{n+2}^{r+1} - 1, r = n$$

□

$$\text{推论5.1.6. } \begin{cases} \sum_{l=2k-1}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{2k+1} = C_{n+2}^{2k+1}, 2k-1 < n-1 \\ \sum_{l=2k-1}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{2k+1} = C_{n+2}^{2k+1} - 1, 2k-1 = n-1 \end{cases}$$

引理5.1.5.

$$\begin{aligned}
&\{[\sigma(s) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s)\} - \sum_{k=0}^{[(n+1)/2]} [\sigma(s) \cdot \hat{p}]^{n+2-2k} \sum_{l=2k-1|0}^n (-1)^{n-l} C_{n+1}^l [C_l^{2k-1} \sigma^2(s) - C_{l+1}^{2k+1}] \\
&= \sum_{k=0}^{[(n+2)/2]} [C_{n+2}^{2k+1} - C_{n+1}^{2k-1} \sigma^2(s)] [\sigma(s) \cdot \hat{p}]^{n+2-2k}
\end{aligned}$$

5.1.3  $i^{-1}\hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\}\}$  的通项公式的数学归纳法证明

$$\text{定理5.1.1. } A(1, n) = \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k+1} - C_n^{2k-1} \sigma^2(s)] [\sigma(s) \cdot \hat{p}]^{n+1-2k}, n \geq 0$$

$$\begin{aligned}
\text{证明: } &A(1, n+1) = \{[\sigma(s) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s)\} - \sum_{l=0}^n C_{n+1}^l A(1, l) [-\sigma(s) \cdot \hat{p}]^{n+1-l} \\
&= \{[\sigma(s) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s)\} - \sum_{l=0}^n C_{n+1}^l \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k+1} - C_l^{2k-1} \sigma^2(s)] [\sigma(s) \cdot \hat{p}]^{l+1-2k} [-\sigma(s) \cdot \hat{p}]^{n+1-l} \\
&= \{[\sigma(s) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s)\} - \sum_{l=0}^n \sum_{k=0}^{[(l+1)/2]} (-1)^{n+1-l} C_{n+1}^l [C_{l+1}^{2k+1} - C_l^{2k-1} \sigma^2(s)] [\sigma(s) \cdot \hat{p}]^{n+2-2k} \\
&= \{[\sigma(s) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s)\} - \sum_{k=0}^{[(n+1)/2]} \sum_{l=2k-1|0}^n (-1)^{n+1-l} C_{n+1}^l [C_{l+1}^{2k+1} - C_l^{2k-1} \sigma^2(s)] [\sigma(s) \cdot \hat{p}]^{n+2-2k} \\
&= \{[\sigma(s) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s)\} - \sum_{k=0}^{[(n+1)/2]} [\sigma(s) \cdot \hat{p}]^{n+2-2k} \sum_{l=2k-1|0}^n (-1)^{n-l} C_{n+1}^l [C_l^{2k-1} \sigma^2(s) - C_{l+1}^{2k+1}] \\
&= \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k+1} - C_n^{2k-1} \sigma^2(s)] [\sigma(s) \cdot \hat{p}]^{n+1-2k}
\end{aligned}$$

□

5.2  $i^{-n}\hat{p} \cdot \{[\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}]\}$  和  $i^{-n}\hat{p} \cdot \{[\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}]\}$  通项公式的另一种求法5.2.1  $i^{-n}\hat{p} \cdot \{[\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}]\}$  的通项公式

$$\text{定义5.2.1. } A_L(n, l) := i^{-n}\hat{p} \cdot \{[\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}]\}, A_L(n, 0) = \sigma(s) \cdot \hat{p}$$

推论5.2.1.

$$i^{-n}\hat{p} \cdot \{[\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}]\} = i^{-n} i^l \hat{p} \cdot \{[\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}]\} \times \hat{p}^l - \sum_{k=0}^{l-1} C_l^k i^{-n}\hat{p} \cdot \{[\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^k \sigma(s)\}]\} [-\sigma(s) \cdot \hat{p}]^{l-k}$$



推论5.2.2.

$$\begin{cases} i^{-n}\hat{p} \cdot |[\sigma(s) \times |]^n i^{2k-1}\sigma(s)[\times\hat{p}]^{2k-1} \\ = -i^{-(n+1)}\hat{p} \cdot |[\sigma(s) \times |]^n \sigma(s) \times \hat{p} = -(a_{n+1} + b_{n+1})[\sigma(s) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s) \\ i^{-n}\hat{p} \cdot |[\sigma(s) \times |]^n i^{2k}\sigma(s)[\times\hat{p}]^{2k} \\ = -i^{-n}\hat{p} \cdot |[\sigma(s) \times |]^n \{[\sigma(s) \cdot \hat{p}]\hat{p} - \sigma(s)\} = -(a_n + b_n)[\sigma(s) \cdot \hat{p}]^3 + [c_n\sigma^2(s) + 1][\sigma(s) \cdot \hat{p}] \end{cases}$$

推论5.2.3.

$$i^{l-n}[\sigma(s) \times |]^n \sigma(s)[\times\hat{p}]^l = \begin{cases} -a_{n+1}\sigma(s)[\sigma(s) \cdot \hat{p}] - b_{n+1}[\sigma(s) \cdot \hat{p}]\sigma(s) + c_{n+1}\sigma^2(s)\hat{p}, l = 2k - 1 \\ -a_n\sigma(s)[\sigma(s) \cdot \hat{p}]^2 - b_n[\sigma(s) \cdot \hat{p}]\sigma(s)[\sigma(s) \cdot \hat{p}] + c_n\sigma^2(s)\hat{p}[\sigma(s) \cdot \hat{p}] + \sigma(s), l = 2k \\ i^{-n}[\sigma(s) \times |]^n \hat{p} = a_n\sigma(s)[\sigma(s) \cdot \hat{p}] + b_n[\sigma(s) \cdot \hat{p}]\sigma(s) - c_n\sigma^2(s)\hat{p} \end{cases}$$

推论5.2.4.

$$i^{l-n}\hat{p} \cdot |[\sigma(s) \times |]^n \sigma(s)[\times\hat{p}]^l = \begin{cases} -(a_{n+1} + b_{n+1})[\sigma(s) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s), l = 2k - 1 \\ -(a_n + b_n)[\sigma(s) \cdot \hat{p}]^3 + [1 + c_n\sigma^2(s)][\sigma(s) \cdot \hat{p}], l = 2k \end{cases}$$

$$\text{定理5.2.1.} \begin{cases} A_L(n, l) = i^{l-n}\hat{p} \cdot |[\sigma(s) \times |]^n \sigma(s)[\times\hat{p}]^l - \sum_{k=0}^{l-1} C_l^k A_L(n, k)[-\sigma(s) \cdot \hat{p}]^{l-k}, A_L(n, 0) = \sigma(s) \cdot \hat{p} \\ i^{l-n}\hat{p} \cdot |[\sigma(s) \times |]^n \sigma(s)[\times\hat{p}]^l = \begin{cases} k_{n+1}[\sigma(s) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s), l = 2k - 1; k_n = -(a_n + b_n) \\ k_n[\sigma(s) \cdot \hat{p}]^3 + [1 + c_n\sigma^2(s)][\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \end{cases}$$

推论5.2.5.

$$\begin{cases} A_L(n, 0) = C_0^0\sigma(s) \cdot \hat{p} \\ A_L(n, 1) = (C_1^1 k_{n+1} + C_1^0)[\sigma(s) \cdot \hat{p}]^2 + C_1^1 c_{n+1}\sigma^2(s), k_{n+1} = -(a_{n+1} + b_{n+1}) \\ A_L(n, 2) = [C_2^1 k_{n+1} + C_2^2 k_n + C_2^0][\sigma(s) \cdot \hat{p}]^3 + [(C_2^1 c_{n+1} + C_2^2 c_n)\sigma^2(s) + C_2^2][\sigma(s) \cdot \hat{p}] \\ A_L(n, 3) = [C_3^1 k_{n+1} + C_3^2 k_n + C_3^0][\sigma(s) \cdot \hat{p}]^4 + [C_3^3 k_{n+1} + (C_3^1 c_{n+1} + C_3^2 c_n)\sigma^2(s) + C_3^2][\sigma(s) \cdot \hat{p}]^2 \\ + C_3^3 c_{n+1}\sigma^2(s) \\ A_L(n, 4) = [C_4^1 k_{n+1} + C_4^2 k_n + C_4^0][\sigma(s) \cdot \hat{p}]^5 + [C_4^3 k_{n+1} + C_4^4 k_n + (C_4^1 c_{n+1} + C_4^2 c_n)\sigma^2(s) + C_4^2][\sigma(s) \cdot \hat{p}]^3 \\ + [(C_4^3 c_{n+1} + C_4^4 c_n)\sigma^2(s) + C_4^4][\sigma(s) \cdot \hat{p}]^1 \end{cases}$$

$$\text{证明: } A_L(n, 1) = k_{n+1}[\sigma(s) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s) + C_1^0 A_L(n, 0)[\sigma(s) \cdot \hat{p}]^1$$

$$= (C_1^1 k_{n+1} + C_1^0)[\sigma(s) \cdot \hat{p}]^2 + C_1^1 c_{n+1}\sigma^2(s) \quad \square$$

$$\text{证明: } A_L(n, 2) = k_n[\sigma(s) \cdot \hat{p}]^3 + [1 + c_n\sigma^2(s)][\sigma(s) \cdot \hat{p}] - C_2^0 A_L(n, 0)[\sigma(s) \cdot \hat{p}]^2 + C_2^1 A_L(n, 1)[\sigma(s) \cdot \hat{p}]$$

$$= [C_2^1 k_{n+1} + C_2^2 k_n + C_2^0][\sigma(s) \cdot \hat{p}]^3 + [(C_2^1 c_{n+1} + C_2^2 c_n)\sigma^2(s) + C_2^2][\sigma(s) \cdot \hat{p}] \quad \square$$

$$\text{证明: } A_L(n, 3) = k_{n+1}[\sigma(s) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s)$$

$$+ C_3^0 A_L(n, 0)[\sigma(s) \cdot \hat{p}]^3 - C_3^1 A_L(n, 1)[\sigma(s) \cdot \hat{p}]^2 + C_3^2 A_L(n, 2)[\sigma(s) \cdot \hat{p}]^1$$

$$= k_{n+1}[\sigma(s) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s)$$

$$+ C_3^0[\sigma(s) \cdot \hat{p}]^4 - C_3^1\{(C_1^1 k_{n+1} + C_1^0)[\sigma(s) \cdot \hat{p}]^2 + C_1^1 c_{n+1}\sigma^2(s)\}[\sigma(s) \cdot \hat{p}]^2$$

$$+ C_3^2\{[C_2^1 k_{n+1} + C_2^2 k_n + C_2^0][\sigma(s) \cdot \hat{p}]^3 + [(C_2^1 c_{n+1} + C_2^2 c_n)\sigma^2(s) + C_2^2][\sigma(s) \cdot \hat{p}]\}[\sigma(s) \cdot \hat{p}]^1$$

$$= [C_3^1 k_{n+1} + C_3^2 k_n + C_3^0][\sigma(s) \cdot \hat{p}]^4 + [C_3^3 k_{n+1} + (C_3^1 c_{n+1} + C_3^2 c_n)\sigma^2(s) + C_3^2][\sigma(s) \cdot \hat{p}]^2 + C_3^3 c_{n+1}\sigma^2(s) \quad \square$$

$$\text{证明: } A_L(n, 4) = (k_n - 1)[\sigma(s) \cdot \hat{p}]^3 + [1 + c_n\sigma^2(s)][\sigma(s) \cdot \hat{p}]$$

$$- C_4^0 A_L(n, 0)[\sigma(s) \cdot \hat{p}]^4 + C_4^1 A_L(n, 1)[\sigma(s) \cdot \hat{p}]^3 - C_4^2 A_L(n, 2)[\sigma(s) \cdot \hat{p}]^2 + C_4^3 A_L(n, 3)[\sigma(s) \cdot \hat{p}]^1$$

$$= [C_4^1 k_{n+1} + C_4^2 k_n + C_4^0][\sigma(s) \cdot \hat{p}]^5 + [C_4^3 k_{n+1} + C_4^4 k_n + (C_4^1 c_{n+1} + C_4^2 c_n)\sigma^2(s) + C_4^4][\sigma(s) \cdot \hat{p}]^3$$

$$+ [(C_4^3 c_{n+1} + C_4^4 c_n)\sigma^2(s) + C_4^4][\sigma(s) \cdot \hat{p}]^1 \quad \square$$

定理5.2.2.

$$\begin{cases} A_L(n, l) = i^{-n} \hat{p} \cdot [|\sigma(s) \times |]^n [\sigma(s) \cdot \hat{p}]^l \sigma(s) \\ = \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} c_{n+1} + C_l^{2k} c_n) \sigma^2(s) + C_l^{2k}] [\sigma(s) \cdot \hat{p}]^{l+1-2k} - c_n \sigma^2(s) [\sigma(s) \cdot \hat{p}]^{l+1} \\ k_n = -(a_n + b_n) = \frac{(s+1)^{n+1} - (-s)^{n+1} - (2s+1)}{s(2s+1)(s+1)}, c_n = -\frac{(s+1)^{n-1} - (-s)^{n-1}}{2s+1}, n \geq 0, l \geq 0 \end{cases}$$

推论5.2.6.

$$\begin{cases} A_L(n, l) = i^{-n} \hat{p} \cdot [|\sigma(s) \times |]^n [\sigma(s) \cdot \hat{p}]^l \sigma(s) \\ = \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s) + C_l^{2k}] [\sigma(s) \cdot \hat{p}]^{l+1-2k} - b_{n-1} \sigma^2(s) [\sigma(s) \cdot \hat{p}]^{l+1} \\ k_n = -(a_n + b_n) = \frac{(s+1)^{n+1} - (-s)^{n+1} - (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0, l \geq 0 \end{cases}$$

可以用数学归纳法严格证明以上定理, 有空再补上.

### 5.2.2 $i^{-n} \hat{p} \cdot |\sigma(s) [\sigma(s) \cdot \hat{p}]^l | \times \sigma(s)^n$ 的通项公式

定义5.2.2.  $A_R(n, l) := i^{-n} \hat{p} \cdot |\sigma(s) [\sigma(s) \cdot \hat{p}]^l | \times \sigma(s)^n, A_R(n, 0) = \sigma(s) \cdot \hat{p}$

性质5.2.1.  $\sigma(s) [\sigma(s) \cdot \hat{p}]^l = i^l [\hat{p} \times |]^l \sigma(s) - \sum_{k=0}^{l-1} C_n^k [-\sigma(s) \cdot \hat{p}]^{l-k} \sigma(s) [\sigma(s) \cdot \hat{p}]^k$

推论5.2.7.

$$i^{-n} \hat{p} \cdot |\sigma(s) [\sigma(s) \cdot \hat{p}]^l | \times \sigma(s)^n = i^{l-n} \hat{p} \cdot [|\hat{p} \times |]^l \sigma(s) | \times \sigma(s)^n - \sum_{k=0}^{l-1} C_n^k [-\sigma(s) \cdot \hat{p}]^{l-k} i^{-n} \hat{p} \cdot |\sigma(s) [\sigma(s) \cdot \hat{p}]^k | \times \sigma(s)^n$$

推论5.2.8.

$$i^{l-n} [|\hat{p} \times |]^l \sigma(s) | \times \sigma(s)^n = \begin{cases} -a_{n+1} [\sigma(s) \cdot \hat{p}] \sigma(s) - b_{n+1} \sigma(s) [\sigma(s) \cdot \hat{p}] + c_{n+1} \sigma^2(s) \hat{p}, l = 2k - 1 \\ -a_n [\sigma(s) \cdot \hat{p}]^2 \sigma(s) - b_n [\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}] + c_n \sigma^2(s) \hat{p} [\sigma(s) \cdot \hat{p}] + \sigma(s), l = 2k \\ i^{-n} \hat{p} | \times \sigma(s)^n = a_n [\sigma(s) \cdot \hat{p}] \sigma(s) + b_n \sigma(s) [\sigma(s) \cdot \hat{p}] - c_n \sigma^2(s) \hat{p} \end{cases}$$

推论5.2.9.

$$i^{l-n} \hat{p} \cdot [|\hat{p} \times |]^l \sigma(s) | \times \sigma(s)^n = \begin{cases} -(a_{n+1} + b_{n+1}) [\sigma(s) \cdot \hat{p}]^2 + c_{n+1} \sigma^2(s), l = 2k - 1 \\ -(a_n + b_n) [\sigma(s) \cdot \hat{p}]^3 + [1 + c_n \sigma^2(s)] [\sigma(s) \cdot \hat{p}], l = 2k \end{cases}$$

$$\text{定理5.2.3. } \begin{cases} A_R(n, l) = i^{l-n} \hat{p} \cdot [|\hat{p} \times |]^l \sigma(s) | \times \sigma(s)^n - \sum_{k=0}^{l-1} C_n^k [-\sigma(s) \cdot \hat{p}]^{l-k} A_R(n, k), A_R(n, 0) = \sigma(s) \cdot \hat{p} \\ i^{l-n} \hat{p} \cdot [|\hat{p} \times |]^l \sigma(s) | \times \sigma(s)^n = \begin{cases} k_{n+1} [\sigma(s) \cdot \hat{p}]^2 + c_{n+1} \sigma^2(s), l = 2k - 1; k_n = -(a_n + b_n) \\ k_n [\sigma(s) \cdot \hat{p}]^3 + [1 + c_n \sigma^2(s)] [\sigma(s) \cdot \hat{p}], l = 2k \end{cases} \end{cases}$$

与上节的离散方程和初始条件完全等价, 故有以下相同的解.

定理5.2.4.

$$\begin{cases} A_R(n, l) = i^{-n} \hat{p} \cdot |\sigma(s) [\sigma(s) \cdot \hat{p}]^l | \times \sigma(s)^n \\ = \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} c_{n+1} + C_l^{2k} c_n) \sigma^2(s) + C_l^{2k}] [\sigma(s) \cdot \hat{p}]^{l+1-2k} - c_n \sigma^2(s) [\sigma(s) \cdot \hat{p}]^{l+1} \\ k_n = -(a_n + b_n) = \frac{(s+1)^{n+1} - (-s)^{n+1} - (2s+1)}{s(2s+1)(s+1)}, c_n = -\frac{(s+1)^{n-1} - (-s)^{n-1}}{2s+1}, n \geq 0, l \geq 0 \end{cases}$$

推论5.2.10.

$$\begin{cases} A_R(n, l) = i^{-n} \hat{p} \cdot |\sigma(s) [\sigma(s) \cdot \hat{p}]^l | \times \sigma(s)^n \\ = \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s) + C_l^{2k}] [\sigma(s) \cdot \hat{p}]^{l+1-2k} - b_{n-1} \sigma^2(s) [\sigma(s) \cdot \hat{p}]^{l+1} \\ k_n = -(a_n + b_n) = \frac{(s+1)^{n+1} - (-s)^{n+1} - (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0, l \geq 0 \end{cases}$$

推论5.2.11.  $A_L(n, l) = A_R(n, l), \hat{p} \cdot [|\sigma(s) \times |]^n [\sigma(s) \cdot \hat{p}]^l \sigma(s) = \hat{p} \cdot |\sigma(s) [\sigma(s) \cdot \hat{p}]^l | \times \sigma(s)^n, n \geq 0$

## 6 各种通项公式汇总小结

### 6.1 基本通项公式小结

定理6.1.1.

$$\begin{cases} i^n \sigma(s) [|\times \hat{p}]^n = \sum_{k=0}^n C_n^k [\sigma(s) \cdot \hat{p}]^k \sigma(s) [-\sigma(s) \cdot \hat{p}]^{n-k}, n \geq 0 = \begin{cases} i\sigma(s) \times \hat{p}, n = 2k - 1, k \geq 1 \\ \sigma(s) - [\sigma(s) \cdot \hat{p}]\hat{p}, n = 2k \end{cases} \\ i^n [\hat{p} \times |]^n \sigma(s) = \sum_{k=0}^n C_n^k [-\sigma(s) \cdot \hat{p}]^{n-k} \sigma(s) [\sigma(s) \cdot \hat{p}]^k, n \geq 0 = \begin{cases} i\hat{p} \times \sigma(s), n = 2k - 1, k \geq 1 \\ \sigma(s) - [\sigma(s) \cdot \hat{p}]\hat{p}, n = 2k \end{cases} \end{cases}$$

定理6.1.2.

$$\begin{cases} i^{-n} [\sigma(s) \times |]^n \hat{p} = a_n \sigma(s) [\sigma(s) \cdot \hat{p}] + b_n [\sigma(s) \cdot \hat{p}] \sigma(s) - b_{n-1} \sigma^2(s) \hat{p}, n \geq 0 \\ i^{-n} \hat{p} [|\times \sigma(s)]^n = a_n [\sigma(s) \cdot \hat{p}] \sigma(s) + b_n \sigma(s) [\sigma(s) \cdot \hat{p}] - b_{n-1} \sigma^2(s) \hat{p}, n \geq 0 \\ a_n = \frac{[(s+1)^{n+2} - (-s)^{n+2}] - 2[(s+1)^{n+1} - (-s)^{n+1}] + (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1} \end{cases}$$

定理6.1.3.

$$\begin{cases} [\sigma(s) \cdot \hat{p}]^n \sigma(s) = i^n \sigma(s) [|\times \hat{p}]^n - \sum_{k=0}^{n-1} C_n^k [\sigma(s) \cdot \hat{p}]^k \sigma(s) [-\sigma(s) \cdot \hat{p}]^{n-k}, n \geq 0 \\ \sigma(s) [\sigma(s) \cdot \hat{p}]^n = i^n [\hat{p} \times |]^n \sigma(s) - \sum_{k=0}^{n-1} C_n^k [-\sigma(s) \cdot \hat{p}]^{n-k} \sigma(s) [\sigma(s) \cdot \hat{p}]^k, n \geq 0 \end{cases}$$

### 6.2 基本叉乘型通项公式小结

定理6.2.1.

$$\begin{cases} i^{-1} \sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\}, n \geq 0 \\ = \sum_{k=0}^{[n/2]} \{(C_n^{2k+1} + C_n^{2k+2})[\sigma(s) \cdot \hat{p}] \sigma(s) + (C_n^{2k} - C_n^{2k+2})\sigma(s) [\sigma(s) \cdot \hat{p}] - C_n^{2k+1} \sigma^2(s) \hat{p}\} [\sigma(s) \cdot \hat{p}]^{n-2k-1} \\ i^{-1} \{[\sigma(s) \cdot \hat{p}]^n \sigma(s)\} \times \sigma(s), n \geq 0 \\ = \sum_{k=0}^{[n/2]} [\sigma(s) \cdot \hat{p}]^{n-2k-1} \{(C_n^{2k+1} + C_n^{2k+2})\sigma(s) [\sigma(s) \cdot \hat{p}] + (C_n^{2k} - C_n^{2k+2})[\sigma(s) \cdot \hat{p}] \sigma(s) - C_n^{2k+1} \sigma^2(s) \hat{p}\} \end{cases}$$

### 6.3 基本叉乘型扩展通项公式小结

定理6.3.1.

$$\begin{cases} i^{-n} [\sigma(s) \times |]^n \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\}, n \geq 1, l \geq 0 \\ = \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1}] \sigma(s) [\sigma(s) \cdot \hat{p}] - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s) b_{n-1}] [\sigma(s) \cdot \hat{p}] \sigma(s) \\ + [C_{l+1}^{2k+2} \sigma^2(s) b_{n-1} + C_l^{2k+1} \sigma^4(s) b_{n-2}] \hat{p}\} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \\ i^{-n} \sigma(s) [\sigma(s) \cdot \hat{p}]^l [|\times \sigma(s)]^n, n \geq 1, l \geq 0 \\ = \sum_{k=0}^{[l/2]} [\sigma(s) \cdot \hat{p}]^{l-2k-1} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s) a_{n-1}] [\sigma(s) \cdot \hat{p}] \sigma(s) \\ - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s) b_{n-1}] \sigma(s) [\sigma(s) \cdot \hat{p}] + [C_{l+1}^{2k+2} \sigma^2(s) b_{n-1} + C_l^{2k+1} \sigma^4(s) b_{n-2}] \hat{p}\} \end{cases}$$

### 6.4 基本标积型通项公式小结

定理6.4.1.  $\sigma(s) \cdot [\sigma(s) \cdot \hat{p}]^n \sigma(s) = \sigma(s) [\sigma(s) \cdot \hat{p}]^n \cdot \sigma(s) = \sum_{k=0}^{[n/2]} [C_n^{2k} \sigma^2(s) - C_{n+1}^{2k+2}] [\sigma(s) \cdot \hat{p}]^{n-2k}, n \geq 0$

### 6.5 基本标积型扩展通项公式小结

定理6.5.1.  $i^n \sigma(s) [|\times \hat{p}]^n \cdot \hat{p} = i^n \hat{p} \cdot [|\hat{p} \times |]^n \sigma(s) = 0, n \geq 1$

定理6.5.2.

$$\begin{cases} i^{-n} \hat{p} \cdot \{[\sigma(s) \times |]^n \hat{p}\} = i^{-n} \{\hat{p} [|\times \sigma(s)]^n\} \cdot \hat{p} = -k_n [\sigma(s) \cdot \hat{p}]^2 + b_{n-1} \sigma^2(s) \\ k_n = -(a_n + b_n) = \frac{(s+1)^{n+1} - (-s)^{n+1} - (2s+1)}{s(2s+1)(s+1)}, b_{n-1} = -\frac{(s+1)^{n-1} - (-s)^{n-1}}{2s+1}, n \geq 0 \end{cases}$$

定理6.5.3.

$$\begin{cases} A_L(n, l) = i^{-n} \hat{p} \cdot [|\sigma(s) \times |]^n [\sigma(s) \cdot \hat{p}]^l \sigma(s) = i^{-n} \hat{p} \cdot |\sigma(s) [\sigma(s) \cdot \hat{p}]^l | \times \sigma(s)^n = A_R(n, l) \\ = \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s) + C_l^{2k} [\sigma(s) \cdot \hat{p}]^{l+1-2k} - b_{n-1} \sigma^2(s) [\sigma(s) \cdot \hat{p}]^{l+1} \\ k_n = -(a_n + b_n) = \frac{(s+1)^{n+1} - (-s)^{n+1} - (2s+1)}{s(2s+1)(s+1)}, b_n = -\frac{(s+1)^n - (-s)^n}{2s+1}, n \geq 0, l \geq 0 \end{cases}$$

定理6.5.4.

$$\begin{cases} i^{-n} \sigma(s) \cdot [|\sigma(s) \times |]^n \{[\sigma(s) \cdot \hat{p}]^l \sigma(s)\} = i^{-n} \{[\sigma(s) [\sigma(s) \cdot \hat{p}]^l | \times \sigma(s)^n \cdot \sigma(s) \\ = \sigma(s) \cdot [\sigma(s) \cdot \hat{p}]^l \sigma(s) = \sum_{k=0}^{[l/2]} [C_l^{2k} \sigma^2(s) - C_{l+1}^{2k+2}] [\sigma(s) \cdot \hat{p}]^{l-2k}, n \geq 0, l \geq 0 \end{cases}$$

## 6.6 更一般各种通项公式的探讨

利用以上基本的通项公式，可以比较容易地严格推导出更多、更复杂的各种通项公式，不再存在原则上的推导困难。

## 7 特殊的通项公式

### 7.1 一种特殊的通项公式

$$\text{性质7.1.1.} \begin{cases} \hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^1 \sigma(s)\}\} = 2i[\sigma(s) \cdot \hat{p}]^2 - i\sigma^2(s) \\ \hat{p} \cdot \{\sigma(s) \times \{[\sigma(s) \cdot \hat{p}]^2 \sigma(s)\}\} = 3i[\sigma(s) \cdot \hat{p}]^2 - i[\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}] \end{cases}$$

$$\text{定理7.1.1.} [\sigma(s) \cdot \hat{p}] \hat{p} = -i[\sigma(s) \times \hat{p}] \times [\sigma(s) \times \hat{p}] = \sigma(s) - \sigma(s)[\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma(s) + 2[\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}]$$

$$\text{证明: } [\sigma(s) \cdot \hat{p}] \hat{p} = -i[\sigma(s) \times \hat{p}] \times [\sigma(s) \times \hat{p}]$$

$$= i\sigma(s) \times [|\sigma(s) \cdot \hat{p} \sigma(s) [\sigma(s) \cdot \hat{p}] + [\sigma(s) \cdot \hat{p}] i\sigma(s) [\sigma(s) \cdot \hat{p}] | \times \sigma(s)$$

$$- i\sigma(s) \times [|\sigma(s) \cdot \hat{p}]^2 \sigma(s) + [\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}]$$

$$= \{-[\sigma(s) \cdot \hat{p}] \sigma(s) - \sigma(s) [\sigma(s) \cdot \hat{p}] + \sigma^2(s) \hat{p}\} [\sigma(s) \cdot \hat{p}] + [\sigma(s) \cdot \hat{p}] \{-[\sigma(s) \cdot \hat{p}] \sigma(s) - \sigma(s) [\sigma(s) \cdot \hat{p}] + \sigma^2(s) \hat{p}\}$$

$$+ 3[\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}] - 2\sigma^2(s) \hat{p} [\sigma(s) \cdot \hat{p}] + \sigma(s) + [\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}]$$

$$= -\{[\sigma(s) \cdot \hat{p}] \sigma(s) + \sigma(s) [\sigma(s) \cdot \hat{p}] - \sigma^2(s) \hat{p}\} [\sigma(s) \cdot \hat{p}] - [\sigma(s) \cdot \hat{p}] \{[\sigma(s) \cdot \hat{p}] \sigma(s) + \sigma(s) [\sigma(s) \cdot \hat{p}] - \sigma^2(s) \hat{p}\}$$

$$+ 3[\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}] - 2\sigma^2(s) \hat{p} [\sigma(s) \cdot \hat{p}] + \sigma(s) + [\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}]$$

$$= \sigma(s) - \sigma(s)[\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma(s) + 2[\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}] \quad \square$$

### 7.2 该特殊通项公式的推论

$$\text{推论7.2.1.} [1 - (h - h')^2] \lambda^+(\hat{p}, h; s) \sigma(s) \lambda(\hat{p}, h'; s) = \delta_{hh'} h \hat{p}$$

$$\text{证明: } \lambda^+(\hat{p}, h; s) [\sigma(s) \cdot \hat{p}] \hat{p} \lambda(\hat{p}, h'; s)$$

$$= \lambda^+(\hat{p}, h; s) \{\sigma(s) - \sigma(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma(s) + 2[\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}]\} \lambda(\hat{p}, h'; s)$$

$$\Leftrightarrow [1 - (h - h')^2] \lambda^+(\hat{p}, h; s) \sigma(s) \lambda(\hat{p}, h'; s) = \delta_{hh'} h \hat{p} \quad \square$$

$$\text{推论7.2.2.} [\sigma(s) \cdot \hat{p}]^n \hat{p}_{i_1} \hat{p}_{i_2} \cdot \hat{p}_{i_n} =$$

$$\{\sigma_{i_1}(s) - \sigma_{i_1}(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma_{i_1}(s) + 2[\sigma(s) \cdot \hat{p}] \sigma_{i_1}(s) [\sigma(s) \cdot \hat{p}]\}$$

$$\{\sigma_{i_2}(s) - \sigma_{i_2}(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma_{i_2}(s) + 2[\sigma(s) \cdot \hat{p}] \sigma_{i_2}(s) [\sigma(s) \cdot \hat{p}]\}$$

.....

$$\{\sigma_{i_n}(s) - \sigma_{i_n}(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma_{i_n}(s) + 2[\sigma(s) \cdot \hat{p}] \sigma_{i_n}(s) [\sigma(s) \cdot \hat{p}]\}$$

$$\text{推论7.2.3.} [2\sigma \cdot \hat{p}]^n \hat{p}_{i_1} \hat{p}_{i_2} \cdot \hat{p}_{i_n} = \{\sigma_{i_1} + [\sigma \cdot \hat{p}] \sigma_{i_1} [\sigma \cdot \hat{p}]\} \{\sigma_{i_2} + [\sigma \cdot \hat{p}] \sigma_{i_2} [\sigma \cdot \hat{p}]\} \cdots \{\sigma_{i_n} + [\sigma \cdot \hat{p}] \sigma_{i_n} [\sigma(s) \cdot \hat{p}]\}$$

$$\text{推论7.2.4.} \frac{1}{4} \hat{p}_i \hat{p}_j$$

$$= \{\frac{1}{4} \sigma_i + \frac{1}{4} [\sigma \cdot \hat{p}] \sigma_i [\sigma \cdot \hat{p}]\} \{\frac{1}{4} \sigma_j + \frac{1}{4} [\sigma \cdot \hat{p}] \sigma_j [\sigma \cdot \hat{p}]\}$$

$$= \frac{1}{16} \{\sigma_i \sigma_j + [\sigma \cdot \hat{p}] \sigma_i [\sigma \cdot \hat{p}] \sigma_j + \sigma_i [\sigma \cdot \hat{p}] \sigma_j [\sigma \cdot \hat{p}] + [\sigma \cdot \hat{p}] \sigma_i \sigma_j [\sigma \cdot \hat{p}]\}$$

$$\begin{aligned}
& \text{证明: } \lambda^+(\hat{p}, h; s) [\sigma(s) \cdot \hat{p}]^n \hat{p}_{i_1} \hat{p}_{i_2} \cdots \hat{p}_{i_n} \lambda(\hat{p}, h'; s) \\
&= \lambda^+(\hat{p}, h; s) \\
& \{ \sigma_{i_1}(s) - \sigma_{i_1}(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma_{i_1}(s) + 2[\sigma(s) \cdot \hat{p}] \sigma_{i_1}(s) [\sigma(s) \cdot \hat{p}] \} \\
& \{ \sigma_{i_2}(s) - \sigma_{i_2}(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma_{i_2}(s) + 2[\sigma(s) \cdot \hat{p}] \sigma_{i_2}(s) [\sigma(s) \cdot \hat{p}] \} \lambda(\hat{p}, h'; s) \\
& \dots \dots \dots \\
& \{ \sigma_{i_n}(s) - \sigma_{i_n}(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma_{i_n}(s) + 2[\sigma(s) \cdot \hat{p}] \sigma_{i_n}(s) [\sigma(s) \cdot \hat{p}] \} \\
& \lambda(\hat{p}, h'; s) \\
&= \lambda^+(\hat{p}, h; s) \\
& \{ \sigma_{i_1}(s) - \sigma_{i_1}(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma_{i_1}(s) + 2[\sigma(s) \cdot \hat{p}] \sigma_{i_1}(s) [\sigma(s) \cdot \hat{p}] \} \lambda(\hat{p}, h'; s) \\
& \sum_{h_1=s}^{-s} \lambda(\hat{p}, h_1; s) \lambda^+(\hat{p}, h_1; s) \\
& \{ \sigma_{i_2}(s) - \sigma_{i_2}(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma_{i_2}(s) + 2[\sigma(s) \cdot \hat{p}] \sigma_{i_2}(s) [\sigma(s) \cdot \hat{p}] \} \\
& \sum_{h_2=s}^{-s} \lambda(\hat{p}, h_2; s) \lambda^+(\hat{p}, h_2; s) \\
& \dots \dots \dots \\
& \{ \sigma_{i_n}(s) - \sigma_{i_n}(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma_{i_n}(s) + 2[\sigma(s) \cdot \hat{p}] \sigma_{i_n}(s) [\sigma(s) \cdot \hat{p}] \} \\
& \lambda(\hat{p}, h'; s) \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{定理7.2.1. } [\sigma(s) \cdot \hat{p}] \hat{p} \\
&= \sigma(s) [\sigma(s) \cdot \hat{p}]^2 - [\sigma(s) \cdot \hat{p}]^2 \sigma(s) + 2[\sigma(s) \cdot \hat{p}] \sigma(s) [\sigma(s) \cdot \hat{p}] \\
&= \sigma(s) - \sum_{k=0}^2 C_2^k [-\sigma(s) \cdot \hat{p}]^k \sigma(s) [\sigma(s) \cdot \hat{p}]^{2-k}
\end{aligned}$$

$$\begin{aligned}
& \text{推论7.2.5. } [\sigma(s) \cdot \hat{p}]^n \hat{p}_{i_1} \hat{p}_{i_2} \cdots \hat{p}_{i_n} = \\
& \{ \sigma_{i_1}(s) - \sum_{k=0}^2 C_2^k [-\sigma(s) \cdot \hat{p}]^k \sigma_{i_1}(s) [\sigma(s) \cdot \hat{p}]^{2-k} \} \\
& \{ \sigma_{i_2}(s) - \sum_{k=0}^2 C_2^k [-\sigma(s) \cdot \hat{p}]^k \sigma_{i_2}(s) [\sigma(s) \cdot \hat{p}]^{2-k} \} \\
& \dots \dots \dots \\
& \{ \sigma_{i_n}(s) - \sum_{k=0}^2 C_2^k [-\sigma(s) \cdot \hat{p}]^k \sigma_{i_n}(s) [\sigma(s) \cdot \hat{p}]^{2-k} \}
\end{aligned}$$

$$\text{推论7.2.6. } \lambda^+(\hat{p}, -s\zeta) \sigma_i(s) [\sigma(s) \cdot \hat{p}]^n \sigma_j(s) \lambda(\hat{p}, -s\zeta) = (-\zeta)^n s^2 s^n \hat{p}_i \hat{p}_j + (-\zeta)^n (s-1)^n \left[ \frac{s}{2} (\delta_{ij} - \hat{p}_i \hat{p}_j - i\zeta \varepsilon_{ij}^k \hat{p}_k) \right]$$

$$\text{推论7.2.7. } \lambda^+(\hat{p}, h; s) \sigma_i(s) [\sigma(s) \cdot \hat{p}]^n \sigma_j(s) \lambda(\hat{p}, h'; s) = ???$$

## 8 $[\sigma(s) \cdot \hat{p}]^{2s+1}$ 的低阶展开

### 8.1 $[\sigma(s) \cdot \hat{p}]^{2s+1}$ 低阶展开的猜想

$$\text{猜想8.1.1. } [\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k}$$

可以通过多项式定理的自然数拆分猜想完全证明以上猜想，先放一放，等我以后有时间再来做这件事。

### 8.2 $[\sigma(s) \cdot \hat{p}]^{2s+1}$ 的低阶展开式系数方程的严格证明

$$\text{定理8.2.1. } [\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \Rightarrow h^{2s+1} = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s) h^{2s+1-2k}, h = s, (s-1), \dots, 1|2$$

$$\text{证明: } [\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k}$$

$$\Rightarrow \lambda^+(s, h) [\sigma(s) \cdot \hat{p}]^{2s+1} \lambda(s, h) = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s) \lambda^+(s, h) [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \lambda(s, h), h = s, (s-1), \dots, -(s-1), -s$$

$$\Rightarrow h^{2s+1} = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s) h^{2s+1-2k}, h = s, (s-1), \dots, -(s-1), -s$$

$$\Rightarrow h^{2s+1} = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s) h^{2s+1-2k}, h = s, (s-1), \dots, \frac{1}{2}|1$$

□

**定理8.2.2.**  $[\sigma(n - \frac{1}{2}) \cdot \hat{p}]^{2n} = \sum_{k=0}^{n-1} \bar{X}_k(n - \frac{1}{2}) [\sigma(n - \frac{1}{2}) \cdot \hat{p}]^{2k}, [\sigma(n) \cdot \hat{p}]^{2n+1} = \sum_{k=0}^{n-1} \bar{X}_k(n) [\sigma(n) \cdot \hat{p}]^{2k+1}$

### 8.3 $[\sigma(s) \cdot \hat{p}]^{2s+1}$ 的低阶展开式求解

**定义8.3.1.**  $C_{\{a_1, a_2, \dots, a_n\}}^k :=$  按组合规律选出  $k$  个  $a$  相乘, 并把所有乘积项全部加起来,  $C_{\{a_1, a_2, \dots, a_n\}}^0 := 1$

**推论8.3.1.**  $C_{\{1_1, 1_2, \dots, 1_n\}}^k = C_n^k$

**猜想8.3.1.**

$$\left\{ \begin{aligned} & \begin{bmatrix} (2n-1)^{2n-2} & (2n-1)^{2n-4} & \dots & (2n-1)^2 & (2n-1)^0 \\ (2n-3)^{2n-2} & (2n-3)^{2n-4} & \dots & (2n-3)^2 & (2n-3)^0 \\ \dots & \dots & \dots & \dots & \dots \\ 3^{2n-2} & 3^{2n-4} & \dots & 3^2 & 3^0 \\ 1^{2n-2} & 1^{2n-4} & \dots & 1^2 & 1^0 \end{bmatrix} Y(n - \frac{1}{2}) = \begin{bmatrix} (2n-1)^{2n} \\ (2n-3)^{2n} \\ \dots \\ 3^{2n} \\ 1^{2n} \end{bmatrix}, Y(n - \frac{1}{2}) = \begin{bmatrix} (-1)^0 [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^1] \\ (-1)^1 [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^2] \\ \dots \\ (-1)^{n-2} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^{n-1}] \\ (-1)^{n-1} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^n] \end{bmatrix} \\ & \begin{bmatrix} (n-\frac{1}{2})^{2n-2} & (n-\frac{1}{2})^{2n-4} & \dots & (n-\frac{1}{2})^2 & (n-\frac{1}{2})^0 \\ (n-\frac{3}{2})^{2n-2} & (n-\frac{3}{2})^{2n-4} & \dots & (n-\frac{3}{2})^2 & (n-\frac{3}{2})^0 \\ \dots & \dots & \dots & \dots & \dots \\ (\frac{3}{2})^{2n-2} & (\frac{3}{2})^{2n-4} & \dots & (\frac{3}{2})^2 & (\frac{3}{2})^0 \\ (\frac{1}{2})^{2n-2} & (\frac{1}{2})^{2n-4} & \dots & (\frac{1}{2})^2 & (\frac{1}{2})^0 \end{bmatrix} X(n - \frac{1}{2}) = \begin{bmatrix} (n-\frac{1}{2})^{2n} \\ (n-\frac{3}{2})^{2n} \\ \dots \\ (\frac{3}{2})^{2n} \\ (\frac{1}{2})^{2n} \end{bmatrix}, X(n - \frac{1}{2}) = - \begin{bmatrix} (-4)^{-1} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^1] \\ (-4)^{-2} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^2] \\ \dots \\ (-4)^{-(n-1)} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^{n-1}] \\ (-4)^{-n} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^n] \end{bmatrix} \\ & \begin{bmatrix} (n-\frac{1}{2})^0 & (n-\frac{1}{2})^2 & \dots & (n-\frac{1}{2})^{2n-4} & (n-\frac{1}{2})^{2n-2} \\ (n-\frac{3}{2})^0 & (n-\frac{3}{2})^2 & \dots & (n-\frac{3}{2})^{2n-4} & (n-\frac{3}{2})^{2n-2} \\ \dots & \dots & \dots & \dots & \dots \\ (\frac{3}{2})^0 & (\frac{3}{2})^2 & \dots & (\frac{3}{2})^{2n-4} & (\frac{3}{2})^{2n-2} \\ (\frac{1}{2})^0 & (\frac{1}{2})^2 & \dots & (\frac{1}{2})^{2n-4} & (\frac{1}{2})^{2n-2} \end{bmatrix} \bar{X}(n - \frac{1}{2}) = \begin{bmatrix} (n-\frac{1}{2})^{2n} \\ (n-\frac{3}{2})^{2n} \\ \dots \\ (\frac{3}{2})^{2n} \\ (\frac{1}{2})^{2n} \end{bmatrix}, \bar{X}(n - \frac{1}{2}) = - \begin{bmatrix} (-4)^{-n} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^n] \\ (-4)^{-(n-1)} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^{n-1}] \\ \dots \\ (-4)^{-2} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^2] \\ (-4)^{-1} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^1] \end{bmatrix} \end{aligned} \right.$$

**猜想8.3.2.**

$$\left\{ \begin{aligned} & \begin{bmatrix} (2n)^{2n-2} & (2n)^{2n-4} & \dots & (2n)^2 & (2n)^0 \\ (2n-2)^{2n-2} & (2n-2)^{2n-4} & \dots & (2n-2)^2 & (2n-2)^0 \\ \dots & \dots & \dots & \dots & \dots \\ 4^{2n-2} & 4^{2n-4} & \dots & 4^2 & 4^0 \\ 2^{2n-2} & 2^{2n-4} & \dots & 2^2 & 2^0 \end{bmatrix} Y(n) = \begin{bmatrix} (2n)^{2n} \\ (2n-2)^{2n} \\ \dots \\ 4^{2n} \\ 2^{2n} \end{bmatrix}, Y(n) = \begin{bmatrix} (-1)^0 [C_{\{2^2, 4^2, \dots, (2n)^2\}}^1] \\ (-1)^1 [C_{\{2^2, 4^2, \dots, (2n)^2\}}^2] \\ \dots \\ (-1)^{n-2} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^{n-1}] \\ (-1)^{n-1} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^n] \end{bmatrix} \\ & \begin{bmatrix} n^{2n-2} & n^{2n-4} & \dots & n^2 & n^0 \\ (n-1)^{2n-2} & (n-1)^{2n-4} & \dots & (n-1)^2 & (n-1)^0 \\ \dots & \dots & \dots & \dots & \dots \\ 2^{2n-2} & 2^{2n-4} & \dots & 2^2 & 2^0 \\ 1^{2n-2} & 1^{2n-4} & \dots & 1^2 & 1^0 \end{bmatrix} X(n) = \begin{bmatrix} n^{2n} \\ (n-1)^{2n} \\ \dots \\ 2^{2n} \\ 1^{2n} \end{bmatrix}, X(n) = - \begin{bmatrix} (-4)^{-1} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^1] \\ (-4)^{-2} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^2] \\ \dots \\ (-4)^{-(n-1)} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^{n-1}] \\ (-4)^{-n} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^n] \end{bmatrix} \\ & \begin{bmatrix} n^0 & n^2 & \dots & n^{2n-4} & n^{2n-2} \\ (n-1)^0 & (n-1)^2 & \dots & (n-1)^{2n-4} & (n-1)^{2n-2} \\ \dots & \dots & \dots & \dots & \dots \\ 2^0 & 2^2 & \dots & 2^{2n-4} & 2^{2n-2} \\ 1^0 & 1^2 & \dots & 1^{2n-4} & 1^{2n-2} \end{bmatrix} \bar{X}(n) = \begin{bmatrix} n^{2n} \\ (n-1)^{2n} \\ \dots \\ 2^{2n} \\ 1^{2n} \end{bmatrix}, \bar{X}(n) = - \begin{bmatrix} (-4)^{-(n-1)} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^{n-1}] \\ (-4)^{-n} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^n] \\ \dots \\ (-4)^{-2} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^2] \\ (-4)^{-1} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^1] \end{bmatrix} \end{aligned} \right.$$

**推论8.3.2.**  $\left\{ \begin{aligned} & X(n - \frac{1}{2}) = \begin{bmatrix} (4)^{-1} Y_1(n - \frac{1}{2}) \\ (4)^{-2} Y_2(n - \frac{1}{2}) \\ \dots \\ (4)^{-n} Y_n(n - \frac{1}{2}) \end{bmatrix}, X(n) = \begin{bmatrix} (4)^{-1} Y_1(n) \\ (4)^{-2} Y_2(n) \\ \dots \\ (4)^{-n} Y_n(n) \end{bmatrix} \\ & Y(n - \frac{1}{2}) = \begin{bmatrix} (4)^1 X_1(n - \frac{1}{2}) \\ (4)^2 X_2(n - \frac{1}{2}) \\ \dots \\ (4)^n X_n(n - \frac{1}{2}) \end{bmatrix}, Y(n) = \begin{bmatrix} (4)^1 X_1(n) \\ (4)^2 X_2(n) \\ \dots \\ (4)^n X_n(n) \end{bmatrix} \end{aligned} \right.$

**推论8.3.3.**  $\left\{ \begin{aligned} & [\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k} = \sum_{k=1}^{[s+\frac{1}{2}]} 4^{-k} Y_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \\ & \Omega^{2s+1}(s) = \sum_{k=1}^{[s+\frac{1}{2}]} Y_k(s) \Omega^{2s+1-2k}(s) = \sum_{k=1}^{[s+\frac{1}{2}]} 4^k X_k(s) \Omega^{2s+1-2k}(s), [\sigma(s) \cdot \hat{p}] \leftrightarrow \frac{1}{2} \Omega(s) \end{aligned} \right.$

猜想8.3.3.

$$\left\{ \begin{aligned} & \begin{bmatrix} (n-\frac{1}{2})^0 & (n-\frac{1}{2})^2 & \cdots & (n-\frac{1}{2})^{2n-4} & (n-\frac{1}{2})^{2n-2} \\ (n-\frac{3}{2})^0 & (n-\frac{3}{2})^2 & \cdots & (n-\frac{3}{2})^{2n-4} & (n-\frac{3}{2})^{2n-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (\frac{3}{2})^0 & (\frac{3}{2})^2 & \cdots & (\frac{3}{2})^{2n-4} & (\frac{3}{2})^{2n-2} \\ (\frac{1}{2})^0 & (\frac{1}{2})^2 & \cdots & (\frac{1}{2})^{2n-4} & (\frac{1}{2})^{2n-2} \end{bmatrix} \bar{X}(n-\frac{1}{2}) = \begin{bmatrix} (n-\frac{1}{2})^{2n} \\ (n-\frac{3}{2})^{2n} \\ \vdots \\ (\frac{3}{2})^{2n} \\ (\frac{1}{2})^{2n} \end{bmatrix}, \bar{X}(n-\frac{1}{2}) = - \begin{bmatrix} (-4)^{-n} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^n] \\ (-4)^{-(n-1)} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^{n-1}] \\ \vdots \\ (-4)^{-2} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^2] \\ (-4)^{-1} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^1] \end{bmatrix} \\ & \begin{bmatrix} n^0 & n^2 & \cdots & n^{2n-4} & n^{2n-2} \\ (n-1)^0 & (n-1)^2 & \cdots & (n-1)^{2n-4} & (n-1)^{2n-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 2^0 & 2^2 & \cdots & 2^{2n-4} & 2^{2n-2} \\ 1^0 & 1^2 & \cdots & 1^{2n-4} & 1^{2n-2} \end{bmatrix} \bar{X}(n) = \begin{bmatrix} n^{2n} \\ (n-1)^{2n} \\ \vdots \\ 2^{2n} \\ 1^{2n} \end{bmatrix}, \bar{X}(n) = - \begin{bmatrix} (-4)^{-n} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^n] \\ (-4)^{-(n-1)} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^{n-1}] \\ \vdots \\ (-4)^{-2} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^2] \\ (-4)^{-1} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^1] \end{bmatrix} \\ & \begin{bmatrix} s^0 & s^2 & \cdots & s^{2[s-\frac{1}{2}]} \\ (s-1)^0 & (s-1)^2 & \cdots & (s-1)^{2[s-\frac{1}{2}]} \\ \vdots & \vdots & \cdots & \vdots \\ (\frac{1}{2}|1)^0 & (\frac{1}{2}|1)^2 & \cdots & (\frac{1}{2}|1)^{2[s-\frac{1}{2}]} \end{bmatrix} \bar{X}(s) = \begin{bmatrix} s^{2[s+\frac{1}{2}]} \\ (s-1)^{2[s+\frac{1}{2}]} \\ \vdots \\ (\frac{1}{2}|1)^{2[s+\frac{1}{2}]} \end{bmatrix}, \bar{X}(s) = - \begin{bmatrix} (-4)^{-[s+\frac{1}{2}]} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^{[s+\frac{1}{2}]}] \\ (-4)^{-[s-\frac{1}{2}]} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^{[s-\frac{1}{2}]}] \\ \vdots \\ (-4)^{-2} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^2] \\ (-4)^{-1} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^1] \end{bmatrix} \end{aligned} \right.$$

$$\text{猜想8.3.4.} \begin{cases} [\sigma(s) \cdot \hat{p}]^{2s+1} = - \sum_{k=1}^{[s+\frac{1}{2}]} (-4)^{-k} C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^k [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \\ \Omega^{2s+1}(s) = - \sum_{k=1}^{[s+\frac{1}{2}]} (-1)^k C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^k \Omega^{2s+1-2k}(s) \end{cases}$$

$$\text{猜想8.3.5.} \begin{cases} \sum_{k=0}^{[s+\frac{1}{2}]} (-4)^{-k} C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^k [\sigma(s) \cdot \hat{p}]^{2s+1-2k} = 0 \\ \sum_{k=0}^{[s+\frac{1}{2}]} (-1)^k C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^k \Omega^{2s+1-2k}(s) = 0 \\ \sum_{k=0}^{[s+\frac{1}{2}]} (-4h^2)^{-k} C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^k = 0, h = s, (s-1), \dots, \frac{1}{2}|1 \end{cases}$$

### 8.4 $C_{\{a_1, a_2, \dots, a_n\}}^k$ 的性质

性质8.4.1.  $C_{\{a_1, a_2, \dots, a_n\}}^k = C_{\{a_1\}}^1 C_{\{a_2, \dots, a_n\}}^{k-1} + C_{\{a_1\}}^0 C_{\{a_2, \dots, a_n\}}^k$

性质8.4.2.  $C_{\{a_1, a_2, \dots, a_n\}}^k = C_{\{a_1, a_2\}}^2 C_{\{a_3, \dots, a_n\}}^{k-2} + C_{\{a_1, a_2\}}^1 C_{\{a_3, \dots, a_n\}}^{k-1} + C_{\{a_1, a_2\}}^0 C_{\{a_3, \dots, a_n\}}^k$

性质8.4.3.  $C_{\{a_1, a_2, \dots, a_n\}}^k = \sum_{i=0}^l C_{\{a_1, a_2, \dots, a_i\}}^i C_{\{a_{i+1}, \dots, a_n\}}^{k-i}$

性质8.4.4.  $C_{\{a_1, a_2, \dots, a_{l_0}\}}^{i_0} = \sum_{i_1=0}^{l_1} \sum_{i_2=0}^{l_2} C_{\{a_1, a_2, \dots, a_{i_2}\}}^{i_2} C_{\{a_{i_2+1}, \dots, a_{i_1}\}}^{i_1-i_2} C_{\{a_{i_1+1}, \dots, a_{l_0}\}}^{i_0-i_1}$

性质8.4.5.  $C_{\{a_1, a_2, \dots, a_{l_0}\}}^{i_0} = \sum_{i_1=0}^{l_1} \sum_{i_2=0}^{l_2} \cdots \sum_{i_3=0}^{l_3} C_{\{a_1, a_2, \dots, a_{i_3}\}}^{i_3} C_{\{a_{i_3+1}, a_2, \dots, a_{i_2}\}}^{i_2-i_3} C_{\{a_{i_2+1}, \dots, a_{i_1}\}}^{i_1-i_2} C_{\{a_{i_1+1}, \dots, a_{l_0}\}}^{i_0-i_1}$

性质8.4.6.  $C_{\{a_1, a_2, \dots, a_{l_0}\}}^{i_0} = \sum_{i_1=0}^{l_1} \sum_{i_2=0}^{l_2} \cdots \sum_{i_r=0}^{l_r} C_{\{a_1, a_2, \dots, a_{i_r}\}}^{i_r} \cdots C_{\{a_{i_r+1}, a_2, \dots, a_{i_2}\}}^{i_2-i_r} C_{\{a_{i_2+1}, \dots, a_{i_1}\}}^{i_1-i_2} C_{\{a_{i_1+1}, \dots, a_{l_0}\}}^{i_0-i_1}$

### 8.5 $[\sigma(s) \cdot \hat{p}]^{2s+1}$ 的低阶展开式的特例验证

定义8.5.1.  $a_i^2 = 1$

性质8.5.1.  $(a_1)^2 = 1, [\sigma(\frac{1}{2}) \cdot \hat{p}]^4 = \frac{1}{4}$

性质8.5.2.  $(a_1 + a_2)^3 = 4(a_1 + a_2), [\sigma(1) \cdot \hat{p}]^3 = [\sigma(1) \cdot \hat{p}]$

推论8.5.1.

$$\begin{bmatrix} 3^2 & 3^0 \\ 1^2 & 1^0 \end{bmatrix} Y = \begin{bmatrix} 3^4 \\ 1^4 \end{bmatrix} \Leftrightarrow Y = \frac{1}{8} \begin{bmatrix} 1^0 & -1^2 \\ -3^0 & 3^2 \end{bmatrix} \begin{bmatrix} 3^4 \\ 1^4 \end{bmatrix} = \begin{bmatrix} 10 \\ -9 \end{bmatrix} = \begin{bmatrix} 1^2+3^2 \\ -1^2 3^2 \end{bmatrix}$$

$$\text{性质8.5.3. } (a_1 + a_2 + a_3)^4 = 10(a_1 + a_2 + a_3)^2 - 9, [\sigma(\frac{3}{2}) \cdot \hat{p}]^4 = \frac{5}{2}[\sigma(\frac{3}{2}) \cdot \hat{p}]^2 - \frac{9}{16}$$

推论8.5.2.

$$\begin{bmatrix} 4^2 & 4^0 \\ 2^2 & 2^0 \end{bmatrix} Y = \begin{bmatrix} 4^4 \\ 2^4 \end{bmatrix} \Leftrightarrow Y = \frac{1}{12} \begin{bmatrix} 2^0 & 2^2 \\ -4^0 & 4^2 \end{bmatrix} \begin{bmatrix} 4^4 \\ 2^4 \end{bmatrix} = \begin{bmatrix} 20 \\ -64 \end{bmatrix} = \begin{bmatrix} 2^2+4^2 \\ -2^2 4^2 \end{bmatrix}$$

$$\text{性质8.5.4. } (a_1 + a_2 + a_3 + a_4)^5 = 20(a_1 + a_2 + a_3 + a_4)^3 - 64(a_1 + a_2 + a_3 + a_4), [\sigma(2) \cdot \hat{p}]^5 = 5[\sigma(2) \cdot \hat{p}]^3 - 4[\sigma(2) \cdot \hat{p}]$$

推论8.5.3.

$$\begin{bmatrix} 5^4 & 5^2 & 5^0 \\ 3^4 & 3^2 & 3^0 \\ 1^4 & 1^2 & 1^0 \end{bmatrix} Y = \begin{bmatrix} 5^6 \\ 3^6 \\ 1^6 \end{bmatrix} \Leftrightarrow Y = \frac{1}{384} \begin{bmatrix} 1 & -3 & 2 \\ -10 & 78 & -68 \\ 9 & -75 & 450 \end{bmatrix} \begin{bmatrix} 5^6 \\ 3^6 \\ 1^6 \end{bmatrix} = \begin{bmatrix} 35 \\ -259 \\ 225 \end{bmatrix} = \begin{bmatrix} 1^2+3^2+5^2 \\ -(1^2 3^2+3^2 5^2+5^2 1^2) \\ 1^2 3^2 5^2 \end{bmatrix}$$

$$\text{性质8.5.5. } \begin{cases} (a_1 + a_2 + \dots + a_5)^6 = 35(a_1 + a_2 + \dots + a_5)^4 - 259(a_1 + a_2 + \dots + a_5)^2 + 225 \\ [\sigma(\frac{5}{2}) \cdot \hat{p}]^6 = \frac{35}{4}[\sigma(\frac{5}{2}) \cdot \hat{p}]^4 - \frac{259}{16}[\sigma(\frac{5}{2}) \cdot \hat{p}]^2 + \frac{225}{64} \end{cases}$$

推论8.5.4.

$$\begin{bmatrix} 6^4 & 6^2 & 6^0 \\ 4^4 & 4^2 & 4^0 \\ 2^4 & 2^2 & 2^0 \end{bmatrix} Y = \begin{bmatrix} 6^6 \\ 4^6 \\ 2^6 \end{bmatrix} \Leftrightarrow Y = \frac{1}{7680} \begin{bmatrix} 12 & -32 & 20 \\ -240 & 1280 & -1040 \\ 768 & -4608 & 11520 \end{bmatrix} \begin{bmatrix} 6^6 \\ 4^6 \\ 2^6 \end{bmatrix} = \begin{bmatrix} 56 \\ -784 \\ 48^2 \end{bmatrix} = \begin{bmatrix} 2^2+4^2+6^2 \\ -(2^2 4^2+4^2 6^2+6^2 2^2) \\ 2^2 4^2 6^2 \end{bmatrix}$$

$$\text{性质8.5.6. } \begin{cases} (a_1 + a_2 + \dots + a_6)^7 = 56(a_1 + a_2 + \dots + a_5)^5 - 784(a_1 + a_2 + \dots + a_5)^3 + 2304(a_1 + a_2 + \dots + a_5) \\ [\sigma(3) \cdot \hat{p}]^7 = 14[\sigma(3) \cdot \hat{p}]^5 - 49[\sigma(3) \cdot \hat{p}]^3 + 36[\sigma(3) \cdot \hat{p}] \end{cases}$$

推论8.5.5.

$$\begin{bmatrix} 3^2 & 3^0 \\ 1^2 & 1^0 \end{bmatrix}^{-1} = \frac{1}{8} \begin{bmatrix} 1^0 & -3^0 \\ -1^2 & 3^2 \end{bmatrix}, \begin{bmatrix} 4^2 & 4^0 \\ 2^2 & 2^0 \end{bmatrix}^{-1} = \frac{1}{12} \begin{bmatrix} 2^0 & -4^0 \\ -2^2 & 4^2 \end{bmatrix}$$

$$\begin{bmatrix} 5^4 & 5^2 & 5^0 \\ 3^4 & 3^2 & 3^0 \\ 1^4 & 1^2 & 1^0 \end{bmatrix}^{-1} = \frac{1}{384} \begin{bmatrix} 1 & -3 & 2 \\ -10 & 78 & -68 \\ 9 & -75 & 450 \end{bmatrix}, \begin{bmatrix} 6^4 & 6^2 & 6^0 \\ 4^4 & 4^2 & 4^0 \\ 2^4 & 2^2 & 2^0 \end{bmatrix}^{-1} = \frac{1}{7680} \begin{bmatrix} 12 & -32 & 20 \\ -240 & 1280 & -1040 \\ 768 & -4608 & 11520 \end{bmatrix}$$

$$\text{性质8.5.7. } (a_1 + a_2 + \dots + a_7)^8 = 84(a_1 + a_2 + \dots + a_5)^6 - 1974(a_1 + a_2 + \dots + a_5)^4 + 12916(a_1 + a_2 + \dots + a_5)^2 - 11025$$

$$84 = C_9^6 = 1^2 + 3^2 + 5^2 + 7^2, 1974 = 1^2 3^2 + 3^2 5^2 + 5^2 7^2 + 7^2 1^2 + 1^2 5^2 + 3^2 7^2$$

$$12916 = 3^2 5^2 7^2 + 5^2 7^2 1^2 + 1^2 3^2 5^2 + 1^2 3^2 7^2, 11025 = 1^2 3^2 5^2 7^2$$

$$[\sigma(\frac{7}{2}) \cdot \hat{p}]^8 = 21[\sigma(\frac{7}{2}) \cdot \hat{p}]^6 - \frac{987}{8}[\sigma(\frac{7}{2}) \cdot \hat{p}]^4 + \frac{3229}{16}[\sigma(\frac{7}{2}) \cdot \hat{p}]^2 - \frac{11025}{256}$$

$$\text{性质8.5.8. } (a_1 + a_2 + \dots + a_8)^9$$

$$= 120(a_1 + a_2 + \dots + a_8)^7 - 4368(a_1 + a_2 + \dots + a_8)^5 + 52480(a_1 + a_2 + \dots + a_8)^3 - 147456(a_1 + a_2 + \dots + a_8)$$

$$120 = C_{10}^7 = 2^2 + 4^2 + 6^2 + 8^2, 4368 = 2^2 4^2 + 4^2 6^2 + 6^2 8^2 + 8^2 2^2 + 2^2 6^2 + 4^2 8^2$$

$$52480 = 4^2 6^2 8^2 + 6^2 8^2 2^2 + 2^2 4^2 6^2 + 2^2 4^2 8^2, 147456 = 2^2 4^2 6^2 8^2$$

$$[\sigma(4) \cdot \hat{p}]^9 = 30[\sigma(4) \cdot \hat{p}]^7 - 273[\sigma(4) \cdot \hat{p}]^5 + 820[\sigma(4) \cdot \hat{p}]^3 - 576[\sigma(4) \cdot \hat{p}]$$

## 8.6 $[\sigma(s) \cdot \hat{p}]^{2s+1}$ 的低阶展开式系数方程的更严格证明

$$\text{定理8.6.1. } [\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=0}^{2s} B_k(s) [\sigma(s) \cdot \hat{p}]^k \Rightarrow h^{2s+1} = \sum_{k=0}^{2s} B_k(s) h^k, h = s, (s-1), \dots, -(s-1), -s$$

$$\text{证明: } [\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=0}^{2s} B_k(s) [\sigma(s) \cdot \hat{p}]^k$$

$$\Rightarrow \lambda^+(s, h) [\sigma(s) \cdot \hat{p}]^{2s+1} \lambda(s, h) = \sum_{k=0}^{2s} B_k(s) \lambda^+(s, h) [\sigma(s) \cdot \hat{p}]^k \lambda(s, h), h = s, (s-1), \dots, -(s-1), -s$$

$$\Rightarrow h^{2s+1} = \sum_{k=0}^{2s} B_k(s) h^k, h = s, (s-1), \dots, -(s-1), -s$$

$$\Rightarrow \begin{bmatrix} s^0 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s} \\ \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s} \end{bmatrix} \begin{bmatrix} B_0(s) \\ B_1(s) \\ \dots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = \begin{bmatrix} (s)^{2s+1} \\ (s-1)^{2s+1} \\ \dots \\ (1-s)^{2s+1} \\ (-s)^{2s+1} \end{bmatrix}$$

$$\Rightarrow h^{2s+1} = \sum_{k=0}^{2s} B_k(s) h^k, h = s, (s-1), \dots, -(s-1), -s$$

$$\Rightarrow \begin{bmatrix} B_0(s) \\ B_1(s) \\ \dots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = \begin{bmatrix} s^0 & s^1 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s-1} & (s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix}^{-1} \begin{bmatrix} (s)^{2s+1} \\ (s-1)^{2s+1} \\ \dots \\ (1-s)^{2s+1} \\ (-s)^{2s+1} \end{bmatrix}$$

□



可以验证  $s = \frac{1}{2}, 1, \frac{3}{2}, 2$  以上解法都是正确的，下面是更具体的解法。

**定理8.6.2.**  $[\sigma(s) \cdot \hat{p}]^{2s+1} = -\frac{1}{2} \sum_{k=0}^{2s} [1 - (-1)^{2s-k}] (-\frac{1}{4})^{[s+\frac{1}{2}]-[\frac{k}{2}]} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^{[s+\frac{1}{2}]-[\frac{k}{2}]}][\sigma(s) \cdot \hat{p}]^k$

证明:

$$\begin{aligned} & \begin{bmatrix} s^0 & s^1 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s-1} & (s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix} \begin{bmatrix} B_0(s) \\ B_1(s) \\ \dots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = \begin{bmatrix} (s)^{2s+1} \\ (s-1)^{2s+1} \\ \dots \\ (1-s)^{2s+1} \\ (-s)^{2s+1} \end{bmatrix} \\ \Leftrightarrow & \frac{1}{2} \begin{bmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & -1 & 0 \\ 1 & 0 & \dots & 0 & -1 \end{bmatrix} \begin{bmatrix} s^0 & s^1 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s-1} & (s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix} \begin{bmatrix} B_0(s) \\ B_1(s) \\ \dots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & -1 & 0 \\ 1 & 0 & \dots & 0 & -1 \end{bmatrix} \begin{bmatrix} (s)^{2s+1} \\ (s-1)^{2s+1} \\ \dots \\ (1-s)^{2s+1} \\ (-s)^{2s+1} \end{bmatrix} \\ \Leftrightarrow & \begin{bmatrix} s^0 & 0 & s^2 & 0 & \dots & \frac{1}{2}[1+(-1)^s]s^{2s} \\ (s-1)^0 & 0 & (s-1)^2 & 0 & \dots & \frac{1}{2}[1+(-1)^{2s}](s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & (s-1)^1 & 0 & (s-1)^3 & \dots & \frac{1}{2}[1-(-1)^{2s}](s-1)^{2s} \\ 0 & s^1 & 0 & s^3 & \dots & \frac{1}{2}[1-(-1)^s]s^{2s} \end{bmatrix} \begin{bmatrix} B_0(s) \\ B_1(s) \\ \dots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[1+(-1)^{2s+1}]s^{2s+1} \\ \frac{1}{2}[1+(-1)^{2s+1}](s-1)^{2s+1} \\ \dots \\ \frac{1}{2}[1-(-1)^{2s+1}](s-1)^{2s+1} \\ \frac{1}{2}[1-(-1)^{2s+1}]s^{2s+1} \end{bmatrix} \\ \Leftrightarrow & \begin{bmatrix} s^0 & s^2 & \dots & s^{2[s]} & 0 & 0 & \dots & 0 \\ (s-1)^0 & (s-1)^2 & \dots & (s-1)^{2[s]} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & (s-1)^1 & (s-1)^3 & \dots & (s-1)^{2[s-\frac{1}{2}]+1} \\ 0 & 0 & \dots & 0 & s^1 & s^3 & \dots & s^{2[s-\frac{1}{2}]+1} \end{bmatrix} \begin{bmatrix} B_0(s) \\ B_2(s) \\ \dots \\ B_{2[s]}(s) \\ B_1(s) \\ B_3(s) \\ \dots \\ B_{2[s-\frac{1}{2}]+1}(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[1+(-1)^{2s+1}]s^{2s+1} \\ \frac{1}{2}[1+(-1)^{2s+1}](s-1)^{2s+1} \\ \dots \\ \frac{1}{2}[1-(-1)^{2s+1}](s-1)^{2s+1} \\ \frac{1}{2}[1-(-1)^{2s+1}]s^{2s+1} \end{bmatrix} \\ \Leftrightarrow & \left\{ \begin{array}{l} \begin{bmatrix} n^0 & n^2 & \dots & n^{2n} & 0 & 0 & \dots & 0 \\ (n-1)^0 & (n-1)^2 & \dots & (n-1)^{2n} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1^0 & 1^2 & \dots & 1^{2n} & 0 & 0 & \dots & 0 \\ 0^0=1 & 0^2 & \dots & 0^{2n} & 0^1 & 0^3 & \dots & 0^{2n-1} \\ \dots & \dots & \dots & \dots & 1^1 & 1^3 & \dots & 1^{2n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & (n-1)^1 & (n-1)^3 & \dots & (n-1)^{2n-1} \\ 0 & 0 & \dots & 0 & n^1 & n^3 & \dots & n^{2n-1} \end{bmatrix} \begin{bmatrix} B_0(n) \\ B_2(n) \\ \dots \\ B_{2n}(n) \\ B_1(n) \\ B_3(n) \\ \dots \\ B_{2n-1}(n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 0^{2n+1} \\ 1^{2n+1} \\ \dots \\ (n-1)^{2n+1} \\ n^{2n+1} \end{bmatrix} \\ \begin{bmatrix} (n-\frac{1}{2})^0 & (n-\frac{1}{2})^2 & \dots & (n-\frac{1}{2})^{2n-2} & 0 & 0 & \dots & 0 \\ (n-\frac{3}{2})^0 & (n-\frac{3}{2})^2 & \dots & (n-\frac{3}{2})^{2n-2} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (\frac{1}{2})^0 & (\frac{1}{2})^2 & \dots & (\frac{1}{2})^{2n-2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & (\frac{1}{2})^1 & (\frac{1}{2})^3 & \dots & (\frac{1}{2})^{2n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & (n-\frac{3}{2})^1 & (n-\frac{3}{2})^3 & \dots & (n-\frac{3}{2})^{2n-1} \\ 0 & 0 & \dots & 0 & (n-\frac{1}{2})^1 & (n-\frac{1}{2})^3 & \dots & (n-\frac{1}{2})^{2n-1} \end{bmatrix} \begin{bmatrix} B_0(n-\frac{1}{2}) \\ B_2(n-\frac{1}{2}) \\ \dots \\ B_{2n-2}(n-\frac{1}{2}) \\ B_1(n-\frac{1}{2}) \\ B_3(n-\frac{1}{2}) \\ \dots \\ B_{2n-1}(n-\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} (n-\frac{1}{2})^{2n} \\ (n-\frac{3}{2})^{2n} \\ \dots \\ (\frac{1}{2})^{2n} \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix} \end{array} \right. \\ \Leftrightarrow & \left\{ \begin{array}{l} \begin{bmatrix} (n-\frac{1}{2})^0 & (n-\frac{1}{2})^2 & \dots & (n-\frac{1}{2})^{2n-2} \\ (n-\frac{3}{2})^0 & (n-\frac{3}{2})^2 & \dots & (n-\frac{3}{2})^{2n-2} \\ \dots & \dots & \dots & \dots \\ (\frac{3}{2})^0 & (\frac{3}{2})^2 & \dots & (\frac{3}{2})^{2n-2} \\ (\frac{1}{2})^0 & (\frac{1}{2})^2 & \dots & (\frac{1}{2})^{2n-2} \end{bmatrix} \begin{bmatrix} B_0(n-\frac{1}{2}) \\ B_2(n-\frac{1}{2}) \\ \dots \\ B_{2n-4}(n-\frac{1}{2}) \\ B_{2n-2}(n-\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} (n-\frac{1}{2})^{2n} \\ (n-\frac{3}{2})^{2n} \\ \dots \\ (\frac{3}{2})^{2n} \\ (\frac{1}{2})^{2n} \end{bmatrix} \\ \begin{bmatrix} n^0 & n^2 & \dots & n^{2n-2} \\ (n-1)^0 & (n-1)^2 & \dots & (n-1)^{2n-2} \\ \dots & \dots & \dots & \dots \\ 2^0 & 2^2 & \dots & 2^{2n-2} \\ 1^0 & 1^2 & \dots & 1^{2n-2} \end{bmatrix} \begin{bmatrix} B_1(n) \\ B_3(n) \\ \dots \\ B_{2n-3}(n) \\ B_{2n-1}(n) \end{bmatrix} = \begin{bmatrix} n^{2n} \\ (n-1)^{2n} \\ \dots \\ 2^{2n} \\ 1^{2n} \end{bmatrix} \end{array} \right. \\ \Leftrightarrow & \begin{bmatrix} B_0(n-\frac{1}{2}) \\ B_1(n-\frac{1}{2}) \\ B_2(n-\frac{1}{2}) \\ B_3(n-\frac{1}{2}) \\ \dots \\ B_{2n-1}(n-\frac{1}{2}) \\ B_{2n}(n-\frac{1}{2}) \end{bmatrix} = - \begin{bmatrix} (-4)^{-n} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^{n, (2n-1)^2}] \\ 0 \\ (-4)^{-(n-1)} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^{n-1, (2n-1)^2}] \\ \dots \\ (-4)^{-2} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^2, (2n-1)^2] \\ 0 \\ (-4)^{-1} [C_{\{1^2, 3^2, \dots, (2n-1)^2\}}^1, (2n-1)^2] \end{bmatrix}, \begin{bmatrix} B_0(n) \\ B_1(n) \\ B_2(n) \\ B_3(n) \\ \dots \\ B_{2n-3}(n) \\ B_{2n-2}(n) \\ B_{2n-1}(n) \\ B_{2n}(n) \end{bmatrix} = - \begin{bmatrix} 0 \\ (-4)^{-n} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^{n, (2n)^2}] \\ 0 \\ (-4)^{-(n-1)} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^{n-1, (2n)^2}] \\ \dots \\ (-4)^{-2} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^2, (2n)^2] \\ 0 \\ (-4)^{-1} [C_{\{2^2, 4^2, \dots, (2n)^2\}}^1, (2n)^2] \end{bmatrix} \\ \Leftrightarrow & \begin{bmatrix} B_0(s) \\ B_1(s) \\ \dots \\ B_k(s) \\ \dots \\ B_{2s-1}(s) \\ B_{2s}(s) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} [1-(-1)^{2s}](-4)^{-[s+\frac{1}{2}]} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^{[s+\frac{1}{2}]}] \\ [1-(-1)^{2s-1}](-4)^{-[s+\frac{1}{2}]} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^{[s+\frac{1}{2}]}] \\ \dots \\ [1-(-1)^{2s-k}](-4)^{[\frac{k}{2}]-[s+\frac{1}{2}]} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^{[s+\frac{1}{2}]-[\frac{k}{2}]}] \\ \dots \\ [1-(-1)^1](-4)^{-1} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^1] \\ [1-(-1)^0](-4)^{-1} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^0] \end{bmatrix} \\ \Rightarrow & [\sigma(s) \cdot \hat{p}]^{2s+1} = -\frac{1}{2} \sum_{k=0}^{2s} [1 - (-1)^{2s-k}] (-\frac{1}{4})^{[s+\frac{1}{2}]-[\frac{k}{2}]} [C_{\{(1|2)^2, \dots, (2s-2)^2, (2s)^2\}}^{[s+\frac{1}{2}]-[\frac{k}{2}]}][\sigma(s) \cdot \hat{p}]^k \quad \square \end{aligned}$$

8.7  $[\sigma(s) \cdot \vec{p}]^{2s+1+m}$ 的低阶展开式系数

$$\text{性质8.7.1. } [\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+\frac{1}{2}]} X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k}, n := [s + \frac{1}{2}]$$

$$\text{性质8.7.2. } [\sigma(s) \cdot \hat{p}]^{2s+2} = \sum_{k=1}^n X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+2-2k}$$

$$\begin{aligned} \text{性质8.7.3. } [\sigma(s) \cdot \hat{p}]^{2s+3} &= \sum_{k=1}^n X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+3-2k} \\ &= X_1(s) [\sigma(s) \cdot \hat{p}]^{2s+1} + \sum_{k=1}^{n-1} X_{k+1}(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \\ &= \sum_{k=1}^n X_1(s) X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k} + \sum_{k=1}^{n-1} X_{k+1}(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \\ &= \sum_{k=1}^n [X_1(s) X_k(s) + X_{k+1}(s)] [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \end{aligned}$$

$$\begin{aligned} \text{性质8.7.4. } [\sigma(s) \cdot \hat{p}]^{2n+5} &= \sum_{k=1}^n [X_1(s) X_k(s) + X_{k+1}(s)] [\sigma(s) \cdot \hat{p}]^{2s+3-2k} \\ &= [X_1^2(s) + X_2(s)] [\sigma(s) \cdot \hat{p}]^{2s+1} + \sum_{k=1}^n [X_1(s) X_{k+1}(s) + X_{k+2}(s)] [\sigma(s) \cdot \hat{p}]^{2s+3-2k} \\ &= \sum_{k=1}^{[s+\frac{1}{2}]} [X_1^2(s) + X_2(s)] X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k} + \sum_{k=1}^n [X_1(s) X_{k+1}(s) + X_{k+2}(s)] [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \\ &= \sum_{k=1}^n \{ [X_1^2(s) + X_2(s)] X_k(s) + X_1(s) X_{k+1}(s) + X_{k+2}(s) \} [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \end{aligned}$$

$$\begin{aligned} \text{性质8.7.5. } [\sigma(s) \cdot \hat{p}]^{2n+7} &= \sum_{k=1}^n [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \\ &\{ [X_1^3(s) + 2X_1(s) X_2(s) + X_3(s)] X_k(s) + [X_1^2(s) + X_2(s)] X_{k+1}(s) + X_1(s) X_{k+2}(s) + X_{k+3}(n) \} \end{aligned}$$

$$\begin{aligned} \text{性质8.7.6. } [\sigma(s) \cdot \hat{p}]^{2n+9} &= \sum_{k=1}^n [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \\ &\{ [X_1^4(s) + 3X_1^2(s) X_2(s) + 2X_1(s) X_3(s) + X_2^2(s) + X_4(s)] X_k(s) \\ &+ [X_1^3(s) + 2X_1(s) X_2(s) + X_3(s)] X_{k+1}(s) + [X_1^2(s) + X_2(s)] X_{k+2}(s) + X_1(s) X_{k+3}(s) + X_{k+4}(n) \} \end{aligned}$$

$$\begin{aligned} \text{性质8.7.7. } [\sigma(s) \cdot \hat{p}]^{2s+1+2m} &= \sum_{k=1}^n \sum_{l=0}^m \{ (l+1-m) X_1^{m-l}(s) + \sum_{i=1}^{=m-l} \sum_{i=1}^n r_i [X_1^{r_1}(s) X_2^{r_1}(s) \cdots X_n^{r_n}(s)] \} u(n-k-l) X_{k+l}(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \\ &= \sum_{k=1}^n \sum_{l=0}^{m|(n-k)} \{ (l+1-m) X_1^{m-l}(s) + \sum_{i=1}^{=m-l} \sum_{i=1}^n r_i [X_1^{r_1}(s) X_2^{r_1}(s) \cdots X_n^{r_n}(s)] \} X_{k+l}(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k} \end{aligned}$$

$$\begin{aligned} \text{性质8.7.8. } [\sigma(s) \cdot \vec{p}]^{2s+1+2m} &= \sum_{k=1}^n \sum_{l=0}^{m|(n-k)} \{ (l+1-m) X_1^{m-l}(s) + \sum_{i=1}^{=m-l} \sum_{i=1}^n r_i [X_1^{r_1}(s) X_2^{r_1}(s) \cdots X_n^{r_n}(s)] \} X_{k+l}(s) (\vec{p}^2)^{k+m} [\sigma(s) \cdot \vec{p}]^{2s+1-2k} \\ &= \sum_{k=1}^n \sum_{l=0}^{m|(n-k)} \{ (l+1-m) X_1^{m-l}(s) + \sum_{i=1}^{=m-l} \sum_{i=1}^n r_i [X_1^{r_1}(s) X_2^{r_1}(s) \cdots X_n^{r_n}(s)] \} X_{k+l}(s) (\vec{p}^2)^{k+m} [\sigma(s) \cdot \vec{p}]^{2s+2-2k} \end{aligned}$$

$$\begin{aligned} \text{性质8.7.9. } [\sigma(s) \cdot \vec{p}]^{2s+2+2m} &= \sum_{k=1}^n \sum_{l=0}^{m|(n-k)} \{ (l+1-m) X_1^{m-l}(s) + \sum_{i=1}^{=m-l} \sum_{i=1}^n r_i [X_1^{r_1}(s) X_2^{r_1}(s) \cdots X_n^{r_n}(s)] \} X_{k+l}(s) (\vec{p}^2)^{k+m} [\sigma(s) \cdot \vec{p}]^{2s+2-2k} \\ &= \sum_{k=1}^n \sum_{l=0}^{m|(n-k)} \{ (l+1-m) X_1^{m-l}(s) + \sum_{i=1}^{=m-l} \sum_{i=1}^n r_i [X_1^{r_1}(s) X_2^{r_1}(s) \cdots X_n^{r_n}(s)] \} X_{k+l}(s) (\vec{p}^2)^{k+m} [\sigma(s) \cdot \vec{p}]^{2s+2-2k} \end{aligned}$$

$$\text{性质8.7.10. } [\sigma(\frac{1}{2}) \cdot \vec{p}]^{2+2m} = X_1^{1+m}(\frac{1}{2}) (\vec{p}^2)^{1+m} [\sigma(\frac{1}{2}) \cdot \vec{p}]^0, [\sigma(\frac{1}{2}) \cdot \vec{p}]^{3+2m} = X_1^{1+m}(\frac{1}{2}) (\vec{p}^2)^{1+m} [\sigma(\frac{1}{2}) \cdot \vec{p}]^1$$

$$\text{性质8.7.11. } [\sigma(1) \cdot \vec{p}]^{3+2m} = X_1^{1+m}(1) (\vec{p}^2)^{1+m} [\sigma(1) \cdot \vec{p}]^1, [\sigma(1) \cdot \vec{p}]^{4+2m} = X_1^{1+m}(1) (\vec{p}^2)^{1+m} [\sigma(1) \cdot \vec{p}]^2$$

## 9 $e^{\vec{\vartheta} \cdot \sigma(s)}$ 的多项式展开

### 9.1 $e^{\vec{\vartheta} \cdot \sigma(s)}$ 多项式展开系数的求解(原则上解决了)

定理9.1.1.  $e^{\vec{\vartheta} \cdot \sigma(s)} = \sum_{k=0}^{2s} A_k(s) [\vec{\vartheta} \cdot \sigma(s)]^k, \vec{\vartheta} = \sqrt{\vartheta^2} \hat{\vartheta}, \hat{\vartheta}^2 = 1 \Rightarrow e^{h\sqrt{\vartheta^2}} = \sum_{k=0}^{2s} A_k(s) [h\sqrt{\vartheta^2}]^k, h = s, \dots, -s$

证明:  $e^{\vec{\vartheta} \cdot \sigma(s)} = \sum_{k=0}^{2s} A_k(s) [\vec{\vartheta} \cdot \sigma(s)]^k, \vec{\vartheta} = \sqrt{\vartheta^2} \hat{\vartheta}, \hat{\vartheta}^2 = 1 (\sqrt{\vartheta^2} \text{有} \pm \text{两个值,任取一个即可, 结论都一样})$

$\Rightarrow \tilde{\lambda}^T(s, h) e^{\vec{\vartheta} \cdot \sigma(s)} \lambda(s, h) = \sum_{k=0}^{2s} A_k(s) \tilde{\lambda}^T(s, h) [\vec{\vartheta} \cdot \sigma(s)]^k \lambda(s, h), h = s, \dots, -s$

$\Rightarrow e^{h\sqrt{\vartheta^2}} = \sum_{k=0}^{2s} A_k(s) [h\sqrt{\vartheta^2}]^k, h = s, \dots, -s$

$$\Rightarrow \begin{bmatrix} s^0 & s^1 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s-1} & (s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix} \begin{bmatrix} (\sqrt{\vartheta^2})^0 A_0(s) \\ (\sqrt{\vartheta^2})^1 A_1(s) \\ \dots \\ (\sqrt{\vartheta^2})^{2s-1} A_{2s-1}(s) \\ (\sqrt{\vartheta^2})^{2s} A_{2s}(s) \end{bmatrix} = \begin{bmatrix} e^{s\sqrt{\vartheta^2}} \\ e^{(s-1)\sqrt{\vartheta^2}} \\ \dots \\ e^{(1-s)\sqrt{\vartheta^2}} \\ e^{(-s)\sqrt{\vartheta^2}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_0(s) \\ A_1(s) \\ \dots \\ A_{2s-1}(s) \\ A_{2s}(s) \end{bmatrix} = \begin{bmatrix} (\sqrt{\vartheta^2})^{-0} & 0 & \dots & 0 & 0 \\ 0 & (\sqrt{\vartheta^2})^{-1} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (\sqrt{\vartheta^2})^{1-2s} & 0 \\ 0 & 0 & \dots & 0 & (\sqrt{\vartheta^2})^{-2s} \end{bmatrix} \begin{bmatrix} s^0 & s^1 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s-1} & (s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix}^{-1} \begin{bmatrix} e^{s\sqrt{\vartheta^2}} \\ e^{(s-1)\sqrt{\vartheta^2}} \\ \dots \\ e^{(1-s)\sqrt{\vartheta^2}} \\ e^{(-s)\sqrt{\vartheta^2}} \end{bmatrix} \quad \square$$

可以验证  $s = \frac{1}{2}, 1, \frac{3}{2}, 2$  以上解法都是正确的。

### 9.2 多项式展开系数求解的关键I

推论9.2.1.  $\begin{bmatrix} (\frac{1}{2})^0 & (\frac{1}{2})^1 \\ (-\frac{1}{2})^0 & (-\frac{1}{2})^1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$

推论9.2.2.  $\begin{bmatrix} 1^0 & 1^1 & 1^2 \\ 0^0 & 0^1 & 0^2 \\ (-1)^0 & (-1)^1 & (-1)^2 \end{bmatrix}^{-1} = \frac{1}{2!} \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$

推论9.2.3.  $\begin{bmatrix} (\frac{3}{2})^0 & (\frac{3}{2})^1 & (\frac{3}{2})^2 & (\frac{3}{2})^3 \\ (\frac{1}{2})^0 & (\frac{1}{2})^1 & (\frac{1}{2})^2 & (\frac{1}{2})^3 \\ (-\frac{1}{2})^0 & (-\frac{1}{2})^1 & (-\frac{1}{2})^2 & (-\frac{1}{2})^3 \\ (-\frac{3}{2})^0 & (-\frac{3}{2})^1 & (-\frac{3}{2})^2 & (-\frac{3}{2})^3 \end{bmatrix}^{-1} = \frac{1}{48} \begin{bmatrix} -3 & 27 & 27 & -3 \\ -2 & 18 & -18 & 2 \\ 12 & -12 & -12 & 12 \\ 8 & -8 & 8 & -8 \end{bmatrix} = \begin{bmatrix} -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \\ -\frac{1}{24} & \frac{9}{24} & -\frac{9}{24} & \frac{1}{24} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \end{bmatrix}$

推论9.2.4.  $\begin{bmatrix} 2^0 & 2^1 & 2^2 & 2^3 & 2^4 \\ 1^0 & 1^1 & 1^2 & 1^3 & 1^4 \\ 0^0 & 0^1 & 0^2 & 0^3 & 0^4 \\ (-1)^0 & (-1)^1 & (-1)^2 & (-1)^3 & (-1)^4 \\ (-2)^0 & (-2)^1 & (-2)^2 & (-2)^3 & (-2)^4 \end{bmatrix}^{-1} = \frac{1}{4!} \begin{bmatrix} 0 & 0 & 24 & 0 & 0 \\ -2 & 16 & 0 & -16 & 2 \\ -1 & 16 & -30 & 16 & -1 \\ 2 & -4 & 0 & 4 & -2 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{12} & \frac{2}{3} & 0 & -\frac{2}{3} & \frac{1}{12} \\ -\frac{1}{24} & \frac{2}{3} & -\frac{5}{4} & \frac{2}{3} & -\frac{1}{24} \\ \frac{1}{12} & -\frac{1}{6} & 0 & \frac{1}{6} & -\frac{1}{12} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{bmatrix}$

推论9.2.5.  $\begin{bmatrix} (\frac{5}{2})^0 & (\frac{5}{2})^1 & (\frac{5}{2})^2 & (\frac{5}{2})^3 & (\frac{5}{2})^4 & (\frac{5}{2})^5 \\ (\frac{3}{2})^0 & (\frac{3}{2})^1 & (\frac{3}{2})^2 & (\frac{3}{2})^3 & (\frac{3}{2})^4 & (\frac{3}{2})^5 \\ (\frac{1}{2})^0 & (\frac{1}{2})^1 & (\frac{1}{2})^2 & (\frac{1}{2})^3 & (\frac{1}{2})^4 & (\frac{1}{2})^5 \\ (-\frac{1}{2})^0 & (-\frac{1}{2})^1 & (-\frac{1}{2})^2 & (-\frac{1}{2})^3 & (-\frac{1}{2})^4 & (-\frac{1}{2})^5 \\ (-\frac{3}{2})^0 & (-\frac{3}{2})^1 & (-\frac{3}{2})^2 & (-\frac{3}{2})^3 & (-\frac{3}{2})^4 & (-\frac{3}{2})^5 \\ (-\frac{5}{2})^0 & (-\frac{5}{2})^1 & (-\frac{5}{2})^2 & (-\frac{5}{2})^3 & (-\frac{5}{2})^4 & (-\frac{5}{2})^5 \end{bmatrix}^{-1} = \frac{1}{96} \begin{bmatrix} 9 & -75 & 225 & -225 & 75 & 9 \\ 8 & -8 & 4 & 4 & -8 & 8 \\ 9 & -25 & 225 & -225 & 25 & -9 \\ 20 & -4 & 2 & -2 & 4 & -20 \\ -5 & 39 & -34 & -34 & 39 & -5 \\ -2 & 26 & -68 & 68 & -26 & 2 \\ 2 & -6 & 4 & 4 & -6 & 2 \\ \frac{4}{5} & -4 & -8 & 8 & 4 & -\frac{4}{5} \end{bmatrix}$

定理9.2.1.

$$\begin{bmatrix} s^0 & s^1 & \dots & s^{2s-1} & s^{2s} \\ (s-1)^0 & (s-1)^1 & \dots & (s-1)^{2s-1} & (s-1)^{2s} \\ \dots & \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^1 & \dots & (1-s)^{2s-1} & (1-s)^{2s} \\ (-s)^0 & (-s)^1 & \dots & (-s)^{2s-1} & (-s)^{2s} \end{bmatrix}^{-1} =? \begin{bmatrix} s^0 & s^2 & \dots & s^{2[s+1/2]} \\ (s-1)^0 & (s-1)^2 & \dots & (s-1)^{2[s+1/2]} \\ \dots & \dots & \dots & \dots \\ (1-s)^0 & (1-s)^2 & \dots & (1-s)^{2[s+1/2]} \\ (-s)^0 & (-s)^2 & \dots & (-s)^{2[s+1/2]} \end{bmatrix}^{-1} =? \begin{bmatrix} s^1 & s^3 & \dots & s^{2[s+1/2]-1} \\ (s-1)^1 & (s-1)^3 & \dots & (s-1)^{2[s+1/2]-1} \\ \dots & \dots & \dots & \dots \\ (1-s)^1 & (1-s)^3 & \dots & (1-s)^{2[s+1/2]-1} \\ (-s)^1 & (-s)^3 & \dots & (-s)^{2[s+1/2]-1} \end{bmatrix}^{-1} =?$$

推论9.2.6.  $\begin{bmatrix} s^0 & s^2 \\ (s-1)^0 & (s-1)^2 \end{bmatrix}^{-1} = -\frac{1}{2s-1} \begin{bmatrix} (s-1)^2 & -s^2 \\ -1 & 1 \end{bmatrix}$

$$\text{推论9.2.7. } \det \begin{bmatrix} s^0 & & & \\ (s-1)^0 & s^2 & & \\ & (s-1)^2 & s^4 & \\ & & (s-2)^2 & (s-2)^4 \end{bmatrix} = -(2s-1), \det \begin{bmatrix} s^0 & s^2 & s^4 \\ (s-1)^0 & (s-1)^2 & (s-1)^4 \\ (s-2)^0 & (s-2)^2 & (s-2)^4 \\ (s-3)^0 & (s-3)^2 & (s-3)^4 \end{bmatrix} = -2(2s-1)(2s-2)(2s-3)$$

$$\det \begin{bmatrix} s^0 & s^2 & s^4 & s^6 \\ (s-1)^0 & (s-1)^2 & (s-1)^4 & (s-1)^6 \\ (s-2)^0 & (s-2)^2 & (s-2)^4 & (s-2)^6 \\ (s-3)^0 & (s-3)^2 & (s-3)^4 & (s-3)^6 \end{bmatrix} = 12(2s-3)(2s-1)(2s-2)(2s-3)(2s-4)(2s-5)$$

推论9.2.8.

$$\begin{bmatrix} s^0 & s^2 & s^4 \\ (s-1)^0 & (s-1)^2 & (s-1)^4 \\ (s-2)^0 & (s-2)^2 & (s-2)^4 \end{bmatrix}^{-1} = -\frac{1}{2(2s-1)(2s-2)(2s-3)} \begin{bmatrix} -(s-1)^2(s-2)^2(2s-3) & s^2(s-2)^2(2s-2) & -s^2(s-1)^2(2s-1) \\ [(s-1)^2+(s-2)^2](2s-3) & -[s^2+(s-2)^2]2(2s-2) & [s^2+(s-1)^2](2s-1) \\ -2(2s-3) & 2(2s-2) & -(2s-1) \end{bmatrix}$$

推论9.2.9.

$$\begin{bmatrix} s^k & s^{k+2} & s^{k+4} \\ (s-1)^k & (s-1)^{k+2} & (s-1)^{k+4} \\ (s-2)^k & (s-2)^{k+2} & (s-2)^{k+4} \end{bmatrix}^{-1} = \begin{bmatrix} s^0 & s^2 & s^4 \\ (s-1)^0 & (s-1)^2 & (s-1)^4 \\ (s-2)^0 & (s-2)^2 & (s-2)^4 \end{bmatrix}^{-1} \begin{bmatrix} s^{-k} & 0 & 0 \\ 0 & (s-1)^{-k} & 0 \\ 0 & 0 & (s-2)^{-k} \end{bmatrix}$$

推论9.2.10.

$$\begin{bmatrix} s^0 & s^2 & s^4 & s^6 \\ (s-1)^0 & (s-1)^2 & (s-1)^4 & (s-1)^6 \\ (s-2)^0 & (s-2)^2 & (s-2)^4 & (s-2)^6 \\ (s-3)^0 & (s-3)^2 & (s-3)^4 & (s-3)^6 \end{bmatrix}^{-1} = \frac{1}{12(2s-3)(2s-1)(2s-2)(2s-3)(2s-4)(2s-5)}$$

$$\begin{bmatrix} -2(s-1)^2(s-2)^2(s-3)^2(2s-3)(2s-4)(2s-5), & 6s^2(s-2)^2(s-3)^2(2s-2)(2s-3)(2s-5), & -6s^2(s-1)^2(s-3)^2(2s-1)(2s-3)(2s-4), & 2s^2(s-1)^2(s-2)^2(2s-1)(2s-2) \end{bmatrix}$$

### 9.3 多项式展开系数求解的关键II

$$\text{推论9.3.1. } \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 1^0 & 1^1 & 1^2 & 1^3 & 1^4 & 1^5 & 1^6 \\ 0^0 & 0^1 & 0^2 & 0^3 & 0^4 & 0^5 & 0^6 \\ (-1)^0 & (-1)^1 & (-1)^2 & (-1)^3 & (-1)^4 & (-1)^5 & (-1)^6 \\ (-2)^0 & (-2)^1 & (-2)^2 & (-2)^3 & (-2)^4 & (-2)^5 & (-2)^6 \\ (-3)^0 & (-3)^1 & (-3)^2 & (-3)^3 & (-3)^4 & (-3)^5 & (-3)^6 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{60} & -\frac{9}{60} & \frac{3}{4} & 0 & -\frac{3}{4} & \frac{9}{60} & -\frac{1}{60} \\ \frac{1}{80} & -\frac{7}{60} & \frac{41}{48} & -\frac{3}{2} & \frac{41}{48} & -\frac{7}{60} & \frac{1}{80} \\ -\frac{1}{48} & \frac{1}{6} & -\frac{13}{48} & 0 & \frac{13}{48} & -\frac{1}{6} & \frac{1}{48} \\ -\frac{1}{72} & \frac{1}{8} & -\frac{3}{8} & \frac{19}{36} & -\frac{3}{8} & \frac{1}{8} & -\frac{1}{72} \\ \frac{1}{240} & -\frac{1}{60} & \frac{1}{48} & 0 & -\frac{1}{48} & \frac{1}{60} & -\frac{1}{240} \\ \frac{1}{720} & -\frac{1}{120} & \frac{1}{48} & -\frac{1}{36} & \frac{1}{48} & -\frac{1}{120} & \frac{1}{720} \end{bmatrix}$$

证明:

$$= \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 1^0 & 1^1 & 1^2 & 1^3 & 1^4 & 1^5 & 1^6 \\ 0^0 & 0^1 & 0^2 & 0^3 & 0^4 & 0^5 & 0^6 \\ (-1)^0 & (-1)^1 & (-1)^2 & (-1)^3 & (-1)^4 & (-1)^5 & (-1)^6 \\ (-2)^0 & (-2)^1 & (-2)^2 & (-2)^3 & (-2)^4 & (-2)^5 & (-2)^6 \\ (-3)^0 & (-3)^1 & (-3)^2 & (-3)^3 & (-3)^4 & (-3)^5 & (-3)^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 1^0 & 1^1 & 1^2 & 1^3 & 1^4 & 1^5 & 1^6 \\ 0^0 & 0^1 & 0^2 & 0^3 & 0^4 & 0^5 & 0^6 \\ 2*1^0 & 0 & 2*1^2 & 0 & 2*1^4 & 0 & 2*1^6 \\ 2*2^0 & 0 & 2*2^2 & 0 & 2*2^4 & 0 & 2*2^6 \\ 2*3^0 & 0 & 2*3^2 & 0 & 2*3^4 & 0 & 2*3^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^0 & 3^1 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 \\ 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\ 1^0 & 1^1 & 1^2 & 1^3 & 1^4 & 1^5 & 1^6 \\ 0^0 & 0^1 & 0^2 & 0^3 & 0^4 & 0^5 & 0^6 \\ 1^0 & 0 & 1^2 & 0 & 1^4 & 0 & 1^6 \\ 2^0 & 0 & 2^2 & 0 & 2^4 & 0 & 2^6 \\ 3^0 & 0 & 3^2 & 0 & 3^4 & 0 & 3^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3^1 & 0 & 3^3 & 0 & 3^5 & 0 \\ 0 & 2^1 & 0 & 2^3 & 0 & 2^5 & 0 \\ 0 & 1^1 & 0 & 1^3 & 0 & 1^5 & 0 \\ 1^0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1^0 & 0 & 1^2 & 0 & 1^4 & 0 & 1^6 \\ 2^0 & 0 & 2^2 & 0 & 2^4 & 0 & 2^6 \\ 3^0 & 0 & 3^2 & 0 & 3^4 & 0 & 3^6 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 3^3-3^1 & 0 & 3^5-3^1 & 0 \\ 0 & 0 & 0 & 2^3-2^1 & 0 & 2^5-2^1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2^4-2^2 & 0 & 2^6-2^2 \\ 0 & 0 & 0 & 0 & 3^4-3^2 & 0 & 3^6-3^2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -\frac{3}{2} & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -1 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -2 & 3 & -2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{9}{2} & 8 & -\frac{9}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 3^3-3^1 & 0 & 3^5-3^1 & 0 \\ 0 & 0 & 0 & 2^3-2^1 & 0 & 2^5-2^1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2^3-2^1 & 0 & 2^5-2^1 \\ 0 & 0 & 0 & 0 & 3^3-3^1 & 0 & 3^5-3^1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -\frac{3}{2} & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -1 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & -2 & \frac{3}{2} & -1 & \frac{1}{4} & 0 \\ \frac{1}{6} & 0 & -\frac{3}{2} & \frac{8}{3} & -\frac{3}{2} & 0 & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 24 & 0 & 240 & 0 \\ 0 & 0 & 0 & 6 & 0 & 30 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 30 & 0 \\ 0 & 0 & 0 & 0 & 24 & 0 & 240 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & -\frac{3}{2} & 0 & \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -1 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & -2 & \frac{3}{2} & -1 & \frac{1}{4} & 0 \\ \frac{1}{6} & 0 & -\frac{3}{2} & \frac{8}{3} & -\frac{3}{2} & 0 & \frac{1}{6} \end{bmatrix}$$





$$\begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \cdots & 0 & 0 & 0 & -\frac{n^1}{2} & 0 & \frac{n^1}{2} & 0 & 0 & 0 & \cdots & -\frac{1}{2} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \frac{1}{2} & 0 & 0 & -\frac{4^1}{2} & 0 & \frac{4^1}{2} & 0 & 0 & -\frac{1}{2} & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{2} & 0 & -\frac{3^1}{2} & 0 & \frac{3^1}{2} & 0 & -\frac{1}{2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \frac{1}{2} & -\frac{2^1}{2} & 0 & \frac{2^1}{2} & -\frac{1}{2} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \frac{1}{2 \cdot 1} & -\frac{1}{1} & \frac{1}{2 \cdot 1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \frac{1}{2 \cdot 2} & -\frac{2^1}{2} & 2^1 - \frac{1}{2} & -\frac{2^1}{2} & \frac{1}{2 \cdot 2} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{2 \cdot 3} & 0 & -\frac{3^1}{2} & 3^1 - \frac{1}{3} & -\frac{3^1}{2} & 0 & \frac{1}{2 \cdot 3} & 0 & \cdots & 0 \\ 0 & \cdots & \frac{1}{2 \cdot 4} & 0 & 0 & -\frac{4^1}{2} & 4^1 - \frac{1}{4} & -\frac{4^1}{2} & 0 & 0 & \frac{1}{2 \cdot 4} & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{2 \cdot n} & \cdots & 0 & 0 & 0 & -\frac{n^1}{2} & n^1 - \frac{1}{n} & -\frac{n^1}{2} & 0 & 0 & 0 & \cdots & \frac{1}{2 \cdot n} \end{bmatrix}$$

□

## 10 $[\sigma(s) \cdot \hat{p}]^{2s+1}$ 低阶展开系数的同构性

### 10.1 一个重要的递推关系

引理10.1.1.  $[\sigma(s) \cdot \hat{p}]^{2s+3} = \sum_{k=1}^n [X_1(s)X_k(s) + X_{k+1}(s)][\sigma(s) \cdot \hat{p}]^{2s+1-2k}$

定理10.1.1.

$$[\sigma(s-1) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s-1) \cdot \hat{p}]^{2s+1-2k} \Leftrightarrow X_{k+1}(s) = X_{k+1}(s-1) - s^2 X_k(s-1)$$

$$; [\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s) \cdot \hat{p}]^{2s+1-2k}, X_1(s) = \frac{1}{4} C_{2s+2}^3, X_{[s+1/2]}(s-1) := 0, k = 1, \dots, [s-1/2]$$

证明:  $[\sigma(s-1) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s-1) \cdot \hat{p}]^{2s+1-2k}$

$$\Leftrightarrow [\sigma(s-1) \cdot \hat{p}]^{2s+1} = X_1(s)[\sigma(s-1) \cdot \hat{p}]^{2s-1} + \sum_{k=2}^{[s+1/2]} X_k(s)[\sigma(s-1) \cdot \hat{p}]^{2s+1-2k}$$

$$\Leftrightarrow [\sigma(s-1) \cdot \hat{p}]^{2s+1} = X_1(s)[\sigma(s-1) \cdot \hat{p}]^{2s-1} + \sum_{k=1}^{[s-1/2]} X_{k+1}(s)[\sigma(s-1) \cdot \hat{p}]^{2s-1-2k}$$

$$\Leftrightarrow \sum_{k=1}^{[s-1/2]} X_{k+1}(s)[\sigma(s-1) \cdot \hat{p}]^{2s-1-2k} = [\sigma(s-1) \cdot \hat{p}]^{2s+1} - X_1(s)[\sigma(s-1) \cdot \hat{p}]^{2s-1}$$

$$\Leftrightarrow \sum_{k=1}^{[s-1/2]} X_{k+1}(s)[\sigma(s-1) \cdot \hat{p}]^{2s-1-2k}$$

$$= \sum_{k=1}^{[s-1/2]} [X_1(s-1)X_k(s-1) + X_{k+1}(s-1)][\sigma(s-1) \cdot \hat{p}]^{2s-1-2k} - X_1(s) \sum_{k=1}^{[s-1/2]} X_k(s-1)[\sigma(s-1) \cdot \hat{p}]^{2s-1-2k}$$

$$\Leftrightarrow \sum_{k=1}^{[s-1/2]} X_{k+1}(s)[\sigma(s-1) \cdot \hat{p}]^{2s-1-2k}$$

$$= [X_1(s-1) - X_1(s)] \sum_{k=1}^{[s-1/2]} X_k(s-1)[\sigma(s-1) \cdot \hat{p}]^{2s-1-2k} + \sum_{k=1}^{[s-1/2]} X_{k+1}(s-1)[\sigma(s-1) \cdot \hat{p}]^{2s-1-2k}$$

$$\Leftrightarrow X_{k+1}(s) = [X_1(s-1) - X_1(s)]X_k(s-1) + X_{k+1}(s-1), k = 1, \dots, [s-1/2]$$

$$\Leftrightarrow X_{k+1}(s) = X_{k+1}(s-1) - s^2 X_k(s-1), k = 1, \dots, [s-1/2]$$

□

定理10.1.2.  $\begin{cases} X_{k+1}(n+1) = X_{k+1}(n) - (n+1)^2 X_k(n) \\ X_1(n+1) = \frac{1}{4} C_{2n+4}^3, X_{n+1}(n) := 0, k = 1, \dots, n \end{cases} \Rightarrow \begin{cases} X_k(n) = -(-1)^k C_{\{1^2, 2^2, \dots, n^2\}}^k \\ k = 1, \dots, n \end{cases}$

证明: 采用数学归纳法证明此定理。

第一步:  $i = 1$ 时成立:

$$X_1(1) = -(-1)^1 C_{\{1^2\}}^1$$

第二步: 假设  $i = n$ 时成立:

$$X_k(n) = -(-1)^k C_{\{1^2, 2^2, \dots, n^2\}}^k, k = 1, \dots, n$$

第三步:  $i = n + 1$ 时,

$$\begin{aligned}
X_{k+1}(n+1) &= X_{k+1}(n) - (n+1)^2 X_k(n) = -(-1)^{k+1} C_{\{1^2, 2^2, \dots, n^2\}}^{k+1} + (n+1)^2 (-1)^k C_{\{1^2, 2^2, \dots, n^2\}}^k, k = 1, \dots, n \\
\Leftrightarrow X_{k+1}(n+1) &= -(-1)^{k+1} [C_{\{1^2, 2^2, \dots, n^2\}}^{k+1} + (n+1)^2 C_{\{1^2, 2^2, \dots, n^2\}}^k], k = 1, \dots, n \\
\Leftrightarrow X_{k+1}(n+1) &= -(-1)^{k+1} C_{\{1^2, 2^2, \dots, n^2, (n+1)^2\}}^{k+1}, k = 1, \dots, n \\
\Rightarrow X_k(n+1) &= -(-1)^k C_{\{1^2, 2^2, \dots, n^2, (n+1)^2\}}^{k+1}, k = 1, \dots, n+1
\end{aligned}$$

此步证明了  $i = n+1$  时成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

**定理10.1.3.**

$$\begin{cases} X_{k+1}(n + \frac{1}{2}) = X_{k+1}(n - \frac{1}{2}) - (n + \frac{1}{2})^2 X_k(n - \frac{1}{2}) \\ X_1(n + \frac{1}{2}) = \frac{1}{4} C_{2n+3}^3, X_{n+1}(n - \frac{1}{2}) := 0, k = 1, \dots, n-1 \end{cases} \Rightarrow \begin{cases} X_k(n - \frac{1}{2}) = -(-1)^k C_{\{(1/2)^2, \dots, (n-1/2)^2\}}^k \\ k = 1, \dots, n \end{cases}$$

**证明:** 采用数学归纳法证明此定理。

第一步:  $i = 1$  时成立:

$$X_1(\frac{1}{2}) = -(-1)^1 C_{\{(1/2)^2\}}^1$$

第二步: 假设  $i = n$  时成立:

$$X_k(n - \frac{1}{2}) = -(-1)^k C_{\{(1/2)^2, \dots, (n-1/2)^2\}}^k, k = 1, \dots, n$$

第三步:  $i = n+1$  时,

$$\begin{aligned}
X_{k+1}(n + \frac{1}{2}) &= X_{k+1}(n - \frac{1}{2}) - (n + \frac{1}{2})^2 X_k(n - \frac{1}{2}) \\
&= -(-1)^{k+1} C_{\{(1/2)^2, \dots, (n-1/2)^2\}}^{k+1} + (n+1)^2 (-1)^k C_{\{(1/2)^2, \dots, (n-1/2)^2\}}^k, k = 1, \dots, n \\
\Leftrightarrow X_{k+1}(n + \frac{1}{2}) &= -(-1)^{k+1} [C_{\{(1/2)^2, \dots, (n-1/2)^2\}}^{k+1} + (n + \frac{1}{2})^2 C_{\{(1/2)^2, \dots, (n-1/2)^2\}}^k], k = 1, \dots, n \\
\Leftrightarrow X_{k+1}(n + \frac{1}{2}) &= -(-1)^{k+1} C_{\{(1/2)^2, \dots, (n-1/2)^2, (n+1/2)^2\}}^{k+1}, k = 1, \dots, n \\
\Rightarrow X_k(n + \frac{1}{2}) &= -(-1)^k C_{\{(1/2)^2, \dots, (n-1/2)^2, (n+1/2)^2\}}^{k+1}, k = 1, \dots, n+1
\end{aligned}$$

此步证明了  $i = n+1$  时成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

$$\text{推论10.1.1. } \begin{cases} X_{k+1}(s) = X_{k+1}(s-1) - s^2 X_k(s-1) \\ X_1(s) = \frac{1}{4} C_{2s+2}^3, X_{[s+1/2]}(s-1) := 0, k = 1, \dots, [s-1] \end{cases} \Leftrightarrow \begin{cases} X_k(s) = (-1)^{k+1} C_{\{(1/2|1)^2, \dots, s^2\}}^k \\ k = 1, \dots, [s + \frac{1}{2}] \end{cases}$$

$$\text{推论10.1.2. } X_k(s) = (-1)^{k+1} C_{\{(1/2|1)^2, \dots, s^2\}}^k, k = 1, \dots, [s + \frac{1}{2}] \Leftrightarrow X(s) = - \begin{bmatrix} (-4)^{-1} [C_{\{(1/2)^2, \dots, (2s)^2\}}^1] \\ (-4)^{-2} [C_{\{(1/2)^2, \dots, (2s)^2\}}^2] \\ \vdots \\ (-4)^{-[s+1/2]} [C_{\{(1/2)^2, \dots, (2s)^2\}}^{[s+1/2]}] \end{bmatrix}$$

**推论10.1.3.**

$$\begin{cases} [\sigma(s-1) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\sigma(s-1) \cdot \hat{p}]^{2s+1-2k} \\ [\sigma(s) \cdot \hat{p}]^{2s+1} := \sum_{k=1}^{[s+1/2]} X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k}, X_1(s) := \frac{1}{4} C_{2s+2}^3 \end{cases} \Leftrightarrow \begin{cases} X_k(s) = (-1)^{k+1} C_{\{(1/2|1)^2, \dots, s^2\}}^k \\ k = 1, \dots, [s + \frac{1}{2}] \end{cases}$$

$$\text{推论10.1.4. } [\sigma(s) \cdot \hat{p}]^{2s+1} := \sum_{k=1}^{[s+1/2]} X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k}, X_1(s) := \frac{1}{4} C_{2s+2}^3 \Leftrightarrow \begin{cases} X_k(s) = (-1)^{k+1} C_{\{(1/2|1)^2, \dots, s^2\}}^k \\ k = 1, \dots, [s + \frac{1}{2}] \end{cases}$$

## 11 展开系数的同构性

### 11.1 $z^{2s+1}$ 展开系数的同构性

**定义11.1.1.**  $X_k(s) := (-1)^{k+1} C_{\{(1/2|1)^2, \dots, s^2\}}^k, k = 1, 2, \dots, [s-1]$

$$\text{引理11.1.1. } z^{2s-1} = \sum_{k=1}^{[s-1/2]} X_k(s-1) z^{2s-1-2k} \Rightarrow z^{2s+1} = \sum_{k=1}^{[s-1/2]} [X_1(s-1) X_k(s-1) + X_{k+1}(s-1)] z^{2s-1-2k}$$



$$\begin{aligned}
\text{证明: } z^{2s+1} &= \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s+1-2k} \\
&= X_1(s-1)z^{2s-1} + \sum_{k=1}^{[s-3/2]} X_{k+1}(s-1)z^{2s-1-2k} \\
&= \sum_{k=1}^{[s-1/2]} X_1(s-1)X_k(s-1)z^{2s-1-2k} + \sum_{k=1}^{[s-3/2]} X_{k+1}(s-1)z^{2s-1-2k} \\
&= \sum_{k=1}^{[s-1/2]} [X_1(s-1)X_k(s-1) + X_{k+1}(s-1)]z^{2s-1-2k}
\end{aligned}$$

□

$$\text{定理11.1.1. } z^{2s-1} = \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s-1-2k} \Rightarrow z^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+1-2k}$$

$$\begin{aligned}
\text{证明: } X_{k+1}(s) &= X_{k+1}(s-1) - s^2 X_k(s-1), k=1, \dots, [s-1] \\
\Leftrightarrow X_{k+1}(s) &= [X_1(s-1) - X_1(s)]X_k(s-1) + X_{k+1}(s-1), k=1, \dots, [s-1] \\
\Rightarrow \sum_{k=1}^{[s-1/2]} X_{k+1}(s)z^{2s-1-2k} &= [X_1(s-1) - X_1(s)] \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s-1-2k} + \sum_{k=1}^{[s-3/2]} X_{k+1}(s-1)z^{2s-1-2k} \\
\Leftrightarrow z^{2s+1} &= X_1(s)z^{2s-1} + \sum_{k=1}^{[s-1/2]} X_{k+1}(s)z^{2s-1-2k} \\
\Leftrightarrow z^{2s+1} &= X_1(s)z^{2s-1} + \sum_{k=2}^{[s+1/2]} X_k(s)z^{2s+1-2k} \\
\Leftrightarrow z^{2s+1} &= \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+1-2k}
\end{aligned}$$

□

$$\text{推论11.1.1. } z^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+1-2k} \Rightarrow z^{2(s+l)+1} = \sum_{k=1}^{[(s+l)+1/2]} X_k(s+l)z^{2(s+l)+1-2k}, l \geq 0$$

$$\text{推论11.1.2. } \begin{cases} [\sigma(\frac{1}{2}) \cdot \hat{p}]^2 = \frac{1}{4} \Rightarrow [\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n} = \sum_{k=1}^n X_k(n - \frac{1}{2})[\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n-2k}, n \geq 1 \\ [\sigma(1) \cdot \hat{p}]^3 = [\sigma(1) \cdot \hat{p}] \Rightarrow [\sigma(1) \cdot \hat{p}]^{2n+1} = \sum_{k=1}^n X_k(n)[\sigma(1) \cdot \hat{p}]^{2n+1-2k}, n \geq 1 \\ [2\sigma(\frac{1}{2}) \cdot \hat{p}]^3 = [2\sigma(\frac{1}{2}) \cdot \hat{p}] \Rightarrow [2\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n+1} = \sum_{k=1}^n X_k(n)[2\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n+1-2k}, n \geq 1 \end{cases}$$

$$\text{推论11.1.3. } \begin{cases} (\frac{1}{2})^2 = \frac{1}{4} \Rightarrow (\frac{1}{2})^{2n} = \sum_{k=1}^n X_k(n - \frac{1}{2})(\frac{1}{2})^{2n-2k}, n \geq 1 \Leftrightarrow \sum_{k=1}^n 4^k X_k(n - \frac{1}{2}) = 1, n \geq 1 \\ 1^3 = 1 \Rightarrow 1^{2n+1} = \sum_{k=1}^n X_k(n)1^{2n+1-2k}, n \geq 1 \Leftrightarrow \sum_{k=1}^n X_k(n) = 1, n \geq 1 \end{cases}$$

## 11.2 $z^{2s+2}$ 展开系数的同构性

定义11.2.1.  $X_k(s) := (-1)^{k+1} C_{\{(1/2|1)^2, \dots, s^2\}}^k, k=1, 2, \dots, [s-1]$

$$\text{引理11.2.1. } z^{2s} = \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s-2k} \Rightarrow z^{2s+2} = \sum_{k=1}^{[s-1/2]} [X_1(s-1)X_k(s-1) + X_{k+1}(s-1)]z^{2s-2k}$$

$$\begin{aligned}
\text{证明: } z^{2s+2} &= \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s+2-2k} \\
&= X_1(s-1)z^{2s} + \sum_{k=1}^{[s-3/2]} X_{k+1}(s-1)z^{2s-2k} \\
&= \sum_{k=1}^{[s-1/2]} X_1(s-1)X_k(s-1)z^{2s-2k} + \sum_{k=1}^{[s-3/2]} X_{k+1}(s-1)z^{2s-2k} \\
&= \sum_{k=1}^{[s-1/2]} [X_1(s-1)X_k(s-1) + X_{k+1}(s-1)]z^{2s-2k}
\end{aligned}$$

□

$$\text{定理11.2.1. } z^{2s} = \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s-2k} \Rightarrow z^{2s+2} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+2-2k}$$

证明:  $X_{k+1}(s) = X_{k+1}(s-1) - s^2 X_k(s-1), k = 1, \dots, [s-1/2]$

$\Leftrightarrow X_{k+1}(s) = [X_1(s-1) - X_1(s)]X_k(s-1) + X_{k+1}(s-1), k = 1, \dots, [s-1/2]$

$\Rightarrow \sum_{k=1}^{[s-1/2]} X_{k+1}(s)z^{2s-2k} = [X_1(s-1) - X_1(s)] \sum_{k=1}^{[s-1/2]} X_k(s-1)z^{2s-2k} + \sum_{k=1}^{[s-3/2]} X_{k+1}(s-1)z^{2s-2k}$

$\Leftrightarrow z^{2s+2} = X_1(s)z^{2s} + \sum_{k=1}^{[s-1/2]} X_{k+1}(s)z^{2s-2k}$

$\Leftrightarrow z^{2s+2} = X_1(s)z^{2s} + \sum_{k=2}^{[s+1/2]} X_k(s)z^{2s+2-2k}$

$\Leftrightarrow z^{2s+2} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+2-2k}$

□

推论11.2.1.  $z^{2s+2} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+2-2k} \Rightarrow z^{2(s+l)+2} = \sum_{k=1}^{[(s+l)+1/2]} X_k(s+l)z^{2(s+l)+2-2k}, l \geq 0$

推论11.2.2.  $\begin{cases} [\sigma(\frac{1}{2}) \cdot \hat{p}]^3 = \frac{1}{4}[\sigma(\frac{1}{2}) \cdot \hat{p}] \Rightarrow [\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n+1} = \sum_{k=1}^n X_k(n - \frac{1}{2})[\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n+1-2k}, n \geq 1 \\ [\frac{1}{2}\sigma(1) \cdot \hat{p}]^3 = \frac{1}{4}[\frac{1}{2}\sigma(1) \cdot \hat{p}] \Rightarrow [\frac{1}{2}\sigma(1) \cdot \hat{p}]^{2n+1} = \sum_{k=1}^n X_k(n - \frac{1}{2})[\frac{1}{2}\sigma(1) \cdot \hat{p}]^{2n+1-2k}, n \geq 1 \\ [\sigma(1) \cdot \hat{p}]^4 = [\sigma(1) \cdot \hat{p}]^2 \Rightarrow [\sigma(1) \cdot \hat{p}]^{2n+2} = \sum_{k=1}^n X_k(n)[\sigma(1) \cdot \hat{p}]^{2n+2-2k}, n \geq 1 \\ [2\sigma(\frac{1}{2}) \cdot \hat{p}]^4 = [2\sigma(\frac{1}{2}) \cdot \hat{p}]^2 \Rightarrow [2\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n+2} = \sum_{k=1}^n X_k(n)[2\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n+2-2k}, n \geq 1 \end{cases}$

### 11.3 $[\sigma(s) \cdot \hat{p}]^{2s+1}$ 展开系数的同构性

推论11.3.1.

$[\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s) \cdot \hat{p}]^{2s+1-2k} [\Rightarrow] [\sigma(s) \cdot \hat{p}]^{2(s+l)+1} = \sum_{k=1}^{[(s+l)+1/2]} X_k(s+l)[\sigma(s) \cdot \hat{p}]^{2(s+l)+1-2k}, l \geq 0$

推论11.3.2.  $[\sigma(s-l) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s-l) \cdot \hat{p}]^{2s+1-2k}; 0 \leq l \leq [s], s \geq \frac{1}{2}$

### 11.4 $[\sigma(s) \cdot \hat{p}]^{2s+2}$ 展开系数的同构性

推论11.4.1.

$[\sigma(s) \cdot \hat{p}]^{2s+2} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s) \cdot \hat{p}]^{2s+2-2k} [\Rightarrow] [\sigma(s) \cdot \hat{p}]^{2(s+l)+2} = \sum_{k=1}^{[(s+l)+1/2]} X_k(s+l)[\sigma(s) \cdot \hat{p}]^{2(s+l)+2-2k}, l \geq 0$

推论11.4.2.  $[\sigma(s-l) \cdot \hat{p}]^{2s+2} = \sum_{k=1}^{[s+1/2]} X_k(s)[\sigma(s-l) \cdot \hat{p}]^{2s+2-2k}; 0 \leq l \leq [s], s \geq \frac{1}{2}$

推论11.4.3.  $\begin{cases} [\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n} = \sum_{k=1}^n X_k(n - \frac{1}{2})[\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n-2k}, [\frac{1}{2}\sigma(1) \cdot \hat{p}]^{2n+1} = \sum_{k=1}^n X_k(n - \frac{1}{2})[\frac{1}{2}\sigma(1) \cdot \hat{p}]^{2n+1-2k}, n \geq 1 \\ [2\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n+1} = \sum_{k=1}^n X_k(n)[2\sigma(\frac{1}{2}) \cdot \hat{p}]^{2n+1-2k}, [\sigma(1) \cdot \hat{p}]^{2n+1} = \sum_{k=1}^n X_k(n)[\sigma(1) \cdot \hat{p}]^{2n+1-2k}, n \geq 1 \end{cases}$

## 12 $e^{\vec{\rho} \cdot \sigma(s)}$ 低阶展开系数的同构性

### 12.1 一个重要定理及其推论

定理12.1.1.

$z^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+1-2k} \Leftrightarrow \begin{cases} z^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+1-2k} \\ z^l = \sum_{k=0}^{2s} c(l, k; s)z^k, l \geq 2s+1 \end{cases} \Rightarrow \begin{cases} e^{\rho z} = \sum_{k=0}^{2s} A_k(s)(\rho z)^k \\ A_k(s) := \frac{1}{k!} + \sum_{l=2s+1}^{+\infty} \frac{\rho^{l-k}}{l!} c(l, k; s) \end{cases}$

证明:  $z^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s)z^{2s+1-2k}, z^l = \sum_{k=0}^{2s} c(l, k; s)z^k, l \geq 2s+1$

$\Rightarrow e^{\rho z} = \sum_{k=0}^{+\infty} \frac{\rho^k}{k!} z^k$

$$\begin{aligned}
&= \sum_{k=0}^{2s} \frac{\rho^k}{k!} z^k + \sum_{k=2s+1}^{+\infty} \frac{\rho^k}{k!} z^k \\
&= \sum_{k=0}^{2s} \frac{\rho^k}{k!} z^k + \sum_{l=2s+1}^{+\infty} \frac{\rho^l}{l!} \sum_{k=0}^{2s} c(l, k; s) z^k \\
&= \sum_{k=0}^{2s} \left[ \frac{\rho^k}{k!} + \sum_{l=2s+1}^{+\infty} \frac{\rho^l}{l!} c(l, k; s) \right] z^k \\
&= \sum_{k=0}^{2s} A_k(s) (\rho z)^k, A_k(s) := \frac{1}{k!} + \sum_{l=2s+1}^{+\infty} \frac{\rho^{l-k}}{l!} c(l, k; s)
\end{aligned}$$

□

$$\text{定理12.1.2. } z^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) z^{2s+1-2k} \Rightarrow e^{\rho z} = \sum_{k=0}^{2(s+l)} A_k(s+l) (\rho z)^k = \sum_{k=0}^{+\infty} \frac{1}{k!} (\rho z)^k$$

## 12.2 一个重要猜想及其推论

$$\text{猜想12.2.1. } [\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k}$$

$$\text{推论12.2.1. } [\sigma(s-l) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\sigma(s-l) \cdot \hat{p}]^{2s+1-2k}, l = 0, 1, \dots, [s]$$

$$\text{推论12.2.2. } e^{\vec{\vartheta} \cdot \sigma(s-l)} = \sum_{k=0}^{2s} A_k(s) [\vec{\vartheta} \cdot \sigma(s-l)]^k, l = 0, 1, \dots, [s]$$

## 12.3 猜想推论: $e^{\vec{\vartheta} \cdot \sigma(n-\frac{1}{2})}$ 多项式展开系数之间的关系

$$\text{定理12.3.1. } e^{\vec{\vartheta} \cdot \sigma(\frac{1}{2})} = \sum_{k=0}^{2n-1} A_k(n-\frac{1}{2}) [\vec{\vartheta} \cdot \sigma(\frac{1}{2})]^k$$

$$\text{推论12.3.1. } \begin{cases} e^{\frac{1}{2}\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n-1} \frac{A_k(n-\frac{1}{2})}{2^k} [\sqrt{\vec{\vartheta}^2}]^k \\ e^{-\frac{1}{2}\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n-1} \frac{A_k(n-\frac{1}{2})}{2^k} [-\sqrt{\vec{\vartheta}^2}]^k \end{cases} \Leftrightarrow \begin{cases} \cosh \frac{\sqrt{\vec{\vartheta}^2}}{2} = \sum_{i=0}^{n-1} \frac{A_{2i}(n-\frac{1}{2})}{2^{2i}} [\sqrt{\vec{\vartheta}^2}]^{2i} \\ \sinh \frac{\sqrt{\vec{\vartheta}^2}}{2} = \sum_{i=0}^{n-1} \frac{A_{2i+1}(n-\frac{1}{2})}{2^{2i+1}} [\sqrt{\vec{\vartheta}^2}]^{2i+1} \end{cases}$$

## 12.4 猜想推论: $e^{\vec{\vartheta} \cdot \sigma(n)}$ 多项式展开系数之间的关系

$$\text{定理12.4.1. } e^{\vec{\vartheta} \cdot \sigma(1)} = \sum_{k=0}^{2n} A_k(n) [\vec{\vartheta} \cdot \sigma(1)]^k, e^{\vec{\vartheta} \cdot \sigma} = \sum_{k=0}^{2s} A_k(n) (\vec{\vartheta} \cdot \sigma)^k$$

$$\text{推论12.4.1. } \begin{cases} e^{\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n} A_k(n) [\sqrt{\vec{\vartheta}^2}]^k \\ A_0(n) = 1 \\ e^{-\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n} A_k(n) [-\sqrt{\vec{\vartheta}^2}]^k \end{cases} \Leftrightarrow \begin{cases} \cosh \sqrt{\vec{\vartheta}^2} = \sum_{i=0}^n A_{2i}(n) [\sqrt{\vec{\vartheta}^2}]^{2i} \\ A_0(n) = 1 \\ \sinh \sqrt{\vec{\vartheta}^2} = \sum_{i=0}^{n-1} A_{2i+1}(n) [\sqrt{\vec{\vartheta}^2}]^{2i+1} \end{cases}$$

$$\text{推论12.4.2. } \begin{cases} e^{\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n} A_k(n) [\sqrt{\vec{\vartheta}^2}]^k \\ e^{-\sqrt{\vec{\vartheta}^2}} = \sum_{k=0}^{2n} A_k(n) [-\sqrt{\vec{\vartheta}^2}]^k \end{cases} \Leftrightarrow \begin{cases} \cosh \sqrt{\vec{\vartheta}^2} = \sum_{i=0}^n A_{2i}(n) [\sqrt{\vec{\vartheta}^2}]^{2i} \\ \sinh \sqrt{\vec{\vartheta}^2} = \sum_{i=0}^{n-1} A_{2i+1}(n) [\sqrt{\vec{\vartheta}^2}]^{2i+1} \end{cases}$$

## 12.5 $e^{\vec{\vartheta} \cdot \sigma(s)}$ 泰勒展开系数的相等性

$$\text{定理12.5.1. } \lim_{s \rightarrow +\infty} A_k(s) = \frac{1}{k!}, e^{\vec{\vartheta} \cdot \sigma(s)} = \sum_{k=0}^{+\infty} \frac{1}{k!} [\vec{\vartheta} \cdot \sigma(s)]^k$$

## 12.6 推理过程的梳理

首先有以下猜想, 可以利用多项式展开定理和自然数拆分的方式加以证明, 但只在低阶情形得到了严格证明, 一般情形仍是猜想, 真正没被严格证明的是这个猜想, 其它均可严格加以证明。

$$\text{猜想12.6.1. } [\sigma(s) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\sigma(s) \cdot \hat{p}]^{2s+1-2k}$$

有了以上猜想后，一是可以通过线性代数方法直接求出  $X_k(s)$ ，但也只在低阶情形得到了严格求解，一般情形本质上仍是猜出来的；二是通过以上推理严格得到递推关系，并可以完全严格解出系数。以下两个同构性的推论，一是可以从以上猜想和系数完全推理出来；二是可以从高级表象变换技术推理，但不能具体推得系数。

$$\text{推论12.6.1. } [\sigma(s-l) \cdot \hat{p}]^{2s+1} = \sum_{k=1}^{[s+1/2]} X_k(s) [\sigma(s-l) \cdot \hat{p}]^{2s+1-2k}, l = 0, 1, \dots, [s]$$

$$\text{推论12.6.2. } e^{\vec{\vartheta} \cdot \sigma(s-l)} = \sum_{k=0}^{2s} A_k(s) [\vec{\vartheta} \cdot \sigma(s-l)]^k, l = 0, 1, \dots, [s]$$

# 第十七章 自旋代数的推广

自我评述：在本章节中为了进一步研究一般自旋粒子的物理内容，只以 $\sigma(s; w) \times \sigma(s; w) = i\sigma(s; w)$ 为唯一前提条件，我进一步推广发展了关于自旋代数更一般的数学分析，为研究各种自旋粒子提供了更广泛的数学基础。

## 1 自旋-单位矢量算符的基本代数性质

### 1.1 $\sigma(s; w)$ 的基本前提性质

定义1.1.1.  $\sigma(s; w) \times \sigma(s; w) = i\sigma(s; w), \sigma(s; 1) := \sigma(s)$

引理1.1.1.  $\sigma(s; w) \times \sigma(s; w) = i\sigma(s; w) \Rightarrow [\sigma_k(s; w), \sigma^2(s; w)] = 0; k = x, y, z$

证明:  $[\sigma_x(s; w), \sigma^2(s; w)]$

$$\begin{aligned}
 &= [\sigma_x(s; w), \sigma_x^2(s; w) + \sigma_y^2(s; w) + \sigma_z^2(s; w)] \\
 &= [\sigma_x(s; w), \sigma_y^2(s; w)] + [\sigma_x(s; w), \sigma_z^2(s; w)] \\
 &= [\sigma_x(s; w)\sigma_y^2(s; w) - \sigma_y^2(s; w)\sigma_x(s; w)] + [\sigma_x(s; w)\sigma_z^2(s; w) - \sigma_z^2(s; w)\sigma_x(s; w)] \\
 &= [\sigma_x(s; w)\sigma_y^2(s; w) - \sigma_y(s; w)\sigma_x(s; w)\sigma_y(s; w) + \sigma_y(s; w)\sigma_x(s; w)\sigma_y(s; w) - \sigma_y^2(s; w)\sigma_x(s; w)] \\
 &+ [\sigma_x(s; w)\sigma_z^2(s; w) - \sigma_z(s; w)\sigma_x(s; w)\sigma_z(s; w) + \sigma_z(s; w)\sigma_x(s; w)\sigma_z(s; w) - \sigma_z^2(s; w)\sigma_x(s; w)] \\
 &= [\sigma_x(s; w)\sigma_y(s; w) - \sigma_y(s; w)\sigma_x(s; w)]\sigma_y(s; w) + \sigma_y(s; w)[\sigma_x(s; w)\sigma_y(s; w) - \sigma_y(s; w)\sigma_x(s; w)] \\
 &+ [\sigma_x(s; w)\sigma_z(s; w) - \sigma_z(s; w)\sigma_x(s; w)]\sigma_z(s; w) + \sigma_z(s; w)[\sigma_x(s; w)\sigma_z(s; w) - \sigma_z(s; w)\sigma_x(s; w)] \\
 &= [\sigma_x(s; w), \sigma_y(s; w)]\sigma_y(s; w) + \sigma_y(s; w)[\sigma_x(s; w), \sigma_y(s; w)] \\
 &+ [\sigma_x(s; w), \sigma_z(s; w)]\sigma_z(s; w) + \sigma_z(s; w)[\sigma_x(s; w), \sigma_z(s; w)] \\
 &= i\sigma_z(s; w)\sigma_y(s; w) + i\sigma_y(s; w)\sigma_z(s; w) \\
 &- i\sigma_y(s; w)\sigma_z(s; w) - i\sigma_z(s; w)\sigma_y(s; w) \\
 &= 0
 \end{aligned}$$

□

### 1.2 $\sigma(s; w), \vec{p}$ 的基本运算规则

性质1.2.1. 
$$\begin{cases} \sigma(s; w) \times \vec{p} = [\sigma(s; w), i\sigma(s; w) \cdot \vec{p}] = i\{\sigma(s; w)[\sigma(s; w) \cdot \vec{p}] - [\sigma(s; w) \cdot \vec{p}]\sigma(s; w)\} \\ \vec{p} \times \sigma(s; w) = -[\sigma(s; w), i\sigma(s; w) \cdot \vec{p}] = i\{[\sigma(s; w) \cdot \vec{p}]\sigma(s; w) - \sigma(s; w)[\sigma(s; w) \cdot \vec{p}]\} \end{cases}$$

性质1.2.2. 
$$\begin{cases} \sigma(s; w) \times \vec{p} = -\vec{p} \times \sigma(s; w) & \begin{cases} \sigma(s; w) \cdot \vec{p} = \vec{p} \cdot \sigma(s; w) \\ [\sigma(s; w), \sigma^2(s; w)] = 0 \end{cases} & \begin{cases} [\sigma(s; w) \times \vec{p}] \cdot \vec{p} = 0 \\ \sigma(s; w) \cdot [\sigma(s; w) \times \vec{p}] = i\sigma(s; w) \cdot \vec{p} \\ [\vec{p} \times \sigma(s; w)] \cdot \sigma(s; w) = i\sigma(s; w) \cdot \vec{p} \end{cases} \end{cases}$$

性质1.2.3.  $\sigma(s; w) \cdot \vec{p} = \sigma^2(s; w)[\sigma(s; w) \cdot \vec{p}] - \sigma(s; w) \cdot [\sigma(s; w) \cdot \vec{p}]\sigma(s; w)$

性质1.2.4. 
$$\begin{cases} [\sigma(s; w) \times \vec{p}] \times \vec{p} = [\sigma(s; w) \cdot \vec{p}]\vec{p} - \sigma(s; w)(\vec{p} \cdot \vec{p}) \\ \sigma(s; w) \times [\sigma(s; w) \times \vec{p}] = [\sigma(s; w) \cdot \vec{p}]\sigma(s; w) - \sigma^2(s; w)\vec{p} \\ [\vec{p} \times \sigma(s; w)] \times \sigma(s; w) = \sigma(s; w)[\sigma(s; w) \cdot \vec{p}] - \sigma^2(s; w)\vec{p} \\ \sigma(s; w) \times \{\sigma(s; w) \times [\sigma(s; w) \times \vec{p}]\} = \sigma(s; w) \times \{[\sigma(s; w) \cdot \vec{p}]\sigma(s; w)\} - \sigma(s; w) \times \vec{p}\sigma^2(s; w) \end{cases}$$

性质1.2.5. 
$$\begin{cases} \{[\sigma(s; w) \times \vec{p}] \times \vec{p}\} \cdot \vec{p} = 0 \\ \{\sigma(s; w) \times [\sigma(s; w) \times \vec{p}]\} \cdot \sigma(s; w) = 0 \\ \{\sigma(s; w) \cdot \{[\vec{p} \times \sigma(s; w)] \times \sigma(s; w)\}\} = 0 \end{cases} \quad \begin{cases} \vec{p} \cdot \{[\sigma(s; w) \times \vec{p}] \times \vec{p}\} = 0 \\ \sigma(s; w) \cdot \{\sigma(s; w) \times [\sigma(s; w) \times \vec{p}]\} = -\sigma(s; w) \cdot \vec{p} \\ \{[\vec{p} \times \sigma(s; w)] \times \sigma(s; w)\} \cdot \sigma(s; w) = -\sigma(s; w) \cdot \vec{p} \end{cases}$$

$$\text{性质1.2.6.} \left\{ \begin{array}{l} \{[\sigma(s; w) \times \vec{p}] \times \vec{p}\} \cdot \sigma(s; w) = [\sigma(s; w) \cdot \vec{p}]^2 - \sigma^2(s; w) \vec{p}^2 \\ \sigma(s; w) \cdot \{[\sigma(s; w) \times \vec{p}] \times \vec{p}\} = [\sigma(s; w) \cdot \vec{p}]^2 - \sigma^2(s; w) \vec{p}^2 \\ \{\sigma(s; w) \times [\sigma(s; w) \times \vec{p}]\} \cdot \vec{p} = [\sigma(s; w) \cdot \vec{p}]^2 - \sigma^2(s; w) \vec{p}^2 \\ \vec{p} \cdot \{[\vec{p} \times \sigma(s; w)] \times \sigma(s; w)\} = [\sigma(s; w) \cdot \vec{p}]^2 - \sigma^2(s; w) \vec{p}^2 \\ -[\sigma(s; w) \times \vec{p}] \cdot [\sigma(s; w) \times \vec{p}] = [\sigma(s; w) \cdot \vec{p}]^2 - \sigma^2(s; w) \vec{p}^2 \end{array} \right.$$

$$\text{性质1.2.7.} \left\{ \begin{array}{l} [\sigma(s; w) \times \vec{p}] \times [\sigma(s; w) \times \vec{p}] = i[\sigma(s; w) \cdot \vec{p}] \vec{p} \\ \{[\sigma(s; w) \times \vec{p}] \times [\sigma(s; w) \times \vec{p}]\} \cdot \vec{p} = i[\sigma(s; w) \cdot \vec{p}] \vec{p}^2 \\ \sigma(s; w) \cdot \{[\sigma(s; w) \times \vec{p}] \times [\sigma(s; w) \times \vec{p}]\} = i[\sigma(s; w) \cdot \vec{p}]^2 \\ \{[\sigma(s; w) \times \vec{p}] \times [\sigma(s; w) \times \vec{p}]\} \cdot \sigma(s; w) = i[\sigma(s; w) \cdot \vec{p}]^2 \end{array} \right.$$

$$\text{性质1.2.8.} \left\{ \begin{array}{l} [\sigma(s; w) \times \vec{p}] \times \{[\sigma(s; w) \times \vec{p}] \times [\sigma(s; w) \times \vec{p}]\} = i\{[\sigma(s; w) \times \vec{p}] \times \vec{p}\} [\sigma(s; w) \cdot \vec{p}] \\ \{[\sigma(s; w) \times \vec{p}] \times [\sigma(s; w) \times \vec{p}]\} \times [\sigma(s; w) \times \vec{p}] = -i[\sigma(s; w) \cdot \vec{p}] \{[\sigma(s; w) \times \vec{p}] \times \vec{p}\} \\ [\sigma(s; w) \times \vec{p}] \cdot \{[\sigma(s; w) \times \vec{p}] \times [\sigma(s; w) \times \vec{p}]\} = 0 \\ \{[\sigma(s; w) \times \vec{p}] \times [\sigma(s; w) \times \vec{p}]\} \cdot [\sigma(s; w) \times \vec{p}] = 0 \end{array} \right.$$

$$\text{性质1.2.9.} \left\{ \begin{array}{l} \{[\sigma(s; w) \times \hat{p}] \times \hat{p}\} \times \vec{p} = -\sigma(s; w) \times \hat{p} \\ i^{2k-1} \hat{p} \cdot \{\sigma(s; w) \times \{\sigma(s; w) [\times \hat{p}]^{2k-1}\}\} = i\{[\sigma(s; w) \cdot \hat{p}]^2 - \sigma^2(s; w)\} \\ i^{2k} \hat{p} \cdot \{\sigma(s; w) \times \{\sigma(s; w) [\times \hat{p}]^{2k}\}\} = i\sigma(s; w) \cdot \hat{p} \end{array} \right.$$

## 2 基本展开式

### 2.1 $\sigma(s; w)[\times \hat{p}]^n$ 和 $i^n[\hat{p} \times ]^n \sigma(s; w)$ 的两种展开式

#### 2.1.1 $i^n \sigma(s; w)[\times \hat{p}]^n$ 和 $i^n[\hat{p} \times ]^n \sigma(s; w)$ 的简单展开式和通项公式

$$\text{性质2.1.1.} \quad \sigma(s; w)[\times \hat{p}]^1 = \sigma(s; w) \times \hat{p}, \sigma(s; w)[\times \hat{p}]^2 = [\sigma(s; w) \cdot \hat{p}] \hat{p} - \sigma(s; w), \sigma(s; w)[\times \hat{p}]^3 = -\sigma(s; w) \times \hat{p}$$

定理2.1.1.  $k \geq 1$

$$i^n \sigma(s; w)[\times \hat{p}]^n = \begin{cases} i\sigma(s; w) \times \hat{p}, n = 2k - 1 \\ \sigma(s; w) - [\sigma(s; w) \cdot \hat{p}] \hat{p}, n = 2k \end{cases}, i^n[\hat{p} \times ]^n \sigma(s; w) = \begin{cases} i\hat{p} \times \sigma(s; w), n = 2k - 1 \\ \sigma(s; w) - [\sigma(s; w) \cdot \hat{p}] \hat{p}, n = 2k \end{cases}$$

$$\text{推论2.1.1.} \quad \sigma(s; w)[\times \hat{p}]^n = (-1)^n[\hat{p} \times ]^n \sigma(s; w), \sigma(s; w)[\times \hat{p}]^n \cdot \hat{p} = (-1)^n \hat{p} \cdot [[\hat{p} \times ]^n \sigma(s; w)] = 0, n \geq 1$$

推论2.1.2.  $k \geq 1$

$$i^n \sigma(s; w) \cdot \{\sigma(s; w)[\times \hat{p}]^n\} = i^n \{[\hat{p} \times ]^n \sigma(s; w)\} \cdot \sigma(s; w) = \begin{cases} -\sigma(s; w) \cdot \hat{p}, n = 2k - 1 \\ -\{[\sigma(s; w) \cdot \hat{p}]^2 - \sigma^2(s; w)\}, n = 2k \end{cases}$$

推论2.1.3.  $n \geq 1$

$$i^n \sigma(s; w) \cdot \{\sigma(s; w)[\times \hat{p}]^n\} = i^n \{[\hat{p} \times ]^n \sigma(s; w)\} \cdot \sigma(s; w) = -\{[\sigma(s; w) \cdot \hat{p}]^{2-n\%2} - (1 - n\%2)\sigma^2(s; w)\}$$

推论2.1.4.

$$\begin{cases} \sigma(s; w)[\times \hat{p}]^{2k-1} = (-1)^{k+1} \sigma(s; w) \times \hat{p} \\ \sigma(s; w)[\times \hat{p}]^{2k} = (-1)^{k+1} \{[\sigma(s; w) \cdot \hat{p}] \hat{p} - \sigma(s; w)\} \end{cases} \Rightarrow \begin{cases} i^{2k-1} i \hat{p} \cdot \{\sigma(s; w) \times \{\sigma(s; w)[\times \hat{p}]^{2k-1}\}\} = -\{[\sigma(s; w) \cdot \hat{p}]^2 - \sigma^2(s; w)\} \\ i^{2k} i \hat{p} \cdot \{\sigma(s; w) \times \{\sigma(s; w)[\times \hat{p}]^{2k}\}\} = -\sigma(s; w) \cdot \hat{p} \end{cases}$$

### 2.1.2 $\sigma(s; w)[\times \hat{p}]^n$ 和 $i^n[\hat{p} \times ]^n \sigma(s; w)$ 的类二项式展开

性质2.1.2.  $\sigma(s; w) \times \hat{p} = [\sigma(s; w), i\sigma(s; w) \cdot \hat{p}] = i\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)\}$   
 $= i\{\sigma(s; w)A + B\sigma(s; w)\}$

性质2.1.3.  $\sigma(s; w) \times \hat{p} \times \hat{p} = i^2\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^2 - 2[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + [\sigma(s; w) \cdot \hat{p}]^2\sigma(s; w)\}$   
 $= i^2[\sigma(s; w)A^2 + 2B\sigma(s; w)A + B^2\sigma(s; w)] \simeq i^2[\sigma^{\frac{1}{2}}(s; w)A + B\sigma^{\frac{1}{2}}(s; w)]_{B \parallel A}^2$

定理2.1.2.  $\sigma(s; w)[\times \hat{p}]^n = i^n \sum_{k=0}^n C_n^k [-\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n-k}, n \geq 1$

证明：采用数学归纳法证明：

第一步： $i = 1$ 时成立，即  $\sigma(s; w)[\times \hat{p}]^1 = i^1 \sum_{k=0}^1 C_1^k [-\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{1-k}$

第二步：假设  $i = n$ 时成立，即  $\sigma(s; w)[\times \hat{p}]^n = i^n \sum_{k=0}^n C_n^k [-\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n-k}$

第三步： $i = n + 1$ 时

$$\begin{aligned} \sigma(s; w)[\times \hat{p}]^{n+1} &= \sigma(s; w)[\times \hat{p}]^n \times \hat{p} \\ &= i^n \sum_{k=0}^n C_n^k [-\sigma(s; w) \cdot \hat{p}]^k [\sigma(s; w) \times \hat{p}] [\sigma(s; w) \cdot \hat{p}]^{n-k} \\ &= i^{n+1} \sum_{k=0}^n C_n^k [-\sigma(s; w) \cdot \hat{p}]^k \{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)\} [\sigma(s; w) \cdot \hat{p}]^{n-k} \\ &= i^{n+1} \sum_{k=0}^n C_n^k [-\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n+1-k} + i^{n+1} \sum_{k=0}^n C_n^k [-\sigma(s; w) \cdot \hat{p}]^{k+1} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n-k} \\ &= i^{n+1} \sum_{k=0}^n C_n^k [-\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n+1-k} + i^{n+1} \sum_{k=1}^{n+1} C_n^{k-1} [-\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n+1-k} \\ &= i^{n+1} \sum_{k=0}^{n+1} C_n^k [-\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n+1-k} + i^{n+1} \sum_{k=0}^{n+1} C_n^{k-1} [-\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n+1-k} \\ &= i^{n+1} \sum_{k=0}^{n+1} (C_n^k + C_n^{k-1}) [-\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n+1-k} \\ &= i^{n+1} \sum_{k=0}^{n+1} C_{n+1}^k [-\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n+1-k} \end{aligned}$$

此步证明了  $i = n + 1$ 时成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

$$\text{推论2.1.5. } \begin{cases} i^n \sigma(s; w)[\times \hat{p}]^n = \sum_{k=0}^n C_n^k [\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [-\sigma(s; w) \cdot \hat{p}]^{n-k}, n \geq 0 \\ i^n [\hat{p} \times ]^n \sigma(s; w) = \sum_{k=0}^n C_n^k [-\sigma(s; w) \cdot \hat{p}]^{n-k} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^k, n \geq 0 \end{cases}$$

### 2.2 $[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)$ 和 $\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^n$ 的递推公式

定理2.2.1.

$$\begin{cases} [\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w) = i^n \sigma(s; w)[\times \hat{p}]^n - \sum_{k=0}^{n-1} C_n^k [\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [-\sigma(s; w) \cdot \hat{p}]^{n-k}, n \geq 0 \\ \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^n = i^n [\hat{p} \times ]^n \sigma(s; w) - \sum_{k=0}^{n-1} C_n^k [-\sigma(s; w) \cdot \hat{p}]^{n-k} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^k, n \geq 0 \end{cases}$$

推论2.2.1.

$$\begin{cases} \sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n \sigma_\alpha(s; w) \\ = i^n \sigma(s; w) \cdot \{\sigma(s; w)[\times \hat{p}]^n\} - \sum_{k=0}^{n-1} C_n^k \sigma^\alpha(s; w) [\sigma(s; w) \cdot \hat{p}]^k \sigma_\alpha(s; w) [-\sigma(s; w) \cdot \hat{p}]^{n-k}, n \geq 0 \\ \sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n \sigma_\alpha(s; w) \\ = i^n \{[\hat{p} \times ]^n \sigma(s; w)\} \cdot \sigma(s; w) - \sum_{k=0}^{n-1} C_n^k [-\sigma(s; w) \cdot \hat{p}]^{n-k} \sigma^\alpha(s; w) [\sigma(s; w) \cdot \hat{p}]^k \sigma_\alpha(s; w), n \geq 0 \end{cases}$$

推论2.2.2.

$$\begin{cases} \sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\} \\ = i^n \sigma(s; w) \times \{\sigma(s; w)[\times \hat{p}]^n\} - \sum_{k=0}^{n-1} C_n^k \sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w)\} [-\sigma(s; w) \cdot \hat{p}]^{n-k}, n \geq 0 \\ \{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^n\} \times \sigma(s; w) \\ = i^n \{[\hat{p} \times ]^n \sigma(s; w)\} \times \sigma(s; w) - \sum_{k=0}^{n-1} C_n^k [-\sigma(s; w) \cdot \hat{p}]^{n-k} \{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^k\} \times \sigma(s; w), n \geq 0 \end{cases}$$

推论2.2.3.  $X(n) = O(n) - \sum_{k=0}^{n-1} C_n^k X(k) [-\sigma(s; w) \cdot \hat{p}]^{n-k}, n \geq 0$

## 2.3 $i^{-n}[\sigma(s; w) \times ]^n \hat{p}$ 和 $i^{-n} \hat{p} [ \times \sigma(s; w) ]^n$ 的通项公式

### 2.3.1 $i^{-n}[\sigma(s; w) \times ]^n \hat{p}$ 的通项公式

性质2.3.1.

$$\begin{cases} i^{-0}[\sigma(s; w) \times ]^0 \hat{p} = \hat{p} \\ i^{-1}[\sigma(s; w) \times ]^1 \hat{p} = \{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)\} \\ i^{-2}[\sigma(s; w) \times ]^2 \hat{p} = -[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + \sigma^2(s; w)\hat{p} \\ i^{-3}[\sigma(s; w) \times ]^3 \hat{p} = -[1 - \sigma^2(s; w)]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - [1 + \sigma^2(s; w)][\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + \sigma^2(s; w)\hat{p} \\ i^{-4}[\sigma(s; w) \times ]^4 \hat{p} = -[2 - \sigma^2(s; w)]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - [1 + 2\sigma^2(s; w)][\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + [1 + \sigma^2(s; w)]\sigma^2(s; w)\hat{p} \\ i^{-5}[\sigma(s; w) \times ]^5 \hat{p} = -[3 - \sigma^4(s; w)]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - [1 + 3\sigma^2(s; w) + \sigma^4(s; w)][\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + [1 + 2\sigma^2(s; w)]\sigma^2(s; w)\hat{p} \end{cases}$$

性质2.3.2.

$$\begin{cases} \hat{p} \cdot [ \sigma(s; w) \times ]^2 \hat{p} = [\sigma(s; w) \cdot \hat{p}]^2 - \sigma^2(s; w) \\ \hat{p} \cdot [ \sigma(s; w) \times ]^3 \hat{p} = 2i[\sigma(s; w) \cdot \hat{p}]^2 - i\sigma^2(s; w) \\ \hat{p} \cdot [ \sigma(s; w) \times ]^4 \hat{p} = i^2[3 + \sigma^2(s; w)][\sigma(s; w) \cdot \hat{p}]^2 - i^2[1 + \sigma^2(s; w)]\sigma^2(s; w) \end{cases}$$

定理2.3.1.

$$\begin{cases} i^{-n}[\sigma(s; w) \times ]^n \hat{p} = a_n \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - c_n \sigma^2(s; w)\hat{p} \\ a_{n+1} = a_n + b_n - c_n \sigma^2(s; w), b_{n+1} = b_n + c_n \sigma^2(s; w), c_{n+1} = b_n, n \geq 0 \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s; w); a_1 = 1, b_1 = -1, c_1 = 0 \end{cases}$$

证明: 采用数学归纳法证明:

第一步:  $i = 0$  时成立, 即

$$\begin{cases} i^{-0}[\sigma(s; w) \times ]^0 \hat{p} = a_0 \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_0 [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - c_0 \sigma^2(s; w)\hat{p} \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s; w) \end{cases}$$

第二步: 假设  $i = n$  时成立, 即

$$i^{-n}[\sigma(s; w) \times ]^n \hat{p} = a_n \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - c_n \sigma^2(s; w)\hat{p}$$

第三步:  $i = n + 1$  时

$$\begin{aligned} & i^{-(n+1)}[\sigma(s; w) \times ]^{n+1} \hat{p} \\ & = a_n \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - i b_n \sigma(s; w) \times [ \sigma(s; w) \cdot \hat{p} ] \sigma(s; w) + i c_n \sigma(s; w) \times \sigma^2(s; w) \hat{p} \\ & = a_n \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_n \{ \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - \sigma^2(s; w)\hat{p} \} - c_n \{ \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) \} \sigma^2(s; w) \\ & = (a_n + b_n - c_n \sigma^2(s; w))\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + (b_n + c_n \sigma^2(s; w))[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - b_n \sigma^2(s; w)\hat{p} \\ & = a_{n+1} \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_{n+1} [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - c_{n+1} \sigma^2(s; w)\hat{p} \end{aligned}$$

此步证明了  $i = n + 1$  时成立.

第四步: 得到如下递推关系



$$\begin{cases} a_{n+1} = a_n + b_n - c_n \sigma^2(s; w), b_{n+1} = b_n + c_n \sigma^2(s; w), c_{n+1} = b_n \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s; w); a_1 = 1, b_1 = -1, c_1 = 0 \end{cases}$$

第五步：根据以上归纳法推理，命题成立，定理得证。  $\square$

推论2.3.1.  $i^{-n}[\sigma(s; w) \times ]^n \hat{p} = a_n \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + b_n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - b_{n-1} \sigma^2(s; w) \hat{p}, n \geq 0$

推论2.3.2.  $i^{-n} \hat{p} \cdot \{[\sigma(s; w) \times ]^n \hat{p}\} = -k_n [\sigma(s; w) \cdot \hat{p}]^2 + b_{n-1} \sigma^2(s; w), n \geq 0, k_n = -(a_n + b_n)$

### 2.3.2 $i^{-n} \hat{p} [ \times \sigma(s; w) ]^n$ 的通项公式

定理2.3.2.

$$\begin{cases} i^{-n} \hat{p} [ \times \sigma(s; w) ]^n = a_n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + b_n \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] - c_n \sigma^2(s; w) \hat{p} \\ a_{n+1} = a_n + b_n - c_n \sigma^2(s; w), b_{n+1} = b_n + c_n \sigma^2(s; w), c_{n+1} = b_n, n \geq 0 \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s; w); a_1 = 1, b_1 = -1, c_1 = 0 \end{cases}$$

证明：采用数学归纳法证明：

第一步： $i = 0$ 时成立，即

$$\begin{cases} i^{-0} \hat{p} [ \times \sigma(s; w) ]^0 = a_0 [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + b_0 \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] - c_0 \sigma^2(s; w) \hat{p} \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s; w) \end{cases}$$

第二步：假设 $i = n$ 时成立，即

$$i^{-n} \hat{p} [ \times \sigma(s; w) ]^n = a_n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + b_n \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] - c_n \sigma^2(s; w) \hat{p}$$

第三步： $i = n + 1$ 时

$$i^{-(n+1)} \hat{p} [ \times \sigma(s; w) ]^{n+1}$$

$$= a_n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - i b_n \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] \times \sigma(s; w) + i c_n \hat{p} \times \sigma(s; w) \sigma^2(s; w)$$

$$= a_n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + b_n \{ [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] - \sigma^2(s; w) \hat{p} \} - c_n \{ [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] \} \sigma^2(s; w)$$

$$= (a_n + b_n - c_n \sigma^2(s; w)) [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + (b_n + c_n \sigma^2(s; w)) \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] - b_n \sigma^2(s; w) \hat{p}$$

$$= a_{n+1} [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + b_{n+1} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] - c_{n+1} \sigma^2(s; w) \hat{p}$$

此步证明了 $i = n + 1$ 时成立。

第四步：得到如下递推关系

$$\begin{cases} a_{n+1} = a_n + b_n - c_n \sigma^2(s; w), b_{n+1} = b_n + c_n \sigma^2(s; w), c_{n+1} = b_n \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s; w); a_1 = 1, b_1 = -1, c_1 = 0 \end{cases}$$

第五步：根据以上归纳法推理，命题成立，定理得证。  $\square$

推论2.3.3.  $i^{-n} \hat{p} [ \times \sigma(s; w) ]^n = a_n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + b_n \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] - b_{n-1} \sigma^2(s; w) \hat{p}, n \geq 0$

推论2.3.4.  $i^{-n} \{ \hat{p} [ \times \sigma(s; w) ]^n \} \cdot \hat{p} = -k_n [\sigma(s; w) \cdot \hat{p}]^2 + b_{n-1} \sigma^2(s; w), n \geq 0, k_n = -(a_n + b_n)$

推论2.3.5.  $\hat{p} \cdot \{ [\sigma(s; w) \times ]^n \hat{p} \} = \{ \hat{p} [ \times \sigma(s; w) ]^n \} \cdot \hat{p}, n \geq 0$

### 2.3.3 参数汇总

推论2.3.6.

$$\begin{cases} a_{n+1} = a_n + b_n - c_n \sigma^2(s; w), b_{n+1} = b_n + c_n \sigma^2(s; w), c_{n+1} = b_n \\ a_0 = 0, b_0 = 0, c_0 = -\sigma^{-2}(s; w); a_1 = 1, b_1 = -1, c_1 = 0; a_2 = 0, b_2 = -1, c_2 = -1 \\ k_n = -(a_n + b_n) \end{cases}$$

### 2.3.4 $i^{l-n}\{[\sigma(s; w) \times ||^n \hat{p}][\times \hat{p}]^l\}$ 的通项公式

$$\text{推论2.3.7. } \begin{cases} i^{l-n}\{[\sigma(s; w) \times ||^n \hat{p}][\times \hat{p}]^l = a_n i^l \sigma(s; w)[\times \hat{p}]^l [\sigma(s; w) \cdot \hat{p}] + b_n [\sigma(s; w) \cdot \hat{p}] i^l \sigma(s; w)[\times \hat{p}]^l \\ = \begin{cases} -a_n \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^2 + (a_n - b_n)[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_n [\sigma(s; w) \cdot \hat{p}]^2 \sigma(s; w), l = 2k - 1 \\ -(a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^2 \hat{p} + a_n \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w), l = 2k \end{cases} \end{cases}$$

$$\text{推论2.3.8. } \begin{cases} i^{l-n}\{[\sigma(s; w) \times ||^n \hat{p}][\times \hat{p}]^l \cdot \sigma(s; w) = \begin{cases} (2a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^2 - a_n \sigma^2(s; w), l = 2k - 1 \\ -(a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s; w) - a_n][\sigma(s; w) \cdot \hat{p}], l = 2k \end{cases} \end{cases}$$

$$\text{推论2.3.9. } \begin{cases} i^{l-n}\sigma(s; w) \cdot | \{ [\sigma(s; w) \times ||^n \hat{p}][\times \hat{p}]^l = \begin{cases} -(a_n + 2b_n)[\sigma(s; w) \cdot \hat{p}]^2 + b_n \sigma^2(s; w), l = 2k - 1 \\ -(a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s; w) - b_n][\sigma(s; w) \cdot \hat{p}], l = 2k \end{cases} \end{cases}$$

$$\text{推论2.3.10. } i^{l-n} \hat{p} \cdot | \{ [\sigma(s; w) \times ||^n \hat{p}][\times \hat{p}]^l = 0, n \geq 0, l \geq 1$$

### 2.3.5 $i^{l-n}[\hat{p} \times ||^l \{ \hat{p}[\times \sigma(s; w)]^n \}$ 的通项公式

$$\text{推论2.3.11. } \begin{cases} i^{l-n}[\hat{p} \times ||^l \{ \hat{p}[\times \sigma(s; w)]^n \} = a_n [\sigma(s; w) \cdot \hat{p}] i^l [\hat{p} \times ||^l \sigma(s; w) + b_n i^l [\hat{p} \times ||^l \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - c_n \sigma^2(s; w) \hat{p}] \\ = \begin{cases} -a_n [\sigma(s; w) \cdot \hat{p}]^2 \sigma(s; w) + (a_n - b_n)[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_n \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^2, l = 2k - 1 \\ -(a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^2 \hat{p} + a_n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + b_n \sigma(s; w)[\sigma(s; w) \cdot \hat{p}], l = 2k \end{cases} \end{cases}$$

$$\text{推论2.3.12. } \begin{cases} i^{l-n}\sigma(s; w) \cdot | \{ \hat{p}[\times \sigma(s; w)]^n \} = \begin{cases} (2a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^2 - a_n \sigma^2(s; w), l = 2k - 1 \\ -(a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s; w) - a_n][\sigma(s; w) \cdot \hat{p}], l = 2k \end{cases} \end{cases}$$

$$\text{推论2.3.13. } \begin{cases} i^{l-n}[\hat{p} \times ||^l \{ \hat{p}[\times \sigma(s; w)]^n \}] \cdot \sigma(s; w) = \begin{cases} -(a_n + 2b_n)[\sigma(s; w) \cdot \hat{p}]^2 + b_n \sigma^2(s; w), l = 2k - 1 \\ -(a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^3 + [(a_n + b_n)\sigma^2(s; w) - b_n][\sigma(s; w) \cdot \hat{p}], l = 2k \end{cases} \end{cases}$$

$$\text{推论2.3.14. } i^{l-n} \hat{p} \cdot | \{ \hat{p}[\times \sigma(s; w)]^n \} = 0, n \geq 0, l \geq 1$$

$$\text{推论2.3.15. } \begin{cases} i^{l-n}\{[\sigma(s; w) \times ||^n \hat{p}][\times \hat{p}]^l \cdot \sigma(s; w) = i^{l-n}\sigma(s; w) \cdot | \{ \hat{p}[\times \sigma(s; w)]^n \} \\ i^{l-n}\sigma(s; w) \cdot | \{ [\sigma(s; w) \times ||^n \hat{p}][\times \hat{p}]^l = i^{l-n}[\hat{p} \times ||^l \{ \hat{p}[\times \sigma(s; w)]^n \}] \cdot \sigma(s; w) \end{cases}$$

## 3 两类基本自旋复合算符的通项公式

### 3.1 $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n \sigma_\alpha(s; w)$ 的通项公式

#### 3.1.1 $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n \sigma_\alpha(s; w)$ 通项公式的试探和猜想

$$\text{推论3.1.1. } \sigma(s; w) \cdot \{ [\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w) \}$$

$$= -\{ [\sigma(s; w) \cdot \hat{p}]^{2-n\%2} - (1 - n\%2)\sigma^2(s; w) \} - \sum_{k=0}^{n-1} C_n^k \{ \sigma(s; w) \cdot \{ [\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) \} \} [-\sigma(s; w) \cdot \hat{p}]^{n-k}$$

性质3.1.1.

$$\begin{cases} \sigma(s; w) \cdot \{ [\sigma(s; w) \cdot \hat{p}]^0 \sigma(s; w) \} = \sigma^2(s; w) \\ \sigma(s; w) \cdot \{ [\sigma(s; w) \cdot \hat{p}]^1 \sigma(s; w) \} = [\sigma^2(s; w) - 1][\sigma(s; w) \cdot \hat{p}] \\ \sigma(s; w) \cdot \{ [\sigma(s; w) \cdot \hat{p}]^2 \sigma(s; w) \} = [\sigma^2(s; w) - 3][\sigma(s; w) \cdot \hat{p}]^2 + \sigma^2(s; w) \\ \sigma(s; w) \cdot \{ [\sigma(s; w) \cdot \hat{p}]^3 \sigma(s; w) \} = [\sigma^2(s; w) - 6][\sigma(s; w) \cdot \hat{p}]^3 + [3\sigma^2(s; w) - 1]\sigma(s; w) \cdot \hat{p} \\ \sigma(s; w) \cdot \{ [\sigma(s; w) \cdot \hat{p}]^4 \sigma(s; w) \} = [\sigma^2(s; w) - 10][\sigma(s; w) \cdot \hat{p}]^4 + [6\sigma^2(s; w) - 5][\sigma(s; w) \cdot \hat{p}]^2 + \sigma^2(s; w) \end{cases}$$

性质3.1.2.

$$\begin{cases} \sigma(s; w) \cdot \{[\sigma(s; w) \cdot \hat{p}]^5 \sigma(s; w)\} = [\sigma^2(s; w) - 15][\sigma(s; w) \cdot \hat{p}]^5 + [10\sigma^2(s; w) - 15][\sigma(s; w) \cdot \hat{p}]^3 \\ + [5\sigma^2(s; w) - 1][\sigma(s; w) \cdot \hat{p}] \\ \sigma(s; w) \cdot \{[\sigma(s; w) \cdot \hat{p}]^6 \sigma(s; w)\} = [\sigma^2(s; w) - 21][\sigma(s; w) \cdot \hat{p}]^6 + [15\sigma^2(s; w) - 35][\sigma(s; w) \cdot \hat{p}]^4 \\ + [15\sigma^2(s; w) - 7][\sigma(s; w) \cdot \hat{p}]^2 + \sigma^2(s; w) \\ \sigma(s; w) \cdot \{[\sigma(s; w) \cdot \hat{p}]^7 \sigma(s; w)\} = [\sigma^2(s; w) - 28][\sigma(s; w) \cdot \hat{p}]^7 + [21\sigma^2(s; w) - 70][\sigma(s; w) \cdot \hat{p}]^5 \\ + [35\sigma^2(s; w) - 28][\sigma(s; w) \cdot \hat{p}]^3 + [7\sigma^2(s; w) - 1][\sigma(s; w) \cdot \hat{p}] \end{cases}$$

猜想3.1.1.  $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n \sigma_\alpha(s; w) = \sum_{k=0}^{[n/2]} [C_n^{2k} \sigma^2(s; w) - C_{n+1}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{n-2k}, n \geq 0$

3.1.2  $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n \sigma_\alpha(s; w)$ 通项公式的有关引理及其证明

引理3.1.1.  $\sum_{l=0}^n \sum_{k=0}^{[l/2]} A(k, l) = \sum_{k=0}^{[n/2]} \sum_{l=2k}^n A(k, l)$

引理3.1.2.  $\sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l C_l^{2k} = C_{n+1}^{2k}, \sum_{l=2k+1}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{2(k+1)} = C_{n+2}^{2(k+1)} - \delta_{n,2k+1}$

定理3.1.1.  $\sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k} \sigma^2(s; w) - C_{n+2}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{n+1-2k} = -[\sigma(s; w) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2]\sigma^2(s; w)$   
 $+ \sum_{k=0}^{[n/2]} \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l [C_l^{2k} \sigma^2(s; w) - C_{l+1}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{n+1-2k}$

证明:  $\sum_{k=0}^{[n/2]} \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l [C_l^{2k} \sigma^2(s; w) - C_{l+1}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{n+1-2k}$   
 $= \sum_{k=0}^{[n/2]} [C_{n+1}^{2k} \sigma^2(s; w) - C_{n+2}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{n+1-2k} + [\sigma(s; w) \cdot \hat{p}]^{2-(n+1)\%2}$   
 $= \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k} \sigma^2(s; w) - C_{n+2}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{n+1-2k} + [\sigma(s; w) \cdot \hat{p}]^{2-(n+1)\%2} + [1 - (n+1)\%2]\sigma^2(s; w) \quad \square$

3.1.3  $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n \sigma_\alpha(s; w)$ 通项公式的数学归纳法证明

定理3.1.2.  $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n \sigma_\alpha(s; w) = \sum_{k=0}^{[n/2]} [C_n^{2k} \sigma^2(s; w) - C_{n+1}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{n-2k}, n \geq 0$

证明:

采用数学归纳法证明此定理。

第一步:  $i = 0$ 时成立:  $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^0 \sigma_\alpha(s; w) = \sum_{k=0}^{[0/2]} [C_0^{2k} \sigma^2(s; w) - C_{0+1}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{0-2k}$

第二步: 假设  $i \leq n$ 时成立:  $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^i \sigma_\alpha(s; w) = \sum_{k=0}^{[i/2]} [C_i^{2k} \sigma^2(s; w) - C_{i+1}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{i-2k}, 0 \leq i \leq n$

第三步:  $i = n + 1$ 时,  $\sigma(s; w) \cdot \{[\sigma(s; w) \cdot \hat{p}]^{n+1} \sigma(s; w)\}$   
 $= -\{[\sigma(s; w) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2]\sigma^2(s; w)\} - \sum_{l=0}^n C_{n+1}^l \{\sigma(s; w) \cdot \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}\} [-\sigma(s; w) \cdot \hat{p}]^{n+1-l}$   
 $= -\{[\sigma(s; w) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2]\sigma^2(s; w)\} - \sum_{l=0}^n C_{n+1}^l \sum_{k=0}^{[l/2]} [C_l^{2k} \sigma^2(s; w) - C_{l+1}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{l-2k} [-\sigma(s; w) \cdot \hat{p}]^{n+1-l}$   
 $= -\{[\sigma(s; w) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2]\sigma^2(s; w)\} - \sum_{l=0}^n \sum_{k=0}^{[l/2]} (-1)^{n+1-l} C_{n+1}^l [C_l^{2k} \sigma^2(s; w) - C_{l+1}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{n+1-2k}$   
 $= -\{[\sigma(s; w) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2]\sigma^2(s; w)\} - \sum_{l=0}^n \sum_{k=0}^{[n/2]} (-1)^{n+1-l} C_{n+1}^l [C_l^{2k} \sigma^2(s; w) - C_{l+1}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{n+1-2k}$   
 $= -\{[\sigma(s; w) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2]\sigma^2(s; w)\} - \sum_{k=0}^{[n/2]} \sum_{l=0}^n (-1)^{n+1-l} C_{n+1}^l [C_l^{2k} \sigma^2(s; w) - C_{l+1}^{2(k+1)}][\sigma(s; w) \cdot \hat{p}]^{n+1-2k}$

$$\begin{aligned}
&= -\{[\sigma(s; w) \cdot \hat{p}]^{2-(n+1)\%2} - [1 - (n+1)\%2]\sigma^2(s; w)\} + \sum_{k=0}^{[n/2]} \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l [C_l^{2k} \sigma^2(s; w) - C_{l+1}^{2(k+1)}] [\sigma(s; w) \cdot \hat{p}]^{n+1-2k} \\
&= \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k} \sigma^2(s; w) - C_{n+2}^{2(k+1)}] [\sigma(s; w) \cdot \hat{p}]^{n+1-2k}
\end{aligned}$$

此步证明了  $i = n + 1$  时成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

### 3.1.4 $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n \sigma_\alpha(s; w)$ 的通项公式的推论

引理3.1.3.  $\sum_{k=0}^{[n/2]} s^{n-2k} [C_n^{2k} \sigma^2(s; w) - C_{n+1}^{2(k+1)}] = s[s^{n+1} + (s-1)^n]$

推论3.1.2.  $\sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{p}]^n \sigma_\alpha(s; w) \lambda(\hat{p}, -s\zeta) = s[s^{n+1} + (s-1)^n][-\zeta]^n \lambda(\hat{p}, -s\zeta)$

推论3.1.3.  $[\frac{1}{s}\sigma(s; w) \cdot \hat{\nabla}]^2 \psi = \psi \Rightarrow \sigma^\alpha(s; w)[\sigma(s; w) \cdot \hat{\nabla}]^n \sigma_\alpha(s; w) \psi = s[s^{n+1} + (s-1)^n][\frac{1}{s}\sigma(s; w) \cdot \hat{\nabla}]^n \psi$

## 3.2 $i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}$ 的通项公式

### 3.2.1 $i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}$ 的通项公式试探

性质3.2.1.

$$\begin{cases} \sigma(s; w) \times \hat{p} = [\sigma(s; w), i\sigma(s; w) \cdot \hat{p}] = i\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)\} \\ \hat{p} \times \sigma(s; w) = -[\sigma(s; w), i\sigma(s; w) \cdot \hat{p}] = i\{[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]\} \end{cases}$$

性质3.2.2.

$$\begin{cases} i^n \sigma(s; w) [|\times \hat{p}]^n = \begin{cases} i\sigma(s; w) \times \hat{p}, n = 2k - 1 \\ -\{[\sigma(s; w) \cdot \hat{p}]\hat{p} - \sigma(s; w)\}, n = 2k \end{cases}, k \geq 1 \\ i^n [\hat{p} \times |]^n \sigma(s; w) = \begin{cases} i\hat{p} \times \sigma(s; w), n = 2k - 1 \\ -\{[\sigma(s; w) \cdot \hat{p}]\hat{p} - \sigma(s; w)\}, n = 2k \end{cases}, k \geq 1 \end{cases}$$

性质3.2.3.

$$\begin{cases} i^n \sigma(s; w) \times \{\sigma(s; w) [|\times \hat{p}]^n\} = \begin{cases} i\{[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - \sigma^2(s; w)\hat{p}\}, n = 2k - 1 \\ -[\sigma(s; w) \times \hat{p}][\sigma(s; w) \cdot \hat{p}] + i\sigma(s; w), n = 2k \end{cases}, k \geq 1 \\ i^n i^{-1}\sigma(s; w) \times \{\sigma(s; w) [|\times \hat{p}]^n\} = \begin{cases} [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - \sigma^2(s; w)\hat{p}, n = 2k - 1 \\ -\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^2 + [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + \sigma(s; w), n = 2k \end{cases}, k \geq 1 \end{cases}$$

推论3.2.1.  $i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}$

$$= i^n i^{-1}\sigma(s; w) \times \{\sigma(s; w) [|\times \hat{p}]^n\} - \sum_{k=0}^{n-1} C_n^k \{i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w)\}\} [-\sigma(s; w) \cdot \hat{p}]^{n-k}$$

$$\Rightarrow \begin{cases} i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^0 \sigma(s; w)\} = \sigma(s; w) \\ i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^1 \sigma(s; w)\} = [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - \sigma^2(s; w)\hat{p} \\ i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^2 \sigma(s; w)\} = 3[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - 2\sigma^2(s; w)\hat{p}[\sigma(s; w) \cdot \hat{p}] + \sigma(s; w) \\ i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^3 \sigma(s; w)\} = 6[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^2 + [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) \\ \quad - 2\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^3 - 3\sigma^2(s; w)\hat{p}[\sigma(s; w) \cdot \hat{p}]^2 + 3\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - \sigma^2(s; w)\hat{p} \\ i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^4 \sigma(s; w)\} = [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)\{10[\sigma(s; w) \cdot \hat{p}]^2 + 5\}[\sigma(s; w) \cdot \hat{p}] \\ \quad + \sigma(s; w)\{-5[\sigma(s; w) \cdot \hat{p}]^4 + 5[\sigma(s; w) \cdot \hat{p}]^2 + 1\} - \sigma^2(s; w)\hat{p}\{4[\sigma(s; w) \cdot \hat{p}]^2 + 4\}[\sigma(s; w) \cdot \hat{p}]^1 \end{cases}$$

猜想3.2.1.  $i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}, n \geq 0$

$$\begin{aligned}
&= [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) \sum_{k=0}^{[(n-1)/2]} (C_n^{2k+1} + C_n^{2k+2})[\sigma(s; w) \cdot \hat{p}]^{n-2k-1} + \sigma(s; w) \sum_{k=0}^{[n/2]} (C_n^{2k} - C_n^{2k+2})[\sigma(s; w) \cdot \hat{p}]^{n-2k} \\
&\quad - \sigma^2(s; w)\hat{p} \sum_{k=0}^{[(n-1)/2]} C_n^{2k+1}[\sigma(s; w) \cdot \hat{p}]^{n-2k-1}
\end{aligned}$$

### 3.2.2 $i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}$ 通项公式的数学归纳法证明

**定理3.2.1.**  $i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}, n \geq 0$

$$= \sum_{k=0}^{[n/2]} \{(C_n^{2k+1} + C_n^{2k+2})[\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + (C_n^{2k} - C_n^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - C_n^{2k+1}\sigma^2(s; w)\hat{p}\}[\sigma(s; w) \cdot \hat{p}]^{n-2k-1}$$

**证明:**

采用数学归纳法证明此定理。

第一步:  $i = 0$  时成立:

$$i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^0 \sigma(s; w)\} \\ = \sum_{k=0}^{[0/2]} \{(C_0^{2k+1} + C_0^{2k+2})[\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + (C_0^{2k} - C_0^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - C_0^{2k+1}\sigma^2(s; w)\hat{p}\}[\sigma(s; w) \cdot \hat{p}]^{0-2k-1}$$

第二步: 假设  $0 \leq l \leq n$  时成立:

$$i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\} \\ = \sum_{k=0}^{[l/2]} \{(C_l^{2k+1} + C_l^{2k+2})[\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + (C_l^{2k} - C_l^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - C_l^{2k+1}\sigma^2(s; w)\hat{p}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1}$$

第三步:  $i = n + 1$  时,  $i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^{n+1} \sigma(s; w)\}$

$$= i^{n+1}i^{-1}\sigma(s; w) \times \{\sigma(s; w)[\times \hat{p}]^{n+1}\} - \sum_{l=0}^n C_{n+1}^l \{i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}\}[-\sigma(s; w) \cdot \hat{p}]^{n+1-l}$$

$$= i^{n+1}i^{-1}\sigma(s; w) \times \{\sigma(s; w)[\times \hat{p}]^{n+1}\} - \sum_{l=0}^n C_{n+1}^l \sum_{k=0}^{[l/2]} \{(C_l^{2k+1} + C_l^{2k+2})[\sigma(s; w) \cdot \hat{p}] \sigma(s; w)$$

$$+ (C_l^{2k} - C_l^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - C_l^{2k+1}\sigma^2(s; w)\hat{p}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1}[-\sigma(s; w) \cdot \hat{p}]^{n+1-l}$$

$$= i^{n+1}i^{-1}\sigma(s; w) \times \{\sigma(s; w)[\times \hat{p}]^{n+1}\} + \sum_{l=0}^n \sum_{k=0}^{[l/2]} (-1)^{n-l} C_{n+1}^l \{(C_l^{2k+1} + C_l^{2k+2})[\sigma(s; w) \cdot \hat{p}] \sigma(s; w)$$

$$+ (C_l^{2k} - C_l^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - C_l^{2k+1}\sigma^2(s; w)\hat{p}\}[\sigma(s; w) \cdot \hat{p}]^{n-2k}$$

$$= i^{n+1}i^{-1}\sigma(s; w) \times \{\sigma(s; w)[\times \hat{p}]^{n+1}\} + \sum_{k=0}^{[n/2]} \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l \{(C_l^{2k+1} + C_l^{2k+2})[\sigma(s; w) \cdot \hat{p}] \sigma(s; w)$$

$$+ (C_l^{2k} - C_l^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - C_l^{2k+1}\sigma^2(s; w)\hat{p}\}[\sigma(s; w) \cdot \hat{p}]^{n-2k}$$

$$= \sum_{k=0}^{[(n+1)/2]}$$

$$\{(C_{n+1}^{2k+1} + C_{n+1}^{2k+2})[\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + (C_{n+1}^{2k} - C_{n+1}^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - C_{n+1}^{2k+1}\sigma^2(s; w)\hat{p}\}[\sigma(s; w) \cdot \hat{p}]^{n-2k}$$

此步证明了  $i = n + 1$  时成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

### 3.2.3 $i^{-1}\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^n\} \times \sigma(s; w)$ 通项公式的归纳法证明

**定理3.2.2.**  $i^{-1}\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^n\} \times \sigma(s; w), n \geq 0$

$$= \left\{ \sum_{k=0}^{[(n-1)/2]} (C_n^{2k+1} + C_n^{2k+2})[\sigma(s; w) \cdot \hat{p}]^{n-2k-1} \right\} \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + \left\{ \sum_{k=0}^{[n/2]} (C_n^{2k} - C_n^{2k+2})[\sigma(s; w) \cdot \hat{p}]^{n-2k} \right\} \sigma(s; w)$$

$$- \left\{ \sum_{k=0}^{[(n-1)/2]} C_n^{2k+1}[\sigma(s; w) \cdot \hat{p}]^{n-2k-1} \right\} \sigma^2(s; w)\hat{p}$$

**定理3.2.3.**  $i^{-1}\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^n\} \times \sigma(s; w), n \geq 0$

$$= \sum_{k=0}^{[n/2]} [\sigma(s; w) \cdot \hat{p}]^{n-2k-1} \{(C_n^{2k+1} + C_n^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + (C_n^{2k} - C_n^{2k+2})[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - C_n^{2k+1}\sigma^2(s; w)\hat{p}\}$$

**证明:**

采用数学归纳法证明此定理。

第一步:  $i = 0$  时成立:

$$i^{-1}\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^0\} \times \sigma(s; w)$$

$$= \sum_{k=0}^{[0/2]} [\sigma(s; w) \cdot \hat{p}]^{0-2k-1} \{(C_0^{2k+1} + C_0^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + (C_0^{2k} - C_0^{2k+2})[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - C_0^{2k+1}\sigma^2(s; w)\hat{p}\}$$

第二步: 假设  $0 \leq l \leq n$  时成立:

$$i^{-1}\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l\} \times \sigma(s; w)$$

$$= \sum_{k=0}^{[l/2]} [\sigma(s; w) \cdot \hat{p}]^{i-2k-1} \{ (C_i^{2k+1} + C_i^{2k+2}) \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + (C_i^{2k} - C_i^{2k+2}) [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - C_i^{2k+1} \sigma^2(s; w) \hat{p} \}$$

第三步:  $i = n + 1$  时,  $i^{-1} \{ \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^{n+1} \} \times \sigma(s; w)$

$$= i^{n+1} i^{-1} \{ [\hat{p} \times ]^{n+1} \sigma(s; w) \} \times \sigma(s; w) - \sum_{l=0}^n C_{n+1}^l [-\sigma(s; w) \cdot \hat{p}]^{n+1-l} \{ i^{-1} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^l \} \times \sigma(s; w)$$

$$= i^{n+1} i^{-1} \{ [\hat{p} \times ]^{n+1} \sigma(s; w) \} \times \sigma(s; w) - \sum_{l=0}^n C_{n+1}^l \sum_{k=0}^{[l/2]} [-\sigma(s; w) \cdot \hat{p}]^{n+1-l} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \{ (C_l^{2k+1} + C_l^{2k+2}) \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + (C_l^{2k} - C_l^{2k+2}) [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - C_l^{2k+1} \sigma^2(s; w) \hat{p} \}$$

$$= i^{n+1} i^{-1} \{ [\hat{p} \times ]^{n+1} \sigma(s; w) \} \times \sigma(s; w) + \sum_{l=0}^n \sum_{k=0}^{[l/2]} [\sigma(s; w) \cdot \hat{p}]^{n-2k} (-1)^{n-l} C_{n+1}^l$$

$$\{ (C_l^{2k+1} + C_l^{2k+2}) \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + (C_l^{2k} - C_l^{2k+2}) [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - C_l^{2k+1} \sigma^2(s; w) \hat{p} \}$$

$$= i^{n+1} i^{-1} \{ [\hat{p} \times ]^{n+1} \sigma(s; w) \} \times \sigma(s; w) + \sum_{k=0}^{[n/2]} \sum_{l=2k}^n [\sigma(s; w) \cdot \hat{p}]^{n-2k} (-1)^{n-l} C_{n+1}^l$$

$$\{ (C_l^{2k+1} + C_l^{2k+2}) \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + (C_l^{2k} - C_l^{2k+2}) [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - C_l^{2k+1} \sigma^2(s; w) \hat{p} \}$$

$$= \sum_{k=0}^{[(n+1)/2]} [\sigma(s; w) \cdot \hat{p}]^{n-2k} \{ (C_{n+1}^{2k+1} + C_{n+1}^{2k+2}) \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + (C_{n+1}^{2k} - C_{n+1}^{2k+2}) [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - C_{n+1}^{2k+1} \sigma^2(s; w) \hat{p} \}$$

此步证明了  $i = n + 1$  时成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

### 3.2.4 $i^{-1} \hat{p} \cdot |\sigma(s; w) \times \{ [\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w) \}$ 的通项公式

$$\text{推论 3.2.2. } i^{-1} \hat{p} \cdot |\sigma(s; w) \times \{ [\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w) \} = \sum_{k=0}^{(n+1)/2} [C_{n+1}^{2k+1} - C_n^{2k-1} \sigma^2(s; w)] [\sigma(s; w) \cdot \hat{p}]^{n+1-2k}, n \geq 0$$

$$\text{证明: } i^{-1} \hat{p} \cdot |\sigma(s; w) \times \{ [\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w) \} = i^{-1} \{ \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^n \} \times \sigma(s; w) \cdot \hat{p}$$

$$= \sum_{k=0}^{[n/2]} \{ (C_n^{2k+1} + C_n^{2k+2}) [\sigma(s; w) \cdot \hat{p}]^2 + (C_n^{2k} - C_n^{2k+2}) [\sigma(s; w) \cdot \hat{p}]^2 - C_n^{2k+1} \sigma^2(s; w) \} [\sigma(s; w) \cdot \hat{p}]^{n-2k-1}$$

$$= \sum_{k=0}^{[n/2]} \{ (C_n^{2k+1} + C_n^{2k+2}) + (C_n^{2k} - C_n^{2k+2}) \} [\sigma(s; w) \cdot \hat{p}]^{n+1-2k} - \sum_{k=0}^{[n/2]} C_n^{2k+1} \sigma^2(s; w) [\sigma(s; w) \cdot \hat{p}]^{n-1-2k}$$

$$= \sum_{k=0}^{[n/2]} C_{n+1}^{2k+1} [\sigma(s; w) \cdot \hat{p}]^{n+1-2k} - \sum_{k=1}^{[n/2]+1} C_n^{2k-1} \sigma^2(s; w) [\sigma(s; w) \cdot \hat{p}]^{n+1-2k}$$

$$= \sum_{k=0}^{[(n+1)/2]} C_{n+1}^{2k+1} [\sigma(s; w) \cdot \hat{p}]^{n+1-2k} - \sum_{k=0}^{[(n-1)/2]+1} C_n^{2k-1} \sigma^2(s; w) [\sigma(s; w) \cdot \hat{p}]^{n+1-2k}$$

$$= \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k+1} - C_n^{2k-1} \sigma^2(s; w)] [\sigma(s; w) \cdot \hat{p}]^{n+1-2k}$$

□

## 4 复杂自旋复合算符的通项公式

### 4.1 $i^{-n} [\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w)$ 和 $i^{-n} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] [ \times \sigma(s; w) ]^n$ 的通项公式

#### 4.1.1 $i^{-n} [\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w)$ 的通项公式

性质 4.1.1.

$$\begin{cases} i^{-0} [\sigma(s; w) \times ]^0 \{ [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \} = [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \\ i^{-1} [\sigma(s; w) \times ]^1 \{ [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \} = \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - \sigma^2(s; w) \hat{p} \\ i^{-2} [\sigma(s; w) \times ]^2 \{ [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \} = [2 - \sigma^2(s; w)] \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + [1 + \sigma^2(s; w)] [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - \sigma^2(s; w) \hat{p} \\ i^{-3} [\sigma(s; w) \times ]^3 \{ [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \} = [3 - \sigma^2(s; w)] \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + [1 + 2\sigma^2(s; w)] [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - [1 + \sigma^2(s; w)] \hat{p} \\ i^{-4} [\sigma(s; w) \times ]^4 \{ [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \} = [4 - \sigma^4(s; w)] \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + [1 + 3\sigma^2(s; w) + \sigma^4(s; w)] [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - [1 + \sigma^2(s; w)] \hat{p} \end{cases}$$

性质 4.1.2.

$$\begin{cases} i^{-n} [\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) = \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] - i^{-(n+1)} [\sigma(s; w) \times ]^{n+1} \hat{p} \\ i^{-n} \hat{p} \cdot [ \sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) = [\sigma(s; w) \cdot \hat{p}]^2 - i^{-(n+1)} \hat{p} \cdot [ \sigma(s; w) \times ]^{n+1} \hat{p} \end{cases}$$

证明:  $i^{-(n+1)}[\sigma(s; w) \times ]^{n+1}\hat{p}$   
 $= i^{-n}[\sigma(s; w) \times ]^n i^{-1}\sigma(s; w) \times \hat{p}$   
 $= i^{-n}[\sigma(s; w) \times ]^n \{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)\}$   
 $= \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - i^{-n}[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)$   
 $\Rightarrow i^{-n}[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) = \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - i^{-(n+1)}[\sigma(s; w) \times ]^{n+1}\hat{p}$   $\square$

推论4.1.1.  $i^{-n}[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)$   
 $= (1 - a_{n+1})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - b_{n+1}[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + c_{n+1}\sigma^2(s; w)\hat{p}$

推论4.1.2.  $i^{-n}[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)$   
 $= (1 - a_{n+1})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - b_{n+1}[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + b_n\sigma^2(s; w)\hat{p}, n \geq 0$

推论4.1.3.  $i^{-n}\hat{p} \cdot |[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) = (1 - a_{n+1} - b_{n+1})[\sigma(s; w) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s; w)\hat{p}, n \geq 0$

推论4.1.4.  $i^{-n}\hat{p} \cdot |[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) = (1 - a_n - 2b_n)[\sigma(s; w) \cdot \hat{p}]^2 + b_n\sigma^2(s; w)\hat{p}, n \geq 0$

#### 4.1.2 $i^{-n}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}][\times \sigma(s; w)]^n$ 的通项公式

性质4.1.3.

$$\begin{cases} i^{-n}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}][\times \sigma(s; w)]^n = [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - i^{-(n+1)}\hat{p}[\times \sigma(s; w)]^{n+1} \\ i^{-n}\hat{p} \cdot |[\sigma(s; w)[\sigma(s; w) \cdot \hat{p}][\times \sigma(s; w)]^n = [\sigma(s; w) \cdot \hat{p}]^2 - i^{-(n+1)}\hat{p}[\times \sigma(s; w)]^{n+1} \cdot \hat{p} \end{cases}$$

证明:  $i^{-(n+1)}\hat{p}[\times \sigma(s; w)]^{n+1}$   
 $= i^{-1}\hat{p} \times \sigma(s; w)[\times \sigma(s; w)]^n i^{-n}$   
 $= \{[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]\}[\times \sigma(s; w)]^n i^{-n}$   
 $= [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - i^{-n}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}][\times \sigma(s; w)]^n$   
 $\Rightarrow i^{-n}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}][\times \sigma(s; w)]^n = [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - i^{-(n+1)}\hat{p}[\times \sigma(s; w)]^{n+1}$   $\square$

推论4.1.5.  $i^{-n}[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)$   
 $= (1 - a_{n+1})[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - b_{n+1}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + c_{n+1}\sigma^2(s; w)\hat{p}, n \geq 0$

推论4.1.6.  $i^{-n}[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)$   
 $= (1 - a_{n+1})[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - b_{n+1}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_n\sigma^2(s; w)\hat{p}, n \geq 0$

推论4.1.7.  $i^{-n}\hat{p} \cdot |[\sigma(s; w)[\sigma(s; w) \cdot \hat{p}][\times \sigma(s; w)]^n = (1 - a_{n+1} - b_{n+1})[\sigma(s; w) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s; w)\hat{p}, n \geq 0$

推论4.1.8.  $i^{-n}\hat{p} \cdot |[\sigma(s; w)[\sigma(s; w) \cdot \hat{p}][\times \sigma(s; w)]^n = (1 - a_n - 2b_n)[\sigma(s; w) \cdot \hat{p}]^2 + b_n\sigma^2(s; w)\hat{p}, n \geq 0$

推论4.1.9.  $i^{-n}\hat{p} \cdot |[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) = i^{-n}\hat{p} \cdot |[\sigma(s; w)[\sigma(s; w) \cdot \hat{p}][\times \sigma(s; w)]^n, n \geq 0$

#### 4.2 $i^{-n}[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)$ 和 $i^{-n}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l [\times \sigma(s; w)]^n$ 的通项公式

##### 4.2.1 $i^{-n}[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)$ 的通项公式

定理4.2.1.  $i^{-n}[\sigma(s; w) \times ]^n \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}, n \geq 1, l \geq 0$   
 $= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s; w) a_{n-1}] \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]$   
 $- [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s; w) b_{n-1}] [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + [C_{l+1}^{2k+2} \sigma^2(s; w) b_{n-1} + C_l^{2k+1} \sigma^4(s; w) b_{n-2}] \hat{p}\} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1}$

证明:  $i^{-n}[\sigma(s; w) \times ]^n \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}$   
 $= i^{-(n-1)}[\sigma(s; w) \times ]^{n-1} i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}$   
 $= i^{-(n-1)}[\sigma(s; w) \times ]^{n-1} \sum_{k=0}^{[l/2]} \{(C_l^{2k+1} + C_l^{2k+2})[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + (C_l^{2k} - C_l^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]$   
 $- C_l^{2k+1}\sigma^2(s; w)\hat{p}\} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1}$

$$\begin{aligned}
&= \sum_{k=0}^{[l/2]} \{(C_l^{2k+1} + C_l^{2k+2})\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - i^{-n}[\sigma(s; w) \times ]^n \hat{p}\} + (C_l^{2k} - C_l^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \\
&- C_l^{2k+1}\sigma^2(s; w)i^{-(n-1)}[\sigma(s; w) \times ]^{n-1} \hat{p}\}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{-C_{l+1}^{2k+2}i^{-n}[\sigma(s; w) \times ]^n \hat{p} + C_{l+1}^{2k+1}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \\
&- C_l^{2k+1}\sigma^2(s; w)i^{-(n-1)}[\sigma(s; w) \times ]^{n-1} \hat{p}\}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{-C_{l+1}^{2k+2}\{a_n\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_n[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - c_n\sigma^2(s; w)\hat{p}\} + C_{l+1}^{2k+1}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \\
&- C_l^{2k+1}\sigma^2(s; w)\{a_{n-1}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_{n-1}[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - c_{n-1}\sigma^2(s; w)\hat{p}\}\}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{[-C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s; w)a_{n-1} + C_{l+1}^{2k+1}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \\
&+ [-C_{l+1}^{2k+2}b_n - C_l^{2k+1}\sigma^2(s; w)b_{n-1}][\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + [C_{l+1}^{2k+2}c_n + C_l^{2k+1}\sigma^2(s; w)c_{n-1}]\sigma^2(s; w)\hat{p}\}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s; w)a_{n-1}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \\
&- [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s; w)b_{n-1}][\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + [C_{l+1}^{2k+2}\sigma^2(s; w)c_n + C_l^{2k+1}\sigma^4(s; w)c_{n-1}]\hat{p}\}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s; w)a_{n-1}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \\
&- [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s; w)b_{n-1}][\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + [C_{l+1}^{2k+2}\sigma^2(s; w)b_{n-1} + C_l^{2k+1}\sigma^4(s; w)b_{n-2}]\hat{p}\}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \quad \square
\end{aligned}$$

**推论4.2.1.**  $i^{-n}\hat{p} \cdot [|\sigma(s; w) \times ]^n \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}, n \geq 1, l \geq 0$

$$\begin{aligned}
&= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n + b_n) - C_l^{2k+1}\sigma^2(s; w)(a_{n-1} + b_{n-1})][\sigma(s; w) \cdot \hat{p}]^2 \\
&+ [C_{l+1}^{2k+2}\sigma^2(s; w)c_n + C_l^{2k+1}\sigma^4(s; w)c_{n-1}]\}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n + b_n) - C_l^{2k+1}(1 + b_n)][\sigma(s; w) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2}\sigma^2(s; w)c_n + C_l^{2k+1}\sigma^4(s; w)c_{n-1}]\}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{[C_l^{2k} - C_{l+1}^{2k+2}(a_n + b_n) - C_l^{2k+1}b_n][\sigma(s; w) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2}\sigma^2(s; w)b_{n-1} + C_l^{2k+1}\sigma^4(s; w)b_{n-2}]\}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1}
\end{aligned}$$

**推论4.2.2.**  $i^{-n}\hat{p} \cdot [|\sigma(s; w) \times ]^n \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}, n \geq 1, l \geq 0$

$$\begin{aligned}
&= \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1}k_{n+1} + C_l^{2k+2}k_n + (C_l^{2k-1}b_n + C_l^{2k}b_{n-1})\sigma^2(s; w) + C_l^{2k}][\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\
&- b_{n-1}\sigma^2(s; w)[\sigma(s; w) \cdot \hat{p}]^{l+1}
\end{aligned}$$

**证明:**  $i^{-n}\hat{p} \cdot [|\sigma(s; w) \times ]^n \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}, n \geq 1, l \geq 0$

$$\begin{aligned}
&= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n + b_n) - C_l^{2k+1}\sigma^2(s; w)(a_{n-1} + b_{n-1})][\sigma(s; w) \cdot \hat{p}]^2 \\
&+ [C_{l+1}^{2k+2}\sigma^2(s; w)c_n + C_l^{2k+1}\sigma^4(s; w)c_{n-1}]\}\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n + b_n) - C_l^{2k+1}\sigma^2(s; w)(a_{n-1} + b_{n-1})][\sigma(s; w) \cdot \hat{p}]^2 [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&+ \sum_{k=0}^{[(l-1)/2]} [C_{l+1}^{2k+2}\sigma^2(s; w)c_n + C_l^{2k+1}\sigma^4(s; w)c_{n-1}][\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n + b_n) - C_l^{2k+1}\sigma^2(s; w)(a_{n-1} + b_{n-1})][\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\
&+ \sum_{k=1}^{[(l+1)/2]} [C_{l+1}^{2k}\sigma^2(s; w)c_n + C_l^{2k-1}\sigma^4(s; w)c_{n-1}][\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n + b_n) - C_l^{2k+1}\sigma^2(s; w)(a_{n-1} + b_{n-1})][\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\
&+ \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k}\sigma^2(s; w)c_n + C_l^{2k-1}\sigma^4(s; w)c_{n-1}][\sigma(s; w) \cdot \hat{p}]^{l+1-2k} - \sigma^2(s; w)c_n[\sigma(s; w) \cdot \hat{p}]^{l+1} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k+1} - C_{l+1}^{2k+2}(a_n + b_n) - C_l^{2k+1}\sigma^2(s; w)(a_{n-1} + b_{n-1}) + C_{l+1}^{2k}\sigma^2(s; w)c_n + C_l^{2k-1}\sigma^4(s; w)c_{n-1}] \\
&[\sigma(s; w) \cdot \hat{p}]^{l+1-2k} - \sigma^2(s; w)c_n[\sigma(s; w) \cdot \hat{p}]^{l+1} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1}k_{n+1} + C_l^{2k+2}k_n + (C_l^{2k-1}c_{n+1} + C_l^{2k}c_n)\sigma^2(s; w) + C_l^{2k}]
\end{aligned}$$



$$\begin{aligned}
& [\sigma(s; w) \cdot \hat{p}]^{l+1-2k} - c_n \sigma^2(s; w) [\sigma(s; w) \cdot \hat{p}]^{l+1} \\
&= \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s; w) + C_l^{2k}] [\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\
&- b_{n-1} \sigma^2(s; w) [\sigma(s; w) \cdot \hat{p}]^{l+1}
\end{aligned} \quad \square$$

**推论4.2.3.**  $i^{-n} \hat{p} \cdot |[\sigma(s; w) \times ]^n \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}, n \geq 0, l \geq 0$

$$\begin{aligned}
&= \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s; w) + C_l^{2k}] [\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\
&- b_{n-1} \sigma^2(s; w) [\sigma(s; w) \cdot \hat{p}]^{l+1}
\end{aligned}$$

#### 4.2.2 $i^{-n} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^l |[\times \sigma(s; w)]^n$ 的通项公式

**定理4.2.2.**  $i^{-n} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^l |[\times \sigma(s; w)]^n, n \geq 1, l \geq 0$

$$\begin{aligned}
&= \sum_{k=0}^{[l/2]} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s; w) a_{n-1}] [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \\
&- [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s; w) b_{n-1}] \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + [C_{l+1}^{2k+2} \sigma^2(s; w) b_{n-1} + C_l^{2k+1} \sigma^4(s; w) b_{n-2}] \hat{p} \}
\end{aligned}$$

**证明:**  $i^{-n} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^l |[\times \sigma(s; w)]^n, n \geq 1, l \geq 0$

$$\begin{aligned}
&= i^{-(n-1)} i^{-1} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^l \times \sigma(s; w) |[\times \sigma(s; w)]^{n-1} \\
&= i^{-(n-1)} \sum_{k=0}^{[l/2]} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \{ (C_l^{2k+1} + C_l^{2k+2}) \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] \\
&+ (C_l^{2k} - C_l^{2k+2}) [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - C_l^{2k+1} \sigma^2(s; w) \hat{p} \} |[\times \sigma(s; w)]^{n-1} \\
&= \sum_{k=0}^{[l/2]} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \{ (C_l^{2k+1} + C_l^{2k+2}) \{ [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - i^{-n} \hat{p} |[\times \sigma(s; w)]^n \} \\
&+ (C_l^{2k} - C_l^{2k+2}) [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - C_l^{2k+1} \sigma^2(s; w) i^{-(n-1)} \hat{p} |[\times \sigma(s; w)]^{n-1} \} \} \\
&= \sum_{k=0}^{[l/2]} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \{ -C_{l+1}^{2k+2} i^{-n} \hat{p} |[\times \sigma(s; w)]^n + C_{l+1}^{2k+1} [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \\
&- C_l^{2k+1} \sigma^2(s; w) i^{-(n-1)} \hat{p} |[\times \sigma(s; w)]^{n-1} \} \\
&= \sum_{k=0}^{[l/2]} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \{ -C_{l+1}^{2k+2} \{ a_n [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + b_n \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] - c_n \sigma^2(s; w) \hat{p} \} \\
&+ C_{l+1}^{2k+1} [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) - C_l^{2k+1} \sigma^2(s; w) \{ a_{n-1} [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + b_{n-1} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] - c_{n-1} \sigma^2(s; w) \hat{p} \} \} \\
&= \sum_{k=0}^{[l/2]} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s; w) a_{n-1}] [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \\
&- [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s; w) b_{n-1}] \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + [C_{l+1}^{2k+2} \sigma^2(s; w) c_n + C_l^{2k+1} \sigma^4(s; w) c_{n-1}] \hat{p} \} \\
&= \sum_{k=0}^{[l/2]} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s; w) a_{n-1}] [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \\
&- [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s; w) b_{n-1}] \sigma(s; w) [\sigma(s; w) \cdot \hat{p}] + [C_{l+1}^{2k+2} \sigma^2(s; w) b_{n-1} + C_l^{2k+1} \sigma^4(s; w) b_{n-2}] \hat{p} \}
\end{aligned} \quad \square$$

**推论4.2.4.**  $i^{-n} \hat{p} \cdot |[\sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^l |[\times \sigma(s; w)]^n, n \geq 1, l \geq 0$

$$\begin{aligned}
&= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} \sigma^2(s; w) (a_{n-1} + b_{n-1})] [\sigma(s; w) \cdot \hat{p}]^2 \\
&+ [C_{l+1}^{2k+2} \sigma^2(s; w) c_n + C_l^{2k+1} \sigma^4(s; w) c_{n-1}] \} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{ [C_{l+1}^{2k+1} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} (1 + b_n)] [\sigma(s; w) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2} \sigma^2(s; w) c_n + C_l^{2k+1} \sigma^4(s; w) c_{n-1}] \} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\
&= \sum_{k=0}^{[l/2]} \{ [C_l^{2k} - C_{l+1}^{2k+2} (a_n + b_n) - C_l^{2k+1} b_n] [\sigma(s; w) \cdot \hat{p}]^2 + [C_{l+1}^{2k+2} \sigma^2(s; w) b_{n-1} + C_l^{2k+1} \sigma^4(s; w) b_{n-2}] \} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1}
\end{aligned}$$

**推论4.2.5.**  $i^{-n} \hat{p} \cdot |[\sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^l |[\times \sigma(s; w)]^n, n \geq 0, l \geq 0$

$$\begin{aligned}
&= \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s; w) + C_l^{2k}] [\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\
&- b_{n-1} \sigma^2(s; w) [\sigma(s; w) \cdot \hat{p}]^{l+1}
\end{aligned}$$

**推论4.2.6.**  $i^{-n} \hat{p} \cdot |[\sigma(s; w) \times ]^n \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\} = i^{-n} \hat{p} \cdot |[\sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^l |[\times \sigma(s; w)]^n, n \geq 0, l \geq 0$

**4.2.3**  $i^{-n}\sigma(s; w) \cdot |[\sigma(s; w) \times |]^n\{[\sigma(s; w) \cdot \hat{p}]^l\sigma(s; w)\}, i^{-n}\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l\}[\times \sigma(s; w)]^n \cdot \sigma(s; w)$  的通项公式**定理4.2.3.**  $i^{-n}\sigma(s; w) \cdot |[\sigma(s; w) \times |]^n\{[\sigma(s; w) \cdot \hat{p}]^l\sigma(s; w)\} = i^{-n}\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l\}[\times \sigma(s; w)]^n \cdot \sigma(s; w)$ 

$$= \sum_{k=0}^{[l/2]} [C_l^{2k}\sigma^2(s; w) - C_{l+1}^{2k+2}][\sigma(s; w) \cdot \hat{p}]^{l-2k}, n \geq 1, l \geq 0$$

**证明:**  $n \geq 1, l \geq 0$ 

$$\begin{aligned} & i^{-n}\sigma(s; w) \cdot |[\sigma(s; w) \times |]^n\{[\sigma(s; w) \cdot \hat{p}]^l\sigma(s; w)\} = i^{-n}\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l\}[\times \sigma(s; w)]^n \cdot \sigma(s; w) \\ &= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s; w)a_{n-1}]\sigma^2(s; w)[\sigma(s; w) \cdot \hat{p}] \\ & - [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s; w)b_{n-1}]\sigma(s; w) \cdot [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) \\ & + [C_{l+1}^{2k+2}\sigma^2(s; w)b_{n-1} + C_l^{2k+1}\sigma^4(s; w)b_{n-2}][\sigma(s; w) \cdot \hat{p}]\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\ &= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s; w)a_{n-1}]\sigma^2(s; w)[\sigma(s; w) \cdot \hat{p}] \\ & - [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s; w)b_{n-1}][\sigma^2(s; w) - 1][\sigma(s; w) \cdot \hat{p}] \\ & + [C_{l+1}^{2k+2}\sigma^2(s; w)b_{n-1} + C_l^{2k+1}\sigma^4(s; w)b_{n-2}][\sigma(s; w) \cdot \hat{p}]\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\ &= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s; w)a_{n-1} - C_{l+1}^{2k+2}b_n - C_l^{2k+1}\sigma^2(s; w)b_{n-1}]\sigma^2(s; w) \\ & + [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s; w)b_{n-1}] + [C_{l+1}^{2k+2}\sigma^2(s; w)b_{n-1} + C_l^{2k+1}\sigma^4(s; w)b_{n-2}]\}[\sigma(s; w) \cdot \hat{p}]^{l-2k} \\ &= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} + C_{l+1}^{2k+2}k_n + C_l^{2k+1}\sigma^2(s; w)k_{n-1}]\sigma^2(s; w) + [C_{l+1}^{2k+2}b_{n+1} + C_l^{2k+1}\sigma^2(s; w)b_n]\}[\sigma(s; w) \cdot \hat{p}]^{l-2k} \\ &= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}\sigma^{-2}(s; w)(b_{n+1} + 1) - C_l^{2k+1}(b_n + 1)]\sigma^2(s; w) + [C_{l+1}^{2k+2}b_{n+1} + C_l^{2k+1}\sigma^2(s; w)b_n]\}[\sigma(s; w) \cdot \hat{p}]^{l-2k} \\ &= \sum_{k=0}^{[l/2]} [C_l^{2k}\sigma^2(s; w) - C_{l+1}^{2k+2}][\sigma(s; w) \cdot \hat{p}]^{l-2k} \quad \square \end{aligned}$$

**推论4.2.7.**  $i^{-n}\sigma(s; w) \cdot |[\sigma(s; w) \times |]^n\{[\sigma(s; w) \cdot \hat{p}]^l\sigma(s; w)\} = i^{-n}\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l\}[\times \sigma(s; w)]^n \cdot \sigma(s; w)$ 

$$= \sigma(s; w) \cdot |[\sigma(s; w) \cdot \hat{p}]^l\sigma(s; w) = \sum_{k=0}^{[l/2]} [C_l^{2k}\sigma^2(s; w) - C_{l+1}^{2k+2}][\sigma(s; w) \cdot \hat{p}]^{l-2k}, n \geq 0, l \geq 0$$

**4.2.4**  $i^{-n}[\sigma(s; w) \times |]^n\{[\sigma(s; w) \cdot \hat{p}]^l\sigma(s; w)\} \cdot \sigma(s; w), i^{-n}\sigma(s; w) \cdot |[\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l\}[\times \sigma(s; w)]^n$  的通项公式**定理4.2.4.**  $i^{-n}[\sigma(s; w) \times |]^n\{[\sigma(s; w) \cdot \hat{p}]^l\sigma(s; w)\} \cdot \sigma(s; w) = i^{-n}\sigma(s; w) \cdot |[\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l\}[\times \sigma(s; w)]^n$ 

$$= \sum_{k=0}^{[l/2]} [C_l^{2k}\sigma^2(s; w) - C_{l+2}^{2k+2} + C_{l+1}^{2k+2}(a_n - b_n) + C_l^{2k+1}\sigma^2(s; w)(a_{n-1} - b_{n-1})][\sigma(s; w) \cdot \hat{p}]^{l-2k}, n \geq 1, l \geq 0$$

**证明:**  $n \geq 1, l \geq 0$ 

$$\begin{aligned} & i^{-n}[\sigma(s; w) \times |]^n\{[\sigma(s; w) \cdot \hat{p}]^l\sigma(s; w)\} \cdot \sigma(s; w) = i^{-n}\sigma(s; w) \cdot |[\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l\}[\times \sigma(s; w)]^n \\ &= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s; w)a_{n-1}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \cdot \sigma(s; w) \\ & - [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s; w)b_{n-1}][\sigma(s; w) \cdot \hat{p}]\sigma^2(s; w) \\ & + [C_{l+1}^{2k+2}\sigma^2(s; w)b_{n-1} + C_l^{2k+1}\sigma^4(s; w)b_{n-2}][\sigma(s; w) \cdot \hat{p}]\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\ &= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s; w)a_{n-1}][\sigma^2(s; w) - 1][\sigma(s; w) \cdot \hat{p}] \\ & - [C_{l+1}^{2k+2}b_n + C_l^{2k+1}\sigma^2(s; w)b_{n-1}]\sigma^2(s; w)[\sigma(s; w) \cdot \hat{p}] \\ & + [C_{l+1}^{2k+2}\sigma^2(s; w)b_{n-1} + C_l^{2k+1}\sigma^4(s; w)b_{n-2}][\sigma(s; w) \cdot \hat{p}]\}[\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\ &= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}a_n - C_l^{2k+1}\sigma^2(s; w)a_{n-1} - C_{l+1}^{2k+2}b_n - C_l^{2k+1}\sigma^2(s; w)b_{n-1}]\sigma^2(s; w) \\ & + [-C_{l+1}^{2k+1} + C_{l+1}^{2k+2}a_n + C_l^{2k+1}\sigma^2(s; w)a_{n-1}] + [C_{l+1}^{2k+2}\sigma^2(s; w)b_{n-1} + C_l^{2k+1}\sigma^4(s; w)b_{n-2}]\}[\sigma(s; w) \cdot \hat{p}]^{l-2k} \\ &= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} + C_{l+1}^{2k+2}k_n + C_l^{2k+1}\sigma^2(s; w)k_{n-1}]\sigma^2(s; w) + [C_{l+1}^{2k+2}b_{n+1} + C_l^{2k+1}\sigma^2(s; w)b_n] \\ & + [-C_{l+1}^{2k+1} + C_{l+1}^{2k+2}(a_n - b_n) + C_l^{2k+1}\sigma^2(s; w)(a_{n-1} - b_{n-1})]\}[\sigma(s; w) \cdot \hat{p}]^{l-2k} \\ &= \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2}\sigma^{-2}(s; w)(b_{n+1} + 1) - C_l^{2k+1}(b_n + 1)]\sigma^2(s; w) + [C_{l+1}^{2k+2}b_{n+1} + C_l^{2k+1}\sigma^2(s; w)b_n] \end{aligned}$$

$$\begin{aligned}
& + [-C_{l+1}^{2k+1} + C_{l+1}^{2k+2}(a_n - b_n) + C_l^{2k+1}\sigma^2(s; w)(a_{n-1} - b_{n-1})][\sigma(s; w) \cdot \hat{p}]^{l-2k} \\
& = \sum_{k=0}^{\lfloor l/2 \rfloor} \{C_l^{2k}\sigma^2(s; w) - C_{l+1}^{2k+2} + [-C_{l+1}^{2k+1} + C_{l+1}^{2k+2}(a_n - b_n) + C_l^{2k+1}\sigma^2(s; w)(a_{n-1} - b_{n-1})]\}[\sigma(s; w) \cdot \hat{p}]^{l-2k} \\
& = \sum_{k=0}^{\lfloor l/2 \rfloor} [C_l^{2k}\sigma^2(s; w) - C_{l+2}^{2k+2} + C_{l+1}^{2k+2}(a_n - b_n) + C_l^{2k+1}\sigma^2(s; w)(a_{n-1} - b_{n-1})][\sigma(s; w) \cdot \hat{p}]^{l-2k}
\end{aligned}$$

□

## 5 复杂自旋复合算符通项公式的另一种独立求法

### 5.1 $i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}\}$ 通项公式的另一种求法

#### 5.1.1 $i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}\}$ 的通项公式的试探和猜想

定义5.1.1.  $A(1, n) := i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}\}$ ,  $A(1, 0) = \sigma(s; w) \cdot \hat{p}$

推论5.1.1. 
$$\begin{cases} \sigma(s; w)[\times \hat{p}]^{2k-1} = (-1)^{k+1}\sigma(s; w) \times \hat{p} \\ \sigma(s; w)[\times \hat{p}]^{2k} = (-1)^{k+1}[\sigma(s; w) \cdot \hat{p}]\hat{p} - \sigma(s; w) \end{cases}$$

$$\Rightarrow \begin{cases} i^{2k-1}i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{\sigma(s; w)[\times \hat{p}]^{2k-1}\}\} = \{[\sigma(s; w) \cdot \hat{p}]^2 - \sigma^2(s; w)\} \\ i^{2k}i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{\sigma(s; w)[\times \hat{p}]^{2k}\}\} = \sigma(s; w) \cdot \hat{p} \end{cases}$$

推论5.1.2.  $i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}\}$

$$= i^{-1}i^n \hat{p} \cdot \{\sigma(s; w) \times \{\sigma(s; w)[\times \hat{p}]^n\}\} - \sum_{k=0}^{n-1} C_n^k i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w)\}\} [-\sigma(s; w) \cdot \hat{p}]^{n-k}$$

推论5.1.3.

$$A(1, n) = \{[\sigma(s; w) \cdot \hat{p}]^{1+n\%2} - (n\%2)\sigma^2(s; w)\} - \sum_{k=0}^{n-1} C_n^k A(1, k) [-\sigma(s; w) \cdot \hat{p}]^{n-k}, A(1, 0) = -\sigma(s; w) \cdot \hat{p}$$

性质5.1.1.

$$\begin{cases} A(1, 0) = 1[\sigma(s; w) \cdot \hat{p}] \\ A(1, 1) = 2[\sigma(s; w) \cdot \hat{p}]^2 - \sigma^2(s; w) \\ A(1, 2) = 3[\sigma(s; w) \cdot \hat{p}]^3 - [2\sigma^2(s; w) - 1][\sigma(s; w) \cdot \hat{p}] \\ A(1, 3) = 4[\sigma(s; w) \cdot \hat{p}]^4 - [3\sigma^2(s; w) - 4][\sigma(s; w) \cdot \hat{p}]^2 - \sigma^2(s; w) \\ A(1, 4) = 5[\sigma(s; w) \cdot \hat{p}]^5 - [4\sigma^2(s; w) - 10][\sigma(s; w) \cdot \hat{p}]^3 - [4\sigma^2(s; w) - 1][\sigma(s; w) \cdot \hat{p}]^1 \\ A(1, 5) = 6[\sigma(s; w) \cdot \hat{p}]^6 - [5\sigma^2(s; w) - 20][\sigma(s; w) \cdot \hat{p}]^4 - [10\sigma^2(s; w) - 6][\sigma(s; w) \cdot \hat{p}]^2 - \sigma^2(s; w) \end{cases}$$

重新整理得到:

性质5.1.2.

$$\begin{cases} A(1, 0) = C_1^1[\sigma(s; w) \cdot \hat{p}] \\ A(1, 1) = C_2^1[\sigma(s; w) \cdot \hat{p}]^2 - C_1^1\sigma^2(s; w) \\ A(1, 2) = C_3^1[\sigma(s; w) \cdot \hat{p}]^3 - [C_2^1\sigma^2(s; w) - C_3^3][\sigma(s; w) \cdot \hat{p}] \\ A(1, 3) = C_4^1[\sigma(s; w) \cdot \hat{p}]^4 - [C_3^1\sigma^2(s; w) - C_4^3][\sigma(s; w) \cdot \hat{p}]^2 - C_3^3\sigma^2(s; w) \\ A(1, 4) = C_5^1[\sigma(s; w) \cdot \hat{p}]^5 - [C_4^1\sigma^2(s; w) - C_5^3][\sigma(s; w) \cdot \hat{p}]^3 - [C_4^3\sigma^2(s; w) - C_5^5][\sigma(s; w) \cdot \hat{p}]^1 \\ A(1, 5) = C_6^1[\sigma(s; w) \cdot \hat{p}]^6 - [C_5^1\sigma^2(s; w) - C_6^3][\sigma(s; w) \cdot \hat{p}]^4 - [C_5^3\sigma^2(s; w) - C_6^5][\sigma(s; w) \cdot \hat{p}]^2 - C_5^5\sigma^2(s; w) \end{cases}$$

猜想5.1.1.

$$A(1, n) = C_{n+1}^1[\sigma(s; w) \cdot \hat{p}]^{n+1} - [C_n^1\sigma^2(s; w) - C_{n+1}^3][\sigma(s; w) \cdot \hat{p}]^{n-1} - [C_n^3\sigma^2(s; w) - C_{n+1}^5][\sigma(s; w) \cdot \hat{p}]^{n-3} + \dots$$

猜想5.1.2.  $i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}\} = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} [C_{n+1}^{2k+1} - C_n^{2k-1}\sigma^2(s; w)][\sigma(s; w) \cdot \hat{p}]^{n+1-2k}, n \geq 0$

下面用数学归纳法严格证明以上猜想。

5.1.2  $i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}\}$ 的通项公式的有关引理及其证明

推论5.1.4.

$$A(1, n) = \{[\sigma(s; w) \cdot \hat{p}]^{1+n\%2} - (n\%2)\sigma^2(s; w)\} - \sum_{l=0}^{n-1} C_n^l A(1, l) [-\sigma(s; w) \cdot \hat{p}]^{n-l}, A(1, 0) = -\sigma(s; w) \cdot \hat{p}$$

$$\text{引理5.1.1. } \sum_{l=0}^n \sum_{k=0}^{[(l+1)/2]} A(k, l) = \sum_{k=0}^{[(n+1)/2]} \sum_{l=2k-1|0}^n A(k, l)$$

$$\text{引理5.1.2. } \sum_{l=2k}^n (-1)^{n-l} C_{n+1}^l C_l^{2k} = C_{n+1}^{2k}, \sum_{l=2k+1}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{2(k+1)} = C_{n+2}^{2(k+1)} - \delta_{n, 2k+1}$$

$$\text{引理5.1.3. } \sum_{l=r}^n (-1)^{n-l} C_{n+1}^l C_l^r = C_{n+1}^r, \sum_{l=r}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{r+1} = C_{n+2}^{r+1} - \delta_{nr}$$

引理5.1.4.

$$\begin{aligned} & \{[\sigma(s; w) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s; w)\} - \sum_{k=0}^{[(n+1)/2]} [\sigma(s; w) \cdot \hat{p}]^{n+2-2k} \sum_{l=2k-1|0}^n (-1)^{n-l} C_{n+1}^l [C_l^{2k-1} \sigma^2(s; w) - C_{l+1}^{2k+1}] \\ &= \sum_{k=0}^{[(n+2)/2]} [C_{n+2}^{2k+1} - C_{n+1}^{2k-1} \sigma^2(s; w)] [\sigma(s; w) \cdot \hat{p}]^{n+2-2k} \end{aligned}$$

5.1.3  $i^{-1}\hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}\}$ 的通项公式的数学归纳法证明

$$\text{定理5.1.1. } A(1, n) = \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k+1} - C_n^{2k-1} \sigma^2(s; w)] [\sigma(s; w) \cdot \hat{p}]^{n+1-2k}, n \geq 0$$

$$\begin{aligned} \text{证明: } & A(1, n+1) = \{[\sigma(s; w) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s; w)\} - \sum_{l=0}^n C_{n+1}^l A(1, l) [-\sigma(s; w) \cdot \hat{p}]^{n+1-l} \\ &= \{[\sigma(s; w) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s; w)\} \\ &- \sum_{l=0}^n C_{n+1}^l \sum_{k=0}^{[(l+1)/2]} [C_{l+1}^{2k+1} - C_l^{2k-1} \sigma^2(s; w)] [\sigma(s; w) \cdot \hat{p}]^{l+1-2k} [-\sigma(s; w) \cdot \hat{p}]^{n+1-l} \\ &= \{[\sigma(s; w) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s; w)\} - \sum_{l=0}^n \sum_{k=0}^{[(l+1)/2]} (-1)^{n+1-l} C_{n+1}^l [C_{l+1}^{2k+1} - C_l^{2k-1} \sigma^2(s; w)] [\sigma(s; w) \cdot \hat{p}]^{n+2-2k} \\ &= \{[\sigma(s; w) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s; w)\} \\ &- \sum_{k=0}^{[(n+1)/2]} \sum_{l=2k-1|0}^n (-1)^{n+1-l} C_{n+1}^l [C_{l+1}^{2k+1} - C_l^{2k-1} \sigma^2(s; w)] [\sigma(s; w) \cdot \hat{p}]^{n+2-2k} \\ &= \{[\sigma(s; w) \cdot \hat{p}]^{1+(n+1)\%2} - (n+1)\%2\sigma^2(s; w)\} - \sum_{k=0}^{[(n+1)/2]} [\sigma(s; w) \cdot \hat{p}]^{n+2-2k} \sum_{l=2k-1|0}^n (-1)^{n-l} C_{n+1}^l [C_l^{2k-1} \sigma^2(s; w) - C_{l+1}^{2k+1}] \\ &= \sum_{k=0}^{[(n+1)/2]} [C_{n+1}^{2k+1} - C_n^{2k-1} \sigma^2(s; w)] [\sigma(s; w) \cdot \hat{p}]^{n+1-2k} \quad \square \end{aligned}$$

5.2  $i^{-n}\hat{p} \cdot \|\sigma(s; w) \times \|\|^n[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)$  和  $i^{-n}\hat{p} \cdot \|\sigma(s; w) \|\sigma(s; w) \cdot \hat{p}\|^l \|\times \sigma(s; w)\|^n$  通项公式的另一种求法5.2.1  $i^{-n}\hat{p} \cdot \|\sigma(s; w) \times \|\|^n[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)$  的通项公式

$$\text{定义5.2.1. } A_L(n, l) := i^{-n}\hat{p} \cdot \|\sigma(s; w) \times \|\|^n[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w), A_L(n, 0) = \sigma(s; w) \cdot \hat{p}$$

推论5.2.1.

$$\begin{aligned} & i^{-n}\hat{p} \cdot \|\sigma(s; w) \times \|\|^n[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w) \\ &= i^{-n}i^l \hat{p} \cdot \|\sigma(s; w) \times \|\|^n \sigma(s; w) \|\times \hat{p}\|^l - \sum_{k=0}^{l-1} C_l^k i^{-n}\hat{p} \cdot \|\sigma(s; w) \times \|\|^n[\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [-\sigma(s; w) \cdot \hat{p}]^{l-k} \end{aligned}$$

推论5.2.2.

$$\begin{cases} i^{-n}\hat{p} \cdot \|\sigma(s; w) \times \|\|^n i^{2k-1} \sigma(s; w) \|\times \hat{p}\|^{2k-1} \\ = -i^{-(n+1)}\hat{p} \cdot \|\sigma(s; w) \times \|\|^n \sigma(s; w) \times \hat{p} = -(a_{n+1} + b_{n+1})[\sigma(s; w) \cdot \hat{p}]^2 + c_{n+1} \sigma^2(s; w) \\ i^{-n}\hat{p} \cdot \|\sigma(s; w) \times \|\|^n i^{2k} \sigma(s; w) \|\times \hat{p}\|^{2k} \\ = -i^{-n}\hat{p} \cdot \|\sigma(s; w) \times \|\|^n \{[\sigma(s; w) \cdot \hat{p}] \hat{p} - \sigma(s; w)\} = -(a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^3 + [c_n \sigma^2(s; w) + 1][\sigma(s; w) \cdot \hat{p}] \end{cases}$$

推论5.2.3.  $i^{l-n}[\sigma(s; w) \times ||^n \sigma(s; w)] \times \hat{p}^l$

$$= \begin{cases} -a_{n+1}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - b_{n+1}[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + c_{n+1}\sigma^2(s; w)\hat{p}, l = 2k - 1 \\ -a_n\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^2 - b_n[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + c_n\sigma^2(s; w)\hat{p}[\sigma(s; w) \cdot \hat{p}] + \sigma(s; w), l = 2k \\ i^{-n}[\sigma(s; w) \times ||^n \hat{p} = a_n\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_n[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - c_n\sigma^2(s; w)\hat{p} \end{cases}$$

推论5.2.4.

$$i^{l-n}\hat{p} \cdot ||[\sigma(s; w) \times ||^n \sigma(s; w)] \times \hat{p}^l = \begin{cases} -(a_{n+1} + b_{n+1})[\sigma(s; w) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s; w), l = 2k - 1 \\ -(a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^3 + [1 + c_n\sigma^2(s; w)][\sigma(s; w) \cdot \hat{p}], l = 2k \end{cases}$$

定理5.2.1.

$$\begin{cases} A_L(n, l) = i^{l-n}\hat{p} \cdot ||[\sigma(s; w) \times ||^n \sigma(s; w)] \times \hat{p}^l - \sum_{k=0}^{l-1} C_l^k A_L(n, k)[- \sigma(s; w) \cdot \hat{p}]^{l-k}, A_L(n, 0) = \sigma(s; w) \cdot \hat{p} \\ i^{l-n}\hat{p} \cdot ||[\sigma(s; w) \times ||^n \sigma(s; w)] \times \hat{p}^l = \begin{cases} k_{n+1}[\sigma(s; w) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s; w), l = 2k - 1; k_n = -(a_n + b_n) \\ k_n[\sigma(s; w) \cdot \hat{p}]^3 + [1 + c_n\sigma^2(s; w)][\sigma(s; w) \cdot \hat{p}], l = 2k \end{cases} \end{cases}$$

推论5.2.5.

$$\begin{cases} A_L(n, 0) = C_0^0 \sigma(s; w) \cdot \hat{p} \\ A_L(n, 1) = (C_1^1 k_{n+1} + C_1^0)[\sigma(s; w) \cdot \hat{p}]^2 + C_1^1 c_{n+1} \sigma^2(s; w), k_{n+1} = -(a_{n+1} + b_{n+1}) \\ A_L(n, 2) = [C_2^1 k_{n+1} + C_2^2 k_n + C_2^0][\sigma(s; w) \cdot \hat{p}]^3 + [(C_2^1 c_{n+1} + C_2^2 c_n) \sigma^2(s; w) + C_2^2][\sigma(s; w) \cdot \hat{p}] \\ A_L(n, 3) = [C_3^1 k_{n+1} + C_3^2 k_n + C_3^0][\sigma(s; w) \cdot \hat{p}]^4 + [C_3^3 k_{n+1} + (C_3^1 c_{n+1} + C_3^2 c_n) \sigma^2(s; w) + C_3^2][\sigma(s; w) \cdot \hat{p}]^2 \\ + C_3^3 c_{n+1} \sigma^2(s; w) \\ A_L(n, 4) = [C_4^1 k_{n+1} + C_4^2 k_n + C_4^0][\sigma(s; w) \cdot \hat{p}]^5 + [C_4^3 k_{n+1} + C_4^4 k_n + (C_4^1 c_{n+1} + C_4^2 c_n) \sigma^2(s; w) + C_4^2][\sigma(s; w) \cdot \hat{p}]^3 \\ + [(C_4^3 c_{n+1} + C_4^4 c_n) \sigma^2(s; w) + C_4^4][\sigma(s; w) \cdot \hat{p}]^1 \end{cases}$$

定理5.2.2.

$$\begin{cases} A_L(n, l) = i^{-n}\hat{p} \cdot ||[\sigma(s; w) \times ||^n [\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)] \\ = \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} c_{n+1} + C_l^{2k} c_n) \sigma^2(s; w) + C_l^{2k}][\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\ - c_n \sigma^2(s; w) [\sigma(s; w) \cdot \hat{p}]^{l+1} \\ k_n = -(a_n + b_n), n \geq 0, l \geq 0 \end{cases}$$

推论5.2.6.

$$\begin{cases} A_L(n, l) = i^{-n}\hat{p} \cdot ||[\sigma(s; w) \times ||^n [\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)] \\ = \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s; w) + C_l^{2k}][\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\ - b_{n-1} \sigma^2(s; w) [\sigma(s; w) \cdot \hat{p}]^{l+1}; k_n = -(a_n + b_n), n \geq 0, l \geq 0 \end{cases}$$

可以用数学归纳法严格证明以上定理, 有空再补上。

### 5.2.2 $i^{-n}\hat{p} \cdot |\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l || \times \sigma(s; w)|^n$ 的通项公式

定义5.2.2.  $A_R(n, l) := i^{-n}\hat{p} \cdot |\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l || \times \sigma(s; w)|^n, A_R(n, 0) = \sigma(s; w) \cdot \hat{p}$

性质5.2.1.  $\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l = i^l [\hat{p} \times ||^l \sigma(s; w) - \sum_{k=0}^{l-1} C_n^k [-\sigma(s; w) \cdot \hat{p}]^{l-k} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^k$

推论5.2.7.  $i^{-n}\hat{p} \cdot |\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l || \times \sigma(s; w)|^n$

$$= i^{l-n}\hat{p} \cdot ||[\hat{p} \times ||^l \sigma(s; w)] \times \sigma(s; w)|^n - \sum_{k=0}^{l-1} C_n^k [-\sigma(s; w) \cdot \hat{p}]^{l-k} i^{-n}\hat{p} \cdot |\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^k || \times \sigma(s; w)|^n$$

推论5.2.8.  $i^{l-n}[\hat{p} \times ]^l \sigma(s; w)[\times \sigma(s; w)]^n$

$$= \begin{cases} -a_{n+1}[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - b_{n+1}\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + c_{n+1}\sigma^2(s; w)\hat{p}, l = 2k - 1 \\ -a_n[\sigma(s; w) \cdot \hat{p}]^2\sigma(s; w) - b_n[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + c_n\sigma^2(s; w)\hat{p}[\sigma(s; w) \cdot \hat{p}] + \sigma(s; w), l = 2k \\ i^{-n}\hat{p}[\times \sigma(s; w)]^n = a_n[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + b_n\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - c_n\sigma^2(s; w)\hat{p} \end{cases}$$

$$\text{推论5.2.9. } i^{l-n}\hat{p} \cdot [[\hat{p} \times ]^l \sigma(s; w)[\times \sigma(s; w)]^n = \begin{cases} -(a_{n+1} + b_{n+1})[\sigma(s; w) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s; w), l = 2k - 1 \\ -(a_n + b_n)[\sigma(s; w) \cdot \hat{p}]^3 + [1 + c_n\sigma^2(s; w)][\sigma(s; w) \cdot \hat{p}], l = 2k \end{cases}$$

定理5.2.3.

$$\begin{cases} A_R(n, l) = i^{l-n}\hat{p} \cdot [[\hat{p} \times ]^l \sigma(s; w)[\times \sigma(s; w)]^n - \sum_{k=0}^{l-1} C_n^k [-\sigma(s; w) \cdot \hat{p}]^{l-k} A_R(n, k), A_R(n, 0) = \sigma(s; w) \cdot \hat{p} \\ i^{l-n}\hat{p} \cdot [[\hat{p} \times ]^l \sigma(s; w)[\times \sigma(s; w)]^n = \begin{cases} k_{n+1}[\sigma(s; w) \cdot \hat{p}]^2 + c_{n+1}\sigma^2(s; w), l = 2k - 1; k_n = -(a_n + b_n) \\ k_n[\sigma(s; w) \cdot \hat{p}]^3 + [1 + c_n\sigma^2(s; w)][\sigma(s; w) \cdot \hat{p}], l = 2k \end{cases} \end{cases}$$

与上节的离散方程和初始条件完全等价, 故有以下相同的解。

定理5.2.4.

$$\begin{cases} A_R(n, l) = i^{-n}\hat{p} \cdot |\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l|[\times \sigma(s; w)]^n \\ = \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1}k_{n+1} + C_l^{2k+2}k_n + (C_l^{2k-1}c_{n+1} + C_l^{2k}c_n)\sigma^2(s; w) + C_l^{2k}][\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\ -c_n\sigma^2(s; w)[\sigma(s; w) \cdot \hat{p}]^{l+1}; k_n = -(a_n + b_n), n \geq 0, l \geq 0 \end{cases}$$

推论5.2.10.

$$\begin{cases} A_R(n, l) = i^{-n}\hat{p} \cdot |\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l|[\times \sigma(s; w)]^n \\ = \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1}k_{n+1} + C_l^{2k+2}k_n + (C_l^{2k-1}b_n + C_l^{2k}b_{n-1})\sigma^2(s; w) + C_l^{2k}][\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\ -b_{n-1}\sigma^2(s; w)[\sigma(s; w) \cdot \hat{p}]^{l+1}; k_n = -(a_n + b_n), n \geq 0, l \geq 0 \end{cases}$$

推论5.2.11.  $A_L(n, l) = A_R(n, l), \hat{p} \cdot [[\sigma(s; w) \times ]^n [\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)] = \hat{p} \cdot |\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l|[\times \sigma(s; w)]^n, n \geq 0$

## 6 各种通项公式汇总小结

### 6.1 基本通项公式小结

定理6.1.1.

$$\begin{cases} i^n \sigma(s; w)[\times \hat{p}]^n = \sum_{k=0}^n C_n^k [\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [-\sigma(s; w) \cdot \hat{p}]^{n-k}, n \geq 0 = \begin{cases} i\sigma(s; w) \times \hat{p}, n = 2k - 1, k \geq 1 \\ \sigma(s; w) - [\sigma(s; w) \cdot \hat{p}]\hat{p}, n = 2k \end{cases} \\ i^n [\hat{p} \times ]^n \sigma(s; w) = \sum_{k=0}^n C_n^k [-\sigma(s; w) \cdot \hat{p}]^{n-k} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^k, n \geq 0 = \begin{cases} i\hat{p} \times \sigma(s; w), n = 2k - 1, k \geq 1 \\ \sigma(s; w) - [\sigma(s; w) \cdot \hat{p}]\hat{p}, n = 2k \end{cases} \end{cases}$$

定理6.1.2.

$$\begin{cases} i^{-n}[\sigma(s; w) \times ]^n \hat{p} = a_n \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + b_n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - b_{n-1}\sigma^2(s; w)\hat{p}, n \geq 0 \\ i^{-n}\hat{p}[\times \sigma(s; w)]^n = a_n [\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + b_n \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - b_{n-1}\sigma^2(s; w)\hat{p}, n \geq 0 \end{cases}$$

定理6.1.3.

$$\begin{cases} [\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w) = i^n \sigma(s; w)[\times \hat{p}]^n - \sum_{k=0}^{n-1} C_n^k [\sigma(s; w) \cdot \hat{p}]^k \sigma(s; w) [-\sigma(s; w) \cdot \hat{p}]^{n-k}, n \geq 0 \\ \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^n = i^n [\hat{p} \times ]^n \sigma(s; w) - \sum_{k=0}^{n-1} C_n^k [-\sigma(s; w) \cdot \hat{p}]^{n-k} \sigma(s; w) [\sigma(s; w) \cdot \hat{p}]^k, n \geq 0 \end{cases}$$

## 6.2 基本叉乘型通项公式小结

定理6.2.1.

$$\begin{cases} i^{-1}\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w)\}, n \geq 0 \\ = \sum_{k=0}^{[n/2]} \{(C_n^{2k+1} + C_n^{2k+2})[\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + (C_n^{2k} - C_n^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - C_n^{2k+1} \sigma^2(s; w) \hat{p}\} [\sigma(s; w) \cdot \hat{p}]^{n-2k-1} \\ i^{-1}\{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^n\} \times \sigma(s; w), n \geq 0 \\ = \sum_{k=0}^{[n/2]} [\sigma(s; w) \cdot \hat{p}]^{n-2k-1} \{(C_n^{2k+1} + C_n^{2k+2})\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + (C_n^{2k} - C_n^{2k+2})[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - C_n^{2k+1} \sigma^2(s; w) \hat{p}\} \end{cases}$$

## 6.3 基本叉乘型扩展通项公式小结

定理6.3.1.

$$\begin{cases} i^{-n}[\sigma(s; w) \times ||^n \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}], n \geq 1, l \geq 0 \\ = \sum_{k=0}^{[l/2]} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s; w) a_{n-1}] \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \\ - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s; w) b_{n-1}] [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) + [C_{l+1}^{2k+2} \sigma^2(s; w) b_{n-1} + C_l^{2k+1} \sigma^4(s; w) b_{n-2}] \hat{p}\} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \\ i^{-n} \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l [|\times \sigma(s; w)|]^n, n \geq 1, l \geq 0 \\ = \sum_{k=0}^{[l/2]} [\sigma(s; w) \cdot \hat{p}]^{l-2k-1} \{[C_{l+1}^{2k+1} - C_{l+1}^{2k+2} a_n - C_l^{2k+1} \sigma^2(s; w) a_{n-1}] [\sigma(s; w) \cdot \hat{p}] \sigma(s; w) \\ - [C_{l+1}^{2k+2} b_n + C_l^{2k+1} \sigma^2(s; w) b_{n-1}] \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + [C_{l+1}^{2k+2} \sigma^2(s; w) b_{n-1} + C_l^{2k+1} \sigma^4(s; w) b_{n-2}] \hat{p}\} \end{cases}$$

## 6.4 基本标积型通项公式小结

定理6.4.1.

$$\sigma(s; w) \cdot [\sigma(s; w) \cdot \hat{p}]^n \sigma(s; w) = \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^n \cdot \sigma(s; w) = \sum_{k=0}^{[n/2]} [C_n^{2k} \sigma^2(s; w) - C_{n+1}^{2k+2}] [\sigma(s; w) \cdot \hat{p}]^{n-2k}, n \geq 0$$

## 6.5 基本标积型扩展通项公式小结

定理6.5.1.  $i^n \sigma(s; w)[|\times \hat{p}|^n] \cdot \hat{p} = i^n \hat{p} \cdot [|\hat{p} \times ||^n \sigma(s; w) = 0, n \geq 1$

定理6.5.2.

$$\{i^{-n} \hat{p} \cdot \{[\sigma(s; w) \times ||^n \hat{p}]\} = i^{-n} \{|\hat{p}| \times \sigma(s; w)\}^n \cdot \hat{p} = -k_n [\sigma(s; w) \cdot \hat{p}]^2 + b_{n-1} \sigma^2(s; w), n \geq 0$$

定理6.5.3.

$$\begin{cases} A_L(n, l) = i^{-n} \hat{p} \cdot [|\sigma(s; w) \times ||^n [\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w) = i^{-n} \hat{p} \cdot |\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l [|\times \sigma(s; w)|]^n = A_R(n, l) \\ = \sum_{k=0}^{[(l+1)/2]} [C_l^{2k+1} k_{n+1} + C_l^{2k+2} k_n + (C_l^{2k-1} b_n + C_l^{2k} b_{n-1}) \sigma^2(s; w) + C_l^{2k}] [\sigma(s; w) \cdot \hat{p}]^{l+1-2k} \\ - b_{n-1} \sigma^2(s; w) [\sigma(s; w) \cdot \hat{p}]^{l+1}, n \geq 0, l \geq 0 \end{cases}$$

定理6.5.4.

$$\begin{cases} i^{-n} \sigma(s; w) \cdot [|\sigma(s; w) \times ||^n \{[\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w)\}] = i^{-n} \{\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^l [|\times \sigma(s; w)|]^n\} \cdot \sigma(s; w) \\ = \sigma(s; w) \cdot [\sigma(s; w) \cdot \hat{p}]^l \sigma(s; w) = \sum_{k=0}^{[l/2]} [C_l^{2k} \sigma^2(s; w) - C_{l+1}^{2k+2}] [\sigma(s; w) \cdot \hat{p}]^{l-2k}, n \geq 0, l \geq 0 \end{cases}$$

## 6.6 更一般各种通项公式的探讨

利用以上基本的通项公式, 可以比较容易地严格推导出更多、更复杂的各种通项公式, 不再存在原则上的推导困难。

## 7 特殊的通项公式

### 7.1 一种特殊的通项公式

$$\text{性质7.1.1. } \begin{cases} \hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^1 \sigma(s; w)\}\} = 2i[\sigma(s; w) \cdot \hat{p}]^2 - i\sigma^2(s; w) \\ \hat{p} \cdot \{\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^2 \sigma(s; w)\}\} = 3i[\sigma(s; w) \cdot \hat{p}]^2 - i[\sigma^2(s; w) - 1][\sigma(s; w) \cdot \hat{p}] \end{cases}$$

$$\begin{aligned} \text{定理7.1.1. } & [\sigma(s; w) \cdot \hat{p}]\hat{p} = -i[\sigma(s; w) \times \hat{p}] \times [\sigma(s; w) \times \hat{p}] \\ & = \sigma(s; w) - \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^2 - [\sigma(s; w) \cdot \hat{p}]^2 \sigma(s; w) + 2[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \end{aligned}$$

$$\begin{aligned} \text{证明: } & [\sigma(s; w) \cdot \hat{p}]\hat{p} = -i[\sigma(s; w) \times \hat{p}] \times [\sigma(s; w) \times \hat{p}] \\ & = i\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + [\sigma(s; w) \cdot \hat{p}]i\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]\} \times \sigma(s; w) \\ & - i\sigma(s; w) \times \{[\sigma(s; w) \cdot \hat{p}]^2 \sigma(s; w) + [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}]\} \\ & = \{-[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + \sigma^2(s; w)\hat{p}\}[\sigma(s; w) \cdot \hat{p}] \\ & + [\sigma(s; w) \cdot \hat{p}]\{-[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) - \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] + \sigma^2(s; w)\hat{p}\} \\ & + 3[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - 2\sigma^2(s; w)\hat{p}[\sigma(s; w) \cdot \hat{p}] + \sigma(s; w) + [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \\ & = -\{[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - \sigma^2(s; w)\hat{p}\}[\sigma(s; w) \cdot \hat{p}] \\ & - [\sigma(s; w) \cdot \hat{p}]\{[\sigma(s; w) \cdot \hat{p}]\sigma(s; w) + \sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - \sigma^2(s; w)\hat{p}\} \\ & + 3[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] - 2\sigma^2(s; w)\hat{p}[\sigma(s; w) \cdot \hat{p}] + \sigma(s; w) + [\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \\ & = \sigma(s; w) - \sigma(s; w)[\sigma(s; w) \cdot \hat{p}]^2 - [\sigma(s; w) \cdot \hat{p}]^2 \sigma(s; w) + 2[\sigma(s; w) \cdot \hat{p}]\sigma(s; w)[\sigma(s; w) \cdot \hat{p}] \end{aligned} \quad \square$$



# 第十八章 特殊拟微分算子和矩阵连乘积

## 1 新数学工具的建立

### 1.1 特殊拟微分算子的引入

平面波解假设：假设满足无质量粒子物理方程的所有平面波解都不含零频解，所以常数解不是无质量粒子方程的平面波解，应另行处理。

$$\text{定义1.1.1. } f(\vec{r}, t) := \int_{\vec{p} \neq 0} f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}$$

$$\text{定义1.1.2. } \begin{cases} \frac{1}{\sqrt{m^2 - \nabla^2}} f(\vec{r}, t) := \frac{1}{(2\pi)^3} \int \frac{1}{E} f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}, \forall f(\vec{r}, t) \\ \sqrt{m^2 - \nabla^2} f(\vec{r}, t) := \frac{1}{(2\pi)^3} \int E f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}, \forall f(\vec{r}, t) \end{cases} \quad \sqrt{m^2 - \nabla^2} \longleftrightarrow E = \sqrt{m^2 + \vec{p}^2}$$

$$\text{定义1.1.3. } \begin{cases} \frac{1}{\sqrt{-\nabla^2}} f(\vec{r}, t) := \frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|} f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}, \forall f(\vec{r}, t) \\ \sqrt{-\nabla^2} f(\vec{r}, t) := \frac{1}{(2\pi)^3} \int |\vec{p}| f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}, \forall f(\vec{r}, t) \end{cases} \quad \sqrt{-\nabla^2} \longleftrightarrow |\vec{p}|$$

### 1.2 有质量特殊拟微分算子的基本性质

$$\text{性质1.2.1. } \begin{cases} (\sqrt{m^2 - \nabla^2})^2 = m^2 - \nabla^2, (\frac{1}{\sqrt{m^2 - \nabla^2}})^2 = \frac{1}{m^2 - \nabla^2} \\ \sqrt{m^2 - \nabla^2} \frac{1}{\sqrt{m^2 - \nabla^2}} = \frac{1}{\sqrt{m^2 - \nabla^2}} \sqrt{m^2 - \nabla^2} = 1 \\ [\sqrt{m^2 - \nabla^2}]^* = \sqrt{m^2 - \nabla^2}, [\frac{1}{\sqrt{m^2 - \nabla^2}}]^* = \frac{1}{\sqrt{m^2 - \nabla^2}} \end{cases}$$

证明： $(\sqrt{m^2 - \nabla^2})^* f(\vec{r}, t)$

$$\begin{aligned} &= [\sqrt{m^2 - \nabla^2} f^*(\vec{r}, t)]^* \\ &= [\frac{1}{(2\pi)^3} \int \sqrt{m^2 + \vec{p}^2} f^*(-\hat{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}]^* \\ &= \frac{1}{(2\pi)^3} \int \sqrt{m^2 + \vec{p}^2} f(-\hat{p}, t) e^{-i\vec{p} \cdot \vec{r}} d^3 \vec{p} \\ &= \frac{1}{(2\pi)^3} \int \sqrt{m^2 + \vec{p}^2} f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} \\ &= \sqrt{m^2 - \nabla^2} f(\vec{r}, t) \end{aligned} \quad \square$$

证明： $(\frac{1}{\sqrt{m^2 - \nabla^2}})^* f(\vec{r}, t)$

$$\begin{aligned} &= [\frac{1}{\sqrt{m^2 - \nabla^2}} f^*(\vec{r}, t)]^* \\ &= [\frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{m^2 + \vec{p}^2}} f^*(-\hat{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}]^* \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{m^2 + \vec{p}^2}} f(-\hat{p}, t) e^{-i\vec{p} \cdot \vec{r}} d^3 \vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{m^2 + \vec{p}^2}} f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} \\ &= \frac{1}{\sqrt{m^2 - \nabla^2}} f(\vec{r}, t) \end{aligned} \quad \square$$

$$\text{性质1.2.2. } (\sqrt{m^2 - \nabla^2})^n f(\vec{r}, t) = \int \sqrt{m^2 + \vec{p}^2}^n f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}, n \in \mathbb{Z}$$

$$\text{性质1.2.3. } \int \sqrt{m^2 - \nabla^2} f(\vec{r}, t) d^3 \vec{r} = m f(\vec{p} = 0, t), \int \frac{1}{\sqrt{m^2 - \nabla^2}} f(\vec{r}, t) d^3 \vec{r} = \frac{1}{m} f(\vec{p} = 0, t)$$

$$\text{性质1.2.4. } (\sqrt{m^2 - \nabla^2})^n f(\vec{r}, t) = \int \sqrt{m^2 + \vec{p}^2}^n f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}, n \in \mathbb{Z}$$

$$\text{性质1.2.5. } \int f(\vec{r}, t) (\sqrt{m^2 - \nabla^2})^n g(\vec{r}, t) d^3 \vec{r} = \int [(\sqrt{m^2 - \nabla^2})^n f(\vec{r}, t)] g(\vec{r}, t) d^3 \vec{r}$$

$$\begin{aligned} \text{证明：} & \int f(\vec{r}, t) (\sqrt{m^2 - \nabla^2})^n g(\vec{r}, t) d^3 \vec{r} \\ &= \frac{1}{(2\pi)^3} \int f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} \frac{1}{(2\pi)^3} \int \sqrt{m^2 + \vec{p}^2}^n g(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} d^3 \vec{r} \\ &= \frac{1}{(2\pi)^3} \int \sqrt{m^2 + \vec{p}^2}^n f(\vec{p}, t) g(\vec{p}, t) \delta^3(\vec{p} + \vec{p}) d^3 \vec{p} d^3 \vec{p} \\ &= \frac{1}{(2\pi)^3} \int \sqrt{m^2 + \vec{p}^2}^n f(-\hat{p}, t) g(\vec{p}, t) d^3 \vec{p} \end{aligned} \quad \square$$

$$\begin{aligned}
& \text{证明: } \int [(\sqrt{m^2 - \nabla^2})^n f(\vec{r}, t)] g(\vec{r}, t) d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int \sqrt{m^2 + \vec{p}^2}^n f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} \frac{1}{(2\pi)^3} \int g(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int \sqrt{m^2 + \vec{p}^2}^n f(\vec{p}, t) g(\vec{p}, t) \delta^3(\vec{p} + \vec{p}) d^3 \vec{p} d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int \sqrt{m^2 + \vec{p}^2}^n f(-\hat{p}, t) g(\vec{p}, t) d^3 \vec{p}
\end{aligned}$$

□

$$\text{性质1.2.6. } (\sqrt{m^2 - \nabla^2})^n \delta^3(\vec{r}) = \frac{1}{(2\pi)^3} \int \sqrt{m^2 + \vec{p}^2}^n e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} = (\sqrt{m^2 - \nabla^2})^n \delta^3(-\vec{r})$$

$$\text{性质1.2.7. } \int f(\vec{r}', t) (\sqrt{-\nabla'^2})^n \delta^3(\vec{r} - \vec{r}') d^3 \vec{r}' = (\sqrt{m^2 - \nabla^2})^n f(\vec{r}, t)$$

### 1.3 无质量特殊拟微分算子的基本性质

$$\text{性质1.3.1. } \begin{cases} (\sqrt{-\nabla^2})^2 = -\nabla^2, (\frac{1}{\sqrt{-\nabla^2}})^2 = \frac{1}{-\nabla^2} \\ \sqrt{-\nabla^2} \frac{1}{\sqrt{-\nabla^2}} = \frac{1}{\sqrt{-\nabla^2}} \sqrt{-\nabla^2} = 1 \\ [\sqrt{-\nabla^2}]^* = \sqrt{-\nabla^2}, [\frac{1}{\sqrt{-\nabla^2}}]^* = \frac{1}{\sqrt{-\nabla^2}} \end{cases}$$

$$\begin{aligned}
& \text{证明: } (\sqrt{-\nabla^2})^* f(\vec{r}, t) \\
&= [\sqrt{-\nabla^2} f^*(\vec{r}, t)]^* \\
&= [\frac{1}{(2\pi)^3} \int |\vec{p}| f^*(-\hat{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}]^* \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}| f(-\hat{p}, t) e^{-i\vec{p} \cdot \vec{r}} d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}| f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} \\
&= \sqrt{-\nabla^2} f(\vec{r}, t)
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } (\frac{1}{\sqrt{-\nabla^2}})^* f(\vec{r}, t) \\
&= [\frac{1}{\sqrt{-\nabla^2}} f^*(\vec{r}, t)]^* \\
&= [\frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|} f^*(-\hat{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}]^* \\
&= \frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|} f(-\hat{p}, t) e^{-i\vec{p} \cdot \vec{r}} d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|} f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} \\
&= \frac{1}{\sqrt{-\nabla^2}} f(\vec{r}, t)
\end{aligned}$$

□

$$\text{性质1.3.2. } (\sqrt{-\nabla^2})^n f(\vec{r}, t) = \int |\vec{p}|^n f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}, n \in Z$$

$$\text{性质1.3.3. } \int \sqrt{-\nabla^2} f(\vec{r}, t) d^3 \vec{r} = 0, \int \frac{1}{\sqrt{-\nabla^2}} f(\vec{r}, t) d^3 \vec{r} = \text{奇异性}$$

$$\text{性质1.3.4. } (\sqrt{-\nabla^2})^n f(\vec{r}, t) = \int |\vec{p}|^n f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}, n \in Z$$

$$\text{性质1.3.5. } \int f(\vec{r}, t) (\sqrt{-\nabla^2})^n g(\vec{r}, t) d^3 \vec{r} = \int [(\sqrt{-\nabla^2})^n f(\vec{r}, t)] g(\vec{r}, t) d^3 \vec{r}$$

$$\begin{aligned}
& \text{证明: } \int f(\vec{r}, t) (\sqrt{-\nabla^2})^n g(\vec{r}, t) d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} \frac{1}{(2\pi)^3} \int |\vec{p}|^n g(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^n f(\vec{p}, t) g(\vec{p}, t) \delta^3(\vec{p} + \vec{p}) d^3 \vec{p} d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^n f(-\hat{p}, t) g(\vec{p}, t) d^3 \vec{p}
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } \int [(\sqrt{-\nabla^2})^n f(\vec{r}, t)] g(\vec{r}, t) d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^n f(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} \frac{1}{(2\pi)^3} \int g(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^n f(\vec{p}, t) g(\vec{p}, t) \delta^3(\vec{p} + \vec{p}) d^3 \vec{p} d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^n f(-\hat{p}, t) g(\vec{p}, t) d^3 \vec{p}
\end{aligned}$$

□

$$\text{性质1.3.6. } (\sqrt{-\nabla^2})^n \delta^3(\vec{r}) = \frac{1}{(2\pi)^3} \int |\vec{p}|^n e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p} = (\sqrt{-\nabla^2})^n \delta^3(-\vec{r})$$

$$\text{性质1.3.7. } \int f(\vec{r}', t) (\sqrt{-\nabla'^2})^n \delta^3(\vec{r} - \vec{r}') d^3 \vec{r}' = (\sqrt{-\nabla^2})^n f(\vec{r}, t)$$

## 2 四维时空中矩阵连乘迹

### 2.1 自旋矩阵连乘迹 $tr[\sigma_{\alpha_1}(s) \cdots \sigma_{\alpha_n}(s)]$ 的性质

推论2.1.1.

$$tr[\sigma_{\alpha'_\zeta}(s)] = 0, tr[\sigma^{\alpha_\zeta}(s)] = 0$$

$$tr[\sigma_{\alpha'_\zeta}(s)\sigma_{\beta'_\zeta}(s)] = \frac{2}{3}s(s + \frac{1}{2})(s + 1)\delta_{\alpha'_\zeta\beta'_\zeta}, tr[\sigma^{\alpha_\zeta}(s)\sigma^{\beta_\zeta}(s)] = \frac{2}{3}s(s + \frac{1}{2})(s + 1)\delta^{\alpha_\zeta\beta_\zeta}$$

### 2.2 泡利矩阵连乘迹 $tr[\sigma_{\alpha_1} \cdots \sigma_{\alpha_n}]$ 的性质

定义2.2.1.  $A_{\alpha_1 \cdots \alpha_n} := tr[\sigma_{\alpha_1} \cdots \sigma_{\alpha_n}]$

性质2.2.1.

$$A_{\alpha_1} = 0$$

$$A_{\alpha_1\alpha_2} = 2\delta_{\alpha_1\alpha_2}$$

$$A_{\alpha_1\alpha_2\alpha_3} = 2i\varepsilon_{\alpha_1\alpha_2\alpha_3}$$

$$A_{\alpha_1\alpha_2\alpha_3\alpha_4} = 2[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} - S_{\alpha_1\alpha_2\alpha_3\alpha_4}]$$

$$A_{\alpha_1\alpha_2\alpha_3\alpha_4} = 2[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4} - \delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4} + \delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}]$$

$$A_{\alpha_1 \cdots \alpha_5} = 2i[\varepsilon_{\alpha_1\alpha_2\alpha_3}\delta_{\alpha_4\alpha_5} - \varepsilon_{\alpha_1\alpha_2\alpha_4}\delta_{\alpha_3\alpha_5} + \varepsilon_{\alpha_1\alpha_2\alpha_5}\delta_{\alpha_3\alpha_4} + \varepsilon_{\alpha_3\alpha_4\alpha_5}\delta_{\alpha_1\alpha_2}]$$

定理2.2.1.  $A_{\alpha_1 \cdots \alpha_n} = i\varepsilon_{\alpha_1\alpha_2}{}^\alpha A_{\alpha\alpha_3 \cdots \alpha_n} + \delta_{\alpha_1\alpha_2} A_{\alpha_3 \cdots \alpha_n}$

### 2.3 Dirac矩阵连乘迹 $tr[\gamma_{a_1} \cdots \gamma_{a_n}]$ 的一般性质

定义2.3.1.  $B_{a_1 \cdots a_n} := tr[\gamma_{a_1} \cdots \gamma_{a_n}], B_{a_1 \cdots a_n}^5 := tr[\gamma^5 \gamma_{a_1} \cdots \gamma_{a_n}]$

性质2.3.1.

$$B_{a_1} = 0, B_{a_1}^5 = 0$$

$$B_{a_1 a_2} = 4\delta_{a_1 a_2}, B_{a_1 a_2}^5 = 0$$

定理2.3.1.

$$\begin{cases} B_{a_1 \cdots a_n} = \varepsilon_{a_1 a_2 a_3}{}^a B_{aa_4 \cdots a_n} + \delta_{a_1 a_2} B_{a_3 \cdots a_n} + \delta_{a_3 [a_2} B_{a_1] a_4 \cdots a_n} \\ B_{a_1 \cdots a_n}^5 = \varepsilon_{a_1 a_2 a_3}{}^a B_{aa_4 \cdots a_n}^5 + \delta_{a_1 a_2} B_{a_3 \cdots a_n}^5 + \delta_{a_3 [a_2} B_{a_1] a_4 \cdots a_n}^5 \end{cases}$$

### 2.4 Dirac矩阵连乘迹 $tr[\gamma_a(\varsigma)\gamma_b(\varsigma) \cdots]$ 的具体性质

$$tr[\gamma_a(\varsigma)] = 0 \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \quad (18.1)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \quad (18.2)$$

$$tr[\gamma_5(\varsigma)] = 0 \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)] = 0 \quad (18.3)$$

$$tr[S_{ab}(e, \varsigma)] = 0 \quad tr[\gamma_c(\varsigma)S_{ab}(e, \varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)] = 0 \quad (18.4)$$

$$tr[\gamma_5(\varsigma)S_{ab}(e, \varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_c(\varsigma)S_{ab}(e, \varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)] = 0 \quad (18.5)$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)] = 4\delta_{ab} \quad tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 4[\delta_{ab}\delta_{cd} - \delta_{a[c}\delta_{d]b}] \quad (18.6)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)] = 0 \quad tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 4\varepsilon_{abcd} \quad (18.7)$$

$$tr[S_{ab}(e, \varsigma)S_{cd}(e, \varsigma)] = S_{abcd} = \delta_{ac}\delta_{db} - \delta_{ad}\delta_{cb} \quad tr[\gamma_5 S_{ab}(e, \varsigma)S_{cd}(e, \varsigma)] = -\varepsilon_{abcd} \quad (18.8)$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)S_{cd}(e, \varsigma)] = 2iS_{abcd} \quad tr[\gamma_5 \gamma_a(\varsigma)\gamma_b(\varsigma)S_{cd}(e, \varsigma)] = -2i\varepsilon_{abcd} \quad (18.9)$$

$$tr[S_{ab}(e, \varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = 2iS_{abcd} \quad tr[\gamma_5 S_{ab}(e, \varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)] = -2i\varepsilon_{abcd} \quad (18.10)$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = 2i\{\delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef}\} \quad (18.11)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = -2i\{\delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \quad (18.12)$$

$$tr[\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)\gamma_f(\varsigma)] = 4\{(\delta_{ab}\delta_{cd} - S_{abcd})\delta_{ef} - (\delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef})\} \quad (18.13)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)\gamma_b(\varsigma)\gamma_c(\varsigma)\gamma_d(\varsigma)\gamma_e(\varsigma)\gamma_f(\varsigma)] = 4\{\varepsilon_{abcd}\delta_{ef} + \delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \quad (18.14)$$

$$tr[\gamma_a(\varsigma)S_{bc}(e, \varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = \delta_{ab}S_{cdef} + \delta_{cd}S_{abef} + \delta_{a[c}S_{d]bef} - \delta_{b[c}S_{d]aef} - \delta_{bc}S_{adef} \quad (18.15)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)S_{bc}(e, \varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = \delta_{bc}\varepsilon_{adef} - \{\delta_{ab}\varepsilon_{cdef} + \delta_{cd}\varepsilon_{abef} + \delta_{a[c}\varepsilon_{d]bef} - \delta_{b[c}\varepsilon_{d]aef}\} \quad (18.16)$$

$$tr[\gamma_a(\varsigma)S_{bc}(e, \varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = \delta_{ad}S_{bcef} + \delta_{a[b}S_{c]def} + \delta_{d[b}S_{c]aef} \quad (18.17)$$

$$tr[\gamma_5(\varsigma)\gamma_a(\varsigma)S_{bc}(e, \varsigma)\gamma_d(\varsigma)S_{ef}(e, \varsigma)] = -\{\delta_{ad}\varepsilon_{bcef} + \delta_{a[b}\varepsilon_{c]def} + \delta_{d[b}\varepsilon_{c]aef}\} \quad (18.18)$$

### 3 N+1=n维时空中的Dirac矩阵连乘积

在2023年9月5日-9月7日这三天灵感突发，我终于把这一章节的猜想都进行了严格的证明，本章节几乎所有的猜想都变成了定理。剩下的那个猜想事实上也可以认为被严格证明了，只是表述上有些复杂而已，需要进一步理清和梳理。

#### 3.1 N+1=n维时空中Dirac矩阵连乘积的一个重要定理及其推论

定理3.1.1.

$$\begin{cases} \frac{1}{(l-1)!}\gamma_{a_1}\gamma_{[a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]} \\ \frac{1}{(l-1)!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\gamma_{[a_1}\cdots\gamma_{a_{l-2}}\delta_{a_{l-1}]a_l} \end{cases}$$

证明：分情况证明：

$$1: a_i = a_j, i > j \geq 2$$

$$0 = \frac{1}{(l-1)!}\gamma_{a_1}\gamma_{[a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]} = 0$$

$$2: a_1 \neq a_2 \neq a_3 \neq \cdots \neq a_l$$

$$\frac{1}{(l-1)!}\gamma_{a_1}\gamma_{[a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]}$$

$$3: a_2 \neq a_3 \neq \cdots \neq a_l; a_1 = a_i, i \geq 2$$

$$\frac{1}{(l-1)!}\gamma_{a_1}\gamma_{[a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l} = \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]} \quad \square$$

证明：分情况证明：

$$1: a_i = a_j, i < j \leq l-1$$

$$0 = \frac{1}{(l-1)!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\gamma_{[a_1}\cdots\gamma_{a_{l-2}}\delta_{a_{l-1}]a_l} = 0$$

$$2: a_1 \neq a_2 \neq a_3 \neq \cdots \neq a_l$$

$$\frac{1}{(l-1)!}\gamma_{a_1}\gamma_{[a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\gamma_{[a_1}\cdots\gamma_{a_{l-2}}\delta_{a_{l-1}]a_l}$$

$$3: a_1 \neq a_2 \neq \cdots \neq a_{l-1}; a_l = a_i, i \leq l-1$$

$$\frac{1}{(l-1)!}\gamma_{a_1}\gamma_{[a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l} = \frac{1}{(l-2)!}\gamma_{[a_1}\cdots\gamma_{a_{l-2}}\delta_{a_{l-1}]a_l} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\gamma_{[a_1}\cdots\gamma_{a_{l-2}}\delta_{a_{l-1}]a_l} \quad \square$$

$$\text{推论3.1.1. } \frac{1}{(l-2)!}\gamma_{a_1}\gamma_{a_2}\gamma_{[a_3}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]} + \frac{1}{(l-3)!}\gamma_{a_1}\delta_{a_2[a_3}\gamma_{a_4}\cdots\gamma_{a_l]}$$

$$\text{证明: } \frac{1}{(l-1)!}\gamma_{a_1}\gamma_{[a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]}$$

$$\Rightarrow \gamma_{a_1}\left[\frac{1}{(l-2)!}\gamma_{a_2}\gamma_{[a_3}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} - \frac{1}{(l-3)!}\delta_{a_2[a_3}\gamma_{a_4}\cdots\gamma_{a_l}]\right] = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]}$$

$$\Leftrightarrow \frac{1}{(l-2)!}\gamma_{a_1}\gamma_{a_2}\gamma_{[a_3}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]} + \frac{1}{(l-3)!}\gamma_{a_1}\delta_{a_2[a_3}\gamma_{a_4}\cdots\gamma_{a_l]}$$

$$\Leftrightarrow \frac{1}{(l-2)!}\gamma_{a_1}\gamma_{a_2}\gamma_{[a_3}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} = \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]} + \frac{1}{(l-3)!}\gamma_{a_1}\delta_{a_2[a_3}\gamma_{a_4}\cdots\gamma_{a_l]} \quad \square$$

$$\text{推论3.1.2. } \frac{1}{(l-k)!}\gamma_{a_1}\cdots\gamma_{a_k}\gamma_{[a_{k+1}}\cdots\gamma_{a_{l-1}}\gamma_{a_l]}$$

$$= \frac{1}{l!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_{l-1}}\gamma_{a_l]} + \frac{1}{(l-2)!}\delta_{a_1[a_2}\gamma_{a_3}\cdots\gamma_{a_l]} + \frac{1}{(l-3)!}\gamma_{a_1}\delta_{a_2[a_3}\gamma_{a_4}\cdots\gamma_{a_l]} + \cdots + \frac{1}{(l-k-1)!}\gamma_{a_1}\cdots\gamma_{a_{k-1}}\delta_{a_k[a_{k+1}}\gamma_{a_{k+2}}\cdots\gamma_{a_l]}$$

**推论3.1.3.**

$$\begin{cases} \frac{1}{4!} \gamma[a\gamma b\gamma c\gamma d]\gamma e = \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d\gamma e] + \frac{1}{3!} \gamma[a\gamma b\gamma c\delta d]e \\ \gamma a\gamma b\gamma c\gamma d = \frac{1}{4!} \gamma[a\gamma b\gamma c\gamma d] + \frac{1}{2!} \langle \delta_{ab}\gamma[c\gamma d], \frac{4!}{1!1!|2!2!} \rangle + \frac{1}{0!} \langle \delta_{ab}\delta_{cd}, \frac{4!}{2!|2!2!} \rangle \\ \Rightarrow \gamma a\gamma b\gamma c\gamma d\gamma e = \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d\gamma e] + \frac{1}{3!} \langle \delta_{ab}\gamma[c\gamma d\gamma e], \frac{5!}{1!1!|2!3!} \rangle + \frac{1}{1!} \langle \delta_{ad}\delta_{bc}\gamma e, \frac{5!}{2!1!|2!2!1!} \rangle \end{cases}$$

**证明:**  $\gamma a\gamma b\gamma c\gamma d\gamma e$ 

$$\begin{aligned} &= \frac{1}{4!} \gamma[a\gamma b\gamma c\gamma d]\gamma e + \frac{1}{2!} (\delta_{ab}\gamma[c\gamma d] - \delta_{ac}\gamma[b\gamma d] + \delta_{ad}\gamma[b\gamma c] + \gamma[a\gamma b]\delta_{cd} - \gamma[a\gamma c]\delta_{bd} + \gamma[a\gamma d]\delta_{bc})\gamma e \\ &+ \frac{1}{0!} (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\gamma e \\ &= \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d\gamma e] + \frac{1}{3!} \gamma[a\gamma b\gamma c\delta d]e \\ &+ \frac{1}{2!} (\delta_{ab}\gamma[c\gamma d] - \delta_{ac}\gamma[b\gamma d] + \delta_{ad}\gamma[b\gamma c] + \gamma[a\gamma b]\delta_{cd} - \gamma[a\gamma c]\delta_{bd} + \gamma[a\gamma d]\delta_{bc})\gamma e \\ &+ \frac{1}{0!} (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\gamma e \\ &= \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d\gamma e] + \frac{1}{3!} \gamma[a\gamma b\gamma c\delta d]e \\ &+ \delta_{ab}(\frac{1}{3!} \gamma[c\gamma d\gamma e] + \frac{1}{1!} \gamma[c\delta d]e) - \delta_{ac}(\frac{1}{3!} \gamma[b\gamma d\gamma e] + \frac{1}{1!} \gamma[b\delta d]e) + \delta_{ad}(\frac{1}{3!} \gamma[b\gamma c\gamma e] + \frac{1}{1!} \gamma[b\delta c]e) + (\frac{1}{3!} \gamma[a\gamma b\gamma e] + \frac{1}{1!} \gamma[a\delta b]e)\delta_{cd} \\ &- (\frac{1}{3!} \gamma[a\gamma c\gamma e] + \frac{1}{1!} \gamma[a\delta c]e)\delta_{bd} + (\frac{1}{3!} \gamma[a\gamma d\gamma e] + \frac{1}{1!} \gamma[a\delta d]e)\delta_{bc} + \frac{1}{0!} (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\gamma e \\ &= \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d\gamma e] \\ &+ \frac{1}{3!} (\gamma[a\gamma b\gamma c]\delta_{de} - \gamma[b\gamma c\gamma d]\delta_{ae} + \gamma[c\gamma d\gamma a]\delta_{be} - \gamma[d\gamma a\gamma b]\delta_{ce}) \\ &+ \frac{1}{3!} (\delta_{ab}\gamma[c\gamma d\gamma e] - \delta_{ac}\gamma[b\gamma d\gamma e] + \delta_{ad}\gamma[b\gamma c\gamma e] + \gamma[a\gamma b\gamma e]\delta_{cd}) - \gamma[a\gamma c\gamma e]\delta_{bd} + \gamma[a\gamma d\gamma e]\delta_{bc} \\ &+ \frac{1}{1!} (\delta_{ab}\gamma[c\delta d]e - \delta_{ac}\gamma[b\delta d]e + \delta_{ad}\gamma[b\delta c]e + \gamma[a\delta b]e\delta_{cd} - \gamma[a\delta c]e\delta_{bd} + \gamma[a\delta d]e\delta_{bc}) + \frac{1}{1!} (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\gamma e \\ &= \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d\gamma e] \\ &+ \frac{1}{3!} [(\delta_{ab}\gamma[c\gamma d\gamma e] - \delta_{ac}\gamma[b\gamma d\gamma e] + \delta_{ad}\gamma[b\gamma c\gamma e] - \delta_{ae}\gamma[b\gamma c\gamma d]) \\ &+ (\delta_{bc}\gamma[a\gamma d\gamma e] - \delta_{bd}\gamma[a\gamma c\gamma e] + \gamma[c\gamma d\gamma a]\delta_{be}) + (\delta_{cd}\gamma[a\gamma b\gamma e] - \delta_{ce}\gamma[d\gamma a\gamma b]) + (\delta_{de}\gamma[a\gamma b\gamma c])] \\ &+ \frac{1}{1!} [(\delta_{bc}\delta_{cd} - \delta_{bd}\delta_{ce} + \delta_{bc}\delta_{de})\gamma a + (-\delta_{ac}\delta_{de} + \delta_{ad}\delta_{ce} - \delta_{ae}\delta_{cd})\gamma b \\ &+ (\delta_{ab}\delta_{de} - \delta_{ad}\delta_{be} + \delta_{ae}\delta_{bd})\gamma c + (-\delta_{ab}\delta_{ce} + \delta_{ac}\delta_{be} - \delta_{ae}\delta_{bc})\gamma d + (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\gamma e] \\ &= \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d\gamma e] + \frac{1}{3!} \langle \delta_{ab}\gamma[c\gamma d\gamma e], C_5^3 \rangle + \langle \frac{1}{1!} \delta_{ad}\delta_{bc}\gamma e, C_5^1 C_4^2 / 2! \rangle \\ &= \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d\gamma e] + \frac{1}{3!} \langle \delta_{ab}\gamma[c\gamma d\gamma e], \frac{5!}{1!1!|2!3!} \rangle + \langle \frac{1}{1!} \delta_{ad}\delta_{bc}\gamma e, \frac{5!}{2!1!|2!2!1!} \rangle \end{aligned}$$

□

**性质3.1.1.**

$$\begin{cases} \gamma a = \frac{1}{1!} \gamma a \\ \frac{1}{1!} \gamma a\gamma b = \frac{1}{2!} \gamma[a\gamma b] + \frac{1}{0!} \delta_{ab} \\ \frac{1}{2!} \gamma a\gamma[b\gamma c] = \frac{1}{3!} \gamma[a\gamma b\gamma c] + \frac{1}{1!} \delta_{a[b\gamma c]}, \frac{1}{2!} \gamma[a\gamma b]\gamma c = \frac{1}{3!} \gamma[a\gamma b\gamma c] + \frac{1}{1!} \gamma[a\delta b]c \\ \frac{1}{3!} \gamma a\gamma[b\gamma c\gamma d] = \frac{1}{4!} \gamma[a\gamma b\gamma c\gamma d] + \frac{1}{2!} \delta_{a[b\gamma c\gamma d]}, \frac{1}{3!} \gamma[a\gamma b\gamma c]\gamma d = \frac{1}{4!} \gamma[a\gamma b\gamma c\gamma d] + \frac{1}{2!} \gamma[a\gamma b\delta c]d \\ \frac{1}{4!} \gamma a\gamma[b\gamma c\gamma d\gamma e] = \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d\gamma e] + \frac{1}{3!} \delta_{a[b\gamma c\gamma d\gamma e]}, \frac{1}{4!} \gamma[a\gamma b\gamma c\gamma d]\gamma e = \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d\gamma e] + \frac{1}{3!} \gamma[a\gamma b\gamma c\delta d]e \\ \frac{1}{5!} \gamma a\gamma[b\gamma c\gamma d\gamma e]\gamma f = \frac{1}{6!} \gamma[a\gamma b\gamma c\gamma d\gamma e\gamma f] + \frac{1}{4!} \delta_{a[b\gamma c\gamma d\gamma e\gamma f]}, \frac{1}{5!} \gamma[a\gamma b\gamma c\gamma d]\gamma e\gamma f = \frac{1}{6!} \gamma[a\gamma b\gamma c\gamma d\gamma e\gamma f] + \frac{1}{4!} \gamma[a\gamma b\gamma c\gamma d\delta e]f \end{cases}$$

**3.2 N+1=n维时空中Dirac矩阵连乘积的重要推论****定理3.2.1.**  $\frac{1}{(l-1)!} \gamma a_1 \gamma[a_2 \cdots \gamma a_{l-1} \gamma a_l] = \frac{1}{0!l!} \gamma[a_1 \cdots \gamma a_{l-1} \gamma a_l] + \frac{1}{1!(l-2)!} \sum_{k_1 \neq 1} (-1)^{k_1} \delta_{a_1 a_{k_1}} \gamma[a_2 \cdots \gamma a_{k_1-1} \gamma a_{k_1+1} \cdots \gamma a_l]$ **定理3.2.2.**  $\frac{1}{(l-2)!} \gamma a_1 \gamma a_2 \gamma[a_3 \cdots \gamma a_{l-1} \gamma a_l] = \frac{1}{0!l!} \gamma[a_1 \cdots \gamma a_{l-1} \gamma a_l] + \frac{1}{1!(l-2)!} \sum_{k_1 \neq 1} (-1)^{k_1} \delta_{a_1 a_{k_1}} \gamma[a_2 \cdots \gamma a_{k_1-1} \gamma a_{k_1+1} \cdots \gamma a_l] + \frac{1}{1!(l-2)!} \sum_{k_2 \neq 2} (-1)^{k_2} \delta_{a_2 a_{k_2}} \gamma[a_1 \gamma a_3 \cdots \gamma a_{k_2-1} \gamma a_{k_2+1} \cdots \gamma a_l] + \frac{1}{2!(l-4)!} \sum_{k_1, k_2 \neq 1, 2} (-1)^{k_1+k_2} \delta_{a_1[a_{k_1} \delta_{a_{k_2}}]a_2} \gamma[a_3 \cdots \gamma a_{k_1-1} \gamma a_{k_1+1} \cdots \gamma a_{k_2-1} \gamma a_{k_2+1} \cdots \gamma a_l]$ **证明:**  $\frac{1}{(l-2)!} \gamma a_1 \gamma a_2 \gamma[a_3 \cdots \gamma a_{l-1} \gamma a_l]$ 

$$\begin{aligned} &= \frac{1}{(l-1)!} \gamma a_1 \gamma[a_2 \cdots \gamma a_{l-1} \gamma a_l] + \frac{1}{(l-3)!} \sum_{k=3}^l (-1)^k \delta_{a_2 a_k} \gamma a_1 \gamma[a_3 \cdots \gamma a_{k-1} \gamma a_{k+1} \cdots \gamma a_l] \\ &= \frac{1}{l!} \gamma[a_1 \cdots \gamma a_{l-1} \gamma a_l] + \frac{1}{(l-2)!} \sum_{k=2}^l (-1)^k \delta_{a_1 a_k} \gamma[a_2 \cdots \gamma a_{k-1} \gamma a_{k+1} \cdots \gamma a_l] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(l-2)!} \sum_{k=3}^l (-1)^k \delta_{a_2 a_k} \gamma_{[a_1 \gamma_{a_3} \cdots \gamma_{a_{k-1}} \gamma_{a_{k+1}} \cdots \gamma_{a_l}] + \frac{1}{(l-4)!} \sum_{k=3}^l (-1)^k \delta_{a_2 a_k} \sum_{k'=3}^l (-1)^{k'+u(k'-k)} \delta_{a_1 a_{k'}} \gamma_{[a_3 \cdots \gamma_{a_{k'-1}} \gamma_{a_{k'+1}} \cdots \gamma_{a_l}] \\
& = \frac{1}{l!} \gamma_{[a_1 \cdots \gamma_{a_{l-1}} \gamma_{a_l}] \\
& + \frac{1}{(l-2)!} \sum_{k \neq 1} (-1)^k \delta_{a_1 a_k} \gamma_{[a_2 \cdots \gamma_{a_{k-1}} \gamma_{a_{k+1}} \cdots \gamma_{a_l}] + \frac{1}{(l-2)!} \sum_{k \neq 2} (-1)^k \delta_{a_2 a_k} \gamma_{[a_1 \gamma_{a_3} \cdots \gamma_{a_{k-1}} \gamma_{a_{k+1}} \cdots \gamma_{a_l}] \\
& + \frac{1}{(l-4)!} \sum_{k \neq 1, 2} \sum_{k' \neq 1, 2, k} (-1)^{k+k'+u(k'-k)} \delta_{a_1 [a_{k_1} \delta_{a_{k_2}}] a_2} \gamma_{[a_3 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_l}] \\
& = \frac{1}{l!} \gamma_{[a_1 \cdots \gamma_{a_{l-1}} \gamma_{a_l}] \\
& + \frac{1}{(l-2)!} \sum_{k_1 \neq 1} (-1)^{k_1} \delta_{a_1 a_{k_1}} \gamma_{[a_2 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_l}] + \frac{1}{(l-2)!} \sum_{k_2 \neq 2} (-1)^{k_2} \delta_{a_2 a_{k_2}} \gamma_{[a_1 \gamma_{a_3} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_l}] \\
& + \frac{1}{(l-4)!} \sum_{k_2 \neq 1, 2} \sum_{k_1 \neq 1, 2, k_2} (-1)^{k_1+k_2+u(k_2-k_1)} \delta_{a_1 [a_{k_1} \delta_{a_{k_2}}] a_2} \gamma_{[a_3 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_l}] \\
& = \frac{1}{0!l!} \gamma_{[a_1 \cdots \gamma_{a_{l-1}} \gamma_{a_l}] \\
& + \frac{1}{1!(l-2)!} \sum_{k_1 \neq 1} (-1)^{k_1} \delta_{a_1 a_{k_1}} \gamma_{[a_2 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_l}] + \frac{1}{1!(l-2)!} \sum_{k_2 \neq 2} (-1)^{k_2} \delta_{a_2 a_{k_2}} \gamma_{[a_1 \gamma_{a_3} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_l}] \\
& + \frac{1}{2!(l-4)!} \sum_{k_2 \neq 1, 2} \sum_{k_1 \neq 1, 2, k_2} (-1)^{k_1+k_2} \delta_{a_1 [a_{k_1} \delta_{a_{k_2}}] a_2} \gamma_{[a_3 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_l}] \\
& = \frac{1}{0!l!} \gamma_{[a_1 \cdots \gamma_{a_{l-1}} \gamma_{a_l}] \\
& + \frac{1}{1!(l-2)!} \sum_{k_1 \neq 1} (-1)^{k_1} \delta_{a_1 a_{k_1}} \gamma_{[a_2 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_l}] + \frac{1}{1!(l-2)!} \sum_{k_2 \neq 2} (-1)^{k_2} \delta_{a_2 a_{k_2}} \gamma_{[a_1 \gamma_{a_3} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_l}] \\
& + \frac{1}{2!(l-4)!} \sum_{k_1, k_2 \neq 1, 2} (-1)^{k_1+k_2} \delta_{a_1 [a_{k_1} \delta_{a_{k_2}}] a_2} \gamma_{[a_3 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_l}] \quad \square
\end{aligned}$$

### 3.3 N+1=n维时空中Dirac矩阵连乘积的重要猜想

**猜想3.3.1.**  $\frac{1}{(l-2)!} \gamma_{a_1} \cdots \gamma_{a_i} \gamma_{[a_{i+1} \cdots \gamma_{a_{l-1}} \gamma_{a_l}] = \frac{1}{0!l!} \gamma_{[a_1 \cdots \gamma_{a_{l-1}} \gamma_{a_l}]$

$$\begin{aligned}
& + \frac{1}{1!(l-2)!} \sum_{j_1}^{=1, \dots, i} \sum_{k_1 \neq}^{j_1} (-1)^{k_1} \delta_{a_{j_1} a_{k_1}} \gamma_{[a_1 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_l}] \\
& + \frac{1}{2!(l-4)!} \sum_{j_1, j_2}^{=1, \dots, i} \sum_{k_1, k_2 \neq}^{j_1, j_2} (-1)^{k_1+k_2} \delta_{[a_{k_1} \delta_{a_{k_2}}] a_2}^{a_{j_1} a_{j_2}} \gamma_{[a_1 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_l}] \\
& + \frac{1}{3!(l-6)!} \sum_{j_1, j_2, j_3}^{=1, \dots, i} \sum_{k_1, k_2, k_3 \neq}^{j_1, j_2, j_3} (-1)^{k_1+k_2+k_3} \delta_{[a_{k_1} \delta_{a_{k_2}} \delta_{a_{k_3}}] a_2}^{a_{j_1} a_{j_2} a_{j_3}} \gamma_{[a_1 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_{k_3-1}} \gamma_{a_{k_3+1}} \cdots \gamma_{a_l}] \\
& + \dots
\end{aligned}$$

**猜想3.3.2.**  $\frac{1}{(l-i)!} \gamma_{a_1} \cdots \gamma_{a_l} = \frac{1}{0!l!} \gamma_{[a_1 \cdots \gamma_{a_{l-1}} \gamma_{a_l}]$

$$\begin{aligned}
& + \frac{1}{1!(l-2)!} \sum_{j_1}^{=1, \dots, l} \sum_{k_1 \neq}^{j_1} (-1)^{k_1} \delta_{a_{j_1} a_{k_1}} \gamma_{[a_1 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_l}] \\
& + \frac{1}{2!(l-4)!} \sum_{j_1, j_2}^{=1, \dots, l} \sum_{k_1, k_2 \neq}^{j_1, j_2} (-1)^{k_1+k_2} \delta_{[a_{k_1} \delta_{a_{k_2}}] a_2}^{a_{j_1} a_{j_2}} \gamma_{[a_1 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_l}] \\
& + \frac{1}{3!(l-6)!} \sum_{j_1, j_2, j_3}^{=1, \dots, l} \sum_{k_1, k_2, k_3 \neq}^{j_1, j_2, j_3} (-1)^{k_1+k_2+k_3} \delta_{[a_{k_1} \delta_{a_{k_2}} \delta_{a_{k_3}}] a_2}^{a_{j_1} a_{j_2} a_{j_3}} \gamma_{[a_1 \cdots \gamma_{a_{k_1-1}} \gamma_{a_{k_1+1}} \cdots \gamma_{a_{k_2-1}} \gamma_{a_{k_2+1}} \cdots \gamma_{a_{k_3-1}} \gamma_{a_{k_3+1}} \cdots \gamma_{a_l}] \\
& + \dots
\end{aligned}$$

### 3.4 N+1=n维时空中Dirac矩阵连乘积猜想的优化表述

**定义3.4.1.**

$$\begin{cases}
\frac{1}{2!} \langle \delta_{ab} \gamma_{[c \gamma_d]}, \frac{4!}{111!|2!2!} \rangle := \frac{1}{2!} (\delta_{ab} \gamma_{[c \gamma_d]} - \delta_{ac} \gamma_{[b \gamma_d]} + \delta_{ad} \gamma_{[b \gamma_c]} + \gamma_{[a \gamma_b]} \delta_{cd} - \gamma_{[a \gamma_c]} \delta_{bd} + \gamma_{[a \gamma_d]} \delta_{bc}) \\
\frac{1}{0!} \langle \delta_{ab} \delta_{cd}, \frac{4!}{2!|2!2!} \rangle := \frac{1}{0!} (\delta_{ab} \delta_{cd} - \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \\
\frac{1}{1!} \langle \delta_{ab} \gamma_c, \frac{3!}{111!|2!1!} \rangle := \frac{1}{1!} (\delta_{ab} \gamma_c + \delta_{bc} \gamma_a - \delta_{ac} \gamma_b) \\
\frac{1}{0!} \langle \delta_{ab}, \frac{2!}{1!|2!} \rangle := \frac{1}{0!} \delta_{ab}
\end{cases}$$

**性质3.4.1.**

$$\begin{cases} \langle \delta_{ab} \gamma_{[c\gamma d]}, \frac{4!}{111!|2!2!} \rangle = \frac{1}{2!} (\langle \delta_{\{ab\}} \gamma_{[c\gamma d]}, \frac{4!}{111!|2!2!} \rangle + \langle \delta_{[ab]} \gamma_{[c\gamma d]}, \frac{4!}{111!|2!2!} \rangle) \\ \langle \delta_{ab} \delta_{cd}, \frac{4!}{2!|2!2!} \rangle = \frac{1}{2!} (\langle \delta_{\{ab\}} \delta_{cd}, \frac{4!}{2!|2!2!} \rangle + \langle \delta_{[ab]} \delta_{cd}, \frac{4!}{2!|2!2!} \rangle) \\ \langle \delta_{ab} \gamma_c, \frac{3!}{111!|2!1!} \rangle = \frac{1}{2!} (\langle \delta_{\{ab\}} \gamma_c, \frac{3!}{111!|2!1!} \rangle + \langle \delta_{[ab]} \gamma_c, \frac{3!}{111!|2!1!} \rangle) \\ \langle \delta_{ab}, \frac{2!}{1!|2!} \rangle = \frac{1}{2!} (\langle \delta_{\{ab\}}, \frac{2!}{1!|2!} \rangle + \langle \delta_{[ab]}, \frac{2!}{1!|2!} \rangle) \end{cases}$$

猜想3.4.1.

$$\begin{cases} \gamma_a = \frac{1}{1!} \gamma_a \\ \gamma_a \gamma_b = \frac{1}{2!} \gamma_{[a\gamma b]} + \frac{1}{0!} \langle \delta_{ab}, \frac{2!}{1!|2!} \rangle \\ \gamma_a \gamma_b \gamma_c = \frac{1}{3!} \gamma_{[a\gamma b\gamma c]} + \frac{1}{1!} \langle \delta_{ab} \gamma_c, \frac{3!}{111!|2!1!} \rangle \\ \gamma_a \gamma_b \gamma_c \gamma_d = \frac{1}{4!} \gamma_{[a\gamma b\gamma c\gamma d]} + \frac{1}{2!} \langle \delta_{ab} \gamma_{[c\gamma d]}, \frac{4!}{111!|2!2!} \rangle + \frac{1}{0!} \langle \delta_{ab} \delta_{cd}, \frac{4!}{2!|2!2!} \rangle \\ \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e = \frac{1}{5!} \gamma_{[a\gamma b\gamma c\gamma d\gamma e]} + \frac{1}{3!} \langle \delta_{ab} \gamma_{[c\gamma d\gamma e]}, \frac{5!}{111!|2!3!} \rangle + \frac{1}{1!} \langle \delta_{ad} \delta_{bc} \gamma_e, \frac{5!}{211!|2!2!1!} \rangle \\ \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \gamma_f = \frac{1}{6!} \gamma_{[a\gamma b\gamma c\gamma d\gamma e\gamma f]} + \frac{1}{4!} \langle \delta_{ab} \gamma_{[c\gamma d\gamma e\gamma f]}, \frac{6!}{111!|2!4!} \rangle + \frac{1}{2!} \langle \delta_{ab} \delta_{cd} \gamma_{[e\gamma f]}, \frac{6!}{211!|2!2!2!} \rangle + \frac{1}{0!} \langle \delta_{ab} \delta_{cd} \delta_{ef}, \frac{6!}{3!|2!2!2!} \rangle \\ \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \gamma_f \gamma_g = \frac{1}{7!} \gamma_{[a\gamma b\gamma c\gamma d\gamma e\gamma f\gamma g]} + \frac{1}{5!} \langle \delta_{ab} \gamma_{[c\gamma d\gamma e\gamma f\gamma g]}, \frac{7!}{111!|2!5!} \rangle + \frac{1}{3!} \langle \delta_{ab} \delta_{cd} \gamma_{[e\gamma f\gamma g]}, \frac{7!}{211!|2!2!3!} \rangle \\ + \frac{1}{1!} \langle \delta_{ab} \delta_{cd} \delta_{ef} \gamma_g, \frac{7!}{311!|2!2!2!1!} \rangle \\ \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \gamma_f \gamma_g \gamma_h = \frac{1}{8!} \gamma_{[a\gamma b\gamma c\gamma d\gamma e\gamma f\gamma g\gamma h]} + \frac{1}{6!} \langle \delta_{ab} \gamma_{[c\gamma d\gamma e\gamma f\gamma g\gamma h]}, \frac{8!}{111!|2!6!} \rangle \\ + \frac{1}{4!} \langle \delta_{ab} \delta_{cd} \gamma_{[e\gamma f\gamma g\gamma h]}, \frac{8!}{211!|2!2!4!} \rangle + \frac{1}{2!} \langle \delta_{ab} \delta_{cd} \delta_{ef} \gamma_{[g\gamma h]}, \frac{8!}{311!|2!2!2!2!} \rangle + \frac{1}{0!} \langle \delta_{ab} \delta_{cd} \delta_{ef} \delta_{gh}, \frac{8!}{4!|2!2!2!2!} \rangle \\ \dots \end{cases}$$

命题3.4.1.

$$\begin{cases} \gamma_a = \frac{1}{1!} \gamma_a \\ \gamma_{\{a\gamma b\}} = \langle \{ \delta_{ab}, \frac{2!}{1!|2!} \} \rangle = \delta_{\{ab\}} \\ \gamma_{\{a\gamma b\gamma c\}} = \langle \{ \delta_{ab} \gamma_c, \frac{3!}{111!|2!1!} \} \rangle = \delta_{\{ab\gamma c\}} \\ \gamma_{\{a\gamma b\gamma c\gamma d\}} = \langle \{ \delta_{ab} \delta_{cd}, \frac{4!}{2!|2!2!} \} \rangle = \delta_{\{ab\delta cd\}} \\ \gamma_{\{a\gamma b\gamma c\gamma d\gamma e\}} = \langle \{ \delta_{ad} \delta_{bc} \gamma_e, \frac{5!}{211!|2!2!1!} \} \rangle = \delta_{\{ad\delta bc\gamma e\}} \\ \gamma_{\{a\gamma b\gamma c\gamma d\gamma e\gamma f\}} = \langle \{ \delta_{ab} \delta_{cd} \delta_{ef}, \frac{6!}{3!|2!2!2!} \} \rangle = \delta_{\{ab\delta cd\delta ef\}} \\ \gamma_{\{a\gamma b\gamma c\gamma d\gamma e\gamma f\gamma g\}} = \langle \{ \delta_{ab} \delta_{cd} \delta_{ef} \gamma_g, \frac{7!}{311!|2!2!2!1!} \} \rangle = \delta_{\{ab\delta cd\delta ef\gamma g\}} \\ \gamma_{\{a\gamma b\gamma c\gamma d\gamma e\gamma f\gamma g\gamma h\}} = \langle \{ \delta_{ab} \delta_{cd} \delta_{ef} \delta_{gh}, \frac{8!}{4!|2!2!2!2!} \} \rangle = \delta_{\{ab\delta cd\delta ef\delta gh\}} \\ \dots \end{cases}$$

## 3.5 N+1=n维时空中前几阶Dirac矩阵连乘积猜想的验证

猜想3.5.1.

$$\begin{cases} \gamma_a = \frac{1}{1!} \gamma_a \\ \gamma_a \gamma_b = \frac{1}{2!} \gamma_{[a\gamma b]} + \frac{1}{0!} \delta_{ab} \\ \gamma_a \gamma_b \gamma_c = \frac{1}{3!} \gamma_{[a\gamma b\gamma c]} + \frac{1}{1!} (\delta_{ab} \gamma_c + \delta_{bc} \gamma_a - \delta_{ac} \gamma_b) \\ \gamma_a \gamma_b \gamma_c \gamma_d = \frac{1}{4!} \gamma_{[a\gamma b\gamma c\gamma d]} + \frac{1}{2!} (\delta_{ab} \gamma_{[c\gamma d]} + \dots) + \frac{1}{0!} (\delta_{ab} \delta_{cd} - \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \\ \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e = \frac{1}{5!} \gamma_{[a\gamma b\gamma c\gamma d\gamma e]} + \frac{1}{3!} (\delta_{ab} \gamma_{[c\gamma d\gamma e]} + \dots) + \frac{1}{1!} (\delta_{ab} \delta_{cd} \gamma_e + \dots) \\ \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \gamma_f = \frac{1}{6!} \gamma_{[a\gamma b\gamma c\gamma d\gamma e\gamma f]} + \frac{1}{4!} (\delta_{ab} \gamma_{[c\gamma d\gamma e\gamma f]} + \dots) + \frac{1}{2!} (\delta_{ab} \delta_{cd} \gamma_{[e\gamma f]} + \dots) + \frac{1}{0!} (\delta_{ab} \delta_{cd} \delta_{ef} + \dots) \\ \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \gamma_f \gamma_g = \frac{1}{7!} \gamma_{[a\gamma b\gamma c\gamma d\gamma e\gamma f\gamma g]} + \frac{1}{5!} (\delta_{ab} \gamma_{[c\gamma d\gamma e\gamma f\gamma g]} + \dots) + \frac{1}{3!} (\delta_{ab} \delta_{cd} \gamma_{[e\gamma f\gamma g]} + \dots) + \frac{1}{1!} (\delta_{ab} \delta_{cd} \delta_{ef} \gamma_g + \dots) \\ \dots \end{cases}$$

证明:  $\gamma_a \gamma_b = \frac{1}{2!} \gamma_{[a\gamma b]} + \frac{1}{0!} \delta_{ab}$  $\Leftrightarrow \gamma_{a_1} \gamma_{a_2} = \frac{1}{2!} \gamma_{[a_1\gamma a_2]} + \frac{1}{0!} \delta_{a_1 a_2}$

$$\begin{aligned} \Rightarrow \gamma^{a_1} \gamma^{a_2} \gamma_{a'_1} \gamma_{a'_2} &= \left( \frac{1}{2!} \gamma^{[a_1} \gamma^{a_2]} + \frac{1}{0!} \delta^{a_1 a_2} \right) \left( \frac{1}{2!} \gamma_{[a'_1} \gamma_{a'_2]} + \frac{1}{0!} \delta_{a'_1 a'_2} \right) \\ \Rightarrow \text{tr} \{ \gamma^{a_1} \gamma^{a_2} \gamma_{a'_1} \gamma_{a'_2} \} &= \frac{2^{[\frac{n}{2}]} (1!)^2}{2!} \delta_{[a'_1}^{[a_1} \delta_{a'_2]}^{a_2]} + \frac{2^{[\frac{n}{2}]} (1!)^2}{0!} \delta^{a_1 a_2} \delta_{a'_1 a'_2} = 2^{[\frac{n}{2}]} (1!)^2 \left( \frac{1}{2!} \delta_{[a'_1}^{[a_1} \delta_{a'_2]}^{a_2]} + \frac{1}{0!} \delta^{a_1 a_2} \delta_{a'_1 a'_2} \right) \end{aligned} \quad \square$$

证明:  $\gamma_a \gamma_b \gamma_c$

$$\begin{aligned} &= \frac{1}{2} (\gamma_a \gamma_b \gamma_c - \gamma_a \gamma_c \gamma_b + 2\gamma_a \delta_{bc}) \\ &= \frac{1}{4} (\gamma_a \gamma_b \gamma_c - \gamma_b \gamma_a \gamma_c + \gamma_c \gamma_a \gamma_b - \gamma_a \gamma_c \gamma_b + 2\delta_{ab} \gamma_c - 2\delta_{ac} \gamma_b + 4\gamma_a \delta_{bc}) \\ &= \frac{1}{8} (\gamma_a \gamma [b \gamma c] + \gamma_b \gamma [c \gamma a] + \gamma_c \gamma [a \gamma b] + 6\delta_{ab} \gamma_c - 6\delta_{ac} \gamma_b + 6\gamma_a \delta_{bc} + 2\gamma_a \gamma_b \gamma_c) \\ \Leftrightarrow \gamma_a \gamma_b \gamma_c &= \frac{1}{3!} \gamma [a \gamma b \gamma c] + (\delta_{ab} \gamma_c + \delta_{bc} \gamma_a - \delta_{ac} \gamma_b) \\ \Leftrightarrow \gamma_a \gamma_b \gamma_c &= \frac{1}{3!} \gamma [a \gamma b \gamma c] + (\delta_{a[b \gamma c]} + \gamma_a \delta_{bc}) \end{aligned} \quad \square$$

$$\text{证明: } \gamma_a \gamma_b \gamma_c = \frac{1}{3!} \gamma [a \gamma b \gamma c] + (\delta_{a[b \gamma c]} + \gamma_a \delta_{bc}) \Rightarrow \begin{cases} \gamma_a \gamma [b \gamma c] = \frac{1}{3} \gamma [a \gamma b \gamma c] + 2\delta_{a[b \gamma c]} \\ \gamma [a \gamma b] \gamma_c = \frac{1}{3} \gamma [a \gamma b \gamma c] + 2\gamma [a \delta_{bc}] \end{cases} \quad \square$$

$$\begin{aligned} \text{证明: } \begin{cases} \gamma_a \gamma [b \gamma c] = \frac{1}{3} \gamma [a \gamma b \gamma c] + 2\delta_{a[b \gamma c]} \\ \gamma [a \gamma b] \gamma_c = \frac{1}{3} \gamma [a \gamma b \gamma c] + 2\gamma [a \delta_{bc}] \end{cases} \\ \Rightarrow \gamma^{a_1} \gamma^{[a_2} \gamma^{a_3]} \gamma_{[a'_1} \gamma_{a'_2]} \gamma_{a'_3} &= \left( \frac{1}{3} \gamma^{[a_1} \gamma^{a_2} \gamma^{a_3]} + 2\delta^{a_1 [a_2} \gamma^{a_3]} \right) \left( \frac{1}{3} \gamma_{[a'_1} \gamma_{a'_2]} \gamma_{a'_3} + 2\gamma_{[a'_1} \delta_{a'_2]} a'_3 \right) \\ \Rightarrow \text{tr} \{ \gamma^{a_1} \gamma^{[a_2} \gamma^{a_3]} \gamma_{[a'_1} \gamma_{a'_2]} \gamma_{a'_3} \} &= \frac{2^{[\frac{n}{2}]} (2!)^2}{3!} \delta_{[a'_1}^{[a_1} \delta_{a'_2]}^{a_2} \delta_{a'_3]}^{a_3]} + 4 \text{tr} \{ \delta^{a_1 [a_2} \gamma^{a_3]} \gamma_{[a'_1} \delta_{a'_2]} a'_3 \} \\ &= \frac{2^{[\frac{n}{2}]} (2!)^2}{3!} \delta_{[a'_1}^{[a_1} \delta_{a'_2]}^{a_2} \delta_{a'_3]}^{a_3]} + \frac{2^{[\frac{n}{2}]} (2!)^2}{1!} \delta^{a_1 [a_2} \delta_{[a'_1}^{a_3]} \delta_{a'_2]} a'_3 \\ &= 2^{[\frac{n}{2}]} (2!)^2 \left( \frac{1}{3!} \delta_{[a'_1}^{[a_1} \delta_{a'_2]}^{a_2} \delta_{a'_3]}^{a_3]} + \frac{1}{1!} \delta^{a_1 [a_2} \delta_{[a'_1}^{a_3]} \delta_{a'_2]} a'_3 \right) \end{aligned} \quad \square$$

$$\begin{aligned} \text{证明: } \gamma_a \gamma_b \gamma_c \gamma_d &= \frac{1}{3!} \gamma_a \gamma [b \gamma c \gamma d] + \gamma_a (\delta_{b[c \gamma d]} + \gamma_b \delta_{cd}) \\ &= \frac{1}{3!} \gamma_a (\gamma_b \gamma [c \gamma d] + \gamma_c \gamma [d \gamma b] + \gamma_d \gamma [b \gamma c]) + \gamma_a (\delta_{b[c \gamma d]} + \gamma_b \delta_{cd}) \\ &= \frac{1}{3!2} [(\gamma_a \gamma_b - \gamma_b \gamma_a) \gamma [c \gamma d] + (\gamma_a \gamma_c - \gamma_c \gamma_a) \gamma [d \gamma b] + (\gamma_a \gamma_d - \gamma_d \gamma_a) \gamma [b \gamma c]] \\ &\quad + \frac{1}{3!} (\delta_{ab} \gamma [c \gamma d] + \delta_{ac} \gamma [d \gamma b] + \delta_{ad} \gamma [b \gamma c]) + \gamma_a (\delta_{b[c \gamma d]} + \gamma_b \delta_{cd}) \\ &= \frac{1}{3!2} \gamma_a \gamma [b \gamma c \gamma d] - \frac{1}{3!2} (\gamma_b \gamma_a \gamma [c \gamma d] + \gamma_c \gamma_a \gamma [d \gamma b] + \gamma_d \gamma_a \gamma [b \gamma c]) \\ &\quad + \frac{1}{3!} (\delta_{ab} \gamma [c \gamma d] + \delta_{ac} \gamma [d \gamma b] + \delta_{ad} \gamma [b \gamma c]) + \gamma_a (\delta_{b[c \gamma d]} + \gamma_b \delta_{cd}) \\ &= \frac{1}{3!3!} \gamma_a \gamma [b \gamma c \gamma d] - \frac{1}{3!} [\gamma_b (\frac{1}{3!} \gamma [a \gamma c \gamma d] + \delta_{a[c \gamma d]}) + \gamma_c (\frac{1}{3!} \gamma [a \gamma d \gamma b] + \delta_{a[d \gamma b]}) + \gamma_d (\frac{1}{3!} \gamma [a \gamma b \gamma c] + \delta_{a[b \gamma c]})] \\ &\quad + \frac{1}{3!3} \gamma_a \gamma [b \gamma c \gamma d] + \frac{1}{3!} (\delta_{ab} \gamma [c \gamma d] + \delta_{ac} \gamma [d \gamma b] + \delta_{ad} \gamma [b \gamma c]) + \gamma_a (\delta_{b[c \gamma d]} + \gamma_b \delta_{cd}) \\ &= \frac{1}{3!3!} \gamma [a \gamma b \gamma c \gamma d] - \frac{1}{3!} (\gamma_b \delta_{a[c \gamma d]} + \gamma_c \delta_{a[d \gamma b]} + \gamma_d \delta_{a[b \gamma c]}) \\ &\quad + \frac{1}{3!3} \gamma_a \gamma [b \gamma c \gamma d] + \frac{1}{3!} (\delta_{ab} \gamma [c \gamma d] + \delta_{ac} \gamma [d \gamma b] + \delta_{ad} \gamma [b \gamma c]) + \gamma_a (\delta_{b[c \gamma d]} + \gamma_b \delta_{cd}) \\ &= \frac{1}{3!3!} \gamma [a \gamma b \gamma c \gamma d] + \frac{1}{3} (\delta_{ab} \gamma [c \gamma d] + \delta_{ac} \gamma [d \gamma b] + \delta_{ad} \gamma [b \gamma c]) + \frac{2}{3} \gamma_a (\delta_{b[c \gamma d]} + \gamma_b \delta_{cd}) + \frac{1}{3} \gamma_a \gamma_b \gamma_c \gamma_d \\ &= \frac{1}{3!3!} \gamma [a \gamma b \gamma c \gamma d] + \frac{1}{3} (\delta_{ab} \gamma [c \gamma d] + \delta_{ac} \gamma [d \gamma b] + \delta_{ad} \gamma [b \gamma c]) + \frac{1}{3} (\gamma [a \gamma b] \delta_{cd} + \gamma [c \gamma a] \delta_{bd} + \gamma [a \gamma d] \delta_{bc}) \\ &\quad + \frac{2}{3} (\delta_{ab} \delta_{cd} - \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) + \frac{1}{3} \gamma_a \gamma_b \gamma_c \gamma_d \\ \Leftrightarrow \gamma_a \gamma_b \gamma_c \gamma_d &= \frac{1}{4!} \gamma [a \gamma b \gamma c \gamma d] + \frac{1}{2!} (\delta_{ab} \gamma [c \gamma d] + \delta_{ac} \gamma [d \gamma b] + \delta_{ad} \gamma [b \gamma c] + \gamma [a \gamma b] \delta_{cd} + \gamma [c \gamma a] \delta_{bd} + \gamma [a \gamma d] \delta_{bc}) \\ &\quad + (\delta_{ab} \delta_{cd} - \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \\ \Leftrightarrow \gamma_a \gamma_b \gamma_c \gamma_d &= \frac{1}{4!} \gamma [a \gamma b \gamma c \gamma d] + \frac{1}{2!} (\delta_{ab} \gamma [c \gamma d] + \dots) + (\delta_{ab} \delta_{cd} - \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \end{aligned} \quad \square$$

$$\text{证明: } \gamma_a \gamma_b \gamma_c \gamma_d = \frac{1}{4!} \gamma [a \gamma b \gamma c \gamma d] + \frac{1}{2!} (\delta_{ab} \gamma [c \gamma d] + \delta_{ac} \gamma [d \gamma b] + \delta_{ad} \gamma [b \gamma c] + \gamma [a \gamma b] \delta_{cd} + \gamma [c \gamma a] \delta_{bd} + \gamma [a \gamma d] \delta_{bc}) + (\delta_{ab} \delta_{cd} - \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc})$$

$$\Rightarrow \begin{cases} \gamma_a \gamma [b \gamma c \gamma d] = \frac{1}{4} \gamma [a \gamma b \gamma c \gamma d] + 3\delta_{a[b \gamma c \gamma d]} \\ \gamma [a \gamma b \gamma c] \gamma_d = \frac{1}{4} \gamma [a \gamma b \gamma c \gamma d] + 3\gamma [a \gamma b \delta_{c]d} \end{cases} \quad \square$$

证明:

$$\begin{cases} \gamma_a \gamma [b \gamma c \gamma d] = \frac{1}{4} \gamma [a \gamma b \gamma c \gamma d] + 3\delta_{a[b \gamma c \gamma d]} \\ \gamma [a \gamma b \gamma c] \gamma_d = \frac{1}{4} \gamma [a \gamma b \gamma c \gamma d] + 3\gamma [a \gamma b \delta_{c]d} \end{cases}$$



$$\begin{aligned}
&\Rightarrow \gamma_{a_1} \gamma_{[a_2 \gamma_{a_3} \gamma_{a_4}] \gamma_{[a'_1 \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4}]} = \left(\frac{1}{4} \gamma_{[a_1 \gamma_{a_2} \gamma_{a_3} \gamma_{a_4}] + 3\delta_{a_1[a_2 \gamma_{a_3} \gamma_{a_4}]}\right) \left(\frac{1}{4} \gamma_{[a'_1 \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4}] + 3\gamma_{[a'_1 \gamma_{a'_2} \delta_{a'_3} a'_4]}\right) \\
&\Rightarrow \text{tr} \{ \gamma^{a_1} \gamma^{[a_2 \gamma^{a_3} \gamma^{a_4}] \gamma_{[a'_1 \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4}]} \} = \left(\frac{1}{4} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4}] + 3\delta^{a_1[a_2 \gamma^{a_3} \gamma^{a_4}]}\right) \left(\frac{1}{4} \gamma_{[a'_1 \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4}] + 3\gamma_{[a'_1 \gamma_{a'_2} \delta_{a'_3} a'_4]}\right) \\
&= \frac{2^{\lfloor \frac{n}{2} \rfloor} (3!)^2}{4!} \delta_{[a'_1 \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \delta_{a'_4}^{a_4}]} + 9 \text{tr} \{ \delta^{a_1[a_2 \gamma^{a_3} \gamma^{a_4}] \gamma_{[a'_1 \gamma_{a'_2} \delta_{a'_3} a'_4]} \} \\
&= \frac{2^{\lfloor \frac{n}{2} \rfloor} (3!)^2}{4!} \delta_{[a'_1 \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \delta_{a'_4}^{a_4}]} + \frac{9}{4} \text{tr} \{ \delta^{a_1[a_2 \gamma^{[a_3} \gamma^{a_4]}] \gamma_{[[a'_1 \gamma_{a'_2} \delta_{a'_3} a'_4]} \} \\
&= \frac{2^{\lfloor \frac{n}{2} \rfloor} (3!)^2}{4!} \delta_{[a'_1 \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \delta_{a'_4}^{a_4}]} + 9 \frac{2^{\lfloor \frac{n}{2} \rfloor}}{2!} \delta^{a_1[a_2 \delta_{[[a'_1 \delta_{a'_2}^{a_2}]] \delta_{a'_3} a'_4]} \\
&= \frac{2^{\lfloor \frac{n}{2} \rfloor} (3!)^2}{4!} \delta_{[a'_1 \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \delta_{a'_4}^{a_4}]} + \frac{2^{\lfloor \frac{n}{2} \rfloor} (3!)^2}{2!} \delta^{a_1[a_2 \delta_{[a'_1 \delta_{a'_2}^{a_2}]} \delta_{a'_3} a'_4]} \\
&= 2^{\lfloor \frac{n}{2} \rfloor} (3!)^2 \left( \frac{1}{4!} \delta_{[a'_1 \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \delta_{a'_4}^{a_4}]} + \frac{1}{2!} \delta^{a_1[a_2 \delta_{[a'_1 \delta_{a'_2}^{a_2}]} \delta_{a'_3} a'_4]} \right) \quad \square
\end{aligned}$$

自我评述：以上Dirac矩阵连乘积展开是根据前几项的具体计算结果，然后归纳合理猜出来的，本质上还没有被严格证明，以后有时间再补上。以上虽然可以一步步严格具体完整写出来，但写法不够紧凑，无法方便运用，必须想一个好的写法表示出来，以便使用。

### 3.6 N+1=n维时空中Dirac矩阵连乘积常数项的具体计算

引理3.6.1.  $\gamma_{a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4}$  的常数项

$$\begin{aligned}
&= \delta_{a_1 a_2} \delta_{a_3 a_4} - \delta_{a_1 a_3} \delta_{a_2 a_4} + \delta_{a_1 a_4} \delta_{a_2 a_3} \\
&= \delta_{a_1 a_2} \delta_{a_3 a_4} - \delta_{a_1 [a_3 \delta_{a_4} a_2]} \\
&= \delta_{a_1 a_2} \delta_{a_3 a_4} + \delta_{a_2 [a_3 \delta_{a_4} a_1]}
\end{aligned}$$

引理3.6.2.  $\gamma_{a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5} \gamma_{a_6}$  的常数项

$$\begin{aligned}
&= \delta_{a_1 a_2} (\delta_{a_3 a_4} \delta_{a_5 a_6} - \delta_{a_3 a_5} \delta_{a_4 a_6} + \delta_{a_3 a_6} \delta_{a_4 a_5}) - \delta_{a_1 a_3} (\delta_{a_2 a_4} \delta_{a_5 a_6} - \delta_{a_2 a_5} \delta_{a_4 a_6} + \delta_{a_2 a_6} \delta_{a_4 a_5}) \\
&+ \delta_{a_1 a_4} (\delta_{a_2 a_3} \delta_{a_5 a_6} - \delta_{a_2 a_5} \delta_{a_3 a_6} + \delta_{a_2 a_6} \delta_{a_3 a_5}) - \delta_{a_1 a_5} (\delta_{a_2 a_3} \delta_{a_4 a_6} - \delta_{a_2 a_4} \delta_{a_3 a_6} + \delta_{a_2 a_6} \delta_{a_3 a_4}) \\
&+ \delta_{a_1 a_6} (\delta_{a_2 a_3} \delta_{a_4 a_5} - \delta_{a_2 a_4} \delta_{a_3 a_5} + \delta_{a_2 a_5} \delta_{a_3 a_4}) \\
&= \delta_{a_1 a_2} (\delta_{a_3 a_4} \delta_{a_5 a_6} - \delta_{a_3 [a_5 \delta_{a_6} a_4]}) - \delta_{a_1 a_3} (\delta_{a_2 a_4} \delta_{a_5 a_6} - \delta_{a_2 [a_5 \delta_{a_6} a_4]}) \\
&+ \delta_{a_1 a_4} (\delta_{a_2 a_3} \delta_{a_5 a_6} - \delta_{a_2 [a_5 \delta_{a_6} a_3]}) - \delta_{a_1 a_5} (\delta_{a_2 a_3} \delta_{a_4 a_6} - \delta_{a_2 [a_4 \delta_{a_6} a_3]}) \\
&+ \delta_{a_1 a_6} (\delta_{a_2 a_3} \delta_{a_4 a_5} - \delta_{a_2 [a_4 \delta_{a_5} a_3]})
\end{aligned}$$

引理3.6.3.  $\gamma_{a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5} \gamma_{a_6} \gamma_{a_7} \gamma_{a_8}$  的常数项

$$\begin{aligned}
&= \delta_{a_1 a_2} (\delta_{a_3 a_4} \delta_{a_5 a_6} - \delta_{a_3 a_5} \delta_{a_4 a_6} + \delta_{a_3 a_6} \delta_{a_4 a_5}) \delta_{a_7 a_8} + \cdots - \delta_{a_1 a_3} (\delta_{a_2 a_4} \delta_{a_5 a_6} - \delta_{a_2 a_5} \delta_{a_4 a_6} + \delta_{a_2 a_6} \delta_{a_4 a_5}) \delta_{a_7 a_8} + \cdots \\
&+ \delta_{a_1 a_4} (\delta_{a_2 a_3} \delta_{a_5 a_6} - \delta_{a_2 a_5} \delta_{a_3 a_6} + \delta_{a_2 a_6} \delta_{a_3 a_5}) \delta_{a_7 a_8} + \cdots - \delta_{a_1 a_5} (\delta_{a_2 a_3} \delta_{a_4 a_6} - \delta_{a_2 a_4} \delta_{a_3 a_6} + \delta_{a_2 a_6} \delta_{a_3 a_4}) \delta_{a_7 a_8} + \cdots \\
&+ \delta_{a_1 a_6} (\delta_{a_2 a_3} \delta_{a_4 a_5} - \delta_{a_2 a_4} \delta_{a_3 a_5} + \delta_{a_2 a_5} \delta_{a_3 a_4}) \delta_{a_7 a_8} + \cdots - \delta_{a_1 a_7} (\delta_{a_2 a_3} \delta_{a_4 a_5} - \delta_{a_2 a_4} \delta_{a_3 a_5} + \delta_{a_2 a_5} \delta_{a_3 a_4}) \delta_{a_6 a_8} + \cdots \\
&+ \delta_{a_1 a_8} (\delta_{a_2 a_3} \delta_{a_4 a_5} - \delta_{a_2 a_4} \delta_{a_3 a_5} + \delta_{a_2 a_5} \delta_{a_3 a_4}) \delta_{a_6 a_7} + \cdots \\
&= \delta_{a_1 a_2} [\delta_{a_3 a_4} (\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8} + \delta_{a_5 a_8} \delta_{a_6 a_7}) - \delta_{a_3 a_5} (\delta_{a_4 a_6} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_6 a_8} + \delta_{a_4 a_8} \delta_{a_6 a_7}) \\
&+ \delta_{a_3 a_6} (\delta_{a_4 a_5} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_5 a_8} + \delta_{a_4 a_8} \delta_{a_5 a_7})] \\
&- \delta_{a_1 a_3} [\delta_{a_2 a_4} (\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8} + \delta_{a_5 a_8} \delta_{a_6 a_7}) - \delta_{a_2 a_5} (\delta_{a_4 a_6} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_6 a_8} + \delta_{a_4 a_8} \delta_{a_6 a_7}) \\
&+ \delta_{a_2 a_6} (\delta_{a_4 a_5} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_5 a_8} + \delta_{a_4 a_8} \delta_{a_5 a_7})] \\
&+ \delta_{a_1 a_4} [\delta_{a_2 a_3} (\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8} + \delta_{a_5 a_8} \delta_{a_6 a_7}) - \delta_{a_2 a_5} (\delta_{a_3 a_6} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_6 a_8} + \delta_{a_3 a_8} \delta_{a_6 a_7}) \\
&+ \delta_{a_2 a_6} (\delta_{a_3 a_5} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_5 a_8} + \delta_{a_3 a_8} \delta_{a_5 a_7})] \\
&- \delta_{a_1 a_5} [\delta_{a_2 a_3} (\delta_{a_4 a_6} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_6 a_8} + \delta_{a_4 a_8} \delta_{a_6 a_7}) - \delta_{a_2 a_4} (\delta_{a_3 a_6} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_6 a_8} + \delta_{a_3 a_8} \delta_{a_6 a_7}) \\
&+ \delta_{a_2 a_6} (\delta_{a_3 a_4} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_4 a_8} + \delta_{a_3 a_8} \delta_{a_4 a_7})] \\
&+ \delta_{a_1 a_6} [\delta_{a_2 a_3} (\delta_{a_4 a_5} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_5 a_8} + \delta_{a_4 a_8} \delta_{a_5 a_7}) - \delta_{a_2 a_4} (\delta_{a_3 a_5} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_5 a_8} + \delta_{a_3 a_8} \delta_{a_5 a_7}) \\
&+ \delta_{a_2 a_5} (\delta_{a_3 a_4} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_4 a_8} + \delta_{a_3 a_8} \delta_{a_4 a_7})] \\
&- \delta_{a_1 a_7} [\delta_{a_2 a_3} (\delta_{a_4 a_5} \delta_{a_6 a_8} - \delta_{a_4 a_6} \delta_{a_5 a_8} + \delta_{a_4 a_8} \delta_{a_5 a_6}) - \delta_{a_2 a_4} (\delta_{a_3 a_5} \delta_{a_6 a_8} - \delta_{a_3 a_6} \delta_{a_5 a_8} + \delta_{a_3 a_8} \delta_{a_5 a_6}) \\
&+ \delta_{a_2 a_5} (\delta_{a_3 a_4} \delta_{a_6 a_8} - \delta_{a_3 a_6} \delta_{a_4 a_8} + \delta_{a_3 a_8} \delta_{a_4 a_6})] \\
&+ \delta_{a_1 a_8} [\delta_{a_2 a_3} (\delta_{a_4 a_5} \delta_{a_6 a_7} - \delta_{a_4 a_6} \delta_{a_5 a_7} + \delta_{a_4 a_7} \delta_{a_5 a_6}) - \delta_{a_2 a_4} (\delta_{a_3 a_5} \delta_{a_6 a_7} - \delta_{a_3 a_6} \delta_{a_5 a_7} + \delta_{a_3 a_7} \delta_{a_5 a_6}) \\
&+ \delta_{a_2 a_5} (\delta_{a_3 a_4} \delta_{a_6 a_7} - \delta_{a_3 a_6} \delta_{a_4 a_7} + \delta_{a_3 a_7} \delta_{a_4 a_6})]
\end{aligned}$$

以上虽然也是可以一步步严格具体完整写出来，但写法也不够紧凑、简洁，无法方便运用，必须想一个好的写法表示出来，以便使用。

## 4 N+1=n维时空Dirac矩阵连乘积正交性和完备性的证明

### 4.1 N+1=n维时空Dirac矩阵连乘积正交性的证明

定义4.1.1.  $\{\gamma_a, \gamma_b\} = 2\delta_{ab}; 1 \leq a, b \leq N+1$

引理4.1.1.  $\Gamma_{a_1 a_2 \cdots a_l} := \frac{1}{l!} \gamma_{[a_1} \cdots \gamma_{a_l]} = \gamma_{a_1} \cdots \gamma_{a_l}; a_1 \neq a_2 \neq \cdots \neq a_l$

定理4.1.1.  $tr\{\gamma_{a_1} \cdots \gamma_{a_{2k+1}}\} = 0, k \geq 0$

证明:  $tr\{\gamma_{a_1} \cdots \gamma_{a_{2k+1}}\} = tr\{\gamma_0^2 \gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_{2k+1}}\}$   
 $= \frac{1}{2} tr\{\gamma_0^2 \gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_{2k+1}}\} + \frac{1}{2} tr\{\gamma_0^2 \gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_{2k+1}}\}$   
 $= \frac{1}{2} tr\{\gamma_0^2 \gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_{2k+1}}\} - \frac{1}{2} tr\{\gamma_0 \gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_{2k+1}} \gamma_0\} = 0$  □

定理4.1.2.  $tr\{\frac{1}{l!} \gamma_{[a_1} \cdots \gamma_{a_l]}\} = 0$

证明: 分情况证明:

1:  $tr\{\frac{1}{l!} \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]}\} = 0, a_i = a_j$

2:  $a_1 \neq a_2 \neq \cdots \neq a_l, l = 2k;$

$tr\{\frac{1}{l!} \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]}\}$   
 $= \frac{1}{2} tr\{\frac{1}{l!} \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]}\} - \frac{1}{2} tr\{\frac{1}{l!} \gamma_{[a_2} \gamma_{a_1} \cdots \gamma_{a_l]}\}$   
 $= \frac{1}{2} tr\{\frac{1}{l!} \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]}\} - \frac{1}{2} tr\{\frac{1}{l!} \gamma_{[a_2} \cdots \gamma_{a_l} \gamma_{a_1}]\} = 0$

3:  $a_1 \neq a_2 \neq \cdots \neq a_l, l = 2k+1;$

$tr\{\frac{1}{l!} \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]}\} = tr\{\frac{1}{l!} \gamma_0^2 \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]}\}$   
 $= \frac{1}{2} tr\{\frac{1}{l!} \gamma_0^2 \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]}\} + \frac{1}{2} tr\{\frac{1}{l!} \gamma_0^2 \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]}\}$   
 $= \frac{1}{2} tr\{\frac{1}{l!} \gamma_0^2 \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]}\} - \frac{1}{2} tr\{\frac{1}{l!} \gamma_0 \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]} \gamma_0\} = 0$  □

推论4.1.1.  $tr\{\gamma_{a_1} \cdots \gamma_{a_l}\} = 0; a_1 \neq a_2 \neq \cdots \neq a_l$

证明:  $a_1 \neq a_2 \neq \cdots \neq a_l$

$tr\{\gamma_{a_1} \cdots \gamma_{a_l}\} = tr\{\frac{1}{l!} \gamma_{[a_1} \cdots \gamma_{a_l]}\} = 0$  □

定理4.1.3.  $tr\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}})\} = 0; a_1 \neq a_2 \neq \cdots \neq a_l; b_1 \neq b_2 \neq \cdots \neq b_{l'}; l \neq l'$

证明: 分情况证明:

1:  $l + l' = 2k + 1;$

$tr\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}})\} = 0$

2:  $l \neq l', l + l' = 2k;$

必有两个 $\gamma_a$ 或 $\gamma_b$ 不同于其他所有 $\gamma$ 矩阵，不妨设就是 $\gamma_{a_1} \gamma_{a_2}$ ，则有：

$tr\{(\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}})\}$   
 $= \frac{1}{2} tr\{(\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}})\} + \frac{1}{2} tr\{(\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}})\}$   
 $= \frac{1}{2} tr\{(\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}})\} + \frac{1}{2} tr\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}}) \gamma_{a_2}\}$   
 $= \frac{1}{2} tr\{(\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}})\} + \frac{1}{2} tr\{(\gamma_{a_2} \gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}})\}$   
 $= \frac{1}{2} tr\{(\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}})\} - \frac{1}{2} tr\{(\gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_{l'}})\}$   
 $= 0$  □

推论4.1.2.  $tr\{\frac{1}{l!} \gamma_{[a_1} \cdots \gamma_{a_l]} \frac{1}{l'!} \gamma_{[b_1} \cdots \gamma_{b_{l'}}]\} = 0, l \neq l'$

定理4.1.4.

$$\begin{aligned} \text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} &= (-1)^{\frac{l(l-1)}{2}} \delta_{a_1 b_1} \delta_{a_2 b_2} \cdots \delta_{a_l b_l}; a_1 \neq a_2 \neq \cdots \neq a_l, b_1 < b_2 < \cdots < b_l, 1 \leq l \leq N+1 \\ \text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma^{[b_1} \cdots \gamma^{b_l]})\} &= (-1)^{\frac{l(l-1)}{2}} \delta_{a_1}^{[b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_l}^{b_l]}; a_1 \neq a_2 \neq \cdots \neq a_l, b_1 < b_2 < \cdots < b_l, 1 \leq l \leq N+1 \\ \text{tr}\{\frac{1}{l!}(\gamma_{[a_1} \cdots \gamma_{a_l]})\frac{1}{l!}(\gamma^{[b_1} \cdots \gamma^{b_l]})\} &= (-1)^{\frac{l(l-1)}{2}} \frac{1}{l!} \delta_{a_1}^{[b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_l}^{b_l]}; a_1 \neq a_2 \neq \cdots \neq a_l, b_1 < b_2 < \cdots < b_l, 1 \leq l \leq N+1 \end{aligned}$$

证明: 分情况证明:

1:  $a_1 < b_1$

$$\begin{aligned} &\text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} \\ &= \frac{1}{2} \text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} + \frac{1}{2} \text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} \\ &= \frac{1}{2} \text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} - \frac{1}{2} \text{tr}\{(\gamma_{a_2} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\gamma_{a_1}\} = 0 \end{aligned}$$

2:  $b_1 < a_1$

$$\begin{aligned} &\text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} \\ &= \frac{1}{2} \text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} + \frac{1}{2} \text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} \\ &= \frac{1}{2} \text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} - \frac{1}{2} \text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})\gamma_{b_1}(\gamma_{b_2} \cdots \gamma_{b_l})\} = 0 \end{aligned}$$

3:  $a_1 = b_1$

$$\begin{aligned} &\text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} \\ &= i^{l-1} \text{tr}\{(\gamma_{a_2} \cdots \gamma_{a_l})\gamma_{a_1}\gamma_{b_1}(\gamma_{b_2} \cdots \gamma_{b_l})\} \\ &= i^{l-1} \text{tr}\{(\gamma_{a_2} \cdots \gamma_{a_l})(\gamma_{b_2} \cdots \gamma_{b_l})\} \end{aligned}$$

4: 反复进行以上三步推理, 可得:  $\text{tr}\{(\gamma_{a_1} \cdots \gamma_{a_l})(\gamma_{b_1} \cdots \gamma_{b_l})\} = i^{\frac{l(l-1)}{2}} 2^{\lfloor \frac{n}{2} \rfloor} \delta_{a_1 b_1} \delta_{a_2 b_2} \cdots \delta_{a_l b_l}$  □

## 4.2 N+1=n维时空Dirac矩阵连乘积正交性汇总

定理4.2.1.  $\text{tr}\{\frac{1}{l!}(\gamma_{[a_1} \cdots \gamma_{a_l]})\frac{1}{l!}(\gamma^{[b_1} \cdots \gamma^{b_l]})\} = (-1)^{\frac{l(l-1)}{2}} 2^{\lfloor \frac{n}{2} \rfloor} \delta_{a_1}^{[b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_l}^{b_l]} \delta_{ll'} = (-1)^{\frac{l(l-1)}{2}} 2^{\lfloor \frac{n}{2} \rfloor} \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_l}^{b_l]} \delta_{ll'}$

## 4.3 N+1=n维时空Dirac矩阵连乘积完备性的证明

定理4.3.1.  $X = I_* \otimes C \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_k \end{bmatrix}, k = 4^{\lfloor n/2 \rfloor} - 1, \forall X; (\Gamma_0, \cdots, \Gamma_k) = \{\Gamma_{a_1 a_2 \cdots a_l} | a_1 \neq a_2 \neq \cdots \neq a_l, 0 \leq l \leq n\}$

证明:  $\begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_k \end{bmatrix} = I_* \otimes A \begin{bmatrix} \tilde{\Gamma}_0 \\ \tilde{\Gamma}_1 \\ \vdots \\ \tilde{\Gamma}_k \end{bmatrix}; (\tilde{\Gamma}_0, \cdots, \tilde{\Gamma}_k) = \{\delta_{i_*} \delta_{j_*} | 0 \leq i, j \leq n\}$

$$\Rightarrow \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_k \end{bmatrix} \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_k \end{bmatrix}^+ = I_* \otimes A \begin{bmatrix} \tilde{\Gamma}_0 \\ \tilde{\Gamma}_1 \\ \vdots \\ \tilde{\Gamma}_k \end{bmatrix} \begin{bmatrix} \tilde{\Gamma}_0 \\ \tilde{\Gamma}_1 \\ \vdots \\ \tilde{\Gamma}_k \end{bmatrix}^+ I_* \otimes A^+$$

$$\Leftrightarrow \begin{bmatrix} \Gamma_0 \Gamma_0^+ & \Gamma_0 \Gamma_1^+ & \cdots & \Gamma_0 \Gamma_k^+ \\ \Gamma_1 \Gamma_0^+ & \Gamma_1 \Gamma_1^+ & \cdots & \Gamma_1 \Gamma_k^+ \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_k \Gamma_0^+ & \Gamma_k \Gamma_1^+ & \cdots & \Gamma_k \Gamma_k^+ \end{bmatrix} = I_* \otimes A \begin{bmatrix} \tilde{\Gamma}_0 \tilde{\Gamma}_0^+ & \tilde{\Gamma}_0 \tilde{\Gamma}_1^+ & \cdots & \tilde{\Gamma}_0 \tilde{\Gamma}_k^+ \\ \tilde{\Gamma}_1 \tilde{\Gamma}_0^+ & \tilde{\Gamma}_1 \tilde{\Gamma}_1^+ & \cdots & \tilde{\Gamma}_1 \tilde{\Gamma}_k^+ \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\Gamma}_k \tilde{\Gamma}_0^+ & \tilde{\Gamma}_k \tilde{\Gamma}_1^+ & \cdots & \tilde{\Gamma}_k \tilde{\Gamma}_k^+ \end{bmatrix} I_* \otimes A^+$$

$$\Rightarrow \begin{bmatrix} \text{tr}(\Gamma_0 \Gamma_0^+) & \text{tr}(\Gamma_0 \Gamma_1^+) & \cdots & \text{tr}(\Gamma_0 \Gamma_k^+) \\ \text{tr}(\Gamma_1 \Gamma_0^+) & \text{tr}(\Gamma_1 \Gamma_1^+) & \cdots & \text{tr}(\Gamma_1 \Gamma_k^+) \\ \vdots & \vdots & \ddots & \vdots \\ \text{tr}(\Gamma_k \Gamma_0^+) & \text{tr}(\Gamma_k \Gamma_1^+) & \cdots & \text{tr}(\Gamma_k \Gamma_k^+) \end{bmatrix} = I_* \otimes A \begin{bmatrix} \text{tr}(\tilde{\Gamma}_0 \tilde{\Gamma}_0^+) & \text{tr}(\tilde{\Gamma}_0 \tilde{\Gamma}_1^+) & \cdots & \text{tr}(\tilde{\Gamma}_0 \tilde{\Gamma}_k^+) \\ \text{tr}(\tilde{\Gamma}_1 \tilde{\Gamma}_0^+) & \text{tr}(\tilde{\Gamma}_1 \tilde{\Gamma}_1^+) & \cdots & \text{tr}(\tilde{\Gamma}_1 \tilde{\Gamma}_k^+) \\ \vdots & \vdots & \ddots & \vdots \\ \text{tr}(\tilde{\Gamma}_k \tilde{\Gamma}_0^+) & \text{tr}(\tilde{\Gamma}_k \tilde{\Gamma}_1^+) & \cdots & \text{tr}(\tilde{\Gamma}_k \tilde{\Gamma}_k^+) \end{bmatrix} I_* \otimes A^+$$

$$\Rightarrow A A^+ = I$$

$$\Rightarrow \begin{bmatrix} \tilde{\Gamma}_0 \\ \tilde{\Gamma}_1 \\ \vdots \\ \tilde{\Gamma}_k \end{bmatrix} = I_* \otimes A^+ \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_k \end{bmatrix}$$

$$\Rightarrow X = I_* \otimes \tilde{C} \begin{bmatrix} \tilde{\Gamma}_0 \\ \tilde{\Gamma}_1 \\ \vdots \\ \tilde{\Gamma}_k \end{bmatrix} = (I_* \otimes \tilde{C})(I_* \otimes A^+) \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_k \end{bmatrix} = I_* \otimes C \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_k \end{bmatrix}, C = \tilde{C} A^+, \forall X \quad \square$$

推论4.3.1.  $X = \sum_{l=0}^{[n/2](2-n\%2)} \frac{1}{(l!)^2} F_{a_1 \cdots a_l} \gamma^{[a_1} \cdots \gamma^{a_l]} = \sum_{l=0}^{[n/2](2-n\%2)} \frac{2^{-[n/2]} i^{l(l-1)}}{(l!)^3} \text{tr}(\gamma_{[a_1} \cdots \gamma_{a_l]} X) \gamma^{[a_1} \cdots \gamma^{a_l]}, \forall X$

定理4.3.2.  $2^{-[n/2]} \sum_{l=0}^{[n/2](2-n\%2)} \frac{i^{l(l-1)}}{(l!)^3} (\gamma_{[a_1} \cdots \gamma_{a_l]})_{\lambda' \mu'} (\gamma^{[a_1} \cdots \gamma^{a_l]})_{\lambda \mu} = \delta_{\lambda \mu'} \delta_{\lambda' \mu}$

$$\begin{aligned}
& \text{证明: } \delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} X_{\mu'}^{\lambda'} = X_{\lambda}^{\mu}, \forall X \\
& = \frac{2^{-k}}{(0!)^3} X_{\lambda'}^{\lambda'} \delta_{\lambda}^{\mu} + \sum_{l=1}^{[n/2](2-n\%2)} \frac{2^{-[n/2]i^{l(l-1)}}}{(l!)^3} \text{tr}(\gamma_{[a_1 \cdots a_l]} X)_{\lambda'}^{\lambda'} (\gamma^{[a_1 \cdots a_l]})_{\lambda}^{\mu} \\
& = \frac{2^{-k}}{(0!)^3} \delta_{\lambda'}^{\mu'} X_{\mu'}^{\lambda'} \delta_{\lambda}^{\mu} + \sum_{l=1}^{[n/2](2-n\%2)} \frac{2^{-[n/2]i^{l(l-1)}}}{(l!)^3} (\gamma_{[a_1 \cdots a_l]}^{\lambda'}^{\mu'})_{\lambda}^{\mu} X_{\mu'}^{\lambda'} (\gamma^{[a_1 \cdots a_l]})_{\lambda}^{\mu} \\
& = \left[ \frac{2^{-k}}{(0!)^3} \delta_{\lambda'}^{\mu'} \delta_{\lambda}^{\mu} + \sum_{l=1}^{[n/2](2-n\%2)} \frac{2^{-[n/2]i^{l(l-1)}}}{(l!)^3} (\gamma_{[a_1 \cdots a_l]}^{\lambda'}^{\mu'})_{\lambda}^{\mu'} (\gamma^{[a_1 \cdots a_l]})_{\lambda}^{\mu} \right] X_{\mu'}^{\lambda'} \\
& \Leftrightarrow \frac{2^{-k}}{(0!)^3} \delta_{\lambda'}^{\mu'} \delta_{\lambda}^{\mu} + \sum_{l=1}^{[n/2](2-n\%2)} \frac{2^{-[n/2]i^{l(l-1)}}}{(l!)^3} (\gamma_{[a_1 \cdots a_l]}^{\lambda'}^{\mu'})_{\lambda}^{\mu'} (\gamma^{[a_1 \cdots a_l]})_{\lambda}^{\mu} = \delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} \\
& \Leftrightarrow 2^{-[n/2]} \sum_{l=0}^{[n/2](2-n\%2)} \frac{i^{l(l-1)}}{(l!)^3} (\gamma_{[a_1 \cdots a_l]}^{\lambda'}^{\mu'})_{\lambda}^{\mu'} (\gamma^{[a_1 \cdots a_l]})_{\lambda}^{\mu} = \delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} \quad \square
\end{aligned}$$

$$\text{推论4.3.2. } 2^{-[n/2]} \sum_{l=0}^{[n/2](2-n\%2)} \frac{i^{l(l-1)}}{l!} \left\{ \left( \frac{1}{l!} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \right)_{\lambda}^{\mu} \left( \frac{1}{l!} \gamma_{[a_1 \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \cdots \gamma_{a_{l-1}} \gamma_{a_l}]} \right)_{\eta}^{\xi} \right\} = \delta_{\lambda}^{\xi} \delta_{\eta}^{\mu}$$

$$\text{推论4.3.3. } 2^{-[n/2]} \sum_{l=0}^{[n/2](2-n\%2)} \frac{1}{l!} \left\{ \left( \frac{1}{l!} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \right)^+_{\lambda}^{\mu} \left( \frac{1}{l!} \gamma_{[a_1 \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \cdots \gamma_{a_{l-1}} \gamma_{a_l}]} \right)_{\eta}^{\xi} \right\} = \delta_{\lambda}^{\xi} \delta_{\eta}^{\mu}$$

$$\text{推论4.3.4. } 2^{-[n/2]} \sum_{l=0}^{[n/2](2-n\%2)} \frac{1}{l!} \left\{ \left( \frac{1}{l!} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \right)_{\lambda}^{\mu} \left( \frac{1}{l!} \gamma_{[a_1 \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \cdots \gamma_{a_{l-1}} \gamma_{a_l}]} \right)^+_{\eta}^{\xi} \right\} = \delta_{\lambda}^{\xi} \delta_{\eta}^{\mu}$$

$$\text{推论4.3.5. } \delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} + \sigma^x \lambda^{\mu'} \sigma^x \lambda^{\mu} + \sigma^y \lambda^{\mu'} \sigma^y \lambda^{\mu} + \sigma^z \lambda^{\mu'} \sigma^z \lambda^{\mu} = 2\delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu}$$

$$\begin{aligned}
& \text{证明: } \frac{2^{-1}}{(0!)^3} \delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} + \sum_{l=1}^2 \frac{2^{-1}i^{l(l-1)}}{(l!)^3} (\gamma_{[a_1 \cdots a_l]}^{\lambda'}^{\mu'})_{\lambda}^{\mu} (\gamma^{[a_1 \cdots a_l]})_{\lambda}^{\mu} = \delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} \\
& \Leftrightarrow \delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} + \gamma_{x\lambda'}^{\mu'} \gamma^x \lambda^{\mu} + \gamma_{y\lambda'}^{\mu'} \gamma^y \lambda^{\mu} - \frac{1}{4} [\gamma_x, \gamma_y]_{\lambda'}^{\mu'} [\gamma^x, \gamma^y]_{\lambda}^{\mu} = 2\delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} \\
& \Leftrightarrow \delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} + \sigma^x \lambda^{\mu'} \sigma^x \lambda^{\mu} + \sigma^y \lambda^{\mu'} \sigma^y \lambda^{\mu} + \sigma^z \lambda^{\mu'} \sigma^z \lambda^{\mu} = 2\delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} \\
& \Leftrightarrow \sigma^x \lambda^{\mu'} \sigma^x \lambda^{\mu} + \sigma^y \lambda^{\mu'} \sigma^y \lambda^{\mu} + \sigma^z \lambda^{\mu'} \sigma^z \lambda^{\mu} = 2\delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} - \delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} = \delta_{\lambda}^{\mu'} \delta_{\lambda'}^{\mu} - \varepsilon_{\lambda\lambda'} \varepsilon^{\mu\mu'} \quad \square
\end{aligned}$$

#### 4.4 N+1=n维时空中Dirac矩阵连乘迹性质

$$\begin{aligned}
& \text{推论4.4.1. } \text{tr} \left\{ \frac{1}{1!} \gamma^{b_1} \frac{1}{1!} \gamma_{a_1} \right\} = 2^{\lfloor \frac{n}{2} \rfloor} \delta_{a_1}^{b_1} \\
& \text{tr} \left\{ \frac{1}{2!} \gamma^{[b_1 \gamma^{b_2}]} \frac{1}{2!} \gamma_{[a_1 \gamma_{a_2}]} \right\} = -2^{\lfloor \frac{n}{2} \rfloor} \delta_{[a_1 \gamma_{a_2}]}^{[b_1 \gamma^{b_2}]}, \text{tr} \left\{ \frac{1}{3!} \gamma^{[b_1 \cdots \gamma^{b_3}]} \frac{1}{3!} \gamma_{[a_1 \cdots \gamma_{a_3}]} \right\} = -2^{\lfloor \frac{n}{2} \rfloor} \delta_{[a_1 \cdots \gamma_{a_3}]}^{[b_1 \cdots \gamma^{b_3}]} \\
& \text{tr} \left\{ \frac{1}{4!} \gamma^{[b_1 \cdots \gamma^{b_4}]} \frac{1}{4!} \gamma_{[a_1 \cdots \gamma_{a_4}]} \right\} = 2^{\lfloor \frac{n}{2} \rfloor} \delta_{[a_1 \cdots \gamma_{a_4}]}^{[b_1 \cdots \gamma^{b_4}]} \\
& \text{tr} \left\{ \frac{1}{5!} \gamma^{[b_1 \cdots \gamma^{b_5}]} \frac{1}{5!} \gamma_{[a_1 \cdots \gamma_{a_5}]} \right\} = 2^{\lfloor \frac{n}{2} \rfloor} \delta_{[a_1 \cdots \gamma_{a_5}]}^{[b_1 \cdots \gamma^{b_5}]} \\
& \dots
\end{aligned}$$

$$\begin{aligned}
& \text{tr} \left\{ \frac{1}{l!} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \frac{1}{l!} \gamma_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]} \right\} = i^{l(l-1)} \frac{2^{\lfloor \frac{n}{2} \rfloor}}{l!} \delta_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]}^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \\
& \text{tr} \left\{ \frac{1}{l!} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \gamma^0 \frac{1}{l!} \gamma_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]} \gamma_0 \right\} = i^{l(l+1)} \frac{2^{\lfloor \frac{n}{2} \rfloor}}{l!} \eta_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]}^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]}
\end{aligned}$$

推论4.4.2.

$$\begin{aligned}
& \text{tr} \left\{ \left( \frac{1}{l!} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \right)^+_{\lambda}^{\mu} \frac{1}{l!} \gamma_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]} \right\} = \frac{2^{\lfloor \frac{n}{2} \rfloor}}{l!} \delta_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]}^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \\
& \text{tr} \left\{ \left( \frac{1}{l!} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \right) + \gamma^0 \frac{1}{l!} \gamma_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]} \gamma_0 \right\} = \frac{2^{\lfloor \frac{n}{2} \rfloor}}{l!} \eta_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]}^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \\
& \text{tr} \left\{ \left( \frac{1}{l!} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \right) \gamma^0 + \frac{1}{l!} \gamma_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]} \gamma_0 \right\} = \frac{2^{\lfloor \frac{n}{2} \rfloor}}{l!} \delta_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]}^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]}
\end{aligned}$$

推论4.4.3.

$$\begin{aligned}
& \text{tr} \left\{ \frac{1}{l!} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \frac{1}{l!} \gamma_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]} \right\} = i^{l(l-1)} \frac{2^{\lfloor \frac{n}{2} \rfloor}}{l!} \delta_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]}^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \delta_{ll'} \\
& \text{tr} \left\{ \frac{1}{l!} \gamma^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \gamma^0 \frac{1}{l!} \gamma_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]} \gamma_0 \right\} = i^{l(l+1)} \frac{2^{\lfloor \frac{n}{2} \rfloor}}{l!} \eta_{[b_1 \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \cdots \gamma_{b_{l-1}} \gamma_{b_l}]}^{[a_1 \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \cdots \gamma^{a_{l-1}} \gamma^{a_l}]} \delta_{ll'}
\end{aligned}$$

#### 4.5 N+1=n维时空中Dirac矩阵连乘迹有关定理的严格证明

定理4.5.1.

$$\begin{aligned}
& \text{tr} \left\{ \gamma^{[a_1 \gamma^{a_2} \cdots \gamma^{a_{l-1}}]} \gamma^{a_l} \gamma_{[a'_1 \gamma_{a'_2} \cdots \gamma_{a'_{l-1}}]} \gamma_{a'_l} \right\} = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1 \gamma_{a'_2} \cdots \gamma_{a'_{l-1}}]}^{[a_1 \gamma^{a_2} \cdots \gamma^{a_{l-1}}]} - \frac{1}{(l-2)!} \delta_{[a'_1 \cdots \gamma_{a'_{l-2}} \gamma^{a_{l-1}}] a_l} \delta_{a'_{l-1} a'_l} \right\} \\
& [\Leftrightarrow] \\
& \text{tr} \left\{ \gamma^{[a_1 \cdots \gamma^{a_{l-1}}]} \gamma^{a_l} \gamma^0 \gamma_{[a'_1 \cdots \gamma_{a'_{l-1}}]} \gamma_{a'_l} \gamma_0 \right\} = i^{l(l+1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \eta_{[a'_1 \gamma_{a'_2} \cdots \gamma_{a'_{l-1}}]}^{[a_1 \gamma^{a_2} \cdots \gamma^{a_{l-1}}]} - \frac{1}{(l-2)!} \eta_{[a'_1 \cdots \gamma_{a'_{l-2}} \gamma^{a_{l-1}}] a_l} \delta_{a'_{l-1} a'_l} \right\}
\end{aligned}$$

证明：分情况证明：

$$1 : a_i = a_j, i \neq j$$

$$tr\{\gamma^{[a_1 \gamma^{a_2} \dots \gamma^{a_{i-1}}] \gamma^{a_i} \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \dots \delta_{a'_{i-2}}^{a_{i-2}} \delta^{a_{i-1}] a_i} \delta_{a'_{i-1}] a'_i} \right\} = 0$$

$$2 : b_i = b_j, i \neq j$$

$$tr\{\gamma^{[a_1 \gamma^{a_2} \dots \gamma^{a_{i-1}}] \gamma^{a_i} \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \dots \delta_{a'_{i-2}}^{a_{i-2}} \delta^{a_{i-1}] a_i} \delta_{a'_{i-1}] a'_i} \right\} = 0$$

$$3 : a_1 \neq a_2 \neq \dots \neq a_{i-1} \neq a_i, a'_1 \neq a'_2 \neq \dots \neq a'_{i-1} \neq a'_i$$

$$tr\{\gamma^{[a_1 \gamma^{a_2} \dots \gamma^{a_{i-1}}] \gamma^{a_i} \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = tr\{\gamma^{[a_1 \gamma^{a_2} \dots \gamma^{a_{i-1}}] \gamma^{a_i} \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} \\ = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \dots \delta_{a'_{i-2}}^{a_{i-2}} \delta^{a_{i-1}] a_i} \delta_{a'_{i-1}] a'_i} \right\}$$

$$4 : a_1 \neq a_2 \neq \dots \neq a_{i-1}, a_i = a_l; a'_1 \neq a'_2 \neq \dots \neq a'_{i-1} \neq a'_i$$

$$tr\{\gamma^{[a_1 \gamma^{a_2} \dots \gamma^{a_{i-1}}] \gamma^{a_i} \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \dots \delta_{a'_{i-2}}^{a_{i-2}} \delta^{a_{i-1}] a_i} \delta_{a'_{i-1}] a'_i} \right\} = 0$$

$$5 : a_1 \neq a_2 \neq \dots \neq a_{i-1} \neq a_i; a'_1 \neq a'_2 \neq \dots \neq a'_{i-1}, a_j = a'_i$$

$$tr\{\gamma^{[a_1 \gamma^{a_2} \dots \gamma^{a_{i-1}}] \gamma^{a_i} \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \dots \delta_{a'_{i-2}}^{a_{i-2}} \delta^{a_{i-1}] a_i} \delta_{a'_{i-1}] a'_i} \right\} = 0$$

$$6 : a_1 \neq a_2 \neq \dots \neq a_{i-1}, a_i = a_l; a'_1 \neq a'_2 \neq \dots \neq a'_{i-1}, a_j = a'_i$$

$$tr\{\gamma^{[a_1 \gamma^{a_2} \dots \gamma^{a_{i-1}}] \gamma^{a_i} \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{(l-2)(l-3)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \dots \delta_{a'_{i-2}}^{a_{i-2}} \delta^{a_{i-1}] a_i} \delta_{a'_{i-1}] a'_i} \\ = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \dots \delta_{a'_{i-2}}^{a_{i-2}} \delta^{a_{i-1}] a_i} \delta_{a'_{i-1}] a'_i} \right\} = 0 \quad \square$$

推论4.5.1.

$$tr\{\gamma^{a_1} \gamma^{[a_2 \dots \gamma^{a_{i-1}}] \gamma^{a_i} \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \dots \delta_{a'_i}^{a_i]} \right\}$$

$$tr\{\gamma^0 \gamma^{a_1} \gamma^{[a_2 \dots \gamma^{a_i}] \gamma_0 \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_i}]}\} = i^{l(l+1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \dots \delta_{a'_i}^{a_i]} \right\}$$

推论4.5.2.

$$tr\{\gamma^{a_1} \gamma^{[a_2 \dots \gamma^{a_{i-1}}] \gamma^{a_i} \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \dots \delta_{a'_i}^{a_i]} \delta_{a'_{i-1}] a'_i} \right\}$$

$$tr\{\gamma^0 \gamma^{a_1} \gamma^{[a_2 \dots \gamma^{a_i}] \gamma_0 \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{l(l+1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \eta_{[a'_1}^{[a_1} \dots \eta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \dots \delta_{a'_i}^{a_i]} \delta_{a'_{i-1}] a'_i} \right\}$$

推论4.5.3.

$$tr\{\gamma^{[a_1 \gamma^{a_2} \dots \gamma^{a_{i-1}}] \gamma^{a_i} \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{l(l-1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \dots \delta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \dots \delta_{a'_i}^{a_i]} \delta_{a'_{i-1}] a'_i} \right\}$$

$$tr\{\gamma^0 \gamma^{[a_1 \gamma^{a_2} \dots \gamma^{a_i}] \gamma_0 \gamma_{[a'_1 \gamma_{a'_2} \dots \gamma_{a'_{i-1}}] \gamma_{a'_i}}\} = i^{l(l+1)} 2^{\lfloor \frac{n}{2} \rfloor} [(l-1)!]^2 \left\{ \frac{1}{l!} \eta_{[a'_1}^{[a_1} \dots \eta_{a'_i}^{a_i]} - \frac{1}{(l-2)!} \delta_{[a'_1}^{[a_1} \delta_{a'_2}^{a_2} \delta_{a'_3}^{a_3} \dots \delta_{a'_i}^{a_i]} \delta_{a'_{i-1}] a'_i} \right\}$$

自我评述：以上Dirac矩阵连乘迹公式是可以通过前几节Dirac矩阵连乘积的相关定理证明出来的，但也可以采用以上直接的方法严格证明它。

## 5 N+1维时空中 $\delta$ 函数积和式的性质

### 5.1 N+1维时空中 $\delta$ 函数积和式的指标单调轮换求和规律

引理5.1.1.

$$\left\{ \begin{aligned} \frac{1}{2} \sum_b^a \{ \delta^{a_1 a_2} \delta_{[b_1}^{[a_3} \delta_{b_2}^{a_4]} \delta_{b_3 b_4} \} &= \sum_b^a \{ \delta^{a_1 a_2} \delta_{b_1}^{[a_3} \delta_{b_2}^{a_4]} \delta_{b_3 b_4} \} = \sum_b^a \{ \delta^{a_1 a_2} \delta_{[b_1}^{[a_3} \delta_{b_2}^{a_4]} \delta_{b_3 b_4} \} \\ &= \frac{1}{2} (\delta^{a_1 [a_2} \delta_{[b_1}^{a_3} \delta_{b_2}^{a_4]} \delta_{b_3] b_4} + \delta^{a_1 [a_2} \delta_{[b_1}^{a_3} \delta_{b_4}^{a_4]} \delta_{b_2] b_3} - \delta^{a_1 [a_2} \delta_{[b_1}^{a_3} \delta_{b_3}^{a_4]} \delta_{b_2] b_4} + \delta^{a_1 [a_2} \delta_{[b_3}^{a_3} \delta_{b_4}^{a_4]} \delta_{b_1] b_2}) \\ &+ \frac{1}{2} (\delta^{a_2 a_3} \delta_{[b_1}^{[a_1} \delta_{b_2}^{a_4]} \delta_{b_3] b_4} + \delta^{a_2 a_3} \delta_{[b_1}^{[a_1} \delta_{b_4}^{a_4]} \delta_{b_2] b_3} - \delta^{a_2 a_3} \delta_{[b_1}^{[a_1} \delta_{b_3}^{a_4]} \delta_{b_2] b_4} + \delta^{a_2 a_3} \delta_{[b_3}^{[a_1} \delta_{b_4}^{a_4]} \delta_{b_1] b_2}) \\ &- \frac{1}{2} (\delta^{a_2 a_4} \delta_{[b_1}^{[a_1} \delta_{b_2}^{a_3]} \delta_{b_3] b_4} + \delta^{a_2 a_4} \delta_{[b_1}^{[a_1} \delta_{b_4}^{a_3]} \delta_{b_2] b_3} - \delta^{a_2 a_4} \delta_{[b_1}^{[a_1} \delta_{b_3}^{a_3]} \delta_{b_2] b_4} + \delta^{a_2 a_4} \delta_{[b_3}^{[a_1} \delta_{b_4}^{a_3]} \delta_{b_1] b_2}) \\ &+ \frac{1}{2} (\delta^{a_3 a_4} \delta_{[b_1}^{[a_1} \delta_{b_2}^{a_2]} \delta_{b_3] b_4} + \delta^{a_3 a_4} \delta_{[b_1}^{[a_1} \delta_{b_4}^{a_2]} \delta_{b_2] b_3} - \delta^{a_3 a_4} \delta_{[b_1}^{[a_1} \delta_{b_3}^{a_2]} \delta_{b_2] b_4} + \delta^{a_3 a_4} \delta_{[b_3}^{[a_1} \delta_{b_4}^{a_2]} \delta_{b_1] b_2}) \end{aligned} \right.$$

引理5.1.2.

$$\left\{ \begin{aligned} \sum_{ab}^a \{ \delta^{a_1 a_2} \delta_{b_1 b_2} \} &= \delta^{a_1 a_2} \delta_{b_1 b_2} - \delta_{[b_1}^{a_1} \delta_{b_2]}^{a_2}, \sum_a \{ \delta_{a_1 a_2} \delta_{a_3 a_4} \} = \delta_{a_1 a_2} \delta_{a_3 a_4} - \delta_{a_1 [a_3} \delta_{a_4]}^{a_2} \\ \sum_b^a \{ \delta^{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_2} \delta_{b_3 b_4} \} &= (\delta^{a_1 a_2} \delta_{a_3 a_4} - \delta_{a_1 [a_3} \delta_{a_4]}^{a_2}) (\delta_{b_1 b_2} \delta_{b_3 b_4} - \delta_{b_1 [b_3} \delta_{b_4]}^{b_2}) \\ \sum_{ab}^a \{ \delta^{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_2} \delta_{b_3 b_4} \} &= \frac{1}{0!} \sum_b^a \{ \delta^{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_2} \delta_{b_3 b_4} \} + \frac{1}{2!} \sum_b^a \{ \delta^{a_1 a_2} \delta_{[b_1}^{[a_3} \delta_{b_2]}^{a_4]} \delta_{b_3 b_4} \} + \frac{1}{4!} \delta_{[b_1}^{[a_1} \delta_{b_2}^{a_2} \delta_{b_3}^{a_3} \delta_{b_4]}^{a_4} \end{aligned} \right.$$

$$\begin{aligned}
& \text{引理5.1.3. } \gamma^{a_1} \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \gamma^{a_5} \gamma^{a_6} \gamma^{a_7} \gamma^{a_8} \gamma_{b_1} \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \gamma_{b_5} \gamma_{b_6} \gamma_{b_7} \gamma_{b_8} \\
& = \dots + \frac{1}{0!} \sum_b^a \{ \delta^{a_1 a_2} \delta^{a_3 a_4} \delta^{a_5 a_6} \delta^{a_7 a_8} \delta_{b_1 b_2} \delta_{b_3 b_4} \delta_{b_5 b_6} \delta_{b_7 b_8} \} + \frac{1}{2!} \sum_b^a \{ \delta^{a_1 a_2} \delta^{a_3 a_4} \delta^{a_5 a_6} \delta_{[b_1}^{a_7} \delta_{b_2]}^{a_8} \} \delta_{b_3 b_4} \delta_{b_5 b_6} \delta_{b_7 b_8} \} \\
& + \frac{1}{4!} \sum_b^a \{ \delta^{a_1 a_2} \delta^{a_3 a_4} \delta_{[b_1}^{a_5} \delta_{b_2}^{a_6} \delta_{b_3}^{a_7} \delta_{b_4]}^{a_8} \} \delta_{b_5 b_6} \delta_{b_7 b_8} \} + \frac{1}{6!} \sum_b^a \{ \delta^{a_1 a_2} \delta_{[b_1}^{a_3} \delta_{b_2}^{a_4} \delta_{b_3}^{a_5} \delta_{b_4}^{a_6} \delta_{b_5}^{a_7} \delta_{b_6]}^{a_8} \} \delta_{b_7 b_8} \} + \frac{1}{8!} \delta_{[b_1}^{a_1} \delta_{b_2}^{a_2} \delta_{b_3}^{a_3} \delta_{b_4}^{a_4} \delta_{b_5}^{a_5} \delta_{b_6}^{a_6} \delta_{b_7}^{a_7} \delta_{b_8]}^{a_8} \}
\end{aligned}$$

$$\begin{aligned}
& \text{引理5.1.4. } \gamma^{a_1} \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \dots \gamma^{a_{2l-3}} \gamma^{a_{2l-2}} \gamma^{a_{2l-1}} \gamma^{a_{2l}} \gamma_{b_1} \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \dots \gamma_{b_{2l-3}} \gamma_{b_{2l-2}} \gamma_{b_{2l-1}} \gamma_{b_{2l}} \\
& = \dots \\
& + \frac{1}{0!} \sum_b^a \{ \delta^{a_1 a_2} \delta^{a_3 a_4} \dots \delta^{a_{2l-3} a_{2l-2}} \delta^{a_{2l-1} a_{2l}} \delta_{b_1 b_2} \delta_{b_3 b_4} \dots \delta_{b_{2l-3} b_{2l-2}} \delta_{b_{2l-1} b_{2l}} \} \\
& + \frac{1}{2!} \sum_b^a \{ \delta^{a_1 a_2} \delta^{a_3 a_4} \dots \delta^{a_{2l-3} a_{2l-2}} \delta_{[b_1}^{a_{2l-1}} \delta_{b_2]}^{a_{2l}} \} \delta_{b_3 b_4} \dots \delta_{b_{2l-3} b_{2l-2}} \delta_{b_{2l-1} b_{2l}} \} \\
& + \frac{1}{4!} \sum_b^a \{ \delta^{a_1 a_2} \delta^{a_3 a_4} \dots \delta_{[b_1}^{a_{2l-3}} \delta_{b_2}^{a_{2l-2}} \delta_{b_3}^{a_{2l-1}} \delta_{b_4]}^{a_{2l}} \} \dots \delta_{b_{2l-3} b_{2l-2}} \delta_{b_{2l-1} b_{2l}} \} + \dots \\
& + \frac{1}{(2l-2)!} \sum_b^a \{ \delta^{a_1 a_2} \delta_{[b_1}^{a_3} \delta_{b_2}^{a_4} \delta_{b_3}^{a_5} \delta_{b_4}^{a_6} \dots \delta_{b_{2l-5}}^{a_{2l-3}} \delta_{b_{2l-4}}^{a_{2l-2}} \delta_{b_{2l-3}}^{a_{2l-1}} \delta_{b_{2l-2}}]^{a_{2l}} \} \delta_{b_{2l-1} b_{2l}} \} \\
& + \frac{1}{(2l)!} \delta_{[b_1}^{a_1} \delta_{b_2}^{a_2} \delta_{b_3}^{a_3} \delta_{b_4}^{a_4} \dots \delta_{b_{2l-3}}^{a_{2l-3}} \delta_{b_{2l-2}}^{a_{2l-2}} \delta_{b_{2l-1}}^{a_{2l-1}} \delta_{b_{2l}}]^{a_{2l}} \}
\end{aligned}$$

$$\begin{aligned}
& \text{引理5.1.5. } \gamma^{a_1} \gamma^{a_2} \gamma^{a_3} \gamma^{a_4} \dots \gamma^{a_{2l-3}} \gamma^{a_{2l-2}} \gamma^{a_{2l-1}} \gamma^{a_{2l}} \gamma^{a_{2l+1}} \gamma_{b_1} \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \dots \gamma_{b_{2l-3}} \gamma_{b_{2l-2}} \gamma_{b_{2l-1}} \gamma_{b_{2l}} \gamma^{b_{2l+1}} \\
& = \dots \\
& + \frac{1}{1!} \sum_b^a \{ \delta^{a_1 a_2} \delta^{a_3 a_4} \dots \delta^{a_{2l-3} a_{2l-2}} \delta^{a_{2l-1} a_{2l}} \delta_{b_1}^{a_{2l+1}} \delta_{b_2 b_3} \delta_{b_4 b_5} \dots \delta_{b_{2l-2} b_{2l-1}} \delta_{b_{2l} b_{2l+1}} \} \\
& + \frac{1}{3!} \sum_b^a \{ \delta^{a_1 a_2} \delta^{a_3 a_4} \dots \delta^{a_{2l-3} a_{2l-2}} \delta_{[b_1}^{a_{2l-1}} \delta_{b_2}^{a_{2l}} \delta_{b_3]}^{a_{2l+1}} \} \delta_{b_4 b_5} \dots \delta_{b_{2l-2} b_{2l-1}} \delta_{b_{2l} b_{2l+1}} \} \\
& + \frac{1}{5!} \sum_b^a \{ \delta^{a_1 a_2} \delta^{a_3 a_4} \dots \delta_{[b_1}^{a_{2l-3}} \delta_{b_2}^{a_{2l-2}} \delta_{b_3}^{a_{2l-1}} \delta_{b_4}^{a_{2l}} \delta_{b_5]}^{a_{2l+1}} \} \dots \delta_{b_{2l-2} b_{2l-1}} \delta_{b_{2l} b_{2l+1}} \} + \dots \\
& + \frac{1}{(2l-1)!} \sum_b^a \{ \delta^{a_1 a_2} \delta_{[b_1}^{a_3} \delta_{b_2}^{a_4} \delta_{b_3}^{a_5} \delta_{b_4}^{a_6} \dots \delta_{b_{2l-3}}^{a_{2l-1}} \delta_{b_{2l-2}}^{a_{2l}} \delta_{b_{2l-1}}]^{a_{2l+1}} \} \delta_{b_{2l} b_{2l+1}} \} \\
& + \frac{1}{(2l+1)!} \delta_{[b_1}^{a_1} \delta_{b_2}^{a_2} \delta_{b_3}^{a_3} \delta_{b_4}^{a_4} \dots \delta_{b_{2l-3}}^{a_{2l-3}} \delta_{b_{2l-2}}^{a_{2l-2}} \delta_{b_{2l-1}}^{a_{2l-1}} \delta_{b_{2l}}^{a_{2l}} \delta_{b_{2l+1}}]^{a_{2l+1}} \}
\end{aligned}$$

$$\begin{aligned}
& \text{引理5.1.6. } \gamma_{a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \dots \gamma_{a_{2l-3}} \gamma_{a_{2l-2}} \gamma_{a_{2l-1}} \gamma_{a_{2l}} \gamma_{b_1} \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \dots \gamma_{b_{2l-3}} \gamma_{b_{2l-2}} \gamma_{b_{2l-1}} \gamma_{b_{2l}} \\
& = \dots \\
& + \frac{1}{0!} \sum_b^a \{ \delta_{a_1 a_2} \delta_{a_3 a_4} \dots \delta_{a_{2l-3} a_{2l-2}} \delta_{a_{2l-1} a_{2l}} \delta_{b_1 b_2} \delta_{b_3 b_4} \dots \delta_{b_{2l-3} b_{2l-2}} \delta_{b_{2l-1} b_{2l}} \} \\
& + \frac{1}{2!} \sum_b^a \{ \delta_{a_1 a_2} \delta_{a_3 a_4} \dots \delta_{a_{2l-3} a_{2l-2}} \delta_{[a_{2l-1} (b_1 \delta_{a_{2l}}) b_2]} \delta_{b_3 b_4} \dots \delta_{b_{2l-3} b_{2l-2}} \delta_{b_{2l-1} b_{2l}} \} \\
& + \frac{1}{4!} \sum_b^a \{ \delta_{a_1 a_2} \delta_{a_3 a_4} \dots \delta_{[a_{2l-3} \langle b_1 \delta_{a_{2l-2}} b_2 \delta_{a_{2l-1}} b_3 \delta_{a_{2l}} \rangle b_4]} \dots \delta_{b_{2l-3} b_{2l-2}} \delta_{b_{2l-1} b_{2l}} \} + \dots \\
& + \frac{1}{(2l-2)!} \sum_b^a \{ \delta_{a_1 a_2} \delta_{[a_3 \langle b_1 \delta_{a_4} b_2 \delta_{a_5} b_3 \delta_{a_6} b_4 \dots \delta_{a_{2l-3} b_{2l-5}} \delta_{a_{2l-2} b_{2l-4}} \delta_{a_{2l-1} b_{2l-3}} \delta_{a_{2l}} \rangle b_{2l-2}] \delta_{b_{2l-1} b_{2l}} \} \\
& + \frac{1}{(2l)!} \delta_{[a_1 \langle b_1 \delta_{a_2} b_2 \delta_{a_3} b_3 \delta_{a_4} b_4 \dots \delta_{a_{2l-3} b_{2l-3}} \delta_{a_{2l-2} b_{2l-2}} \delta_{a_{2l-1} b_{2l-1}} \delta_{a_{2l}} \rangle b_{2l}] \}
\end{aligned}$$

$$\begin{aligned}
& \text{引理5.1.7. } \gamma_{a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \dots \gamma_{a_{2l-3}} \gamma_{a_{2l-2}} \gamma_{a_{2l-1}} \gamma_{a_{2l}} \gamma_{a_{2l+1}} \gamma_{b_1} \gamma_{b_2} \gamma_{b_3} \gamma_{b_4} \dots \gamma_{b_{2l-3}} \gamma_{b_{2l-2}} \gamma_{b_{2l-1}} \gamma_{b_{2l}} \gamma_{b_{2l+1}} \\
& = \dots \\
& + \frac{1}{1!} \sum_b^a \{ \delta_{a_1 a_2} \delta_{a_3 a_4} \dots \delta_{a_{2l-3} a_{2l-2}} \delta_{a_{2l-1} a_{2l}} \delta_{a_{2l+1} b_1} \delta_{b_2 b_3} \delta_{b_4 b_5} \dots \delta_{b_{2l-2} b_{2l-1}} \delta_{b_{2l} b_{2l+1}} \} \\
& + \frac{1}{3!} \sum_b^a \{ \delta_{a_1 a_2} \delta_{a_3 a_4} \dots \delta_{a_{2l-3} a_{2l-2}} \delta_{[a_{2l-1} (b_1 \delta_{a_{2l}} b_2 \delta_{a_{2l+1}}) b_3]} \delta_{b_4 b_5} \dots \delta_{b_{2l-2} b_{2l-1}} \delta_{b_{2l} b_{2l+1}} \} \\
& + \frac{1}{5!} \sum_b^a \{ \delta_{a_1 a_2} \delta_{a_3 a_4} \dots \delta_{[a_{2l-3} \langle b_1 \delta_{a_{2l-2}} b_2 \delta_{a_{2l-1}} b_3 \delta_{a_{2l}} b_4 \delta_{a_{2l+1}} \rangle b_5]} \dots \delta_{b_{2l-2} b_{2l-1}} \delta_{b_{2l} b_{2l+1}} \} + \dots \\
& + \frac{1}{(2l-1)!} \sum_b^a \{ \delta_{a_1 a_2} \delta_{[a_3 \langle b_1 \delta_{a_4} b_2 \delta_{a_5} b_3 \delta_{a_6} b_4 \dots \delta_{a_{2l-1} b_{2l-3}} \delta_{a_{2l} b_{2l-2}} \delta_{a_{2l+1}} \rangle b_{2l-1}] \delta_{b_{2l} b_{2l+1}} \} \\
& + \frac{1}{(2l+1)!} \delta_{[a_1 \langle b_1 \delta_{a_2} b_2 \delta_{a_3} b_3 \delta_{a_4} b_4 \dots \delta_{a_{2l-3} b_{2l-3}} \delta_{a_{2l-2} b_{2l-2}} \delta_{a_{2l-1} b_{2l-1}} \delta_{a_{2l} b_{2l}} \delta_{a_{2l+1}} \rangle b_{2l+1}] \}
\end{aligned}$$

以上写法虽然已经紧凑了些,也是可以一步步严格具体完整写出来,但还是不够简洁,仍无法方便运用,必须想一个更好的写法表示出来,以便使用。

## 5.2 $w + 1$ 阶全对称张量的展开与缩并

性质5.2.1.  $A_{(a_1 a_2 a_3 a_4 \cdots a_{2s})} = A_{a_1(a_2 a_3 a_4 \cdots a_{2s})} + A_{a_2(a_1 a_3 a_4 \cdots a_{2s})} + A_{a_3(a_1 a_2 a_4 \cdots a_{2s})} + \cdots$

性质5.2.2.  $A_{(a_1 a_2 a_3 a_4 \cdots a_{2s})} = \langle A_{(a_1 \cdots a_l)\{a_{l+1} \cdots a_{2s}\}}, \frac{(2s)!}{l!(2s-l)!} \rangle$   
 $= \langle A_{(\underbrace{a_1 \cdots a_{l_1}}_{l_1})(\underbrace{a_{l_1+1} \cdots a_{l_1+l_2}}_{l_2}) \cdots (\underbrace{a_{l_1+\cdots+l_{n-1}+1} \cdots a_{l_1+\cdots+l_n}}_{l_n})}, \frac{(2s)!}{l_1!l_2! \cdots l_n!} \rangle, l_1 + l_2 + \cdots + l_n = 2s$

性质5.2.3.  $\Gamma_{A_{1\zeta} A_{2\zeta} \cdots A_{2s\zeta}}^{k_\zeta}(s; w) \Gamma_{k_\zeta}^{B_{1\zeta} B_{2\zeta} \cdots B_{2s\zeta}}(s; w) = \frac{1}{(2s)!} \delta_{A_{1\zeta}}^{B_{1\zeta}} \delta_{A_{2\zeta}}^{B_{2\zeta}} \cdots \delta_{A_{2s\zeta}}^{B_{2s\zeta}} = \frac{1}{(2s)!} \delta_{(A_{1\zeta} A_{2\zeta} \cdots A_{2s\zeta})}^{(B_{1\zeta} B_{2\zeta} \cdots B_{2s\zeta})}$

性质5.2.4.  $\delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) = \delta_{a_1}^{b_1} \delta_{(a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) + \delta_{a_2}^{b_2} \delta_{(a_1}^{b_1} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) + \delta_{a_3}^{b_3} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_{2s}}^{b_{2s}}) + \cdots$

性质5.2.5.  $\delta_{b_1}^{a_1} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) = (2s + w) \delta_{(a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}})$

证明:  $\delta_{b_1}^{a_1} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}})$   
 $= \delta_{b_1}^{a_1} \delta_{a_1}^{b_1} \delta_{(a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) + \delta_{b_1}^{a_1} \delta_{a_2}^{b_2} \delta_{(a_1}^{b_1} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) + \delta_{b_1}^{a_1} \delta_{a_3}^{b_3} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_{2s}}^{b_{2s}}) + \cdots$   
 $= n \delta_{(a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) + \delta_{(a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) + \delta_{(a_3}^{b_3} \delta_{a_2}^{b_2} \cdots \delta_{a_{2s}}^{b_{2s}}) + \cdots$   
 $= (2s + w) \delta_{(a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}})$  □

性质5.2.6.

$$\begin{cases} \delta_{b_1}^{a_1} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) = 1! C_{2s+w}^1 \delta_{(a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}), & \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) = 2! C_{2s+w}^2 \delta_{(a_3}^{b_3} \delta_{a_4}^{b_4} \cdots \delta_{a_{2s}}^{b_{2s}}) \\ \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} \delta_{b_3}^{a_3} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) = 3! C_{2s+w}^3 \delta_{(a_4}^{b_4} \cdots \delta_{a_{2s}}^{b_{2s}}) \cdots \cdots \\ \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} \cdots \delta_{b_{2s-1}}^{a_{2s-1}} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) = (2s-1)! C_{2s+w}^{2s-1} \delta_{a_{2s}}^{b_{2s}}, & \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} \cdots \delta_{b_{2s}}^{a_{2s}} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) = (2s)! C_{2s+w}^{2s} \end{cases}$$

性质5.2.7.

$$\begin{cases} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_{2s}}^{b_{2s}}) \delta_{b_{2s}}^{a_{2s}} = 1! C_{2s+w}^1 \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_{2s-1}}^{b_{2s-1}}), & \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_{2s}}^{b_{2s}}) \delta_{b_{2s-1}}^{a_{2s-1}} \delta_{b_{2s}}^{a_{2s}} = 2! C_{2s+w}^2 \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_{2s-2}}^{b_{2s-2}}) \\ \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) \delta_{b_{2s-2}}^{a_{2s-2}} \cdots \delta_{b_{2s}}^{a_{2s}} = 3! C_{2s+w}^3 \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_{2s-3}}^{b_{2s-3}}) \cdots \cdots \\ \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) \delta_{b_{2s-1}}^{a_{2s-1}} \cdots \delta_{b_{2s}}^{a_{2s}} \delta_{b_1}^{a_1} = (2s-1)! C_{2s+w}^{2s-1} \delta_{a_{2s}}^{b_{2s}}, & \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_{2s}}^{b_{2s}}) \delta_{b_{2s}}^{a_{2s}} \cdots \delta_{b_2}^{a_2} \delta_{b_1}^{a_1} = (2s)! C_{2s+w}^{2s} \end{cases}$$

## 5.3 $N+1=n$ 维时空中反对称张量的展开与缩并

性质5.3.1.  $A_{[a_1 a_2 a_3 a_4 \cdots a_n]} = A_{a_1[a_2 a_3 a_4 \cdots a_n]} - A_{a_2[a_1 a_3 a_4 \cdots a_n]} + A_{a_3[a_1 a_2 a_4 \cdots a_n]} + \cdots$

性质5.3.2.  $A_{[a_1 a_2 a_3 a_4 \cdots a_{2s}]} = \langle A_{[a_1 \cdots a_l][a_{l+1} \cdots a_{2s}]}, \frac{(2s)!}{l!(2s-l)!} \rangle$   
 $= \langle A_{([\underbrace{a_1 \cdots a_{l_1}}_{l_1}][\underbrace{a_{l_1+1} \cdots a_{l_1+l_2}}_{l_2}]) \cdots ([\underbrace{a_{l_1+\cdots+l_{n-1}+1} \cdots a_{l_1+\cdots+l_n}}_{l_n}]}, \frac{(2s)!}{l_1!l_2! \cdots l_n!} \rangle, l_1 + l_2 + \cdots + l_n = 2s$

性质5.3.3.  $\varepsilon_{a_1 a_2 \cdots a_n} \varepsilon^{b_1 b_2 \cdots b_n} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_n}^{b_n} = \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_n}^{b_n}]$

性质5.3.4.  $\delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] = \delta_{a_1}^{b_1} \delta_{[a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] - \delta_{a_2}^{b_2} \delta_{[a_1}^{b_1} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] + \delta_{a_3}^{b_3} \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_n}^{b_n}] + \cdots$

性质5.3.5.  $\delta_{b_1}^{a_1} \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] = \delta_{[a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}]$

证明:  $\delta_{b_1}^{a_1} \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}]$   
 $= \delta_{b_1}^{a_1} \delta_{a_1}^{b_1} \delta_{[a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] - \delta_{b_1}^{a_1} \delta_{a_2}^{b_2} \delta_{[a_1}^{b_1} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] + \delta_{b_1}^{a_1} \delta_{a_3}^{b_3} \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_n}^{b_n}] + \cdots$   
 $= n \delta_{[a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] - \delta_{[a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] + \delta_{[a_3}^{b_3} \delta_{a_2}^{b_2} \cdots \delta_{a_n}^{b_n}] + \cdots$   
 $= \delta_{[a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}]$  □

性质5.3.6.

$$\begin{cases} \delta_{b_1}^{a_1} \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] = 1! \delta_{[a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}], & \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] = 2! \delta_{[a_3}^{b_3} \delta_{a_4}^{b_4} \cdots \delta_{a_n}^{b_n}] \\ \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} \delta_{b_3}^{a_3} \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] = 3! \delta_{[a_4}^{b_4} \cdots \delta_{a_n}^{b_n}] \cdots \cdots \\ \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} \cdots \delta_{b_{n-1}}^{a_{n-1}} \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] = (n-1)! \delta_{a_n}^{b_n}, & \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} \cdots \delta_{b_n}^{a_n} \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}] = n! \end{cases}$$

性质5.3.7.

$$\begin{cases} \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_n}^{b_n}) \delta_{b_n}^{a_n} = 1! \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_{n-1}}^{b_{n-1}}], & \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_n}^{b_n}) \delta_{b_{n-1}}^{a_{n-1}} \delta_{b_n}^{a_n} = 2! \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_{n-2}}^{b_{n-2}}] \\ \delta_{(a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}) \delta_{b_{n-2}}^{a_{n-2}} \cdots \delta_{b_n}^{a_n} = 3! \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \cdots \delta_{a_{n-3}}^{b_{n-3}}] \cdots \cdots \\ \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}) \delta_{b_n}^{a_n} \cdots \delta_{b_2}^{a_2} = (n-1)! \delta_{a_1}^{b_1}, & \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \cdots \delta_{a_n}^{b_n}) \delta_{b_n}^{a_n} \cdots \delta_{b_2}^{a_2} \delta_{b_1}^{a_1} = n! \end{cases}$$

## 6 N+1=n维时空中的Q积

### 6.1 N+1=n维时空中Q积的具体计算

定义6.1.1.  $K := (m - \gamma_a \partial^a) \gamma_0, \tilde{K} := CK^T \tilde{C} = -\gamma_0 (m + \gamma_a \partial^a), Q := (m - \gamma_a \partial^a), \tilde{Q} := (m + \gamma_a \partial^a)$

性质6.1.1.  $\Gamma_0 Q = \tilde{Q} \Gamma_0, Q \Gamma_0 = \Gamma_0 \tilde{Q}; \Gamma_0 Q \Gamma_0 = \tilde{Q}, \Gamma_0 \tilde{Q} \Gamma_0 = Q$

$$\text{性质6.1.2.} \begin{cases} \gamma_{a_1} Q \gamma_{a_1'} \tilde{Q} = -\Gamma_0 \gamma_{a_1} \tilde{Q} \Gamma_0 \gamma_{a_1'} \tilde{Q} = -\gamma_{a_1} \Gamma_0 \tilde{Q} \gamma_{a_1'} \Gamma_0 \tilde{Q} \\ \gamma_{a_1} Q \gamma_{a_1'} \tilde{Q} = -\gamma_{a_1} Q \Gamma_0 \gamma_{a_1'} Q \Gamma_0 = \gamma_{a_1} Q \gamma_{a_1'} \Gamma_0 Q \Gamma_0 \\ \gamma_{a_1} Q \gamma_{a_1'} \tilde{Q} \cdots \gamma_{a_l} Q \gamma_{a_l'} \tilde{Q} = (-1)^l \gamma_{a_1} \Gamma_0 \tilde{Q} \gamma_{a_1'} \Gamma_0 \tilde{Q} \cdots \gamma_{a_l} \Gamma_0 \tilde{Q} \gamma_{a_l'} \Gamma_0 \tilde{Q} \\ \gamma_{a_1} Q \gamma_{a_1'} \tilde{Q} \cdots \gamma_{a_l} Q \gamma_{a_l'} \tilde{Q} = (-1)^{l-1} \gamma_{a_1} Q \gamma_{a_1'} \Gamma_0 Q \gamma_{a_2} \Gamma_0 Q \gamma_{a_2'} \Gamma_0 Q \cdots \gamma_{a_l} \Gamma_0 Q \gamma_{a_l'} \Gamma_0 Q \Gamma_0 \end{cases}$$

$$\text{性质6.1.3.} \begin{cases} tr(\gamma_{a_1} Q \gamma_{a_1'} \tilde{Q}) = -tr(\Gamma_0 \gamma_{a_1} \tilde{Q} \Gamma_0 \gamma_{a_1'} \tilde{Q}) = -tr(\gamma_{a_1} \Gamma_0 \tilde{Q} \gamma_{a_1'} \Gamma_0 \tilde{Q}) \\ tr(\gamma_{a_1} Q \gamma_{a_1'} \tilde{Q}) = -tr(\Gamma_0 \gamma_{a_1} Q \Gamma_0 \gamma_{a_1'} Q) = -tr(\gamma_{a_1} \Gamma_0 Q \gamma_{a_1'} \Gamma_0 Q) \\ tr(\gamma_{a_1} Q \gamma_{a_1'} \tilde{Q} \cdots \gamma_{a_l} Q \gamma_{a_l'} \tilde{Q}) = (-1)^l tr(\gamma_{a_1} \Gamma_0 \tilde{Q} \gamma_{a_1'} \Gamma_0 \tilde{Q} \cdots \gamma_{a_l} \Gamma_0 \tilde{Q} \gamma_{a_l'} \Gamma_0 \tilde{Q}) \\ tr(\gamma_{a_1} Q \gamma_{a_1'} \tilde{Q} \cdots \gamma_{a_l} Q \gamma_{a_l'} \tilde{Q}) = (-1)^l tr(\gamma_{a_1} \Gamma_0 Q \gamma_{a_1'} \Gamma_0 Q \cdots \gamma_{a_l} \Gamma_0 Q \gamma_{a_l'} \Gamma_0 Q) \end{cases}$$

$$\text{性质6.1.4.} \begin{cases} tr[\gamma_a Q \gamma_{a'} \tilde{Q}] = 8(m^2 \delta_{aa'} - \partial_a \partial_{a'}) = 8m^2(\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \\ tr[\gamma_a \Gamma_0 \tilde{Q} \gamma_{a'} \Gamma_0 \tilde{Q}] = -8(m^2 \delta_{aa'} - \partial_a \partial_{a'}) = -8m^2(\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \\ tr[\gamma_a \Gamma_0 Q \gamma_{a'} \Gamma_0 Q] = -8(m^2 \delta_{aa'} - \partial_a \partial_{a'}) = -8m^2(\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \end{cases}$$

$$\text{性质6.1.5.} \begin{cases} tr[\gamma_a \tilde{Q} \gamma_{a'} \tilde{Q}] = tr[\gamma_a Q \gamma_{a'} Q] = 8\partial_a \partial_{a'} \\ tr[\gamma_a \Gamma_0 Q \gamma_{a'} \Gamma_0 \tilde{Q}] = tr[\gamma_a \Gamma_0 \tilde{Q} \gamma_{a'} \Gamma_0 \tilde{Q}] = -8\partial_a \partial_{a'} \end{cases}$$

证明:  $tr[\gamma_a Q \gamma_{a'} \tilde{Q}] = tr[\gamma_a (m - \gamma_{a_1} \partial^{a_1}) \gamma_{a'} (m + \gamma_{a_2} \partial^{a_2})]$   
 $= m^2 tr(\gamma_a \gamma_{a'}) - tr(\gamma_a \gamma_{a_1} \gamma_{a'} \gamma_{a_2}) \partial^{a_1} \partial^{a_2}$   
 $= 4m^2 \delta_{aa'} - 4(\delta_{aa_1} \delta_{a' a_2} - \delta_{aa'} \delta_{a_1 a_2} + \delta_{aa_2} \delta_{a_1 a'}) \partial^{a_1} \partial^{a_2}$   
 $= 4m^2 \delta_{aa'} - 4(2\partial_a \partial_{a'} - \delta_{aa'} m^2)$   
 $= 8(m^2 \delta_{aa'} - \partial_a \partial_{a'})$  □

证明:  $tr[\gamma_a \tilde{Q} \gamma_{a'} \tilde{Q}] = tr[\gamma_a (m + \gamma_{a_1} \partial^{a_1}) \gamma_{a'} (m + \gamma_{a_2} \partial^{a_2})]$   
 $= m^2 tr(\gamma_a \gamma_{a'}) + tr(\gamma_a \gamma_{a_1} \gamma_{a'} \gamma_{a_2}) \partial^{a_1} \partial^{a_2}$   
 $= 4m^2 \delta_{aa'} + 4(\delta_{aa_1} \delta_{a' a_2} - \delta_{aa'} \delta_{a_1 a_2} + \delta_{aa_2} \delta_{a_1 a'}) \partial^{a_1} \partial^{a_2}$   
 $= 4m^2 \delta_{aa'} + 4(2\partial_a \partial_{a'} - \delta_{aa'} m^2)$   
 $= 8\partial_a \partial_{a'}$  □

证明:  $tr[\gamma_a Q \gamma_{a'} Q] = tr[\gamma_a (m - \gamma_{a_1} \partial^{a_1}) \gamma_{a'} (m - \gamma_{a_2} \partial^{a_2})]$   
 $= m^2 tr(\gamma_a \gamma_{a'}) + tr(\gamma_a \gamma_{a_1} \gamma_{a'} \gamma_{a_2}) \partial^{a_1} \partial^{a_2}$   
 $= 4m^2 \delta_{aa'} + 4(\delta_{aa_1} \delta_{a' a_2} - \delta_{aa'} \delta_{a_1 a_2} + \delta_{aa_2} \delta_{a_1 a'}) \partial^{a_1} \partial^{a_2}$   
 $= 4m^2 \delta_{aa'} + 4(2\partial_a \partial_{a'} - \delta_{aa'} m^2)$   
 $= 8\partial_a \partial_{a'}$  □



势对易规则可以从场对易规则通过Q积计算得到，原则上可以用来严格证明Behrends-Fronsdal猜想公式，但十分繁琐，难以使用，事实上还是很难证明Behrends-Fronsdal猜想公式。

# 第十九章 非相对论性粒子的量子化

自我评述：在本章我充分运用四维傅里叶变换技巧，汇聚讨论了非相对论性粒子的二次量子化细节。

## 1 傅里叶分析技巧和平面波解展开 [39]

### 1.1 波函数的傅里叶展开

基本思想：将波函数按傅里叶分析完备唯一展开，将方程和约束看作对波函数的选取条件，最后得到的是符合方程和约束的完备平面波解。

定义1.1.1.  $k \equiv (\vec{k}, iE), x \equiv (\vec{r}, it), d^4x \equiv d^3\vec{r}dt, \omega_k \equiv \sqrt{\vec{k}^2 + m^2} > 0, p_a(\omega_k) \equiv (\vec{k}, i\omega_k)_a$

定义1.1.2.  $\int_{\vec{r}=-\infty}^{+\infty} \equiv \int_{r_x=-\infty}^{+\infty} \int_{r_y=-\infty}^{+\infty} \int_{r_z=-\infty}^{+\infty}, \int_{x=-\infty}^{+\infty} \equiv \int_{r_x=-\infty}^{+\infty} \int_{r_y=-\infty}^{+\infty} \int_{r_z=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty}$

定义1.1.3.  $\int_{\vec{k}=-\infty}^{+\infty} \equiv \int_{p_x=-\infty}^{+\infty} \int_{p_y=-\infty}^{+\infty} \int_{p_z=-\infty}^{+\infty}, \int_{k=-\infty}^{+\infty} \equiv \int_{p_x=-\infty}^{+\infty} \int_{p_y=-\infty}^{+\infty} \int_{p_z=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty}$

波函数的傅里叶展开：

$$\phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \Phi(\vec{k}, E) e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k}dE, \Phi(\vec{k}, E) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \phi(\vec{r}, t) e^{-i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{r}dt$$

波函数的傅里叶展开紧凑形式：

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int_{k=-\infty}^{+\infty} \Phi(k) e^{ik\cdot x} d^4k \Leftrightarrow \Phi(k) = \frac{1}{(2\pi)^{3/2}} \int_{x=-\infty}^{+\infty} \phi(x) e^{-ik\cdot x} d^4x \quad (19.1)$$

### 1.2 波函数的傅里叶展开与洛伦兹协变性

定义1.2.1.  $\Phi(k) \equiv \frac{1}{(2\pi)^{3/2}} \int_{x=-\infty}^{+\infty} \phi(x) e^{-ik\cdot x} d^4x, \Phi'(k') \equiv \frac{1}{(2\pi)^{3/2}} \int_{x'=-\infty}^{+\infty} \phi'(x') e^{-ik'\cdot x'} d^4x'$

对于以上两式数学上只是随意的两个定义，可以有随意的数学含义，但可以赋予其明确的物理意义，将前式看作在参考系 $O$ 中的表达，后式看作在参考系 $O'$ 中的表达。详细地说，将 $x$ 看作时空坐标在参考系 $O$ 中的表达，将 $k$ 看作四维动量在参考系 $O$ 中的表达，将 $\phi(x)$ 看作时空波函数在参考系 $O$ 中的表达，将 $\Phi(k)$ 看作动量相空间波函数在参考系 $O$ 中的表达；将 $x'$ 看作时空坐标在参考系 $O'$ 中的表达，将 $k'$ 看作四维动量在参考系 $O'$ 中的表达，将 $\phi'(x')$ 看作时空波函数在参考系 $O'$ 中的表达，将 $\Phi'(k')$ 看作动量相空间波函数在参考系 $O'$ 中的表达。参考系 $O'$ 中的物理量与参考系 $O$ 中的对应物理量由洛伦兹变换相联系。在这个联系中 $k\cdot x, d^4x$ 表现为标量， $\phi(x), \Phi(k)$ 表现为协变量。

定理1.2.1.  $\phi'(e^\varepsilon x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\phi(x) \Leftrightarrow \Phi'(e^\varepsilon k) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\Phi(k)$

证明： $\phi'(e^\varepsilon x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\phi(x)$

$$\Leftrightarrow \frac{i}{(2\pi)^{3/2}} \int_{x=-\infty}^{+\infty} \phi'(e^\varepsilon x) e^{-ik\cdot x} d^4x = \frac{i}{(2\pi)^{3/2}} \int_{x=-\infty}^{+\infty} e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\phi(x) e^{-ik\cdot x} d^4x$$

$$\Leftrightarrow \frac{i}{(2\pi)^{3/2}} \int_{e^{-\varepsilon}x'=-\infty}^{+\infty} \phi'(x') e^{-ik e^{-\varepsilon}\cdot x'} d^4x' = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\Phi(k)$$

$$\Leftrightarrow \frac{i}{(2\pi)^{3/2}} \int_{x'=-\infty}^{+\infty} \phi'(x') e^{-ik e^{-\varepsilon}\cdot x'} d^4x' = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\Phi(k)$$

$$\Leftrightarrow \Phi'(e^\varepsilon k) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\Phi(k) \quad \square$$

以上定理表明，如果时空波函数是个协变量，那么其傅里叶展开系数全是同类型的协变量，反之亦然。以上结论从数学上看，等价于作一个变量代换 $x' = e^\varepsilon x, k' = e^\varepsilon k$ ，且满足 $\Phi'(e^\varepsilon x) = e^{\frac{i}{2}\varepsilon^{ab}S_{ab}}\Phi(x)$ ，这个变换就是洛伦兹变换。

### 1.3 特殊函数的洛伦兹协变性

$$\text{推论1.3.1. } \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{\omega} d^3\vec{k} = \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \frac{1}{2\omega} [\delta(E-\omega) + \delta(E+\omega)] d^3\vec{k} dE = \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \delta(E^2 - \omega^2) d^3\vec{k} dE \text{ (为标量)}$$

得到一个重要的数学技巧:  $\frac{1}{\omega} d^3\vec{k} = \delta(E^2 - \omega^2) d^3\vec{k} dE$ , 它是一个标量。有没有一个更直观的证法?

$$\text{推论1.3.2. } \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega} d^3\vec{k} = \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \frac{1}{2\omega} \delta(E-\omega) d^3\vec{k} dE = \int_{\vec{k}=-\infty}^{+\infty} \int_{E=0}^{+\infty} \delta(E^2 - \omega^2) d^3\vec{k} dE \text{ (为标量)}$$

$$\text{推论1.3.3. } \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega} d^3\vec{k} = \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \frac{1}{2\omega} \delta(E+\omega) d^3\vec{k} dE = \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^0 \delta(E^2 - \omega^2) d^3\vec{k} dE \text{ (为标量)}$$

$$\text{推论1.3.4. } \delta(E^2 - \omega^2) = \frac{1}{2\omega} [\delta(E-\omega) + \delta(E+\omega)]$$

$$\text{推论1.3.5. } \delta(E^2 - \omega^2) U(E) = \frac{1}{2\omega} \delta(E-\omega), \delta(E^2 - \omega^2) U(-E) = \frac{1}{2\omega} \delta(E+\omega)$$

$$\text{推论1.3.6. } \delta(E^2 - \omega^2) U(E-\omega) = \frac{1}{2\omega} \delta(E-\omega), \delta(E^2 - \omega^2) U(-E-\omega) = \frac{1}{2\omega} \delta(E+\omega)$$

$$\text{推论1.3.7. } \delta(-k_a k^a - m^2) \text{ 为标量。}$$

$$\text{推论1.3.8. } \frac{1}{\omega} \delta(E-\omega), \frac{1}{\omega} \delta(E+\omega) \text{ 为标量。}$$

$$\text{推论1.3.9. } \delta^4(k' - k), \delta^4(x' - x) \text{ 为标量。}$$

$$\text{推论1.3.10. } d^4k, d^4x, \frac{1}{\omega} d^3\vec{k} \text{ 为标量。}$$

## 2 粒子守恒协变量

### 2.1 电流源守恒方程和守恒协变量

$$\text{推论2.1.1. } Q = \int_{\vec{r}=-\infty}^{+\infty} \rho(\vec{r}, t_0) d^3\vec{r}, \quad \partial_a J^a(\vec{r}, t) = 0 \Rightarrow Q = \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \partial_a [J^a(\vec{r}, t) U(t-t_0)] d^3\vec{r} dt$$

$$\begin{aligned} \text{证明: } Q &= \int_{\vec{r}=-\infty}^{+\infty} \rho(\vec{r}, t_0) d^3\vec{r} = \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \rho(\vec{r}, t) \delta(t-t_0) d^3\vec{r} dt \\ &= \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \rho(\vec{r}, t) \partial_t U(t-t_0) d^3\vec{r} dt = \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} J^a(\vec{r}, t) \partial_a U(t-t_0) d^3\vec{r} dt \end{aligned}$$

$$\partial_a J^a(\vec{r}, t) = 0 \Rightarrow \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \partial_a [J^a(\vec{r}, t) U(t-t_0)] d^3\vec{r} dt \quad \square$$

$$\text{推论2.1.2. } P^a = \int_{\vec{r}=-\infty}^{+\infty} T^{a0}(\vec{r}, t_0) d^3\vec{r}, \quad \partial_b T^{ab}(\vec{r}, t) = 0 \Rightarrow P^a = \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \partial_b [T^{ab}(\vec{r}, t) U(t-t_0)] d^3\vec{r} dt,$$

$$\begin{aligned} \text{证明: } P^a &= \int_{\vec{r}=-\infty}^{+\infty} T^{a0}(\vec{r}, t_0) d^3\vec{r} = \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} T^{a0}(\vec{r}, t) \delta(t-t_0) d^3\vec{r} dt \\ &= \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} T^{a0}(\vec{r}, t) \partial_t U(t-t_0) d^3\vec{r} dt = \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} T^{ab}(\vec{r}, t) \partial_b U(t-t_0) d^3\vec{r} dt \end{aligned}$$

$$\partial_b T^{ab}(\vec{r}, t) = 0 \Rightarrow \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \partial_b [T^{ab}(\vec{r}, t) U(t-t_0)] d^3\vec{r} dt, \quad \square$$

$$\text{推论2.1.3. } M^{ab} = \int_{\vec{r}=-\infty}^{+\infty} J^{ab0}(\vec{r}, t_0) d^3\vec{r}, \quad \partial_c J^{abc}(\vec{r}, t) = 0 \Rightarrow M^{ab} = \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \partial_c [J^{abc}(\vec{r}, t) U(t-t_0)] d^3\vec{r} dt$$

$$\begin{aligned}
\text{证明: } M^{ab} &= \int_{\vec{r}=-\infty}^{+\infty} J^{ab0}(\vec{r}, t_0) d^3\vec{r} = \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} J^{ab0}(\vec{r}, t) \delta(t-t_0) d^3\vec{r} dt \\
&= \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} J^{ab0}(\vec{r}, t) \partial_t U(t-t_0) d^3\vec{r} dt = \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} J^{abc}(\vec{r}, t) \partial_c U(t-t_0) d^3\vec{r} dt \\
\partial_c J^{abc}(\vec{r}, t) &\stackrel{=0}{=} \int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \partial_c [J^{abc}(\vec{r}, t) U(t-t_0)] d^3\vec{r} dt \quad \square
\end{aligned}$$

数学技巧：我这里只提供关键的技巧，并没有给出完整的证明。温伯格书中利用电流源守恒方程、物理函数在空间无穷远处为零和洛伦兹变换下物理时间前后不变的特性，从上面命题就可以证明  $Q, P^a, M^{ab}$  的洛伦兹协变性。

### 3 非相对论性粒子

#### 3.1 复标量场方程的平面波解

$$\text{复标量场方程: } (\partial_a \partial^a - m^2) \phi(\vec{r}, t) = 0 \Leftrightarrow (\nabla^2 - \partial_t^2 - m^2) \phi(\vec{r}, t) = 0 \quad (19.2)$$

$$\text{定理3.1.1. } (\partial_a \partial^a - m^2) \phi(\vec{r}, t) = 0 \Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k}, -\omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k}$$

$$\text{证明: } (\partial_a \partial^a - m^2) \phi(\vec{r}, t) = 0 \Rightarrow \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \phi(\vec{k}, E) (-\vec{k}^2 + E^2 - m^2) e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE = 0$$

$$\Leftrightarrow \phi(\vec{k}, E) (E^2 - \vec{k}^2 - m^2) = 0 \Leftrightarrow \phi(\vec{k}, E) = a(\vec{k}, E) \delta(E^2 - \vec{k}^2 - m^2) + \phi_0(\vec{k}, E) \delta_{E^2, \vec{k}^2+m^2}$$

$$\Rightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} [a(\vec{k}, E) \delta(E^2 - \vec{k}^2 - m^2) + \phi_0(\vec{k}, E) \delta_{E^2, \vec{k}^2+m^2}] e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} a(\vec{k}, E) \delta(E^2 - \vec{k}^2 - m^2) e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE, \text{ 明显洛伦兹协变}$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \frac{1}{2\omega_k} a(\vec{k}, E) [\delta(E - \omega_k) + \delta(E + \omega_k)] e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \frac{1}{2\omega_k} a(\vec{k}, E) [\delta(E - \omega_k) + \delta(E + \omega_k)] e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(\vec{k}, -\omega_k) e^{i(\vec{k}\cdot\vec{r}+\omega_k t)}] d^3\vec{k}$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k}, -\omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k} \quad \square$$

这里用了一种与一般书中不同的方法，采用了四维而非三维的傅里叶展开，清晰地展现出粒子离壳和在壳的物理概念，洛伦兹协变性也明显表现于其中，并且包含了狄拉克函数解的代数新解法。在证明的过程也看到了正能解和负能解的分解，并且负能解可以按两种含义进行理解，一是将负能解理解为一种负质量粒子，二是将负能解还是理解为正质量粒子，不过要理解为反射波，正能解理解为入射波。

$$\text{推论3.1.1. } a'(e^\varepsilon[\vec{k}, E]) \delta(E^2 - \vec{k}^2 - m^2) = e^{\frac{i}{2}\varepsilon^{ab} S_{ab}} a(\vec{k}, E) \delta(E^2 - \vec{k}^2 - m^2)$$

$$\Rightarrow a'(e^\varepsilon[\vec{k}, \omega_k]) = e^{\frac{i}{2}\varepsilon^{ab} S_{ab}} a(\vec{k}, \omega_k), a'(e^\varepsilon[\vec{k}, -\omega_k]) = e^{\frac{i}{2}\varepsilon^{ab} S_{ab}} a(\vec{k}, -\omega_k)$$

$$\text{推论3.1.2. } a(\vec{k}, E) \delta(E^2 - \vec{k}^2 - m^2) = \frac{1}{2\omega_k} a(\vec{k}, E) [\delta(E - \omega_k) + \delta(E + \omega_k)], |\vec{k}| \ll m$$

$$\approx \frac{1}{2(m + \frac{\vec{k}^2}{2m})} [a(\vec{k}, m + \frac{\vec{k}^2}{2m}) \delta(E - m - \frac{\vec{k}^2}{2m}) + a(\vec{k}, -m - \frac{\vec{k}^2}{2m}) \delta(E + m + \frac{\vec{k}^2}{2m})]$$

$$\text{推论3.1.3. } \phi(\vec{r}, t) \approx \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2(m + \frac{\vec{k}^2}{2m})} [a(\vec{k}, m + \frac{\vec{k}^2}{2m}) e^{i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m} t)} e^{-imt} + a(-\vec{k}, -m - \frac{\vec{k}^2}{2m}) e^{-i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m} t)} e^{imt}] d^3\vec{k}$$

从上可知，在非相对论极限下，复标量场平面波解分成两个非相对论性的正反粒子，可以同时存在。这可以继续分析下去，是否可以证明正、负能解独自守恒？

### 3.2 复标量场方程的两个非相对论性分支

$$\text{定理3.2.1. } (\partial_a \partial^a - m^2)\phi(\vec{r}, t) = 0 \Rightarrow \text{两个分支: } \begin{cases} \text{正能解: } (i\partial_t + \frac{1}{2m}\nabla^2)\phi_+(\vec{r}, t) = 0 \\ \text{负能解: } (-i\partial_t + \frac{1}{2m}\nabla^2)\phi_-(\vec{r}, t) = 0 \end{cases}$$

$$\text{定理3.2.2. 正能解: } (i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r}, t) = 0 \Leftrightarrow \text{负能解: } (-i\partial_t + \frac{1}{2m}\nabla^2)\phi^*(\vec{r}, t) = 0$$

### 3.3 正能解薛定谔方程的作用量及其泊松括号

$$\text{推论3.3.1. 拉氏密度: } \mathcal{L} = \frac{1}{2}[i\phi^*(\vec{r}, t)\partial_t\phi(\vec{r}, t) - i\phi(\vec{r}, t)\partial_t\phi^*(\vec{r}, t) - \frac{1}{m}\nabla\phi^*(\vec{r}, t) \cdot \nabla\phi(\vec{r}, t)]$$

$$\text{拉氏密度: } \mathcal{L} = i\phi^*(\vec{r}, t)\partial_t\phi(\vec{r}, t) + \frac{1}{2m}\phi^*(\vec{r}, t)\nabla^2\phi(\vec{r}, t)$$

$$\text{运动方程: } (i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r}, t) = 0$$

$$\text{推论3.3.2. 正则变量: } \pi(\vec{r}, t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = i\phi^*(\vec{r}, t)$$

$$\text{推论3.3.3. 哈密顿密度: } \mathcal{H} = \frac{\partial\mathcal{L}}{\partial\phi}\dot{\phi} - \mathcal{L} = -\frac{1}{2m}\phi^*(\vec{r}, t)\nabla^2\phi(\vec{r}, t) = \frac{i}{2m}\pi(\vec{r}, t)\nabla^2\phi(\vec{r}, t)$$

$$\text{推论3.3.4. 动量密度: } \mathcal{P} = -\frac{\partial\mathcal{L}}{\partial\phi}\nabla\phi = -i\phi^*(\vec{r}, t)\nabla\phi(\vec{r}, t) = -\pi(\vec{r}, t)\nabla\phi(\vec{r}, t)$$

$$\text{推论3.3.5. 拉氏密度: } \mathcal{L}_H = \pi(\vec{r}, t)\partial_t\phi(\vec{r}, t) - \frac{i}{2m}\pi(\vec{r}, t)\nabla^2\phi(\vec{r}, t)$$

正则变量之间的基本关系:

$$\text{推论3.3.6. } \{\phi(\vec{r}', t), \phi(\vec{r}, t)\}_p = 0, \{\pi(\vec{r}', t), \pi(\vec{r}, t)\}_p = 0, \{\phi(\vec{r}', t), \pi(\vec{r}, t)\}_p = \delta^3(\vec{r}' - \vec{r})$$

哈密顿运动方程:

$$\text{推论3.3.7. } \begin{cases} \dot{\phi}(\vec{r}, t) = \frac{i}{2m}\nabla^2\phi(\vec{r}, t) = \{\phi(\vec{r}, t), H\}_p \\ \dot{\pi}(\vec{r}, t) = -\frac{i}{2m}\nabla^2\pi(\vec{r}, t) = \{\pi(\vec{r}, t), H\}_p \end{cases}$$

几率流守恒方程和守恒荷:

$$\text{推论3.3.8. } i\partial_t[\phi^*(\vec{r}, t)\phi(\vec{r}, t)] + \frac{1}{2m}\nabla \cdot [\phi^*(\vec{r}, t)\nabla\phi(\vec{r}, t) - \phi(\vec{r}, t)\nabla\phi^*(\vec{r}, t)] = 0 \\ \Rightarrow \dot{Q} = 0, Q = \int_{\vec{r}=-\infty}^{+\infty} \phi^*(\vec{r}, t)\phi(\vec{r}, t)d^3\vec{r} \in R$$

以上守恒量的存在, 说明总几率守恒, 几率解释有数学基础, 但其实也可以有另外的解释, 比如电荷。这就是数学和物理的联系与区别, 一个明确的数学结论可以有几种合理的物理解释。

### 3.4 正能解薛定谔方程的平面波解

薛定谔方程正能解分支:

$$\text{定理3.4.1. } (i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r}, t) = 0 \Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} a(\vec{k}, \frac{\vec{k}^2}{2m}) e^{i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)} d^3\vec{k}$$

$$\text{证明: } (i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r}, t) = 0 \Rightarrow \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \phi(\vec{k}, E)(E - \frac{\vec{k}^2}{2m}) e^{i(\vec{k}\cdot\vec{r} - Et)} d^3\vec{k}dE = 0$$

$$\Leftrightarrow \phi(\vec{k}, E)(E - \frac{\vec{k}^2}{2m}) = 0 \Leftrightarrow \phi(\vec{k}, E) = a(\vec{k}, E)\delta(E - \frac{\vec{k}^2}{2m}) + \phi_0(\vec{k}, E)\delta_{E, \frac{\vec{k}^2}{2m}}$$

$$\Rightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} [a(\vec{k}, E)\delta(E - \frac{\vec{k}^2}{2m}) + \phi_0(\vec{k}, E)\delta_{E, \frac{\vec{k}^2}{2m}}] e^{i(\vec{k}\cdot\vec{r} - Et)} d^3\vec{k}dE$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} a(\vec{k}, E)\delta(E - \frac{\vec{k}^2}{2m}) e^{i(\vec{k}\cdot\vec{r} - Et)} d^3\vec{k}dE$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} a(\vec{k}, \frac{\vec{k}^2}{2m}) e^{i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)} d^3\vec{k}$$

$$\Leftrightarrow a(\vec{k}) \equiv a(\vec{k}, \frac{\vec{k}^2}{2m}) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{r}=-\infty}^{+\infty} \phi(\vec{r}, t) e^{-i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)} d^3\vec{r}$$

□

$$\text{推论3.4.1. } \phi(-\vec{r}, -t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} a(\vec{k}) e^{-i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)} d^3\vec{k}$$

$$\text{推论3.4.2. } H = \int_{\vec{k}=-\infty}^{+\infty} \frac{\vec{k}^2}{2m} a^*(\vec{k}) a(\vec{k}) d^3\vec{k}, \vec{P} = \int_{\vec{k}=-\infty}^{+\infty} \vec{k} a^*(\vec{k}) a(\vec{k}) d^3\vec{k}, Q = \int_{\vec{k}=-\infty}^{+\infty} a^*(\vec{k}) a(\vec{k}) d^3\vec{k}$$

### 3.5 负能解薛定谔方程的作用量及其泊松括号

$$\text{推论3.5.1. 拉氏密度: } \mathcal{L} = \frac{1}{2} [-i\phi^*(\vec{r}, t)\partial_t\phi(\vec{r}, t) + i\phi(\vec{r}, t)\partial_t\phi^*(\vec{r}, t) - \frac{1}{m}\nabla\phi^*(\vec{r}, t) \cdot \nabla\phi(\vec{r}, t)]$$

$$\text{拉氏密度: } \mathcal{L} = -i\phi^*(\vec{r}, t)\partial_t\phi(\vec{r}, t) + \frac{1}{2m}\phi^*(\vec{r}, t)\nabla^2\phi(\vec{r}, t)$$

$$\text{运动方程: } (-i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r}, t) = 0$$

$$\text{推论3.5.2. 正则变量: } \pi(\vec{r}, t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = -i\phi^*(\vec{r}, t)$$

$$\text{推论3.5.3. 哈密顿密度: } \mathcal{H} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - \mathcal{L} = -\frac{1}{2m}\phi^*(\vec{r}, t)\nabla^2\phi(\vec{r}, t) = -\frac{i}{2m}\pi(\vec{r}, t)\nabla^2\phi(\vec{r}, t)$$

$$\text{推论3.5.4. 动量密度: } \mathcal{P} = -\frac{\partial\mathcal{L}}{\partial\phi}\nabla\phi = i\phi^*(\vec{r}, t)\nabla\phi(\vec{r}, t) = -\pi(\vec{r}, t)\nabla\phi(\vec{r}, t)$$

$$\text{推论3.5.5. 拉氏密度: } \mathcal{L}_H = \pi(\vec{r}, t)\partial_t\phi(\vec{r}, t) + \frac{i}{2m}\pi(\vec{r}, t)\nabla^2\phi(\vec{r}, t)$$

正则变量之间的基本关系:

$$\text{推论3.5.6. } \{\phi(\vec{r}', t), \phi(\vec{r}, t)\}_p = 0, \{\pi(\vec{r}', t), \pi(\vec{r}, t)\}_p = 0, \{\phi(\vec{r}', t), \pi(\vec{r}, t)\}_p = \delta^3(\vec{r}' - \vec{r})$$

哈密顿运动方程:

$$\text{推论3.5.7. } \begin{cases} \dot{\phi}(\vec{r}, t) = -\frac{i}{2m}\nabla^2\phi(\vec{r}, t) = \{\phi(\vec{r}, t), H\}_p \\ \dot{\pi}(\vec{r}, t) = \frac{i}{2m}\nabla^2\pi(\vec{r}, t) = \{\pi(\vec{r}, t), H\}_p \end{cases}$$

几率流守恒方程和守恒荷:

$$\text{推论3.5.8. } -i\partial_t[\phi^*(\vec{r}, t)\phi(\vec{r}, t)] + \frac{1}{2m}\nabla \cdot [\phi^*(\vec{r}, t)\nabla\phi(\vec{r}, t) - \phi(\vec{r}, t)\nabla\phi^*(\vec{r}, t)] = 0 \\ \Rightarrow \dot{Q} = 0, Q = \int_{\vec{r}=-\infty}^{+\infty} \phi^*(\vec{r}, t)\phi(\vec{r}, t)d^3\vec{r} \in R$$

以上守恒量的存在, 说明总几率守恒, 几率解释有数学基础, 但其实也可以有另外的解释, 比如电荷。这就是数学和物理的联系与区别, 一个明确的数学结论可以有几种合理的物理解释。

### 3.6 负能解薛定谔方程的平面波解

薛定谔方程负能解分支:

$$\text{定理3.6.1. } (-i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r}, t) = 0 \Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} a(-\vec{k}, -\frac{\vec{k}^2}{2m}) e^{-i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)} d^3\vec{k}$$

$$\text{证明: } (-i\partial_t + \frac{1}{2m}\nabla^2)\phi(\vec{r}, t) = 0 \Rightarrow \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \phi(\vec{k}, E)(E + \frac{\vec{k}^2}{2m}) e^{i(\vec{k}\cdot\vec{r} - Et)} d^3\vec{k}dE = 0$$

$$\Leftrightarrow \phi(\vec{k}, E)(E + \frac{\vec{k}^2}{2m}) = 0 \Leftrightarrow \phi(\vec{k}, E) = a(\vec{k}, E)\delta(E + \frac{\vec{k}^2}{2m}) + \phi_0(\vec{k}, E)\delta_{E, -\frac{\vec{k}^2}{2m}}$$

$$\Rightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} [a(\vec{k}, E)\delta(E + \frac{\vec{k}^2}{2m}) + \phi_0(\vec{k}, E)\delta_{E, -\frac{\vec{k}^2}{2m}}] e^{i(\vec{k}\cdot\vec{r} - Et)} d^3\vec{k}dE$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} a(\vec{k}, E)\delta(E + \frac{\vec{k}^2}{2m}) e^{i(\vec{k}\cdot\vec{r} - Et)} d^3\vec{k}dE$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} a(\vec{k}, -\frac{\vec{k}^2}{2m}) e^{i(\vec{k}\cdot\vec{r} + \frac{\vec{k}^2}{2m}t)} d^3\vec{k}$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} a(-\vec{k}, -\frac{\vec{k}^2}{2m}) e^{-i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)} d^3\vec{k}$$

$$\Leftrightarrow a(\vec{k}) \equiv a(-\vec{k}, -\frac{\vec{k}^2}{2m}) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{r}=-\infty}^{+\infty} \phi(\vec{r}, t) e^{i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)} d^3\vec{r}$$

□

$$\text{推论3.6.1. } H = \int_{\vec{k}=-\infty}^{+\infty} \frac{\vec{k}^2}{2m} a^*(\vec{k})a(\vec{k})d^3\vec{k}, \vec{P} = \int_{\vec{k}=-\infty}^{+\infty} \vec{k}a^*(\vec{k})a(\vec{k})d^3\vec{k}, Q = \int_{\vec{k}=-\infty}^{+\infty} a^*(\vec{k})a(\vec{k})d^3\vec{k}$$

从上可知，在非相对论极限下，正能量分支和负能量分支薛定谔方程各自描述一个非相对论性粒子，不能同时描述两个正反粒子，若出现在同一个方程中就变为零，无意义。并且正能量分支的解共轭就是负能量分支的解，反之亦然，即正粒子的共轭就表征反粒子。另外从上面两个定理的证明可以看出，只有认为 $m$ 可以取正负就可以统一表述两个分支，形式上与正能量分支一样。 $m > 0$ 描述正分支， $m < 0$ 描述反分支。对于负能解有两种理解，一种是描述负质量的粒子，另一种仍是描述正质量的粒子，量子化后与正能解是等价的。

## 4 正、负能解薛定谔方程的二次量子化

### 4.1 正、负能解薛定谔方程的二次量子化讨论

$$\text{推论4.1.1. } H = \int_{\vec{k}=-\infty}^{+\infty} \frac{\vec{k}^2}{2m} a^+(\vec{k})a(\vec{k})d^3\vec{k}, \vec{P} = \int_{\vec{k}=-\infty}^{+\infty} \vec{k}a^+(\vec{k})a(\vec{k})d^3\vec{k}, Q = \hat{N} = \int_{\vec{k}=-\infty}^{+\infty} a^+(\vec{k})a(\vec{k})d^3\vec{k}$$

以上关系不依赖于对易或反对易关系，也不依赖于正能解或负能解。

定义4.1.1. 量子方程： $i\partial_t|\Psi\rangle = H|\Psi\rangle$ ， $|\Psi\rangle$  为量子波函数。

$$\text{推论4.1.2. } -i\nabla|\Psi\rangle = \vec{P}|\Psi\rangle$$

按对易子理解：描述非相对论性玻色子

$$\text{推论4.1.3. } \begin{cases} [\phi(\vec{r}', t), \phi(\vec{r}, t)] = 0 \\ [\phi^+(\vec{r}', t), \phi^+(\vec{r}, t)] = 0 \\ [\phi(\vec{r}', t), \phi^+(\vec{r}, t)] = \delta^3(\vec{r}' - \vec{r}) \end{cases} \Leftrightarrow \begin{cases} [a(\vec{k}'), a(\vec{k})] = 0 \\ [a^+(\vec{k}'), a^+(\vec{k})] = 0 \\ [a(\vec{k}'), a^+(\vec{k})] = \delta^3(\vec{k}' - \vec{k}) \end{cases}$$

按反对易子理解：描述非相对论性费米子

$$\text{推论4.1.4. } \begin{cases} \{\phi(\vec{r}', t), \phi(\vec{r}, t)\} = 0 \\ \{\phi^+(\vec{r}', t), \phi^+(\vec{r}, t)\} = 0 \\ \{\phi(\vec{r}', t), \phi^+(\vec{r}, t)\} = \delta^3(\vec{r}' - \vec{r}) \end{cases} \Leftrightarrow \begin{cases} \{a(\vec{k}'), a(\vec{k})\} = 0 \\ \{a^+(\vec{k}'), a^+(\vec{k})\} = 0 \\ \{a(\vec{k}'), a^+(\vec{k})\} = \delta^3(\vec{k}' - \vec{k}) \end{cases}$$

### 4.2 粒子的量子描述

$$\text{定义4.2.1. } \hat{N} = \int_{\vec{k}=-\infty}^{+\infty} a^+(\vec{k})a(\vec{k})d^3\vec{k}, P^a = \int_{\vec{k}=-\infty}^{+\infty} k^a a^+(\vec{k})a(\vec{k})d^3\vec{k}$$

定义4.2.2. 若 $|0\rangle \neq 0$ ，且 $a(\vec{k})|0\rangle = 0, \forall \vec{k}$ ，则 $|0\rangle$  为真空态或基态。

$$\text{推论4.2.1. } \hat{N}|0\rangle = 0$$

$$\text{证明: } a(\vec{k})|0\rangle = 0, \forall \vec{k} \Rightarrow a^+(\vec{k})a(\vec{k})|0\rangle = 0, \forall \vec{k} \Rightarrow \int_{\vec{k}=-\infty}^{+\infty} a^+(\vec{k})a(\vec{k})|0\rangle d^3\vec{k} = 0$$

$$\Rightarrow \int_{\vec{k}=-\infty}^{+\infty} a^+(\vec{k})a(\vec{k})d^3\vec{k}|0\rangle = 0 \Rightarrow \hat{N}|0\rangle = 0 \quad \square$$

$$\text{推论4.2.2. } \langle 0|0\rangle > 0, \text{归一化: } \langle 0|0\rangle = 1$$

## 5 玻色子的量子描述

### 5.1 玻色子的基本对易关系

$$\text{定义5.1.1. } [a(\vec{k}'), a(\vec{k})] = 0, [a^+(\vec{k}'), a^+(\vec{k})] = 0, [a(\vec{k}'), a^+(\vec{k})] = \delta^3(\vec{k}' - \vec{k})$$

$$\text{定义5.1.2. } \hat{N}(\vec{k}) \equiv a^+(\vec{k})a(\vec{k}), k^a \equiv (\vec{k}, i\omega_k), \omega_k \equiv \frac{\vec{k}^2}{2m}$$

## 5.2 玻色子的粒子数算符性质

推论5.2.1.  $[\hat{N}, a(\vec{k})] = -a(\vec{k})$

$$\begin{aligned} \text{证明: } [\hat{N}, a(\vec{k})] &= \int_{\vec{k}'=-\infty}^{+\infty} [a^+(\vec{k}')a(\vec{k}')a(\vec{k}) - a(\vec{k})a^+(\vec{k}')a(\vec{k}')]d^3\vec{k}' \\ &= \int_{\vec{k}'=-\infty}^{+\infty} [a^+(\vec{k}')a(\vec{k})a(\vec{k}') - a(\vec{k})a^+(\vec{k}')a(\vec{k}')]d^3\vec{k}' \\ &= \int_{\vec{k}'=-\infty}^{+\infty} [a^+(\vec{k}'), a(\vec{k})]a(\vec{k}')d^3\vec{k}' \\ &= \int_{\vec{k}'=-\infty}^{+\infty} -\delta^3(\vec{k}' - \vec{k})a(\vec{k}')d^3\vec{k}' = -a(\vec{k}) \end{aligned} \quad \square$$

推论5.2.2.  $[\hat{N}, a(\vec{k})] = -a(\vec{k}) \Leftrightarrow [\hat{N}, a^+(\vec{k})] = a^+(\vec{k})$

推论5.2.3.  $[\hat{N}, a^+(\vec{k})a(\vec{k})] = 0, \forall \vec{k}$

$$\begin{aligned} \text{证明: } [\hat{N}, a^+(\vec{k})a(\vec{k})] &= \hat{N}a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})a(\vec{k})\hat{N} \\ &= \hat{N}a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})\hat{N}a(\vec{k}) + a^+(\vec{k})\hat{N}a(\vec{k}) - a^+(\vec{k})a(\vec{k})\hat{N} \\ &= [\hat{N}, a^+(\vec{k})]a(\vec{k}) + a^+(\vec{k})[\hat{N}, a(\vec{k})] = a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})a(\vec{k}) = 0 \end{aligned} \quad \square$$

可以用数学归纳法证明如下结论。

推论5.2.4.  $[\hat{N}, a^n(\vec{k})] = -na^n(\vec{k}), [\hat{N}, a^{+n}(\vec{k})] = na^{+n}(\vec{k})$

推论5.2.5.  $[\hat{N}, \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)] = (\sum_{i=1}^{\infty} n_i) \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)$

推论5.2.6.  $[\hat{N}, a^{+n}(\vec{k})a^n(\vec{k})] = 0, [\hat{N}, a^n(\vec{k})a^{+n}(\vec{k})] = 0, [\hat{N}, \hat{N}^n(k)] = 0, [\hat{N}, [a(\vec{k})a^+(\vec{k})]^n] = 0$

## 5.3 玻色子的能量动量算符性质

推论5.3.1.  $[P^a, a(\vec{k})] = -k^a a(\vec{k})$

$$\begin{aligned} \text{证明: } [P^a, a(\vec{k})] &= \int_{\vec{k}'=-\infty}^{+\infty} k'^a [a^+(\vec{k}')a(\vec{k}')a(\vec{k}) - a(\vec{k})a^+(\vec{k}')a(\vec{k}')]d^3\vec{k}' \\ &= \int_{\vec{k}'=-\infty}^{+\infty} k'^a [a^+(\vec{k}')a(\vec{k})a(\vec{k}') - a(\vec{k})a^+(\vec{k}')a(\vec{k}')]d^3\vec{k}' \\ &= \int_{\vec{k}'=-\infty}^{+\infty} k'^a [a^+(\vec{k}'), a(\vec{k})]a(\vec{k}')d^3\vec{k}' \\ &= \int_{\vec{k}'=-\infty}^{+\infty} -k'^a \delta^3(\vec{k}' - \vec{k})a(\vec{k}')d^3\vec{k}' = -k^a a(\vec{k}) \end{aligned} \quad \square$$

推论5.3.2.  $[P^a, a(\vec{k})] = -k^a a(\vec{k}) \Leftrightarrow [P^a, a^+(\vec{k})] = k^a a^+(\vec{k})$

推论5.3.3.  $[P^a, a^+(\vec{k})a(\vec{k})] = 0, \forall \vec{k}$

$$\begin{aligned} \text{证明: } [P^a, a^+(\vec{k})a(\vec{k})] &= P^a a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})a(\vec{k})P^a \\ &= P^a a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})P^a a(\vec{k}) + a^+(\vec{k})P^a a(\vec{k}) - a^+(\vec{k})a(\vec{k})P^a \\ &= [P^a, a^+(\vec{k})]a(\vec{k}) + a^+(\vec{k})[P^a, a(\vec{k})] = k^a a^+(\vec{k})a(\vec{k}) - k^a a^+(\vec{k})a(\vec{k}) = 0 \end{aligned} \quad \square$$

可以用数学归纳法证明如下结论。

推论5.3.4.  $[P^a, a^n(\vec{k})] = -nk^a a^n(\vec{k}) \Leftrightarrow [P^a, a^{+n}(\vec{k})] = nk^a a^{+n}(\vec{k})$

推论5.3.5.  $[P^a, \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)] = (\sum_{i=1}^{\infty} n_i k_i^a) \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)$

推论5.3.6.  $[P^a, \hat{N}] = 0$



## 5.4 玻色子解的一般构造

定义5.4.1.  $a(\vec{k}, t) \equiv a(k)e^{-i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)}$ ,  $a^+(k, t) \equiv a^+(k)e^{i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)}$

推论5.4.1.  $\dot{a}(\vec{k}, t) = i[H, a(\vec{k}, t)]$ ,  $\dot{a}^+(\vec{k}, t) = i[H, a^+(\vec{k}, t)]$

定理5.4.1.  $i\partial_t|\Psi\rangle = H|\Psi\rangle \Rightarrow i\partial_t[a(\vec{k}, t)|\Psi\rangle] = Ha(\vec{k}, t)|\Psi\rangle$

定理5.4.2.  $i\partial_t|\Psi\rangle = H|\Psi\rangle \Rightarrow i\partial_t[a^+(\vec{k}, t)|\Psi\rangle] = Ha^+(\vec{k}, t)|\Psi\rangle$

## 5.5 玻色子量子态的构造一

推论5.5.1.  $i\partial_t|0\rangle = H|0\rangle \Rightarrow i\partial_t[\prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i, t)|0\rangle] = H\prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i, t)|0\rangle$ ,  $n_i = 0, 1, 2, \dots, \infty$

定义5.5.1.  $|n_1, n_2, \dots, n_\infty\rangle \equiv \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)|0\rangle$ ,  $n_i = 0, 1, 2, \dots, \infty$

定义5.5.2.  $|n_1, n_2, \dots, n_\infty, t\rangle \equiv \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i, t)|0\rangle = \exp\{i\sum_{i=1}^{\infty} n_i(\vec{k}_i \cdot \vec{r} - \frac{\vec{k}_i^2}{2m}t)\}|n_1, n_2, \dots, n_\infty\rangle$

此量子态含义如下：动量空间的每一个数学点 $\vec{k}_i$ 对应一个能级，这个能级上填充了 $n_i$ 个质量为 $m$ ，动量为 $\vec{k}_i$ 的粒子，这样类似的能级的有无穷个。由于物理粒子总数是有限的，所以很多个能级上粒子数为零。这个量子态表示多个粒子在动量空间的一个分布，每个量子态粒子总数是可变的。它是粒子数算符、能量动量算符的共同本征态。

推论5.5.2. 
$$\begin{cases} \hat{N}|n_1, n_2, \dots, n_\infty, t\rangle = \sum_{i=1}^{\infty} n_i|n_1, n_2, \dots, n_\infty, t\rangle \\ H|n_1, n_2, \dots, n_\infty, t\rangle = \sum_{i=1}^{\infty} n_i \frac{\vec{k}_i^2}{2m}|n_1, n_2, \dots, n_\infty, t\rangle \\ \vec{P}|n_1, n_2, \dots, n_\infty, t\rangle = \sum_{i=1}^{\infty} n_i \vec{k}_i|n_1, n_2, \dots, n_\infty, t\rangle \end{cases}$$

推论5.5.3.  $\phi(-\vec{r}, -t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} a(\vec{k})e^{-i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)} d^3\vec{k}$ ,  $\phi^+(-\vec{r}, -t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} a^+(\vec{k})e^{i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m}t)} d^3\vec{k}$

推论5.5.4.  $i\partial_t|\Psi\rangle = H|\Psi\rangle \Rightarrow i\partial_t[\hat{\phi}(-\vec{r}, -t)|\Psi\rangle] = H[\hat{\phi}(-\vec{r}, -t)|\Psi\rangle]$

推论5.5.5.  $i\partial_t|\Psi\rangle = H|\Psi\rangle \Rightarrow i\partial_t[\hat{\phi}^+(-\vec{r}, -t)|\Psi\rangle] = H[\hat{\phi}^+(-\vec{r}, -t)|\Psi\rangle]$

## 5.6 玻色子量子态的构造二

定义5.6.1.

$|n\rangle = [n! \int_{\vec{k}=-\infty}^{+\infty} |F(\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n)|^2 d^n\vec{k}]^{-\frac{1}{2}} \int_{\vec{k}=-\infty}^{+\infty} d^n\vec{k} F(\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n) a^+(\vec{k}_1) a^+(\vec{k}_2) \dots a^+(\vec{k}_n) |0\rangle$

$F(\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n)$ 对玻色子全对称，对费米子全反对称。

定义5.6.2.  $|n, t\rangle = \exp\{i\sum_{i=1}^n (\vec{k}_i \cdot \vec{r} - \frac{\vec{k}_i^2}{2m}t)\}|n\rangle$

此量子态含义如下： $n$ 个粒子填充在动量空间中所有可能分布的一个混合动量态，这个量子态粒子总数是固定的。每个粒子的动量可能取任何值，它是粒子数算符的本征态，但不是能量动量算符的本征态。

推论5.6.1.  $i\partial_t|0\rangle = H|0\rangle \Rightarrow i\partial_t|n, t\rangle = H|n, t\rangle$

推论5.6.2.  $\hat{N}|n\rangle = n|n\rangle$ ,  $\hat{N}|n, t\rangle = n|n, t\rangle$ ,  $\langle n|n\rangle = 1$

## 5.7 玻色子坐标空间和动量空间的对应

推论5.7.1.  $[P^a, a(\vec{k})] = -k^a a(\vec{k}) \Leftrightarrow [P^a, \phi(\vec{r}, t)] = i\partial^a \phi(\vec{r}, t)$

推论5.7.2.  $[P^a, a^+(\vec{k})] = k^a a^+(\vec{k}) \Leftrightarrow [P^a, \phi^+(\vec{r}, t)] = -i\partial^a \phi^+(\vec{r}, t)$

## 5.8 玻色子量子态的存在性

推论5.8.1.  $\langle 0|0 \rangle = 1 \Rightarrow |0 \rangle \neq 0$

推论5.8.2.  $a^n(\vec{k})a^{+n}(\vec{k}) = a^{+n}(\vec{k})a^n(\vec{k}) + n!\delta^n(0), n \geq 1$

推论5.8.3.  $\prod_{i=1}^{\infty} a^{n_i}(\vec{k}_i) \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i) = \prod_{i=1, n_i \geq 1}^{\infty} [a^{+n_i}(\vec{k}_i)a^{n_i}(\vec{k}_i) + n_i!\delta^{n_i}(0)]$

推论5.8.4.  $\prod_{i=1}^{\infty} a^{n_i}(\vec{k}_i) \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i)|0 \rangle = \prod_{i=1, n_i \geq 1}^{\infty} n_i!\delta^{n_i}(0)|0 \rangle \neq 0, \forall \vec{k}_1 \neq \vec{k}_2 \neq \dots \neq \vec{k}_n$

## 6 费米子的量子描述

### 6.1 费米子的基本对易关系

定义6.1.1.  $\{a(\vec{k}'), a(\vec{k})\} = 0, \{a^+(\vec{k}'), a^+(\vec{k})\} = 0, \{a(\vec{k}'), a^+(\vec{k})\} = \delta^3(\vec{k}' - \vec{k})$

推论6.1.1.  $a(\vec{k}')a(\vec{k}) = 0, a^+(\vec{k}')a^+(\vec{k}) = 0$

### 6.2 费米子的粒子数算符性质

推论6.2.1.  $[\hat{N}, a(\vec{k})] = -a(\vec{k})$

证明:  $[\hat{N}, a(\vec{k})] = \int_{\vec{k}'=-\infty}^{+\infty} [a^+(\vec{k}')a(\vec{k}')a(\vec{k}) - a(\vec{k})a^+(\vec{k}')a(\vec{k}')]d^3\vec{k}'$   
 $= \int_{\vec{k}'=-\infty}^{+\infty} [-a^+(\vec{k}')a(\vec{k})a(\vec{k}') - a(\vec{k})a^+(\vec{k}')a(\vec{k}')]d^3\vec{k}'$   
 $= \int_{\vec{k}'=-\infty}^{+\infty} -\{a^+(\vec{k}'), a(\vec{k})\}a(\vec{k}')d^3\vec{k}'$   
 $= \int_{\vec{k}'=-\infty}^{+\infty} -\delta^3(\vec{k}' - \vec{k})a(\vec{k}')d^3\vec{k}' = -a(\vec{k})$  □

推论6.2.2.  $[\hat{N}, a(\vec{k})] = -a(\vec{k}) \Leftrightarrow [\hat{N}, a^+(\vec{k})] = a^+(\vec{k})$

推论6.2.3.  $[\hat{N}, a^+(\vec{k})a(\vec{k})] = 0, \forall \vec{k}$

证明:  $[\hat{N}, a^+(\vec{k})a(\vec{k})] = \hat{N}a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})a(\vec{k})\hat{N}$   
 $= \hat{N}a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})\hat{N}a(\vec{k}) + a^+(\vec{k})\hat{N}a(\vec{k}) - a^+(\vec{k})a(\vec{k})\hat{N}$   
 $= [\hat{N}, a^+(\vec{k})]a(\vec{k}) + a^+(\vec{k})[\hat{N}, a(\vec{k})] = a^+(\vec{k})a(\vec{k}) - a^+(\vec{k})a(\vec{k}) = 0$  □

可以用数学归纳法证明如下结论。

推论6.2.4.  $\hat{N}a^+(\vec{k}_1)a^+(\vec{k}_2)\dots a^+(\vec{k}_n)|0 \rangle = na^+(\vec{k}_1)a^+(\vec{k}_2)\dots a^+(\vec{k}_n)|0 \rangle, \forall \vec{k}_1 \neq \vec{k}_2 \neq \dots \neq \vec{k}_n$

推论6.2.5.  $\hat{N}a^+(\vec{k}_1)a^+(\vec{k}_2)\dots a^+(\vec{k}_n)|0 \rangle = na^+(\vec{k}_1)a^+(\vec{k}_2)\dots a^+(\vec{k}_n)|0 \rangle, \forall \vec{k}_1\vec{k}_2\dots\vec{k}_n$

推论6.2.6.  $[\hat{N}, a^n(\vec{k})] = -na^n(\vec{k}), [\hat{N}, a^{+n}(\vec{k})] = na^{+n}(\vec{k}), n = 0, 1$

推论6.2.7.  $[\hat{N}, \prod_{i=1}^{\infty} a^{n_i}(\vec{k}_i)] = (\sum_{i=1}^{\infty} n_i) \prod_{i=1}^{\infty} a^{n_i}(\vec{k}_i), n_i = 0, 1$

推论6.2.8.  $[\hat{N}, a^{+n}(\vec{k})a^n(\vec{k})] = 0, [\hat{N}, a^n(\vec{k})a^{+n}(\vec{k})] = 0, [\hat{N}, \hat{N}^n(k)] = 0, [\hat{N}, [a(\vec{k})a^+(\vec{k})]^n] = 0$

### 6.3 费米子量子态的存在性

推论6.3.1.  $\langle 0|0 \rangle = 1 \Rightarrow |0 \rangle \neq 0$

推论6.3.2.  $a(\vec{k})a^+(\vec{k}) = -a^+(\vec{k})a(\vec{k}) + \delta(0), n = 0, 1$

推论6.3.3.  $\prod_{i=1}^{\infty} a^{n_i}(\vec{k}_i) \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i) = \prod_{i=1, n_i=1}^{\infty} [a^{+n_i}(\vec{k})a^{n_i}(\vec{k}) + n_i! \delta^{n_i}(0)]$

推论6.3.4.  $\prod_{i=1}^{\infty} a^{n_i}(\vec{k}_i) \prod_{i=1}^{\infty} a^{+n_i}(\vec{k}_i) |0 \rangle = \pm \prod_{i=1, n_i=1}^{\infty} n_i! \delta^{n_i}(0) |0 \rangle \neq 0, \forall \vec{k}_1 \neq \vec{k}_2 \neq \dots \neq \vec{k}_n$

推论6.3.5.  $a^+(\vec{k}_1)a^+(\vec{k}_2) \dots a^+(\vec{k}_n) |0 \rangle \neq 0, \forall \vec{k}_1 \neq \vec{k}_2 \neq \dots \neq \vec{k}_n$

### 6.4 占有数表象

定义6.4.1.  $|n_1 n_2 \dots n_k \dots \rangle = |n_1 \rangle \otimes |n_2 \rangle \otimes \dots \otimes |n_k \rangle \otimes \dots, \langle n_1 n_2 \dots n_k \dots | = |n_1 n_2 \dots n_k \dots \rangle^+$

推论6.4.1.  $|n_1 n_2 \dots n_k \dots \rangle = \frac{1}{n_1! n_2! \dots n_k! \dots} (a_1^+)^{n_1} \otimes (a_2^+)^{n_2} \dots \otimes (a_k^+)^{n_k} \dots |0_1 0_2 \dots 0_k \dots \rangle$

推论6.4.2. 正交性:  $\langle n'_1 n'_2 \dots n'_k \dots | n_1 n_2 \dots n_k \dots \rangle = \delta_{n'_1, n_1} \delta_{n'_2, n_2} \dots \delta_{n'_k, n_k} \dots$

推论6.4.3. 完备性:  $\sum |n_1 n_2 \dots n_k \dots \rangle \langle n_1 n_2 \dots n_k \dots | = 1, \sum |n_k \rangle \langle n_k | = 1$

# 第二十章 Majorana粒子和中微子的量子化

自我评述：由于大多量子场论的书均没有详细论述Majorana粒子和中微子的量子化，我也一直没找到相应内容，为了弥补这个缺憾，决定自己动手推导演算，在本章中我应用洛伦兹推动变换先给出了Dirac粒子的量子化，然后在此基础上采用类似的技巧又给出了Majorana粒子和中微子的详细量子化细节。

## 1 洛伦兹推动变换应用—Dirac方程平面波的求解 [26, 27]

### 1.1 一般表象下Dirac旋量的洛伦兹推动变换

Dirac方程:

$$\text{定义1.1.1. } (\gamma^a \partial_a + m)\psi = 0, \gamma^a p_a = \gamma \cdot \vec{p} + \gamma_4 iE, E = \sqrt{\vec{p}^2 + m^2} > 0$$

Dirac旋量推动变换:

$$\text{推论1.1.1. } D_{\vec{v}} = e^{-\ln[\gamma_v(1+v)]\hat{v} \cdot (\frac{i}{2}\vec{\gamma}\gamma_4)} = \frac{1+\gamma_v-i\gamma_v\vec{v}\cdot\vec{\gamma}\gamma_4}{\sqrt{2(\gamma_v+1)}} = \frac{E+m-i\vec{p}\cdot\vec{\gamma}\gamma_4}{\sqrt{2m(E+m)}} = \frac{m-i\gamma^a p_a \gamma_4}{\sqrt{2m(E+m)}}$$

Dirac旋量洛伦兹推动变换因子的性质

$$\text{性质1.1.1. } (m - i\gamma^a p_a \gamma_4)^+ = (m - i\gamma^a p_a \gamma_4)$$

$$\text{性质1.1.2. } (m - i\gamma^a p_a \gamma_4)^+ \gamma_4 (m - i\gamma^a p_a \gamma_4) = 2m(E + m)\gamma_4$$

$$\text{性质1.1.3. } (E + m + i\vec{p} \cdot \vec{\gamma}\gamma_4)(E + m - i\vec{p} \cdot \vec{\gamma}\gamma_4) = 2m(E + m)$$

$$\text{性质1.1.4. } (m - i\gamma^a p_a \gamma_4)^+ (m - i\gamma^a p_a \gamma_4) = 2(E + m)(E - i\vec{p} \cdot \vec{\gamma}\gamma_4)$$

$$\text{性质1.1.5. } (m + i\gamma^a p_a \gamma_4)^+ (m - i\gamma^a p_a \gamma_4) = 2m^2 - 2E(E - i\vec{p} \cdot \vec{\gamma}\gamma_4)$$

### 1.2 一般表象下Dirac方程的静止解和运动解

静止电子解

$$\text{推论1.2.1. } \partial_{t_0}\psi(\vec{0}) = -im\gamma_4\psi(\vec{0}) \Leftrightarrow \psi(\vec{0}) = e^{-i\gamma_4 m t_0}\psi_0, \forall \psi_0$$

动量 $\vec{p}$ 电子解

$$\text{推论1.2.2. } \psi(\vec{p}) = \frac{m-i\gamma^a p_a \gamma_4}{\sqrt{2m(E+m)}} e^{i\gamma_4(\vec{p}\cdot\vec{r}-Et)}\psi_p = \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_4}{E+m}\right) e^{i\gamma_4(\vec{p}\cdot\vec{r}-Et)}\psi_p, \bar{\psi}(\vec{p})\psi(\vec{p}) = \bar{\psi}_p\psi_p$$

### 1.3 特殊表象下Dirac方程的洛伦兹推动变换和平面波解

#### 1.3.1 特殊表象下Dirac方程的洛伦兹推动变换

特殊表象:  $(\gamma_a, \gamma_5) := [(\sigma \otimes \sigma_y, \zeta I \otimes \sigma_z), -\zeta I \otimes \sigma_x]$

$$\text{推论1.3.1. } \gamma^a p_a = i \begin{bmatrix} \zeta E & -\sigma \cdot \vec{p} \\ \sigma \cdot \vec{p} & -\zeta E \end{bmatrix}, E = \sqrt{\vec{p}^2 + m^2} > 0$$

$$\text{推论1.3.2. } S_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$S_y(\sigma_x, \sigma_y, \sigma_z)S_y^+ = (-\sigma_z, \sigma_y, \sigma_x), S_y^+(\sigma_x, \sigma_y, \sigma_z)S_y = (\sigma_z, \sigma_y, -\sigma_x)$$

$$I \otimes S_y[(\sigma \otimes \sigma_y, \zeta I \otimes \sigma_z), -\zeta I \otimes \sigma_x]I \otimes S_y^+ = [(\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x), \zeta I \otimes \sigma_z]$$

$$I \otimes S_y^+[(\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x), \zeta I \otimes \sigma_x]I \otimes S_y = [(\sigma \otimes \sigma_y, \zeta I \otimes \sigma_z), -\zeta I \otimes \sigma_x]$$

Dirac旋量推动变换

$$\text{推论1.3.3. } D_{\vec{v}} = \frac{m-i\gamma^a p_a \gamma_4}{\sqrt{2m(E+m)}} = \frac{E+m+\zeta\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}} = \frac{1}{\sqrt{2m(E+m)}} \begin{bmatrix} E+m & \zeta\sigma \cdot \vec{p} \\ \zeta\sigma \cdot \vec{p} & E+m \end{bmatrix}$$

### 1.3.2 特殊表象下Dirac方程的静止解和运动解

Dirac方程:

定义1.3.1.  $(\gamma^a \partial_a + m)\psi = 0, \gamma^a = (\sigma \otimes \sigma_y, \zeta I \otimes \sigma_z)$

静止电子解:

推论1.3.4.  $\partial_{t_0} \psi(\vec{0}) = -im\gamma_4 \psi(\vec{0}) \Leftrightarrow \psi(\vec{0}) = e^{-i\gamma_4 m t_0} \psi_0 = \begin{bmatrix} \xi_0 e^{-i\zeta m t_0} \\ \eta_0 e^{i\zeta m t_0} \end{bmatrix}, \forall \xi_0, \eta_0$

动量 $\vec{p}$ 电子解:

推论1.3.5.

$\psi(\vec{p}) = \sqrt{\frac{E+m}{2m}} (1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_4}{E+m}) e^{i\gamma_4(\vec{p}\cdot\vec{r}-Et)} \psi_p = \sqrt{\frac{E+m}{2m}} \left[ \begin{bmatrix} \xi(\vec{p}) \\ \frac{\zeta\sigma\cdot\vec{p}}{E+m} \xi(\vec{p}) \end{bmatrix} e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + \begin{bmatrix} \frac{\zeta\sigma\cdot\vec{p}}{E+m} \eta(\vec{p}) \\ \eta(\vec{p}) \end{bmatrix} e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} \right], \forall \xi(\vec{p}), \eta(\vec{p})$

### 1.3.3 特殊表象下Dirac方程按z-自旋本征态展开的 $\vec{p}$ 动量平面波解

按z-自旋本征态展开的动量 $\vec{p}$ 电子平面波解:

推论1.3.6.

$\psi(\vec{p}) = \frac{E+m+\zeta\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}} \left\{ [a_\zeta(\vec{p}, \frac{1}{2}) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_\zeta(\vec{p}, -\frac{1}{2}) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}] e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + [b_\zeta^+(\vec{p}, \frac{1}{2}) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b_\zeta^+(\vec{p}, -\frac{1}{2}) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}] e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} \right\}$

推论1.3.7.  $\psi(\vec{p}) = \sum_h [a_\zeta(\vec{p}, h) u_\zeta(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + b_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}]$  (在一般表象下也成立)

### 1.3.4 特殊表象下Dirac方程自旋基

定义1.3.2.  $\xi_+ = \eta_+ := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \xi_- = \eta_- := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

四个自旋基:

定义1.3.3.  $u_\zeta(\vec{p}, \frac{1}{2}) \equiv \frac{E+m+\zeta\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{m-i\zeta\gamma^a p_a}{\sqrt{2m(E+m)}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \xi_+ \\ \frac{\zeta\sigma\cdot\vec{p}}{E+m} \xi_+ \end{bmatrix}$

定义1.3.4.  $u_\zeta(\vec{p}, -\frac{1}{2}) \equiv \frac{E+m+\zeta\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{m-i\zeta\gamma^a p_a}{\sqrt{2m(E+m)}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \xi_- \\ \frac{\zeta\sigma\cdot\vec{p}}{E+m} \xi_- \end{bmatrix}$

定义1.3.5.  $v_\zeta(\vec{p}, \frac{1}{2}) \equiv \frac{E+m+\zeta\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{m+i\zeta\gamma^a p_a}{\sqrt{2m(E+m)}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{\zeta\sigma\cdot\vec{p}}{E+m} \eta_+ \\ \eta_+ \end{bmatrix}$

定义1.3.6.  $v_\zeta(\vec{p}, -\frac{1}{2}) \equiv \frac{E+m+\zeta\vec{p}\cdot\sigma\otimes\sigma_x}{\sqrt{2m(E+m)}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{m+i\zeta\gamma^a p_a}{\sqrt{2m(E+m)}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} \frac{\zeta\sigma\cdot\vec{p}}{E+m} \eta_- \\ \eta_- \end{bmatrix}$

推论1.3.8.  $u_\zeta(\vec{p}, h) \equiv -\zeta\gamma_5 v_\zeta(\vec{p}, h), u_\zeta(\vec{p}, h) \equiv i\gamma_2\gamma_4\gamma_5 u_\zeta(\vec{p}, -h), v_\zeta(\vec{p}, h) \equiv i\gamma_2\gamma_4\gamma_5 v_\zeta(\vec{p}, -h)$

推论1.3.9.  $(E+m+\zeta\vec{p}\cdot\sigma\otimes\sigma_x) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (E+m+\zeta\vec{p}\cdot\sigma\otimes\sigma_x)^+ \zeta I \otimes \sigma_z = (E+m)(\zeta m - i\gamma^a p_a)$

推论1.3.10.  $(E+m+\zeta\vec{p}\cdot\sigma\otimes\sigma_x) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (E+m+\zeta\vec{p}\cdot\sigma\otimes\sigma_x)^+ \zeta I \otimes \sigma_z = (E+m)(-\zeta m - i\gamma^a p_a)$

### 1.3.5 一般表象下Dirac方程自旋基性质

推论1.3.11.  $u_\zeta(\vec{p}, h) = -\zeta\gamma_5 v_\zeta(\vec{p}, h), v_\zeta(\vec{p}, h) = -\zeta\gamma_5 u_\zeta(\vec{p}, h), h = -\frac{1}{2}, \frac{1}{2}$

推论1.3.12.  $\bar{u}_\zeta(\vec{p}, h) u_\zeta(\vec{p}, h') = \zeta \delta_{hh'}, \bar{v}_\zeta(\vec{p}, h) v_\zeta(\vec{p}, h') = -\zeta \delta_{hh'}, \bar{u}_\zeta(\vec{p}, h) v_\zeta(\vec{p}, h') = 0, \bar{v}_\zeta(\vec{p}, h) u_\zeta(\vec{p}, h') = 0$

推论1.3.13.  $u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}, h') = \frac{E}{m} \delta_{hh'}, v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}, h') = \frac{E}{m} \delta_{hh'}, u_\zeta^+(\vec{p}, h) v_\zeta(-\vec{p}, h') = 0, v_\zeta^+(\vec{p}, h) u_\zeta(-\vec{p}, h') = 0$

$$\text{推论1.3.14. } \sum_h u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h) = \frac{\zeta m - i\gamma^0 p_a}{2m}, \sum_h v_\zeta(\vec{p}, h) \bar{v}_\zeta(\vec{p}, h) = \frac{-\zeta m - i\gamma^0 p_a}{2m}$$

$$\text{推论1.3.15. } \begin{cases} \sum_h u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h) - v_\zeta(\vec{p}, h) \bar{v}_\zeta(\vec{p}, h) = \zeta \\ \sum_h u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h) + v_\zeta(\vec{p}, h) \bar{v}_\zeta(\vec{p}, h) = \frac{-i\gamma^0 p_a}{m} \\ \sum_h u_\zeta(\vec{p}, h) u_\zeta^+(\vec{p}, h) + v_\zeta(-\vec{p}, h) v_\zeta^+(-\vec{p}, h) = \frac{E}{m} \end{cases}$$

#### 1.4 一般表象下Dirac方程的平面波解

$$\text{推论1.4.1. } \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h [a_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + b_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} v_\zeta(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\text{推论1.4.2. } \psi^+(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} + b_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} v_\zeta^+(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

推论1.4.3.

$$\begin{cases} a_\zeta(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h) \psi(\vec{r}, t) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u^{+\lambda_\zeta}(\vec{p}, h) \psi_{\lambda_\zeta}(\vec{r}, t) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \\ b_\zeta^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} v_\zeta^+(\vec{p}, h) \psi(\vec{r}, t) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} v^{+\lambda_\zeta}(\vec{p}, h) \psi_{\lambda_\zeta}(\vec{r}, t) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \end{cases}$$

推论1.4.4.

$$\begin{cases} a_\zeta^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u^{\lambda_\zeta}(\vec{p}, h) \psi_{\lambda_\zeta}^+(\vec{r}, t) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \\ b_\zeta(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} v^{\lambda_\zeta}(\vec{p}, h) \psi_{\lambda_\zeta}^+(\vec{r}, t) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \end{cases}$$

#### 1.5 一般表象下Dirac方程的自旋基及其性质

$$\text{定义1.5.1. } \tilde{a}(\vec{p}, h) := \begin{cases} a_\zeta(\vec{p}, h), \zeta = 1 \\ b_\zeta^+(\vec{p}, h), \zeta = -1 \end{cases}, \tilde{b}(\vec{p}, h) := \begin{cases} b_\zeta(\vec{p}, h), \zeta = 1 \\ a_\zeta^+(\vec{p}, h), \zeta = -1 \end{cases}$$

$$\text{定义1.5.2. } u(\vec{p}, h) := \begin{cases} u_+(\vec{p}, h), \zeta = 1 \\ v_-(\vec{p}, h), \zeta = -1 \end{cases}, v(\vec{p}, h) := \begin{cases} v_+(\vec{p}, h), \zeta = 1 \\ u_-(\vec{p}, h), \zeta = -1 \end{cases}$$

推论1.5.1.

$$\psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h [\tilde{a}(\vec{p}, h) \sqrt{\frac{m}{E}} u(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + \tilde{b}^+(\vec{p}, h) \sqrt{\frac{m}{E}} v(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\begin{cases} \tilde{a}(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u^+(\vec{p}, h) \psi(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \\ \tilde{b}^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} v^+(\vec{p}, h) \psi(\vec{r}, t) e^{i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \end{cases}$$

自旋基之间的性质(在一般表象下也成立):

$$\text{推论1.5.2. } \bar{u}(\vec{p}, h) u(\vec{p}, h') = \delta_{hh'}, \bar{v}(\vec{p}, h) v(\vec{p}, h') = -\delta_{hh'}, \bar{u}(\vec{p}, h) v(\vec{p}, h') = 0, \bar{v}(\vec{p}, h) u(\vec{p}, h') = 0$$

$$\text{推论1.5.3. } u^+(\vec{p}, h) u(\vec{p}, h') = \frac{E}{m} \delta_{hh'}, v^+(\vec{p}, h) v(\vec{p}, h') = \frac{E}{m} \delta_{hh'}, u^+(\vec{p}, h) v(-\vec{p}, h') = 0, v^+(\vec{p}, h) u(-\vec{p}, h') = 0$$

$$\text{推论1.5.4. } \sum_h u(\vec{p}, h) \bar{u}(\vec{p}, h) = \frac{m - i\gamma^0 p_a}{2m}, \sum_h v(\vec{p}, h) \bar{v}(\vec{p}, h) = \frac{-m - i\gamma^0 p_a}{2m}$$

$$\text{推论1.5.5. } \begin{cases} \sum_h u(\vec{p}, h) \bar{u}(\vec{p}, h) - v(\vec{p}, h) \bar{v}(\vec{p}, h) = 1 \\ \sum_h u(\vec{p}, h) \bar{u}(\vec{p}, h) + v(\vec{p}, h) \bar{v}(\vec{p}, h) = \frac{-i\gamma^0 p_a}{m} \\ \sum_h u(\vec{p}, h) u^+(\vec{p}, h) + v(-\vec{p}, h) v^+(-\vec{p}, h) = \frac{E}{m} \end{cases}$$

## 1.6 一般表象下Dirac方程的等时量子化

$$\text{推论1.6.1.} \quad \begin{cases} \{\psi_{\lambda_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta}^\dagger(\vec{r}', t)\} = \delta_{\lambda_\zeta \lambda'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\psi_{\lambda_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta}(\vec{r}', t)\} = 0 \\ \{\psi_{\lambda_\zeta}^\dagger(\vec{r}, t), \psi_{\lambda'_\zeta}^\dagger(\vec{r}', t)\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^\dagger(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^\dagger(\vec{p}, h), a_\zeta^\dagger(\vec{p}', h')\} = 0 \end{cases}$$

证明:  $\{\psi_{\lambda_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta}^\dagger(\vec{r}', t)\}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int_{\vec{p}, \vec{p}'=-\infty}^{+\infty} \frac{m}{E} \sum_{h, h'} [u_{\zeta \lambda_\zeta}(\vec{p}, h) u_{\zeta \lambda'_\zeta}^*(\vec{p}', h') e^{i\zeta(\vec{p}\cdot\vec{r} - Et - \vec{p}'\cdot\vec{r}' + E't)} \{a_\zeta(\vec{p}, h), a_\zeta^\dagger(\vec{p}', h')\} \\ &+ v_{\zeta \lambda_\zeta}(\vec{p}, h) v_{\zeta \lambda'_\zeta}^*(\vec{p}', h') e^{-i\zeta(\vec{p}\cdot\vec{r} - Et - \vec{p}'\cdot\vec{r}' + E't)} \{b_\zeta^+(\vec{p}, h), b_\zeta(\vec{p}', h')\}] d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int_{\vec{p}, \vec{p}'=-\infty}^{+\infty} \frac{m}{E} \sum_{h, h'} [u_{\zeta \lambda_\zeta}(\vec{p}, h) u_{\zeta \lambda'_\zeta}^*(\vec{p}', h') e^{i\zeta(\vec{p}\cdot\vec{r} - Et - \vec{p}'\cdot\vec{r}' + E't)} \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ &+ v_{\zeta \lambda_\zeta}(\vec{p}, h) v_{\zeta \lambda'_\zeta}^*(\vec{p}', h') e^{-i\zeta(\vec{p}\cdot\vec{r} - Et - \vec{p}'\cdot\vec{r}' + E't)} \delta_{hh'} \delta^3(\vec{p} - \vec{p}')] d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int_{\vec{p}=-\infty}^{+\infty} \frac{m}{E} \sum_h [u_{\zeta \lambda_\zeta}(\vec{p}, h) u_{\zeta \lambda'_\zeta}^*(\vec{p}, h) e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} + v_{\zeta \lambda_\zeta}(\vec{p}, h) v_{\zeta \lambda'_\zeta}^*(\vec{p}, h) e^{-i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int_{\vec{p}=-\infty}^{+\infty} \frac{m}{E} \sum_h [u_{\zeta \lambda_\zeta}(\vec{p}, h) u_{\zeta \lambda'_\zeta}^*(\vec{p}, h) + v_{\zeta \lambda_\zeta}(-\vec{p}, h) v_{\zeta \lambda'_\zeta}^*(-\vec{p}, h)] e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\ &= \delta_{\lambda_\zeta \lambda'_\zeta} \frac{1}{(2\pi)^3} \int_{\vec{p}=-\infty}^{+\infty} e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\ &= \delta_{\lambda_\zeta \lambda'_\zeta} \delta^3(\vec{r} - \vec{r}') \end{aligned} \quad \square$$

证明:  $\{a_\zeta(\vec{p}, h), a_\zeta^\dagger(\vec{p}', h')\}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \frac{m}{E} \int_{\vec{r}, \vec{r}'=-\infty}^{+\infty} u_{\lambda_\zeta}^*(\vec{p}, h) u_{\lambda'_\zeta}(\vec{p}', h') \{\psi_{\lambda_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta}^\dagger(\vec{r}', t)\} e^{-i\zeta(\vec{p}\cdot\vec{r} - Et)} e^{i\zeta(\vec{p}'\cdot\vec{r}' - E't)} d^3\vec{r} d^3\vec{r}' \\ &= \frac{1}{(2\pi)^3} \frac{m}{E} \int_{\vec{r}, \vec{r}'=-\infty}^{+\infty} u_{\lambda_\zeta}^*(\vec{p}, h) u_{\lambda'_\zeta}(\vec{p}', h') \delta_{\lambda_\zeta \lambda'_\zeta} \delta^3(\vec{r} - \vec{r}') e^{-i\zeta(\vec{p}\cdot\vec{r} - Et)} e^{i\zeta(\vec{p}'\cdot\vec{r}' - E't)} d^3\vec{r} d^3\vec{r}' \\ &= \frac{m}{E} u_{\lambda_\zeta}^*(\vec{p}, h) u_{\lambda'_\zeta}(\vec{p}', h') e^{-i\zeta(E't - Et)} \frac{1}{(2\pi)^3} \int_{\vec{r}=-\infty}^{+\infty} e^{i\zeta(\vec{p}' - \vec{p})\cdot\vec{r}} d^3\vec{r} \\ &= \frac{m}{E} u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}', h') e^{-i\zeta(E't - Et)} \delta^3(\vec{p} - \vec{p}') \\ &= \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \end{aligned} \quad \square$$

$$\text{推论1.6.2.} \quad \begin{cases} : P_u := \int -i\psi^\dagger \partial_u \psi d^3r := \int \sum_h p_u [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) + b_\zeta^+(\vec{p}, h) b_\zeta(\vec{p}, h)] d^3\vec{p} \\ : Q := \int \psi^\dagger \psi d^3r := \int \sum_h [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - b_\zeta^+(\vec{p}, h) b_\zeta(\vec{p}, h)] d^3\vec{p} \stackrel{=1}{=} 0 \end{cases}$$

## 1.7 一般表象下Dirac方程的协变反对易规则

$$\text{推论1.7.1.} \quad \begin{cases} \psi_{\lambda_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h [a_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} u_{\zeta \lambda_\zeta}(\vec{p}, h) e^{i\zeta p x} + b_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} v_{\zeta \lambda_\zeta}(\vec{p}, h) e^{-i\zeta p x}] d^3\vec{p} \\ \bar{\psi}_{\lambda'_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} \bar{u}_{\zeta \lambda'_\zeta}(\vec{p}, h) e^{-i\zeta p x} + b_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} \bar{v}_{\zeta \lambda'_\zeta}(\vec{p}, h) e^{i\zeta p x}] d^3\vec{p} \end{cases}$$

推论1.7.2.  $\{\psi_{\lambda_\zeta}(x), \bar{\psi}_{\lambda'_\zeta}(x')\} = i(m - \gamma^a \partial_a)_{\lambda_\zeta \lambda'_\zeta} \Delta(x - x')$

证明:  $\{\psi_{\lambda_\zeta}(x), \bar{\psi}_{\lambda'_\zeta}(x')\} = \frac{1}{(2\pi)^3} \int \sum_{h, h'} \sqrt{\frac{m}{E}} \sqrt{\frac{m}{E'}}$

$$\begin{aligned} &[\{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} u_{\zeta \lambda_\zeta}(\vec{p}, h) \bar{u}_{\zeta \lambda'_\zeta}(\vec{p}', h') e^{i\zeta(p x - p' x')} + \{b_\zeta^+(\vec{p}, h), b_\zeta(\vec{p}', h')\} v_{\zeta \lambda_\zeta}(\vec{p}, h) \bar{v}_{\zeta \lambda'_\zeta}(\vec{p}', h') e^{-i\zeta(p x - p' x')}] d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \sum_{h, h'} \sqrt{\frac{m}{E}} \sqrt{\frac{m}{E'}} \delta_{hh'} \delta^3(\vec{p} - \vec{p}') [u_{\zeta \lambda_\zeta}(\vec{p}, h) \bar{u}_{\zeta \lambda'_\zeta}(\vec{p}', h') e^{i\zeta(p x - p' x')} + v_{\zeta \lambda_\zeta}(\vec{p}, h) \bar{v}_{\zeta \lambda'_\zeta}(\vec{p}', h') e^{-i\zeta(p x - p' x')}] d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \sum_h \frac{m}{E} [u_{\zeta \lambda_\zeta}(\vec{p}, h) \bar{u}_{\zeta \lambda'_\zeta}(\vec{p}, h) e^{i\zeta p(x-x')} + v_{\zeta \lambda_\zeta}(\vec{p}, h) \bar{v}_{\zeta \lambda'_\zeta}(\vec{p}, h) e^{-i\zeta p(x-x')}] d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{m}{E} \left[ \frac{(\zeta m - i\gamma^a p_a)_{\lambda_\zeta \lambda'_\zeta}}{2m} e^{i\zeta p(x-x')} + \frac{(-\zeta m - i\gamma^a p_a)_{\lambda_\zeta \lambda'_\zeta}}{2m} e^{-i\zeta p(x-x')} \right] d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{2E} [(\zeta m - i\gamma^a p_a)_{\lambda_\zeta \lambda'_\zeta} e^{i\zeta p(x-x')} + (-\zeta m - i\gamma^a p_a)_{\lambda_\zeta \lambda'_\zeta} e^{-i\zeta p(x-x')}] d^3\vec{p} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int \frac{1}{2E} [\zeta(m - \gamma^a \partial_a)_{\lambda_c \lambda'_c} e^{i\zeta p(x-x')} - \zeta(m - \gamma^a \partial_a)_{\lambda_c \lambda'_c} e^{-i\zeta p(x-x')}] d^3 \vec{p} \\
&= i(m - \gamma^a \partial_a)_{\lambda_c \lambda'_c} \Delta(x - x')
\end{aligned}$$

□

$$\text{推论1.7.3. } \{\psi_{\lambda_c}(x), \bar{\psi}_{\lambda'_c}(x')\} = i(m - \gamma^a \partial_a)_{\lambda_c \lambda'_c} \Delta(x - x') \Leftrightarrow \{\psi_{\lambda_c}(x), \psi_{\lambda'_c}^+(x')\} = i[(m - \gamma^a \partial_a) \gamma^4]_{\lambda_c \lambda'_c} \Delta(x - x')$$

$$\text{推论1.7.4. } \{\psi_{\lambda_c}(\vec{r}, t), \bar{\psi}_{\lambda'_c}(\vec{r}', t)\} = \gamma_{\lambda_c \lambda'_c}^4 \delta^3(\vec{r} - \vec{r}') \Leftrightarrow \{\psi_{\lambda_c}(\vec{r}, t), \psi_{\lambda'_c}^+(\vec{r}', t)\} = \delta_{\lambda_c \lambda'_c} \delta^3(\vec{r} - \vec{r}')$$

## 1.8 一般表象下Dirac方程的守恒荷

$$\text{推论1.8.1. } Q = \int \psi^+ \psi d^3 r = \int \sum_h [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) + b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^3 \vec{p}$$

$$\text{证明: } Q = \int \psi^+ \psi d^3 r$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int \sum_{h, h'} [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} + b_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} v_\zeta^+(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}] \\
&[a_\zeta(\vec{p}', h') \sqrt{\frac{m}{E'}} u_\zeta(\vec{p}', h') e^{i\zeta(\vec{p}'\cdot\vec{r}-E't)} + b_\zeta^+(\vec{p}', h') \sqrt{\frac{m}{E'}} v_\zeta(\vec{p}', h') e^{-i\zeta(\vec{p}'\cdot\vec{r}-E't)}] d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r} \\
&= \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}', h') + b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}', h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}', h')] \delta^3(\vec{p} - \vec{p}') d^3 \vec{p}' d^3 \vec{p} \\
&= \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}, h') + b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}, h')] d^3 \vec{p} \\
&= \int \sum_h [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) + b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^3 \vec{p}
\end{aligned}$$

□

$$\text{推论1.8.2. } H = i \int \psi^+ \partial_t \psi d^3 r = \zeta \int \sum_h E(\vec{p}) [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^3 \vec{p}$$

$$\text{证明: } H = i \int \psi^+ \partial_t \psi d^3 r$$

$$\begin{aligned}
&= i \frac{1}{(2\pi)^3} \int \sum_{h, h'} [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} + b_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} v_\zeta^+(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}] \\
&(-i\zeta E') [a_\zeta(\vec{p}', h') \sqrt{\frac{m}{E'}} u_\zeta(\vec{p}', h') e^{i\zeta(\vec{p}'\cdot\vec{r}-E't)} - b_\zeta^+(\vec{p}', h') \sqrt{\frac{m}{E'}} v_\zeta(\vec{p}', h') e^{-i\zeta(\vec{p}'\cdot\vec{r}-E't)}] d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r} \\
&= -i \int \sum_{h, h'} \frac{m}{E} (-i\zeta E') [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}', h') - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}', h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}', h')] \delta^3(\vec{p} - \vec{p}') d^3 \vec{p}' d^3 \vec{p} \\
&= -i \int \sum_{h, h'} \frac{m}{E} (-i\zeta E') [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}, h') - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}, h')] d^3 \vec{p} \\
&= \zeta \int \sum_h E(\vec{p}) [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^3 \vec{p}
\end{aligned}$$

□

$$\text{推论1.8.3. } \vec{P} = -i \int \psi^+ \nabla \psi d^3 r = \zeta \int \sum_h \vec{p} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^3 \vec{p}$$

$$\text{证明: } \vec{P} = -i \int \psi^+ \nabla \psi d^3 r$$

$$\begin{aligned}
&= -i \frac{1}{(2\pi)^3} \int \sum_{h, h'} [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} + b_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} v_\zeta^+(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}] \\
&(i\zeta \vec{p}') [a_\zeta(\vec{p}', h') \sqrt{\frac{m}{E'}} u_\zeta(\vec{p}', h') e^{i\zeta(\vec{p}'\cdot\vec{r}-E't)} - b_\zeta^+(\vec{p}', h') \sqrt{\frac{m}{E'}} v_\zeta(\vec{p}', h') e^{-i\zeta(\vec{p}'\cdot\vec{r}-E't)}] d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r} \\
&= -i \int \sum_{h, h'} \frac{m}{E} (i\zeta \vec{p}') [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}', h') - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}', h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}', h')] \delta^3(\vec{p} - \vec{p}') d^3 \vec{p}' d^3 \vec{p} \\
&= -i \int \sum_{h, h'} \frac{m}{E} (i\zeta \vec{p}') [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}, h') - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}, h')] d^3 \vec{p} \\
&= \zeta \int \sum_h \vec{p} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^3 \vec{p}
\end{aligned}$$

□

$$\text{推论1.8.4. } P_u = -i \int \psi^+ \partial_u \psi d^3 r = \zeta \int \sum_h p_u [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^3 \vec{p}$$

$$\text{证明: } P_u = -i \int \psi^+ \partial_u \psi d^3 r$$

$$\begin{aligned}
&= i \frac{1}{(2\pi)^3} \int \sum_{h, h'} [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} + b_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} v_\zeta^+(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}] \\
&(i\zeta p'_u) [a_\zeta(\vec{p}', h') \sqrt{\frac{m}{E'}} u_\zeta(\vec{p}', h') e^{i\zeta(\vec{p}'\cdot\vec{r}-E't)} - b_\zeta^+(\vec{p}', h') \sqrt{\frac{m}{E'}} v_\zeta(\vec{p}', h') e^{-i\zeta(\vec{p}'\cdot\vec{r}-E't)}] d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r} \\
&= -i \int \sum_{h, h'} \frac{m}{E} (i\zeta p'_u) [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}', h') - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}', h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}', h')] \delta^3(\vec{p} - \vec{p}') d^3 \vec{p}' d^3 \vec{p} \\
&= -i \int \sum_{h, h'} \frac{m}{E} (i\zeta p'_u) [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}, h') - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}, h')] d^3 \vec{p} \\
&= \zeta \int \sum_h p_u [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^3 \vec{p}
\end{aligned}$$

□



## 2 实表象下Majorana方程的平面波解和量子化

### 2.1 严格求解Majorana方程的平面波解 [26]

#### 2.1.1 两种表象下Majorana方程单动量平面波解之间的关系

实表象和Dirac表象下的Majorana方程:

$$\text{定义2.1.1. } \begin{cases} (\gamma_s^a \partial_a + m)\psi_s = 0, \gamma_s^a = (\sigma_{-z}, \sigma_{zy}, \zeta \sigma_{zx}), \psi_s^* = \psi_s \\ (\gamma^a \partial_a + m)\psi = 0, \gamma^a = (\sigma \otimes \sigma_y, \zeta I \otimes \sigma_z), \psi^* = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi, S_{em}^T(\zeta) S_{em}(\zeta) = -\sigma_y \otimes \sigma_y \end{cases}$$

$$\text{推论2.1.1. } \begin{cases} \psi_s(\vec{p}) := e^{i\theta} S_{em}(\zeta) \psi(\vec{p}), \psi_s(\vec{p}) = \psi_s^*(\vec{p}) \\ \psi_s(\vec{0}) := e^{i\theta} S_{em}(\zeta) \psi(\vec{0}), \psi_s(\vec{0}) = \psi_s^*(\vec{0}) \end{cases} \quad S_{em}(\zeta) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & -i & -i & 0 \\ 0 & -\zeta & \zeta & 0 \end{bmatrix}$$

$$\text{推论2.1.2. } \psi_s(\vec{p}) = \psi_s^*(\vec{p}) \Leftrightarrow \psi^*(\vec{p}) = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi(\vec{p})$$

$$\text{推论2.1.3. } \psi(\vec{0}) = \begin{bmatrix} \xi_0 e^{-i\zeta m t_0} \\ \eta_0 e^{i\zeta m t_0} \end{bmatrix}; \psi^*(\vec{0}) = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi(\vec{0}) \Leftrightarrow \eta_0 = -ie^{-2i\theta} \sigma_y \xi_0^* \Leftrightarrow \psi(\vec{0}) = \begin{bmatrix} \xi_0 e^{-i\zeta m t_0} \\ -ie^{-2i\theta} \sigma_y \xi_0^* e^{i\zeta m t_0} \end{bmatrix}$$

$$\text{推论2.1.4. } \psi(\vec{0}) = \begin{bmatrix} \xi e^{-i\zeta m t_0} \\ -ie^{-2i\theta} \sigma_y \xi^* e^{i\zeta m t_0} \end{bmatrix} \Leftrightarrow \psi_s(\vec{0}) = \frac{1}{\sqrt{2}} \begin{bmatrix} i(e^{i\theta} \xi_1 e^{-i\zeta m t_0} - e^{-i\theta} \xi_1^* e^{i\zeta m t_0}) \\ -(e^{i\theta} \xi_1 e^{-i\zeta m t_0} + e^{-i\theta} \xi_1^* e^{i\zeta m t_0}) \\ -i(e^{i\theta} \xi_2 e^{-i\zeta m t_0} - e^{-i\theta} \xi_2^* e^{i\zeta m t_0}) \\ -\zeta(e^{i\theta} \xi_2 e^{-i\zeta m t_0} + e^{-i\theta} \xi_2^* e^{i\zeta m t_0}) \end{bmatrix} \in R; \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

$$\text{推论2.1.5. } \begin{cases} \psi(\vec{p}) = \frac{E+m+\zeta \vec{p} \cdot \sigma \otimes \sigma_x}{\sqrt{2m(E+m)}} \psi(\vec{0}) = \frac{E+m+\zeta \vec{p} \cdot \sigma \otimes \sigma_x}{\sqrt{2m(E+m)}} \begin{bmatrix} \xi_0 e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} \\ -ie^{-2i\theta} \sigma_y \xi_0^* e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} \end{bmatrix} \\ \psi_s(\vec{p}) = \frac{m-i\gamma_s^a p_a \gamma_{s4}}{\sqrt{2m(E+m)}} \psi_s(\vec{0}) = \frac{m-i\gamma_s^a p_a \gamma_{s4}}{\sqrt{2m(E+m)}} e^{i\gamma_{s4}(\vec{p} \cdot \vec{r} - Et)} \psi_{s0} = \psi_s^*(\vec{p}), \psi_{s0} = e^{i\theta} S_{em}(\zeta) \psi_0 \end{cases}$$

#### 2.1.2 Dirac表象下Majorana方程的具体单动量平面波解

$$\text{推论2.1.6. } \psi(\vec{p}) = \sum_h [a_\zeta(\vec{p}, h) u_\zeta(\vec{p}, h) e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} - e^{-2i\theta} \sigma_y \otimes \sigma_y a_\zeta^+(\vec{p}, h) u_\zeta^*(\vec{p}, h) e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)}]$$

$$\text{证明: } \psi(\vec{p}) = \frac{E+m+\zeta \vec{p} \cdot \sigma \otimes \sigma_x}{\sqrt{2m(E+m)}} \psi(\vec{0}) = \frac{E+m+\zeta \vec{p} \cdot \sigma \otimes \sigma_x}{\sqrt{2m(E+m)}}$$

$$\begin{aligned} & \{ [a_\zeta(\vec{p}, \frac{1}{2}) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_\zeta(\vec{p}, -\frac{1}{2}) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}] e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} + [-e^{-2i\theta} a_\zeta^+(\vec{p}, -\frac{1}{2}) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + e^{-2i\theta} a_\zeta^+(\vec{p}, \frac{1}{2}) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}] e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} \} \\ & = [a_\zeta(\vec{p}, \frac{1}{2}) u_\zeta(\vec{p}, \frac{1}{2}) + a_\zeta(\vec{p}, -\frac{1}{2}) u_\zeta(\vec{p}, -\frac{1}{2})] e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} + e^{-2i\theta} [-a_\zeta^+(\vec{p}, -\frac{1}{2}) v_\zeta(\vec{p}, \frac{1}{2}) + a_\zeta^+(\vec{p}, \frac{1}{2}) v_\zeta(\vec{p}, -\frac{1}{2})] e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} \\ & = \sum_h [a_\zeta(\vec{p}, h) u_\zeta(\vec{p}, h) e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} - e^{-2i\theta} \sigma_y \otimes \sigma_y a_\zeta^+(\vec{p}, h) u_\zeta^*(\vec{p}, h) e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)}] \\ & = \sum_h [a_\zeta(\vec{p}, h) u_\zeta(\vec{p}, h) e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} + [e^{i\theta} S_{em}(\zeta)]^+ [e^{i\theta} S_{em}(\zeta)]^* a_\zeta^+(\vec{p}, h) u_\zeta^*(\vec{p}, h) e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)}] \quad \square \end{aligned}$$

$$\text{推论2.1.7. } \begin{cases} u^*(\vec{p}, h) = (-1)^{s+\frac{1}{2}} \sigma_y \otimes \sigma_y v_\zeta(\vec{p}, -h) \\ v^*(\vec{p}, h) = (-1)^{h-\frac{1}{2}} \sigma_y \otimes \sigma_y u_\zeta(\vec{p}, -h) \end{cases}$$

$$\text{推论2.1.8. } \psi^+(\vec{p}) =$$

$$[a_\zeta^+(\vec{p}, \frac{1}{2}) u_\zeta^+(\vec{p}, \frac{1}{2}) + a_\zeta^+(\vec{p}, -\frac{1}{2}) u_\zeta^+(\vec{p}, -\frac{1}{2})] e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} + e^{2i\theta} [-a_\zeta(\vec{p}, -\frac{1}{2}) v_\zeta^+(\vec{p}, \frac{1}{2}) + a_\zeta(\vec{p}, \frac{1}{2}) v_\zeta^+(\vec{p}, -\frac{1}{2})] e^{i\zeta(\vec{p} \cdot \vec{r} - Et)}$$

#### 2.1.3 实表象下Majorana方程的具体单动量平面波解

$$\text{推论2.1.9. } \psi_s(\vec{p}) = \sum_h [a_\zeta(\vec{p}, h) u_s(\vec{p}, h) e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} + a_\zeta^+(\vec{p}, h) u_s^*(\vec{p}, h) e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)}]$$

$$\text{证明: } \psi_s(\vec{p}) = \frac{m-i\gamma_s^a p_a \gamma_{s4}}{\sqrt{2m(E+m)}} \psi_s(\vec{0}) = e^{i\theta} S_{em}(\zeta) \psi(\vec{p})$$

$$\begin{aligned} & = e^{i\theta} S_{em}(\zeta) [a_\zeta(\vec{p}, \frac{1}{2}) u_\zeta(\vec{p}, \frac{1}{2}) + a_\zeta(\vec{p}, -\frac{1}{2}) u_\zeta(\vec{p}, -\frac{1}{2})] e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} + e^{-2i\theta} [-a_\zeta^+(\vec{p}, -\frac{1}{2}) v_\zeta(\vec{p}, \frac{1}{2}) + a_\zeta^+(\vec{p}, \frac{1}{2}) v_\zeta(\vec{p}, -\frac{1}{2})] e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} \\ & = [a_\zeta(\vec{p}, \frac{1}{2}) u_s(\vec{p}, \frac{1}{2}) + a_\zeta(\vec{p}, -\frac{1}{2}) u_s(\vec{p}, -\frac{1}{2})] e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} + e^{-2i\theta} [-a_\zeta^+(\vec{p}, -\frac{1}{2}) v_s(\vec{p}, \frac{1}{2}) + a_\zeta^+(\vec{p}, \frac{1}{2}) v_s(\vec{p}, -\frac{1}{2})] e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} \\ & = \frac{m-i\gamma_s^a p_a \gamma_{s4}}{\sqrt{2m(E+m)}} \{ [a_\zeta(\vec{p}, \frac{1}{2}) e^{i\theta} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a_\zeta(\vec{p}, -\frac{1}{2}) e^{i\theta} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}] e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} + e^{-i\theta} [-a_\zeta^+(\vec{p}, -\frac{1}{2}) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_\zeta^+(\vec{p}, \frac{1}{2}) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}] e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} \} \\ & = \sum_h [a_\zeta(\vec{p}, h) u_s(\vec{p}, h) e^{i\zeta(\vec{p} \cdot \vec{r} - Et)} + a_\zeta^+(\vec{p}, h) u_s^*(\vec{p}, h) e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)}] \quad \square \end{aligned}$$

$$\text{推论2.1.10. } \psi_s^+(\vec{p}) = \sum_h [a_\zeta^+(\vec{p}, h) u_s^+(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} + a_\zeta(\vec{p}, h) u_s^T(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}]$$

$$\text{推论2.1.11. } \bar{\psi}_s(\vec{p}) = \sum_h [a_\zeta^+(\vec{p}, h) \bar{u}_s(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} - a_\zeta(\vec{p}, h) \bar{u}_s^*(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}]$$

$$\text{推论2.1.12. } \begin{cases} u_s(\vec{p}, \frac{1}{2}) = e^{i\theta} S_{em}(\zeta) u_\zeta(\vec{p}, \frac{1}{2}) = \frac{m-i\gamma_s^a p_a \gamma_{s4}}{\sqrt{2m(E+m)}} e^{i\theta} \begin{bmatrix} i \\ -1 \\ 0 \\ 0 \end{bmatrix} = e^{2i\theta} v_s^*(\vec{p}, -\frac{1}{2}) \\ u_s(\vec{p}, -\frac{1}{2}) = e^{i\theta} S_{em}(\zeta) u_\zeta(\vec{p}, -\frac{1}{2}) = \frac{m-i\gamma_s^a p_a \gamma_{s4}}{\sqrt{2m(E+m)}} e^{i\theta} \begin{bmatrix} 0 \\ 0 \\ -i \\ -\zeta \end{bmatrix} = -e^{2i\theta} v_s^*(\vec{p}, \frac{1}{2}) \\ v_s(\vec{p}, \frac{1}{2}) = e^{i\theta} S_{em}(\zeta) v_\zeta(\vec{p}, \frac{1}{2}) = \frac{m-i\gamma_s^a p_a \gamma_{s4}}{\sqrt{2m(E+m)}} e^{i\theta} \begin{bmatrix} 0 \\ 0 \\ -i \\ \zeta \end{bmatrix} = -e^{2i\theta} u_s^*(\vec{p}, -\frac{1}{2}) \\ v_s(\vec{p}, -\frac{1}{2}) = e^{i\theta} S_{em}(\zeta) v_\zeta(\vec{p}, -\frac{1}{2}) = \frac{m-i\gamma_s^a p_a \gamma_{s4}}{\sqrt{2m(E+m)}} e^{i\theta} \begin{bmatrix} 0 \\ 0 \\ i \\ -\zeta \end{bmatrix} = e^{2i\theta} u_s^*(\vec{p}, \frac{1}{2}) \end{cases}$$

$$\text{推论2.1.13. } \begin{cases} u_s^*(\vec{p}, h) = (-1)^{h-\frac{1}{2}} e^{-2i\theta} v_s(\vec{p}, -h) \\ v_s^*(\vec{p}, h) = (-1)^{h+\frac{1}{2}} e^{2i\theta} u_s(\vec{p}, -h) \end{cases}$$

#### 2.1.4 Majorana方程和中微子方程单动量平面波解之间的等价变换关系

推论2.1.14.

$$(\gamma^a \partial_a + m)\psi(\vec{p}) = 0, \gamma^a = (\sigma \otimes \sigma_y, \zeta I \otimes \sigma_z), \psi^*(\vec{p}) = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi(\vec{p})$$

$$\psi(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}} \begin{bmatrix} (E+m)\xi_0 e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} - \zeta \vec{p} \cdot \sigma (ie^{-2i\theta} \sigma_y \xi_0^*) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} \\ -(E+m)(ie^{-2i\theta} \sigma_y \xi_0^*) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} + \zeta \vec{p} \cdot \sigma \xi_0 e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} \end{bmatrix} = \begin{bmatrix} \lambda(\vec{p}) \\ -ie^{-2i\theta} \sigma_y \lambda^*(\vec{p}) \end{bmatrix}$$

$$[\Downarrow]$$

推论2.1.15.

$$(\sigma, -i\zeta)_a \partial^a \nu(\vec{p}) - me^{-2i\theta} \sigma_y \nu^*(\vec{p}) = 0$$

$$\nu(\vec{p}) = \frac{1}{\sqrt{2}} [\lambda(\vec{p}) + ie^{-2i\theta} \sigma_y \lambda^*(\vec{p})] = \frac{E+m-\zeta \vec{p} \cdot \sigma}{\sqrt{2m(E+m)}} \frac{1}{\sqrt{2}} (\xi_0 e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + ie^{-2i\theta} \sigma_y \xi_0^* e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)})$$

$$[\Downarrow]$$

推论2.1.16.

$$(\sigma, i\zeta)_a \partial^a [-ie^{-2i\theta} \sigma_y \nu^*(\vec{p})] - me^{-2i\theta} \sigma_y [-ie^{-2i\theta} \sigma_y \nu^*(\vec{p})]^* = 0$$

$$-ie^{-2i\theta} \sigma_y \nu^*(\vec{p}) = \frac{1}{\sqrt{2}} [\lambda(\vec{p}) - ie^{-2i\theta} \sigma_y \lambda^*(\vec{p})] = \frac{E+m+\zeta \vec{p} \cdot \sigma}{\sqrt{2m(E+m)}} \frac{1}{\sqrt{2}} (\xi_0 e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} - ie^{-2i\theta} \sigma_y \xi_0^* e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)})$$

按螺旋度展开的平面波解:

$$\text{推论2.1.17. } \psi(\vec{p}) = [a(p, +) \begin{bmatrix} \lambda(p, +) \\ \zeta \sqrt{\frac{E-m}{E+m}} \lambda(p, +) \end{bmatrix} + a(p, -) \begin{bmatrix} \lambda(p, -) \\ -\zeta \sqrt{\frac{E-m}{E+m}} \lambda(p, -) \end{bmatrix}] e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}$$

$$+ [b(p, +) \begin{bmatrix} \zeta \sqrt{\frac{E-m}{E+m}} \lambda(p, +) \\ \lambda(p, +) \end{bmatrix} + b(p, -) \begin{bmatrix} -\zeta \sqrt{\frac{E-m}{E+m}} \lambda(p, -) \\ \lambda(p, -) \end{bmatrix}] e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}, \frac{\sigma \cdot \vec{p}}{|\vec{p}|} \lambda(+) = \lambda(+), \frac{\sigma \cdot \vec{p}}{|\vec{p}|} \lambda(-) = -\lambda(-)$$

#### 2.1.5 从特殊表象Dirac方程出发构造平面波解<sup>[26]</sup>

$$\text{推论2.1.18. } (\gamma^a \partial_a + m)\psi(\vec{p}) = 0, \gamma^a = (\sigma \otimes \sigma_y, \zeta I \otimes \sigma_z), \psi^*(\vec{p}) = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi(\vec{p})$$

$$\text{推论2.1.19. } \lambda(\vec{p}) = \psi_1(\vec{p}) = \frac{1}{\sqrt{2m(E+m)}} [(E+m)\xi_0 e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} - i\zeta e^{-2i\theta} \sigma \cdot \vec{p} \sigma_y \xi_0^* e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}]$$

$$\text{推论2.1.20. } \psi(\vec{p}) = \begin{bmatrix} \lambda(\vec{p}) \\ -i\sigma_y e^{-2i\theta} \lambda^*(\vec{p}) \end{bmatrix}, \psi_s(\vec{p}) = S_{em}(\zeta) \begin{bmatrix} e^{i\theta} \lambda(\vec{p}) \\ -i\sigma_y [e^{i\theta} \lambda(\vec{p})]^* \end{bmatrix}, \nu(\vec{p}) = \frac{1}{\sqrt{2}} [\lambda(\vec{p}) + ie^{-2i\theta} \sigma_y \lambda^*(\vec{p})]$$

### 2.1.6 从中微子方程出发构造平面波解

推论2.1.21.  $(\sigma, -i\zeta)_a \partial^a \nu(\vec{p}) - m e^{-2i\theta} \sigma_y \nu^*(\vec{p}) = 0$

推论2.1.22.  $\nu(\vec{p}) = \frac{E+m-\zeta\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}} \frac{1}{\sqrt{2}} (\xi_0 e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + i e^{-2i\theta} \sigma_y \xi_0^* e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)})$

推论2.1.23.  $\psi(\vec{p}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \nu(\vec{p}) - i e^{-2i\theta} \sigma_y \nu^*(\vec{p}) \\ -\nu(\vec{p}) - i e^{-2i\theta} \sigma_y \nu^*(\vec{p}) \end{bmatrix}, \psi_s(\vec{p}) = \frac{1}{\sqrt{2}} S_{em}(\zeta) \begin{bmatrix} e^{i\theta} \nu(\vec{p}) - i \sigma_y [e^{i\theta} \nu(\vec{p})]^* \\ -e^{i\theta} \nu(\vec{p}) - i \sigma_y [e^{i\theta} \nu(\vec{p})]^* \end{bmatrix}$

## 2.2 实表象下Majorana方程自旋基的性质

**Majorana方程:**  $(\gamma_s^a \partial_a + m)\psi = 0, \gamma_s^a = (\sigma_{-\zeta} \sigma_{\zeta y}, \zeta \sigma_{\zeta z}), \psi_s^* = \psi_s$

实表象下两个自旋基之间的性质

性质2.2.1.  $\bar{u}_s(\vec{p}, h) u_s(\vec{p}, h') = \zeta \delta_{hh'}, \bar{u}_s(\vec{p}, h) u_s^*(\vec{p}, h') = 0$

性质2.2.2.  $\sum_h u_s(\vec{p}, h) \bar{u}_s(\vec{p}, h) = \frac{\zeta m - i \gamma_s^a p_a}{2m}$

性质2.2.3.  $\begin{cases} \sum_h u_s(\vec{p}, h) \bar{u}_s(\vec{p}, h) - [\sum_h u_s(\vec{p}, h) \bar{u}_s(\vec{p}, h)]^* = \zeta \\ \sum_h u_s(\vec{p}, h) \bar{u}_s(\vec{p}, h) + [\sum_h u_s(\vec{p}, h) \bar{u}_s(\vec{p}, h)]^* = \frac{-i \gamma_s^a p_a}{m} \end{cases}$

性质2.2.4.  $u_s^+(\vec{p}, h) u_s(\vec{p}, h') = \frac{E}{m} \delta_{hh'}, u_s^+(\vec{p}, h) u_s^*(-\vec{p}, h') = 0$

性质2.2.5.  $\sum_h u_s(\vec{p}, h) u_s^+(\vec{p}, h) + [\sum_h u_s(-\vec{p}, h) u_s^+(\vec{p}, h)]^* = \frac{E}{m}$

## 2.3 实表象下Majorana方程的平面波解

推论2.3.1.  $\psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h [a_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} u_s(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_s^*(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$

推论2.3.2.  $\nabla \psi(\vec{r}, t) = i\zeta \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \vec{p} \sum_h [a_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} u_s(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} - a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_s^*(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$

推论2.3.3.  $\psi^*(\vec{r}, t) = \psi(\vec{r}, t)$

推论2.3.4.

$$\begin{cases} a_\zeta(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u_s^+(\vec{p}, h) \psi(\vec{r}, t) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u_{s\lambda_\zeta}^*(\vec{p}, h) \psi^{\lambda_\zeta}(\vec{r}, t) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \\ a_\zeta^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u_s^T(\vec{p}, h) \psi(\vec{r}, t) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u_{s\lambda_\zeta}(\vec{p}, h) \psi^{\lambda_\zeta}(\vec{r}, t) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \end{cases}$$

## 2.4 实表象下Majorana方程的守恒荷

**Majorana作用量:**  $L = -\frac{1}{2} \int \bar{\psi}(\gamma_s^a \partial_a + m)\psi dr^3$ , **Majorana哈密顿量:**  $H = \frac{1}{2} \int \bar{\psi}(\gamma_s \cdot \nabla + m)\psi dr^3$

推论2.4.1.  $\bar{\psi}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} \bar{u}_s(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} - a_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} \bar{u}_s^*(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$

推论2.4.2.  $H = i \int \psi^+ \partial_t \psi dr^3 = \int \sum_h \zeta E [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}, h)] d^3\vec{p}$

证明:  $H = \int [\bar{\psi}(\gamma_s \cdot \nabla) + m]\psi dr^3 = i \int \psi^+ \partial_t \psi dr^3$

$$= \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h') \bar{u}_s(\vec{p}, h) (m + i\zeta \gamma_s \cdot \vec{p}) u_s(\vec{p}, h') - a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}, h') \bar{u}_s^*(\vec{p}, h) (m - i\zeta \gamma_s \cdot \vec{p}) u_s^*(\vec{p}, h')] d^3\vec{p}$$

$$= \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h') \bar{u}_s(\vec{p}, h) \{2m\zeta [\sum_{s''} u^*(\vec{p}, s'') \bar{u}^*(\vec{p}, s'')] + \zeta E \gamma_s^4\} u_s(\vec{p}, h') - a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}, h') \bar{u}_s^*(\vec{p}, h) \{2m\zeta [\sum_{s''} u_\zeta(\vec{p}, s'') \bar{u}_\zeta(\vec{p}, s'')] - \zeta E \gamma_s^4\} u_s^*(\vec{p}, h')] d^3\vec{p}$$

$$= \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h') u_s^+(\vec{p}, h) \zeta E u_s(\vec{p}, h') - a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}, h') u_s^T(\vec{p}, h) \zeta E u_s^*(\vec{p}, h')] d^3\vec{p}$$

$$= \int \sum_h \zeta E [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}, h)] d^3\vec{p}$$

□

$$\text{推论2.4.3. } \vec{P} = \int -i\psi^+\nabla\psi dr^3 = \int \sum_h \varsigma \vec{p} [a_\zeta^+(\vec{p}, h)a_\zeta(\vec{p}, h) - a_\zeta(\vec{p}, h)a_\zeta^+(\vec{p}, h)] d^3\vec{p}$$

$$\begin{aligned} \text{证明: } \vec{P} &= \int -i\psi^+\nabla\psi dr^3 \\ &= -i \frac{1}{(2\pi)^3} \int \sum_{h, h'} [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_s^+(\vec{p}, h) e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} + a_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} u_s^T(\vec{p}, h) e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}] (i\varsigma\vec{p}) \\ &\quad [a_\zeta(\vec{p}', h') \sqrt{\frac{m}{E'}} u_{s'}(\vec{p}', h') e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)} - a_\zeta^+(\vec{p}', h') \sqrt{\frac{m}{E'}} u_{s'}^*(\vec{p}', h') e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}] d^3\vec{p}' d^3\vec{p} dr^3 \\ &= -i \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_s^+(\vec{p}, h) (i\varsigma\vec{p}) u_s(\vec{p}', h') - a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}', h') u_s^T(\vec{p}, h) (i\varsigma\vec{p}') u_{s'}^*(\vec{p}', h')] \delta^3(\vec{p}-\vec{p}') d^3\vec{p}' d^3\vec{p} \\ &= -i \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_s^+(\vec{p}, h) (i\varsigma\vec{p}) u_s(\vec{p}', h') - a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}', h') u_s^T(\vec{p}, h) (i\varsigma\vec{p}') u_{s'}^*(\vec{p}', h')] d^3\vec{p} \\ &= \int \sum_h \varsigma \vec{p} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}, h)] d^3\vec{p} \quad \square \end{aligned}$$

$$\text{推论2.4.4. } Q = \int \psi^+\psi dr^3 = \int \sum_h [a_\zeta^+(\vec{p}, h)a_\zeta(\vec{p}, h) + a_\zeta(\vec{p}, h)a_\zeta^+(\vec{p}, h)] d^3\vec{p}$$

$$\begin{aligned} \text{证明: } Q &= \int \psi^+\psi dr^3 \\ &= \frac{1}{(2\pi)^3} \int \sum_{h, h'} [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_s^+(\vec{p}, h) e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} + a_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} u_s^T(\vec{p}, h) e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)}] \\ &\quad [a_\zeta(\vec{p}', h') \sqrt{\frac{m}{E'}} u_{s'}(\vec{p}', h') e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)} + a_\zeta^+(\vec{p}', h') \sqrt{\frac{m}{E'}} u_{s'}^*(\vec{p}', h') e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}] d^3\vec{p}' d^3\vec{p} dr^3 \\ &= \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_s^+(\vec{p}, h) u_s(\vec{p}', h') + a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}', h') u_s^T(\vec{p}, h) u_{s'}^*(\vec{p}', h')] \delta^3(\vec{p}-\vec{p}') d^3\vec{p}' d^3\vec{p} \\ &= \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_s^+(\vec{p}, h) u_s(\vec{p}', h') + a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}', h') u_s^T(\vec{p}, h) (i\varsigma\vec{p}') u_{s'}^*(\vec{p}', h')] d^3\vec{p} \\ &= \int \sum_h [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) + a_\zeta(\vec{p}, h) a_\zeta^+(\vec{p}, h)] d^3\vec{p} \quad \square \end{aligned}$$

$$\text{推论2.4.5. } L = -\frac{1}{2} \int \bar{\psi}(\gamma_s^a \partial_a + m)\psi dr^3 = 0$$

## 2.5 实表象下Majorana方程的量子化

运用以上结论和性质，可以得到如下对易关系：

$$\text{推论2.5.1. } \begin{cases} \{\psi_{\lambda_\zeta}(\vec{r}, t), \psi_{\mu_\zeta}(\vec{r}', t)\} = \delta_{\lambda_\zeta\mu_\zeta} \delta^3(\vec{r}-\vec{r}') \\ \psi^*(\vec{r}, t) = \psi(\vec{r}, t) \end{cases} \Leftrightarrow \begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p}-\vec{p}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^+(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = 0 \end{cases}$$

$$\text{推论2.5.2. } \begin{cases} :H := \frac{1}{2} \int i\psi^+ \partial_t \psi dr^3 := \frac{1}{2} \int [\bar{\psi}(\gamma_s \cdot \nabla + m)\psi] dr^3 \stackrel{\text{S}}{=} \int \sum_h E(p) a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) d^3\vec{p} \\ :P := \frac{1}{2} \int -i\psi^+ \nabla \psi dr^3 \stackrel{\text{S}}{=} \int \sum_h \vec{p} a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) d^3\vec{p} \end{cases}$$

$$\text{推论2.5.3. } \begin{cases} :P_u := \frac{1}{2} \int -i\psi^+ \partial_u \psi dr^3 \stackrel{\text{S}}{=} \int \sum_h p_u a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) d^3\vec{p} \\ :Q := \int \psi^+ \psi dr^3 := \int \sum_h 0 a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) d^3\vec{p} \stackrel{\text{S}}{=} 0 \end{cases}$$

$$\text{推论2.5.4. } [P_u, P_v] = 0, [Q, P_u] = 0$$

## 3 任意表象下Majorana方程的平面波解和量子化

### 3.1 任意表象下Majorana方程自旋基的性质

任意表象下Majorana方程:  $(\gamma^a \partial_a + m)\psi = 0, \psi_s = S\psi, \psi^* = S^T S\psi, \gamma^a = S^+(\sigma_{-\zeta} \sigma_{+\zeta y}, \sigma_{+\zeta z}) S$

任意表象下两个自旋基之间的性质

$$\text{性质3.1.1. } \bar{u}_\zeta(\vec{p}, h) u_\zeta(\vec{p}, h') = \varsigma \delta_{hh'}, \bar{u}_\zeta(\vec{p}, h) (S^+ S^*) u^*(\vec{p}, h') = 0$$

$$\text{性质3.1.2. } \sum_h u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h) = \frac{\varsigma m - i\gamma^a p_a}{2m}$$

$$\text{性质3.1.3.} \quad \begin{cases} \sum_h u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h) - [\sum_h u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h)]^* = \varsigma \\ \sum_h u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h) + [\sum_h u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h)]^* = \frac{-i\gamma^a p_a}{m} \end{cases}$$

$$\text{性质3.1.4.} \quad u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}, h') = \frac{E}{m} \delta_{hh'}, u_\zeta^+(\vec{p}, h) (S^+ S^*) u^*(-\vec{p}, h') = 0$$

$$\text{性质3.1.5.} \quad \sum_h u_\zeta(\vec{p}, h) u_\zeta^+(\vec{p}, h) + [\sum_h u_\zeta(-\vec{p}, h) u_\zeta^+(-\vec{p}, h)]^* = \frac{E}{m}$$

### 3.2 任意表象下Majorana方程的平面波解

推论3.2.1.

$$\psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h [a_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta(\vec{p}, h) e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} + S^+ S^* a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u^*(\vec{p}, h) e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

推论3.2.2.

$$\nabla\psi(\vec{r}, t) = i\varsigma \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \vec{p} \sum_h [a_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta(\vec{p}, h) e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} - S^+ S^* a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u^*(\vec{p}, h) e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\text{推论3.2.3.} \quad \psi^*(\vec{r}, t) = S^T S \psi(\vec{r}, t)$$

推论3.2.4.

$$\begin{cases} a_\zeta(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}', h) \psi(\vec{r}, t) e^{-i\varsigma(\vec{p}'\cdot\vec{r}-Et)} d^3\vec{r} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u^{+\lambda_\zeta}(\vec{p}', h) \psi_{\lambda_\zeta}(\vec{r}, t) e^{-i\varsigma(\vec{p}'\cdot\vec{r}-Et)} d^3\vec{r} \\ a_\zeta^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u_\zeta^T(\vec{p}', h) \psi^*(\vec{r}, t) e^{i\varsigma(\vec{p}'\cdot\vec{r}-Et)} d^3\vec{r} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u_\zeta^{\lambda'_\zeta}(\vec{p}', h) \psi_{\lambda'_\zeta}^+(\vec{r}, t) e^{i\varsigma(\vec{p}'\cdot\vec{r}-Et)} d^3\vec{r} \end{cases}$$

### 3.3 任意表象下Majorana方程的量子化

$$\text{推论3.3.1.} \quad \begin{cases} \{\psi_{\lambda_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta}^+(\vec{r}', t)\} = \delta_{\lambda_\zeta \lambda'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \psi^*(\vec{r}, t) = S^T S \psi(\vec{r}, t) \end{cases} \Leftrightarrow \begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^+(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = 0 \end{cases}$$

$$\text{推论3.3.2.} \quad L = -\frac{1}{2} \int \bar{\psi} (\gamma_s^a \partial_a + m) \psi d^3r = 0$$

$$\text{推论3.3.3.} \quad \begin{cases} :H :=: \frac{1}{2} \int i\psi^+ \partial_t \psi d^3r :=: \frac{1}{2} \int [\bar{\psi} (\gamma_s \cdot \nabla + m) \psi] d^3r \stackrel{\text{S}}{=} \int \sum_h E(p) a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) d^3\vec{p} \\ :P :=: \frac{1}{2} \int -i\psi^+ \nabla \psi d^3r \stackrel{\text{S}}{=} \int \sum_h \vec{p} a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) d^3\vec{p} \end{cases}$$

$$\text{推论3.3.4.} \quad \begin{cases} :P_u :=: \frac{1}{2} \int -i\psi^+ \partial_u \psi d^3r \stackrel{\text{S}}{=} \int \sum_h p_u a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) d^3\vec{p} \\ :Q :=: \int \psi^+ \psi d^3r := \int \sum_h 0 a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) d^3\vec{p} \stackrel{\text{S}}{=} 0 \end{cases}$$

$$\text{推论3.3.5.} \quad [P_u, P_v] = 0, [Q, P_u] = 0$$

在表象变换下，湮灭产生算符和其对易关系是标量、不变量。系统能量动量算符、守恒荷也是标量、不变量。波函数算符和其对易关系是表象协变量。

## 4 Majorana方程和有质量中微子方程的等价关系

### 4.1 Majorana方程和有质量中微子方程等价的正则反对易关系

Dirac表象下Majorana方程反对易关系

$$\text{推论4.1.1.} \quad \begin{cases} \{\psi_{\lambda_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta}^+(\vec{r}', t)\} = \delta_{\lambda_\zeta \lambda'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \psi^*(\vec{r}, t) = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi(\vec{r}, t) \end{cases} \Leftrightarrow \begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^+(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = 0 \end{cases}$$

$$\text{推论4.1.2. } \psi^*(\vec{r}, t) = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi(\vec{r}, t) \Leftrightarrow \psi(\vec{r}, t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \nu(\vec{r}, t) - ie^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \\ -\nu(\vec{r}, t) - ie^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \end{bmatrix}$$

Majorana方程和中微子方程正则反对易关系的等价变换:

$$\text{推论4.1.3. } \begin{cases} \{\psi_{\lambda_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta}^+(\vec{r}', t)\} = \delta_{\lambda_\zeta \lambda'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \psi(\vec{r}, t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \nu(\vec{r}, t) - ie^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \\ -\nu(\vec{r}, t) - ie^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \end{bmatrix} \end{cases} \Leftrightarrow \begin{cases} \{\nu_{A_\zeta}(\vec{r}, t), \nu_{A'_\zeta}^+(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\nu_{A_\zeta}(\vec{r}, t), \nu_{B_\zeta}(\vec{r}', t)\} = 0 \\ \{\nu_{A'_\zeta}^+(\vec{r}, t), \nu_{B'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases}$$

证明:

$$\begin{cases} \{\psi_{\lambda_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta}^+(\vec{r}', t)\} = \delta_{\lambda_\zeta \lambda'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \psi(\vec{r}, t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \nu(\vec{r}, t) - ie^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \\ -\nu(\vec{r}, t) - ie^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \end{bmatrix} \end{cases} \Leftrightarrow \begin{cases} \frac{1}{2} \{\nu_{A_\zeta}(\vec{r}, t) - e^{-2i\theta} (\varepsilon \nu)_{A_\zeta}^*(\vec{r}, t), \nu_{A'_\zeta}^*(\vec{r}', t) - e^{2i\theta} (\varepsilon \nu)_{A'_\zeta}(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \frac{1}{2} \{\nu_{A_\zeta}(\vec{r}, t) - e^{-2i\theta} (\varepsilon \nu)_{A_\zeta}^*(\vec{r}, t), -\nu_{A'_\zeta}^*(\vec{r}', t) - e^{2i\theta} (\varepsilon \nu)_{A'_\zeta}(\vec{r}', t)\} = 0 \\ \frac{1}{2} \{-\nu_{A_\zeta}(\vec{r}, t) - e^{-2i\theta} (\varepsilon \nu)_{A_\zeta}^*(\vec{r}, t), \nu_{A'_\zeta}^*(\vec{r}', t) - e^{2i\theta} (\varepsilon \nu)_{A'_\zeta}(\vec{r}', t)\} = 0 \\ \frac{1}{2} \{-\nu_{A_\zeta}(\vec{r}, t) - e^{-2i\theta} (\varepsilon \nu)_{A_\zeta}^*(\vec{r}, t), -\nu_{A'_\zeta}^*(\vec{r}', t) - e^{2i\theta} (\varepsilon \nu)_{A'_\zeta}(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \end{cases} \Leftrightarrow \begin{cases} \{\nu_{A_\zeta}(\vec{r}, t) - e^{-2i\theta} (\varepsilon \nu)_{A_\zeta}^*(\vec{r}, t), \nu_{A'_\zeta}^*(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\nu_{A_\zeta}(\vec{r}, t) - e^{-2i\theta} (\varepsilon \nu)_{A_\zeta}^*(\vec{r}, t), -e^{2i\theta} (\varepsilon \nu)_{A'_\zeta}(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{-\nu_{A_\zeta}(\vec{r}, t) - e^{-2i\theta} (\varepsilon \nu)_{A_\zeta}^*(\vec{r}, t), -e^{2i\theta} (\varepsilon \nu)_{A'_\zeta}(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\nu_{A_\zeta}(\vec{r}, t) + e^{-2i\theta} (\varepsilon \nu)_{A_\zeta}^*(\vec{r}, t), \nu_{A'_\zeta}^*(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \end{cases} \Leftrightarrow \begin{cases} \{\nu_{A_\zeta}(\vec{r}, t), \nu_{A'_\zeta}^*(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\nu_{A_\zeta}(\vec{r}, t), (\varepsilon \nu)_{A'_\zeta}(\vec{r}', t)\} = 0 \\ \{(\varepsilon \nu)_{A_\zeta}^*(\vec{r}, t), (\varepsilon \nu)_{A'_\zeta}(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{(\varepsilon \nu)_{A_\zeta}^*(\vec{r}, t), \nu_{A'_\zeta}^*(\vec{r}', t)\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{\nu_{A_\zeta}(\vec{r}, t), \nu_{A'_\zeta}^+(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\nu_{A_\zeta}(\vec{r}, t), \nu_{B_\zeta}(\vec{r}', t)\} = 0 \\ \{\nu_{A'_\zeta}^+(\vec{r}, t), \nu_{B'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases}$$

□

$$\text{推论4.1.4. } \begin{cases} \{\nu_{A_\zeta}(\vec{r}, t), \nu_{A'_\zeta}^+(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\nu_{A_\zeta}(\vec{r}, t), \nu_{B_\zeta}(\vec{r}', t)\} = 0 \\ \{\nu_{A'_\zeta}^+(\vec{r}, t), \nu_{B'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{r} - \vec{r}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^+(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = 0 \end{cases}$$

## 4.2 Dirac表象下Majorana作用量和中微子作用量

Majorana拉氏量:  $L = -\frac{1}{2} \int \bar{\psi}(\gamma^\alpha \partial_\alpha + m)\psi dr^3$ , Majorana哈密顿量:  $H = \frac{1}{2} \int \bar{\psi}(\gamma \cdot \nabla + m)\psi dr^3$

$$\text{推论4.2.1. } \gamma^\alpha \partial_\alpha = \begin{bmatrix} \varsigma \partial_\pi & -i\sigma \cdot \nabla \\ i\sigma \cdot \nabla & -\varsigma \partial_\pi \end{bmatrix}, \gamma^4 \gamma^\alpha \partial_\alpha = \begin{bmatrix} \partial_\pi & -i\varsigma \sigma \cdot \nabla \\ -i\varsigma \sigma \cdot \nabla & \partial_\pi \end{bmatrix}, \gamma^\alpha = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_z)$$

$$\text{推论4.2.2. } \bar{\psi}(\vec{r}, t)\psi(\vec{r}, t) = \varsigma \{\nu^+(\vec{r}, t)[-ie^{-2i\theta} \sigma_y \nu^*(\vec{r}, t)] + [ie^{2i\theta} \nu^T(\vec{r}, t)\sigma_y]\nu(\vec{r}, t)\}$$

证明:  $\bar{\psi}(\vec{r}, t)\psi(\vec{r}, t) = \psi^+(\vec{r}, t)\gamma^4\psi(\vec{r}, t)$

$$= \frac{1}{2} \varsigma \left[ \nu^+(\vec{r}, t) + ie^{2i\theta} \nu^T(\vec{r}, t)\sigma_y, -\nu^+(\vec{r}, t) + ie^{2i\theta} \nu^T(\vec{r}, t)\sigma_y \right] \begin{bmatrix} \nu(\vec{r}, t) - ie^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \\ \nu(\vec{r}, t) + ie^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \end{bmatrix}$$

$$\begin{aligned}
&= \frac{1}{2}\zeta\{[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y][\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] - [\nu^+(\vec{r},t) - ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y][\nu(\vec{r},t) + ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)]\} \\
&= \zeta\{\nu^+(\vec{r},t)[-ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] + [ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y]\nu(\vec{r},t)\} \quad \square
\end{aligned}$$

推论4.2.3.  $\bar{\psi}(\vec{r},t)\gamma^a\partial_a\psi(\vec{r},t) = i\zeta[\nu^+(\vec{r},t)(\sigma, -i\zeta)^a\partial_a\nu(\vec{r},t) - \nu^T(\vec{r},t)(\sigma, i\zeta)^a\partial_a\nu^*(\vec{r},t)]$

证明:  $\bar{\psi}(\vec{r},t)\gamma^a\partial_a\psi(\vec{r},t) = \psi^+(\vec{r},t)\gamma^4\gamma^a\partial_a\psi(\vec{r},t)$

$$\begin{aligned}
&= \frac{1}{2}\left[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y, -\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y\right]\left[\begin{array}{c} \partial_\pi \\ -i\zeta\sigma\cdot\nabla \end{array} \begin{array}{c} -i\zeta\sigma\cdot\nabla \\ \partial_\pi \end{array}\right]\left[\begin{array}{c} \nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \\ -\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \end{array}\right] \\
&= \frac{1}{2}\left[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y, -\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y\right]\partial_\pi\left[\begin{array}{c} \nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \\ -\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \end{array}\right] \\
&+ \frac{1}{2}\left[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y, -\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y\right](-i\zeta\sigma\cdot\nabla)\left[\begin{array}{c} -\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \\ \nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t) \end{array}\right] \\
&= \frac{1}{2}\{[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y]\partial_\pi[\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] \\
&+ [\nu^+(\vec{r},t) - ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y]\partial_\pi[\nu(\vec{r},t) + ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)]\} \\
&+ \frac{1}{2}\{[\nu^+(\vec{r},t) + ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y](i\zeta\sigma\cdot\nabla)[\nu(\vec{r},t) + ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] \\
&+ [\nu^+(\vec{r},t) - ie^{2i\theta}\nu^T(\vec{r},t)\sigma_y](i\zeta\sigma\cdot\nabla)[\nu(\vec{r},t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r},t)]\} \\
&= [\nu^+(\vec{r},t)\partial_\pi\nu(\vec{r},t) + \nu^T(\vec{r},t)\partial_\pi\nu^*(\vec{r},t)] + [\nu^+(\vec{r},t)(i\zeta\sigma\cdot\nabla)\nu(\vec{r},t) - \nu^T(\vec{r},t)\sigma_y(i\zeta\sigma\cdot\nabla)\sigma_y\nu^*(\vec{r},t)] \\
&= i\zeta[\nu^+(\vec{r},t)(\sigma, -i\zeta)^a\partial_a\nu(\vec{r},t) - \nu^T(\vec{r},t)\sigma_y(\sigma, i\zeta)^a\partial_a\sigma_y\nu^*(\vec{r},t)] \quad \square
\end{aligned}$$

中微子拉氏量:

推论4.2.4.  $L = -\frac{1}{2}\int\bar{\psi}(\vec{r},t)(\gamma^a\partial_a + m)\psi(\vec{r},t)$   
 $= -\frac{1}{2}i\zeta\int\nu^+(\vec{r},t)[(\sigma, -i\zeta)^a\partial_a\nu(\vec{r},t) - me^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] - \nu^T(\vec{r},t)[(-\sigma^*, i\zeta)^a\partial_a\nu^*(\vec{r},t) - me^{2i\theta}\sigma_y\nu(\vec{r},t)]$

证明:  $L = -\frac{1}{2}\int\bar{\psi}(\vec{r},t)(\gamma^a\partial_a + m)\psi(\vec{r},t)$   
 $= -\frac{1}{2}\int i\zeta[\nu^+(\vec{r},t)(\sigma, -i\zeta)^a\partial_a\nu(\vec{r},t) - \nu^T(\vec{r},t)\sigma_y(\sigma, i\zeta)^a\partial_a\sigma_y\nu^*(\vec{r},t)]$   
 $+ mi\zeta\{\nu^+(\vec{r},t)[-e^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] + [e^{2i\theta}\nu^T(\vec{r},t)\sigma_y]\nu(\vec{r},t)\}$   
 $= -\frac{1}{2}i\zeta\int\nu^+(\vec{r},t)[(\sigma, -i\zeta)^a\partial_a\nu(\vec{r},t) - me^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] - \nu^T(\vec{r},t)\sigma_y[(\sigma, i\zeta)^a\partial_a\sigma_y\nu^*(\vec{r},t) + me^{2i\theta}\sigma_y[\sigma_y\nu^*(\vec{r},t)]^*]$   
 $= -\frac{1}{2}i\zeta\int\nu^+(\vec{r},t)[(\sigma, -i\zeta)^a\partial_a\nu(\vec{r},t) - me^{-2i\theta}\sigma_y\nu^*(\vec{r},t)] - \nu^T(\vec{r},t)[(-\sigma^*, i\zeta)^a\partial_a\nu^*(\vec{r},t) - me^{2i\theta}\sigma_y\nu(\vec{r},t)] \quad \square$

中微子哈密顿量:

推论4.2.5.  $H = \frac{1}{2}\int\bar{\psi}(\gamma\cdot\nabla + m)\psi dr^3$   
 $= i\zeta\frac{1}{2}\int[\nu^+(\vec{r},t)\sigma\cdot\nabla\nu(\vec{r},t) + \nu^T(\vec{r},t)\sigma^*\cdot\nabla\nu^*(\vec{r},t)] - m[e^{-2i\theta}\nu^+(\vec{r},t)\sigma_y\nu^*(\vec{r},t) - e^{2i\theta}\nu^T(\vec{r},t)\sigma_y\nu(\vec{r},t)]dr^3$

中微子的电荷:

推论4.2.6.  $Q = \int\psi^+\psi dr^3 = \int\nu^+(\vec{r},t)\nu(\vec{r},t) + \nu^T(\vec{r},t)\nu^*(\vec{r},t)dr^3 \simeq \int\nu^+(\vec{r},t)\nu(\vec{r},t) + \nu^T(\vec{r},t)\nu^*(\vec{r},t)dr^3$

中微子的能量动量:

推论4.2.7.  $P_u = -i\int\psi^+\partial_u\psi dr^3 = -i\int\nu^+(\vec{r},t)\partial_u\nu(\vec{r},t) + \nu^T(\vec{r},t)\partial_u\nu^*(\vec{r},t)dr^3$

推论4.2.8.  $[P_u, P_v] = 0, [Q, P_u] = 0$

## 5 有质量中微子方程的平面波解和直接量子化 [40]

### 5.1 有质量中微子方程自旋基的性质

推论5.1.1.  $(\sigma, -i\zeta)_a\partial^a\nu(x) - me^{-2i\theta}\sigma_y\nu^*(x) = 0$

$$\text{推论5.1.2. } \begin{cases} \eta(\vec{p}, \frac{1}{2}) := \frac{E+m-\zeta\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = u_1(\vec{p}, \frac{1}{2}) - u_2(\vec{p}, \frac{1}{2}) \\ \eta(\vec{p}, -\frac{1}{2}) := \frac{E+m-\zeta\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = u_1(\vec{p}, -\frac{1}{2}) - u_2(\vec{p}, -\frac{1}{2}) \end{cases}$$

$$\text{推论5.1.3. } \eta(\vec{p}, h) = u_1(\vec{p}, h) - u_2(\vec{p}, h), \eta^+(\vec{p}, h)\eta(-\vec{p}, h') = \delta_{hh'}, \eta^T(\vec{p}, h)\eta^*(-\vec{p}, h') = \delta_{hh'}$$

$$\text{推论5.1.4. } \begin{cases} \sum_h \eta(\vec{p}, h)\eta^+(\vec{p}, h) = \frac{E-\zeta\sigma\cdot\vec{p}}{m} = \frac{-\zeta(\sigma\cdot i\zeta)^a p_a}{m} \\ \sum_h (-1)^{h-\frac{1}{2}} \eta(\vec{p}, h)\eta^+(\vec{p}, h) = i\sigma_y \end{cases}$$

$$\text{推论5.1.5. } \begin{cases} \sum_h [\eta(\vec{p}, h)\eta^+(\vec{p}, h) + \eta(-\vec{p}, h)\eta^+(-\vec{p}, h)] = \frac{2E}{m} \\ \sum_h [\eta^*(\vec{p}, h)\eta^T(\vec{p}, h) + \eta^*(-\vec{p}, h)\eta^T(-\vec{p}, h)] = \frac{2E}{m} \\ \sum_h (-1)^{h-\frac{1}{2}} [\eta(\vec{p}, h)\eta^T(\vec{p}, -h) + \eta(-\vec{p}, -h)\eta^T(-\vec{p}, h)] = 0 \\ \sum_h (-1)^{h-\frac{1}{2}} [\eta^*(\vec{p}, h)\eta^+(\vec{p}, -h) + \eta^*(-\vec{p}, -h)\eta^+(-\vec{p}, h)] = 0 \end{cases}$$

$$\text{推论5.1.6. } \eta^+(\vec{p}, h)\eta(\vec{p}', h') = -(-1)^{h+h'} \eta^T(-\vec{p}, -h)\eta^*(-\vec{p}', -h')$$

$$\text{推论5.1.7. } \begin{cases} \eta^+(\vec{p}, h)\eta(\vec{p}, h') - (-1)^{h+h'} \eta^T(\vec{p}, -h)\eta^*(\vec{p}, -h') = \frac{2E}{m} \delta_{hh'} \\ \eta^+(\vec{p}, h)\eta(\vec{p}, h') + \eta^+(-\vec{p}, h)\eta(-\vec{p}, h') = \frac{2E}{m} \delta_{hh'} \end{cases}$$

$$\text{推论5.1.8. } \begin{cases} \eta^+(\vec{p}, h)\eta(-\vec{p}, h') - (-1)^{h+h'} \eta^T(\vec{p}, -h)\eta^*(-\vec{p}, -h') = 0 \\ \eta^+(\vec{p}, h)\eta(-\vec{p}, h') - \eta^+(-\vec{p}, h)\eta(\vec{p}, h') = 0 \end{cases}$$

$$\text{推论5.1.9. } \begin{cases} \eta^+(\vec{p}, h)\eta(-\vec{p}, -h') - (-1)^{h'-h} \eta^T(\vec{p}, -h)\eta^*(-\vec{p}, h') = 0 \\ \eta^+(\vec{p}, h)\eta(-\vec{p}, -h') - \eta^+(-\vec{p}, h)\eta(\vec{p}, -h') = 0 \end{cases}$$

## 5.2 从Dirac表象下Majorana方程变换得到有质量中微子方程平面波解

$$\text{推论5.2.1. } \nu(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \sqrt{\frac{m}{2E}} \sum_h \frac{E+m-\zeta\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}} [a_\zeta(\vec{p}, h)\xi(h)e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + a_\zeta^+(\vec{p}, h)ie^{-2i\theta}\sigma_y\xi(h)e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\text{推论5.2.2. } \nu(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \sqrt{\frac{m}{2E}} \sum_h [a_\zeta(\vec{p}, h)\eta(\vec{p}, h)e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + (-1)^{h-\frac{1}{2}} e^{-2i\theta} a_\zeta^+(\vec{p}, h)\eta(\vec{p}, -h)e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\text{推论5.2.3. } a_\zeta(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h)\psi(\vec{r}, t)e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r}$$

$$\Leftrightarrow a_\zeta(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{E}{2m}} \int [\eta^+(\vec{p}, h)\nu(\vec{r}, t) - (-1)^{h-\frac{1}{2}} e^{-2i\theta} \eta^T(\vec{p}, -h)\nu^*(\vec{r}, t)] e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r}$$

$$\text{证明: } a_\zeta(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h)\psi(\vec{r}, t)e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r}$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} [u_1^+(\vec{p}, h), u_2^+(\vec{p}, h)] \frac{1}{\sqrt{2}} \begin{bmatrix} \nu(\vec{r}, t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r}, t) \\ -\nu(\vec{r}, t) - ie^{-2i\theta}\sigma_y\nu^*(\vec{r}, t) \end{bmatrix} e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r}$$

$$= \frac{1}{(2\pi)^{3/2}} \int \sqrt{\frac{m}{2E}} [\eta^+(\vec{p}, h)\nu(\vec{r}, t) - ie^{-2i\theta}\eta^+(-\vec{p}, h)\sigma_y\nu^*(\vec{r}, t)] e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r}$$

$$= \frac{1}{(2\pi)^{3/2}} \int \sqrt{\frac{m}{2E}} [\eta^+(\vec{p}, h)\nu(\vec{r}, t) - (-1)^{h-\frac{1}{2}} e^{-2i\theta} \eta^T(\vec{p}, -h)\nu^*(\vec{r}, t)] e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \quad \square$$

## 5.3 有质量中微子方程平面波解系数和量子化条件的直接验证

$$\text{推论5.3.1. } a_\zeta(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{E}{2m}} \int [\eta^+(\vec{p}, h)\nu(\vec{r}, t) - (-1)^{h-\frac{1}{2}} e^{-2i\theta} \eta^T(\vec{p}, -h)\nu^*(\vec{r}, t)] e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r}$$



证明: 
$$\begin{aligned} & \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{m}{2E}} \int [\eta^+(\vec{p}, h) \nu(\vec{r}, t) - (-1)^{h-\frac{1}{2}} e^{-2i\theta} \eta^+(\vec{p}, -h) \nu^*(\vec{r}, t)] e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \\ &= \frac{1}{(2\pi)^3} \sqrt{\frac{m}{2E}} \int d^3\vec{r} d^3\vec{p}' \sqrt{\frac{m}{2E'}} \\ & \sum_{h'} \eta^+(\vec{p}, h) [a_\varsigma(\vec{p}', h') \eta(\vec{p}', h') e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)} + (-1)^{h'-\frac{1}{2}} e^{-2i\theta} a_\varsigma^+(\vec{p}', h') \eta(\vec{p}', -h') e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)}] e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} \\ & - (-1)^{h-\frac{1}{2}} e^{-2i\theta} \eta^T(\vec{p}, -h) [a_\varsigma^+(\vec{p}', h') \eta^*(\vec{p}', h') e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)} + (-1)^{h'-\frac{1}{2}} e^{2i\theta} a_\varsigma(\vec{p}', h') \eta^*(\vec{p}', -h') e^{i\varsigma(\vec{p}'\cdot\vec{r}-E't)}] e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} \\ &= \sqrt{\frac{m}{2E}} \int \sqrt{\frac{m}{2E'}} \sum_{h'} [a_\varsigma(\vec{p}', h') \eta^+(\vec{p}, h) \eta(\vec{p}', h') \delta^3(\vec{p}-\vec{p}') + (-1)^{h-\frac{1}{2}} e^{-2i\theta} a_\varsigma^+(\vec{p}', h') \eta^+(\vec{p}, h) \eta(\vec{p}', -h') \delta^3(\vec{p}+\vec{p}') e^{2i\varsigma E t}] \\ & - (-1)^{h'-\frac{1}{2}} e^{-2i\theta} a_\varsigma^+(\vec{p}', h') \eta^T(\vec{p}, -h) \eta^*(\vec{p}', h') \delta^3(\vec{p}+\vec{p}') e^{2i\varsigma E t} - (-1)^{h+h'} a_\varsigma(\vec{p}', h') \eta^T(\vec{p}, -h) \eta^*(\vec{p}', -h') \delta^3(\vec{p}-\vec{p}')] d^3\vec{p}' \\ &= \frac{m}{2E} \sum_{h'} [a_\varsigma(\vec{p}, h') \eta^+(\vec{p}, h) \eta(\vec{p}, h') + (-1)^{h-\frac{1}{2}} e^{-2i\theta} a^+(\vec{p}, h') \eta^+(\vec{p}, h) \eta(-\vec{p}, -h') e^{2i\varsigma E t}] \\ & - (-1)^{h'-\frac{1}{2}} e^{-2i\theta} a^+(\vec{p}, h') \eta^T(\vec{p}, -h) \eta^*(-\vec{p}, h') e^{2i\varsigma E t} - (-1)^{h+h'} a_\varsigma(\vec{p}, h') \eta^T(\vec{p}, -h) \eta^*(\vec{p}, -h')] \\ &= \frac{m}{2E} \sum_{h'} [a_\varsigma(\vec{p}, h') [\eta^+(\vec{p}, h) \eta(\vec{p}, h') - (-1)^{h+h'} \eta^T(\vec{p}, -h) \eta^*(\vec{p}, -h')] \\ & + (-1)^{h-\frac{1}{2}} e^{2i\varsigma E t} e^{-2i\theta} a^+(\vec{p}, h') [\eta^+(\vec{p}, h) \eta(-\vec{p}, -h') - (-1)^{h'-h} \eta^T(\vec{p}, -h) \eta^*(-\vec{p}, h')]] \\ &= a_\varsigma(\vec{p}, h) \end{aligned}$$
 □

推论5.3.2. 
$$a_\varsigma^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{E}{2m}} \int [\eta^T(\vec{p}, h) \nu^*(\vec{r}, t) - (-1)^{h-\frac{1}{2}} e^{2i\theta} \eta^+(\vec{p}, -h) \nu(\vec{r}, t)] e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r}$$

很容易利用以上两个系数展开式可直接推得如下正则对易关系:

推论5.3.3. 
$$\begin{cases} \{\nu_{A_\varsigma}(\vec{r}, t), \nu_{A'_\varsigma}^+(\vec{r}', t)\} = \delta_{A_\varsigma A'_\varsigma} \delta^3(\vec{r}-\vec{r}') \\ \{\nu_{A_\varsigma}(\vec{r}, t), \nu_{A'_\varsigma}(\vec{r}', t)\} = 0 \\ \{\nu_{A'_\varsigma}^+(\vec{r}, t), \nu_{B'_\varsigma}^+(\vec{r}', t)\} = 0 \end{cases} \Rightarrow \begin{cases} \{a_\varsigma(\vec{p}, h), a_\varsigma^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p}-\vec{p}') \\ \{a_\varsigma(\vec{p}, h), a_\varsigma(\vec{p}', h')\} = 0 \\ \{a_\varsigma^+(\vec{p}, h), a_\varsigma^+(\vec{p}', h')\} = 0 \end{cases}$$

现在反过来可利用波函数展开式直接推得如下正则对易关系:

推论5.3.4. 
$$\begin{cases} \{a_\varsigma(\vec{p}, h), a_\varsigma^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p}-\vec{p}') \\ \{a_\varsigma(\vec{p}, h), a_\varsigma(\vec{p}', h')\} = 0 \\ \{a_\varsigma^+(\vec{p}, h), a_\varsigma^+(\vec{p}', h')\} = 0 \end{cases} \Rightarrow \begin{cases} \{\nu_{A_\varsigma}(x), \nu_{A'_\varsigma}^+(x')\} = -\varsigma(\sigma, i\varsigma)^a \partial_a \Delta(x-x') \\ \{\nu_{A_\varsigma}(\vec{r}, t), \nu_{A'_\varsigma}^+(\vec{r}', t)\} = \delta_{A_\varsigma A'_\varsigma} \delta^3(\vec{r}-\vec{r}') \end{cases}$$

证明:

$$\begin{aligned} & \{\nu_{A_\varsigma}(x), \nu_{A'_\varsigma}^+(x')\} \\ &= \frac{1}{(2\pi)^3} \int \frac{m}{2E} \sum_{h, h'} \{ [a_\varsigma(\vec{p}, h) \eta_{A_\varsigma}(\vec{p}, h) e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} + e^{-2i\theta} a_\varsigma^+(\vec{p}, h) (-1)^{h-\frac{1}{2}} \eta_{A_\varsigma}(\vec{p}, -h) e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)] \\ & [a_\varsigma^+(\vec{p}', h') \eta_{A'_\varsigma}^+(\vec{p}', h') e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't')} + e^{2i\theta} a_\varsigma(\vec{p}', h') (-1)^{h'-\frac{1}{2}} \eta_{A'_\varsigma}^+(\vec{p}', -h') e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}] \} d^3\vec{p}' d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{m}{2E} \sum_{h, h'} [\eta_{A_\varsigma}(\vec{p}, h) \eta_{A'_\varsigma}^+(\vec{p}', h')] \{a_\varsigma(\vec{p}, h), a_\varsigma^+(\vec{p}', h')\} [e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}] \\ & + \eta_{A_\varsigma}(\vec{p}, -h) \eta_{A'_\varsigma}^+(\vec{p}', -h') (-1)^{h-\frac{1}{2}} \{a_\varsigma^+(\vec{p}, h), a_\varsigma(\vec{p}', h')\} (-1)^{h'-\frac{1}{2}} [e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}] d^3\vec{p}' d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{m}{2E} \sum_{h, h'} [\eta_{A_\varsigma}(\vec{p}, h) \eta_{A'_\varsigma}^+(\vec{p}', h')] \delta_{hh'} \delta^3(\vec{p}-\vec{p}') [e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}] \\ & + \eta_{A_\varsigma}(\vec{p}, -h) \eta_{A'_\varsigma}^+(\vec{p}', -h') (-1)^{h-\frac{1}{2}} \delta_{hh'} \delta^3(\vec{p}-\vec{p}') (-1)^{h'-\frac{1}{2}} [e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}] d^3\vec{p}' d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{m}{2E} \sum_h \eta_{A_\varsigma}(\vec{p}, h) \eta_{A'_\varsigma}^+(\vec{p}, h) e^{i\varsigma p\cdot(x-x')} + \eta_{A_\varsigma}(\vec{p}, -h) \eta_{A'_\varsigma}^+(\vec{p}, -h) e^{-i\varsigma p\cdot(x-x')} d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{m}{2E} \sum_h \eta_{A_\varsigma}(\vec{p}, h) \eta_{A'_\varsigma}^+(\vec{p}, h) [e^{i\varsigma p\cdot(x-x')} + e^{-i\varsigma p\cdot(x-x')}] d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{m}{2E} \frac{-\varsigma(\sigma, i\varsigma)^a p_a}{m} [e^{i\varsigma p\cdot(x-x')} + e^{-i\varsigma p\cdot(x-x')}] d^3\vec{p} \\ &= -(\sigma, i\varsigma)^a \partial_a \frac{-i}{(2\pi)^3} \int \frac{1}{2E} [e^{i\varsigma p\cdot(x-x')} - e^{-i\varsigma p\cdot(x-x')}] d^3\vec{p} = -\varsigma(\sigma, i\varsigma)^a \partial_a \Delta(x-x') \end{aligned}$$
 □

证明:

$$\begin{aligned} & \{\nu_{A_\varsigma}(\vec{r}, t), \nu_{A'_\varsigma}^+(\vec{r}', t)\} \\ &= \frac{1}{(2\pi)^3} \int \frac{m}{2E} \sum_{h, h'} \{ [a_\varsigma(\vec{p}, h) \eta_{A_\varsigma}(\vec{p}, h) e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} + e^{-2i\theta} a_\varsigma^+(\vec{p}, h) (-1)^{h-\frac{1}{2}} \eta_{A_\varsigma}(\vec{p}, -h) e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)] \\ & [a_\varsigma^+(\vec{p}', h') \eta_{A'_\varsigma}^+(\vec{p}', h') e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't')} + e^{2i\theta} a_\varsigma(\vec{p}', h') (-1)^{h'-\frac{1}{2}} \eta_{A'_\varsigma}^+(\vec{p}', -h') e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't')}] \} d^3\vec{p}' d^3\vec{p} \end{aligned}$$

$$\begin{aligned}
& [a_\zeta^+(\vec{p}', h')\eta_{A'_\zeta}^+(\vec{p}', h')e^{-i\zeta(\vec{p}'\cdot\vec{r}'-E't)} + e^{2i\theta}a_\zeta(\vec{p}', h')(-1)^{h'-\frac{1}{2}}\eta_{A'_\zeta}^+(\vec{p}', -h')e^{i\zeta(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{p}'d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\int_{h,h'}[\eta_{A_\zeta}(\vec{p}, h)\eta_{A'_\zeta}^+(\vec{p}', h')]\{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\}[e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{-i\zeta(\vec{p}'\cdot\vec{r}'-E't)}] \\
&+ \eta_{A_\zeta}(\vec{p}, -h)\eta_{A'_\zeta}^+(\vec{p}', -h')(-1)^{h-\frac{1}{2}}\{a_\zeta^+(\vec{p}, h), a_\zeta(\vec{p}', h)\}(-1)^{h'-\frac{1}{2}}[e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{i\zeta(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{p}'d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}[\eta_{A_\zeta}(\vec{p}, h)\eta_{A'_\zeta}^+(\vec{p}', h')]\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{-i\zeta(\vec{p}'\cdot\vec{r}'-E't)} \\
&+ \eta_{A_\zeta}(\vec{p}, -h)\eta_{A'_\zeta}^+(\vec{p}', -h')(-1)^{h-\frac{1}{2}}\delta_{hh'}\delta^3(\vec{p}-\vec{p}')(-1)^{h'-\frac{1}{2}}[e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{i\zeta(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{p}'d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_h\eta_{A_\zeta}(\vec{p}, h)\eta_{A'_\zeta}^+(\vec{p}, h)e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} + \eta_{A_\zeta}(\vec{p}, -h)\eta_{A'_\zeta}^+(\vec{p}, -h)e^{-i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_h\eta_{A_\zeta}(\vec{p}, h)\eta_{A'_\zeta}^+(\vec{p}, h)e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} + \eta_{A_\zeta}(-\vec{p}, h)\eta_{A'_\zeta}^+(-\vec{p}, h)e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_h[\eta_{A_\zeta}(\vec{p}, h)\eta_{A'_\zeta}^+(\vec{p}, h) + \eta_{A_\zeta}(-\vec{p}, h)\eta_{A'_\zeta}^+(-\vec{p}, h)]e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_h[\eta(\vec{p}, h)\eta^+(\vec{p}, h) + \eta(-\vec{p}, h)\eta^+(-\vec{p}, h)]_{A_\zeta A'_\zeta}e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p} \\
&= \delta_{A_\zeta A'_\zeta}\delta^3(\vec{r}-\vec{r}')
\end{aligned}$$

□

$$\text{推论5.3.5. } \begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = \delta_{hh'}\delta^3(\vec{p}-\vec{p}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^+(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = 0 \end{cases} \Rightarrow \begin{cases} \{\nu_{A_\zeta}(x), \nu_{B_\zeta}(x')\} = i\zeta m e^{-2i\theta} \varepsilon_{A_\zeta B_\zeta} \Delta(x-x') \\ \{\nu_{A'_\zeta}^+(x), \nu_{B'_\zeta}^+(x')\} = -i\zeta m e^{2i\theta} \varepsilon_{A'_\zeta B'_\zeta} \Delta(x-x') \\ \{\nu_{A_\zeta}(\vec{r}, t), \nu_{B_\zeta}(\vec{r}', t)\} = 0 \\ \{\nu_{A'_\zeta}^+(\vec{r}, t), \nu_{B'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases}$$

证明:

$$\begin{aligned}
& \{\nu_{A_\zeta}(x), \nu_{B_\zeta}(x')\} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}\{[a_\zeta(\vec{p}, h)\eta_{A_\zeta}(\vec{p}, h)e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + e^{-2i\theta}a_\zeta^+(\vec{p}, h)(-1)^{h-\frac{1}{2}}\eta_{A_\zeta}(\vec{p}, -h)e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}], \\
&[a_\zeta(\vec{p}', h')\eta_{B_\zeta}(\vec{p}', h')e^{i\zeta(\vec{p}'\cdot\vec{r}'-E't)} + e^{-2i\theta}a_\zeta^+(\vec{p}', h')(-1)^{h'-\frac{1}{2}}\eta_{B_\zeta}(\vec{p}', -h')e^{-i\zeta(\vec{p}'\cdot\vec{r}'-E't)}]\}d^3\vec{p}'d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}(-1)^{h'-\frac{1}{2}}e^{-2i\theta}[\eta_{A_\zeta}(\vec{p}, h)\eta_{B_\zeta}(\vec{p}', -h')]\{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\}[e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{-i\zeta(\vec{p}'\cdot\vec{r}'-E't)}] \\
&+ (-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta_{A_\zeta}(\vec{p}, -h)\eta_{B_\zeta}(\vec{p}', h')\{a_\zeta^+(\vec{p}, h), a_\zeta(\vec{p}', h')\}[e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{i\zeta(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{p}'d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}(-1)^{h'-\frac{1}{2}}e^{-2i\theta}[\eta_{A_\zeta}(\vec{p}, h)\eta_{B_\zeta}(\vec{p}', -h')]\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{-i\zeta(\vec{p}'\cdot\vec{r}'-E't)} \\
&+ (-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta_{A_\zeta}(\vec{p}, -h)\eta_{B_\zeta}(\vec{p}', h')\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{i\zeta(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{p}'d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}e^{-2i\theta}\sum_h(-1)^{h-\frac{1}{2}}[\eta_{A_\zeta}(\vec{p}, h)\eta_{B_\zeta}(\vec{p}, -h)]e^{i\zeta p\cdot(x-x')} + (-1)^{h-\frac{1}{2}}\eta_{A_\zeta}(\vec{p}, -h)\eta_{B_\zeta}(\vec{p}, h)e^{-i\zeta p\cdot(x-x')}d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{E}e^{-2i\theta}\sum_h[h\eta_{A_\zeta}(\vec{p}, h)\eta_{B_\zeta}(\vec{p}, -h)][e^{i\zeta p\cdot(x-x')} - e^{-i\zeta p\cdot(x-x')}]d^3\vec{p} \\
&= i\zeta m \varepsilon_{A_\zeta B_\zeta} e^{-2i\theta} \frac{-i\zeta}{(2\pi)^3} \int \frac{1}{2E} [e^{i\zeta p\cdot(x-x')} - e^{-i\zeta p\cdot(x-x')}] d^3\vec{p} \\
&= i\zeta m e^{-2i\theta} \varepsilon_{A_\zeta B_\zeta} \Delta(x-x')
\end{aligned}$$

□

证明:

$$\begin{aligned}
& \{\nu_{A_\zeta}(\vec{r}, t), \nu_{B_\zeta}(\vec{r}', t)\} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}\{[a_\zeta(\vec{p}, h)\eta_{A_\zeta}(\vec{p}, h)e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + e^{-2i\theta}a_\zeta^+(\vec{p}, h)(-1)^{h-\frac{1}{2}}\eta_{A_\zeta}(\vec{p}, -h)e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}], \\
&[a_\zeta(\vec{p}', h')\eta_{B_\zeta}(\vec{p}', h')e^{i\zeta(\vec{p}'\cdot\vec{r}'-E't)} + e^{-2i\theta}a_\zeta^+(\vec{p}', h')(-1)^{h'-\frac{1}{2}}\eta_{B_\zeta}(\vec{p}', -h')e^{-i\zeta(\vec{p}'\cdot\vec{r}'-E't)}]\}d^3\vec{p}'d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}(-1)^{h'-\frac{1}{2}}e^{-2i\theta}[\eta_{A_\zeta}(\vec{p}, h)\eta_{B_\zeta}(\vec{p}', -h')]\{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\}[e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{-i\zeta(\vec{p}'\cdot\vec{r}'-E't)}] \\
&+ (-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta_{A_\zeta}(\vec{p}, -h)\eta_{B_\zeta}(\vec{p}', h')\{a_\zeta^+(\vec{p}, h), a_\zeta(\vec{p}', h')\}[e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{i\zeta(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{p}'d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}\sum_{h,h'}(-1)^{h'-\frac{1}{2}}e^{-2i\theta}[\eta_{A_\zeta}(\vec{p}, h)\eta_{B_\zeta}(\vec{p}', -h')]\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{-i\zeta(\vec{p}'\cdot\vec{r}'-E't)} \\
&+ (-1)^{h-\frac{1}{2}}e^{-2i\theta}\eta_{A_\zeta}(\vec{p}, -h)\eta_{B_\zeta}(\vec{p}', h')\delta_{hh'}\delta^3(\vec{p}-\vec{p}')e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}e^{i\zeta(\vec{p}'\cdot\vec{r}'-E't)}]d^3\vec{p}'d^3\vec{p} \\
&= \frac{1}{(2\pi)^3}\int\frac{m}{2E}e^{-2i\theta}\sum_h(-1)^{h-\frac{1}{2}}[\eta_{A_\zeta}(\vec{p}, h)\eta_{B_\zeta}(\vec{p}, -h)]e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} + (-1)^{h-\frac{1}{2}}\eta_{A_\zeta}(\vec{p}, -h)\eta_{B_\zeta}(\vec{p}, h)e^{-i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}
\end{aligned}$$

$$= \frac{1}{(2\pi)^3} \int \frac{m}{2E} e^{-2i\theta} e^{i\mathbf{s}\vec{p}\cdot(\vec{r}-\vec{r}')} \sum_h (-1)^{h-\frac{1}{2}} [\eta_{A_\zeta}(\vec{p}, h) \eta_{B_\zeta}(\vec{p}, -h) + \eta_{A_\zeta}(-\vec{p}, -h) \eta_{B_\zeta}(-\vec{p}, h)] d^3\vec{p}$$

$$= 0$$

□

## 5.4 有质量中微子方程反对易规则小结

推论5.4.1.

$$\begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^+(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{\psi_{s\lambda_\zeta}(x), \psi_{s\lambda'_\zeta}(x')\} = i[(m - \gamma_s^a \partial_a) \gamma_s^4]_{\lambda_\zeta \lambda'_\zeta} \Delta(x - x') \\ \{\psi_{s\lambda_\zeta}(\vec{r}, t), \psi_{s\lambda'_\zeta}(\vec{r}', t)\} = \delta_{\lambda_\zeta \lambda'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \psi_s^*(\vec{r}, t) = \psi_s(\vec{r}, t) \end{cases}$$

推论5.4.2.

$$\begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^+(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{\psi_{\lambda_\zeta}(x), \psi_{\lambda'_\zeta}^+(x')\} = i[(m - \gamma^a \partial_a) \gamma^4]_{\lambda_\zeta \lambda'_\zeta} \Delta(x - x') \\ \{\psi_{\lambda_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta}^+(\vec{r}', t)\} = \delta_{\lambda_\zeta \lambda'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \psi^*(\vec{r}, t) = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi(\vec{r}, t) \end{cases}$$

推论5.4.3.

$$\begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^+(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{\nu_{A_\zeta}(x), \nu_{A'_\zeta}^+(x')\} = -\varsigma(\sigma, i\varsigma)_{A_\zeta A'_\zeta}^a \partial_a \Delta(x - x') \\ \{\nu_{A_\zeta}(x), \nu_{B_\zeta}(x')\} = i\varsigma m e^{-2i\theta} \varepsilon_{A_\zeta B_\zeta} \Delta(x - x') \\ \{\nu_{A'_\zeta}^+(x), \nu_{B'_\zeta}^+(x')\} = -i\varsigma m e^{2i\theta} \varepsilon_{A'_\zeta B'_\zeta} \Delta(x - x') \end{cases}$$

推论5.4.4.

$$\begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^+(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{\nu_{A_\zeta}(\vec{r}, t), \nu_{A'_\zeta}^+(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\nu_{A_\zeta}(\vec{r}, t), \nu_{B_\zeta}(\vec{r}', t)\} = 0 \\ \{\nu_{A'_\zeta}^+(\vec{r}, t), \nu_{B'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases}$$

## 5.5 有质量中微子方程的三种等价描述小结

### 5.5.1 从有质量中微子方程出发构造平面波解

推论5.5.1.

$$\begin{cases} (\sigma, -i\varsigma)_a \partial^a \nu(x) - m e^{-2i\theta} \sigma_y \nu^*(x) = 0 \\ \psi(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} \nu(x) - i e^{-2i\theta} \sigma_y \nu^*(x) \\ -\nu(x) - i e^{-2i\theta} \sigma_y \nu^*(x) \end{bmatrix} \end{cases} \Leftrightarrow \begin{cases} (\gamma^a \partial_a + m) \psi(x) = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_z) \\ \psi^*(x) = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi(x) \\ \nu(x) = \frac{1}{\sqrt{2}} [\psi_1(x) + i e^{-2i\theta} \sigma_y \psi_1^*(x)] \end{cases}$$

$$\begin{cases} \nu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{E+m-\varsigma\vec{p}\cdot\sigma}{\sqrt{2m(E+m)}} \frac{1}{\sqrt{2}} (\xi_0 e^{i\varsigma p \cdot x} + i e^{-2i\theta} \sigma_y \xi_0^* e^{-i\varsigma p \cdot x}) d^3\vec{p} \\ \psi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{E+m+\varsigma\vec{p}\cdot\sigma \otimes \sigma_x}{\sqrt{2m(E+m)}} \begin{bmatrix} \xi_0 e^{i\varsigma p \cdot x} \\ -i e^{-2i\theta} \sigma_y \xi_0^* e^{-i\varsigma p \cdot x} \end{bmatrix} d^3\vec{p} = \frac{1}{(2\pi)^{3/2}} \int \begin{bmatrix} \frac{(E+m)\xi_0 e^{i\varsigma p \cdot x} - \varsigma\vec{p}\cdot\sigma (i e^{-2i\theta} \sigma_y \xi_0^*) e^{-i\varsigma p \cdot x}}{\sqrt{2m(E+m)}} \\ \frac{-(E+m)(i e^{-2i\theta} \sigma_y \xi_0^*) e^{-i\varsigma p \cdot x} + \varsigma\vec{p}\cdot\sigma \xi_0 e^{i\varsigma p \cdot x}}{\sqrt{2m(E+m)}} \end{bmatrix} d^3\vec{p} \end{cases}$$

$$\xi_0 = a(\vec{p}, \frac{1}{2}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a(\vec{p}, -\frac{1}{2}) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ 或 } \xi_0 = a(\vec{p}, -\frac{\varsigma}{2}) \lambda(\hat{p}, -\frac{\varsigma}{2}) + a(\vec{p}, \frac{\varsigma}{2}) \lambda(\hat{p}, \frac{\varsigma}{2})$$

## 6 无质量中微子方程的平面波解和量子化初步(后面章节会详细展开)

### 6.1 无质量中微子方程的平面波解

推论6.1.1.  $(\sigma, -i\varsigma)^a \partial_a \nu(\vec{r}, t) = 0$ 

$$\text{推论6.1.2. } \nu_{A_\zeta}(\vec{r}, t) = \int_{\vec{p} \neq 0} \frac{1}{2} (1 - \varsigma \frac{\vec{p}\cdot\sigma}{|\vec{p}|}) [\xi(\vec{p}) e^{i\varsigma(\vec{p}\cdot\vec{r} - Et)} + \eta(\vec{p}) e^{-i\varsigma(\vec{p}\cdot\vec{r} - Et)}] d^3\vec{p}$$

$$= \int_{\vec{p} \neq 0} \lambda(p, -\varsigma) [a_+(\vec{p}) e^{i\varsigma(\vec{p}\cdot\vec{r} - Et)} + a_-(\vec{p}) e^{-i\varsigma(\vec{p}\cdot\vec{r} - Et)}] d^3\vec{p}, \frac{\vec{p}\cdot\sigma}{|\vec{p}|} \lambda(p, -\varsigma) = -\varsigma \lambda(p, -\varsigma)$$

推论6.1.3.  $\nabla \nu(\vec{r}, t) = \int_{\vec{p} \neq 0} i\varsigma \vec{p} \lambda(p, -\varsigma) [a_+(\vec{p}) e^{i\varsigma(\vec{p}\cdot\vec{r} - Et)} - a_-(\vec{p}) e^{-i\varsigma(\vec{p}\cdot\vec{r} - Et)}] d^3\vec{p}$

$$\text{推论6.1.4. } \nu^+(\vec{r}, t) = \int_{\vec{p} \neq 0} \lambda^+(p, -\varsigma) [a_+^\dagger(\vec{p}) e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} + a_-(\vec{p}) e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)}] d^3 \vec{p}$$

$$\text{推论6.1.5. } a_+(\vec{p}) = \int \lambda^+(p, -\varsigma) \nu(\vec{r}, t) e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} d^3 \vec{r}, a_-(\vec{p}) = \int \lambda^+(p, -\varsigma) \nu(\vec{r}, t) e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)} d^3 \vec{r}$$

推论6.1.6.

$$\begin{cases} L = -i\varsigma \int \nu^+(\vec{r}, t) (\sigma, -i\varsigma) \partial_a \nu(\vec{r}, t) d^3 \vec{r} = 0 \\ H = i \int \nu^+(\vec{r}, t) \partial_t \nu(\vec{r}, t) d^3 \vec{r} = i\varsigma \int \nu^+(\vec{r}, t) \sigma \cdot \nabla \nu(\vec{r}, t) d^3 \vec{r} = \varsigma \int E(\vec{p}) [a_+^\dagger(\vec{p}) a_+(\vec{p}) - a_-(\vec{p}) a_-(\vec{p})] d^3 \vec{p} \\ \vec{P} = -i \int \nu^+(\vec{r}, t) \nabla \nu(\vec{r}, t) d^3 \vec{r} = \varsigma \int \vec{p} [a_+^\dagger(\vec{p}) a_+(\vec{p}) - a_-(\vec{p}) a_-(\vec{p})] d^3 \vec{p} \\ Q = \varsigma \int \nu^+(\vec{r}, t) \nu(\vec{r}, t) d^3 \vec{r} = \varsigma \int [a_+^\dagger(\vec{p}) a_+(\vec{p}) + a_-(\vec{p}) a_-(\vec{p})] d^3 \vec{p} \end{cases}$$

证明:

$$\begin{aligned} H &= i\varsigma \int \nu^+(\vec{r}, t) \sigma \cdot \nabla \nu(\vec{r}, t) d^3 \vec{r} = i \int \nu^+(\vec{r}, t) \partial_t \nu(\vec{r}, t) d^3 \vec{r} \\ &= i\varsigma \int d^3 \vec{r} d^3 \vec{p} d^3 \vec{p}' \\ & i\varsigma \lambda^+(\vec{p}, -\varsigma) [a_+^\dagger(\vec{p}) e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} + a_-(\vec{p}) e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)}] \sigma \cdot \vec{p}' \lambda(\vec{p}', -\varsigma) [a_+(\vec{p}') e^{i\varsigma(\vec{p}' \cdot \vec{r} - E't)} - a_-(\vec{p}') e^{-i\varsigma(\vec{p}' \cdot \vec{r} - E't)}] \\ &= \varsigma \int E(\vec{p}) [a_+^\dagger(\vec{p}) a_+(\vec{p}) - a_-(\vec{p}) a_-(\vec{p})] d^3 \vec{p} \quad \square \end{aligned}$$

证明:

$$\begin{aligned} \vec{P} &= -i \int \nu^+(\vec{r}, t) \nabla \nu(\vec{r}, t) d^3 \vec{r} \\ &= -i \int d^3 \vec{r} d^3 \vec{p} d^3 \vec{p}' \\ & i\varsigma \vec{p}' \lambda^+(\vec{p}, -\varsigma) \lambda(\vec{p}', -\varsigma) [a_+^\dagger(\vec{p}) e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} + a_-(\vec{p}) e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)}] [a_+(\vec{p}') e^{i\varsigma(\vec{p}' \cdot \vec{r} - E't)} - a_-(\vec{p}') e^{-i\varsigma(\vec{p}' \cdot \vec{r} - E't)}] \\ &= \varsigma \int \vec{p} [a_+^\dagger(\vec{p}) a_+(\vec{p}) - a_-(\vec{p}) a_-(\vec{p})] d^3 \vec{p} \quad \square \end{aligned}$$

证明:

$$\begin{aligned} Q &= \varsigma \int \nu^+(\vec{r}, t) \nu(\vec{r}, t) d^3 \vec{r} \\ &= \varsigma \int \lambda^+(\vec{p}, -\varsigma) \lambda(\vec{p}', -\varsigma) [a_+^\dagger(\vec{p}) e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} + a_-(\vec{p}) e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)}] [a_+(\vec{p}') e^{i\varsigma(\vec{p}' \cdot \vec{r} - E't)} + a_-(\vec{p}') e^{-i\varsigma(\vec{p}' \cdot \vec{r} - E't)}] d^3 \vec{r} d^3 \vec{p} d^3 \vec{p}' \\ &= \varsigma \int [a_+^\dagger(\vec{p}) a_+(\vec{p}) + a_-(\vec{p}) a_-(\vec{p})] d^3 \vec{p} \quad \square \end{aligned}$$

## 6.2 无质量中微子方程的量子化

$$\text{推论6.2.1. } \begin{cases} \{\nu_{A_\varsigma}(\vec{r}, t), \nu_{A'_\varsigma}^+(\vec{r}', t)\} = \delta_{A_\varsigma A'_\varsigma} \delta^3(\vec{r} - \vec{r}') \\ \{\nu_{A_\varsigma}(\vec{r}, t), \nu_{A'_\varsigma}(\vec{r}', t)\} = 0 \\ \{\nu_{A_\varsigma}^+(\vec{r}, t), \nu_{A'_\varsigma}^+(\vec{r}', t)\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{a_s(\vec{p}), a_{s'}^+(\vec{p}')\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ \{a_s(\vec{p}), a_{s'}(\vec{p}')\} = 0 \\ \{a_s^+(\vec{p}), a_{s'}^+(\vec{p}')\} = 0 \end{cases}$$

$$\text{推论6.2.2. } \begin{cases} : P_u : \stackrel{\text{c}}{=} \int p_u [a_+^\dagger(\vec{p}) a_+(\vec{p}) + a_-(\vec{p}) a_-(\vec{p})] d^3 \vec{p} \\ : Q : \stackrel{\text{c}}{=} \int [a_+^\dagger(\vec{p}) a_+(\vec{p}) - a_-(\vec{p}) a_-(\vec{p})] d^3 \vec{p} \end{cases}$$

$$\text{推论6.2.3. } S_a = \varepsilon_{abcd} S_{bc} P_d = \varsigma P_a$$

## 6.3 中微子Weyl方程出发回到Dirac表象

推论6.3.1.

$$(\sigma, -i\varsigma)^a \partial_a \nu(\vec{r}, t) - m e^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) = 0 \Leftrightarrow (\gamma^a \partial_a + m) \begin{bmatrix} \nu(\vec{r}, t) \\ -i e^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \end{bmatrix} = 0, \gamma_a := (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x)$$

$$\text{推论6.3.2. } (\sigma, -i\varsigma)^a \partial_a \nu(\vec{r}, t) = 0 \Leftrightarrow \gamma^a \partial_a \begin{bmatrix} \nu(\vec{r}, t) \\ -i e^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \end{bmatrix} = 0, (\gamma_a, \gamma_5) := [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_z]$$

$$\text{推论6.3.3. } \sigma_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \sigma_z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \sigma_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \sigma_y \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\text{推论6.3.4. } \gamma^a \partial_a \begin{bmatrix} \nu(\vec{r}, t) \\ 0 \end{bmatrix} = 0, (\gamma_a, \gamma_5) := [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_z],$$

$$\text{推论6.3.5. } \begin{cases} \gamma^a \partial_a \psi_M(\vec{r}, t) = 0, \psi_M(\vec{r}, t) = -e^{-2i\theta} \sigma_y \otimes \sigma_y \psi_M^*(\vec{r}, t) = \begin{bmatrix} \nu(\vec{r}, t) \\ -ie^{-2i\theta} \sigma_y \nu^*(\vec{r}, t) \end{bmatrix} \\ \gamma^a \partial_a \psi_W(\vec{r}, t) = 0, \psi_W(\vec{r}, t) = \varsigma \gamma_5 \psi_W(\vec{r}, t) = \begin{bmatrix} \nu(\vec{r}, t) \\ 0 \end{bmatrix}; (\gamma_a, \gamma_5) := [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_z] \end{cases}$$

## 7 s-自旋方程的平面波解和另类量子化方案

### 7.1 s-自旋方程的平面波解

$$\text{定理7.1.1. } [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(\vec{r}, t) = 0, S_{ab}(s, \varsigma) = i\sigma_{\varsigma ab} \sigma_{\alpha\varsigma}(s)$$

推论7.1.1.

$$\psi(\vec{r}, t) = \int_{\vec{p} \neq 0} \lambda(p, -s\varsigma) [\eta(\vec{p}, -s\varsigma) e^{i\varsigma(\vec{p}\cdot\vec{r} - Et)} + d^+(\vec{p}, -s\varsigma) e^{-i\varsigma(\vec{p}\cdot\vec{r} - Et)}] d^3\vec{p}, \frac{\sigma(s)\cdot\vec{p}}{|\vec{p}|} \lambda(p, -s\varsigma) = -s\varsigma \lambda(p, -s\varsigma)$$

### 7.2 s-自旋方程的能量动量算符

$$\text{定义7.2.1. } H := \frac{i\varsigma}{s} \int \psi^+(\vec{r}, t) \sigma(s) \cdot \nabla \psi(\vec{r}, t) d^3\vec{r} \quad \vec{P} := -i \int \psi^+(\vec{r}, t) \nabla \psi(\vec{r}, t) d^3\vec{r}$$

### 7.3 s-自旋方程的量子洛伦兹不变性

$$\text{推论7.3.1. } [\psi_A(\vec{r}, t), H] = i\frac{\varsigma}{s} \sigma(s) \cdot \nabla \psi_A(\vec{r}, t)$$

证明:  $[\psi_A(\vec{r}, t), H]$

$$\begin{aligned} &= [\psi_A(\vec{r}, t), i \int \psi_B^+(\vec{r}', t) \frac{\varsigma}{s} \sigma(s) \cdot \nabla' \delta^{BC} \psi_C(\vec{r}', t) d^3\vec{r}'] \\ &= i \int [\psi_A(\vec{r}, t), \psi_B^+(\vec{r}', t) \frac{\varsigma}{s} \sigma(s) \cdot \nabla' \delta^{BC} \psi_C(\vec{r}', t)] d^3\vec{r}' \\ &= i \int \{\psi_A(\vec{r}, t), \psi_B^+(\vec{r}', t)\} \frac{\varsigma}{s} \sigma(s) \cdot \nabla' \delta^{BC} \psi_C(\vec{r}', t) d^3\vec{r}' = i \int [\psi_A(\vec{r}, t), \psi_B^+(\vec{r}', t)] \frac{\varsigma}{s} \sigma(s) \cdot \nabla' \delta^{BC} \psi_C(\vec{r}', t) d^3\vec{r}' \\ &= i \int \delta_{AB} \delta^3(\vec{r} - \vec{r}') \frac{\varsigma}{s} \sigma(s) \cdot \nabla' \delta^{BC} \psi_C(\vec{r}', t) d^3\vec{r}' \\ &= i \frac{\varsigma}{s} \sigma(s) \cdot \nabla \psi_A(\vec{r}, t) \end{aligned} \quad \square$$

$$\text{推论7.3.2. } [\psi_A(\vec{r}, t), \vec{P}] = -i \nabla \psi_A(\vec{r}, t)$$

证明:  $[\psi_A(\vec{r}, t), \vec{P}]$

$$\begin{aligned} &= [\psi_A(\vec{r}, t), -i \int \psi_B^+(\vec{r}', t) \nabla \delta^{BC} \psi_C(\vec{r}', t) d^3\vec{r}'] \\ &= -i \int [\psi_A(\vec{r}, t), \psi_B^+(\vec{r}', t) \nabla \delta^{BC} \psi_C(\vec{r}', t)] d^3\vec{r}' \\ &= -i \int \{\psi_A(\vec{r}, t), \psi_B^+(\vec{r}', t)\} \nabla \delta^{BC} \psi_C(\vec{r}', t) d^3\vec{r}' = -i \int [\psi_A(\vec{r}, t), \psi_B^+(\vec{r}', t)] \nabla \delta^{BC} \psi_C(\vec{r}', t) d^3\vec{r}' \\ &= -i \int \delta_{AB} \delta^3(\vec{r} - \vec{r}') \nabla \delta^{BC} \psi_C(\vec{r}', t) d^3\vec{r}' \\ &= -i \nabla \psi_A(\vec{r}, t) \end{aligned} \quad \square$$

$$\text{推论7.3.3. } [\sigma(s), -is\varsigma]^a \partial_a \psi(\vec{r}, t) = 0 \Leftrightarrow -i\partial_a \psi(\vec{r}, t) = [\psi(\vec{r}, t), P_a]; \begin{cases} [\psi_A(\vec{r}, t), \psi_B^+(\vec{r}', t)]_{\pm} = \delta_{AB} \delta^3(\vec{r} - \vec{r}') \\ [\psi_A(\vec{r}, t), \psi_B(\vec{r}', t)]_{\pm} = 0 \\ [\psi_A^+(\vec{r}, t), \psi_B^+(\vec{r}', t)]_{\pm} = 0 \end{cases}$$

### 7.4 s-自旋方程与量子洛伦兹不变性的自洽性

$$\text{推论7.4.1. } [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(\vec{r}, t) = 0 \stackrel{s=\frac{1}{2}}{\Leftrightarrow} -i\partial_a \psi(\vec{r}, t) = [\psi(\vec{r}, t), P_a]; \begin{cases} \{\psi_A(\vec{r}, t), \psi_B^+(\vec{r}', t)\} = \delta_{AB} \delta^3(\vec{r} - \vec{r}') \\ \{\psi_A(\vec{r}, t), \psi_B(\vec{r}', t)\} = 0 \\ \{\psi_A^+(\vec{r}, t), \psi_B^+(\vec{r}', t)\} = 0 \end{cases}$$

只有自旋为 $\frac{1}{2}$ 时, 此方案才是自洽的。所以只有中微子才可以按此方案量子化, 其它自旋粒子不能按此量子化, 所以被排除。

# 第二十一章 标量场协变量子化方案

自我评述：在本章特别对标量场单独一章进行了详细讨论，对比经典结果印证了新量子化程式的合理性和正确性。

## 1 标量场的经典正则量子化方案 [26, 27, 39, 40]

### 1.1 实标量场的经典描述

#### 1.1.1 实标量场的拉氏密度和哈密顿密度

性质1.1.1. 拉氏密度： $\mathcal{L} = -\frac{1}{2}\partial_a\phi\partial^a\phi - \frac{1}{2}m^2\phi^2$

性质1.1.2. 能量动量张量密度： $T^{ab} = i\frac{\partial\mathcal{L}}{\partial(\partial_b\phi)}\partial^a\phi - ig^{ab}\mathcal{L}, T^{a\pi} = (\mathcal{P}, i\mathcal{H})^a, \partial_b T^{ab} = 0$

性质1.1.3. 哈密顿密度： $\mathcal{H} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - \mathcal{L} = \frac{1}{2}[\dot{\phi}^2(\vec{r}, t) + \nabla\phi(\vec{r}, t) \cdot \nabla\phi(\vec{r}, t) + m^2\phi^2(\vec{r}, t)]$

性质1.1.4. 动量密度： $\mathcal{P} = -\frac{\partial\mathcal{L}}{\partial\phi}\nabla\phi = -\dot{\phi}\nabla\phi$

#### 1.1.2 实标量场的拉氏密度和运动方程

性质1.1.5. 拉氏密度： $\mathcal{L} = -\frac{1}{2}\partial_a\phi(\vec{r}, t)\partial^a\phi(\vec{r}, t) - \frac{1}{2}m^2\phi^2(\vec{r}, t), \phi(\vec{r}, t) = \phi^*(\vec{r}, t)$

性质1.1.6. 运动方程： $(\partial_a\partial^a - m^2)\phi(\vec{r}, t) = 0, \phi(\vec{r}, t) = \phi^*(\vec{r}, t)$

#### 1.1.3 实标量场的哈密顿描述

性质1.1.7.  $\mathcal{H} = \frac{1}{2}[\pi^2(\vec{r}, t) + \partial_i\phi(\vec{r}, t)\partial^i\phi(\vec{r}, t) + m^2\phi^2(\vec{r}, t)], \pi(\vec{r}, t) = \dot{\phi}(\vec{r}, t), \phi(\vec{r}, t) = \phi^*(\vec{r}, t)$

性质1.1.8.  $\mathcal{L} = \pi(\vec{r}, t)\dot{\phi}(\vec{r}, t) - \frac{1}{2}[\pi^2(\vec{r}, t) + \partial_i\phi(\vec{r}, t)\partial^i\phi(\vec{r}, t) + m^2\phi^2(\vec{r}, t)], \pi(\vec{r}, t) = \dot{\phi}(\vec{r}, t), \phi(\vec{r}, t) = \phi^*(\vec{r}, t)$

性质1.1.9. 对易关系：
$$\begin{cases} \{\phi(\vec{r}, t), \pi(\vec{r}, t)\}_p \\ \dot{\pi}(\vec{r}, t) = \nabla^2\phi(\vec{r}, t) - m^2\phi(\vec{r}, t) = \{\pi(\vec{r}, t), H\}_p \end{cases}$$

性质1.1.10.  $\phi(\vec{k}, E)(E^2 - \vec{k}^2 - m^2) = 0 \Leftrightarrow \phi(\vec{k}, E) = a(\vec{k}, E)\delta(E^2 - \vec{k}^2 - m^2) + \phi_0(\vec{k}, E)\delta_{E^2, \vec{k}^2 + m^2}$

性质1.1.11. 运动方程：
$$\begin{cases} \dot{\phi}(\vec{r}, t) = \pi(\vec{r}, t) = \{\phi(\vec{r}, t), H\}_p \\ \dot{\pi}(\vec{r}, t) = \nabla^2\phi(\vec{r}, t) - m^2\phi(\vec{r}, t) = \{\pi(\vec{r}, t), H\}_p \end{cases}$$

#### 1.1.4 实标量场方程的平面波解 [39]

实标量场方程： $(\partial_a\partial^a - m^2)\phi(\vec{r}, t) = 0, \phi(\vec{r}, t) = \phi^*(\vec{r}, t)$  (21.1)

定理1.1.1.  $(\partial_a\partial^a - m^2)\phi(\vec{r}, t) = 0, \phi(\vec{r}, t) = \phi^*(\vec{r}, t)$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a^*(\vec{k}, \omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k}$$

证明： $(\partial_a\partial^a - m^2)\phi(\vec{r}, t) = 0, \phi(\vec{r}, t) = \phi^*(\vec{r}, t)$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k}, -\omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k}, \phi(\vec{r}, t) = \phi^*(\vec{r}, t)$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a^*(\vec{k}, \omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k}, a(-\vec{k}, -\omega_k) = a^*(\vec{k}, \omega_k) \quad \square$$

推论1.1.1.  $\phi_+(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} a(\vec{k}, \omega_k)e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} d^3\vec{k}, \phi_-(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} a^*(\vec{k}, \omega_k)e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} d^3\vec{k}$

## 1.2 实标量场的量子描述

### 1.2.1 实标量场的正则对易关系

$$\text{定理1.2.1. } \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a^+(\vec{k}, \omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k}$$

$$\Leftrightarrow \begin{cases} a(\vec{k}, \omega_k) = \frac{1}{(2\pi)^{3/2}} \int_{p=-\infty}^{+\infty} [i\dot{\phi}(\vec{r}, t) + \omega_k \phi(\vec{r}, t)] e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} d^3\vec{r} \\ a^+(\vec{k}, \omega_k) = \frac{1}{(2\pi)^{3/2}} \int_{p=-\infty}^{+\infty} [-i\dot{\phi}(\vec{r}, t) + \omega_k \phi(\vec{r}, t)] e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} d^3\vec{r} \end{cases}$$

$$\text{证明: } \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a^+(\vec{k}, \omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k}$$

$$\Leftrightarrow \begin{cases} \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a^+(\vec{k}, \omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k} \\ \dot{\phi}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{-i}{2} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} - a^+(\vec{k}, \omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{(2\pi)^{3/2}} \int_{p=-\infty}^{+\infty} \phi(\vec{r}, t) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} d^3\vec{r} = \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) + a^+(-\vec{k}, \omega_k) e^{2i\omega_k t}] \\ \frac{1}{(2\pi)^{3/2}} \int_{p=-\infty}^{+\infty} \dot{\phi}(\vec{r}, t) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} d^3\vec{r} = \frac{-i}{2} [a(\vec{k}, \omega_k) - a^+(-\vec{k}, \omega_k) e^{2i\omega_k t}] \end{cases}$$

$$\Leftrightarrow \begin{cases} a(\vec{k}, \omega_k) = \frac{1}{(2\pi)^{3/2}} \int_{p=-\infty}^{+\infty} [i\dot{\phi}(\vec{r}, t) + \omega_k \phi(\vec{r}, t)] e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} d^3\vec{r} \\ a^+(\vec{k}, \omega_k) = \frac{1}{(2\pi)^{3/2}} \int_{p=-\infty}^{+\infty} [-i\dot{\phi}(\vec{r}, t) + \omega_k \phi(\vec{r}, t)] e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} d^3\vec{r} \end{cases} \quad \square$$

从以上定理可以证明得到以下实标量粒子的正则对易关系

$$\text{推论1.2.1. } \begin{cases} [\phi(\vec{r}', t), \phi(\vec{r}, t)] = 0 \\ [\dot{\phi}(\vec{r}', t), \dot{\phi}(\vec{r}, t)] = 0 \\ [\phi(\vec{r}', t), \dot{\phi}(\vec{r}, t)] = i\delta^3(\vec{r}' - \vec{r}) \end{cases} \Leftrightarrow \begin{cases} [a(\vec{k}', \omega_k'), a(\vec{k}, \omega_k)] = 0 \\ [a^+(\vec{k}', \omega_k'), a^+(\vec{k}, \omega_k)] = 0 \\ [a(\vec{k}', \omega_k'), a^+(\vec{k}, \omega_k)] = 2\omega_k \delta^3(\vec{k}' - \vec{k}) \end{cases}$$

$$\text{定义1.2.1. } a(k) \equiv \frac{1}{\sqrt{2\omega_k}} a(\vec{k}, \omega_k), a^+(k) \equiv \frac{1}{\sqrt{2\omega_k}} a^+(\vec{k}, \omega_k)$$

$$\text{推论1.2.2. } [a(k'), a(k)] = 0, [a^+(k'), a^+(k)] = 0, [a(k'), a^+(k)] = \delta^3(\vec{k}' - \vec{k})$$

### 1.2.2 实标量场的能量算符和动量算符

$$\text{推论1.2.3. } H = \int_{\vec{k}=-\infty}^{+\infty} \omega_k \hat{N}(k) d^3\vec{k} + E(0), H^+ = H$$

$$\text{证明: } H = \int_{\vec{r}=-\infty}^{+\infty} \frac{1}{2} [\dot{\phi}^2(\vec{r}, t) + \nabla\phi(\vec{r}, t) \cdot \nabla\phi(\vec{r}, t) + m^2\phi^2(\vec{r}, t)] d^3\vec{r}$$

$$= \frac{1}{4} \int_{\vec{k}=-\infty}^{+\infty} [a(\vec{k}, \omega_k) a^+(\vec{k}, \omega_k) + a^+(\vec{k}, \omega_k) a(\vec{k}, \omega_k)] d^3\vec{k}$$

$$= \frac{1}{2} \int_{\vec{k}=-\infty}^{+\infty} \omega_k [a(k) a^+(k) + a^+(k) a(k)] d^3\vec{k} = \int_{\vec{k}=-\infty}^{+\infty} \omega_k [a^+(k) a(k) + \frac{1}{2} \delta^3(0)] d^3\vec{k}$$

$$= \int_{\vec{k}=-\infty}^{+\infty} \omega_k \hat{N}(k) d^3\vec{k} + E(0) \quad \square$$

$$\text{推论1.2.4. } P = \int_{\vec{k}=-\infty}^{+\infty} \vec{k} \hat{N}(k) d^3\vec{k}, P^+ = P$$

$$\text{证明: } P = \int_{\vec{k}=-\infty}^{+\infty} -\dot{\phi}(\vec{r}, t) \nabla \phi(\vec{r}, t) d^3 \vec{r}$$

$$= \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{4\omega_k} \vec{k} [a(\vec{k}, \omega_k) a^+(\vec{k}, \omega_k) + a^+(\vec{k}, \omega_k) a(\vec{k}, \omega_k)] d^3 \vec{k}$$

$$= \frac{1}{2} \int_{\vec{k}=-\infty}^{+\infty} \vec{k} [a(k) a^+(k) + a^+(k) a(k)] d^3 \vec{k} = \int_{\vec{k}=-\infty}^{+\infty} \vec{k} [a^+(k) a(k) + \frac{1}{2} \delta^3(0)] d^3 \vec{k}$$

$$= \int_{\vec{k}=-\infty}^{+\infty} \vec{k} a^+(k) a(k) d^3 \vec{k} = \int_{\vec{k}=-\infty}^{+\infty} \vec{k} \hat{N}(k) d^3 \vec{k} \quad \square$$

$$\text{推论1.2.5. } L(t) = -\frac{1}{4} \int_{\vec{k}=-\infty}^{+\infty} [a(\vec{k}, \omega_k) a(-\vec{k}, \omega_k) e^{-2i\omega_k t} + a^+(\vec{k}, \omega_k) a^+(-\vec{k}, \omega_k) e^{2i\omega_k t}] d^3 \vec{k}$$

### 1.2.3 实标量粒子的量子理论小结

$$\text{推论1.2.6. } \begin{cases} \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a^+(\vec{k}, \omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3 \vec{k} \\ \dot{\phi}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{-i}{2} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} - a^+(\vec{k}, \omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3 \vec{k} \\ \nabla \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{i}{2\omega_k} \vec{k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} - a^+(\vec{k}, \omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3 \vec{k} \end{cases}$$

$$\text{推论1.2.7. } H = \int_{\vec{k}=-\infty}^{+\infty} \omega_k \hat{N}(k) d^3 \vec{k} + E(0), \vec{P} = \int_{\vec{k}=-\infty}^{+\infty} \vec{k} \hat{N}(k) d^3 \vec{k}, P^a = (\vec{P}, iH)^a, \dot{P}^a = 0$$

$$\text{推论1.2.8. } [P_a, \phi(\vec{r}, t)] = i\partial_a \phi(\vec{r}, t) \Leftrightarrow \partial_a \phi(\vec{r}, t) = i[\phi(\vec{r}, t), P_a]$$

$$\text{推论1.2.9. } \dot{\phi}(\vec{r}, t) = -i[\phi(\vec{r}, t), H], \dot{\dot{\phi}}(\vec{r}, t) = -i[\dot{\phi}(\vec{r}, t), H]$$

$$\text{推论1.2.10. } \dot{\phi}(\vec{r}, t) = -i[\phi(\vec{r}, t), H] \Leftrightarrow \omega_k a(\vec{k}, \omega_k) = [a(\vec{k}, \omega_k), H] \Leftrightarrow \omega_k a^+(\vec{k}, \omega_k) = -[a^+(\vec{k}, \omega_k), H]$$

$$\text{证明: } \dot{\phi}(\vec{r}, t) = -i[\phi(\vec{r}, t), H]$$

$$\Leftrightarrow \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{-i}{2} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} - a^+(\vec{k}, \omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3 \vec{k}$$

$$= \frac{-i}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} \{ [a(\vec{k}, \omega_k), H] e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + [a^+(\vec{k}, \omega_k), H] e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)} \} d^3 \vec{k}$$

$$\Leftrightarrow \omega_k a(\vec{k}, \omega_k) = [a(\vec{k}, \omega_k), H] \Leftrightarrow \omega_k a^+(\vec{k}, \omega_k) = -[a^+(\vec{k}, \omega_k), H] \quad \square$$

$$\text{推论1.2.11. } \dot{\phi}(\vec{r}, t) = -i[\phi(\vec{r}, t), H] \Leftrightarrow a(\vec{k}, \omega_k) = [a(\vec{k}, \omega_k), H] \Leftrightarrow \omega_k a^+(\vec{k}, \omega_k) = -[a^+(\vec{k}, \omega_k), H]$$

$$\text{定义1.2.2. } a(k, t) \equiv a(k) e^{-i\omega_k t}$$

$$\text{推论1.2.12. } \dot{\phi}(\vec{r}, t) = -i[\phi(\vec{r}, t), H] \Leftrightarrow \dot{a}(k, t) = -i[a(k, t), H], \dot{a}^+(k, t) = -i[a^+(k, t), H]$$

$$\text{推论1.2.13. } \begin{cases} \dot{\phi}(\vec{r}, t) = \pi(\vec{r}, t) = \{\phi(\vec{r}, t), H\}_{\hat{p}} = -i[\phi(\vec{r}, t), H] \\ \dot{\pi}(\vec{r}, t) = \nabla^2 \phi(\vec{r}, t) - m^2 \phi(\vec{r}, t) = \{\pi(\vec{r}, t), H\}_{\hat{p}} = -i[\dot{\phi}(\vec{r}, t), H] \end{cases}$$

对于玻色子场, 经典理论的运动方程和量子理论的算符方程形式上完全一样, 但物理含义不同, 可以将前者看作后者的经典极限或量子平均。经典理论的运动方程可以写成泊松括号形式, 但无法写成对易子形式(实际上为零, 与运动方程不符); 量子理论的算符方程既可以写成泊松括号形式, 也可以写成对易子形式, 即算符形式的泊松括号就等于对易子。



### 1.3 复标量粒子的量子理论

#### 1.3.1 复标量场方程的平面波解 [39]

$$\text{复标量场方程: } (\partial_a \partial^a - m^2)\phi(\vec{r}, t) = 0 \Leftrightarrow (\nabla^2 - \partial_t^2 - m^2)\phi(\vec{r}, t) = 0 \quad (21.2)$$

$$\text{定理1.3.1. } (\partial_a \partial^a - m^2)\phi(\vec{r}, t) = 0 \Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k}, -\omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k}$$

$$\text{证明: } (\partial_a \partial^a - m^2)\phi(\vec{r}, t) = 0 \Rightarrow \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \phi(\vec{k}, E) (-\vec{k}^2 + E^2 - m^2) e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE = 0$$

$$\Leftrightarrow \phi(\vec{k}, E) (E^2 - \vec{k}^2 - m^2) = 0 \Leftrightarrow \phi(\vec{k}, E) = a(\vec{k}, E) \delta(E^2 - \vec{k}^2 - m^2) + \phi_0(\vec{k}, E) \delta_{E^2, \vec{k}^2+m^2}$$

$$\Rightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} [a(\vec{k}, E) \delta(E^2 - \vec{k}^2 - m^2) + \phi_0(\vec{k}, E) \delta_{E^2, \vec{k}^2+m^2}] e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} a(\vec{k}, E) \delta(E^2 - \vec{k}^2 - m^2) e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE, \text{ 明显洛伦兹协变}$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \frac{1}{2\omega_k} a(\vec{k}, E) [\delta(E - \omega_k) + \delta(E + \omega_k)] e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \frac{1}{2\omega_k} a(\vec{k}, E) [\delta(E - \omega_k) + \delta(E + \omega_k)] e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k}, -\omega_k) e^{i(\vec{k}\cdot\vec{r}+\omega_k t)}] d^3\vec{k}$$

$$\Leftrightarrow \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} \frac{1}{2\omega_k} [a(\vec{k}, \omega_k) e^{i(\vec{k}\cdot\vec{r}-\omega_k t)} + a(-\vec{k}, -\omega_k) e^{-i(\vec{k}\cdot\vec{r}-\omega_k t)}] d^3\vec{k} \quad \square$$

这里用了一种与一般书中不同的方法，采用了四维而非三维的傅里叶展开，清晰地展现出粒子离壳和在壳的物理概念，洛伦兹协变性也明显表现于其中，并且包含了狄拉克函数解的代数新解法。在证明的过程也看到了正能解和负能解的分解，并且负能解可以按两种含义进行理解，一是将负能解理解为一种负质量粒子，二是将负能解还是理解为正质量粒子，不过要理解为反射波，正能解理解为入射波。

$$\text{推论1.3.1. } a'(e^\varepsilon[\vec{k}, E]) \delta(E^2 - \vec{k}^2 - m^2) = e^{\frac{1}{2}\varepsilon^{ab} S_{ab}} a(\vec{k}, E) \delta(E^2 - \vec{k}^2 - m^2)$$

$$\Rightarrow a'(e^\varepsilon[\vec{k}, \omega_k]) = e^{\frac{1}{2}\varepsilon^{ab} S_{ab}} a(\vec{k}, \omega_k), a'(e^\varepsilon[\vec{k}, -\omega_k]) = e^{\frac{1}{2}\varepsilon^{ab} S_{ab}} a(\vec{k}, -\omega_k)$$

$$\text{推论1.3.2. } a(\vec{k}, E) \delta(E^2 - \vec{k}^2 - m^2) = \frac{1}{2\omega_k} a(\vec{k}, E) [\delta(E - \omega_k) + \delta(E + \omega_k)], |\vec{k}| \ll m$$

$$\approx \frac{1}{2(m + \frac{\vec{k}^2}{2m})} [a(\vec{k}, m + \frac{\vec{k}^2}{2m}) \delta(E - m - \frac{\vec{k}^2}{2m}) + a(\vec{k}, -m - \frac{\vec{k}^2}{2m}) \delta(E + m + \frac{\vec{k}^2}{2m})]$$

$$\text{推论1.3.3. } \phi(\vec{r}, t) \approx \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-K}^{+K} \frac{1}{2(m + \frac{\vec{k}^2}{2m})} [a(\vec{k}, m + \frac{\vec{k}^2}{2m}) e^{i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m} t)} e^{-imt} + a(-\vec{k}, -m - \frac{\vec{k}^2}{2m}) e^{-i(\vec{k}\cdot\vec{r} - \frac{\vec{k}^2}{2m} t)} e^{imt}] d^3\vec{k}$$

从上可知，在非相对论极限下，复标量场平面波解分成两个非相对论性的正反粒子，可以同时存在。这可以继续分析下去，是否可以证明正、负能解独自守恒？

#### 1.3.2 复标量场坐标空间基本对易关系

复标量场可以看作两个实标量场的相加，不同的是这时有内部SO(2)对称性，所以可以带守恒荷。

$$\text{定义1.3.1. } \phi(\vec{r}, t) = \frac{1}{\sqrt{2}} [\phi_1(\vec{r}, t) + i\phi_2(\vec{r}, t)], \phi^+(\vec{r}, t) = \frac{1}{\sqrt{2}} [\phi_1(\vec{r}, t) - i\phi_2(\vec{r}, t)]$$

$$\text{推论1.3.4. } \begin{cases} [\phi_1(\vec{r}', t), \phi_1(\vec{r}, t)] = 0 \\ [\dot{\phi}_1(\vec{r}', t), \dot{\phi}_1(\vec{r}, t)] = 0 \\ [\phi_1(\vec{r}', t), \dot{\phi}_1(\vec{r}, t)] = i\delta^3(\vec{r}' - \vec{r}) \end{cases}, \begin{cases} [\phi_2(\vec{r}', t), \phi_2(\vec{r}, t)] = 0 \\ [\dot{\phi}_2(\vec{r}', t), \dot{\phi}_2(\vec{r}, t)] = 0 \\ [\phi_2(\vec{r}', t), \dot{\phi}_2(\vec{r}, t)] = i\delta^3(\vec{r}' - \vec{r}) \end{cases}, \begin{cases} [\phi_1(\vec{r}', t), \phi_2(\vec{r}, t)] = 0 \\ [\dot{\phi}_1(\vec{r}', t), \dot{\phi}_2(\vec{r}, t)] = 0 \\ [\phi_1(\vec{r}', t), \dot{\phi}_2(\vec{r}, t)] = 0 \end{cases}$$

$$\text{推论1.3.5. } \begin{cases} [\phi(\vec{r}', t), \phi(\vec{r}, t)] = 0 \\ [\dot{\phi}(\vec{r}', t), \dot{\phi}(\vec{r}, t)] = 0 \\ [\phi(\vec{r}', t), \dot{\phi}(\vec{r}, t)] = 0 \end{cases}, \begin{cases} [\phi^+(\vec{r}', t), \phi^+(\vec{r}, t)] = 0 \\ [\dot{\phi}^+(\vec{r}', t), \dot{\phi}^+(\vec{r}, t)] = 0 \\ [\phi^+(\vec{r}', t), \dot{\phi}^+(\vec{r}, t)] = 0 \end{cases}, \begin{cases} [\phi(\vec{r}', t), \phi^+(\vec{r}, t)] = 0 \\ [\dot{\phi}(\vec{r}', t), \dot{\phi}^+(\vec{r}, t)] = 0 \\ [\phi(\vec{r}', t), \dot{\phi}^+(\vec{r}, t)] = i\delta^3(\vec{r}' - \vec{r}) \end{cases}$$

### 1.3.3 复标量场动量空间基本对易关系

$$\text{定义1.3.2. } a(k) = \frac{1}{\sqrt{2}}[a_1(k) + ia_2(k)], b(k) = \frac{1}{\sqrt{2}}[a_1(k) - ia_2(k)],$$

$$\text{推论1.3.6. } \begin{cases} [a_1(k'), a_1(k)] = 0, [a_1^+(k'), a_1^+(k)] = 0, [a_1(k'), a_1^+(k)] = \delta^3(\vec{k}' - \vec{k}) \\ [a_2(k'), a_2(k)] = 0, [a_2^+(k'), a_2^+(k)] = 0, [a_2(k'), a_2^+(k)] = \delta^3(\vec{k}' - \vec{k}) \\ [a_1(k'), a_2(k)] = 0, [a_1^+(k'), a_2^+(k)] = 0, [a_1(k'), a_2^+(k)] = 0 \end{cases}$$

$$\text{推论1.3.7. } \begin{cases} [a(k'), a(k)] = 0, [a^+(k'), a^+(k)] = 0, [a(k'), a^+(k)] = \delta^3(\vec{k}' - \vec{k}) \\ [b(k'), b(k)] = 0, [b^+(k'), b^+(k)] = 0, [b(k'), b^+(k)] = \delta^3(\vec{k}' - \vec{k}) \\ [a(k'), b(k)] = 0, [a^+(k'), b^+(k)] = 0, [a(k'), b^+(k)] = 0 \end{cases}$$

### 1.3.4 复标量场守恒荷

$$\text{推论1.3.8. } Q = \int_{\vec{k}=-\infty}^{+\infty} [a^+(k)a(k) - b^+(k)b(k)]d^3\vec{k} = \int_{\vec{k}=-\infty}^{+\infty} [\hat{N}_+(k) - \hat{N}_-(k)]d^3\vec{k}$$

$$\text{证明: } Q = \int_{\vec{r}=-\infty}^{+\infty} [\phi_1(\vec{r}, t)\dot{\phi}_2(\vec{r}, t) - \phi_2(\vec{r}, t)\dot{\phi}_1(\vec{r}, t)]d^3\vec{r}$$

$$\Leftrightarrow Q = \int_{\vec{k}=-\infty}^{+\infty} \frac{i}{2\omega_k} [a_1(\vec{k}, \omega_k)a_2^+(\vec{k}, \omega_k) - a_2(\vec{k}, \omega_k)a_1^+(\vec{k}, \omega_k)]d^3\vec{k}$$

$$\Leftrightarrow Q = \int_{\vec{k}=-\infty}^{+\infty} i[a_1(k)a_2^+(k) - a_2(k)a_1^+(k)]d^3\vec{k}$$

$$\Leftrightarrow Q = \int_{\vec{k}=-\infty}^{+\infty} [a^+(k)a(k) - b^+(k)b(k)]d^3\vec{k} = \int_{\vec{k}=-\infty}^{+\infty} [\hat{N}_+(k) - \hat{N}_-(k)]d^3\vec{k} \quad \square$$

$$\text{推论1.3.9. } [Q, \phi(\vec{r}, t)] = -\phi(\vec{r}, t), [Q, \phi^+(\vec{r}, t)] = \phi^+(\vec{r}, t),$$

### 1.3.5 复标量场能量动量算符

$$\text{推论1.3.10. } P^a = \int_{\vec{k}=-\infty}^{+\infty} k^a [\hat{N}_+(k) + \hat{N}_-(k)]d^3\vec{k}, [P^a, \phi(\vec{r}, t)] = i\partial^a \phi(\vec{r}, t)$$

$$\text{推论1.3.11. } [Q, P^a] = 0, [\hat{N}, P^a] = 0, [\hat{N}, Q] = 0$$

## 2 标量场的协变量子化方案 [26, 27, 39, 40]

### 2.1 标量场的守恒量

推论2.1.1.

$$\begin{aligned} H &= \int \frac{1}{2} [\dot{\phi}^+(\vec{r}, t)\dot{\phi}(\vec{r}, t) + \partial_i \phi^+(\vec{r}, t)\partial^i \phi(\vec{r}, t) + m^2 \phi^+(\vec{r}, t)\phi(\vec{r}, t)]d^3\vec{r} \\ &= \int \frac{1}{2} [\dot{\phi}(\vec{r}, t)\dot{\phi}(\vec{r}, t) + \partial_i \phi(\vec{r}, t)\partial^i \phi(\vec{r}, t) + m^2 \phi(\vec{r}, t)\phi(\vec{r}, t)]d^3\vec{r} \\ &= \int \frac{1}{2} [\dot{\phi}(\vec{r}, t)\dot{\phi}(\vec{r}, t) - \phi(\vec{r}, t)\partial_i \partial^i \phi(\vec{r}, t) + m^2 \phi(\vec{r}, t)\phi(\vec{r}, t)]d^3\vec{r} \\ &= \int \frac{1}{2} [\dot{\phi}(\vec{r}, t)\dot{\phi}(\vec{r}, t) - \phi(\vec{r}, t)\partial_t^2 \phi(\vec{r}, t)]d^3\vec{r} \\ &= \int \{\dot{\phi}(\vec{r}, t)\dot{\phi}(\vec{r}, t) - \partial_t [\dot{\phi}(\vec{r}, t)\phi(\vec{r}, t)]\}d^3\vec{r} \end{aligned}$$

$$\text{推论2.1.2. } P = - \int \dot{\phi}(\vec{r}, t)\nabla\phi(\vec{r}, t)d^3\vec{r}$$

$$\text{推论2.1.3. } M_{ij} = - \int \dot{\phi}(\vec{r}, t)(x_i \partial_j - x_j \partial_i)\phi(\vec{r}, t)d^3\vec{r}$$

## 2.2 标量场方程及其平面波解

定义2.2.1.  $(\partial_a \partial^a - m^2)\phi_\sigma(\vec{r}, t) = 0, \phi_\sigma(\vec{r}, t) = \phi_\sigma^+(\vec{r}, t)$

推论2.2.1.  $\phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2E}} [a_\sigma(\vec{p}, 0)e^{i(\vec{p}\cdot\vec{r}-Et)} + a_\sigma^+(\vec{p}, 0)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$

$$\Leftrightarrow \begin{cases} \sqrt{2E}a_\sigma(\vec{p}, 0) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} [E\phi_\sigma(\vec{r}, t) + i\dot{\phi}_\sigma(\vec{r}, t)]e^{-i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \\ \sqrt{2E}a_\sigma^+(\vec{p}, 0) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} [E\phi_\sigma(\vec{r}, t) - i\dot{\phi}_\sigma(\vec{r}, t)]e^{i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \end{cases}$$

推论2.2.2.  $\partial_t \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{-iE}{\sqrt{2E}} [a_\sigma(\vec{p}, 0)e^{i(\vec{p}\cdot\vec{r}-Et)} - a_\sigma^+(\vec{p}, 0)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$

推论2.2.3.  $\partial_i \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{ip_i}{\sqrt{2E}} [a_\sigma(\vec{p}, 0)e^{i(\vec{p}\cdot\vec{r}-Et)} - a_\sigma^+(\vec{p}, 0)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$

## 2.3 标量场数学上一般的协变对易规则

定理2.3.1.

$$\begin{cases} [a_\sigma(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)]_\pm = \delta_\sigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)]_\pm = 0, [a_\sigma^+(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)]_\pm = 0 \end{cases} \Rightarrow [\phi_\sigma(x), \phi_{\sigma'}(x')]_\pm = i\delta_{\sigma\sigma'} \Delta(x - x')$$

证明:  $[\phi_\sigma^{(+)}(x), \phi_{\sigma'}^{(+)}(x')]_\pm = [\phi_\sigma^{(+)}(x), \phi_{\sigma'}^{(-)}(x')]_\pm$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \left[ \frac{1}{\sqrt{2E}} a_\sigma(\vec{p}, 0) e^{ipx}, \frac{1}{\sqrt{2E_{p'}}} a_{\sigma'}^+(\vec{p}', 0) e^{-ip'x'} \right]_\pm d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E_{p'}}} [a_\sigma(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)]_\pm e^{ipx} e^{-ip'x'} d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E_{p'}}} \delta_\sigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') e^{ipx} e^{-ip'x'} d^3\vec{p} d^3\vec{p}' \\ &= i\delta_\sigma \delta_{\sigma\sigma'} \frac{-i}{(2\pi)^3} \int \frac{1}{2E} e^{ip(x-x')} d^3\vec{p} \\ &= i\delta_\sigma \delta_{\sigma\sigma'} \Delta^{(+)}(x - x') \end{aligned}$$

□

证明:  $[\phi_\sigma^{(-)}(x), \phi_{\sigma'}^{(+)}(x')]_\pm = [\phi_\sigma^{(-)}(x), \phi_{\sigma'}^{(+)}(x')]_\pm$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \left[ \frac{1}{\sqrt{2E}} a_\sigma^+(\vec{p}, 0) e^{-ipx}, \frac{1}{\sqrt{2E_{p'}}} a_{\sigma'}(\vec{p}', 0) e^{ip'x'} \right]_\pm d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E_{p'}}} [a_\sigma^+(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)]_\pm e^{-ipx} e^{ip'x'} d^3\vec{p} d^3\vec{p}' \\ &= \pm \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E_{p'}}} \delta_\sigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') e^{-ipx} e^{ip'x'} d^3\vec{p} d^3\vec{p}' \\ &= -\pm i\delta_\sigma \delta_{\sigma\sigma'} \frac{i}{(2\pi)^3} \int \frac{1}{2E} e^{-ip(x-x')} d^3\vec{p} \\ &= -\pm i\delta_\sigma \delta_{\sigma\sigma'} \Delta^{(-)}(x - x') \end{aligned}$$

□

证明:

$$\begin{aligned} [\phi_\sigma(x), \phi_{\sigma'}(x')]_\pm &= \frac{1}{(2\pi)^3} \int \left[ \frac{1}{\sqrt{2E}} [a_\sigma(\vec{p}, 0)e^{ipx} + a_\sigma^+(\vec{p}, 0)e^{-ipx}], \frac{1}{\sqrt{2E_{p'}}} [a_{\sigma'}(\vec{p}', 0)e^{ip'x'} + a_{\sigma'}^+(\vec{p}', 0)e^{-ip'x'}] \right]_\pm d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E_{p'}}} \{ [a_\sigma(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)]_\pm e^{ipx} e^{-ip'x'} + [a_\sigma^+(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)]_\pm e^{-ipx} e^{ip'x'} \} d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E_{p'}}} [\delta_\sigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') e^{ipx} e^{-ip'x'} \pm \delta_\sigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') e^{-ipx} e^{ip'x'}] d^3\vec{p} d^3\vec{p}' \\ &= i\delta_\sigma \delta_{\sigma\sigma'} \frac{-i}{(2\pi)^3} \int \frac{1}{2E} [e^{ip(x-x')} \pm e^{-ip(x-x')}] d^3\vec{p} \\ &= i\delta_\sigma \delta_{\sigma\sigma'} [\Delta^{(+)}(x - x') - \pm \Delta^{(-)}(x - x')] \\ &= i\delta_\sigma \delta_{\sigma\sigma'} [(1 \pm 1)\Delta^{(+)}(x - x') - \pm \Delta(x - x')] \end{aligned}$$

□

从上式可知, 只有 $1 \pm 1 = 0$ 时, 才满足微观因果性, 同时只有 $\delta_\sigma \geq 0$ 时, 才满足几率非负性。所以数学上多种协变对易或反对易方案中, 物理上合理的只有一种: 即 $\delta_\sigma = 1$ (如果要求各标量场间平等, 不是1就可以统一归一化), 且满足对易关系。其实还有两种, 即 $\delta_\sigma = 0$ , 且满足对易或反对易关系, 就是经典情形。

## 2.4 标量场物理的协变对易规则

$$\text{定理2.4.1. } \begin{cases} [a_\sigma(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)] = 0, [a_\sigma^+(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = 0 \end{cases} \Rightarrow [\phi_\sigma(x), \phi_{\sigma'}(x')] = i\delta_{\sigma\sigma'} \Delta(x - x')$$

证明:

$$\begin{aligned} [\phi_\sigma(x), \phi_{\sigma'}(x')] &= \frac{1}{(2\pi)^3} \int \left[ \frac{1}{\sqrt{2E}} [a_\sigma(\vec{p}, 0) e^{ipx} + a_\sigma^+(\vec{p}, 0) e^{-ipx}], \frac{1}{\sqrt{2E'}} [a_{\sigma'}(\vec{p}', 0) e^{ip'x'} + a_{\sigma'}^+(\vec{p}', 0) e^{-ip'x'}] \right] d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E'}} \{ [a_\sigma(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] e^{ipx} e^{-ip'x'} + [a_\sigma^+(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)] e^{-ipx} e^{ip'x'} \} d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E'}} \{ \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') e^{ipx} e^{-ip'x'} - \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') e^{-ipx} e^{ip'x'} \} d^3\vec{p} d^3\vec{p}' \\ &= i\delta_{\sigma\sigma'} \frac{-i}{(2\pi)^3} \int \frac{1}{2E} [e^{ip(x-x')} - e^{-ip(x-x')}] d^3\vec{p} \\ &= i\delta_{\sigma\sigma'} \Delta(x - x') \end{aligned} \quad \square$$

## 2.5 标量场的等时对易规则

$$\text{推论2.5.1. } [\phi_\sigma(x), \phi_{\sigma'}(x')] = i\delta_{\sigma\sigma'} \Delta(x - x') \Rightarrow \begin{cases} [\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] = i\delta_{\sigma\sigma'} \delta^3(\vec{r} - \vec{r}') \\ [\phi_\sigma(\vec{r}, t), \phi_{\sigma'}(\vec{r}', t)] = 0, [\dot{\phi}_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] = 0 \end{cases}$$

推论2.5.2.

$$\begin{cases} [\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] = i\delta_{\sigma\sigma'} \delta^3(\vec{r} - \vec{r}') \\ [\phi_\sigma(\vec{r}, t), \phi_{\sigma'}(\vec{r}', t)] = 0, [\dot{\phi}_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] = 0 \end{cases} \Rightarrow \begin{cases} [a_\sigma(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)] = 0, [a_\sigma^+(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = 0 \end{cases}$$

证明:  $[a_\sigma(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)]$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E'}} \{ [E\phi_\sigma(\vec{r}, t), -i\dot{\phi}_{\sigma'}(\vec{r}', t)] + [i\dot{\phi}_\sigma(\vec{r}, t), E'_{\sigma'}\phi_{\sigma'}(\vec{r}', t)] \} e^{-i(\vec{p}\cdot\vec{r} - E_{\sigma'}t)} e^{i(\vec{p}'\cdot\vec{r}' - E'_{\sigma'}t)} d^3\vec{r} d^3\vec{r}' \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E'}} \{ E\delta_{\sigma\sigma'} \delta^3(\vec{r} - \vec{r}') + E'_{\sigma'} \delta_{\sigma\sigma'} \delta^3(\vec{r} - \vec{r}') \} e^{-i(\vec{p}\cdot\vec{r} - E_{\sigma'}t)} e^{i(\vec{p}'\cdot\vec{r}' - E'_{\sigma'}t)} d^3\vec{r} d^3\vec{r}' \\ &= \frac{1}{(2\pi)^3} \int \delta_{\sigma\sigma'} \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E'}} (E + E'_{\sigma'}) e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}} e^{iE'_{\sigma'}t} e^{-iE_{\sigma'}t} d^3\vec{r} \\ &= \delta_{\sigma\sigma'} \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E'}} (E + E'_{\sigma'}) \delta^3(\vec{p} - \vec{p}') e^{iE'_{\sigma'}t} e^{-iE_{\sigma'}t} \\ &= \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \end{aligned} \quad \square$$

证明:  $[a_\sigma(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)]$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E'}} \{ [E\phi_\sigma(\vec{r}, t), -i\dot{\phi}_{\sigma'}(\vec{r}', t)] + [-i\dot{\phi}_\sigma(\vec{r}, t), E'_{\sigma'}\phi_{\sigma'}(\vec{r}', t)] \} e^{-i(\vec{p}\cdot\vec{r} - E_{\sigma'}t)} e^{-i(\vec{p}'\cdot\vec{r}' - E'_{\sigma'}t)} d^3\vec{r} d^3\vec{r}' \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E'}} \{ E\delta_{\sigma\sigma'} \delta^3(\vec{r} - \vec{r}') - E'_{\sigma'} \delta_{\sigma\sigma'} \delta^3(\vec{r} - \vec{r}') \} e^{-i(\vec{p}\cdot\vec{r} - E_{\sigma'}t)} e^{-i(\vec{p}'\cdot\vec{r}' - E'_{\sigma'}t)} d^3\vec{r} d^3\vec{r}' \\ &= \frac{1}{(2\pi)^3} \int \delta_{\sigma\sigma'} \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E'}} (E - E'_{\sigma'}) e^{-i(\vec{p}+\vec{p}')\cdot\vec{r}} e^{iE'_{\sigma'}t} e^{-iE_{\sigma'}t} d^3\vec{r} \\ &= \delta_{\sigma\sigma'} \frac{1}{\sqrt{2E}} \frac{1}{\sqrt{2E'}} (E - E'_{\sigma'}) \delta^3(\vec{p} + \vec{p}') e^{iE'_{\sigma'}t} e^{-iE_{\sigma'}t} \\ &= 0 \end{aligned} \quad \square$$

## 2.6 标量场对易规则小结

以上几个小节的证明正好形成一个逻辑闭环, 故有如下性质:

$$\text{推论2.6.1. } \begin{cases} [a_\sigma(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)] = 0, [a_\sigma^+(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = 0 \end{cases} \Leftrightarrow \begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0, [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')] = 0 \end{cases}$$

$$\quad \Updownarrow \quad \quad \quad \Updownarrow$$

$$\text{推论2.6.2. } \begin{cases} [\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] = i\delta_{\sigma\sigma'} \delta^3(\vec{r} - \vec{r}') \\ [\phi_\sigma(\vec{r}, t), \phi_{\sigma'}(\vec{r}', t)] = 0, [\dot{\phi}_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] = 0 \end{cases} \Leftrightarrow [\phi_\sigma(x), \phi_{\sigma'}(x')] = i\delta_{\sigma\sigma'} \Delta(x - x')$$

## 2.7 单复标量场方程及其平面波解

定义2.7.1.  $(\partial_a \partial^a - m^2)\phi(\vec{r}, t) = 0$

推论2.7.1.  $\phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2E}} [b_1(\vec{p}, 0)e^{i(\vec{p}\cdot\vec{r}-Et)} + b_2^+(\vec{p}, 0)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$

$$\Leftrightarrow \begin{cases} \sqrt{2E}b_1(\vec{p}, 0) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} [E\phi(\vec{r}, t) + i\dot{\phi}(\vec{r}, t)]e^{-i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \\ \sqrt{2E}b_2^+(\vec{p}, 0) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} [E\phi(\vec{r}, t) - i\dot{\phi}(\vec{r}, t)]e^{i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \end{cases}$$

## 2.8 单个复标量场对易规则

$$\text{推论2.8.1.} \begin{cases} [b_\sigma(\vec{p}, 0), b_\sigma^+(\vec{p}', 0)] = \delta_{\sigma\sigma'}\delta^3(\vec{p} - \vec{p}') \\ [b_\sigma(\vec{p}, 0), b_{\sigma'}(\vec{p}', 0)] = 0, [b_\sigma^+(\vec{p}, 0), b_{\sigma'}^+(\vec{p}', 0)] = 0 \\ b_1(\vec{p}, 0) = \frac{1}{\sqrt{2}}[a_1(\vec{p}, 0) + ia_2(\vec{p}, 0)] \\ b_2(\vec{p}, 0) = \frac{1}{\sqrt{2}}[a_1(\vec{p}, 0) - ia_2(\vec{p}, 0)] \end{cases} \Leftrightarrow \begin{cases} [b_\sigma(\vec{p}), b_\sigma^+(\vec{p}')] = \delta_{\sigma\sigma'}\delta^3(\vec{p} - \vec{p}') \\ [b_\sigma(\vec{p}), b_{\sigma'}(\vec{p}')] = 0, [b_\sigma^+(\vec{p}), b_{\sigma'}^+(\vec{p}')] = 0 \\ b_1(\vec{p}) = \frac{1}{\sqrt{2}}[a_1(\vec{p}) + ia_2(\vec{p})] \\ b_2(\vec{p}) = \frac{1}{\sqrt{2}}[a_1(\vec{p}) - ia_2(\vec{p})] \end{cases}$$

⇕

$$\text{推论2.8.2.} \begin{cases} [a_\sigma(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = \delta_{\sigma\sigma'}\delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)] = 0 \\ [a_\sigma^+(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = 0 \end{cases} \Leftrightarrow \begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')] = \delta_{\sigma\sigma'}\delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')] = 0 \end{cases}$$

⇕

$$\text{推论2.8.3.} \begin{cases} [\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] = i\delta_{\sigma\sigma'}\delta^3(\vec{r} - \vec{r}') \\ [\phi_\sigma(\vec{r}, t), \phi_{\sigma'}(\vec{r}', t)] = 0 \\ [\dot{\phi}_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] = 0 \end{cases} \Leftrightarrow \begin{cases} [\phi_\sigma(x), \phi_{\sigma'}(x')] = i\delta_{\sigma\sigma'}\Delta(x - x') \\ \phi_1(x) = \frac{1}{\sqrt{2}}[\phi(x) + \phi^+(x)] \\ \phi_2(x) = \frac{1}{i\sqrt{2}}[\phi(x) - \phi^+(x)] \end{cases}$$

⇕

$$\text{推论2.8.4.} \begin{cases} [\phi(\vec{r}, t), \dot{\phi}^+(\vec{r}', t)] = i\delta^3(\vec{r} - \vec{r}') \\ [\phi(\vec{r}, t), \phi(\vec{r}', t)] = 0, [\phi^+(\vec{r}, t), \phi^+(\vec{r}', t)] = 0 \\ [\dot{\phi}(\vec{r}, t), \dot{\phi}(\vec{r}', t)] = 0, [\dot{\phi}^+(\vec{r}, t), \dot{\phi}^+(\vec{r}', t)] = 0 \\ [\phi(\vec{r}, t), \phi^+(\vec{r}', t)] = 0, [\dot{\phi}(\vec{r}, t), \dot{\phi}^+(\vec{r}', t)] = 0 \end{cases} \Leftrightarrow \begin{cases} [\phi(x), \phi^+(x')] = i\Delta(x - x') \\ [\phi(x), \phi(x')] = 0, [\phi^+(x), \phi^+(x')] = 0 \\ \phi(x) = \frac{1}{\sqrt{2}}[\phi_1(x) + i\phi_2(x)] \\ \phi^+(x) = \frac{1}{\sqrt{2}}[\phi_1(x) - i\phi_2(x)] \end{cases}$$

⇕

$$\text{推论2.8.5.} \begin{cases} [\psi(\vec{r}, t), \dot{\psi}^+(\vec{r}', t)] = 2i\delta^3(\vec{r} - \vec{r}') \\ [\psi(\vec{r}, t), \psi(\vec{r}', t)] = 0, [\psi^+(\vec{r}, t), \psi^+(\vec{r}', t)] = 0 \\ [\dot{\psi}(\vec{r}, t), \dot{\psi}(\vec{r}', t)] = 0, [\dot{\psi}^+(\vec{r}, t), \dot{\psi}^+(\vec{r}', t)] = 0 \\ [\psi(\vec{r}, t), \psi^+(\vec{r}', t)] = 0, [\dot{\psi}(\vec{r}, t), \dot{\psi}^+(\vec{r}', t)] = 0 \end{cases} \Leftrightarrow \begin{cases} [\psi(x), \psi^+(x')] = 2i\Delta(x - x') \\ [\psi(x), \psi(x')] = 0, [\psi^+(x), \psi^+(x')] = 0 \\ \psi(x) = \phi_1(x) + i\phi_2(x) \\ \psi^+(x) = \phi_1(x) - i\phi_2(x) \end{cases}$$

## 2.9 无质量标量场的因果函数

$$\text{定义2.9.1.} \begin{cases} \Delta^{(+)}(x) := \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} e^{ip\cdot x} d^3\vec{p}, i\Delta^{(+)}(\vec{r}, 0) \leftrightarrow \frac{1}{2|\vec{p}|} \\ \Delta^{(-)}(x) := -\frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} e^{-ip\cdot x} d^3\vec{p}, \Delta^{(-)}(x) = -\Delta^{(+)}(-x) \\ \Delta(x) := \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} [e^{ip\cdot x} - e^{-ip\cdot x}] d^3\vec{p}, \Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) \end{cases}$$

$$\text{性质2.9.1.} \quad \begin{cases} \Delta^*(x) = \Delta(x), \Delta(-x) = -\Delta(x), (\nabla^2 - \partial_t^2)\Delta(x) = 0 \\ \partial_t \Delta(x)|_{t=0} = -\delta^3(\vec{r}), \partial_k \partial_t \Delta(x)|_{t=0} = \partial_t \partial_k \Delta(x)|_{t=0} = -\partial_k \delta^3(\vec{r}) \\ \partial_k \Delta(x)|_{t=0} = 0, \partial_k \partial_l \Delta(x)|_{t=0} = 0, \partial_t^2 \Delta(x)|_{t=0} = 0 \end{cases}$$

$$\text{性质2.9.2.} \quad \Delta(x-x') := \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3 \vec{p}$$

$$\begin{cases} \partial_u \Delta(x-x') = -\partial'_u \Delta(x-x') & \begin{cases} (\sqrt{-\nabla^2})^n \Delta(x-x') = (\sqrt{-\nabla'^2})^n \Delta(x-x') \\ \frac{1}{(\sqrt{-\nabla^2})^n} \Delta(x-x') = \frac{1}{(\sqrt{-\nabla'^2})^n} \Delta(x-x') \\ \partial_\pi^{2n} \Delta(x-x') = \partial_\pi'^{2n} \Delta(x-x') \end{cases} \\ \nabla \Delta(x-x') = -\nabla' \Delta(x-x') \\ \partial_\pi \Delta(x-x') = -\partial'_\pi \Delta(x-x') \end{cases}$$

## 2.10 标量场的对易函数、因果函数和费曼传播子

定义2.10.1.

$$\begin{cases} \Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) = \Delta^{ret}(x) - \Delta^{adv}(x) & \begin{cases} \Delta^{(c)}(x) = \theta(t)\Delta^{(+)}(x) - \theta(-t)\Delta^{(-)}(x) \\ \Delta^{ret}(x) = \theta(t)\Delta(x) = \Delta^{(c)}(x) + \Delta^{(-)}(x) \\ \Delta^{adv}(x) = -\theta(-t)\Delta(x) = \Delta^{(c)}(x) - \Delta^{(+)}(x) \end{cases} \\ \Delta^{(l)}(x) = i[\Delta^{(-)}(x) - \Delta^{(+)}(x)] \\ \Delta_F(x) = \langle T\varphi(x)\varphi(x') \rangle_0 = i\Delta^{(c)}(x-x') \end{cases}$$

推论2.10.1.

$$\begin{cases} (\partial_a \partial^a - m^2)\Delta(x) = 0 & \begin{cases} (\partial_a \partial^a - m^2)\Delta^{(c)}(x) = \delta^4(x) \\ (\partial_a \partial^a - m^2)\Delta^{ret}(x) = \delta^4(x) \\ (\partial_a \partial^a - m^2)\Delta^{adv}(x) = \delta^4(x) \\ (\partial_a \partial^a - m^2)\Delta_F(x) = i\delta^4(x) \end{cases} \\ (\partial_a \partial^a - m^2)\Delta^{(+)}(x) = 0 \\ (\partial_a \partial^a - m^2)\Delta^{(-)}(x) = 0 \\ (\partial_a \partial^a - m^2)\Delta^{(l)}(x) = 0 \end{cases}$$

## 3 标量场各种算符的提取

### 3.1 标量场能量动量算符的提取

推论3.1.1.

$$H = \frac{1}{2} \int \sum_{\vec{\sigma}} E[a^+(\vec{p}, 0)a(\vec{p}, 0) + a(\vec{p}, 0)a^+(\vec{p}, 0)] d^3 \vec{p} = \frac{1}{2} \int \sum_{\vec{\sigma}} [\nabla \phi_{\sigma}(\vec{r}, t)]^2 + \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2 \phi_{\sigma}^2(\vec{r}, t)] d^3 \vec{r}$$

$$\vec{P} = \frac{1}{2} \int \sum_{\vec{\sigma}} \vec{p} [a_{\sigma}^+(\vec{p}, 0)a_{\sigma}(\vec{p}, 0) + a_{\sigma}(\vec{p}, 0)a_{\sigma}^+(\vec{p}, 0)] d^3 \vec{p} = \int \sum_{\vec{\sigma}} -\dot{\phi}_{\sigma}(\vec{r}, t) \nabla \phi_{\sigma}(\vec{r}, t) d^3 \vec{r}$$

$$\begin{aligned} \text{证明: } H &= \frac{1}{2} \int \sum_{\vec{\sigma}} E[a_{\sigma}^+(\vec{p}, 0)a_{\sigma}(\vec{p}, 0) + a_{\sigma}(\vec{p}, 0)a_{\sigma}^+(\vec{p}, 0)] d^3 \vec{p} \\ &= \frac{1}{2} \frac{1}{(2\pi)^3} \int \sum_{\vec{\sigma}} \frac{1}{2} \{ [E\phi_{\sigma}(\vec{r}, t) - i\dot{\phi}_{\sigma}(\vec{r}, t)][E\phi_{\sigma}(\vec{r}', t) + i\dot{\phi}_{\sigma}(\vec{r}', t)] e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} \\ &\quad + [E\phi_{\sigma}(\vec{r}, t) + i\dot{\phi}_{\sigma}(\vec{r}, t)][E\phi_{\sigma}(\vec{r}', t) - i\dot{\phi}_{\sigma}(\vec{r}', t)] e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')} \} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{4} \frac{1}{(2\pi)^3} \int \sum_{\vec{\sigma}} \{ [E_p^2 \phi_{\sigma}(\vec{r}, t)\phi_{\sigma}(\vec{r}', t) + \dot{\phi}_{\sigma}(\vec{r}, t)\dot{\phi}_{\sigma}(\vec{r}', t)] [e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} + e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')}] \\ &\quad + iE[\phi_{\sigma}(\vec{r}, t)\dot{\phi}_{\sigma}(\vec{r}', t) - \dot{\phi}_{\sigma}(\vec{r}, t)\phi_{\sigma}(\vec{r}', t)] [e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} - e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')}] \} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{4} \frac{1}{(2\pi)^3} \int \sum_{\vec{\sigma}} \{ [m^2 \phi_{\sigma}(\vec{r}, t)\phi_{\sigma}(\vec{r}', t) + \dot{\phi}_{\sigma}(\vec{r}, t)\dot{\phi}_{\sigma}(\vec{r}', t)] [e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} + e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')}] \\ &\quad - \phi_{\sigma}(\vec{r}, t)\phi_{\sigma}(\vec{r}', t) \nabla^2 [e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} + e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')}] + 0 \} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{4} \frac{1}{(2\pi)^3} \int \sum_{\vec{\sigma}} \{ [m^2 \phi_{\sigma}(\vec{r}, t)\phi_{\sigma}(\vec{r}', t) + \dot{\phi}_{\sigma}(\vec{r}, t)\dot{\phi}_{\sigma}(\vec{r}', t)] \delta^3(\vec{r}-\vec{r}') - \phi_{\sigma}(\vec{r}, t)\phi_{\sigma}(\vec{r}', t) \nabla^2 \delta^3(\vec{r}-\vec{r}') \} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{2} \int \sum_{\vec{\sigma}} [-\nabla^2 \phi_{\sigma}(\vec{r}, t)\phi_{\sigma}(\vec{r}, t) + \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2 \phi_{\sigma}^2(\vec{r}, t)] d^3 \vec{r} \\ &= \frac{1}{2} \int \sum_{\vec{\sigma}} [-\partial^i [\partial_i \phi_{\sigma}(\vec{r}, t)\phi_{\sigma}(\vec{r}, t)] + [\nabla \phi_{\sigma}(\vec{r}, t)]^2 + \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2 \phi_{\sigma}^2(\vec{r}, t)] d^3 \vec{r} \\ &= \frac{1}{2} \int \sum_{\vec{\sigma}} [\nabla \phi_{\sigma}(\vec{r}, t)]^2 + \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2 \phi_{\sigma}^2(\vec{r}, t)] d^3 \vec{r} \quad \square \end{aligned}$$

$$\begin{aligned} \text{证明: } \vec{P} &= \frac{1}{2} \int \sum_{\vec{\sigma}} \vec{p} [a_{\sigma}^+(\vec{p}, 0)a_{\sigma}(\vec{p}, 0) + a_{\sigma}(\vec{p}, 0)a_{\sigma}^+(\vec{p}, 0)] d^3 \vec{p} \\ &= \frac{1}{2} \frac{1}{(2\pi)^3} \int \sum_{\vec{\sigma}} \frac{\vec{p}}{2E} \{ [E\phi_{\sigma}(\vec{r}, t) - i\dot{\phi}_{\sigma}(\vec{r}, t)][E\phi_{\sigma}(\vec{r}', t) + i\dot{\phi}_{\sigma}(\vec{r}', t)] e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} \end{aligned}$$

$$\begin{aligned}
& + [E\phi_\sigma(\vec{r}, t) + i\dot{\phi}_\sigma(\vec{r}, t)][E\phi_\sigma(\vec{r}', t) - i\dot{\phi}_\sigma(\vec{r}', t)]e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} \} d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \sum_{\sigma} \frac{\vec{p}}{E} \{ [E_p^2\phi_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t) + \dot{\phi}_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t)] [e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] \\
& + iE[\phi_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t) - \dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)] [e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] \} d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \sum_{\sigma} i\vec{p} [\phi_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t) - \dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)] [e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{2} \frac{1}{(2\pi)^3} \int \sum_{\sigma} i\vec{p} [\phi_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t) - \dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{2} \frac{1}{(2\pi)^3} \int \sum_{\sigma} [\phi_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t) - \dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)] \nabla e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{2} \int \sum_{\sigma} [\phi_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t) - \dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)] \nabla \delta^3(\vec{r}-\vec{r}') d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{2} \int \sum_{\sigma} [-\nabla\phi_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}, t) + \nabla\dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}, t)] d^3\vec{r} \\
& = \int \sum_{\sigma} -\dot{\phi}_\sigma(\vec{r}, t)\nabla\phi_\sigma(\vec{r}, t) d^3\vec{r} + \frac{1}{2} \int \sum_{\sigma} \nabla[\dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}, t)] d^3\vec{r} \\
& = \int \sum_{\sigma} -\dot{\phi}_\sigma(\vec{r}, t)\nabla\phi_\sigma(\vec{r}, t) d^3\vec{r}
\end{aligned}$$

□

### 3.2 标量场自旋算符和粒子数算符的提取

$$\text{推论3.2.1. } \hat{S} = \frac{1}{2} \int \sum_{\sigma} E [a_{\sigma}^+(\vec{p}, 0)a_{\sigma}(\vec{p}, 0) - a_{\sigma}(\vec{p}, 0)a_{\sigma}^+(\vec{p}, 0)] d^3\vec{p} = \frac{i}{2} \int \sum_{\sigma} [\phi_\sigma(\vec{r}, t), \dot{\phi}_\sigma(\vec{r}, t)] d^3\vec{r}$$

$$\begin{aligned}
\text{证明: } \hat{S} & = \frac{1}{2} \int \sum_{\sigma} E [a_{\sigma}^+(\vec{p}, 0)a_{\sigma}(\vec{p}, 0) - a_{\sigma}(\vec{p}, 0)a_{\sigma}^+(\vec{p}, 0)] d^3\vec{p} \\
& = \frac{1}{2} \frac{1}{(2\pi)^3} \int \sum_{\sigma} \frac{1}{2E} \{ [E\phi_\sigma(\vec{r}, t) - i\dot{\phi}_\sigma(\vec{r}, t)][E\phi_\sigma(\vec{r}', t) + i\dot{\phi}_\sigma(\vec{r}', t)] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} \\
& - [E\phi_\sigma(\vec{r}, t) + i\dot{\phi}_\sigma(\vec{r}, t)][E\phi_\sigma(\vec{r}', t) - i\dot{\phi}_\sigma(\vec{r}', t)] e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} \} d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \sum_{\sigma} \frac{1}{E} \{ [E_p^2\phi_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t) + \dot{\phi}_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t)] [e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] \\
& + iE[\phi_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t) - \dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)] [e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] \} d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \sum_{\sigma} i [\phi_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t) - \dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)] [e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{2} \int \sum_{\sigma} i [\phi_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t) - \dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)] \delta^3(\vec{r}-\vec{r}') d^3\vec{r}d^3\vec{r}' \\
& = \frac{i}{2} \int \sum_{\sigma} [\phi_\sigma(\vec{r}, t), \dot{\phi}_\sigma(\vec{r}, t)] d^3\vec{r}
\end{aligned}$$

□

$$\text{推论3.2.2. } \hat{N} = \frac{1}{2} \int \sum_{\sigma} E [a_{\sigma}^+(\vec{p}, 0)a_{\sigma}(\vec{p}, 0) + a_{\sigma}(\vec{p}, 0)a_{\sigma}^+(\vec{p}, 0)] d^3\vec{p}$$

$$= \frac{1}{2} \int \sum_{\sigma} \frac{1}{\sqrt{m^2-\nabla^2}} [\nabla\phi_\sigma(\vec{r}, t)]^2 + \dot{\phi}_\sigma^2(\vec{r}, t) + m^2\phi_\sigma^2(\vec{r}, t)] d^3\vec{r}$$

$$\begin{aligned}
\text{证明: } \hat{N} & = \frac{1}{2} \int \sum_{\sigma} E [a_{\sigma}^+(\vec{p}, 0)a_{\sigma}(\vec{p}, 0) + a_{\sigma}(\vec{p}, 0)a_{\sigma}^+(\vec{p}, 0)] d^3\vec{p} \\
& = \frac{1}{2} \frac{1}{(2\pi)^3} \int \sum_{\sigma} \frac{1}{2E} \{ [E\phi_\sigma(\vec{r}, t) - i\dot{\phi}_\sigma(\vec{r}, t)][E\phi_\sigma(\vec{r}', t) + i\dot{\phi}_\sigma(\vec{r}', t)] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} \\
& + [E\phi_\sigma(\vec{r}, t) + i\dot{\phi}_\sigma(\vec{r}, t)][E\phi_\sigma(\vec{r}', t) - i\dot{\phi}_\sigma(\vec{r}', t)] e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} \} d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \sum_{\sigma} \frac{1}{E} \{ [E_p^2\phi_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t) + \dot{\phi}_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t)] [e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] \\
& + iE[\phi_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t) - \dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)] [e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] \} d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \sum_{\sigma} \frac{1}{E} \{ [m^2\phi_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t) + \dot{\phi}_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t)] [e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] \\
& - \phi_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)\nabla^2 [e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] + 0 \} d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \sum_{\sigma} \frac{1}{\sqrt{m^2-\nabla^2}} \{ [m^2\phi_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t) + \dot{\phi}_\sigma(\vec{r}, t)\dot{\phi}_\sigma(\vec{r}', t)] \delta^3(\vec{r}-\vec{r}') - \phi_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}', t)\nabla^2 \delta^3(\vec{r}-\vec{r}') \} d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{2} \int \sum_{\sigma} \frac{1}{\sqrt{m^2-\nabla^2}} [-\nabla^2\phi_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}, t) + \dot{\phi}_\sigma^2(\vec{r}, t) + m^2\phi_\sigma^2(\vec{r}, t)] d^3\vec{r} \\
& = \frac{1}{2} \int \sum_{\sigma} \frac{1}{\sqrt{m^2-\nabla^2}} [-\partial^i[\partial_i\phi_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}, t)] + [\nabla\phi_\sigma(\vec{r}, t)]^2 + \dot{\phi}_\sigma^2(\vec{r}, t) + m^2\phi_\sigma^2(\vec{r}, t)] d^3\vec{r} \\
& = \frac{1}{2} \int \sum_{\sigma} \frac{1}{\sqrt{m^2-\nabla^2}} [\nabla\phi_\sigma(\vec{r}, t)]^2 + \dot{\phi}_\sigma^2(\vec{r}, t) + m^2\phi_\sigma^2(\vec{r}, t)] d^3\vec{r}
\end{aligned}$$

□

### 3.3 标量场空间角动量算符的提取

$$\begin{aligned} \text{定理3.3.1. } M_{ij} &= - \int \sum_{\sigma} \dot{\phi}_{\sigma}(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \phi_{\sigma}(\vec{r}, t) d^3 \vec{r} \\ &= - \frac{i}{2} \int \sum_{\sigma} [a_{\sigma}^{+}(\vec{p}, 0) (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_{\sigma}(\vec{p}, 0) - a_{\sigma}(\vec{p}, 0) (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_{\sigma}^{+}(\vec{p}, 0)] d^3 \vec{p} \end{aligned}$$

证明:  $M_{ij}$

$$\begin{aligned} &= - \int \sum_{\sigma} \dot{\phi}_{\sigma}(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \phi_{\sigma}(\vec{r}, t) d^3 \vec{r} \\ &= - \int \sum_{\sigma} \dot{\phi}_{\sigma}(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \phi_{\sigma}(\vec{r}, t) d^3 \vec{r} \\ &= - \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \sum_{\sigma} \\ &\quad \frac{-iE'}{\sqrt{2E'}} \frac{i(r_i p_j - r_j p_i)}{\sqrt{2E}} [a_{\sigma}(\vec{p}', 0) e^{i(\vec{p}' \cdot \vec{r} - E' t)} - a_{\sigma}^{+}(\vec{p}', 0) e^{-i(\vec{p}' \cdot \vec{r} - E' t)}] [a_{\sigma}(\vec{p}, 0) e^{i(\vec{p} \cdot \vec{r} - E t)} - a_{\sigma}^{+}(\vec{p}, 0) e^{-i(\vec{p} \cdot \vec{r} - E t)}] \\ &= - \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \sum_{\sigma} \\ &\quad \frac{E'(r_i p_j - r_j p_i)}{2\sqrt{E'E}} [a_{\sigma}(\vec{p}', 0) a_{\sigma}(\vec{p}, 0) e^{i[(\vec{p}' + \vec{p}) \cdot \vec{r} - (E' + E)t]} + a_{\sigma}^{+}(\vec{p}', 0) a_{\sigma}^{+}(\vec{p}, 0) e^{-i[(\vec{p}' + \vec{p}) \cdot \vec{r} - (E' + E)t]}] \\ &\quad - \\ &\quad \frac{E'(r_i p_j - r_j p_i)}{2\sqrt{E'E}} [a_{\sigma}(\vec{p}', 0) a_{\sigma}^{+}(\vec{p}, 0) e^{i[(\vec{p}' - \vec{p}) \cdot \vec{r} - (E' - E)t]} + a_{\sigma}^{+}(\vec{p}', 0) a_{\sigma}(\vec{p}, 0) e^{-i[(\vec{p}' - \vec{p}) \cdot \vec{r} - (E' - E)t]}] \\ &= - \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \sum_{\sigma} \frac{-iE'}{2\sqrt{E'E}} \\ &\quad \{ [a_{\sigma}(\vec{p}', 0) a_{\sigma}(\vec{p}, 0) e^{-i(E'+E)t} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) e^{i(\vec{p}' + \vec{p}) \cdot \vec{r}} - a_{\sigma}^{+}(\vec{p}', 0) a_{\sigma}^{+}(\vec{p}, 0) e^{i(E'+E)t} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) e^{-i(\vec{p}' + \vec{p}) \cdot \vec{r}}] \\ &\quad + \\ &\quad [a_{\sigma}(\vec{p}', 0) a_{\sigma}^{+}(\vec{p}, 0) e^{-i(E'-E)t} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) e^{i(\vec{p}' - \vec{p}) \cdot \vec{r}} - a_{\sigma}^{+}(\vec{p}', 0) a_{\sigma}(\vec{p}, 0) e^{i(E'-E)t} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) e^{-i(\vec{p}' - \vec{p}) \cdot \vec{r}}] \} \\ &= i \int d^3 \vec{p} d^3 \vec{p}' \sum_{\sigma} \frac{E'}{2\sqrt{E'E}} \\ &\quad \{ [a_{\sigma}(\vec{p}', 0) a_{\sigma}(\vec{p}, 0) e^{-i(E'+E)t} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) \delta^3(\vec{p} + \vec{p}') - a_{\sigma}^{+}(\vec{p}', 0) a_{\sigma}^{+}(\vec{p}, 0) e^{i(E'+E)t} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) \delta^3(\vec{p} + \vec{p}')] \\ &\quad + \\ &\quad [a_{\sigma}(\vec{p}', 0) a_{\sigma}^{+}(\vec{p}, 0) e^{-i(E'-E)t} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) \delta^3(\vec{p} - \vec{p}') - a_{\sigma}^{+}(\vec{p}', 0) a_{\sigma}(\vec{p}, 0) e^{i(E'-E)t} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) \delta^3(\vec{p} - \vec{p}')] \} \\ &= \frac{i}{2} \int d^3 \vec{p} \sum_{\sigma} \\ &\quad \{ -a_{\sigma}(\vec{p}, 0) \sqrt{E} e^{-iEt} (\tilde{\partial}_i [p_j a_{\sigma}(-\vec{p}, 0) \frac{1}{\sqrt{E}} e^{-iEt}] - \tilde{\partial}_j [p_i a_{\sigma}(-\vec{p}, 0) \frac{1}{\sqrt{E}} e^{-iEt}]) \\ &\quad + \\ &\quad a_{\sigma}^{+}(\vec{p}, 0) \sqrt{E} e^{iEt} (\tilde{\partial}_i [p_j a_{\sigma}^{+}(-\vec{p}, 0) \frac{1}{\sqrt{E}} e^{iEt}] - \tilde{\partial}_j [p_i a_{\sigma}^{+}(-\vec{p}, 0) \frac{1}{\sqrt{E}} e^{iEt}]) \\ &\quad - \\ &\quad a_{\sigma}(\vec{p}, 0) \sqrt{E} e^{-iEt} (\tilde{\partial}_i [p_j a_{\sigma}^{+}(\vec{p}, 0) \frac{1}{\sqrt{E}} e^{iEt}] - \tilde{\partial}_j [p_i a_{\sigma}^{+}(\vec{p}, 0) \frac{1}{\sqrt{E}} e^{iEt}]) \\ &\quad + \\ &\quad a_{\sigma}^{+}(\vec{p}, 0) \sqrt{E} e^{iEt} (\tilde{\partial}_i [p_j a_{\sigma}(\vec{p}, 0) \frac{1}{\sqrt{E}} e^{-iEt}] - \tilde{\partial}_j [p_i a_{\sigma}(\vec{p}, 0) \frac{1}{\sqrt{E}} e^{-iEt}]) \} \\ &= \frac{i}{2} \int d^3 \vec{p} \sum_{\sigma} \\ &\quad \{ -a_{\sigma}(\vec{p}, 0) \sqrt{E} e^{-iEt} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) [a_{\sigma}(-\vec{p}, 0) \frac{1}{\sqrt{E}} e^{-iEt}] + a_{\sigma}^{+}(\vec{p}, 0) \sqrt{E} e^{iEt} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) [a_{\sigma}^{+}(-\vec{p}, 0) \frac{1}{\sqrt{E}} e^{iEt}] \\ &\quad - a_{\sigma}(\vec{p}, 0) \sqrt{E} e^{-iEt} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) [a_{\sigma}^{+}(\vec{p}, 0) \frac{1}{\sqrt{E}} e^{iEt}] + a_{\sigma}^{+}(\vec{p}, 0) \sqrt{E} e^{iEt} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) [a_{\sigma}(\vec{p}, 0) \frac{1}{\sqrt{E}} e^{-iEt}] \} \\ &= \frac{i}{2} \int d^3 \vec{p} \sum_{\sigma} \\ &\quad \{ -a_{\sigma}(\vec{p}, 0) e^{-2iEt} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) a_{\sigma}(-\vec{p}, 0) + a_{\sigma}^{+}(\vec{p}, 0) e^{2iEt} (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) a_{\sigma}^{+}(-\vec{p}, 0) \\ &\quad - a_{\sigma}(\vec{p}, 0) (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) a_{\sigma}^{+}(\vec{p}, 0) + a_{\sigma}^{+}(\vec{p}, 0) (p_j \tilde{\partial}_i - p_i \tilde{\partial}_j) a_{\sigma}(\vec{p}, 0) \} \\ &= - \frac{i}{2} \int d^3 \vec{p} \sum_{\sigma} [a_{\sigma}^{+}(\vec{p}, 0) (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_{\sigma}(\vec{p}, 0) - a_{\sigma}(\vec{p}, 0) (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_{\sigma}^{+}(\vec{p}, 0)] \quad \square \end{aligned}$$

$$\text{推论3.3.1. } \partial_t \phi_{\sigma}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{-iE}{\sqrt{2E}} [a_{\sigma}(\vec{p}, 0) e^{i(\vec{p} \cdot \vec{r} - Et)} - a_{\sigma}^{+}(\vec{p}, 0) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^3 \vec{p}$$

$$\text{推论3.3.2. } (r_i \partial_j - r_j \partial_i) \phi_{\sigma}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{i(r_i p_j - r_j p_i)}{\sqrt{2E}} [a_{\sigma}(\vec{p}, 0) e^{i(\vec{p} \cdot \vec{r} - Et)} - a_{\sigma}^{+}(\vec{p}, 0) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^3 \vec{p}$$



推论3.3.3.

$$H = \frac{1}{2} \int \sum_{\sigma} E[a_{\sigma}^{+}(\vec{p}, 0)a(\vec{p}, 0) + a(\vec{p}, 0)a_{\sigma}^{+}(\vec{p}, 0)]d^3\vec{p} = \frac{1}{2} \int \sum_{\sigma} [\nabla\phi_{\sigma}(\vec{r}, t)]^2 + \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2\phi_{\sigma}^2(\vec{r}, t)]d^3\vec{r}$$

$$\vec{P} = \frac{1}{2} \int \sum_{\sigma} \vec{p}[a_{\sigma}^{+}(\vec{p}, 0)a_{\sigma}(\vec{p}, 0) + a_{\sigma}(\vec{p}, 0)a_{\sigma}^{+}(\vec{p}, 0)]d^3\vec{p} = \int \sum_{\sigma} -\dot{\phi}_{\sigma}(\vec{r}, t)\nabla\phi_{\sigma}(\vec{r}, t)d^3\vec{r}$$

定理3.3.2.  $M_{i\pi} = i \int \sum_{\sigma} \{\frac{1}{2}r_i[\nabla\phi_{\sigma}(\vec{r}, t)]^2 + \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2\phi_{\sigma}^2(\vec{r}, t) + t\partial_i\dot{\phi}_{\sigma}(\vec{r}, t)\phi_{\sigma}(\vec{r}, t)\}d^3\vec{r}$

### 3.4 对易和反对易公式

推论3.4.1.  $\begin{cases} [A, BC] = [A, B]C + B[A, C], [A, CB] = [A, C]B + C[A, B] \\ [A, BC] = \{A, B\}C - B\{A, C\}, [A, CB] = \{A, C\}B - C\{A, B\} \end{cases}$

推论3.4.2.  $\begin{cases} [A, \{B, C\}] = \{[A, B], C\} + \{B, [A, C]\} \\ [A, [B, C]] = \{\{A, B\}, C\} - \{B, \{A, C\}\} \end{cases}$

### 3.5 标量场的彭加莱代数

推论3.5.1.

$$H = \frac{1}{2} \int \sum_{\sigma} [\nabla\phi_{\sigma}(\vec{r}, t)]^2 + \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2\phi_{\sigma}^2(\vec{r}, t)]d^3\vec{r}$$

$$= \frac{1}{2} \int \sum_{\sigma} [\dot{\phi}_{\sigma}^2(\vec{r}, t) - \phi_{\sigma}(\vec{r}, t)\partial_t^2\phi_{\sigma}(\vec{r}, t)]d^3\vec{r}$$

$$\vec{P} = \int \sum_{\sigma} -\dot{\phi}_{\sigma}(\vec{r}, t)\nabla\phi_{\sigma}(\vec{r}, t)d^3\vec{r}$$

证明:  $[P_i(t), P_{\pi}(t)]$

$$= \int \sum_{\sigma, \sigma'} [\dot{\phi}_{\sigma}(\vec{r}, t)\partial_i\phi_{\sigma}(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)\partial_{\pi}\phi_{\sigma'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}'$$

$$= \int \sum_{\sigma, \sigma'} [\dot{\phi}_{\sigma}(\vec{r}, t)\partial_i\phi_{\sigma}(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\partial_{\pi}\phi_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)[\dot{\phi}_{\sigma}(\vec{r}, t)\partial_i\phi_{\sigma}(\vec{r}, t), \partial_{\pi}\phi_{\sigma'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}'$$

$$= \int \sum_{\sigma, \sigma'} \dot{\phi}_{\sigma}(\vec{r}, t)[\partial_i\phi_{\sigma}(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\partial_{\pi}\phi_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)\dot{\phi}_{\sigma}(\vec{r}, t)[\partial_i\phi_{\sigma}(\vec{r}, t), \partial_{\pi}\phi_{\sigma'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}'$$

$$= \int \sum_{\sigma} \dot{\phi}_{\sigma}(\vec{r}, t)\partial_i\delta^3(\vec{r} - \vec{r}')\dot{\phi}_{\sigma}(\vec{r}', t) + \dot{\phi}_{\sigma}(\vec{r}', t)\dot{\phi}_{\sigma}(\vec{r}, t)\partial_i\delta^3(\vec{r} - \vec{r}')d^3\vec{r}d^3\vec{r}'$$

$$= - \int \sum_{\sigma} [\partial_i\dot{\phi}_{\sigma}(\vec{r}, t)\dot{\phi}_{\sigma}(\vec{r}, t) + \dot{\phi}_{\sigma}(\vec{r}, t)\partial_i\dot{\phi}_{\sigma}(\vec{r}, t)]d^3\vec{r}$$

$$= - \int \sum_{\sigma} \partial_i[\dot{\phi}_{\sigma}(\vec{r}, t)\dot{\phi}_{\sigma}(\vec{r}, t)]d^3\vec{r} = 0$$

□

证明:

$$= \int \sum_{\sigma, \sigma'} [\dot{\phi}_{\sigma}(\vec{r}, t)\partial_i\phi_{\sigma}(\vec{r}, t), \phi_{\sigma'}(\vec{r}', t)\partial_t^2\phi_{\sigma'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}'$$

$$= \int \sum_{\sigma, \sigma'} [\dot{\phi}_{\sigma}(\vec{r}, t)\partial_i\phi_{\sigma}(\vec{r}, t), \phi_{\sigma'}(\vec{r}', t)]\partial_t^2\phi_{\sigma'}(\vec{r}', t) + \phi_{\sigma'}(\vec{r}', t)[\dot{\phi}_{\sigma}(\vec{r}, t)\partial_i\phi_{\sigma}(\vec{r}, t), \partial_t^2\phi_{\sigma'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}'$$

$$= \int \sum_{\sigma, \sigma'} [\dot{\phi}_{\sigma}(\vec{r}, t), \phi_{\sigma'}(\vec{r}', t)]\partial_i\phi_{\sigma}(\vec{r}, t)\partial_t^2\phi_{\sigma'}(\vec{r}', t) + \phi_{\sigma'}(\vec{r}', t)[\dot{\phi}_{\sigma}(\vec{r}, t), \partial_t^2\phi_{\sigma'}(\vec{r}', t)]\partial_i\phi_{\sigma}(\vec{r}, t)d^3\vec{r}d^3\vec{r}'$$

$$= \int \sum_{\sigma, \sigma'} -\delta^3(\vec{r} - \vec{r}')\partial_i\phi_{\sigma}(\vec{r}, t)\partial_t^2\phi_{\sigma'}(\vec{r}', t) - \phi_{\sigma'}(\vec{r}', t)(m^2 - \nabla^2)\delta^3(\vec{r} - \vec{r}')\partial_i\phi_{\sigma}(\vec{r}, t)d^3\vec{r}d^3\vec{r}'$$

$$= \int \sum_{\sigma, \sigma'} -\partial_i\phi_{\sigma}(\vec{r}, t)\partial_t^2\phi_{\sigma'}(\vec{r}, t) - \phi_{\sigma'}(\vec{r}, t)(m^2 - \nabla^2)\partial_i\phi_{\sigma}(\vec{r}, t)d^3\vec{r}$$

$$= \int \sum_{\sigma, \sigma'} -\partial_i\phi_{\sigma}(\vec{r}, t)\partial_t^2\phi_{\sigma'}(\vec{r}, t) - \phi_{\sigma'}(\vec{r}, t)\partial_t^2\partial_i\phi_{\sigma}(\vec{r}, t)d^3\vec{r}$$

$$= - \int \sum_{\sigma, \sigma'} \partial_i[\phi_{\sigma}(\vec{r}, t)\partial_t^2\phi_{\sigma'}(\vec{r}, t)]d^3\vec{r} = 0$$

□

推论3.5.2.  $[P_a(t), P_b(t)] = 0$

证明:  $[M_{ij}(t), P_{\pi}(t)]$

$$= \int \sum_{\sigma, \sigma'} [\dot{\phi}_{\sigma}(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\phi_{\sigma}(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)\partial_{\pi}\phi_{\sigma'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}'$$

$$= \int \sum_{\sigma, \sigma'} d^3\vec{r}d^3\vec{r}'$$

$$\begin{aligned}
& [\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\partial_\pi\phi_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)[\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t), \partial_\pi\phi_{\sigma'}(\vec{r}', t)] \\
&= \int \sum_{\sigma, \sigma'} d^3\vec{r}d^3\vec{r}' \\
&\dot{\phi}_\sigma(\vec{r}, t)[(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\partial_\pi\phi_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)\dot{\phi}_\sigma(\vec{r}, t)[(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t), \partial_\pi\phi_{\sigma'}(\vec{r}', t)] \\
&= -i \int \sum_{\sigma, \sigma'} d^3\vec{r}d^3\vec{r}' \\
&\dot{\phi}_\sigma(\vec{r}, t)[(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\dot{\phi}_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)\dot{\phi}_\sigma(\vec{r}, t)[(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] \\
&= -i \int \sum_{\sigma} \dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\delta^3(\vec{r} - \vec{r}')\dot{\phi}_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\delta^3(\vec{r} - \vec{r}')d^3\vec{r}' \\
&= i \int \sum_{\sigma} \{\partial_j[r_i\dot{\phi}_\sigma(\vec{r}, t)] - \partial_i[r_j\dot{\phi}_\sigma(\vec{r}, t)]\}\dot{\phi}_\sigma(\vec{r}, t) + \dot{\phi}_\sigma(\vec{r}, t)\{\partial_j[r_i\dot{\phi}_\sigma(\vec{r}, t)] - \partial_i[r_j\dot{\phi}_\sigma(\vec{r}, t)]\}d^3\vec{r}' \\
&= i \int \sum_{\sigma} \{\partial_j[r_i\dot{\phi}_\sigma(\vec{r}, t)] - \partial_i[r_j\dot{\phi}_\sigma(\vec{r}, t)]\}\dot{\phi}_\sigma(\vec{r}, t) + \dot{\phi}_\sigma(\vec{r}, t)[r_i\partial_j - r_j\partial_i]\dot{\phi}_\sigma(\vec{r}, t)d^3\vec{r}' \\
&= i \int \sum_{\sigma} \{\partial_j[r_i\dot{\phi}_\sigma(\vec{r}, t)]\dot{\phi}_\sigma(\vec{r}, t) - \partial_i[r_j\dot{\phi}_\sigma(\vec{r}, t)]\dot{\phi}_\sigma(\vec{r}, t)\}d^3\vec{r}' = 0
\end{aligned}$$

□

证明:  $[M_{ij}(t), P_k(t)]$ 

$$\begin{aligned}
&= \int \sum_{\sigma, \sigma'} [\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\partial'_k\phi_{\sigma'}(\vec{r}', t)d^3\vec{r}d^3\vec{r}' \\
&= \int \sum_{\sigma, \sigma'} d^3\vec{r}d^3\vec{r}' \\
&[\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\partial'_j\phi_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)[\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t), \partial'_k\phi_{\sigma'}(\vec{r}', t)] \\
&= \int \sum_{\sigma, \sigma'} d^3\vec{r}d^3\vec{r}' \\
&\dot{\phi}_\sigma(\vec{r}, t)[(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\partial'_k\phi_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)[\dot{\phi}_\sigma(\vec{r}, t), \partial'_k\phi_{\sigma'}(\vec{r}', t)](r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t) \\
&= i \int \sum_{\sigma, \sigma'} \dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\delta^3(\vec{r} - \vec{r}')\partial'_k\phi_{\sigma'}(\vec{r}', t) - \dot{\phi}_{\sigma'}(\vec{r}', t)\partial'_k\delta^3(\vec{r} - \vec{r}')\phi_\sigma(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= -i \int \sum_{\sigma} \{\partial_j[r_i\dot{\phi}_\sigma(\vec{r}, t)] - \partial_i[r_j\dot{\phi}_\sigma(\vec{r}, t)]\}\partial_k\phi_\sigma(\vec{r}, t) - \partial_k\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t)d^3\vec{r}' \\
&= i \int \sum_{\sigma} r_i\dot{\phi}_\sigma(\vec{r}, t)\partial_j\partial_k\phi_\sigma(\vec{r}, t) - r_j\dot{\phi}_\sigma(\vec{r}, t)\partial_i\partial_k\phi_\sigma(\vec{r}, t) - \dot{\phi}_\sigma(\vec{r}, t)\partial_k(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t)d^3\vec{r}' \\
&= i \int \sum_{\sigma} \dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\partial_k\phi_\sigma(\vec{r}, t) - \dot{\phi}_\sigma(\vec{r}, t)\partial_k(r_i\partial_j - r_j\partial_i)\phi_\sigma(\vec{r}, t)d^3\vec{r}' \\
&= -i \int \sum_{\sigma} \dot{\phi}_\sigma(\vec{r}, t)[-i(r_i\partial_j - r_j\partial_i), -i\partial_k]\phi_\sigma(\vec{r}, t)d^3\vec{r}' \\
&= -i \int \sum_{\sigma} \dot{\phi}_\sigma(\vec{r}, t)(\delta_{ik}\partial_j - \delta_{jk}\partial_i)\phi_\sigma(\vec{r}, t)d^3\vec{r}' \\
&= -i[P_i(t)\delta_{jk} - P_j(t)\delta_{ik}]
\end{aligned}$$

□

证明:  $[M_{i\pi}(t), P_\pi(t)]$ 

$$\begin{aligned}
&= \int \sum_{\sigma, \sigma'} [\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_\pi - it\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\partial_\pi\phi_{\sigma'}(\vec{r}', t)d^3\vec{r}d^3\vec{r}' \\
&= \int \sum_{\sigma, \sigma'} d^3\vec{r}d^3\vec{r}' \\
&[\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_\pi - it\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\partial_\pi\phi_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)[\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_\pi - it\partial_i)\phi_\sigma(\vec{r}, t), \partial_\pi\phi_{\sigma'}(\vec{r}', t)] \\
&= - \int \sum_{\sigma, \sigma'} [\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_t + t\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\dot{\phi}_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)[\dot{\phi}_\sigma(\vec{r}, t)(r_i\partial_t + t\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}' \\
&= - \int \sum_{\sigma, \sigma'} \dot{\phi}_\sigma(\vec{r}, t)[(r_i\partial_t + t\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\dot{\phi}_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)\dot{\phi}_\sigma(\vec{r}, t)[(r_i\partial_t + t\partial_i)\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}' \\
&= - \int \sum_{\sigma, \sigma'} \dot{\phi}_\sigma(\vec{r}, t)t[\partial_i\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\dot{\phi}_{\sigma'}(\vec{r}', t) + \dot{\phi}_{\sigma'}(\vec{r}', t)\dot{\phi}_\sigma(\vec{r}, t)t[\partial_i\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}'
\end{aligned}$$

□

### 3.6 标量场的彭加莱代数严格证明

$$\text{推论3.6.1.} \quad \begin{cases} [A, BC] = [A, B]C + B[A, C] \\ [BC, A] = [B, A]C + B[C, A] \\ [\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] = i\delta_{\sigma\sigma'}\delta^3(\vec{r} - \vec{r}') \end{cases}$$

推论3.6.2.

$$P_i = \int \sum_{\sigma} -\dot{\phi}_\sigma(\vec{r}, t)\partial_i\phi_\sigma(\vec{r}, t)d^3\vec{r}$$

$$P_\pi = \frac{i}{2} \int \sum_{\sigma} [\nabla \phi_{\sigma}(\vec{r}, t)]^2 + \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2 \phi_{\sigma}^2(\vec{r}, t)] d^3 \vec{r}$$

定理3.6.1.

$$M_{ij} = - \int \sum_{\sigma} \dot{\phi}_{\sigma}(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \phi_{\sigma}(\vec{r}, t) d^3 \vec{r}$$

$$\begin{aligned} M_{i\pi} &= i \int \sum_{\sigma} \left\{ \frac{1}{2} r_i [\nabla \phi_{\sigma}(\vec{r}, t)]^2 + \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2 \phi_{\sigma}^2(\vec{r}, t) \right\} + t \partial_i \dot{\phi}_{\sigma}(\vec{r}, t) \phi_{\sigma}(\vec{r}, t) \Big\} d^3 \vec{r} \\ &= i \int \sum_{\sigma} \left\{ \frac{1}{2} [-r_i \nabla^2 \phi_{\sigma}(\vec{r}, t) \phi_{\sigma}(\vec{r}, t) - \partial_i \phi_{\sigma}(\vec{r}, t) \phi_{\sigma}(\vec{r}, t) + r_i \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2 r_i \phi_{\sigma}^2(\vec{r}, t)] + t \partial_i \dot{\phi}_{\sigma}(\vec{r}, t) \phi_{\sigma}(\vec{r}, t) \right\} d^3 \vec{r} \\ &= i \int \sum_{\sigma} \left\{ \frac{1}{2} [-r_i \nabla^2 \phi_{\sigma}(\vec{r}, t) \phi_{\sigma}(\vec{r}, t) + r_i \dot{\phi}_{\sigma}^2(\vec{r}, t) + m^2 r_i \phi_{\sigma}^2(\vec{r}, t)] + t \partial_i \dot{\phi}_{\sigma}(\vec{r}, t) \phi_{\sigma}(\vec{r}, t) \right\} d^3 \vec{r} \end{aligned}$$

### 3.6.1 引理-数学准备

$$\text{引理3.6.1. } [\dot{\phi}_{\sigma}(\vec{r}, t), [\nabla' \phi_{\sigma'}(\vec{r}', t)]^2] = -2i \delta_{\sigma\sigma'} \nabla' \phi_{\sigma'}(\vec{r}', t) \cdot \nabla' \delta^3(\vec{r} - \vec{r}')$$

$$\begin{aligned} \text{证明: } & [\dot{\phi}_{\sigma}(\vec{r}, t), [\nabla' \phi_{\sigma'}(\vec{r}', t)]^2] \\ &= 2 \nabla' \phi_{\sigma'}(\vec{r}', t) [\dot{\phi}_{\sigma}(\vec{r}, t), \nabla' \phi_{\sigma'}(\vec{r}', t)] \\ &= -2i \delta_{\sigma\sigma'} \nabla' \phi_{\sigma'}(\vec{r}', t) \cdot \nabla' \delta^3(\vec{r} - \vec{r}') \end{aligned} \quad \square$$

$$\text{引理3.6.2. } [\dot{\phi}_{\sigma}(\vec{r}, t), \dot{\phi}_{\sigma'}^2(\vec{r}', t)] = 0$$

$$\text{引理3.6.3. } [\dot{\phi}_{\sigma}(\vec{r}, t), m^2 \phi_{\sigma'}^2(\vec{r}', t)] = -2im^2 \delta_{\sigma\sigma'} \phi_{\sigma'}(\vec{r}', t) \delta^3(\vec{r} - \vec{r}')$$

$$\begin{aligned} \text{证明: } & [\dot{\phi}_{\sigma}(\vec{r}, t), \phi_{\sigma'}^2(\vec{r}', t)] \\ &= 2 \phi_{\sigma'}(\vec{r}', t) [\dot{\phi}_{\sigma}(\vec{r}, t), \phi_{\sigma'}(\vec{r}', t)] \\ &= -2i \delta_{\sigma\sigma'} \phi_{\sigma'}(\vec{r}', t) \delta^3(\vec{r} - \vec{r}') \end{aligned} \quad \square$$

$$\text{引理3.6.4. } [\dot{\phi}_{\sigma}(\vec{r}, t), [\nabla' \phi_{\sigma'}(\vec{r}', t)]^2 + \dot{\phi}_{\sigma'}^2(\vec{r}', t) + m^2 \phi_{\sigma'}^2(\vec{r}', t)] = -2i \delta_{\sigma\sigma'} [\nabla' \phi_{\sigma'}(\vec{r}', t) \cdot \nabla' + m^2 \phi_{\sigma'}(\vec{r}', t)] \delta^3(\vec{r} - \vec{r}')$$

$$\text{引理3.6.5. } [\phi_{\sigma}(\vec{r}, t), [\nabla' \phi_{\sigma'}(\vec{r}', t)]^2 + \dot{\phi}_{\sigma'}^2(\vec{r}', t) + m^2 \phi_{\sigma'}^2(\vec{r}', t)] = 2i \delta_{\sigma\sigma'} \dot{\phi}_{\sigma'}(\vec{r}', t) \delta^3(\vec{r} - \vec{r}')$$

$$\text{引理3.6.6. } [\nabla \phi_{\sigma}(\vec{r}, t), [\nabla' \phi_{\sigma'}(\vec{r}', t)]^2 + \dot{\phi}_{\sigma'}^2(\vec{r}', t) + m^2 \phi_{\sigma'}^2(\vec{r}', t)] = 2i \delta_{\sigma\sigma'} \dot{\phi}_{\sigma'}(\vec{r}', t) \nabla \delta^3(\vec{r} - \vec{r}')$$

### 3.6.2 标量场动量对易规则

$$\text{定理3.6.2. } [P_a(t), P_b(t)] = 0$$

$$\begin{aligned} \text{证明: } & [P_i(t), P_j(t)] \\ &= \int \sum_{\sigma, \sigma'} [\dot{\phi}_{\sigma}(\vec{r}, t) \partial_i \phi_{\sigma}(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t) \partial'_j \phi_{\sigma'}(\vec{r}', t)] d^3 \vec{r} d^3 \vec{r}' \\ &= \int \sum_{\sigma} [\dot{\phi}_{\sigma}(\vec{r}, t) \partial_i \phi_{\sigma}(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t) \partial'_j \phi_{\sigma'}(\vec{r}', t)] + \dot{\phi}_{\sigma'}(\vec{r}', t) [\dot{\phi}_{\sigma}(\vec{r}, t) \partial_i \phi_{\sigma}(\vec{r}, t), \partial'_j \phi_{\sigma'}(\vec{r}', t)] d^3 \vec{r} d^3 \vec{r}' \\ &= \int \sum_{\sigma} \dot{\phi}_{\sigma}(\vec{r}, t) [\partial_i \phi_{\sigma}(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t) \partial'_j \phi_{\sigma'}(\vec{r}', t)] + \dot{\phi}_{\sigma'}(\vec{r}', t) [\dot{\phi}_{\sigma}(\vec{r}, t), \partial'_j \phi_{\sigma'}(\vec{r}', t)] \partial_i \phi_{\sigma}(\vec{r}, t) d^3 \vec{r} d^3 \vec{r}' \\ &= i \int \sum_{\sigma} \dot{\phi}_{\sigma}(\vec{r}, t) \partial_i \delta^3(\vec{r} - \vec{r}') \partial'_j \phi_{\sigma'}(\vec{r}', t) - \dot{\phi}_{\sigma'}(\vec{r}', t) \partial'_j \delta^3(\vec{r} - \vec{r}') \partial_i \phi_{\sigma}(\vec{r}, t) d^3 \vec{r} d^3 \vec{r}' \\ &= -i \int \sum_{\sigma} \dot{\phi}_{\sigma}(\vec{r}, t) \partial'_i \delta^3(\vec{r} - \vec{r}') \partial'_j \phi_{\sigma'}(\vec{r}', t) - \dot{\phi}_{\sigma'}(\vec{r}', t) \partial_j \delta^3(\vec{r} - \vec{r}') \partial_i \phi_{\sigma}(\vec{r}, t) d^3 \vec{r} d^3 \vec{r}' \\ &= i \int \sum_{\sigma} \dot{\phi}_{\sigma}(\vec{r}, t) \partial_i \partial_j \phi_{\sigma}(\vec{r}, t) - \dot{\phi}_{\sigma}(\vec{r}, t) \partial_j \partial_i \phi_{\sigma}(\vec{r}, t) d^3 \vec{r} = 0 \end{aligned} \quad \square$$

$$\text{证明: } [P_i(t), P_\pi(t)]$$

$$\begin{aligned} &= -\frac{i}{2} \int \sum_{\sigma, \sigma'} [\dot{\phi}_{\sigma}(\vec{r}, t) \partial_i \phi_{\sigma}(\vec{r}, t), [\nabla' \phi_{\sigma'}(\vec{r}', t)]^2 + \dot{\phi}_{\sigma'}^2(\vec{r}', t) + m^2 \phi_{\sigma'}^2(\vec{r}', t)] d^3 \vec{r} d^3 \vec{r}' \\ &= -\frac{i}{2} \int \sum_{\sigma, \sigma'} \{ [\dot{\phi}_{\sigma}(\vec{r}, t), [\nabla' \phi_{\sigma'}(\vec{r}', t)]^2 + \dot{\phi}_{\sigma'}^2(\vec{r}', t) + m^2 \phi_{\sigma'}^2(\vec{r}', t)] \partial_i \phi_{\sigma}(\vec{r}, t) \\ &\quad + \dot{\phi}_{\sigma}(\vec{r}, t) [\partial_i \phi_{\sigma}(\vec{r}, t), [\nabla' \phi_{\sigma'}(\vec{r}', t)]^2 + \dot{\phi}_{\sigma'}^2(\vec{r}', t) + m^2 \phi_{\sigma'}^2(\vec{r}', t)] \} d^3 \vec{r} d^3 \vec{r}' \\ &= -\frac{i}{2} \int \sum_{\sigma, \sigma'} \{ [\dot{\phi}_{\sigma}(\vec{r}, t), [\nabla' \phi_{\sigma'}(\vec{r}', t)]^2 + m^2 \phi_{\sigma'}^2(\vec{r}', t)] \partial_i \phi_{\sigma}(\vec{r}, t) + \dot{\phi}_{\sigma}(\vec{r}, t) [\partial_i \phi_{\sigma}(\vec{r}, t), \dot{\phi}_{\sigma'}^2(\vec{r}', t)] \} d^3 \vec{r} d^3 \vec{r}' \\ &= -i \int \sum_{\sigma, \sigma'} \{ [\dot{\phi}_{\sigma}(\vec{r}, t), \nabla' \phi_{\sigma'}(\vec{r}', t)] \cdot \nabla' \phi_{\sigma'}(\vec{r}', t) \partial_i \phi_{\sigma}(\vec{r}, t) + [\dot{\phi}_{\sigma}(\vec{r}, t), \phi_{\sigma'}(\vec{r}', t)] m^2 \phi_{\sigma'}(\vec{r}', t) \partial_i \phi_{\sigma}(\vec{r}, t) \} \end{aligned}$$

$$\begin{aligned}
& + \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_{\sigma'}(\vec{r}', t) [\partial_i \phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] d^3 \vec{r} d^3 \vec{r}' \\
& = -i \int \sum_{\sigma, \sigma'} \{ -i \delta_{\sigma\sigma'} \nabla' \delta^3(\vec{r} - \vec{r}') \cdot \nabla' \dot{\phi}_{\sigma'}(\vec{r}', t) \partial_i \phi_\sigma(\vec{r}, t) - i \delta_{\sigma\sigma'} \delta^3(\vec{r} - \vec{r}') m^2 \dot{\phi}_{\sigma'}(\vec{r}', t) \partial_i \phi_\sigma(\vec{r}, t) \\
& + \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_{\sigma'}(\vec{r}', t) i \delta_{\sigma\sigma'} \partial_i \delta^3(\vec{r} - \vec{r}') \} d^3 \vec{r} d^3 \vec{r}' \\
& = \int \{ \nabla^2 \phi_\sigma(\vec{r}, t) \partial_i \phi_\sigma(\vec{r}, t) - m^2 \phi_\sigma(\vec{r}, t) \partial_i \phi_\sigma(\vec{r}, t) - \partial_i \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_\sigma(\vec{r}, t) \} d^3 \vec{r} \\
& = \int \{ \nabla \phi_\sigma(\vec{r}, t) \partial_i \cdot \nabla \phi_\sigma(\vec{r}, t) - m^2 \phi_\sigma(\vec{r}, t) \partial_i \phi_\sigma(\vec{r}, t) - \partial_i \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_\sigma(\vec{r}, t) \} d^3 \vec{r} \\
& = \frac{1}{2} \int \{ -\partial_i [\nabla \phi_\sigma(\vec{r}, t) \cdot \nabla \phi_\sigma(\vec{r}, t)] - m^2 \partial_i \phi_\sigma^2(\vec{r}, t) - \partial_i \dot{\phi}_\sigma^2(\vec{r}, t) \} d^3 \vec{r} \\
& = -\frac{1}{2} \int \partial_i \{ [\nabla \phi_\sigma(\vec{r}, t)]^2 + m^2 \phi_\sigma^2(\vec{r}, t) + \dot{\phi}_\sigma^2(\vec{r}, t) \} d^3 \vec{r} = 0
\end{aligned}$$

□

### 3.6.3 标量场角动量对易规则

证明:  $[M_{ij}(t), M_{kl}(t)]$

$$\begin{aligned}
& = \int \sum_{\sigma, \sigma'} [\dot{\phi}_\sigma(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t) (r'_k \partial'_l - r'_l \partial'_k) \phi_{\sigma'}(\vec{r}', t)] d^3 \vec{r} d^3 \vec{r}' \\
& = \int \sum_{\sigma, \sigma'} d^3 \vec{r} d^3 \vec{r}' \{ [\dot{\phi}_\sigma(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] (r'_k \partial'_l - r'_l \partial'_k) \phi_{\sigma'}(\vec{r}', t) \\
& + \dot{\phi}_{\sigma'}(\vec{r}', t) [\dot{\phi}_\sigma(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t), (r'_k \partial'_l - r'_l \partial'_k) \phi_{\sigma'}(\vec{r}', t)] \} \\
& = \int \sum_{\sigma, \sigma'} d^3 \vec{r} d^3 \vec{r}' \{ \dot{\phi}_\sigma(\vec{r}, t) [(r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] (r'_k \partial'_l - r'_l \partial'_k) \phi_{\sigma'}(\vec{r}', t) \\
& + \dot{\phi}_{\sigma'}(\vec{r}', t) [\dot{\phi}_\sigma(\vec{r}, t), (r'_k \partial'_l - r'_l \partial'_k) \phi_{\sigma'}(\vec{r}', t)] (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t) \} \\
& = i \int \sum_{\sigma, \sigma'} d^3 \vec{r} d^3 \vec{r}' \\
& \{ \dot{\phi}_\sigma(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \delta^3(\vec{r} - \vec{r}') (r'_k \partial'_l - r'_l \partial'_k) \phi_{\sigma'}(\vec{r}', t) - \dot{\phi}_{\sigma'}(\vec{r}', t) (r'_k \partial'_l - r'_l \partial'_k) \delta^3(\vec{r} - \vec{r}') (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t) \} \\
& = -i \int \sum_{\sigma} d^3 \vec{r} \\
& \{ \{ \partial_j [r_i \dot{\phi}_\sigma(\vec{r}, t)] - \partial_i [r_j \dot{\phi}_\sigma(\vec{r}, t)] \} (r_k \partial_l - r_l \partial_k) \phi_\sigma(\vec{r}, t) - \{ \partial_l [r_k \dot{\phi}_\sigma(\vec{r}, t)] - \partial_k [r_l \dot{\phi}_\sigma(\vec{r}, t)] \} (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t) \} \\
& = -i \int \sum_{\sigma} \dot{\phi}_\sigma(\vec{r}, t) [-i (r_i \partial_j - r_j \partial_i), -i (r_k \partial_l - r_l \partial_k)] \phi_\sigma(\vec{r}, t) d^3 \vec{r} \\
& = -i [\delta_{il} M_{jk}(t) - \delta_{ik} M_{jl}(t) + \delta_{jk} M_{il}(t) - \delta_{jl} M_{ik}(t)]
\end{aligned}$$

□

证明:  $[M_{ij}(t), M_{k\pi}(t)]$

$$\begin{aligned}
& = i \int \sum_{\sigma, \sigma'} [\dot{\phi}_\sigma(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t), \{ \frac{1}{2} r'_k [|\nabla' \phi_{\sigma'}(\vec{r}', t)|^2 + \dot{\phi}_{\sigma'}^2(\vec{r}', t) + m^2 \phi_{\sigma'}^2(\vec{r}', t)] + t \partial'_k \dot{\phi}_{\sigma'}(\vec{r}', t) \phi_{\sigma'}(\vec{r}', t) \} d^3 \vec{r} d^3 \vec{r}' \\
& = i \int \sum_{\sigma, \sigma'} [\dot{\phi}_\sigma(\vec{r}, t), \frac{1}{2} r'_k [|\nabla' \phi_{\sigma'}(\vec{r}', t)|^2 + m^2 \phi_{\sigma'}^2(\vec{r}', t)] + t \partial'_k \dot{\phi}_{\sigma'}(\vec{r}', t) \phi_{\sigma'}(\vec{r}', t)] (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t) \\
& + \dot{\phi}_\sigma(\vec{r}, t) [(r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t), \{ \frac{1}{2} r'_k \dot{\phi}_{\sigma'}^2(\vec{r}', t) + t \partial'_k \dot{\phi}_{\sigma'}(\vec{r}', t) \phi_{\sigma'}(\vec{r}', t) \}] d^3 \vec{r} d^3 \vec{r}' \\
& = \int \sum_{\sigma, \sigma'} \delta_{\sigma\sigma'} [r'_k \nabla' \phi_{\sigma'}(\vec{r}', t) \cdot \nabla' + m^2 r'_k \phi_{\sigma'}(\vec{r}', t) + t \partial'_k \dot{\phi}_{\sigma'}(\vec{r}', t)] \delta^3(\vec{r} - \vec{r}') (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t) \\
& - \delta_{\sigma\sigma'} [r'_k \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_{\sigma'}(\vec{r}', t) + t \dot{\phi}_\sigma(\vec{r}, t) \phi_{\sigma'}(\vec{r}', t) \partial'_k] (r_i \partial_j - r_j \partial_i) \delta^3(\vec{r} - \vec{r}') d^3 \vec{r} d^3 \vec{r}' \\
& = \int [-\partial_k \phi_\sigma(\vec{r}, t) - r_k \nabla^2 \phi_\sigma(\vec{r}, t) + m^2 r_k \phi_\sigma(\vec{r}, t) + t \partial_k \dot{\phi}_\sigma(\vec{r}, t)] (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t) \\
& + [r_k (r_i \partial_j - r_j \partial_i) \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_\sigma(\vec{r}, t) - t (r_i \partial_j - r_j \partial_i) \dot{\phi}_\sigma(\vec{r}, t) \partial_k \phi_\sigma(\vec{r}, t)] d^3 \vec{r} \\
& = \int [-\partial_k \phi_\sigma(\vec{r}, t) - r_k \nabla^2 \phi_\sigma(\vec{r}, t) + m^2 r_k \phi_\sigma(\vec{r}, t)] (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t) \\
& + r_k (r_i \partial_j - r_j \partial_i) \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_\sigma(\vec{r}, t) d^3 \vec{r} \\
& = \int [-\partial_k \phi_\sigma(\vec{r}, t) - r_k \dot{\phi}_\sigma^2(\vec{r}, t)] (r_i \partial_j - r_j \partial_i) \phi_\sigma(\vec{r}, t) \\
& + r_k (r_i \partial_j - r_j \partial_i) \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_\sigma(\vec{r}, t) d^3 \vec{r} \\
& = \\
& = \int \sum_{\sigma} g_{jk} \{ \frac{1}{2} r_i [\nabla \phi_\sigma(\vec{r}, t)]^2 + \dot{\phi}_\sigma^2(\vec{r}, t) + m^2 \phi_\sigma^2(\vec{r}, t) \} + t \partial_i \dot{\phi}_\sigma(\vec{r}, t) \phi_\sigma(\vec{r}, t) \} \\
& - g_{ik} \{ \frac{1}{2} r_j [\nabla \phi_\sigma(\vec{r}, t)]^2 + \dot{\phi}_\sigma^2(\vec{r}, t) + m^2 \phi_\sigma^2(\vec{r}, t) \} + t \partial_j \dot{\phi}_\sigma(\vec{r}, t) \phi_\sigma(\vec{r}, t) \} d^3 \vec{r}
\end{aligned}$$

□

$$M_{ab} = L_{ab} + S_{ab}, L_{ab} = x_a p_b - x_b p_a, g_{ab} = \delta_{ab} \quad (21.3)$$

$$\begin{cases} [M_{ab}, M_{cd}] = -i(g_{ad}M_{bc} - g_{ac}M_{bd} + g_{bc}M_{ad} - g_{bd}M_{ac}) \\ [M_{ij}, M_{k\pi}] = -i(g_{jk}M_{i\pi} - g_{ik}M_{j\pi}) \\ [M_{ab}, p_c] = -i(g_{bc}p_a - g_{ac}p_b), [p_a, p_b] = 0 \end{cases} \quad (21.4)$$

证明:  $[M_{i\pi}(t), M_{j\pi}(t)]$

$$\begin{aligned} &= - \int \sum_{\sigma\sigma'} [\{\frac{1}{2}r_i [[\nabla\phi_\sigma(\vec{r}, t)]^2 + \dot{\phi}_\sigma^2(\vec{r}, t) + m^2\phi_\sigma^2(\vec{r}, t)] + t\partial_i\dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}, t)\} \\ &\quad , \{\frac{1}{2}r'_j [[\nabla'\phi_{\sigma'}(\vec{r}', t)]^2 + \dot{\phi}_{\sigma'}^2(\vec{r}', t) + m^2\phi_{\sigma'}^2(\vec{r}', t)] + t\partial'_j\dot{\phi}_{\sigma'}(\vec{r}', t)\phi_{\sigma'}(\vec{r}', t)\}] d^3\vec{r}d^3\vec{r}' \\ &= - \int \sum_{\sigma\sigma'} \{\frac{1}{4}r_i r'_j [[\nabla\phi_\sigma(\vec{r}, t)]^2 + \dot{\phi}_\sigma^2(\vec{r}, t) + m^2\phi_\sigma^2(\vec{r}, t), [\nabla'\phi_{\sigma'}(\vec{r}', t)]^2 + \dot{\phi}_{\sigma'}^2(\vec{r}', t) + m^2\phi_{\sigma'}^2(\vec{r}', t)] \\ &\quad + t^2[\partial_i\dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}, t), \partial'_j\dot{\phi}_{\sigma'}(\vec{r}', t)\phi_{\sigma'}(\vec{r}', t)] \\ &\quad + \frac{1}{2}r_i t [[\nabla\phi_\sigma(\vec{r}, t)]^2 + \dot{\phi}_\sigma^2(\vec{r}, t) + m^2\phi_\sigma^2(\vec{r}, t), \partial'_j\dot{\phi}_{\sigma'}(\vec{r}', t)\phi_{\sigma'}(\vec{r}', t)] \\ &\quad + \frac{1}{2}r'_j t [\partial_i\dot{\phi}_\sigma(\vec{r}, t)\phi_\sigma(\vec{r}, t), [\nabla'\phi_{\sigma'}(\vec{r}', t)]^2 + \dot{\phi}_{\sigma'}^2(\vec{r}', t) + m^2\phi_{\sigma'}^2(\vec{r}', t)]\} d^3\vec{r}d^3\vec{r}' \\ &= - \int \sum_{\sigma\sigma'} d^3\vec{r}d^3\vec{r}' \\ &\quad \{\frac{1}{4}r_i r'_j [[\nabla\phi_\sigma(\vec{r}, t)]^2 + m^2\phi_\sigma^2(\vec{r}, t), \dot{\phi}_{\sigma'}^2(\vec{r}', t)] + \frac{1}{4}r_i r'_j [\dot{\phi}_\sigma^2(\vec{r}, t), [\nabla'\phi_{\sigma'}(\vec{r}', t)]^2 + m^2\phi_{\sigma'}^2(\vec{r}', t)] \\ &\quad + t^2\partial_i\dot{\phi}_\sigma(\vec{r}, t)[\phi_\sigma(\vec{r}, t), \partial'_j\dot{\phi}_{\sigma'}(\vec{r}', t)]\phi_{\sigma'}(\vec{r}', t) + t^2\partial'_j\dot{\phi}_{\sigma'}(\vec{r}', t)[\partial_i\dot{\phi}_\sigma(\vec{r}, t), \phi_\sigma(\vec{r}, t)]\phi_\sigma(\vec{r}, t) \\ &\quad + \frac{1}{2}r_i t [[\nabla\phi_\sigma(\vec{r}, t)]^2 + m^2\phi_\sigma^2(\vec{r}, t), \partial'_j\dot{\phi}_{\sigma'}(\vec{r}', t)]\phi_{\sigma'}(\vec{r}', t) + \frac{1}{2}r_i t \partial'_j\dot{\phi}_{\sigma'}(\vec{r}', t)[\dot{\phi}_\sigma^2(\vec{r}, t), \phi_{\sigma'}(\vec{r}', t)] \\ &\quad + \frac{1}{2}r'_j t \partial_i\dot{\phi}_\sigma(\vec{r}, t)[\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}^2(\vec{r}', t)] + \frac{1}{2}r'_j t [\partial_i\dot{\phi}_\sigma(\vec{r}, t), [\nabla'\phi_{\sigma'}(\vec{r}', t)]^2 + m^2\phi_{\sigma'}^2(\vec{r}', t)]\phi_\sigma(\vec{r}, t)\} \\ &= - \int \sum_{\sigma\sigma'} d^3\vec{r}d^3\vec{r}' \\ &\quad \{\frac{1}{4}r_i r'_j [[\nabla\phi_\sigma(\vec{r}, t)]^2 + m^2\phi_\sigma^2(\vec{r}, t), \dot{\phi}_{\sigma'}^2(\vec{r}', t)] + \frac{1}{4}r_i r'_j [\dot{\phi}_\sigma^2(\vec{r}, t), [\nabla'\phi_{\sigma'}(\vec{r}', t)]^2 + m^2\phi_{\sigma'}^2(\vec{r}', t)] \\ &\quad + t^2\partial_i\dot{\phi}_\sigma(\vec{r}, t)i\delta_{\sigma\sigma'}\partial'_j\delta^3(\vec{r} - \vec{r}')\phi_{\sigma'}(\vec{r}', t) - t^2\partial'_j\dot{\phi}_{\sigma'}(\vec{r}', t)i\delta_{\sigma\sigma'}\partial_i\delta^3(\vec{r} - \vec{r}')\phi_\sigma(\vec{r}, t) \\ &\quad + r_i t [\nabla\phi_\sigma(\vec{r}, t) \cdot \nabla + m^2\phi_\sigma(\vec{r}, t)]i\delta_{\sigma\sigma'}\partial'_j\delta^3(\vec{r} - \vec{r}')\phi_{\sigma'}(\vec{r}', t) - r_i t \partial'_j\dot{\phi}_{\sigma'}(\vec{r}', t)\phi_\sigma(\vec{r}, t)i\delta_{\sigma\sigma'}\delta^3(\vec{r} - \vec{r}') \\ &\quad + r'_j t \partial_i\dot{\phi}_\sigma(\vec{r}, t)\phi_{\sigma'}(\vec{r}', t)i\delta_{\sigma\sigma'}\delta^3(\vec{r} - \vec{r}') - r'_j t [\nabla'\phi_{\sigma'}(\vec{r}', t) \cdot \nabla' + m^2\phi_{\sigma'}(\vec{r}', t)]i\delta_{\sigma\sigma'}\partial_i\delta^3(\vec{r} - \vec{r}')\phi_\sigma(\vec{r}, t)\} \\ &= - \int \sum_{\sigma\sigma'} d^3\vec{r}d^3\vec{r}' \frac{1}{4}r_i r'_j \\ &\quad \{\frac{1}{4}r_i r'_j [[\nabla\phi_\sigma(\vec{r}, t)]^2 + m^2\phi_\sigma^2(\vec{r}, t), \dot{\phi}_{\sigma'}^2(\vec{r}', t)] + \frac{1}{4}r_i r'_j [\dot{\phi}_\sigma^2(\vec{r}, t), [\nabla'\phi_{\sigma'}(\vec{r}', t)]^2 + m^2\phi_{\sigma'}^2(\vec{r}', t)] \\ &\quad - i t^2 \partial_i \dot{\phi}_\sigma(\vec{r}, t) \partial_j \phi_\sigma(\vec{r}, t) + i t^2 \partial_j \dot{\phi}_\sigma(\vec{r}, t) \partial_i \phi_\sigma(\vec{r}, t) \\ &\quad - i r_i t [\nabla\phi_\sigma(\vec{r}, t) \cdot \nabla + m^2\phi_\sigma(\vec{r}, t)] \partial_j \phi_\sigma(\vec{r}, t) - i r_i t \partial_j \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_\sigma(\vec{r}, t) \\ &\quad + i r_j t \partial_i \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_\sigma(\vec{r}, t) + i r_j t [\nabla\phi_\sigma(\vec{r}, t) \cdot \nabla + m^2\phi_\sigma(\vec{r}, t)] \partial_i \phi_\sigma(\vec{r}, t)\} \\ &= - \int \sum_{\sigma\sigma'} d^3\vec{r}d^3\vec{r}' \frac{1}{4}r_i r'_j \\ &\quad \{[[\nabla\phi_\sigma(\vec{r}, t)]^2 + m^2\phi_\sigma^2(\vec{r}, t), \dot{\phi}_{\sigma'}^2(\vec{r}', t)] + [\dot{\phi}_\sigma^2(\vec{r}, t), [\nabla'\phi_{\sigma'}(\vec{r}', t)]^2 + m^2\phi_{\sigma'}^2(\vec{r}', t)] \\ &\quad - i t^2 \partial_i \dot{\phi}_\sigma(\vec{r}, t) \partial_j \phi_\sigma(\vec{r}, t) + i t^2 \partial_j \dot{\phi}_\sigma(\vec{r}, t) \partial_i \phi_\sigma(\vec{r}, t) \\ &\quad - i r_i t [\nabla\phi_\sigma(\vec{r}, t) \cdot \nabla + m^2\phi_\sigma(\vec{r}, t)] \partial_j \phi_\sigma(\vec{r}, t) - i r_i t \partial_j \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_\sigma(\vec{r}, t) \\ &\quad + i r_j t \partial_i \dot{\phi}_\sigma(\vec{r}, t) \dot{\phi}_\sigma(\vec{r}, t) + i r_j t [\nabla\phi_\sigma(\vec{r}, t) \cdot \nabla + m^2\phi_\sigma(\vec{r}, t)] \partial_i \phi_\sigma(\vec{r}, t)\} \\ &= - \int \sum_{\sigma\sigma'} d^3\vec{r}d^3\vec{r}' \frac{1}{4}r_i r'_j \\ &\quad \{2\dot{\phi}_{\sigma'}(\vec{r}', t)\nabla\phi_\sigma(\vec{r}, t) \cdot [\nabla\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] + 2m^2\dot{\phi}_{\sigma'}(\vec{r}', t)\phi_\sigma(\vec{r}, t)[\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] \\ &\quad + 2[\nabla\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)] \cdot \nabla\phi_\sigma(\vec{r}, t)\dot{\phi}_{\sigma'}(\vec{r}', t) + 2m^2[\phi_\sigma(\vec{r}, t), \dot{\phi}_{\sigma'}(\vec{r}', t)]\phi_\sigma(\vec{r}, t)\dot{\phi}_{\sigma'}(\vec{r}', t) \\ &\quad + [\dot{\phi}_\sigma^2(\vec{r}, t), [\nabla'\phi_{\sigma'}(\vec{r}', t)]^2 + m^2\phi_{\sigma'}^2(\vec{r}', t)] \\ &\quad - 2\dot{\phi}_\sigma(\vec{r}, t)\nabla'\phi_{\sigma'}(\vec{r}', t) \cdot i\delta_{\sigma\sigma'}\nabla\delta^3(\vec{r} - \vec{r}') + 2m^2\dot{\phi}_{\sigma'}(\vec{r}', t)\phi_\sigma(\vec{r}, t)i\delta_{\sigma\sigma'}\delta^3(\vec{r} - \vec{r}') \\ &\quad + 2i\delta_{\sigma\sigma'}\nabla\delta^3(\vec{r} - \vec{r}') \cdot \nabla\phi_\sigma(\vec{r}, t)\dot{\phi}_{\sigma'}(\vec{r}', t) + 2m^2i\delta_{\sigma\sigma'}\delta^3(\vec{r} - \vec{r}')\phi_\sigma(\vec{r}, t)\dot{\phi}_{\sigma'}(\vec{r}', t) \\ &\quad - 2\dot{\phi}_\sigma(\vec{r}, t)\nabla'\phi_{\sigma'}(\vec{r}', t) \cdot i\delta_{\sigma\sigma'}\nabla'\delta^3(\vec{r}' - \vec{r}) - 2m^2\dot{\phi}_\sigma(\vec{r}, t)\phi_{\sigma'}(\vec{r}', t)i\delta_{\sigma\sigma'}\delta^3(\vec{r}' - \vec{r}) \} \end{aligned}$$

$$\begin{aligned}
& -2i\delta_{\sigma'\sigma}\nabla'\delta^3(\vec{r}'-\vec{r})\cdot\nabla'\phi_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t)-2m^2i\delta_{\sigma'\sigma}\delta^3(\vec{r}'-\vec{r})\phi_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t) \\
& = -\int\sum_{\sigma\sigma'}d^3\vec{r}d^3\vec{r}'\frac{1}{4}r_i r'_j \\
& \{2\dot{\phi}_{\sigma'}(\vec{r}',t)\nabla\phi_\sigma(\vec{r},t)\cdot i\delta_{\sigma\sigma'}\nabla\delta^3(\vec{r}-\vec{r}')+2i\delta_{\sigma\sigma'}\nabla\delta^3(\vec{r}-\vec{r}')\cdot\nabla\phi_\sigma(\vec{r},t)\dot{\phi}_{\sigma'}(\vec{r}',t) \\
& -2\dot{\phi}_\sigma(\vec{r},t)\nabla'\phi_{\sigma'}(\vec{r}',t)\cdot i\delta_{\sigma'\sigma}\nabla'\delta^3(\vec{r}'-\vec{r})-2i\delta_{\sigma'\sigma}\nabla'\delta^3(\vec{r}'-\vec{r})\cdot\nabla'\phi_{\sigma'}(\vec{r}',t)\dot{\phi}_\sigma(\vec{r},t) \\
& = -\int d^3\vec{r}d^3\vec{r}'\frac{1}{2} \\
& \{r'_j\dot{\phi}_\sigma(\vec{r}',t)r_i\nabla\phi_\sigma(\vec{r},t)\cdot i\nabla\delta^3(\vec{r}-\vec{r}')+i\nabla\delta^3(\vec{r}-\vec{r}')\cdot r_i\nabla\phi_\sigma(\vec{r},t)r'_j\dot{\phi}_\sigma(\vec{r}',t) \\
& -r_i\dot{\phi}_\sigma(\vec{r},t)r'_j\nabla'\phi_{\sigma'}(\vec{r}',t)\cdot i\nabla'\delta^3(\vec{r}'-\vec{r})-i\nabla'\delta^3(\vec{r}'-\vec{r})\cdot r'_j\nabla'\phi_{\sigma'}(\vec{r}',t)r_i\dot{\phi}_\sigma(\vec{r},t) \\
& = -\int d^3\vec{r}d^3\vec{r}'\frac{i}{2} \\
& \{\nabla[r_j\dot{\phi}_\sigma(\vec{r},t)]r_i\cdot\nabla\phi_\sigma(\vec{r},t)+r_i\nabla\phi_\sigma(\vec{r},t)\cdot\nabla[r_j\dot{\phi}_\sigma(\vec{r},t)] \\
& -\nabla[r_i\dot{\phi}_\sigma(\vec{r},t)]r_j\cdot\nabla\phi_\sigma(\vec{r},t)-r_j\nabla\phi_\sigma(\vec{r},t)\cdot\nabla[r_i\dot{\phi}_\sigma(\vec{r},t)] \\
& = -\int d^3\vec{r}\frac{i}{2} \\
& \{\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t)+(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t)\dot{\phi}_\sigma(\vec{r},t) \\
& = -i\int d^3\vec{r}\dot{\phi}_\sigma(\vec{r},t)(r_i\partial_j-r_j\partial_i)\phi_\sigma(\vec{r},t) \\
& = iM_{ij}(t)
\end{aligned}$$

□

# 第二十二章 电磁场协变量子化新方案

本章主要从传统电磁场势量子化方案出发导出电磁场场强协变量子化新方案，主要体现其导出过程，没有体现其完整性，主要是为了验证新协变量子化程式的正确性。后面章节会在两种表象下分别直接给出其完整的场强协变量子化方案。

## 1 电磁场方程的规范势分析 [23, 25]

### 1.1 有质量电磁场方程的规范势描述

$$\text{定理1.1.1.1.} \quad \begin{cases} \nabla \cdot \vec{E} = m^2\phi - \rho, \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = m^2 \vec{A} - \vec{J} + \partial_t \vec{E} \\ \nabla \cdot \vec{J} + \partial_t \rho = 0 \\ \vec{E} = -\partial_t \vec{A} - \nabla \phi, \vec{B} = \nabla \times \vec{A} \end{cases} \Leftrightarrow \begin{cases} (\nabla^2 - \partial_t^2 - m^2)\phi = \rho \\ (\nabla^2 - \partial_t^2 - m^2)\vec{A} = \vec{J} \\ \nabla \cdot \vec{A} + \partial_t \phi = 0 \\ \vec{E} = -\partial_t \vec{A} - \nabla \phi, \vec{B} = \nabla \times \vec{A} \end{cases}$$

### 1.2 电磁场方程的一般规范势描述

$$\text{引理1.2.1.} \quad \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\text{引理1.2.2.} \quad \nabla \cdot \vec{B} = 0, \vec{A} = \frac{\nabla \times \vec{B}}{-\nabla^2} + \nabla \theta \Leftrightarrow \vec{B} = \nabla \times \vec{A}, \nabla \cdot \vec{A} = \nabla^2 \theta$$

正向证明:

$$\begin{aligned} \text{证明: } \nabla \cdot \vec{B} = 0, \vec{A} &= \frac{\nabla \times \vec{B}}{-\nabla^2} + \nabla \theta \\ \Rightarrow \nabla \times \vec{A} &= \frac{\nabla \times (\nabla \times \vec{B})}{-\nabla^2} + \nabla \times \nabla \theta, \nabla \cdot \vec{A} = \frac{\nabla \cdot (\nabla \times \vec{B})}{-\nabla^2} + \nabla \cdot \nabla \theta \\ \Rightarrow \nabla \times \vec{A} &= \frac{\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B}}{-\nabla^2}, \nabla \cdot \vec{A} = \nabla^2 \theta \\ \Rightarrow \vec{B} &= \nabla \times \vec{A}, \nabla \cdot \vec{A} = \nabla^2 \theta \end{aligned} \quad \square$$

反向证明:

$$\begin{aligned} \text{证明: } \vec{B} &= \nabla \times \vec{A}, \nabla \cdot \vec{A} = \nabla^2 \theta \\ \Rightarrow \nabla \times \vec{B} &= \nabla \times (\nabla \times \vec{A}), \nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) \\ \Rightarrow \nabla \times \vec{B} &= \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}, \nabla \cdot \vec{B} = 0 \\ \Rightarrow \nabla \cdot \vec{B} &= 0, \vec{A} = \frac{\nabla \times \vec{B}}{-\nabla^2} + \nabla \theta \end{aligned} \quad \square$$

$$\text{定理1.2.1.} \quad \begin{cases} \nabla \cdot \vec{E} = -\rho, \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = -\vec{J} + \partial_t \vec{E} \\ \vec{A} = \frac{\nabla \times \vec{B}}{-\nabla^2} + \nabla \theta, \phi = \frac{\nabla \cdot \vec{E}}{-\nabla^2} - \partial_t \theta \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \phi = \rho - \partial_t \nabla^2 \theta, \nabla \cdot \vec{A} = \nabla^2 \theta \\ \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J} + \nabla(\partial_t \phi + \nabla^2 \theta) \\ \vec{E} = -\partial_t \vec{A} - \nabla \phi, \vec{B} = \nabla \times \vec{A} \end{cases}$$

正向证明:

$$\text{证明: } \nabla^2 \phi = \nabla^2 \frac{\nabla \cdot \vec{E}}{-\nabla^2} - \nabla^2 \partial_t \theta = -\nabla \cdot \vec{E} - \nabla^2 \partial_t \theta = \rho - \partial_t \nabla^2 \theta \quad \square$$

$$\text{证明: } \nabla \cdot \vec{A} = \frac{\nabla \cdot \nabla \times \vec{B}}{-\nabla^2} + \nabla \cdot \nabla \theta = \nabla^2 \theta \quad \square$$

$$\begin{aligned} \text{证明: } \nabla^2 \vec{A} - \partial_t^2 \vec{A} &= (\nabla^2 - \partial_t^2) \frac{\nabla \times \vec{B}}{-\nabla^2} + (\nabla^2 - \partial_t^2) \nabla \theta \\ &= -\nabla \times \vec{B} + \partial_t \frac{\nabla \times \partial_t \vec{B}}{-\nabla^2} + (\nabla^2 - \partial_t^2) \nabla \theta \\ &= -\nabla \times \vec{B} - \partial_t \frac{\nabla \times \nabla \times \vec{E}}{-\nabla^2} + (\nabla^2 - \partial_t^2) \nabla \theta \\ &= -\nabla \times \vec{B} - \partial_t \frac{\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}}{-\nabla^2} + (\nabla^2 - \partial_t^2) \nabla \theta \\ &= -\nabla \times \vec{B} + \partial_t \vec{E} - \partial_t \frac{\nabla(\nabla \cdot \vec{E})}{-\nabla^2} + (\nabla^2 - \partial_t^2) \nabla \theta \end{aligned}$$

$$= \vec{J} + \partial_t \nabla(\phi + \partial_t \theta) + (\nabla^2 - \partial_t^2) \nabla \theta$$

$$= \vec{J} + \nabla(\partial_t \phi + \nabla^2 \theta) \quad \square$$

$$\text{证明: } \nabla \times \vec{A} = \frac{\nabla \times \nabla \times \vec{B}}{-\nabla^2} + \nabla \times \nabla \theta = \frac{\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B}}{-\nabla^2} = \frac{0 - \nabla^2 \vec{B}}{-\nabla^2} = \vec{B} \quad \square$$

$$\text{证明: } -\partial_t \vec{A} - \nabla \phi = \partial_t \frac{\nabla \times \vec{B}}{\nabla^2} - \partial_t \nabla \theta + \nabla \frac{\nabla \cdot \vec{E}}{\nabla^2} + \nabla \partial_t \theta$$

$$= \frac{\nabla \times \partial_t \vec{B}}{\nabla^2} + \nabla \frac{\nabla \cdot \vec{E}}{\nabla^2} = -\frac{\nabla \times \nabla \times \vec{E}}{\nabla^2} + \nabla \frac{\nabla \cdot \vec{E}}{\nabla^2}$$

$$= -\frac{\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}}{\nabla^2} + \frac{\nabla(\nabla \cdot \vec{E})}{\nabla^2} = \vec{E} \quad \square$$

反向证明:

$$\text{证明: } \nabla \cdot \vec{E} = -\nabla \cdot \partial_t \vec{A} - \nabla \cdot \nabla \phi = -\partial_t(\nabla \cdot \vec{A}) - \nabla^2 \phi = -\partial_t \nabla^2 \theta - \rho + \partial_t \nabla^2 \theta = -\rho \quad \square$$

$$\text{证明: } \nabla \times \vec{E} = -\nabla \times \partial_t \vec{A} - \nabla \times \nabla \phi = -\partial_t \nabla \times \vec{A} - 0 = -\partial_t \vec{B} \quad \square$$

$$\text{证明: } \nabla \cdot \vec{B} = \nabla \cdot \nabla \times \vec{A} = 0 \quad \square$$

$$\text{证明: } \nabla \times \vec{B} - \partial_t \vec{E}$$

$$= \nabla \times \nabla \times \vec{A} + \partial_t^2 \vec{A} + \partial_t \nabla \phi = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} + \partial_t^2 \vec{A} + \partial_t \nabla \phi$$

$$= \nabla(\nabla^2 \theta) - \nabla^2 \vec{A} + \partial_t^2 \vec{A} + \nabla \partial_t \phi = -\nabla^2 \vec{A} + \partial_t^2 \vec{A} + \nabla(\partial_t \phi + \nabla^2 \theta)$$

$$= -\vec{J} \quad \square$$

$$\text{证明: } \frac{\nabla \times \vec{B}}{-\nabla^2} + \nabla \theta = \frac{\nabla \times \nabla \times \vec{A}}{-\nabla^2} + \nabla \theta = \frac{\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}}{-\nabla^2} + \nabla \theta = \vec{A} + \frac{\nabla(\nabla^2 \theta)}{-\nabla^2} + \nabla \theta = \vec{A} \quad \square$$

$$\text{证明: } \frac{\nabla \cdot \vec{E}}{-\nabla^2} - \partial_t \theta = \frac{\nabla \cdot (\partial_t \vec{A} + \nabla \phi)}{\nabla^2} - \partial_t \theta = \frac{\partial_t(\nabla \cdot \vec{A}) + \nabla^2 \phi}{\nabla^2} - \partial_t \theta = \phi + \frac{\partial_t \nabla^2 \theta}{\nabla^2} - \partial_t \theta = \phi \quad \square$$

证明完毕。

$$\text{推论1.2.1. } \begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b] \Psi = -i\sigma_{\varsigma ab}^{[\beta \varsigma]} J^b \\ \vec{A} = \frac{-i\varsigma}{\sqrt{2}} \frac{\nabla \times (\Psi - \Psi^*)}{\nabla^2} + \nabla \theta \\ \phi = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi + \Psi^*)}{\nabla^2} - \partial_t \theta \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \phi = \rho - \partial_t \nabla^2 \theta, \nabla \cdot \vec{A} = \nabla^2 \theta \\ \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J} + \nabla(\partial_t \phi + \nabla^2 \theta) \\ \Psi = -\partial_t \vec{A} - \nabla \phi - i\varsigma \nabla \times \vec{A} \end{cases}$$

$$\text{推论1.2.2. } \vec{A} = \vec{A} - \nabla \theta, \phi = \tilde{\phi} + \partial_t \theta, \vec{A} := \frac{\nabla \times \vec{B}}{-\nabla^2}, \tilde{\phi} := \frac{\nabla \cdot \vec{E}}{-\nabla^2}$$

当 $\theta = 0$ 时, 便是辐射规范; 当 $\theta = \frac{\partial_t \phi}{-\nabla^2}$ 时, 便是洛伦兹规范。

### 1.3 电磁场方程的辐射规范势描述( $\theta = 0$ )

$$\text{定理1.3.1. } \begin{cases} \nabla \cdot \vec{E} = -\rho, \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = -\vec{J} + \partial_t \vec{E} \\ \vec{A} = \frac{\nabla \times \vec{B}}{-\nabla^2}, \tilde{\phi} = \frac{\nabla \cdot \vec{E}}{-\nabla^2} \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J} + \partial_t \nabla \tilde{\phi} \\ \nabla^2 \tilde{\phi} = \rho, \nabla \cdot \vec{A} = 0 \\ \vec{E} = -\partial_t \vec{A} - \nabla \tilde{\phi}, \vec{B} = \nabla \times \vec{A} \end{cases}$$

$$\text{推论1.3.1. } \begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b] \Psi = -i\sigma_{\varsigma ab}^{[\beta \varsigma]} J^b \\ \vec{A} = \frac{-i\varsigma}{\sqrt{2}} \frac{\nabla \times (\Psi - \Psi^*)}{\nabla^2}, \tilde{\phi} = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi + \Psi^*)}{\nabla^2} \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J} + \partial_t \nabla \tilde{\phi} \\ \nabla^2 \tilde{\phi} = \rho, \nabla \cdot \vec{A} = 0 \\ \sqrt{2} \Psi = -\partial_t \vec{A} - \nabla \tilde{\phi} - i\varsigma \nabla \times \vec{A} \end{cases}$$

推论1.3.2.

$$[\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \Rightarrow \begin{cases} [\tilde{A}_i(x), \partial_{t'} \tilde{A}_j(x')] = -i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \partial_t \Delta(x - x') \\ [\partial_t \tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \partial_t \Delta(x - x') \\ [\tilde{A}_i(x), (\nabla' \times \tilde{A})_j(x')] = -i\varepsilon_{ij}^k \partial_k \Delta(x - x') \\ [(\nabla \times \tilde{A})_i(x), \tilde{A}_j(x')] = -i\varepsilon_{ij}^k \partial_k \Delta(x - x') \end{cases}$$



## 推论1.3.3.

$$[\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \Rightarrow \begin{cases} [\partial_t \tilde{A}_i(x), \partial_{t'} \tilde{A}_j(x')] = -i(\delta_{ij} \nabla^2 - \partial_i \partial_j) \Delta(x - x') \\ [(\nabla \times \tilde{A})_i(x), (\nabla' \times \tilde{A})_j(x')] = -i(\delta_{ij} \nabla^2 - \partial_i \partial_j) \Delta(x - x') \\ [\partial_t \tilde{A}_i(x), (\nabla' \times \tilde{A})_j(x')] = -i \varepsilon_{ij}^k \partial_k \partial_t \Delta(x - x') \\ [(\nabla \times \tilde{A})_i(x), \partial_{t'} \tilde{A}_j(x')] = i \varepsilon_{ij}^k \partial_k \partial_t \Delta(x - x') \end{cases}$$

## 定理1.3.2.

$$\begin{cases} [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')] = i \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha_\zeta}(x), \Psi_{\beta_\zeta}(x')] = 0, [\Psi_{\alpha'_\zeta}^+(x), \Psi_{\beta'_\zeta}^+(x')] = 0 \\ [\partial_a + i S_{ab}(\gamma, \varsigma) \partial^b] \Psi = -i \sigma_{\varsigma ab}^{[\beta_\zeta]} J^b \\ \tilde{A} = \frac{-i\varsigma}{\sqrt{2}} \frac{\nabla \times (\Psi - \Psi^*)}{\nabla^2}, \tilde{\phi} = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi + \Psi^*)}{\nabla^2} \end{cases} \Leftrightarrow \begin{cases} [\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \\ [\tilde{A}_i(x), \tilde{\phi}(x')] = 0, [\tilde{\phi}(x), \tilde{\phi}(x')] = 0 \\ \nabla^2 \tilde{A} - \partial_t^2 \tilde{A} = \vec{J} + \partial_t \nabla \tilde{\phi}, \nabla^2 \tilde{\phi} = \rho, \nabla \cdot \tilde{A} = 0 \\ \sqrt{2} \Psi = -\partial_t \tilde{A} - \nabla \tilde{\phi} - i\varsigma \nabla \times \tilde{A} \end{cases}$$

## 原始详细证明:(只给一个)

$$\begin{aligned} \text{证明: } & [\tilde{A}_i(x), \tilde{A}_j(x')] \\ &= \frac{i\varsigma}{\sqrt{2}} \frac{1}{\nabla^2} \frac{i\varsigma}{\sqrt{2}} \frac{1}{\nabla'^2} [\varepsilon_i^{kl} \partial_k [\Psi_l(x) - \Psi_l^+(x)], \varepsilon_j^{mn} \partial'_m [\Psi_n(x') - \Psi_n^+(x')]] \\ &= \frac{-1}{2} \frac{1}{\nabla^2 \nabla'^2} \varepsilon_i^{kl} \varepsilon_j^{mn} \partial_k \partial'_m [\Psi_l(x) - \Psi_l^+(x), \Psi_n(x') - \Psi_n^+(x')] \\ &= \frac{1}{2} \frac{1}{\nabla^2 \nabla'^2} \varepsilon_i^{kl} \varepsilon_j^{mn} \partial_k \partial'_m \{ [\Psi_l(x), \Psi_n^+(x')] + [\Psi_l^+(x), \Psi_n(x')] \} \\ &= \frac{1}{2} \frac{1}{\nabla^2 \nabla'^2} \varepsilon_i^{kl} \varepsilon_j^{mn} \partial_k \partial'_m [i \sigma_{ln}^{ab} \partial_a \partial_b \Delta(x - x') - i \sigma_{nl}^{ab} \partial'_a \partial'_b \Delta(x' - x)] \\ &= -\frac{1}{2} \frac{1}{\nabla^4} \varepsilon_i^{kl} \varepsilon_j^{mn} \partial_k \partial'_m [i \sigma_{ln}^{ab} \partial_a \partial_b + i \sigma_{nl}^{ab} \partial'_a \partial'_b] \Delta(x - x') \\ &= \frac{i}{2} \frac{1}{\nabla^4} \varepsilon_i^{kl} \varepsilon_j^{mn} \partial_k \partial'_m [(\nabla^2 - \partial_\pi^2) \delta_{ln} - 2 \partial_l \partial_n] \Delta(x - x') \\ &= \frac{i}{2} \frac{1}{\nabla^4} \varepsilon_i^{kl} \delta_{ln} \varepsilon_j^{mn} \partial_k \partial'_m (\nabla^2 - \partial_\pi^2) \Delta(x - x') \\ &= i \frac{1}{\nabla^4} \varepsilon_i^{kl} \delta_{ln} \varepsilon_j^{mn} \partial_k \partial'_m \nabla^2 \Delta(x - x') \\ &= i \frac{1}{\nabla^2} \varepsilon_i^{kl} \delta_{ln} \varepsilon_j^{mn} \partial_k \partial'_m \Delta(x - x') \\ &= i \frac{1}{\nabla^2} (\delta_{ij} \delta_{km} - \delta_i^k \delta_j^m) \partial_k \partial'_m \Delta(x - x') \\ &= i \frac{1}{\nabla^2} (\delta_{ij} \nabla^2 - \partial_i \partial_j) \Delta(x - x') \\ &= i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \quad \square \end{aligned}$$

## 简洁证明:

$$\begin{aligned} \text{证明: } & [\tilde{A}_i(x), \tilde{A}_j(x')] \\ &= [\frac{(\nabla \times \vec{B})_i}{-\nabla^2}(x), \frac{(\nabla' \times \vec{B})_j}{-\nabla'^2}(x')] = \frac{1}{\nabla^2 \nabla'^2} [(\nabla \times \vec{B})_i(x), (\nabla' \times \vec{B})_j(x')] \\ &= \frac{1}{\nabla^4} i(\delta_{ij} \nabla^2 - \partial_i \partial_j) \nabla^2 \Delta(x - x') = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \quad \square \end{aligned}$$

$$\begin{aligned} \text{证明: } & [\tilde{A}_i(x), \tilde{\phi}(x')] = [\frac{(\nabla \times \vec{B})_i}{-\nabla^2}(x), \frac{\nabla' \cdot \vec{E}}{-\nabla'^2}(x')] = \frac{1}{\nabla^2 \nabla'^2} [(\nabla \times \vec{B})_i(x), \nabla' \cdot \vec{E}(x')] \\ &= \frac{1}{\nabla^4} [\varepsilon_i^{jk} \partial_j B_k(x), \nabla' \cdot \vec{E}(x')] = \frac{1}{\nabla^4} \varepsilon_i^{jk} \partial_j [B_k(x), \nabla' \cdot \vec{E}(x')] = 0 \quad \square \end{aligned}$$

$$\text{证明: } [\tilde{\phi}(x), \tilde{\phi}(x')] = [\frac{\nabla \cdot \vec{E}}{-\nabla^2}(x), \frac{\nabla' \cdot \vec{E}}{-\nabla'^2}(x')] = \frac{1}{\nabla^2 \nabla'^2} [\nabla \cdot \vec{E}(x), \nabla' \cdot \vec{E}(x')] = 0 \quad \square$$

## 反向证明:

$$\begin{aligned} \text{证明: } & [\Psi_i(x), \Psi_j^+(x')] \\ &= \frac{1}{2} [-\partial_t \tilde{A}_i(x) - \partial_i \tilde{\phi}(x) - i\varsigma (\nabla \times \tilde{A})_i(x), -\partial_{t'} \tilde{A}_j(x') - \partial'_j \tilde{\phi}(x') + i\varsigma (\nabla' \times \tilde{A})_j(x')] \\ &= \frac{1}{2} [-\partial_t \tilde{A}_i(x) - i\varsigma (\nabla \times \tilde{A})_i(x), -\partial_{t'} \tilde{A}_j(x') + i\varsigma (\nabla' \times \tilde{A})_j(x')] \\ &= \frac{1}{2} \{ [\partial_t \tilde{A}_i(x), \partial_{t'} \tilde{A}_j(x')] + [(\nabla \times \tilde{A})_i(x), (\nabla' \times \tilde{A})_j(x')] - i\varsigma [\partial_t \tilde{A}_i(x), (\nabla' \times \tilde{A})_j(x')] + i\varsigma [(\nabla \times \tilde{A})_i(x), \partial_{t'} \tilde{A}_j(x')] \} \\ &= -i(\delta_{ij} \nabla^2 - \partial_i \partial_j) \Delta(x - x') - \varsigma \varepsilon_{ij}^k \partial_k \partial_t \Delta(x - x') \\ &= i \sigma_{ij}^{ab} \partial_a \partial_b \Delta(x - x') \quad \square \end{aligned}$$

证明:  $[\Psi_i(x), \Psi_j(x')]$

$$\begin{aligned}
&= \frac{1}{2}[-\partial_t \tilde{A}_i(x) - \partial_i \tilde{\phi}(x) - i\zeta(\nabla \times \tilde{A})_i(x), -\partial_{t'} \tilde{A}_j(x') - \partial'_j \tilde{\phi}(x') - i\zeta(\nabla' \times \tilde{A})_j(x')] \\
&= \frac{1}{2}[-\partial_t \tilde{A}_i(x) - i\zeta(\nabla \times \tilde{A})_i(x), -\partial_{t'} \tilde{A}_j(x') - i\zeta(\nabla' \times \tilde{A})_j(x')] \\
&= \frac{1}{2}\{[\partial_t \tilde{A}_i(x), \partial_{t'} \tilde{A}_j(x')] - [(\nabla \times \tilde{A})_i(x), (\nabla' \times \tilde{A})_j(x')] + i\zeta[\partial_t \tilde{A}_i(x), (\nabla' \times \tilde{A})_j(x')] + i\zeta[(\nabla \times \tilde{A})_i(x), \partial_{t'} \tilde{A}_j(x')]\} \\
&= 0
\end{aligned}$$

证明:  $[\Psi_i^+(x), \Psi_j^+(x')]$

$$\begin{aligned}
&= \frac{1}{2}[-\partial_t \tilde{A}_i(x) - \partial_i \tilde{\phi}(x) + i\zeta(\nabla \times \tilde{A})_i(x), -\partial_{t'} \tilde{A}_j(x') - \partial'_j \tilde{\phi}(x') + i\zeta(\nabla' \times \tilde{A})_j(x')] \\
&= \frac{1}{2}[-\partial_t \tilde{A}_i(x) + i\zeta(\nabla \times \tilde{A})_i(x), -\partial_{t'} \tilde{A}_j(x') + i\zeta(\nabla' \times \tilde{A})_j(x')] \\
&= \frac{1}{2}\{[\partial_t \tilde{A}_i(x), \partial_{t'} \tilde{A}_j(x')] - [(\nabla \times \tilde{A})_i(x), (\nabla' \times \tilde{A})_j(x')] - i\zeta[\partial_t \tilde{A}_i(x), (\nabla' \times \tilde{A})_j(x')] - i\zeta[(\nabla \times \tilde{A})_i(x), \partial_{t'} \tilde{A}_j(x')]\} \\
&= 0
\end{aligned}$$

推论1.3.4.

$$\begin{cases} [\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2})\Delta(x-x') \\ [\tilde{A}_i(x), \tilde{\phi}(x')] = 0, [\tilde{\phi}(x), \tilde{\phi}(x')] = 0 \\ \sqrt{2}\Psi = -\partial_t \tilde{A} - \nabla \tilde{\phi} - i\zeta \nabla \times \tilde{A} \end{cases} \Rightarrow \begin{cases} [\tilde{A}_i(x), E_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2})\partial_t \Delta(x-x') \\ [E_i(x), \tilde{A}_j(x')] = -i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2})\partial_t \Delta(x-x') \\ [\tilde{A}_i(x), B_j(x')] = -i\varepsilon_{ij}^k \partial_k \Delta(x-x') \\ [B_i(x), \tilde{A}_j(x')] = -i\varepsilon_{ij}^k \partial_k \Delta(x-x') \end{cases}$$

证明:  $[B_i(x), \tilde{A}_j(x')]$

$$\begin{aligned}
&= [\varepsilon_i^{kl} \partial_k \tilde{A}_l(x), \tilde{A}_j(x')] \\
&= i\varepsilon_i^{kl} \partial_k (\delta_{lj} - \frac{\partial_l \partial_j}{\nabla^2})\Delta(x-x') \\
&= i\varepsilon_i^{kl} \partial_k \delta_{lj} \Delta(x-x') \\
&= -i\varepsilon_{ij}^k \partial_k \Delta(x-x')
\end{aligned}$$

证明:  $[E_i(x) - i\zeta B_i(x), \tilde{A}_j(x')]$

$$\begin{aligned}
&= [-i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2})\partial_t - \zeta \varepsilon_{ij}^k \partial_k]\Delta(x-x') \\
&= i\frac{\partial_t}{\nabla^2}(\partial_i \partial_j - \delta_{ij} \nabla^2 + i\zeta \varepsilon_{ij}^k \partial_k \partial_t)\Delta(x-x') \\
&= i\sigma_{ij}^{ab} \partial_a \partial_b \frac{\partial_t}{\nabla^2} \Delta(x-x')
\end{aligned}$$

推论1.3.5.  $\sigma_{\alpha\zeta\alpha'}^{ab} \partial_a \partial_b = \partial_{\alpha\zeta} \partial_{\alpha'} - \frac{1}{2} \delta_{\alpha\zeta\alpha'} (\nabla^2 + \partial_t^2) + i\zeta \varepsilon^k{}_{\alpha\zeta\alpha'} \partial_k \partial_t$

推论1.3.6.

$$\begin{cases} [\Psi_{\alpha\zeta}(x), \Psi_{\alpha'\zeta'}^+(x')] = i\sigma_{\alpha\zeta\alpha'}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\Psi_{\alpha\zeta}(x), \Psi_{\beta\zeta}(x')] = 0, [\Psi_{\alpha'\zeta'}^+(x), \Psi_{\beta'\zeta'}^+(x')] = 0 \\ \tilde{A} = \frac{-i\zeta}{\sqrt{2}} \frac{\nabla \times (\Psi - \Psi^*)}{\nabla^2}, \tilde{\phi} = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi + \Psi^*)}{\nabla^2} \end{cases} \Rightarrow \begin{cases} [\Psi_i(x), \tilde{A}_j(x')] = \frac{i}{\sqrt{2}} \sigma_{ij}^{ab} \partial_a \partial_b \frac{\partial_t}{\nabla^2} \Delta(x-x') \\ [\tilde{A}_i(x), \Psi_j(x')] = -\frac{i}{\sqrt{2}} \sigma_{ji}^{ab} \partial_a \partial_b \frac{\partial_t}{\nabla^2} \Delta(x-x') \end{cases}$$

推论1.3.7.

$$\begin{cases} [\Psi_{\alpha\zeta}(x), \Psi_{\alpha'\zeta'}^+(x')] = i\sigma_{\alpha\zeta\alpha'}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\Psi_{\alpha\zeta}(x), \Psi_{\beta\zeta}(x')] = 0, [\Psi_{\alpha'\zeta'}^+(x), \Psi_{\beta'\zeta'}^+(x')] = 0 \\ [\partial_a + iS_{ab}(\gamma, \zeta) \partial^b] \Psi = -i\sigma_{\zeta}^{[\beta\zeta]} J^b \\ \tilde{A} = \frac{-i\zeta}{\sqrt{2}} \frac{\nabla \times (\Psi - \Psi^*)}{\nabla^2}, \tilde{\phi} = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi + \Psi^*)}{\nabla^2} \end{cases} \Rightarrow \begin{cases} [\tilde{\phi}(x), \tilde{A}(x')] = 0, [\tilde{\phi}(x), \tilde{\phi}(x')] = 0 \\ [\tilde{\phi}(x), \Psi(x')] = 0, [\tilde{\phi}(x), \Psi^+(x')] = 0 \\ [J_a(x), \tilde{A}(x')] = 0, [J_a(x), \tilde{\phi}(x')] = 0 \\ [J_a(x), \Psi(x')] = 0, [J_a(x), \Psi^+(x')] = 0 \\ [J_a(x), J_b(x')] = 0 \end{cases}$$

从上可知, 电磁场方程和辐射规范势方程、约束条件与协变对易关系是相容的, 且标势 $\tilde{\phi}(x)$ 和源 $J_a(x)$ 相对于电磁场是c-数, 不是算子。从这个意义上知标势即静电场不可量子化, 因为它连算子都不是。

## 1.4 一般电磁场场强协变对易关系分析

定理1.4.1.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]\Psi(x) = -i\sigma_{\varsigma ab}^{[\beta\zeta]}J^b(x) \\ \Psi(x) = \frac{1}{\sqrt{2}}[\vec{E}(x) - i\varsigma\vec{B}(x)] \end{cases} \Leftrightarrow \begin{cases} \nabla \cdot \vec{E}(x) = -\rho(x), \nabla \times \vec{E}(x) = -\partial_t \vec{B}(x) \\ \nabla \cdot \vec{B}(x) = 0, \nabla \times \vec{B}(x) = -\vec{J}(x) + \partial_t \vec{E}(x) \end{cases}$$

定理1.4.2.

$$\begin{cases} [\Psi_{\alpha\zeta}(x), \Psi_{\alpha'\zeta}^+(x')] = i\sigma_{\alpha\zeta\alpha'\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\Psi_{\alpha\zeta}(x), \Psi_{\beta\zeta}(x')] = 0, [\Psi_{\alpha\zeta}^+(x), \Psi_{\beta\zeta}^+(x')] = 0 \\ \Psi(x) = \frac{1}{\sqrt{2}}[\vec{E}(x) - i\varsigma\vec{B}(x)] \end{cases} \Leftrightarrow \begin{cases} [E_i(x), E_j(x')] = -i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\Delta(x-x') \\ [B_i(x), B_j(x')] = -i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\Delta(x-x') \\ [E_i(x), B_j(x')] = i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x') \\ [B_i(x), E_j(x')] = -i\varepsilon_{ij}{}^k\partial_k\partial_t\Delta(x-x') \end{cases}$$

$$\text{推论1.4.1.} \begin{cases} [\nabla \cdot \vec{E}(x), \vec{E}(x')] = 0 & [\nabla \cdot \vec{E}(x), \nabla' \cdot \vec{E}(x')] = 0 \\ [\nabla \cdot \vec{B}(x), \vec{B}(x')] = 0 & [\nabla \cdot \vec{B}(x), \nabla' \cdot \vec{B}(x')] = 0 \\ [\nabla \cdot \vec{E}(x), \vec{B}(x')] = 0 & [\nabla \cdot \vec{E}(x), \nabla' \cdot \vec{B}(x')] = 0 \\ [\nabla \cdot \vec{B}(x), \vec{E}(x')] = 0 & [\nabla \cdot \vec{B}(x), \nabla' \cdot \vec{E}(x')] = 0 \end{cases}$$

$$\text{推论1.4.2.} \begin{cases} [\partial_t E_i(x) - (\nabla \times \vec{B})_i(x), \vec{E}(x')] = 0 \\ [\partial_t E_i(x) - (\nabla \times \vec{B})_i(x), \vec{B}(x')] = 0 \\ [\partial_t B_i(x) + (\nabla \times \vec{E})_i(x), \vec{E}(x')] = 0 \\ [\partial_t B_i(x) + (\nabla \times \vec{E})_i(x), \vec{B}(x')] = 0 \end{cases}$$

$$\text{推论1.4.3.} \begin{cases} [J_i(x), \vec{E}(x')] = 0, [J_i(x), \vec{B}(x')] = 0 \\ [\rho(x), \vec{E}(x')] = 0, [\rho(x), \vec{B}(x')] = 0 \\ [J_a(x), \vec{E}(x')] = 0, [J_a(x), \vec{B}(x')] = 0 \\ [J_a(x), J_b(x')] = 0 \end{cases}$$

$$\text{推论1.4.4.} \begin{cases} [(\nabla \times \vec{E})_i(x), (\nabla' \times \vec{E})_j(x')] = i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\nabla^2\Delta(x-x') \\ [(\nabla \times \vec{B})_i(x), (\nabla' \times \vec{B})_j(x')] = i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\nabla^2\Delta(x-x') \\ [(\nabla \times \vec{E})_i(x), (\nabla' \times \vec{B})_j(x')] = -i\varepsilon_{ij}{}^k\partial_k\partial_t\nabla^2\Delta(x-x') \\ [(\nabla \times \vec{B})_i(x), (\nabla' \times \vec{E})_j(x')] = i\varepsilon_{ij}{}^k\partial_k\partial_t\nabla^2\Delta(x-x') \end{cases}$$

$$\text{推论1.4.5.} \begin{cases} [\partial_t E_i(x), \partial_{t'} E_j(x')] = i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\nabla^2\Delta(x-x') \\ [\partial_t B_i(x), \partial_{t'} B_j(x')] = i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\nabla^2\Delta(x-x') \\ [\partial_t E_i(x), \partial_{t'} B_j(x')] = -i\varepsilon_{ij}{}^k\partial_k\partial_t\nabla^2\Delta(x-x') \\ [\partial_t B_i(x), \partial_{t'} E_j(x')] = i\varepsilon_{ij}{}^k\partial_k\partial_t\nabla^2\Delta(x-x') \end{cases}$$

$$\text{推论1.4.6.} \begin{cases} [\partial_t E_i(x), (\nabla' \times \vec{B})_j(x')] = i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\nabla^2\Delta(x-x') \\ [(\nabla \times \vec{B})_i(x), \partial_{t'} E_j(x')] = i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\nabla^2\Delta(x-x') \\ [\partial_t B_i(x), (\nabla' \times \vec{E})_j(x')] = -i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\nabla^2\Delta(x-x') \\ [(\nabla \times \vec{E})_i(x), \partial_{t'} B_j(x')] = -i(\delta_{ij}\nabla^2 - \partial_i\partial_j)\nabla^2\Delta(x-x') \end{cases}$$

$$\text{推论1.4.7.} \quad \begin{cases} [\partial_t E_i(x), (\nabla' \times \vec{E})_j(x')] = i\varepsilon_{ij}^k \partial_k \partial_t \Delta(x - x') \\ [(\nabla \times \vec{E})_i(x), \partial_{t'} E_j(x')] = -i\varepsilon_{ij}^k \partial_k \partial_t \Delta(x - x') \\ [\partial_t B_i(x), (\nabla' \times \vec{B})_j(x')] = i\varepsilon_{ij}^k \partial_k \partial_t \Delta(x - x') \\ [(\nabla \times \vec{B})_i(x), \partial_{t'} B_j(x')] = -i\varepsilon_{ij}^k \partial_k \partial_t \Delta(x - x') \end{cases}$$

从上可知，一般电磁场方程和约束条件与协变对易关系是相容的，且源 $J_a(x)$ 相对于电磁场是c-数。

### 1.5 电磁场方程的洛伦兹 $\lambda$ -规范势描述( $\theta = \frac{\partial_t \phi}{-\nabla^2}$ )有内在矛盾

$$\text{定理1.5.1.} \quad \begin{cases} \nabla \cdot \vec{E} = -\rho, \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = -\vec{J} + \partial_t \vec{E} \\ \vec{A} = \frac{\nabla \times \vec{B} + \nabla \partial_t \phi}{-\nabla^2}, \phi = \frac{\nabla \cdot \vec{E} - \partial_t^2 \phi}{-\nabla^2} \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J}, \nabla^2 \phi - \partial_t^2 \phi = \rho \\ \nabla \cdot \vec{A} + \partial_t \phi = 0 \\ \vec{E} = -\partial_t \vec{A} - \nabla \phi, \vec{B} = \nabla \times \vec{A} \end{cases}$$

$$\text{定理1.5.2.} \quad \begin{cases} \langle |\nabla \cdot \vec{E} = -\rho| \rangle, \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{B} = 0, \langle |\nabla \times \vec{B} = -\vec{J} + \partial_t \vec{E}| \rangle \\ \vec{A} = \frac{\nabla \times \vec{B} + \nabla \partial_t \phi}{-\nabla^2}, \phi = \frac{\nabla \cdot \vec{E} - \partial_t^2 \phi}{-\nabla^2} \end{cases}$$

$$\Leftrightarrow \begin{cases} \nabla \cdot \vec{E} = -\rho - \partial_t (\nabla \cdot \vec{A} + \partial_t \phi), \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = -\vec{J} + \partial_t \vec{E} + \nabla (\nabla \cdot \vec{A} + \partial_t \phi) \\ \langle |\vec{A} = \frac{\nabla \times \vec{B} + \nabla \partial_t \phi}{-\nabla^2}| \rangle, \phi = \frac{\nabla \cdot \vec{E} - \partial_t^2 \phi}{-\nabla^2} \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J}, \nabla^2 \phi - \partial_t^2 \phi = \rho \\ \langle |\nabla \cdot \vec{A} + \partial_t \phi| \rangle = 0 \\ \vec{E} = -\partial_t \vec{A} - \nabla \phi, \vec{B} = \nabla \times \vec{A} \end{cases}$$

$$\text{推论1.5.1.} \quad \begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b] \Psi = -i\sigma_{\alpha\beta}^{[\beta\varsigma]} J^b \\ \vec{A} = \frac{-i\varsigma}{\sqrt{2}} \frac{\nabla \times (\Psi - \Psi^*)}{\nabla^2} - \frac{\nabla \partial_t \phi}{\nabla^2} \\ \phi = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi + \Psi^*)}{\nabla^2} + \frac{\partial_t^2 \phi}{\nabla^2} \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J}, \nabla^2 \phi - \partial_t^2 \phi = \rho \\ \nabla \cdot \vec{A} + \partial_t \phi = 0 \\ \sqrt{2} \Psi = -\partial_t \vec{A} - \nabla \phi - i\varsigma \nabla \times \vec{A} \end{cases}$$

推论1.5.2.

$$\begin{cases} [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\lambda-1}{\lambda} \frac{\partial_a \partial_b}{\square + i\varepsilon}) \Delta(x - x') \\ \phi = -iA_0, \sqrt{2} \Psi = -\partial_t \vec{A} - \nabla \phi - i\varsigma \nabla \times \vec{A} \end{cases} \Rightarrow \begin{cases} [\Psi_{\alpha\varsigma}(x), \Psi_{\alpha'\varsigma'}^+(x')] = i\sigma_{\alpha\alpha'}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha\varsigma}(x), \Psi_{\beta\varsigma'}(x')] = 0, [\Psi_{\alpha'\varsigma'}^+(x), \Psi_{\beta'\varsigma'}^+(x')] = 0 \\ [\Psi_i(x), \phi(x')] = [\Psi_i^+(x), \phi(x')] = \frac{i}{\sqrt{2}} \partial_i \Delta(x - x') \\ [\phi(x), \phi(x')] = -i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square + i\varepsilon}) \Delta(x - x') \end{cases}$$

证明:  $[\Psi_i(x), \Psi_j(x')]$

$$\begin{aligned} &= \frac{1}{2} \{ [-\partial_t A_i(x) - \partial_i \phi(x) - i\varsigma (\nabla \times \vec{A})_i(x), -\partial_{t'} A_j(x') - \partial_j' \phi(x') - i\varsigma (\nabla' \times \vec{A})_j(x')] \\ &= \frac{1}{2} \{ [-\partial_t A_i(x) - i\varsigma (\nabla \times \vec{A})_i(x), -\partial_{t'} A_j(x') - i\varsigma (\nabla' \times \vec{A})_j(x')] \\ &+ [\partial_i \phi(x), \partial_j' \phi(x')] + [\partial_i \phi(x), \partial_{t'} A_j(x')] + [\partial_t A_i(x), \partial_j' \phi(x')] \} \\ &= \frac{1}{2} \{ [\partial_t A_i(x), \partial_{t'} A_j(x')] - [(\nabla \times \vec{A})_i(x), (\nabla' \times \vec{A})_j(x')] + i\varsigma [\partial_t A_i(x), (\nabla' \times \vec{A})_j(x')] + i\varsigma [(\nabla \times \vec{A})_i(x), \partial_{t'} A_j(x')] \\ &+ [\partial_i \phi(x), \partial_j' \phi(x')] + [\partial_i \phi(x), \partial_{t'} A_j(x')] + [\partial_t A_i(x), \partial_j' \phi(x')] \} \\ &= \frac{1}{2} \{ [\partial_t A_i(x), \partial_{t'} A_j(x')] - [(\nabla \times \vec{A})_i(x), (\nabla' \times \vec{A})_j(x')] + [\partial_i \phi(x), \partial_j' \phi(x')] + [\partial_i \phi(x), \partial_{t'} A_j(x')] + [\partial_t A_i(x), \partial_j' \phi(x')] \} \\ &= \frac{1}{2} \{ i \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_j \nabla^2}{\square + i\varepsilon} - i \partial_i \partial_j + i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square + i\varepsilon}) \partial_i \partial_j - i \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_j \nabla^2}{\square + i\varepsilon} - i \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_j \nabla^2}{\square + i\varepsilon} \} \Delta(x - x') \\ &= 0 \end{aligned}$$

□

证明:  $[\Psi_i^+(x), \Psi_j^+(x')]$

$$\begin{aligned} &= \frac{1}{2} \{ [-\partial_t A_i(x) - \partial_i \phi(x) + i\varsigma (\nabla \times \vec{A})_i(x), -\partial_{t'} A_j(x') - \partial_j' \phi(x') + i\varsigma (\nabla' \times \vec{A})_j(x')] \\ &= \frac{1}{2} \{ [-\partial_t A_i(x) + i\varsigma (\nabla \times \vec{A})_i(x), -\partial_{t'} A_j(x') + i\varsigma (\nabla' \times \vec{A})_j(x')] \\ &+ [\partial_i \phi(x), \partial_j' \phi(x')] + [\partial_i \phi(x), \partial_{t'} A_j(x')] + [\partial_t A_i(x), \partial_j' \phi(x')] \} \\ &= \frac{1}{2} \{ [\partial_t A_i(x), \partial_{t'} A_j(x')] - [(\nabla \times \vec{A})_i(x), (\nabla' \times \vec{A})_j(x')] - i\varsigma [\partial_t A_i(x), (\nabla' \times \vec{A})_j(x')] - i\varsigma [(\nabla \times \vec{A})_i(x), \partial_{t'} A_j(x')] \\ &+ [\partial_i \phi(x), \partial_j' \phi(x')] + [\partial_i \phi(x), \partial_{t'} A_j(x')] + [\partial_t A_i(x), \partial_j' \phi(x')] \} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \{ [\partial_t A_i(x), \partial_{t'} A_j(x')] - [(\nabla \times \vec{A})_i(x), (\nabla' \times \vec{A})_j(x')] + [\partial_i \phi(x), \partial'_j \phi(x')] + [\partial_i \phi(x), \partial_{t'} A_j(x')] + [\partial_t A_i(x), \partial'_j \phi(x')] \} \\
&= \frac{1}{2} \{ i \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_j \nabla^2}{\square+i\varepsilon} - i \partial_i \partial_j + i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \partial_i \partial_j - i \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_j \nabla^2}{\square+i\varepsilon} - i \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_j \nabla^2}{\square+i\varepsilon} \} \Delta(x-x') \\
&= 0
\end{aligned} \quad \square$$

证明:  $[\Psi_i(x), \Psi_j^+(x')]$

$$\begin{aligned}
&= \frac{1}{2} \{ [-\partial_t A_i(x) - \partial_i \phi(x) - i\zeta(\nabla \times \vec{A})_i(x), -\partial_{t'} A_j(x') - \partial'_j \phi(x') + i\zeta(\nabla' \times \vec{A})_j(x')] \\
&= \frac{1}{2} [-\partial_t A_i(x) - i\zeta(\nabla \times \vec{A})_i(x), -\partial_{t'} A_j(x') + i\zeta(\nabla' \times \vec{A})_j(x')] \\
&+ [\partial_i \phi(x), \partial'_j \phi(x')] + [\partial_i \phi(x), \partial_{t'} A_j(x')] + [\partial_t A_i(x), \partial'_j \phi(x')] \} \\
&= \frac{1}{2} \{ [\partial_t A_i(x), \partial_{t'} A_j(x')] + [(\nabla \times \vec{A})_i(x), (\nabla' \times \vec{A})_j(x')] - i\zeta[\partial_t A_i(x), (\nabla' \times \vec{A})_j(x')] + i\zeta[(\nabla \times \vec{A})_i(x), \partial_{t'} A_j(x')] \\
&+ [\partial_i \phi(x), \partial'_j \phi(x')] + [\partial_i \phi(x), \partial_{t'} A_j(x')] + [\partial_t A_i(x), \partial'_j \phi(x')] \} \\
&= [(\nabla \times \vec{A})_i(x), (\nabla' \times \vec{A})_j(x')] - i\zeta[\partial_t A_i(x), (\nabla' \times \vec{A})_j(x')] \\
&+ \frac{1}{2} \{ [\partial_t A_i(x), \partial_{t'} A_j(x')] - [(\nabla \times \vec{A})_i(x), (\nabla' \times \vec{A})_j(x')] + i\zeta[\partial_t A_i(x), (\nabla' \times \vec{A})_j(x')] + i\zeta[(\nabla \times \vec{A})_i(x), \partial_{t'} A_j(x')] \\
&+ [\partial_i \phi(x), \partial'_j \phi(x')] + [\partial_i \phi(x), \partial_{t'} A_j(x')] + [\partial_t A_i(x), \partial'_j \phi(x')] \} \\
&= [(\nabla \times \vec{A})_i(x), (\nabla' \times \vec{A})_j(x')] - i\zeta[\partial_t A_i(x), (\nabla' \times \vec{A})_j(x')] \\
&+ \frac{1}{2} \{ [\partial_t A_i(x), \partial_{t'} A_j(x')] - [(\nabla \times \vec{A})_i(x), (\nabla' \times \vec{A})_j(x')] + [\partial_i \phi(x), \partial'_j \phi(x')] + [\partial_i \phi(x), \partial_{t'} A_j(x')] + [\partial_t A_i(x), \partial'_j \phi(x')] \} \\
&= [(\nabla \times \vec{A})_i(x), (\nabla' \times \vec{A})_j(x')] - i\zeta[\partial_t A_i(x), (\nabla' \times \vec{A})_j(x')] \\
&= -i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \nabla^2 \Delta(x-x') - \zeta \varepsilon_{ij}^k \partial_k \partial_t \Delta(x-x') \\
&= i\sigma_{ij}^{ab} \partial_a \partial_b \Delta(x-x')
\end{aligned} \quad \square$$

$$\text{证明: } [\phi(x), \phi(x')] = -[A_0(x), A_0(x')] = -i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x-x') \quad \square$$

证明:  $[\Psi_i(x), \phi(x')]$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} [-\partial_t A_i(x) - \partial_i \phi(x) - i\zeta(\nabla \times \vec{A})_i(x), \phi(x')] \\
&= -\frac{1}{\sqrt{2}} [\partial_t A_i(x) + \partial_i \phi(x), \phi(x')] \\
&= -\frac{1}{\sqrt{2}} [i\partial_t \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_t}{\square+i\varepsilon} \Delta(x-x') - i\partial_i (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x-x')] \\
&= -\frac{i}{\sqrt{2}} [\frac{\lambda-1}{\lambda} \frac{\partial_i \nabla^2}{\square+i\varepsilon} - \partial_i (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon})] \Delta(x-x') \\
&= \frac{i}{\sqrt{2}} \partial_i \Delta(x-x')
\end{aligned} \quad \square$$

证明:  $[\Psi_i^+(x), \phi(x')]$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} [-\partial_t A_i(x) - \partial_i \phi(x) + i\zeta(\nabla \times \vec{A})_i(x), \phi(x')] \\
&= -\frac{1}{\sqrt{2}} [\partial_t A_i(x) + \partial_i \phi(x), \phi(x')] \\
&= -\frac{1}{\sqrt{2}} [i\partial_t \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_t}{\square+i\varepsilon} \Delta(x-x') - i\partial_i (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x-x')] \\
&= -\frac{i}{\sqrt{2}} [\frac{\lambda-1}{\lambda} \frac{\partial_i \nabla^2}{\square+i\varepsilon} - \partial_i (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon})] \Delta(x-x') \\
&= \frac{i}{\sqrt{2}} \partial_i \Delta(x-x')
\end{aligned} \quad \square$$

推论1.5.3.

$$\left\{ \begin{aligned} [A_a(x), A_b(x')] &= i(\delta_{ab} - \frac{\lambda-1}{\lambda} \frac{\partial_a \partial_b}{\square+i\varepsilon}) \Delta(x-x') \\ \nabla^2 \vec{A} - \partial_t^2 \vec{A} &= \vec{J}, \nabla^2 \phi - \partial_t^2 \phi = \rho \\ ?\nabla \cdot \vec{A} + \partial_t \phi &= 0?, \phi = -iA_0 \\ \sqrt{2}\Psi &= -\partial_t \vec{A} - \nabla \phi - i\zeta \nabla \times \vec{A} \end{aligned} \right. \Rightarrow \left\{ \begin{aligned} [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')] &= i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\Psi_{\alpha_\zeta}(x), \Psi_{\beta_\zeta}(x')] &= 0, [\Psi_{\alpha'_\zeta}^+(x), \Psi_{\beta'_\zeta}^+(x')] = 0 \\ [\Psi_i(x), \phi(x')] &= [\Psi_i^+(x), \phi(x')] = \frac{i}{\sqrt{2}} \partial_i \Delta(x-x') \\ [\phi(x), \phi(x')] &= -i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x-x') \\ [\partial_a + iS_{ab}(\gamma, \zeta) \partial^b] \Psi &= -i\sigma_{\zeta ab}^{[\beta \zeta]} J^b, A_0(x) = i\phi(x) \\ \vec{A} &= \frac{-i\zeta}{\sqrt{2}} \frac{\nabla \times (\Psi - \Psi^*)}{\nabla^2} - \frac{\nabla \partial_t}{\nabla^2} \phi, \phi = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi + \Psi^*)}{\nabla^2} + \frac{\partial_t^2}{\nabla^2} \phi \end{aligned} \right.$$

推论1.5.4.

$$\begin{cases} [\Psi_{\alpha\zeta}(x), \Psi_{\alpha'\zeta'}^+(x')] = i\sigma_{\alpha\zeta\alpha'\zeta'}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\Psi_{\alpha\zeta}(x), \Psi_{\beta\zeta'}^+(x')] = 0, [\Psi_{\alpha'\zeta'}^+(x), \Psi_{\beta\zeta}^+(x')] = 0 \\ [\Psi_i(x), \phi(x')] = [\Psi_i^+(x), \phi(x')] = \frac{i}{\sqrt{2}} \partial_i \Delta(x-x') \\ [\phi(x), \phi(x')] = -i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\epsilon}) \Delta(x-x') \\ \vec{A} = \frac{-i\zeta}{\sqrt{2}} \frac{\nabla \times (\Psi - \Psi^*)}{\nabla^2} - \frac{\nabla \partial_t}{\nabla^2} \phi, \phi = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi + \Psi^*)}{\nabla^2} + \frac{\partial_t^2}{\nabla^2} \phi \end{cases} \Rightarrow \begin{cases} [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\lambda-1}{\lambda} \frac{\partial_a \partial_b}{\square+i\epsilon}) \Delta(x-x') \\ \phi = -iA_0 \end{cases}$$

证明:  $[A_i(x), A_j(x')]$

$$\begin{aligned} &= [\frac{-i\zeta}{\sqrt{2}} \frac{(\nabla \times (\Psi(x) - \Psi^*(x)))_i}{\nabla^2} - \frac{\partial_i \partial_t}{\nabla^2} \phi(x), \frac{-i\zeta}{\sqrt{2}} \frac{(\nabla' \times (\Psi(x') - \Psi^*(x')))_j}{\nabla'^2} - \frac{\partial'_j \partial_{t'}}{\nabla'^2} \phi(x')] \\ &= [\frac{-i\zeta}{\sqrt{2}} \frac{(\nabla \times (\Psi(x) - \Psi^*(x)))_i}{\nabla^2}, \frac{-i\zeta}{\sqrt{2}} \frac{(\nabla' \times (\Psi(x') - \Psi^*(x')))_j}{\nabla'^2}] + [\frac{\partial_i \partial_t}{\nabla^2} \phi(x), \frac{\partial'_j \partial_{t'}}{\nabla'^2} \phi(x')] \\ &+ [\frac{-i\zeta}{\sqrt{2}} \frac{(\nabla \times (\Psi(x) - \Psi^*(x)))_i}{\nabla^2}, -\frac{\partial'_j \partial_{t'}}{\nabla'^2} \phi(x')] + [-\frac{\partial_i \partial_t}{\nabla^2} \phi(x), \frac{-i\zeta}{\sqrt{2}} \frac{(\nabla' \times (\Psi(x') - \Psi^*(x')))_j}{\nabla'^2}] \\ &= [\frac{-i\zeta}{\sqrt{2}} \frac{(\nabla \times (\Psi(x) - \Psi^*(x)))_i}{\nabla^2}, \frac{-i\zeta}{\sqrt{2}} \frac{(\nabla' \times (\Psi(x') - \Psi^*(x')))_j}{\nabla'^2}] + [\frac{\partial_i \partial_t}{\nabla^2} \phi(x), \frac{\partial'_j \partial_{t'}}{\nabla'^2} \phi(x')] \\ &= [\frac{(\nabla \times \vec{B})_i}{-\nabla^2}, \frac{(\nabla' \times \vec{B})_j}{-\nabla'^2}] + [\frac{\partial_i \partial_t}{\nabla^2} \phi(x), \frac{\partial'_j \partial_{t'}}{\nabla'^2} \phi(x')] \\ &= \frac{1}{\nabla^2 \nabla'^2} [(\nabla \times \vec{B})_i, (\nabla' \times \vec{B})_j] + \frac{\partial_i \partial_t}{\nabla^2} \frac{\partial'_j \partial_{t'}}{\nabla'^2} [\phi(x), \phi(x')] \\ &= \frac{1}{\nabla^2 \nabla'^2} i(\delta_{ij} \nabla^2 - \partial_i \partial_j) \nabla^2 \Delta(x-x') - i \frac{\partial_i \partial_t}{\nabla^2} \frac{\partial'_j \partial_{t'}}{\nabla'^2} (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\epsilon}) \Delta(x-x') \\ &= \frac{1}{\nabla^2} i(\delta_{ij} \nabla^2 - \partial_i \partial_j) \Delta(x-x') - i \frac{\partial_i \partial_t}{\nabla^2} (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\epsilon}) \Delta(x-x') \\ &= \frac{1}{\nabla^2} i \delta_{ij} \nabla^2 \Delta(x-x') - i \frac{\partial_i \partial_t}{\nabla^2} (2 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\epsilon}) \Delta(x-x') \\ &= i(\delta_{ij} - \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_j}{\square+i\epsilon}) \Delta(x-x') - 2i \frac{\partial_i \partial_t}{\nabla^2} \Delta(x-x') \end{aligned} \quad \square$$

证明:  $[A_i(x), \phi(x')]$

$$\begin{aligned} &= [\frac{i\zeta}{\sqrt{2}} \frac{(\nabla \times (\Psi(x) - \Psi^*(x)))_i}{\nabla^2} - \frac{\partial_i \partial_t}{\nabla^2} \phi(x), \phi(x')] \\ &= -[\frac{\partial_i \partial_t}{\nabla^2} \phi(x), \phi(x')] \\ &= i \frac{\partial_i \partial_t}{\nabla^2} (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\epsilon}) \Delta(x-x') \\ &= i \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_t}{\square+i\epsilon} \Delta(x-x') + i \frac{\partial_i \partial_t}{\nabla^2} \Delta(x-x') \end{aligned} \quad \square$$

$$\text{推论1.5.5. } [\tilde{A}_i(x), \phi(x')] = -i \frac{\partial_i \partial_t}{\nabla^2} \Delta(x-x') \Leftrightarrow \begin{cases} [\Psi_i(x), \phi(x')] = \frac{i}{\sqrt{2}} \partial_i \Delta(x-x') \\ [\Psi_i^+(x), \phi(x')] = \frac{i}{\sqrt{2}} \partial_i \Delta(x-x') \end{cases}$$

从上可知, 电磁场方程约束条件与协变对易关系不相容的, 如何合理重新选择额外引入的 $\phi$ 的对易关系去解决这个问题? 虽然传统是把约束条件不看作算符方程, 作为对物理态的挑选, 但这样做不自然, 所以有必要寻求更合理的势协变方案。

## 1.6 洛伦兹规范势和辐射规范势两种描述的等价转换

### 1.6.1 两种规范势方程的等价性

定理1.6.1.

$$\begin{cases} \nabla^2 \tilde{A} - \partial_t^2 \tilde{A} = \vec{J} + \partial_t \nabla \tilde{\phi}, \nabla^2 \tilde{\phi} = \rho \\ \vec{A} = \tilde{A} - \nabla \frac{\partial_t}{\nabla^2} \phi, \phi = \tilde{\phi} + \partial_t \frac{\partial_t}{\nabla^2} \phi \\ \vec{E} = -\partial_t \tilde{A} - \nabla \tilde{\phi}, \vec{B} = \nabla \times \tilde{A} \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J}, \nabla^2 \phi - \partial_t^2 \phi = \rho \\ \vec{A} = \vec{A} + \nabla \frac{\partial_t}{\nabla^2} \phi, \tilde{\phi} = \phi - \partial_t \frac{\partial_t}{\nabla^2} \phi \\ \vec{E} = -\partial_t \vec{A} - \nabla \phi, \vec{B} = \nabla \times \vec{A} \end{cases}$$

定理1.6.2.

$$\begin{cases} \nabla^2 \tilde{A} - \partial_t^2 \tilde{A} = \vec{J} + \partial_t \nabla \tilde{\phi}, \nabla^2 \tilde{\phi} = \rho \\ \nabla \cdot \tilde{A} = 0 \\ \vec{A} = \tilde{A} - \nabla \frac{\partial_t}{\nabla^2} \phi, \phi = \tilde{\phi} + \partial_t \frac{\partial_t}{\nabla^2} \phi \\ \vec{E} = -\partial_t \tilde{A} - \nabla \tilde{\phi}, \vec{B} = \nabla \times \tilde{A} \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J}, \nabla^2 \phi - \partial_t^2 \phi = \rho \\ \nabla \cdot \vec{A} + \partial_t \phi = 0 \\ \vec{A} = \vec{A} + \nabla \frac{\partial_t}{\nabla^2} \phi, \tilde{\phi} = \phi - \partial_t \frac{\partial_t}{\nabla^2} \phi \\ \vec{E} = -\partial_t \vec{A} - \nabla \phi, \vec{B} = \nabla \times \vec{A} \end{cases}$$

定理1.6.3.

$$\begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J} + \partial_t \nabla \tilde{\phi}, \nabla^2 \tilde{\phi} = \rho \\ \langle |\nabla \cdot \vec{A}| \rangle = 0 \\ \vec{A} = \tilde{A} - \nabla \frac{\partial_t}{\nabla^2} \phi, \phi = \tilde{\phi} + \partial_t \frac{\partial_t}{\nabla^2} \phi \\ \vec{E} = -\partial_t \vec{A} - \nabla \tilde{\phi}, \vec{B} = \nabla \times \vec{A} \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J}, \nabla^2 \phi - \partial_t^2 \phi = \rho \\ \langle |\nabla \cdot \vec{A} + \partial_t \phi| \rangle = 0 \\ \tilde{A} = \vec{A} + \nabla \frac{\partial_t}{\nabla^2} \phi, \tilde{\phi} = \phi - \partial_t \frac{\partial_t}{\nabla^2} \phi \\ \vec{E} = -\partial_t \vec{A} - \nabla \phi, \vec{B} = \nabla \times \vec{A} \end{cases}$$

### 1.6.2 两种规范势对易关系的等价性

定理1.6.4.

$$\begin{cases} [\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \\ [\tilde{A}_i(x), \tilde{\phi}(x')] = [\tilde{\phi}(x), \tilde{\phi}(x')] = 0 \\ \phi(x) = \tilde{\phi}(x) + \partial_t \frac{\partial_t}{\nabla^2} \phi(x) \\ A_i(x) = \tilde{A}_i(x) - \frac{\partial_i \partial_t}{\nabla^2} \phi(x), A_0(x) = i\phi(x) \\ [\phi(x), \phi(x')] = -i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x') \\ [\tilde{A}_i(x), \phi(x')] = -i \frac{\partial_i \partial_t}{\nabla^2} \Delta(x - x'), [\tilde{\phi}(x), \phi(x')] = 0 \end{cases} \Leftrightarrow \begin{cases} [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\lambda-1}{\lambda} \frac{\partial_a \partial_b}{\square+i\varepsilon}) \Delta(x - x') \\ \tilde{A}_i(x) = A_i(x) + \frac{\partial_i \partial_t}{\nabla^2} \phi(x) \\ \tilde{\phi}(x) = \phi(x) - \partial_t \frac{\partial_t}{\nabla^2} \phi(x) \\ \phi(x) = -iA_0(x) \end{cases}$$

证明:  $[A_i(x), A_j(x')]$

$$\begin{aligned} &= [\tilde{A}_i(x), \tilde{A}_j(x')] + [\frac{\partial_i \partial_t}{\nabla^2} \phi(x), \frac{\partial_j \partial_t}{\nabla^2} \phi(x')] - [\tilde{A}_i(x), \frac{\partial_j \partial_t}{\nabla^2} \phi(x')] - [\frac{\partial_i \partial_t}{\nabla^2} \phi(x), \tilde{A}_j(x')] \\ &= [\tilde{A}_i(x), \tilde{A}_j(x')] + \frac{\partial_i \partial_t}{\nabla^2} \frac{\partial_j \partial_t}{\nabla^2} [\phi(x), \phi(x')] - \frac{\partial_j \partial_t}{\nabla^2} [\tilde{A}_i(x), \phi(x')] - \frac{\partial_i \partial_t}{\nabla^2} [\phi(x), \tilde{A}_j(x')] \\ &= [\tilde{A}_i(x), \tilde{A}_j(x')] - i \frac{\partial_i \partial_t}{\nabla^2} \frac{\partial_j \partial_t}{\nabla^2} (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x') + i \frac{\partial_j \partial_t}{\nabla^2} \frac{\partial_i \partial_t}{\nabla^2} \Delta(x - x') - i \frac{\partial_i \partial_t}{\nabla^2} \frac{\partial_j \partial_t}{\nabla^2} \Delta(x' - x) \\ &= [\tilde{A}_i(x), \tilde{A}_j(x')] + i \frac{\partial_i \partial_t}{\nabla^2} \frac{\partial_j \partial_t}{\nabla^2} (1 - \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x') \\ &= [\tilde{A}_i(x), \tilde{A}_j(x')] + i \frac{\partial_i \partial_j}{\nabla^2} (1 - \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x') \\ &= i(\delta_{ij} - \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_j}{\square+i\varepsilon}) \Delta(x - x') \quad \square \end{aligned}$$

证明:  $[A_0(x), A_0(x')] = -[\phi(x), \phi(x')] = i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x')$  □

$$\begin{aligned} &\text{证明: } [A_i(x), A_0(x')] = i[A_i(x), \phi(x')] = i[\tilde{A}_i(x), \phi(x')] + i[-\frac{\partial_i \partial_t}{\nabla^2} \phi(x), \phi(x')] \\ &= \frac{\partial_i \partial_t}{\nabla^2} \Delta(x - x') - \frac{\partial_i \partial_t}{\nabla^2} (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x') \\ &= -\frac{\lambda-1}{\lambda} \frac{\partial_i \partial_t}{\square+i\varepsilon} \Delta(x - x') = i(\delta_{i\pi} - \frac{\lambda-1}{\lambda} \frac{\partial_i \partial_\pi}{\square+i\varepsilon}) \Delta(x - x') \quad \square \end{aligned}$$

反向证明:

证明:  $[\phi(x), \phi(x')] = -[A_0(x), A_0(x')] = -i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x')$  □

证明:  $[\tilde{A}_i(x), \phi(x')] = [A_i(x), \phi(x')] + [\frac{\partial_i \partial_t}{\nabla^2} \phi(x), \phi(x')] = -i \frac{\partial_i \partial_t}{\nabla^2} \Delta(x - x')$  □

证明:  $[\phi(x), \tilde{A}_i(x')] = [\phi(x), A_i(x')] + [\phi(x), \frac{\partial_i \partial_t}{\nabla^2} \phi(x')] = -i \frac{\partial_i \partial_t}{\nabla^2} \Delta(x - x')$  □

证明:  $[\tilde{A}_i(x), \tilde{A}_j(x')]$

$$\begin{aligned} &= [A_i(x), A_j(x')] + [\frac{\partial_i \partial_t}{\nabla^2} \phi(x), \frac{\partial_j \partial_t}{\nabla^2} \phi(x')] + [A_i(x), \frac{\partial_j \partial_t}{\nabla^2} \phi(x')] + [\frac{\partial_i \partial_t}{\nabla^2} \phi(x), A_j(x')] \\ &= [A_i(x), A_j(x')] + \frac{\partial_i \partial_t}{\nabla^2} \frac{\partial_j \partial_t}{\nabla^2} [\phi(x), \phi(x')] + \frac{\partial_j \partial_t}{\nabla^2} [A_i(x), \phi(x')] + \frac{\partial_i \partial_t}{\nabla^2} [\phi(x), A_j(x')] \\ &= [A_i(x), A_j(x')] - i \frac{\partial_i \partial_t}{\nabla^2} \frac{\partial_j \partial_t}{\nabla^2} (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x') + i \frac{\partial_j \partial_t}{\nabla^2} \frac{\partial_i \partial_t}{\nabla^2} \Delta(x - x') - i \frac{\partial_i \partial_t}{\nabla^2} \frac{\partial_j \partial_t}{\nabla^2} \Delta(x' - x) \\ &= [A_i(x), A_j(x')] - i \frac{\partial_i \partial_j}{\nabla^2} (1 - \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x') \\ &= [A_i(x), A_j(x')] - i \frac{\partial_i \partial_t}{\nabla^2} \frac{\partial_j \partial_t}{\nabla^2} (1 - \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x') \\ &= [A_i(x), A_j(x')] - i \frac{\partial_i \partial_j}{\nabla^2} (1 - \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\varepsilon}) \Delta(x - x') \\ &= i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \quad \square \end{aligned}$$

### 1.6.3 两种规范势方程和对易关系联立后的等价性

推论1.6.1.

$$\left\{ \begin{array}{l} \nabla^2 \tilde{A} - \partial_t^2 \tilde{A} = \vec{J} + \partial_t \nabla \tilde{\phi}, \nabla^2 \tilde{\phi} = \rho \\ \vec{E} = -\partial_t \tilde{A} - \nabla \tilde{\phi}, \vec{B} = \nabla \times \tilde{A} \\ [\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \\ [\tilde{A}_i(x), \tilde{\phi}(x')] = [\tilde{\phi}(x), \tilde{\phi}(x')] = 0 \\ \phi(x) = \tilde{\phi}(x) + \partial_t \frac{\partial_t}{\nabla^2} \phi(x) \\ A_i(x) = \tilde{A}_i(x) - \frac{\partial_i \partial_t}{\nabla^2} \phi(x), A_0(x) = i\phi(x) \\ [\phi(x), \phi(x')] = -i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square + i\epsilon}) \Delta(x - x') \\ [\tilde{A}_i(x), \phi(x')] = -i \frac{\partial_i \partial_t}{\nabla^2} \Delta(x - x'), [\tilde{\phi}(x), \phi(x')] = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J}, \nabla^2 \phi - \partial_t^2 \phi = \rho \\ \vec{E} = -\partial_t \vec{A} - \nabla \phi, \vec{B} = \nabla \times \vec{A} \\ [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\lambda-1}{\lambda} \frac{\partial_a \partial_b}{\square + i\epsilon}) \Delta(x - x') \\ \tilde{A}_i(x) = A_i(x) + \frac{\partial_i \partial_t}{\nabla^2} \phi(x) \\ \tilde{\phi}(x) = \phi(x) - \partial_t \frac{\partial_t}{\nabla^2} \phi(x) \\ \phi(x) = -iA_0(x) \end{array} \right.$$

### 1.6.4 规范条件与对易关系的不相容性

推论1.6.2.

$$\left\{ \begin{array}{l} \nabla \cdot \tilde{A}(x) = 0 \\ [\tilde{A}_i(x), \phi(x')] = -i \frac{\partial_i \partial_t}{\nabla^2} \Delta(x - x') \\ A_i(x) = \tilde{A}_i(x) - \frac{\partial_i \partial_t}{\nabla^2} \phi(x) \\ A_0(x) = i\phi(x) \end{array} \right. \text{不相容。} \Leftrightarrow \left\{ \begin{array}{l} \partial^a A_a(x) = 0 \\ [A_i(x) + \frac{\partial_i \partial_t}{\nabla^2} \phi(x), \phi(x')] = -i \frac{\partial_i \partial_t}{\nabla^2} \Delta(x - x') \\ \tilde{A}_i(x) = A_i(x) + \frac{\partial_i \partial_t}{\nabla^2} \phi(x) \\ \phi(x) = -iA_0(x) \end{array} \right. \text{不相容。}$$

从上可知, 规范条件与某个对易关系的不相容。不相容性本质来源于非物理引入的 $\phi$ 。

### 1.6.5 规范条件与对易关系不相容性的解决

推论1.6.3.

$$\left\{ \begin{array}{l} \nabla^2 \tilde{A} - \partial_t^2 \tilde{A} = \vec{J} + \partial_t \nabla \tilde{\phi}, \nabla^2 \tilde{\phi} = \rho \\ \vec{E} = -\partial_t \tilde{A} - \nabla \tilde{\phi}, \vec{B} = \nabla \times \tilde{A} \\ \langle |\nabla \cdot \tilde{A}| \rangle = 0 \\ [\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \\ [\tilde{A}_i(x), \tilde{\phi}(x')] = [\tilde{\phi}(x), \tilde{\phi}(x')] = 0 \\ \phi(x) = \tilde{\phi}(x) + \partial_t \frac{\partial_t}{\nabla^2} \phi(x) \\ A_i(x) = \tilde{A}_i(x) - \frac{\partial_i \partial_t}{\nabla^2} \phi(x), A_0(x) = i\phi(x) \\ [\phi(x), \phi(x')] = -i(1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square + i\epsilon}) \Delta(x - x') \\ [\tilde{A}_i(x), \phi(x')] = -i \frac{\partial_i \partial_t}{\nabla^2} \Delta(x - x'), [\tilde{\phi}(x), \phi(x')] = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J}, \nabla^2 \phi - \partial_t^2 \phi = \rho \\ \vec{E} = -\partial_t \vec{A} - \nabla \phi, \vec{B} = \nabla \times \vec{A} \\ \langle |\nabla \cdot \vec{A} + \partial_t \phi| \rangle = 0 \\ [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\lambda-1}{\lambda} \frac{\partial_a \partial_b}{\square + i\epsilon}) \Delta(x - x') \\ \tilde{A}_i(x) = A_i(x) + \frac{\partial_i \partial_t}{\nabla^2} \phi(x) \\ \tilde{\phi}(x) = \phi(x) - \partial_t \frac{\partial_t}{\nabla^2} \phi(x) \\ \phi(x) = -iA_0(x) \end{array} \right.$$

只要将规范条件不再看作算符方程, 而是看作对物理态的选择, 则方程和对易关系就完全相容, 不再出现矛盾, 其中 $\lambda = 1$ 是费曼规范,  $\lambda = \infty$ 是朗道规范。

## 2 辐射规范下的电磁场方程

### 2.1 无源电磁场方程的辐射规范势描述

推论2.1.1.

$$\left\{ \begin{array}{l} \nabla \cdot \vec{E} = 0, \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = \partial_t \vec{E} \\ \vec{A} = \frac{\nabla \times \vec{B}}{-\nabla^2} = \frac{\partial_t \vec{E}}{-\nabla^2} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = 0, \nabla \cdot \vec{A} = 0 \\ \vec{E} = -\partial_t \vec{A}, \vec{B} = \nabla \times \vec{A} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \partial^a F_{ab} = 0, F_{ab} = \partial_a \tilde{A}_b - \partial_b \tilde{A}_a \\ \nabla \cdot \tilde{A} = 0, \tilde{A}_0 = 0 \end{array} \right.$$

性质2.1.1.  $\tilde{A}(\vec{r}, t) = \frac{\partial_t}{-\nabla^2} \vec{E}(\vec{r}, t) \Leftrightarrow \vec{E}(\vec{r}, t) = -\partial_t \tilde{A}(\vec{r}, t)$

性质2.1.2.  $\tilde{A}(\vec{r}, t) = \frac{\nabla \times \vec{B}(\vec{r}, t)}{-\nabla^2} \Leftrightarrow \vec{B}(\vec{r}, t) = \nabla \times \tilde{A}(\vec{r}, t)$



## 2.2 辐射规范势的洛伦兹变换性质

$$\text{定义2.2.1. } \begin{cases} \nabla' = \nabla - \gamma_v \vec{v} \partial_t + (\gamma_v - 1) \vec{v} / v^2 (\vec{v} \cdot \nabla) \\ \partial_{t'} = \gamma_v (\partial_t - \vec{v} \cdot \nabla), \gamma_v \equiv (1 - v^2)^{-\frac{1}{2}} \end{cases}$$

$$\text{推论2.2.1. } \vec{E}' = \gamma_v (\vec{E} - \vec{v} \times \vec{B}) - (\gamma_v - 1) (\vec{v} \cdot \vec{E}) \vec{v} / v^2, \vec{B}' = \gamma_v (\vec{B} + \vec{v} \times \vec{E}) - (\gamma_v - 1) (\vec{v} \cdot \vec{B}) \vec{v} / v^2$$

$$\text{推论2.2.2. } \tilde{A}' = \frac{\nabla' \times \vec{B}'}{-\nabla'^2} = -\frac{[\nabla - \gamma_v \vec{v} \partial_t + (\gamma_v - 1) \vec{v} / v^2 (\vec{v} \cdot \nabla)] \times [\gamma_v (\vec{B} + \vec{v} \times \vec{E}) - (\gamma_v - 1) (\vec{v} \cdot \vec{B}) \vec{v} / v^2]}{[\nabla - \gamma_v \vec{v} \partial_t + (\gamma_v - 1) \vec{v} / v^2 (\vec{v} \cdot \nabla)]^2} = ?$$

## 2.3 电磁场方程势解的分析与探讨

$$\text{定义2.3.1. } \partial^a F_{ab} = 0, F_{ab} = \partial_a A_b - \partial_b A_a$$

如果得到一个解  $A_a$ , 那么  $A_a + \partial_a \theta$  也是它的另一个解。由于  $\theta$  的任意性, 电磁场方程有无穷多组的势解。但是这无穷多的势解只对应  $F_{ab}$  的同一个解。如果固定规范则相当于从无穷多势解中选取一组解  $A_a$ , 这样势解就没有冗余解了, 并且这时它与场强解  $F_{ab}$  可以一一对应。考虑解的完备性, 对于完备场强解  $F_{ab}$ , 可以由一个不完备的势解  $A_a$  完全得到, 而且这个不完备的势解  $A_a$  也可以由完备场强解  $F_{ab}$  完全确定得到。这时完备场强解  $F_{ab}$  与不完备的势解  $A_a$  一一对应, 并且这时电磁场自旋方程与带规范条件的电磁场势方程完全等价。

## 2.4 辐射规范下电磁场方程z-方向的势与场平面波解

$$\text{推论2.4.1. } \partial^a \partial_a \tilde{A} = 0, \nabla \cdot \tilde{A} = 0$$

$$\Rightarrow \tilde{A}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \frac{1}{(2\pi)^{3/2}} \frac{-i}{\sqrt{2|\vec{p}|}} \{ \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, -1) [a_1(\vec{p}) e^{ip \cdot x} - a_2^+(\vec{p}) e^{-ip \cdot x}] + \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1) [a_2(\vec{p}) e^{ip \cdot x} - a_1^+(\vec{p}) e^{-ip \cdot x}] \}$$

推论2.4.2.

$$\vec{E}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2|\vec{p}|}} \{ |\vec{p}| \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, -1) [a_1(\vec{p}) e^{ip \cdot x} + a_2^+(\vec{p}) e^{-ip \cdot x}] + |\vec{p}| \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1) [a_2(\vec{p}) e^{ip \cdot x} + a_1^+(\vec{p}) e^{-ip \cdot x}] \}$$

$$\text{证明: } \vec{E}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = -\partial_t \tilde{A}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2|\vec{p}|}} \{ |\vec{p}| \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, -1) [a_1(\vec{p}) e^{ip \cdot x} + a_2^+(\vec{p}) e^{-ip \cdot x}] + |\vec{p}| \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1) [a_2(\vec{p}) e^{ip \cdot x} + a_1^+(\vec{p}) e^{-ip \cdot x}] \} \quad \square$$

推论2.4.3.

$$\vec{B}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \frac{1}{(2\pi)^{3/2}} \frac{-i}{\sqrt{2|\vec{p}|}} \{ -|\vec{p}| \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, -1) [a_1(\vec{p}) e^{ip \cdot x} + a_2^+(\vec{p}) e^{-ip \cdot x}] + |\vec{p}| \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1) [a_2(\vec{p}) e^{ip \cdot x} + a_1^+(\vec{p}) e^{-ip \cdot x}] \}$$

$$\text{证明: } \vec{B}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \nabla \times \tilde{A}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{-i}{\sqrt{2|\vec{p}|}} \{ i\vec{p} \times \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, -1) [a_1(\vec{p}) e^{ip \cdot x} + a_2^+(\vec{p}) e^{-ip \cdot x}] + i\vec{p} \times \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1) [a_2(\vec{p}) e^{ip \cdot x} + a_1^+(\vec{p}) e^{-ip \cdot x}] \}$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{-i}{\sqrt{2|\vec{p}|}} \{ i|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, -1) [a_1(\vec{p}) e^{ip \cdot x} + a_2^+(\vec{p}) e^{-ip \cdot x}] + i|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1) [a_2(\vec{p}) e^{ip \cdot x} + a_1^+(\vec{p}) e^{-ip \cdot x}] \}$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{-i}{\sqrt{2|\vec{p}|}} \{ -|\vec{p}| \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, -1) [a_1(\vec{p}) e^{ip \cdot x} + a_2^+(\vec{p}) e^{-ip \cdot x}] + |\vec{p}| \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1) [a_2(\vec{p}) e^{ip \cdot x} + a_1^+(\vec{p}) e^{-ip \cdot x}] \} \quad \square$$

$$\text{推论2.4.4. } \begin{cases} \frac{1}{\sqrt{2}} [\vec{E}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix}) - i\vec{B}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix})] = \frac{1}{(2\pi)^{3/2}} \sqrt{|\vec{p}|} \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, -1) [a_1(\vec{p}) e^{ip \cdot x} + a_2^+(\vec{p}) e^{-ip \cdot x}] \\ \frac{1}{\sqrt{2}} [\vec{E}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix}) + i\vec{B}(|\vec{p}| \begin{bmatrix} 0 \\ 1 \end{bmatrix})] = \frac{1}{(2\pi)^{3/2}} \sqrt{|\vec{p}|} \lambda_m(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1) [a_2(\vec{p}) e^{ip \cdot x} + a_1^+(\vec{p}) e^{-ip \cdot x}] \end{cases}$$

## 2.5 辐射规范下电磁场方程的势与场平面波通解

$$\text{定义2.5.1. } \begin{cases} a_1(\vec{p}, -1) := a_1(\vec{p}) & \begin{cases} a_2(\vec{p}, -1) := a_2(\vec{p}) \\ a_2(\vec{p}, 1) := a_2^+(\vec{p}) \end{cases} & \begin{cases} a_1(\vec{p}, -1) = a_1^+(\vec{p}, 1) = a_1(\vec{p}) \\ a_2(\vec{p}, -1) = a_2^+(\vec{p}, 1) = a_2(\vec{p}) \end{cases} \end{cases}$$

$$\text{推论2.5.1. } \partial^a \partial_a \tilde{A}(\vec{r}, t) = 0, \nabla \cdot \tilde{A}(\vec{r}, t) = 0$$

$$\Rightarrow \tilde{A}(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \frac{-i}{\sqrt{2|\vec{p}|}} \{ \lambda_m(\hat{p}, -1) [a_1(\vec{p}) e^{ip \cdot x} - a_2^+(\vec{p}) e^{-ip \cdot x}] + \lambda_m(\hat{p}, 1) [a_2(\vec{p}) e^{ip \cdot x} - a_1^+(\vec{p}) e^{-ip \cdot x}] \}$$

推论2.5.2.

$$\begin{cases} \Psi(\vec{p}, 1) = \frac{1}{\sqrt{2}} [\vec{E}(\vec{p}) - i\vec{B}(\vec{p})] = \frac{1}{(2\pi)^{3/2}} \sqrt{|\vec{p}|} \lambda_m(\hat{p}, -1) [a_1(\vec{p}) e^{ip \cdot x} + a_2^+(\vec{p}) e^{-ip \cdot x}] \\ \Psi(\vec{p}, -1) = \frac{1}{\sqrt{2}} [\vec{E}(\vec{p}) + i\vec{B}(\vec{p})] = \frac{1}{(2\pi)^{3/2}} \sqrt{|\vec{p}|} \lambda_m(\hat{p}, 1) [a_2(\vec{p}) e^{ip \cdot x} + a_1^+(\vec{p}) e^{-ip \cdot x}] \end{cases} \quad \Psi(\vec{p}, -1) = \Psi^*(\vec{p}, 1)$$

推论2.5.3.

$$\begin{cases} \tilde{A}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \frac{-i}{\sqrt{2|\vec{p}|}} \{ \lambda_m(\hat{p}, -1) [a_1(\vec{p}) e^{i\vec{p}\cdot\vec{x}} - a_2^+(\vec{p}) e^{-i\vec{p}\cdot\vec{x}}] + \lambda_m(\hat{p}, 1) [a_2(\vec{p}) e^{i\vec{p}\cdot\vec{x}} - a_1^+(\vec{p}) e^{-i\vec{p}\cdot\vec{x}}] \} d^3\vec{p} \\ \Psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \sqrt{|\vec{p}|} \lambda_m(\vec{p}, -\varsigma) [a_1(\vec{p}, -\varsigma) e^{i\varsigma\vec{p}\cdot\vec{x}} + a_2^+(\vec{p}, -\varsigma) e^{-i\varsigma\vec{p}\cdot\vec{x}}] d^3\vec{p} \\ \tilde{A}(\vec{r}, t) = \frac{1}{2} \frac{\partial_t [\Psi(\vec{r}, t) + \Psi^+(\vec{r}, t)]}{-\nabla^2}, \Psi(\vec{r}, t) = -\partial_t \tilde{A}(\vec{r}, t) - i\varsigma \nabla \times \tilde{A}(\vec{r}, t) \end{cases}$$

## 2.6 辐射规范势解的详细分析

推论2.6.1.

$$\begin{cases} \dot{\tilde{A}}(\vec{r}, t) \\ = \frac{-1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \sqrt{|\vec{p}|} \{ [\lambda_m(\hat{p}, -1) a_1(\vec{p}) + \lambda_m(\hat{p}, 1) a_2(\vec{p})] e^{i\vec{p}\cdot\vec{x}} + [\lambda_m(\hat{p}, 1) a_1^+(\vec{p}) + \lambda_m(\hat{p}, -1) a_2^+(\vec{p})] e^{-i\vec{p}\cdot\vec{x}} \} d^3\vec{p} \\ \nabla \times \tilde{A}(\vec{r}, t) \\ = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \frac{\vec{p}}{\sqrt{|\vec{p}|}} \times \{ [\lambda_m(\hat{p}, -1) a_1(\vec{p}) + \lambda_m(\hat{p}, 1) a_2(\vec{p})] e^{i\vec{p}\cdot\vec{x}} + [\lambda_m(\hat{p}, 1) a_1^+(\vec{p}) + \lambda_m(\hat{p}, -1) a_2^+(\vec{p})] e^{-i\vec{p}\cdot\vec{x}} \} d^3\vec{p} \end{cases}$$

推论2.6.2.

$$\begin{aligned} \tilde{A}(\vec{r}, t) &= \frac{-i}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \frac{1}{\sqrt{|\vec{p}|}} \{ [\lambda_m(\hat{p}, -1) a_1(\vec{p}) + \lambda_m(\hat{p}, 1) a_2(\vec{p})] e^{i\vec{p}\cdot\vec{x}} - [\lambda_m(\hat{p}, 1) a_1^+(\vec{p}) + \lambda_m(\hat{p}, -1) a_2^+(\vec{p})] e^{-i\vec{p}\cdot\vec{x}} \} d^3\vec{p} \\ \Leftrightarrow \lambda_m(\hat{p}, -1) a_1(\vec{p}) + \lambda_m(\hat{p}, 1) a_2(\vec{p}) &= \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \{ i \tilde{A}(\vec{r}, t) \sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r}, t) \frac{1}{\sqrt{|\vec{p}|}} \} e^{-i\vec{p}\cdot\vec{x}} d^3\vec{r} \\ \Leftrightarrow \begin{cases} a_1(\vec{p}) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \lambda_m^+(\hat{p}, -1) \{ i \tilde{A}(\vec{r}, t) \sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r}, t) \frac{1}{\sqrt{|\vec{p}|}} \} e^{-i\vec{p}\cdot\vec{x}} d^3\vec{r} \\ a_2(\vec{p}) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \lambda_m^+(\hat{p}, 1) \{ i \tilde{A}(\vec{r}, t) \sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r}, t) \frac{1}{\sqrt{|\vec{p}|}} \} e^{-i\vec{p}\cdot\vec{x}} d^3\vec{r} \end{cases} \end{aligned}$$

$$\text{推论2.6.3.} \begin{cases} a_1^+(\vec{p}) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \lambda_m^T(\hat{p}, -1) \{ -i \tilde{A}(\vec{r}, t) \sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r}, t) \frac{1}{\sqrt{|\vec{p}|}} \} e^{i\vec{p}\cdot\vec{x}} d^3\vec{r} \\ a_2^+(\vec{p}) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \lambda_m^T(\hat{p}, 1) \{ -i \tilde{A}(\vec{r}, t) \sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r}, t) \frac{1}{\sqrt{|\vec{p}|}} \} e^{i\vec{p}\cdot\vec{x}} d^3\vec{r} \end{cases}$$

## 3 辐射规范下电磁势的对易规则 [26, 27, 39, 40]

### 3.1 辐射规范下电磁势的对易-反对易关系

$$\begin{aligned} \text{推论3.1.1.} \quad & \begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')]_{\pm} = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')]_{\pm} = 0, [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')]_{\pm} = 0 \end{cases} \\ \Leftrightarrow & \begin{cases} [\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm} = \frac{i(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} [\lambda_{mi}(\hat{p}, -1) \lambda_{mj}^+(\hat{p}, -1) \delta_1 + \lambda_{mi}(\hat{p}, 1) \lambda_{mj}^+(\hat{p}, 1) \delta_2] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\ [\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)]_{\pm} = \frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} [\lambda_{mi}(\hat{p}, -1) \lambda_{mj}^+(\hat{p}, -1) \delta_1 + \lambda_{mi}(\hat{p}, 1) \lambda_{mj}^+(\hat{p}, 1) \delta_2] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\ [\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm} = \frac{1\pm 1}{2(2\pi)^3} \int_{\vec{p} \neq 0} |\vec{p}| [\lambda_{mi}(\hat{p}, -1) \lambda_{mj}^+(\hat{p}, -1) \delta_1 + \lambda_{mi}(\hat{p}, 1) \lambda_{mj}^+(\hat{p}, 1) \delta_2] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \end{cases} \end{aligned}$$

证明:  $[\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm}$

$$\begin{aligned} &= i \cdot \frac{1}{2(2\pi)^3} \\ & \int_{\vec{p} \neq 0} \frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}|}} \{ [\lambda_{mi}(\hat{p}, -1) \lambda_{mj}(\hat{p}', 1) [a_1(\vec{p}), a_1^+(\vec{p}')]_{\pm} + \lambda_{mi}(\hat{p}, 1) \lambda_{mj}(\hat{p}', -1) [a_2(\vec{p}), a_2^+(\vec{p}')]_{\pm}] e^{i(\vec{p}\cdot\vec{r} - \vec{p}'\cdot\vec{r}')} e^{i(-Et + E't)} \\ & - \{ \lambda_{mi}(\hat{p}, 1) \lambda_{mj}(\hat{p}', -1) [a_1^+(\vec{p}), a_1(\vec{p}')]_{\pm} + \lambda_{mi}(\hat{p}, -1) \lambda_{mj}(\hat{p}', 1) [a_2^+(\vec{p}), a_2(\vec{p}')]_{\pm} \} e^{-i(\vec{p}\cdot\vec{r} - \vec{p}'\cdot\vec{r}')} e^{-i(-Et + E't)} \} d^3\vec{p} d^3\vec{p}' \\ &= \frac{i}{2(2\pi)^3} \int_{\vec{p} \neq 0} \delta^3(\vec{p} - \vec{p}') \{ [\lambda_{mi}(\hat{p}, -1) \lambda_{mj}(\hat{p}, 1) \delta_1 + \lambda_{mi}(\hat{p}, 1) \lambda_{mj}(\hat{p}, -1) \delta_2] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} \\ & - \pm [\lambda_{mi}(\hat{p}, 1) \lambda_{mj}(\hat{p}, -1) \delta_1 + \lambda_{mi}(\hat{p}, -1) \lambda_{mj}(\hat{p}, 1) \delta_2] e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} \} d^3\vec{p} d^3\vec{p}' \\ &= \frac{i}{2(2\pi)^3} \int_{\vec{p} \neq 0} \{ [\lambda_{mi}(\hat{p}, -1) \lambda_{mj}(\hat{p}, 1) \delta_1 + \lambda_{mi}(\hat{p}, 1) \lambda_{mj}(\hat{p}, -1) \delta_2] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} \\ & - \pm [\lambda_{mi}(\hat{p}, 1) \lambda_{mj}(\hat{p}, -1) \delta_1 + \lambda_{mi}(\hat{p}, -1) \lambda_{mj}(\hat{p}, 1) \delta_2] e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} \} d^3\vec{p} \end{aligned}$$



### 3.2 辐射规范下复场强的对易-反对易关系

$$\text{推论3.2.1.} \quad \begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')]_{\pm} = \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')]_{\pm} = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')]_{\pm} = 0 \end{cases} \Leftrightarrow \begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)]_{\pm} = \varsigma^0 \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)]_{\pm} = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)]_{\pm} = 0 \end{cases}$$

推论3.2.2.

$$\begin{cases} \Psi(\vec{r}, t) = -\partial_t \tilde{A}(\vec{r}, t) - i\varsigma \nabla \times \tilde{A}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \lambda_m(\vec{p}, -\varsigma) \sqrt{|\vec{p}|} [a_1(\vec{p}, -\varsigma) e^{i\varsigma p \cdot x} + a_2^+(\vec{p}, -\varsigma) e^{-i\varsigma p \cdot x}] d^3 \vec{p} \\ \Psi^*(\vec{r}, t) = -\partial_t \tilde{A}(\vec{r}, t) + i\varsigma \nabla \times \tilde{A}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \lambda_m^*(\vec{p}, -\varsigma) \sqrt{|\vec{p}|} [a_1^+(\vec{p}, -\varsigma) e^{-i\varsigma p \cdot x} + a_2(\vec{p}, -\varsigma) e^{i\varsigma p \cdot x}] d^3 \vec{p} \end{cases}$$

推论3.2.3.

$$\begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)]_{\pm} \\ = \varsigma^0 \delta_{\sigma\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)]_{\pm} = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)]_{\pm} = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)]_{\pm} = \frac{1}{(2\pi)^3} \varsigma^0 \\ \int_{\vec{p} \neq 0} |\vec{p}| [\delta_1 \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}, -\varsigma) \pm \delta_2 \lambda_{mi}(\vec{p}, \varsigma) \lambda_{mj}^+(\vec{p}, \varsigma)] e^{i\varsigma \vec{p} \cdot (\vec{r}-\vec{r}')} d^3 \vec{p} \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)]_{\pm} = 0 \\ [\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)]_{\pm} = 0 \end{cases}$$

$$\begin{aligned} \text{证明: } [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)]_{\pm} &= \frac{1}{(2\pi)^3} \int_{\vec{p}, \vec{p}' \neq 0} \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}', -\varsigma) \sqrt{|\vec{p}||\vec{p}'|} \\ &\{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)]_{\pm} e^{i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')} e^{i(-Et+E't)} + [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)]_{\pm} e^{-i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')} e^{-i(-Et+E't)} \} d^3 \vec{p} d^3 \vec{p}' \\ &= \frac{1}{(2\pi)^3} \varsigma^0 \int_{\vec{p}, \vec{p}' \neq 0} \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}', -\varsigma) |\vec{p}| \delta^3(\vec{p}-\vec{p}') [\delta_1 e^{i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')} \pm \delta_2 e^{-i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}] d^3 \vec{p} d^3 \vec{p}' \\ &= \frac{1}{(2\pi)^3} \varsigma^0 \int_{\vec{p} \neq 0} \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}, -\varsigma) |\vec{p}| [\delta_1 e^{i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')} \pm \delta_2 e^{-i\varsigma(\vec{p}\cdot\vec{r}-\vec{p}'\cdot\vec{r}')}] d^3 \vec{p} \\ &= \frac{1}{(2\pi)^3} \varsigma^0 \int_{\vec{p} \neq 0} \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}, -\varsigma) |\vec{p}| \delta_1 e^{i\varsigma \vec{p} \cdot (\vec{r}-\vec{r}')} \pm \lambda_{mi}(-\vec{p}, -\varsigma) \lambda_{mj}^+(-\vec{p}, -\varsigma) |\vec{p}| \delta_2 e^{i\varsigma \vec{p} \cdot (\vec{r}-\vec{r}')} d^3 \vec{p} \\ &= \frac{1}{(2\pi)^3} \varsigma^0 \int_{\vec{p} \neq 0} |\vec{p}| [\delta_1 \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}, -\varsigma) \pm \delta_2 \lambda_{mi}(\vec{p}, \varsigma) \lambda_{mj}^+(\vec{p}, \varsigma)] e^{i\varsigma \vec{p} \cdot (\vec{r}-\vec{r}')} d^3 \vec{p} \end{aligned} \quad \square$$

### 3.3 辐射规范下电磁势和场的同型对易-反对易关系

#### 3.3.1 辐射规范下势同型对易-反对易关系

推论3.3.1.

$$\begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')]_{\pm} = \kappa \delta_{\sigma\sigma'} \delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')]_{\pm} = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')]_{\pm} = 0 \end{cases} \Leftrightarrow \begin{cases} [\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm} = \frac{(1-\pm 1)}{2} i\kappa (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r}-\vec{r}') \\ [\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)]_{\pm} = \frac{(1\pm 1)}{2} \kappa \frac{1}{\sqrt{-\nabla^2}} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r}-\vec{r}') \\ [\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm} = \frac{(1\pm 1)}{2} \kappa \sqrt{-\nabla^2} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r}-\vec{r}') \end{cases}$$

$$\begin{aligned} \text{证明: } [\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm} &= \frac{i(1-\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa [\lambda_{mi}(\hat{p}, -1) \lambda_{mj}^+(\hat{p}, -1) + \lambda_{mi}(\hat{p}, 1) \lambda_{mj}^+(\hat{p}, 1)] e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} d^3 \vec{p} \\ &= \frac{i(1-\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa [\lambda_{mi}(\hat{p}, -1) \lambda_{mj}^+(\hat{p}, -1) + \lambda_{mi}(\hat{p}, 1) \lambda_{mj}^+(\hat{p}, 1)] e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} d^3 \vec{p} \\ &= \frac{i(1-\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa \sum_{h=1}^{-1} \lambda_{mi}(\hat{p}, h) \lambda_{mj}^+(\hat{p}, h) - \lambda_{mi}(\hat{p}, 0) \lambda_{mj}^+(\hat{p}, 0) e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} d^3 \vec{p} \\ &= \frac{i(1-\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa [\delta_{ij} - \lambda_{mi}(\hat{p}, 0) \lambda_{mj}^+(\hat{p}, 0)] e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} d^3 \vec{p} \\ &= \frac{i(1-\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa (\delta_{ij} - \hat{p}_i \hat{p}_j) e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} d^3 \vec{p} \\ &= \frac{i(1-\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa (\delta_{ij} - \frac{p_i p_j}{p^2}) e^{i\vec{p} \cdot (\vec{r}-\vec{r}')} d^3 \vec{p} \\ &= \frac{(1-\pm 1)}{2} i\kappa (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r}-\vec{r}') \end{aligned} \quad \square$$

$$\begin{aligned}
& \text{证明: } [\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)]_{\pm} \\
&= \frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa \frac{1}{|\vec{p}|} [\lambda_{mi}(\hat{p}, -1)\lambda_{mj}^+(\hat{p}, -1) + \lambda_{mi}(\hat{p}, 1)\lambda_{mj}^+(\hat{p}, 1)] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa \frac{1}{|\vec{p}|} (\delta_{ij} - \frac{p_i p_j}{p^2}) e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{(1\pm 1)}{2} \kappa \frac{1}{\sqrt{-\nabla^2}} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } [\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm} \\
&= \frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa |\vec{p}| [\lambda_{mi}(\hat{p}, -1)\lambda_{mj}^+(\hat{p}, -1) + \lambda_{mi}(\hat{p}, 1)\lambda_{mj}^+(\hat{p}, 1)] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa |\vec{p}| (\delta_{ij} - \frac{p_i p_j}{p^2}) e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{(1\pm 1)}{2} \kappa \sqrt{-\nabla^2} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \quad \square
\end{aligned}$$

### 3.3.2 辐射规范下复场强同型对易关系

$$\text{推论3.3.2. } \begin{cases} [a_{\sigma}(\vec{p}), a_{\sigma'}^{\pm}(\vec{p}')]_{-} = \kappa \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_{\sigma}(\vec{p}), a_{\sigma'}(\vec{p}')]_{-} = 0 \\ [a_{\sigma}^{\pm}(\vec{p}), a_{\sigma'}^{\pm}(\vec{p}')]_{-} = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^{\pm}(\vec{r}', t)]_{-} = i\zeta \kappa \gamma^k_{ij} p_k \delta^3(\vec{r} - \vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)]_{-} = 0 \\ [\Psi_i^{\pm}(\vec{r}, t), \Psi_j^{\pm}(\vec{r}', t)]_{-} = 0 \end{cases}$$

$$\begin{aligned}
& \text{证明: } [\Psi_i(\vec{r}, t), \Psi_j^{\pm}(\vec{r}', t)]_{-} \\
&= \frac{1}{(2\pi)^3} \zeta \kappa \int_{\vec{p} \neq 0} |\vec{p}| [\lambda_{mi}(\vec{p}, -\zeta)\lambda_{mj}^+(\vec{p}, -\zeta) - \lambda_{mi}(\vec{p}, \zeta)\lambda_{mj}^+(\vec{p}, \zeta)] e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{1}{(2\pi)^3} \zeta \kappa \int_{\vec{p} \neq 0} [-\zeta |\vec{p}| \lambda_{mi}(\vec{p}, -\zeta)\lambda_{mj}^+(\vec{p}, -\zeta) \cdot 0 - |\vec{p}| \lambda_{mi}(\vec{p}, 0)\lambda_{mj}^+(\vec{p}, 0) + \zeta |\vec{p}| \lambda_{mi}(\vec{p}, \zeta)\lambda_{mj}^+(\vec{p}, \zeta)] e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{1}{(2\pi)^3} \kappa \int_{\vec{p} \neq 0} \gamma^k i^l p_k \sum_{h=1}^{-1} \lambda_{ml}(\vec{p}, h)\lambda_{mj}^+(\vec{p}, h) e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{1}{(2\pi)^3} \kappa \int_{\vec{p} \neq 0} \gamma^k i^l p_k \delta_{lj} e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{1}{(2\pi)^3} \kappa \int_{\vec{p} \neq 0} \gamma^k_{ij} p_k e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= i\zeta \kappa \gamma^k_{ij} p_k \delta^3(\vec{r} - \vec{r}') \quad \square
\end{aligned}$$

### 3.3.3 辐射规范下复场强同型反对易关系

$$\text{推论3.3.3. } \begin{cases} [a_{\sigma}(\vec{p}), a_{\sigma'}^{\pm}(\vec{p}')]_{+} = \kappa \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_{\sigma}(\vec{p}), a_{\sigma'}(\vec{p}')]_{+} = 0 \\ [a_{\sigma}^{\pm}(\vec{p}), a_{\sigma'}^{\pm}(\vec{p}')]_{+} = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^{\pm}(\vec{r}', t)]_{+} = \kappa \sqrt{-\nabla^2} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)]_{+} = 0 \\ [\Psi_i^{\pm}(\vec{r}, t), \Psi_j^{\pm}(\vec{r}', t)]_{+} = 0 \end{cases}$$

$$\begin{aligned}
& \text{证明: } [\Psi_i(\vec{r}, t), \Psi_j^{\pm}(\vec{r}', t)]_{+} = \frac{1}{(2\pi)^3} \kappa \int_{\vec{p} \neq 0} |\vec{p}| [\lambda_{mi}(\vec{p}, -\zeta)\lambda_{mj}^+(\vec{p}, -\zeta) + \lambda_{mi}(\vec{p}, \zeta)\lambda_{mj}^+(\vec{p}, \zeta)] e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{1}{(2\pi)^3} \kappa \int_{\vec{p} \neq 0} |\vec{p}| [\sum_{h=1}^{-1} \lambda_{mi}(\hat{p}, h)\lambda_{mj}^+(\hat{p}, h) - \lambda_{mi}(\hat{p}, 0)\lambda_{mj}^+(\hat{p}, 0)] e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{1}{(2\pi)^3} \kappa \int_{\vec{p} \neq 0} |\vec{p}| (\delta_{ij} - \hat{p}_i \hat{p}_j) e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{1}{(2\pi)^3} \kappa \int_{\vec{p} \neq 0} |\vec{p}| (\delta_{ij} - \frac{p_i p_j}{p^2}) e^{i\zeta\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \kappa \sqrt{-\nabla^2} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \quad \square
\end{aligned}$$

## 3.4 辐射规范下电磁势和场的异型对易-反对易关系

### 3.4.1 辐射规范下势异型对易-反对易关系

$$\text{推论3.4.1. } \begin{cases} [a_{\sigma}(\vec{p}), a_{\sigma'}^{\pm}(\vec{p}')]_{\pm} = \kappa (-1)^{\sigma+1} \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_{\sigma}(\vec{p}), a_{\sigma'}(\vec{p}')]_{\pm} = 0 \\ [a_{\sigma}^{\pm}(\vec{p}), a_{\sigma'}^{\pm}(\vec{p}')]_{\pm} = 0 \end{cases} \Leftrightarrow \begin{cases} [\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm} = -\frac{(1\pm 1)}{2} \kappa \frac{\gamma^k_{ij} \partial_k}{\sqrt{-\nabla^2}} \delta^3(\vec{r} - \vec{r}') \\ [\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)]_{\pm} = -\frac{(1\pm 1)}{2} i\kappa \frac{\gamma^k_{ij} \partial_k}{\nabla^2} \delta^3(\vec{r} - \vec{r}') \\ [\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm} = \frac{(1\pm 1)}{2} i\kappa \gamma^k_{ij} \partial_k \delta^3(\vec{r} - \vec{r}') \end{cases}$$

$$\begin{aligned}
& \text{证明: } [\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm} \\
&= \frac{i(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa [\lambda_{mi}(\hat{p}, -1)\lambda_{mj}^+(\hat{p}, -1) - \lambda_{mi}(\hat{p}, 1)\lambda_{mj}^+(\hat{p}, 1)] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{-i(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa [1 \cdot \lambda_{mi}(\hat{p}, 1)\lambda_{mj}^+(\hat{p}, 1) + 0 \cdot \lambda_{mi}(\hat{p}, 0)\lambda_{mj}^+(\hat{p}, 0) - 1 \cdot \lambda_{mi}(\hat{p}, -1)\lambda_{mj}^+(\hat{p}, -1)] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{-i(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa \frac{\gamma^k{}_i p_k}{|\vec{p}|} \sum_{h=1}^{-1} \lambda_{mi}(\hat{p}, h)\lambda_{mj}^+(\hat{p}, h) e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{-i(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa \frac{\gamma^k{}_i p_k}{|\vec{p}|} \delta_{ij} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{-i(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa \frac{\gamma^k{}_{ij} p_k}{|\vec{p}|} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{(1\pm 1)}{2} \kappa \frac{\gamma^k{}_{ij} \partial_k}{\sqrt{-\nabla^2}} \delta^3(\vec{r} - \vec{r}') \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } [\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)]_{\pm} \\
&= \frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \kappa [\lambda_{mi}(\hat{p}, -1)\lambda_{mj}^+(\hat{p}, -1) - \lambda_{mi}(\hat{p}, 1)\lambda_{mj}^+(\hat{p}, 1)\delta_2] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \kappa [1 \cdot \lambda_{mi}(\hat{p}, 1)\lambda_{mj}^+(\hat{p}, 1) + 0 \cdot \lambda_{mi}(\hat{p}, 0)\lambda_{mj}^+(\hat{p}, 0) - 1 \cdot \lambda_{mi}(\hat{p}, -1)\lambda_{mj}^+(\hat{p}, -1)] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \kappa \frac{\gamma^k{}_i p_k}{|\vec{p}|} \sum_{h=1}^{-1} \lambda_{mi}(\hat{p}, h)\lambda_{mj}^+(\hat{p}, h) e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \kappa \frac{\gamma^k{}_i p_k}{|\vec{p}|} \delta_{ij} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa \frac{\gamma^k{}_{ij} p_k}{p^2} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{(1\pm 1)}{2} i \kappa \frac{\gamma^k{}_{ij} \partial_k}{\nabla^2} \delta^3(\vec{r} - \vec{r}') \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } [\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)]_{\pm} \\
&= \frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} |\vec{p}| \kappa [\lambda_{mi}(\hat{p}, -1)\lambda_{mj}^+(\hat{p}, -1) - \lambda_{mi}(\hat{p}, 1)\lambda_{mj}^+(\hat{p}, 1)\delta_2] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} |\vec{p}| \kappa [1 \cdot \lambda_{mi}(\hat{p}, 1)\lambda_{mj}^+(\hat{p}, 1) + 0 \cdot \lambda_{mi}(\hat{p}, 0)\lambda_{mj}^+(\hat{p}, 0) - 1 \cdot \lambda_{mi}(\hat{p}, -1)\lambda_{mj}^+(\hat{p}, -1)] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} |\vec{p}| \kappa \frac{\gamma^k{}_i p_k}{|\vec{p}|} \sum_{h=1}^{-1} \lambda_{mi}(\hat{p}, h)\lambda_{mj}^+(\hat{p}, h) e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa \gamma^k{}_i p_k \delta_{ij} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= -\frac{(1\pm 1)}{2(2\pi)^3} \int_{\vec{p} \neq 0} \kappa \gamma^k{}_{ij} p_k e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{(1\pm 1)}{2} i \kappa \gamma^k{}_{ij} \partial_k \delta^3(\vec{r} - \vec{r}') \quad \square
\end{aligned}$$

### 3.4.2 辐射规范下复场强异型对易关系

推论3.4.2.

$$\begin{cases} [a_{\sigma}(\vec{p}), a_{\sigma'}^{\dagger}(\vec{p}')]_{-} = \kappa(-1)^{\sigma+1} \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_{\sigma}(\vec{p}), a_{\sigma'}(\vec{p}')]_{-} = 0 \\ [a_{\sigma}^{\dagger}(\vec{p}), a_{\sigma'}^{\dagger}(\vec{p}')]_{-} = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^{\dagger}(\vec{r}', t)]_{-} = \varsigma \kappa \sqrt{-\nabla^2} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)]_{-} = 0 \\ [\Psi_i^{\dagger}(\vec{r}, t), \Psi_j^{\dagger}(\vec{r}', t)]_{-} = 0 \end{cases}$$

$$\begin{aligned}
& \text{证明: } [\Psi_i(\vec{r}, t), \Psi_j^{\dagger}(\vec{r}', t)]_{-} = \frac{1}{(2\pi)^3} \varsigma \kappa \int_{\vec{p} \neq 0} |\vec{p}| [\lambda_{mi}(\vec{p}, -\varsigma)\lambda_{mj}^+(\vec{p}, -\varsigma) + \lambda_{mi}(\vec{p}, \varsigma)\lambda_{mj}^+(\vec{p}, \varsigma)] e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{1}{(2\pi)^3} \varsigma \kappa \int_{\vec{p} \neq 0} |\vec{p}| \left[ \sum_{h=1}^{-1} \lambda_{mi}(\hat{p}, h)\lambda_{mj}^+(\hat{p}, h) - \lambda_{mi}(\hat{p}, 0)\lambda_{mj}^+(\hat{p}, 0) \right] e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{1}{(2\pi)^3} \varsigma \kappa \int_{\vec{p} \neq 0} |\vec{p}| (\delta_{ij} - \hat{p}_i \hat{p}_j) e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \frac{1}{(2\pi)^3} \varsigma \kappa \int_{\vec{p} \neq 0} |\vec{p}| (\delta_{ij} - \frac{p_i p_j}{p^2}) e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} \\
&= \varsigma \kappa \sqrt{-\nabla^2} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \quad \square
\end{aligned}$$

### 3.4.3 辐射规范下复场强异型反对易关系

$$\text{推论3.4.3.} \quad \begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')]_+ = \kappa(-1)^{\sigma+1} \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')]_+ = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')]_+ = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)]_+ = i\kappa \gamma^k{}_{ij} \partial_k \delta^3(\vec{r} - \vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)]_+ = 0 \\ [\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)]_+ = 0 \end{cases}$$

证明:  $[\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)]_+$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \kappa \int_{\vec{p} \neq 0} |\vec{p}| [\lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}, -\varsigma) - \lambda_{mi}(\vec{p}, \varsigma) \lambda_{mj}^+(\vec{p}, \varsigma)] e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} \\ &= -\frac{1}{(2\pi)^3} \varsigma \kappa \int_{\vec{p} \neq 0} [-\varsigma |\vec{p}| \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}, -\varsigma) 0 - |\vec{p}| \lambda_{mi}(\vec{p}, 0) \lambda_{mj}^+(\vec{p}, 0) + \varsigma |\vec{p}| \lambda_{mi}(\vec{p}, \varsigma) \lambda_{mj}^+(\vec{p}, \varsigma)] e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} \\ &= -\frac{1}{(2\pi)^3} \varsigma \kappa \int_{\vec{p} \neq 0} \gamma^k{}_{ij} p_k \sum_{h=1}^{-1} \lambda_{mi}(\vec{p}, h) \lambda_{mj}^+(\vec{p}, h) e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} \\ &= -\frac{1}{(2\pi)^3} \varsigma \kappa \int_{\vec{p} \neq 0} \gamma^k{}_{ij} p_k \delta_{ij} e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} \\ &= -\frac{1}{(2\pi)^3} \varsigma \kappa \int_{\vec{p} \neq 0} \gamma^k{}_{ij} p_k e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} \\ &= i\kappa \gamma^k{}_{ij} p_k \delta^3(\vec{r} - \vec{r}') \end{aligned} \quad \square$$

### 3.5 辐射规范下势和复场强正则对易-反对易关系汇总分析

从电磁场协变量子化方案可知, 只有  $\delta_1 \pm \delta_2 = 0$  时, 才满足微观因果性, 同时只有  $\delta_1, \delta_2 \geq 0$  时, 才满足几率非负性。所以数学上八种协变对易或反对易方案中, 物理上合理的只有一种: 即  $\delta_1 = \delta_2 = 1$ , 且满足对易关系。其实还有两种, 即  $\delta_1 = \delta_2 = 0$ , 且满足对易或反对易关系, 就是经典情形。

#### 3.5.1 辐射规范下同型对易关系

$$\text{推论3.5.1.} \quad \begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')] = \kappa \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)] = i\kappa (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \\ [\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)] = 0 \\ [\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)] = 0 \end{cases}$$

[ $\Downarrow$ ] [ $\Downarrow$ ]

$$\text{推论3.5.2.} \quad \begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = \kappa \varsigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = i\varsigma \kappa \gamma^k{}_{ij} \partial_k \delta^3(\vec{r} - \vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)] = 0 \\ [\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = 0 \end{cases}$$

$\kappa = 1$  时, 同时满足微观因果性和几率非负性, 所以是物理的;  $\kappa = -1$  时, 满足微观因果性但违反几率非负性, 所以是非物理的。

#### 3.5.2 辐射规范下同型反对易关系

推论3.5.3.

$$\begin{cases} \{a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')\} = \kappa \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')\} = 0 \\ \{a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)\} = 0 \\ \{\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)\} = \kappa \frac{1}{\sqrt{-\nabla^2}} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \\ \{\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)\} = \kappa \sqrt{-\nabla^2} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \end{cases}$$

[ $\Downarrow$ ] [ $\Downarrow$ ]

推论3.5.4.

$$\begin{cases} \{a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)\} = \kappa \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)\} = 0 \\ \{a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)\} = \kappa \sqrt{-\nabla^2} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \\ \{\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)\} = 0 \\ \{\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)\} = 0 \end{cases}$$

$\kappa = 1$  时, 满足几率非负性但违反微观因果性, 所以是非物理的;  $\kappa = -1$  时, 同时违反微观因果性和几率非负性, 所以是非物理的。

### 3.5.3 辐射规范下异型对易关系

推论3.5.5.

$$\begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')] = \kappa(-1)^{\sigma+1} \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)] = -\kappa \frac{\gamma^k_{ij} \partial_k}{\sqrt{-\nabla^2}} \delta^3(\vec{r} - \vec{r}') \\ [\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)] = 0 \\ [\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)] = 0 \end{cases}$$

[⇕] [⇕]

推论3.5.6.

$$\begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] \\ = \kappa\varsigma(-1)^{\sigma+1} \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = \kappa\varsigma \sqrt{-\nabla^2} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)] = 0 \\ [\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = 0 \end{cases}$$

无论 $\kappa$ 为何值，均同时违反微观因果性和几率非负性，所以是非物理的。

### 3.5.4 辐射规范下异型反对易关系

推论3.5.7.

$$\begin{cases} \{a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')\} = \kappa(-1)^{\sigma+1} \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')\} = 0 \\ \{a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)\} = 0 \\ \{\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)\} = -i\kappa \frac{\gamma^k_{ij} \partial_k}{\nabla^2} \delta^3(\vec{r} - \vec{r}') \\ \{\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)\} = i\kappa \gamma^k_{ij} \partial_k \delta^3(\vec{r} - \vec{r}') \end{cases}$$

[⇕] [⇕]

推论3.5.8.

$$\begin{cases} \{a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)\} = \kappa(-1)^{\sigma+1} \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)\} = 0 \\ \{a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)\} = i\kappa \gamma^k_{ij} \partial_k \delta^3(\vec{r} - \vec{r}') \\ \{\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)\} = 0 \\ \{\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)\} = 0 \end{cases}$$

无论 $\kappa$ 为何值，均满足微观因果性但违反几率非负性，所以是非物理的。

### 3.6 辐射规范下物理的势和复场强对易关系汇总

$$\text{推论3.6.1.} \begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \\ [\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)] = 0 \\ [\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)] = 0 \end{cases}$$

[⇕] [⇕]

$$\text{推论3.6.2.} \begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = \varsigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = i\varsigma \gamma^k_{ij} \partial_k \delta^3(\vec{r} - \vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)] = 0 \\ [\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = 0 \end{cases}$$

[⇕] [⇕]

$$\text{推论3.6.3.} \begin{cases} [\Psi_{\alpha_\varsigma}(x), \Psi_{\alpha'_\varsigma}^+(x')] = i\sigma_{\alpha_\varsigma \alpha'_\varsigma}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha_\varsigma}(x), \Psi_{\beta_\varsigma}(x')] = 0 \\ [\Psi_{\alpha'_\varsigma}^+(x), \Psi_{\beta'_\varsigma}^+(x')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_{\alpha_\varsigma}(\vec{r}, t), \Psi_{\alpha'_\varsigma}^+(\vec{r}', t)] = \varsigma \varepsilon^k_{\alpha_\varsigma \alpha'_\varsigma} \partial_k \delta^3(\vec{r} - \vec{r}') \\ [\Psi_{\alpha_\varsigma}(\vec{r}, t), \Psi_{\beta_\varsigma}(\vec{r}', t)] = 0 \\ [\Psi_{\alpha'_\varsigma}^+(\vec{r}, t), \Psi_{\beta'_\varsigma}^+(\vec{r}', t)] = 0 \end{cases}$$

电磁场只有这一种情形，才同时满足微观因果性和几率非负性，才是物理的、正确的。



### 3.7 辐射规范下经典反对易关系(零模态)

$$\text{推论3.7.1. } \begin{cases} \{\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)\} = \kappa(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \\ \{\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)\} = 0 \\ \{\dot{\tilde{A}}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)\} = 0 \end{cases}$$

$$\text{推论3.7.2. } \lambda_m(\hat{p}, -1)a_1(\vec{p}) + \lambda_m(\hat{p}, 1)a_2(\vec{p}) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \{i\tilde{A}(\vec{r}, t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r}, t)\frac{1}{\sqrt{|\vec{p}|}}\} e^{-ip \cdot x} d^3\vec{r}$$

$$\Rightarrow \begin{cases} a_1(\vec{p}) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \lambda_m^+(\hat{p}, -1) \{i\tilde{A}(\vec{r}, t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r}, t)\frac{1}{\sqrt{|\vec{p}|}}\} e^{-ip \cdot x} d^3\vec{r} \\ a_1^+(\vec{p}) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \lambda_m^T(\hat{p}, -1) \{-i\tilde{A}(\vec{r}, t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r}, t)\frac{1}{\sqrt{|\vec{p}|}}\} e^{ip \cdot x} d^3\vec{r} \\ a_2(\vec{p}) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \lambda_m^+(\hat{p}, 1) \{i\tilde{A}(\vec{r}, t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r}, t)\frac{1}{\sqrt{|\vec{p}|}}\} e^{-ip \cdot x} d^3\vec{r} \\ a_2^+(\vec{p}) = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \int \lambda_m^T(\hat{p}, 1) \{-i\tilde{A}(\vec{r}, t)\sqrt{|\vec{p}|} - \dot{\tilde{A}}(\vec{r}, t)\frac{1}{\sqrt{|\vec{p}|}}\} e^{ip \cdot x} d^3\vec{r} \end{cases}$$

证明:  $\{a_1(\vec{p}), a_1^+(\vec{p}')\}$

$$\begin{aligned} &= \frac{1}{2} \frac{1}{(2\pi)^3} \int \lambda_m^+(\hat{p}, -1) \lambda_m^j(\hat{p}', -1) [-i \frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} \{\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)\} + i \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}} \{\tilde{A}_j(\vec{r}', t), \dot{\tilde{A}}_i(\vec{r}, t)\}] e^{-ip \cdot x} e^{ip' \cdot x'} d^3\vec{r} d^3\vec{r}' \\ &= -\frac{i}{2} \frac{1}{(2\pi)^3} \int \lambda_m^+(\hat{p}, -1) \lambda_m^j(\hat{p}', -1) \kappa(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') [\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} - \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}] e^{-ip \cdot x} e^{ip' \cdot x'} d^3\vec{r} \\ &= -\frac{i}{2} \frac{1}{(2\pi)^3} \int \lambda_m^+(\hat{p}', -1) \lambda_m^j(\hat{p}, -1) \kappa(\delta_{ij} - \frac{p_i p_j}{p^2}) [\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} - \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}] e^{-ip \cdot x} e^{ip' \cdot x} d^3\vec{r} \\ &= -\frac{i}{2} \lambda_m^+(\hat{p}, -1) \lambda_m^j(\hat{p}, -1) \kappa(\delta_{ij} - \frac{p_i p_j}{p^2}) [\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} - \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}] \delta^3(\vec{p} - \vec{p}') \\ &= 0 \end{aligned} \quad \square$$

证明:  $\{a_1(\vec{p}), a_1(\vec{p}')\}$

$$\begin{aligned} &= \frac{1}{2} \frac{1}{(2\pi)^3} \int \lambda_m^+(\hat{p}, -1) \lambda_m^j(\hat{p}', -1) [-i \frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} \{\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)\} - i \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}} \{\tilde{A}_j(\vec{r}', t), \dot{\tilde{A}}_i(\vec{r}, t)\}] e^{-ip \cdot x} e^{-ip' \cdot x'} d^3\vec{r} d^3\vec{r}' \\ &= -\frac{i}{2} \frac{1}{(2\pi)^3} \int \lambda_m^+(\hat{p}, -1) \lambda_m^j(\hat{p}', -1) \kappa(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') [\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} + \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}] e^{-ip \cdot x} e^{-ip' \cdot x'} d^3\vec{r} d^3\vec{r}' \\ &= -\frac{i}{2} \frac{1}{(2\pi)^3} \int \lambda_m^+(\hat{p}, -1) \lambda_m^j(\hat{p}', -1) \kappa(\delta_{ij} - \frac{p_i p_j}{p^2}) [\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} + \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}] e^{-ip \cdot x} e^{-ip' \cdot x} d^3\vec{r} \\ &= -\frac{i}{2} \lambda_m^+(\hat{p}, -1) \lambda_m^j(-\hat{p}, -1) \kappa(\delta_{ij} - \frac{p_i p_j}{p^2}) [\frac{\sqrt{|\vec{p}|}}{\sqrt{|\vec{p}'|}} + \frac{\sqrt{|\vec{p}'|}}{\sqrt{|\vec{p}|}}] \delta^3(\vec{p} + \vec{p}') e^{2iEt} \\ &= -i \kappa \lambda_m^+(\hat{p}, -1) \lambda_m^j(-\hat{p}, 1) \delta_{ij} e^{2iEt} \delta^3(\vec{p} + \vec{p}') \\ &= -i \kappa \frac{p_+}{p_-} e^{2iEt} \delta^3(\vec{p} + \vec{p}') \end{aligned} \quad \square$$

产生了零模态, 所以是非物理的。

## 4 从传统辐射规范量子化方案导出新方案 [26, 27, 39, 40]

### 4.1 从传统辐射规范势等时对易关系得到复场强的等时对易关系

$$\text{推论4.1.1. } \mathcal{L} = -\frac{1}{4} F^{uv} F_{uv} \Rightarrow \pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \partial_t \tilde{A}_i + \partial_i \phi = -E_i, \pi_4 = 0$$

从 $(\tilde{A}_i, E_i)$ 为正则变量的等时对易关系推得 $(\Psi_i, \Psi_i^\dagger)$ 为基本变量的等时对易关系。

$$\text{推论4.1.2. } \begin{cases} [\tilde{A}_i(\vec{r}, t), E_j(\vec{r}', t)] = -i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \\ [E_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(\vec{r} - \vec{r}') \\ [\tilde{A}_i(\vec{r}, t), \tilde{A}_j(\vec{r}', t)] = 0 \\ [E_i(\vec{r}, t), E_j(\vec{r}', t)] = 0 \end{cases} \Rightarrow \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^\dagger(\vec{r}', t)] = \varsigma \varepsilon_{ij}^k \partial_k \delta^3(\vec{r} - \vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)] = 0 \\ [\Psi_i^\dagger(\vec{r}, t), \Psi_j^\dagger(\vec{r}', t)] = 0 \end{cases}$$

证明:  $[\Psi_i(\vec{r}, t), \Psi_j^\dagger(\vec{r}', t)]$

$$\begin{aligned} &= -\frac{1}{2} i \varsigma \varepsilon_i^{kl} \partial_{x_k} [\tilde{A}_l(\vec{r}, t), E_j(\vec{r}', t)] + \frac{1}{2} i \varsigma \varepsilon_j^{kl} \partial_{x'_k} [E_i(\vec{r}, t), \tilde{A}_l(\vec{r}', t)] \\ &= \frac{1}{2} \varsigma \varepsilon_{ij}^k (\partial_{x_k} - \partial_{x'_k}) \delta^3(\vec{r} - \vec{r}') \end{aligned}$$

$$\begin{aligned}
&= \varsigma \varepsilon_{ij}^k \partial_{(x_k - x'_k)} \delta^3(\vec{r} - \vec{r}') \\
&= i \varsigma \gamma_{ij}^k \partial_k \delta^3(\vec{r} - \vec{r}') \\
&= i \varsigma \gamma^k_{ij} \partial_k \delta^3(\vec{r} - \vec{r}')
\end{aligned}$$

□

证明:  $[\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)]$

$$\begin{aligned}
&= \frac{1}{2} i \varsigma \varepsilon_i^{kl} \partial_{x_k} [\tilde{A}_l(\vec{r}, t), E_j(\vec{r}', t)] + \frac{1}{2} i \varsigma \varepsilon_j^{kl} \partial_{x'_k} [E_i(\vec{r}, t), \tilde{A}_l(\vec{r}', t)] \\
&= -\frac{1}{2} \varsigma \varepsilon_{ij}^k (\partial_{x_k} + \partial_{x'_k}) \delta^3(\vec{r} - \vec{r}') \\
&= 0
\end{aligned}$$

□

证明:  $[\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)]$

$$\begin{aligned}
&= -\frac{1}{2} i \varsigma \varepsilon_i^{kl} \partial_{x_k} [\tilde{A}_l(\vec{r}, t), E_j(\vec{r}', t)] - \frac{1}{2} i \varsigma \varepsilon_j^{kl} \partial_{x'_k} [E_i(\vec{r}, t), \tilde{A}_l(\vec{r}', t)] \\
&= \frac{1}{2} \varsigma \varepsilon_{ij}^k (\partial_{x_k} + \partial_{x'_k}) \delta^3(\vec{r} - \vec{r}') \\
&= 0
\end{aligned}$$

□

## 4.2 在辐射规范下势对易关系与复场强协变对易关系的等价性

推论4.2.1. 
$$\begin{cases} [\Psi_{\alpha\varsigma}(x), \Psi_{\alpha'\varsigma'}^+(x')] = i \sigma_{\alpha\varsigma\alpha'\varsigma'}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha\varsigma}(x), \Psi_{\beta\varsigma}(x')] = 0, [\Psi_{\alpha'\varsigma'}^+(x), \Psi_{\beta'\varsigma'}^+(x')] = 0 \end{cases} \Leftrightarrow [\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x')$$

证明:  $[\tilde{A}_i(x), \tilde{A}_j(x')]$

$$\begin{aligned}
&= \frac{1}{2} \frac{\partial_t}{-\nabla^2} \frac{\partial_{t'}}{-\nabla'^2} [\psi_i(x) + \psi_i^+(x), \psi_j(x') + \psi_j^+(x')] \\
&= \frac{1}{2} \frac{\partial_t}{-\nabla^2} \frac{\partial_{t'}}{-\nabla'^2} \{[\psi_i(x), \psi_j^+(x')] + [\psi_i^+(x), \psi_j(x')]\} \\
&= \frac{1}{2} \frac{\partial_t}{-\nabla^2} \frac{\partial_{t'}}{-\nabla'^2} \{[\psi_i(x), \psi_j^+(x')] - [\psi_j(x'), \psi_i^+(x)]\} \\
&= \frac{1}{2} \frac{\partial_t}{-\nabla^2} \frac{\partial_{t'}}{-\nabla'^2} \{i \sigma_{ij}^{ab} \partial_a \partial_b \Delta(x - x') - [i \sigma_{ji}^{ab} \partial'_a \partial'_b \Delta(x' - x)]\} \\
&= \frac{1}{2} \frac{\partial_t}{-\nabla^2} \frac{\partial_{t'}}{-\nabla'^2} \{i \sigma_{ij}^{ab} \partial_a \partial_b \Delta(x - x') + [i \sigma_{ji}^{ab} \partial_a \partial_b \Delta(x - x')]\} \\
&= -\frac{1}{2} \frac{\partial_t^2}{\nabla^4} \{-i[\frac{1}{2}(\nabla^2 - \partial_\pi^2) \delta_{ij} - \varsigma \varepsilon_{ij}^k \partial_k \partial_\pi - \partial_i \partial_j] - i[\frac{1}{2}(\nabla^2 - \partial_\pi^2) \delta_{ji} - \varsigma \varepsilon_{ji}^k \partial_k \partial_\pi - \partial_j \partial_i]\} \Delta(x - x') \\
&= i \frac{1}{\nabla^2} [\frac{1}{2}(\nabla^2 - \partial_\pi^2) \delta_{ij} - \partial_i \partial_j] \Delta(x - x') \\
&= i \frac{1}{\nabla^2} [\nabla^2 \delta_{ij} - \partial_i \partial_j] \Delta(x - x') \\
&= i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x')
\end{aligned}$$

□

反向证明:

证明:  $[\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x')$

$$\begin{aligned}
&\Rightarrow i[\Psi_i(x), \Psi_j^+(x')] \\
&= \frac{i}{2} [-i \partial_\pi \tilde{A}_i(x) + i \varsigma \varepsilon_i^{kl} \partial_k \tilde{A}_l(x), -i \partial'_\pi \tilde{A}_j(x') - i \varsigma \varepsilon_j^{mn} \partial'_m \tilde{A}_n(x')] \\
&= \frac{1}{2} [-\partial_\pi^2 (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') - \varsigma \varepsilon_j^{mn} (\delta_{in} - \frac{\partial_i \partial_n}{\nabla^2}) \partial_\pi \partial_m \Delta(x - x') + \varsigma \varepsilon_i^{kl} (\delta_{lj} - \frac{\partial_l \partial_j}{\nabla^2}) \partial_k \partial_\pi \Delta(x - x') \\
&\quad + \varepsilon_i^{kl} (\delta_{ln} - \frac{\partial_l \partial_n}{\nabla^2}) \varepsilon_j^{mn} \partial_k \partial_m \Delta(x - x')] \\
&= \frac{1}{2} [-\partial_\pi^2 (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') - 2 \varsigma \varepsilon_{ij}^k \partial_k \partial_\pi \Delta(x - x') + (\delta_{ij} \delta^{km} - \delta_i^m \delta_j^k) \partial_k \partial_m \Delta(x - x')] \\
&= \frac{1}{2} [(-\partial_\pi^2 \delta_{ij} - \partial_i \partial_j) \Delta(x - x') - 2 \varsigma \varepsilon_{ij}^k \partial_k \partial_\pi \Delta(x - x') + (\delta_{ij} \nabla^2 - \partial_i \partial_j) \Delta(x - x')] \\
&= [\frac{1}{2}(\nabla^2 - \partial_\pi^2) \delta_{ij} - \varsigma \varepsilon_{ij}^k \partial_k \partial_\pi - \partial_i \partial_j] \Delta(x - x') \\
&= -\sigma_{ij}^{ab} \partial_a \partial_b \Delta(x - x')
\end{aligned}$$

□

证明:  $[\tilde{A}_i(x), \tilde{A}_j(x')] = -i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x')$

$$\begin{aligned}
&\Rightarrow i[\Psi_i(x), \Psi_j(x')] \\
&= \frac{i}{2} [-i \partial_\pi \tilde{A}_i(x) + i \varsigma \varepsilon_i^{kl} \partial_k \tilde{A}_l(x), -i \partial'_\pi \tilde{A}_j(x') + i \varsigma \varepsilon_j^{mn} \partial'_m \tilde{A}_n(x')] \\
&= \frac{1}{2} [-\partial_\pi^2 (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') + \varsigma \varepsilon_j^{mn} (\delta_{in} - \frac{\partial_i \partial_n}{\nabla^2}) \partial_\pi \partial_m \Delta(x - x') + \varsigma \varepsilon_i^{kl} (\delta_{lj} - \frac{\partial_l \partial_j}{\nabla^2}) \partial_k \partial_\pi \Delta(x - x') \\
&\quad - \varepsilon_i^{kl} (\delta_{ln} - \frac{\partial_l \partial_n}{\nabla^2}) \varepsilon_j^{mn} \partial_k \partial_m \Delta(x - x')] \\
&= \frac{1}{2} [-\partial_\pi^2 (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') - (\delta_{ij} \delta^{km} - \delta_i^m \delta_j^k) \partial_k \partial_m \Delta(x - x')]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}[(-\partial_\pi^2 \delta_{ij} - \partial_i \partial_j) \Delta(x - x') - (\delta_{ij} \nabla^2 - \partial_i \partial_j) \Delta(x - x')] \\
&= -\frac{1}{2} \delta_{ij} (\nabla^2 + \partial_\pi^2) \Delta(x - x') \\
&= 0
\end{aligned}$$

□

证明:  $[\tilde{A}_i(x), \tilde{A}_j(x')] = -i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x')$

$$\begin{aligned}
&\Rightarrow i[\Psi_i^+(x), \Psi_j^+(x')] \\
&= \frac{i}{2}[-i\partial_\pi \tilde{A}_i(x) - i\epsilon \epsilon_i^{kl} \partial_k \tilde{A}_l(x), -i\partial_\pi \tilde{A}_j(x') - i\epsilon \epsilon_j^{mn} \partial'_m \tilde{A}_n(x')] \\
&= \frac{1}{2}[-\partial_\pi^2 (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') - \epsilon \epsilon_j^{mn} (\delta_{in} - \frac{\partial_i \partial_n}{\nabla^2}) \partial_\pi \partial_m \Delta(x - x') - \epsilon \epsilon_i^{kl} (\delta_{lj} - \frac{\partial_l \partial_j}{\nabla^2}) \partial_k \partial_\pi \Delta(x - x') \\
&\quad - \epsilon_i^{kl} (\delta_{ln} - \frac{\partial_l \partial_n}{\nabla^2}) \epsilon_j^{mn} \partial_k \partial_m \Delta(x - x')] \\
&= \frac{1}{2}[-\partial_\pi^2 (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') - (\delta_{ij} \delta^{km} - \delta_i^m \delta_j^k) \partial_k \partial_m \Delta(x - x')] \\
&= \frac{1}{2}[(-\partial_\pi^2 \delta_{ij} - \partial_i \partial_j) \Delta(x - x') - (\delta_{ij} \nabla^2 - \partial_i \partial_j) \Delta(x - x')] \\
&= -\frac{1}{2} \delta_{ij} (\nabla^2 + \partial_\pi^2) \Delta(x - x') \\
&= 0
\end{aligned}$$

□

推论4.2.2.  $[\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \Rightarrow [\tilde{A}_i(\vec{r}, t), \dot{\tilde{A}}_j(\vec{r}', t)] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \delta^3(x - x')$

### 4.3 从传统量子化方案得到复场强 $\Psi(\vec{r}, t)$ 表达的能量动量算符

以  $(\Psi_i, \Psi_i^+)$  为基本变量表达的能量动量算符。

推论4.3.1.

$$\begin{cases}
\mathcal{H} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) = \frac{1}{2}[\Psi^+(\vec{r}, t)\Psi(\vec{r}, t) + \Psi^T(\vec{r}, t)\Psi^*(\vec{r}, t)] = \frac{1}{2}\delta^{ij}\{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t)\} = \Psi^+(\vec{r}, t)\Psi(\vec{r}, t) \\
\vec{\mathcal{P}} = \vec{E} \times \vec{B} = -\frac{1}{2}\varsigma[\Psi^+(\vec{r}, t)\gamma\Psi(\vec{r}, t) - \Psi^T(\vec{r}, t)\gamma\Psi^*(\vec{r}, t)] = \frac{\varsigma}{2}\gamma^{ij}\{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t)\} \\
\vec{\mathcal{M}} = \vec{r} \times (\vec{E} \times \vec{B}) = \frac{1}{2}i\varsigma[\{\Psi_k(\vec{r}, t), x^j\Psi_j^+(\vec{r}, t)\} - \{x^i\Psi_i(\vec{r}, t), \Psi_k^+(\vec{r}, t)\}]
\end{cases}$$

证明:  $\vec{\mathcal{M}} = \vec{r} \times (\vec{E} \times \vec{B})$

$$\begin{aligned}
&= \frac{1}{2}i\varsigma\epsilon_k^{lm}x_l\epsilon_m^{ij}\{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t)\} \\
&= \frac{1}{2}i\varsigma(\delta_k^i\delta^{lj} - \delta_k^j\delta^{li})x_l\{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t)\} \\
&= \frac{1}{2}i\varsigma[\{\Psi_k(\vec{r}, t), x^j\Psi_j^+(\vec{r}, t)\} - \{x^i\Psi_i(\vec{r}, t), \Psi_k^+(\vec{r}, t)\}]
\end{aligned}$$

□

证明:  $\vec{\mathcal{M}} = \vec{r} \times (\vec{E} \times \vec{B}) = \vec{E}(\vec{r} \cdot \vec{B}) - (\vec{r} \cdot \vec{E})\vec{B}$

$$\begin{aligned}
&= -i\varsigma\epsilon_k^{lm}x_l\epsilon_m^{ij}\Psi_i^+(\vec{r}, t)\Psi_j(\vec{r}, t) \\
&= -i\varsigma(\delta_k^i\delta^{lj} - \delta_k^j\delta^{li})x_l\Psi_i^+(\vec{r}, t)\Psi_j(\vec{r}, t) \\
&= -i\varsigma[\Psi_k^+(\vec{r}, t)x^j\Psi_j(\vec{r}, t) - x^i\Psi_i^+(\vec{r}, t)\Psi_k(\vec{r}, t)]
\end{aligned}$$

□

推论4.3.2.  $\begin{cases} H = \frac{1}{2}\delta^{ij} \int \{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t)\} d^3\vec{r} = \int \Psi^+(\vec{r}, t)\Psi(\vec{r}, t) d^3\vec{r} \\ \vec{P} = \frac{\varsigma}{2}\gamma^{ij} \int \{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t)\} d^3\vec{r}, P_\alpha = \frac{\varsigma}{2}(\gamma, -i\varsigma)_\alpha^{ij} \int \{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t)\} d^3\vec{r} \end{cases}$

### 4.4 从传统量子化方案得到复场强 $\Psi(\vec{r}, t)$ 算符的运动方程

从  $(\tilde{A}_i, E_i)$  为正则变量的算符运动方程推得  $(\Psi_i, \Psi_i^+)$  为基本变量的算符运动方程。

推论4.4.1.  $\begin{cases} \dot{\tilde{A}}(\vec{r}, t) = -i[\tilde{A}(\vec{r}, t), H] \\ \dot{E}(\vec{r}, t) = -i[E(\vec{r}, t), H] \end{cases} \Rightarrow \begin{cases} \dot{\Psi}(\vec{r}, t) = -i[\Psi(\vec{r}, t), H] \\ \dot{\Psi}^+(\vec{r}, t) = -i[\Psi^+(\vec{r}, t), H] \end{cases}$

### 4.5 复场强 $\Psi(\vec{r}, t)$ 算符演化方程和约束方程

推论4.5.1.

$$\begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = i\varsigma\gamma^k{}_{ij}\partial_k\delta^3(\vec{r} - \vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)] = 0, [\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = 0 \end{cases} \Rightarrow \begin{cases} [\Psi(\vec{r}, t), H] = i\varsigma\gamma^k\partial_k\Psi(\vec{r}, t) \\ [\Psi_i(\vec{r}, t), P_j] = -i\partial_j\Psi_i(\vec{r}, t) + i\delta_{ij}\nabla \cdot \Psi(\vec{r}, t) \end{cases}$$

证明:  $[\Psi_i(\vec{r}, t), H]$

$$\begin{aligned}
&= [\Psi_i(\vec{r}, t), \frac{1}{2}\delta^{jl} \int \{\Psi_j(\vec{r}', t), \Psi_l^+(\vec{r}', t)\} d^3\vec{r}'] \\
&= \frac{1}{2}\delta^{jl} \int [\Psi_i(\vec{r}, t), \{\Psi_j(\vec{r}', t), \Psi_l^+(\vec{r}', t)\}] d^3\vec{r}' \\
&= \frac{1}{2}\delta^{jl} \int \{\Psi_j(\vec{r}', t), [\Psi_i(\vec{r}, t), \Psi_l^+(\vec{r}', t)]\} + \{\Psi_l^+(\vec{r}', t), [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)]\} d^3\vec{r}' \\
&= \frac{1}{2}\delta^{jl} \int [\{\Psi_j(\vec{r}', t), i\varsigma\gamma^k{}_{il}\partial_k\delta^3(\vec{r}-\vec{r}')\} + 0] d^3\vec{r}' \\
&= i\varsigma\gamma^k{}_{ij}\partial_k\Psi_j(\vec{r}, t) \\
&\succ i\varsigma\gamma^k\partial_k\Psi(\vec{r}, t)
\end{aligned}$$

□

证明:  $[\Psi_i(\vec{r}, t), \vec{P}]$

$$\begin{aligned}
&= [\Psi_i(\vec{r}, t), \frac{\varsigma}{2}\gamma^{jl} \int \{\Psi_j(\vec{r}', t), \Psi_l^+(\vec{r}', t)\} d^3\vec{r}'] \\
&= \frac{\varsigma}{2}\gamma^{jl} \int [\Psi_i(\vec{r}, t), \{\Psi_j(\vec{r}', t), \Psi_l^+(\vec{r}', t)\}] d^3\vec{r}' \\
&= \frac{\varsigma}{2}\gamma^{jl} \int \{\Psi_j(\vec{r}', t), [\Psi_i(\vec{r}, t), \Psi_l^+(\vec{r}', t)]\} + \{\Psi_l^+(\vec{r}', t), [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)]\} d^3\vec{r}' \\
&= \frac{\varsigma}{2}\gamma^{jl} \int [\{\Psi_j(\vec{r}', t), i\varsigma\gamma^k{}_{il}\partial_k\delta^3(\vec{r}-\vec{r}')\} + 0] d^3\vec{r}' \\
&= i\gamma^{jl}\gamma^k{}_{il}\partial_k\Psi_j(\vec{r}, t) \succ -i(\gamma \cdot \nabla)\gamma\psi(\vec{r}, t) \\
&\prec -i(\delta_n{}^k\delta^j{}_i - \delta_{ni}\delta^{jk})\partial_k\Psi_j(\vec{r}, t) \\
&= -i\partial_n\Psi_i(\vec{r}, t) + i\delta_{ni}\nabla \cdot \Psi(\vec{r}, t)
\end{aligned}$$

□

推论4.5.2.  $[\psi_i(\vec{r}, t), P_j] = -i\partial_j\psi_i(\vec{r}, t) + iS_m^+(1)_{ij}\nabla \cdot [S_m(1)\psi(\vec{r}, t)]$

推论4.5.3. 由此可以唯一确定能量动量算符的系数

$$\begin{cases} (\gamma, -i\varsigma)^a\partial_a\Psi(\vec{r}, t) = 0 \\ \nabla \cdot \Psi(\vec{r}, t) = 0 \end{cases} \Leftrightarrow \begin{cases} \dot{\Psi}(\vec{r}, t) = -i[\Psi(\vec{r}, t), H] \\ \partial_i\Psi(\vec{r}, t) = i[\Psi(\vec{r}, t), P_i] \\ [P_a, \Psi(\vec{r}, t)] = i\partial_a\Psi(\vec{r}, t) \end{cases} ; \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = i\varsigma\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)] = 0 \\ [\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = 0 \end{cases}$$

推论4.5.4.

$$[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]\Psi = 0 \Leftrightarrow [P_a, \Psi(\vec{r}, t)] = i\partial_a\Psi(\vec{r}, t); \begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = i\varsigma\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)] = 0, [\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = 0 \end{cases}$$

推论4.5.5. 电磁场约束与对易关系自洽:

$$\begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = i\varsigma\gamma^k{}_{ij}\partial_k\delta^3(\vec{r}-\vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)] = 0, [\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = 0 \end{cases} \Rightarrow \begin{cases} \nabla \cdot \Psi(\vec{r}, t) = 0 \\ \nabla \cdot \Psi^+(\vec{r}, t) = 0 \end{cases}$$

推论4.5.6.  $[P_a, \Psi(\vec{r}, t)] = S_{ab}(\gamma, \varsigma)\partial^b\Psi(\vec{r}, t)$

## 4.6 复场强 $\Psi(\vec{r}, t)$ 算符的量子标积方程(是否最基本?)

定义4.6.1.  $\langle\eta|\dot{\Psi}(\vec{r}, t) + i[\Psi(\vec{r}, t), H]|\varphi\rangle = 0, \langle\eta|\nabla \cdot \Psi(\vec{r}, t)|\varphi\rangle = 0$

定义4.6.2.  $\langle\eta|\partial_a\Psi(\vec{r}, t) - i[\Psi(\vec{r}, t), P_a]|\varphi\rangle = 0 \Leftrightarrow \langle\eta|[P_a, \Psi(\vec{r}, t)] - i\partial_a\Psi(\vec{r}, t)|\varphi\rangle = 0$

它有两个解, 一解是个是算符方程 $\dot{\Psi}(\vec{r}, t) = -i[\Psi(\vec{r}, t), H], \nabla \cdot \Psi(\vec{r}, t) = 0$ 决定的解。另一个解是对所有物理态都是它的真空态的解:  $\langle\eta|\Psi(\vec{r}, t)|\varphi\rangle = 0$ , 所以合起来就是完整的傅里叶展开解。

## 4.7 复场强能量动量算符的Fock表示

能量动量算符:

推论4.7.1. 
$$\begin{cases} H = \frac{1}{2}\delta^{ij} \int \{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t)\} d^3\vec{r} = \frac{1}{2} \int_{\vec{p}\neq 0} |\vec{p}| [\{a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}, -\varsigma)\} + \{a_2(\vec{p}, -\varsigma), a_2^+(\vec{p}, -\varsigma)\}] d^3\vec{p} \\ \vec{P} = \frac{\varsigma}{2}\gamma^{ij} \int \{\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t)\} d^3\vec{r} = \frac{1}{2} \int_{\vec{p}\neq 0} \vec{p} [\{a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}, -\varsigma)\} + \{a_2(\vec{p}, -\varsigma), a_2^+(\vec{p}, -\varsigma)\}] d^3\vec{p} \end{cases}$$

证明:

$$\begin{aligned}
H &= \frac{1}{2} \delta^{ij} \int \{ \Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t) \} d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \frac{1}{2} \delta^{ij} \int_{\vec{p}, \vec{p}' \neq 0} d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}', -\varsigma) \\
&\quad \{ \sqrt{|\vec{p}|} [a_1(\vec{p}, -\varsigma) e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)} + a_2^+(\vec{p}, -\varsigma) e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)}], \sqrt{|\vec{p}'|} [a_1^+(\vec{p}', -\varsigma) e^{-i\varsigma(\vec{p}' \cdot \vec{r} - E't)} + a_2(\vec{p}', -\varsigma) e^{i\varsigma(\vec{p}' \cdot \vec{r} - E't)}] \} \\
&= \frac{1}{2} \int_{\vec{p}, \vec{p}' \neq 0} \lambda_m^+(\vec{p}, -\varsigma) \lambda_m(\vec{p}, -\varsigma) \delta^3(\vec{p} - \vec{p}') |\vec{p}| \{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}, -\varsigma)] + [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}, -\varsigma)] \} + \\
&\quad \lambda_m^+(\vec{p}, -\varsigma) \lambda_m(\vec{p}, -\varsigma) \delta^3(\vec{p} + \vec{p}') |\vec{p}| \{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}, -\varsigma)] + [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}, -\varsigma)] \} e^{-2i\varsigma Et} + \{ a_2^+(\vec{p}, -\varsigma), c^+(-\vec{p}, -\varsigma) \} e^{2i\varsigma Et} \} d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{2} \int_{\vec{p} \neq 0} |\vec{p}| \{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}, -\varsigma)] + [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}, -\varsigma)] + 0 \} d^3 \vec{p} \\
&= \frac{1}{2} \int_{\vec{p} \neq 0} |\vec{p}| \{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}, -\varsigma)] + [a_2(\vec{p}, -\varsigma), a_2^+(\vec{p}, -\varsigma)] \} d^3 \vec{p} \quad \square
\end{aligned}$$

证明:

$$\begin{aligned}
P_k &= \frac{\varsigma}{2} \gamma_k^{ij} \int \{ \Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}, t) \} d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \frac{\varsigma}{2} \gamma_k^{ij} \int_{\vec{p}, \vec{p}' \neq 0} d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}', -\varsigma) \\
&\quad \{ \{ \sqrt{|\vec{p}|} [a_1(\vec{p}, -\varsigma) e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)} + a_2^+(\vec{p}, -\varsigma) e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)}], \{ \sqrt{|\vec{p}'|} [a_1^+(\vec{p}', -\varsigma) e^{-i\varsigma(\vec{p}' \cdot \vec{r} - E't)} + a_2(\vec{p}', -\varsigma) e^{i\varsigma(\vec{p}' \cdot \vec{r} - E't)}] \} \} \\
&= -\frac{\varsigma}{2} \int_{\vec{p}, \vec{p}' \neq 0} \lambda_m^+(\vec{p}, -\varsigma) \gamma_k \lambda_m(\vec{p}, -\varsigma) |\vec{p}| \delta^3(\vec{p} - \vec{p}') \{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}, -\varsigma)] + [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}, -\varsigma)] \} + \\
&\quad \lambda_m^+(\vec{p}, -\varsigma) \gamma_k \lambda_m(\vec{p}, -\varsigma) |\vec{p}| \delta^3(\vec{p} + \vec{p}') \{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}, -\varsigma)] + [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}, -\varsigma)] \} e^{-2i\varsigma Et} + \{ a_2^+(\vec{p}, -\varsigma), c^+(-\vec{p}, -\varsigma) \} e^{2i\varsigma Et} \} d^3 \vec{p} d^3 \vec{p}' \\
&= -\frac{\varsigma}{2} \int_{\vec{p} \neq 0} -\varsigma \vec{p}_k \{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}, -\varsigma)] + [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}, -\varsigma)] \} + 0 d^3 \vec{p} \\
&= \frac{1}{2} \int_{\vec{p} \neq 0} \vec{p}_k \{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}, -\varsigma)] + [a_2(\vec{p}, -\varsigma), a_2^+(\vec{p}, -\varsigma)] \} d^3 \vec{p} \quad \square
\end{aligned}$$

#### 4.8 复场强 $\Psi(\vec{r}, t)$ 算符的 Fock 对易关系

推论 4.8.1.

$$\begin{cases} [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = i\varsigma \gamma_k^{ij} \partial_k \delta^3(\vec{r} - \vec{r}') \\ [\Psi_i(\vec{r}, t), \Psi_j(\vec{r}', t)] = 0 \\ [\Psi_i^+(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = 0 \end{cases} \Leftrightarrow \begin{cases} [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)] = \varsigma \delta^3(\vec{p} - \vec{p}') \\ [a_2(\vec{p}, -\varsigma), a_2^+(\vec{p}', -\varsigma)] = \varsigma \delta^3(\vec{p} - \vec{p}') \\ [a_1(\vec{p}, -\varsigma), a_1(\vec{p}', -\varsigma)] = 0, [a_2(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)] = 0 \\ [a_1(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)] = 0, [a_1(\vec{p}, -\varsigma), a_2^+(\vec{p}', -\varsigma)] = 0 \end{cases}$$

推论 4.8.2.  $[a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)] = \varsigma \delta^3(\vec{p} - \vec{p}')$

证明:  $[a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)]$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int [\lambda_m^{+i}(\vec{p}, -\varsigma) \Psi_i(\vec{r}, t) e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)}, \lambda_m^j(\vec{p}', -\varsigma) \Psi_j^+(\vec{r}', t) e^{i\varsigma(\vec{p}' \cdot \vec{r}' - E't)}] d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda_m^{+i}(\vec{p}, -\varsigma) \lambda_m^j(\vec{p}', -\varsigma) [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r} d^3 \vec{r}' \\
&= i\varsigma \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda_m^{+i}(\vec{p}, -\varsigma) \lambda_m^j(\vec{p}', -\varsigma) \gamma_k^{ij} \partial_k \delta^3(\vec{r} - \vec{r}') e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r} d^3 \vec{r}' \\
&= i\varsigma \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda_m^{+i}(\vec{p}, -\varsigma) \lambda_m^j(\vec{p}', -\varsigma) \gamma_k^{ij} i\varsigma p'_k e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r} \\
&= -\frac{1}{|\vec{p}|} \lambda_m^{+i}(\vec{p}, -\varsigma) \lambda_m^j(\vec{p}', h') \gamma_k^{ij} p_k \delta^3(\vec{p} - \vec{p}') \\
&= -\lambda_m^+(\vec{p}, -\varsigma) \frac{\gamma_k^{ij} p_k}{|\vec{p}|} \lambda_m(\vec{p}, -\varsigma) \delta^3(\vec{p} - \vec{p}') \\
&= \varsigma \lambda_m^+(\vec{p}, -\varsigma) \lambda_m(\vec{p}, -\varsigma) \delta^3(\vec{p} - \vec{p}') \\
&= \varsigma \delta^3(\vec{p} - \vec{p}') \quad \square
\end{aligned}$$

推论 4.8.3.  $[a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)] = -\varsigma \delta^3(\vec{p} - \vec{p}')$

证明:  $[a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)]$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int [\lambda_m^{+i}(\vec{p}, -\varsigma) \Psi_i(\vec{r}, t) e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)}, \lambda_m^j(\vec{p}', -\varsigma) \Psi_j^+(\vec{r}', t) e^{-i\varsigma(\vec{p}' \cdot \vec{r}' - E't)}] d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda_m^{+i}(\vec{p}, -\varsigma) \lambda_m^j(\vec{p}', -\varsigma) [\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{-i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r} d^3 \vec{r}' \\
&= i\varsigma \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda_m^{+i}(\vec{p}, -\varsigma) \lambda_m^j(\vec{p}', -\varsigma) \gamma_k^{ij} \partial_k \delta^3(\vec{r} - \vec{r}') e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{-i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r} d^3 \vec{r}'
\end{aligned}$$

$$\begin{aligned}
&= i\varsigma \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}'||\vec{p}|}} \int \lambda_m^{+i}(\vec{p}, -\varsigma) \lambda_m^j(\vec{p}', -\varsigma) \gamma^k_{ij} (-i\varsigma p'_k) e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} e^{-i\varsigma(\vec{p}'\cdot\vec{r}-E't)} d^3\vec{r}' \\
&= \frac{1}{|\vec{p}'|} \lambda_m^{+i}(\vec{p}, -\varsigma) \lambda_m^j(\vec{p}, h') \gamma^k_{ij} p_k \delta^3(\vec{p}' - \vec{p}') \\
&= \lambda_m^+(\vec{p}, -\varsigma) \frac{\gamma^k_{ij} p_k}{|\vec{p}'|} \lambda_m(\vec{p}, h') \delta^3(\vec{p}' - \vec{p}') \\
&= -\varsigma \lambda_m^+(\vec{p}, -\varsigma) \lambda_m(\vec{p}, h') \delta^3(\vec{p}' - \vec{p}') \\
&= -\varsigma \delta^3(\vec{p}' - \vec{p}')
\end{aligned}$$

□

推论4.8.4.  $[\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)] = i\varsigma \gamma^k_{ij} \partial_k \delta^3(\vec{r} - \vec{r}')$

证明:  $[\Psi_i(\vec{r}, t), \Psi_j^+(\vec{r}', t)]$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} d^3\vec{p} d^3\vec{p}' \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}', -\varsigma) \sqrt{|\vec{p}'||\vec{p}|} \\
&\{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)] e^{i\varsigma(\vec{p}\cdot\vec{r}-Et)} e^{-i\varsigma(\vec{p}'\cdot\vec{r}'-E't)} + [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)] e^{-i\varsigma(\vec{p}\cdot\vec{r}-Et)} e^{i\varsigma(\vec{p}'\cdot\vec{r}'-E't)} \} \\
&= \frac{1}{(2\pi)^3} \int \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}', -\varsigma) [\varsigma \delta^3(\vec{p}' - \vec{p}') e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} - \varsigma \delta^3(\vec{p}' - \vec{p}') e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p} d^3\vec{p}' \\
&= \frac{1}{(2\pi)^3} \int \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}', -\varsigma) \varsigma [|\vec{p}'| e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{-i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p}' \\
&= -\frac{1}{(2\pi)^3} \int [(-\varsigma|\vec{p}'|) \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}', -\varsigma) + (\varsigma|\vec{p}'|) \lambda_{mi}(-\vec{p}, -\varsigma) \lambda_{mj}^+(-\vec{p}, -\varsigma)] e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p}' \\
&= -\frac{1}{(2\pi)^3} \int [(-\varsigma|\vec{p}'|) \lambda_{mi}(\vec{p}, -\varsigma) \lambda_{mj}^+(\vec{p}', -\varsigma) + (0|\vec{p}'|) \lambda_{mi}(\vec{p}, \varsigma) \lambda_{mj}^+(\vec{p}, \varsigma) + (\varsigma|\vec{p}'|) \lambda_{mi}(\vec{p}, \varsigma) \lambda_{mj}^+(\vec{p}, \varsigma)] e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p}' \\
&= -\frac{1}{(2\pi)^3} \int \gamma^k_{ij} p_k \sum_h \lambda_{mi}(\vec{p}, h) \lambda_{mj}^+(\vec{p}, h) e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p}' \\
&= \frac{1}{(2\pi)^3} \int i\varsigma \gamma^k_{ij} p_k \delta_{ij} e^{i\varsigma\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p}' \\
&= i\varsigma \gamma^k_{ij} \partial_k \delta^3(\vec{r} - \vec{r}')
\end{aligned}$$

□

## 4.9 正规化能量动量算符的Fock表示

推论4.9.1.  $: H := \int \sum_{\sigma} |\vec{p}| a_{\sigma}^{\dagger}(\vec{p}) a_{\sigma}(\vec{p}) d^3\vec{p}, : \vec{P} := \int \sum_{\sigma} \vec{p} a_{\sigma}^{\dagger}(\vec{p}) a_{\sigma}(\vec{p}) d^3\vec{p}, : P_a := \int \sum_{\sigma} p_a a_{\sigma}^{\dagger}(\vec{p}) a_{\sigma}(\vec{p}) d^3\vec{p}$

推论4.9.2.  $a_{\sigma}(\vec{p}, \varsigma)|\varphi\rangle = 0, a_{\sigma}(\vec{p}, 0)|\varphi\rangle = 0, \forall \varphi \in Phys$

## 5 量子电动力学之场方案

### 5.1 电磁相互作用之场表述方案

定理5.1.1.

$$\begin{cases} [\Psi_{\alpha\varsigma}(x), \Psi_{\alpha'\varsigma'}^+(x')] = i\sigma_{\alpha\varsigma\alpha'\varsigma'}^{\alpha\beta} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha\varsigma}(x), \Psi_{\beta\varsigma}(x')] = 0, [\Psi_{\alpha'\varsigma'}^+(x), \Psi_{\beta'\varsigma'}^+(x')] = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b] \Psi = -i\sigma_{\varsigma ab}^{[\beta\varsigma]} J^b \\ \tilde{A} = \frac{-i\varsigma}{\sqrt{2}} \frac{\nabla \times (\Psi - \Psi^*)}{\nabla^2}, \tilde{\phi} = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi + \Psi^*)}{\nabla^2} \end{cases} \Leftrightarrow \begin{cases} [\tilde{A}_i(x), \tilde{A}_j(x')] = i(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \\ [\tilde{A}_i(x), \tilde{\phi}(x')] = 0, [\tilde{\phi}(x), \tilde{\phi}(x')] = 0 \\ \nabla^2 \tilde{A} - \partial_t^2 \tilde{A} = \tilde{J} + \partial_t \nabla \tilde{\phi}, \nabla^2 \tilde{\phi} = \rho, \nabla \cdot \tilde{A} = 0 \\ \sqrt{2} \Psi = -\partial_t \tilde{A} - \nabla \tilde{\phi} - i\varsigma \nabla \times \tilde{A} \end{cases}$$

推论5.1.1.  $L = -\int \bar{\psi}(\gamma^a \partial_a + m)\psi dr^3, H = \int \bar{\psi}(\gamma \cdot \nabla + m)\psi dr^3$

推论5.1.2.  $L = -\int \bar{\psi}[\gamma^a(\partial_a - ieA_a) + m]\psi dr^3 = -\int \bar{\psi}(\gamma^a \partial_a + m)\psi dr^3 + \int ie\bar{\psi}\gamma^a A_a \psi dr^3$

推论5.1.3.  $H = \int \bar{\psi}[\gamma \cdot (\nabla - ie\tilde{A}) + \gamma^4 e\tilde{\phi} + m]\psi dr^3 = \int \bar{\psi}(\gamma \cdot \nabla + m)\psi dr^3 - \int ie\bar{\psi}\gamma^a A_a \psi dr^3$

推论5.1.4.  $H_i = -L_i = -\int ie\bar{\psi}\gamma^a A_a \psi dr^3$

$$= -\frac{e\varsigma}{\sqrt{2}} \int \bar{\psi} \frac{\gamma \cdot [\nabla \times (\Psi - \Psi^*)]}{\nabla^2} \psi dr^3 - \frac{e}{\sqrt{2}} \int \bar{\psi} \frac{\gamma^4 [\nabla \cdot (\Psi + \Psi^*)]}{\nabla^2} \psi dr^3 + \int ie\bar{\psi} \frac{\gamma^a \partial_a \partial_t \phi}{\nabla^2} \psi dr^3$$

定理5.1.2.  $H_i = -L_i = -\int ie\bar{\psi}\gamma^a A_a \psi dr^3$

$$= -ie \int \frac{\bar{\psi}}{\sqrt{-\nabla^2}} (\vec{\gamma} \cdot \partial_t \vec{E}) \frac{\psi}{\sqrt{-\nabla^2}} dr^3 + e^2 \int \frac{\bar{\psi} \gamma^a \psi}{\sqrt{-\nabla^2}} \frac{\bar{\psi} \gamma_a \psi}{\sqrt{-\nabla^2}} dr^3 + \int \partial_a [ie\bar{\psi}\gamma^a \psi (\frac{\partial_t \phi}{\nabla^2})] dr^3$$

证明:  $H_i = -L_i = -\int ie\bar{\psi}\gamma^a A_a \psi dr^3$

$$\begin{aligned}
&= -\frac{e\varsigma}{\sqrt{2}} \int \bar{\psi} \frac{\gamma \cdot [\nabla \times (\Psi - \Psi^*)]}{\nabla^2} \psi dr^3 - \frac{e}{\sqrt{2}} \int \bar{\psi} \frac{\gamma^4 [\nabla \cdot (\Psi + \Psi^*)]}{\nabla^2} \psi dr^3 + \int ie\bar{\psi} \frac{\gamma^a \partial_a \partial_t \phi}{\nabla^2} \psi dr^3 \\
&= \frac{ie}{\sqrt{2}} \int \bar{\psi} \frac{\gamma \cdot [\partial_t (\Psi + \Psi^*) - \sqrt{2}\tilde{J}]}{\nabla^2} \psi dr^3 + e \int \bar{\psi} \frac{\gamma^4 \rho}{\nabla^2} \psi dr^3 + \int ie\bar{\psi} \frac{\gamma^a \partial_a \partial_t \phi}{\nabla^2} \psi dr^3 \\
&= \frac{ie}{\sqrt{2}} \int \bar{\psi} \frac{\gamma \cdot [\partial_t (\Psi + \Psi^*)]}{\nabla^2} \psi dr^3 - ie \int \bar{\psi} \gamma^a \psi (\frac{J_a}{\nabla^2}) dr^3 + \int ie\bar{\psi} \gamma^a \psi \partial_a (\frac{\partial_t \phi}{\nabla^2}) dr^3
\end{aligned}$$

$$\begin{aligned}
&= \frac{ie}{\sqrt{2}} \int \bar{\psi} \vec{\gamma} \cdot [\partial_t (\Psi + \Psi^*)] \psi dr^3 + \int J^a \frac{1}{\nabla^2} J_a dr^3 - \int \partial_a [J^a (\frac{\partial_t \phi}{\nabla^2})] dr^3 \\
&= \frac{ie}{\sqrt{2}} \int \bar{\psi} \vec{\gamma} \cdot [\partial_t (\Psi + \Psi^*)] \psi dr^3 - \int \frac{J^a}{\sqrt{-\nabla^2}} \frac{J_a}{\sqrt{-\nabla^2}} dr^3 - \int \partial_a [J^a (\frac{\partial_t \phi}{\nabla^2})] dr^3 \\
&= \frac{ie}{\sqrt{2}} \int \bar{\psi} \vec{\gamma} \cdot [\partial_t (\Psi + \Psi^*)] \psi dr^3 + e^2 \int \frac{\bar{\psi} \gamma^a \psi}{\sqrt{-\nabla^2}} \frac{\bar{\psi} \gamma_a \psi}{\sqrt{-\nabla^2}} dr^3 + \int \partial_a [ie \bar{\psi} \gamma^a \psi (\frac{\partial_t \phi}{\nabla^2})] dr^3 \\
&= ie \int \bar{\psi} \frac{\partial_t (\vec{\gamma} \cdot \vec{E})}{\nabla^2} \psi dr^3 + e^2 \int \frac{\bar{\psi} \gamma^a \psi}{\sqrt{-\nabla^2}} \frac{\bar{\psi} \gamma_a \psi}{\sqrt{-\nabla^2}} dr^3 + \int \partial_a [ie \bar{\psi} \gamma^a \psi (\frac{\partial_t \phi}{\nabla^2})] dr^3 \\
&= -ie \int \frac{\bar{\psi}}{\sqrt{-\nabla^2}} (\vec{\gamma} \cdot \partial_t \vec{E}) \frac{\psi}{\sqrt{-\nabla^2}} dr^3 + e^2 \int \frac{\bar{\psi} \gamma^a \psi}{\sqrt{-\nabla^2}} \frac{\bar{\psi} \gamma_a \psi}{\sqrt{-\nabla^2}} dr^3 + \int \partial_a [ie \bar{\psi} \gamma^a \psi (\frac{\partial_t \phi}{\nabla^2})] dr^3 \quad \square
\end{aligned}$$

以上定理把协变性和非协变性、辐射规范和洛仑兹规范统一表述在一起了，十分美妙。尤其第二项明显描述电子之间的排斥自作用能，大于零。同时完全可以按场量进行微扰展开，而无须用电磁势展开，这样整个展开过程都是物理的，没有非物理因素，且规范无关。是否这样可以避免无穷大？无须再重整化？还需进一步探索。第三项是全微分项，可以去掉(?)，此项十分微妙、美妙，它是洛仑兹协变性的保障，也是统一描述的关键之所在。

## 5.2 电磁相互作用之场表述的S矩阵

$$\text{推论5.2.1. } U(t, t_0) = 1 - i \int_{t_0}^t H_i(t_1) U(t_1, t_0) dt_1, S = U(+\infty, -\infty) = T \exp \left\{ -i \int_{-\infty}^{+\infty} H_i(t) dt \right\}$$

$$\begin{aligned}
\text{推论5.2.2. } S &= U(+\infty, -\infty) = T \exp \left\{ -i \int_{-\infty}^{+\infty} H_i(t) dt \right\} \\
&= T \exp \left\{ -i \int_{-\infty}^{+\infty} \left[ -ie \frac{\bar{\psi}}{\sqrt{-\nabla^2}} (\vec{\gamma} \cdot \partial_t \vec{E}) \frac{\psi}{\sqrt{-\nabla^2}} + e^2 \frac{\bar{\psi} \gamma^a \psi}{\sqrt{-\nabla^2}} \frac{\bar{\psi} \gamma_a \psi}{\sqrt{-\nabla^2}} \right] dx^4 \right\}
\end{aligned}$$

# 第二十三章 无质量粒子的协变量子化方案

自我评述：在本章我应用前面章节创立的数学工具和常数不变张量分析，对各种无质量自旋粒子按相同、统一的新协变量子化程式成功进行了量子化。

## 1 全新的协变量子化程式

### 1.1 新量子化程式

- 1、首先根据常数不变张量分析，合理猜测得到协变对易规则。
- 2、根据微观因果性原理和排除负模态负几率的反常粒子，进一步确定合理的协变对易规则。
- 3、根据得到的协变对易规则，进一步得到福克表象的对易规则。
- 4、根据量子场论中能量动量的普遍福克表示，反推出能量动量算符，并验证它们是否是真正的能量动量，进一步确定自旋表示，角动量表示。
- 5、根据能量算符，重新得到与经典方程形式一样的量子算符方程，并验证量子彭加莱代数是否成立。
- 6、考虑相互作用，进行散射矩阵计算等，并与实验对比。
- 7、向高维空间推广，向弦理论推广
- 8、如何用场传播子代替势传播子
- 9、把弦理论看作势理论，那么其对应的场理论是什么？
- 10、能解决无穷大问题不？
- 11、经典平面波解与量子平面波解的区别，模不存在与模没激发的重大区别，可能隐含物理大发现？

## 2 无质量复标量场协变量子化方案 [26, 27, 39, 40]

### 2.1 复标量场方程及其平面波解

定义2.1.1.  $\partial_\alpha \partial^\alpha \psi(\vec{r}, t) = 0$

推论2.1.1.  $\psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} |\vec{p}|^{-\frac{1}{2}} \lambda(\hat{p}, 0) [a_1(\vec{p}, 0) e^{ip \cdot x} + a_2^+(\vec{p}, 0) e^{-ip \cdot x}] d^3 \vec{p}$

$$\Leftrightarrow \begin{cases} |\vec{p}|^{-\frac{1}{2}} a_1(\vec{p}, 0) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \lambda^+(\hat{p}, 0) [\phi(\vec{r}, t) + \frac{i}{|\vec{p}|} \dot{\phi}(\vec{r}, t)] e^{-ip \cdot x} d^3 \vec{r} \\ |\vec{p}|^{-\frac{1}{2}} a_2^+(\vec{p}, 0) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \lambda^+(\hat{p}, 0) [\phi(\vec{r}, t) - \frac{i}{|\vec{p}|} \dot{\phi}(\vec{r}, t)] e^{ip \cdot x} d^3 \vec{r} \\ \lambda(\hat{p}, 0) := \frac{1}{\sqrt{2}}, \Gamma(0) := \lambda(\hat{p}, 0) \lambda^+(\hat{p}, 0) = \frac{1}{2}, \lambda^+(\hat{p}, 0) \lambda(\hat{p}, 0) = \frac{1}{2} \end{cases}$$

定义2.1.2. 定义投影算子:  $\hat{P}(0) := 2\lambda(\hat{p}, 0) \lambda^+(\hat{p}, 0) = 1$

### 2.2 复标量场协变常数不变张量 $\Gamma(0)$ 的性质

定义2.2.1.  $\lambda(\hat{p}, 0) := \frac{1}{\sqrt{2}}, \Gamma(0) := \lambda(\hat{p}, 0) \lambda^+(\hat{p}, 0) = \frac{1}{2}, \lambda^+(\hat{p}, 0) \lambda(\hat{p}, 0) = \frac{1}{2}$

### 2.3 复标量场数学上一般的协变对易规则

定理2.3.1.

$$\begin{cases} [a_\sigma(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)]_{\pm} = \delta_\sigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)]_{\pm} = 0 \\ [a_\sigma^+(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)]_{\pm} = 0 \end{cases} \Rightarrow \begin{cases} [\psi(x), \psi^+(x')]_{\pm} \\ = i2\Gamma(0) [(\delta_1 \pm \delta_2) \Delta^{(+)}(x - x') - \pm \delta_2 \Delta(x - x')] \\ [\psi(x), \psi(x')]_{\pm} = 0 \\ [\psi^+(x), \psi^+(x')]_{\pm} = 0 \end{cases}$$

证明:  $[\psi^{(+)}(x), \psi^{(+)}(x')]_{\pm}$

$$= \frac{1}{(2\pi)^3} \int |\vec{p}|^{-\frac{1}{2}} |\vec{p}'|^{-\frac{1}{2}} \lambda(\hat{p}, 0) \lambda^+(\hat{p}', 0) [a_1(\vec{p}, 0), a_1^+(\vec{p}', 0)]_{\pm} e^{i\vec{p} \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}'$$



$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^{-1} \delta_1 \lambda(\hat{p}, 0) \lambda^+(\hat{p}, 0) \delta^3(\vec{p} - \vec{p}') e^{i\vec{p} \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^{-1} \delta_1 \lambda(\hat{p}, 0) \lambda^+(\hat{p}, 0) e^{i\vec{p} \cdot (x-x')} d^3 \vec{p} \\
&= i \frac{1}{(2\pi)^3} \int \delta_1 \frac{-i}{2|\vec{p}|} 2\Gamma(0) e^{i\vec{p} \cdot (x-x')} d^3 \vec{p} \\
&= i \delta_1 2\Gamma(0) \Delta^{(+)}(x-x')
\end{aligned}$$

□

$$\begin{aligned}
&\text{证明: } [\psi^{(-)}(x), \psi^{(-)+}(x')]_{\pm} \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^{-\frac{1}{2}} |\vec{p}'|^{-\frac{1}{2}} \lambda(\hat{p}, 0) \lambda^+(\hat{p}, 0) [a_2^+(\vec{p}, 0), a_2(\vec{p}', 0)]_{\pm} e^{-i\vec{p} \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \pm \frac{1}{(2\pi)^3} \int |\vec{p}|^{-1} \delta_2 \lambda(\hat{p}, 0) \lambda^+(\hat{p}, 0) \delta^3(\vec{p} - \vec{p}') e^{-i\vec{p} \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \pm \frac{1}{(2\pi)^3} \int |\vec{p}|^{-1} \delta_2 \lambda(\hat{p}, 0) \lambda^+(\hat{p}, 0) e^{-i\vec{p} \cdot (x-x')} d^3 \vec{p} \\
&= -\pm i \frac{1}{(2\pi)^3} \int \delta_2 \frac{i}{2|\vec{p}|} 2\Gamma(0) e^{-i\vec{p} \cdot (x-x')} d^3 \vec{p} \\
&= -\pm i \delta_2 2\Gamma(0) \Delta^{(-)}(x-x')
\end{aligned}$$

□

$$\begin{aligned}
&\text{证明: } [\psi(x), \psi^+(x')]_{\pm} \\
&= [\psi^{(+)}(x), \psi^{(++)}(x')]_{\pm} + [\psi^{(-)}(x), \psi^{(-)+}(x')]_{\pm} \\
&= i\delta_1 2\Gamma(0) \Delta^{(+)}(x-x') - \pm i\delta_2 2\Gamma(0) \Delta^{(-)}(x-x') \\
&= i2\Gamma(0) [\delta_1 \Delta^{(+)}(x-x') - \pm \delta_2 \Delta^{(-)}(x-x')] \\
&= i2\Gamma(0) [(\delta_1 \pm \delta_2) \Delta^{(+)}(x-x') - \pm \delta_2 \Delta(x-x')]
\end{aligned}$$

□

从上式可知，只有 $\delta_1 \pm \delta_2 = 0$ 时，才满足微观因果性，同时只有 $\delta_1, \delta_2 \geq 0$ 时，才满足几率非负性。所以数学上八种协变对易或反对易方案中，物理上合理的只有一种：即 $\delta_1 = \delta_2 = 1$ ，且满足对易关系。其实还有两种，即 $\delta_1 = \delta_2 = 0$ ，且满足对易或反对易关系，就是经典情形。

## 2.4 复标量场物理的协变对易规则

定理2.4.1.

$$\begin{cases} [a_{\sigma}(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_{\sigma}(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)] = 0 \\ [a_{\sigma}^+(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = 0 \end{cases} \Rightarrow \begin{cases} [\psi(x), \psi^+(x')] = i2\Gamma(0) \Delta(x-x') \\ [\psi(x), \psi(x')] = 0 \\ [\psi^+(x), \psi^+(x')] = 0 \end{cases}$$

推论2.4.1.

$$\begin{cases} [a_{\sigma}(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_{\sigma}(\vec{p}, 0), a_{\sigma'}(\vec{p}', 0)] = 0 \\ [a_{\sigma}^+(\vec{p}, 0), a_{\sigma'}^+(\vec{p}', 0)] = 0 \end{cases} \Rightarrow \begin{cases} [\psi^{(+)}(x), \psi^{(++)}(x')] = i2\Gamma(0) \Delta^{(+)}(x-x') \\ [\psi^{(-)}(x), \psi^{(-)+}(x')] = i2\Gamma(0) \Delta^{(-)}(x-x') \\ [\psi^{(+)}(x), \psi^{(-)+}(x')] = 0 \end{cases}$$

推论2.4.2.

$$\begin{cases} [\psi(x), \psi^+(x')] = i2\Gamma(0) \Delta(x-x') \\ [\psi(x), \psi(x')] = 0 \\ [\psi^+(x), \psi^+(x')] = 0 \end{cases} \Rightarrow \begin{cases} [\psi(\vec{r}, t), \psi^+(\vec{r}', t)] = 0 \\ [\psi(\vec{r}, t), \psi(\vec{r}', t)] = 0 \\ [\psi^+(x), \psi^+(\vec{r}', t)] = 0 \end{cases}$$

## 2.5 复标量场的对易函数、因果函数和费曼传播子

$$\text{定义2.5.1. } \begin{cases} \Delta^{(+)}(x) := \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} e^{i\vec{p} \cdot x} d^3 \vec{p}, i\Delta^{(+)}(\vec{r}, 0) \leftrightarrow \frac{1}{2|\vec{p}|} \\ \Delta^{(-)}(x) := -\frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} e^{-i\vec{p} \cdot x} d^3 \vec{p}, \Delta^{(-)}(x) = -\Delta^{(+)}(-x) \\ \Delta(x) := \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} [e^{i\vec{p} \cdot x} - e^{-i\vec{p} \cdot x}] d^3 \vec{p}, \Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) \end{cases}$$

$$\text{性质2.5.1. } \begin{cases} \Delta^*(x) = \Delta(x), \Delta(-x) = -\Delta(x), (\nabla^2 - \partial_t^2) \Delta(x) = 0 \\ \partial_t \Delta(x)|_{t=0} = -\delta^3(\vec{r}), \partial_k \partial_t \Delta(x)|_{t=0} = \partial_t \partial_k \Delta(x)|_{t=0} = -\partial_k \delta^3(\vec{r}) \\ \partial_k \Delta(x)|_{t=0} = 0, \partial_k \partial_t \Delta(x)|_{t=0} = 0, \partial_t^2 \Delta(x)|_{t=0} = 0 \end{cases}$$

性质2.5.2.  $\Delta(x-x') := \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3\vec{p}$

$$\begin{cases} \partial_u \Delta(x-x') = -\partial'_u \Delta(x-x') & \begin{cases} (\sqrt{-\nabla^2})^n \Delta(x-x') = (\sqrt{-\nabla'^2})^n \Delta(x-x') \\ \frac{1}{(\sqrt{-\nabla^2})^n} \Delta(x-x') = \frac{1}{(\sqrt{-\nabla'^2})^n} \Delta(x-x') \end{cases} \\ \nabla \Delta(x-x') = -\nabla' \Delta(x-x') \\ \partial_\pi \Delta(x-x') = -\partial'_\pi \Delta(x-x') & \partial_\pi^2 \Delta(x-x') = \partial'^2_\pi \Delta(x-x') \end{cases}$$

定义2.5.2.

$$\begin{cases} \Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) = \Delta^{ret}(x) - \Delta^{adv}(x) \\ \Delta^{(l)}(x) = i[\Delta^{(-)}(x) - \Delta^{(+)}(x)] \\ \Delta_F(x) = \langle T\varphi(x)\varphi(x') \rangle_0 = i\Delta^{(c)}(x) = \frac{1}{(2\pi)^4} \int \Delta_F(p) e^{ipx} d^4p \\ \Delta_F(p) = \frac{-i}{p^2 - i\varepsilon} \\ \Delta^{(c)}(x) = \theta(t)\Delta^{(+)}(x) - \theta(-t)\Delta^{(-)}(x) \\ \Delta^{ret}(x) = \theta(t)\Delta(x) = \Delta^{(c)}(x) + \Delta^{(-)}(x) \\ \Delta^{adv}(x) = -\theta(-t)\Delta(x) = \Delta^{(c)}(x) - \Delta^{(+)}(x) \end{cases}$$

推论2.5.1.

$$\begin{cases} \partial_a \partial^a \Delta(x) = 0 & \begin{cases} \partial_a \partial^a \Delta^{(c)}(x) = \delta^4(x) \\ \partial_a \partial^a \Delta^{ret}(x) = \delta^4(x) \\ \partial_a \partial^a \Delta^{adv}(x) = \delta^4(x) \\ \partial_a \partial^a \Delta_F(x) = i\delta^4(x) \end{cases} \\ \partial_a \partial^a \Delta^{(+)}(x) = 0 \\ \partial_a \partial^a \Delta^{(-)}(x) = 0 \\ \partial_a \partial^a \Delta^{(l)}(x) = 0 \end{cases}$$

推论2.5.2.  $\Delta(x)\partial_t\delta(t) = -\partial_t\Delta(x)\delta(t) = \delta^4(x)$

证明:  $\int f(t)\Delta(x)\partial_t\delta(t)dt = -\partial_t[f(t)\Delta(x)]|_{t=0} = f(0)\delta^3(\vec{r})$  □

证明:  $\int f(t)\partial_t\Delta(x)\delta(t)dt = f(t)\partial_t\Delta(x)|_{t=0} = -f(0)\delta^3(\vec{r})$  □

推论2.5.3.  $\partial_t^2[\theta(t)\Delta(x)] = -\delta^4(x) + \theta(t)\partial_t^2\Delta(x)$

证明:  $\partial_t^2[\theta(t)\Delta(x)]$

$$\begin{aligned} &= \partial_t[\partial_t\theta(t)\Delta(x) + \theta(t)\partial_t\Delta(x)] \\ &= \partial_t^2\theta(t)\Delta(x) + 2\partial_t\theta(t)\partial_t\Delta(x) + \theta(t)\partial_t^2\Delta(x) \\ &= \partial_t\delta(t)\Delta(x) + 2\delta(t)\partial_t\Delta(x) + \theta(t)\partial_t^2\Delta(x) \\ &= \delta(t)\partial_t\Delta(x) + \theta(t)\partial_t^2\Delta(x) \\ &= -\delta^4(x) + \theta(t)\partial_t^2\Delta(x) \end{aligned}$$
 □

推论2.5.4.  $\Delta(x)\partial_t^n\delta(t) = \sum_{l=0}^{[(n-1)/2]} C_n^{2l+1} \nabla^{2l} \partial_t^{n-2l-1} \delta^4(x)$

证明:  $\int f(t)\Delta(x)\partial_t^n\delta(t)dt$

$$\begin{aligned} &= (-1)^n \partial_t^n [f(t)\Delta(x)]|_{t=0} = f(0)\delta^3(\vec{r}) \\ &= (-1)^n \sum_{k=0}^n C_n^k \partial_t^{n-k} f(t) \partial_t^k \Delta(x)|_{t=0} = f(0)\delta^3(\vec{r}) \\ &= (-1)^n \sum_{l=0}^{[(n-1)/2]} C_n^{2l+1} \partial_t^{n-2l-1} f(t) \partial_t^{2l+1} \Delta(x)|_{t=0} \\ &= (-1)^{n+1} \sum_{l=0}^{[(n-1)/2]} C_n^{2l+1} \partial_t^{n-2l-1} f(t)|_{t=0} \nabla^{2l} \delta^3(\vec{r}) \\ &= (-1)^{n+1} \sum_{l=0}^{[(n-1)/2]} C_n^{2l+1} \nabla^{2l} \delta^3(\vec{r}) \int \partial_t^{n-2l-1} f(t) \delta(t) dt \end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{[(n-1)/2]} C_n^{2l+1} \nabla^{2l} \delta^3(\vec{r}) \int f(t) \partial_t^{n-2l-1} \delta(t) dt \\
&= \int f(t) \sum_{l=0}^{[(n-1)/2]} C_n^{2l+1} \nabla^{2l} \partial_t^{n-2l-1} \delta^4(x) dt
\end{aligned}$$

□

推论2.5.5.  $\Delta(x) \partial_t^2 \delta(t) = 2 \partial_t \delta^4(x)$

推论2.5.6.  $\Delta(x) \partial_t^3 \delta(t) = 3 \partial_t^2 \delta^4(x) + \nabla^2 \delta^4(x)$

## 2.6 复标量场能量动量算符的提取

推论2.6.1.  $H = \int |\vec{p}| [a_1^+(\vec{p}, 0) a_1(\vec{p}, 0) + a_2(\vec{p}, 0) a_2^+(\vec{p}, 0)] d^3 \vec{p}$   
 $= i \int \psi^+(\vec{r}, t) \sigma \cdot \nabla \psi(\vec{r}, t) d^3 \vec{r} = i \int \psi^+(\vec{r}, t) \partial_t \psi(\vec{r}, t) d^3 \vec{r}$

证明:  $H = \int |\vec{p}| [a_1^+(\vec{p}, 0) a_1(\vec{p}, 0) + a_2(\vec{p}, 0) a_2^+(\vec{p}, 0)] d^3 \vec{p}$   
 $= \frac{1}{(2\pi)^3} \int [\lambda(\hat{p}, 0) \psi^+(\vec{r}', t) e^{i\vec{p} \cdot \vec{x}'} \lambda^+(\hat{p}, 0) \psi(\vec{r}, t) e^{-i\vec{p} \cdot \vec{x}} + \lambda(\hat{p}, 0) \psi^+(\vec{r}', t) e^{-i\vec{p} \cdot \vec{x}'} \lambda^+(\hat{p}, 0) \psi(\vec{r}, t) e^{i\vec{p} \cdot \vec{x}}] d^3 \vec{p} d^3 \vec{r}' d^3 \vec{r}$   
 $= \frac{1}{(2\pi)^3} \int \lambda^+(\hat{p}, 0) \lambda(\hat{p}, 0) \psi^+(\vec{r}', t) \psi(\vec{r}, t) [e^{-i\vec{p} \cdot (\vec{r}' - \vec{r})} + e^{i\vec{p} \cdot (\vec{r}' - \vec{r})}] d^3 \vec{p} d^3 \vec{r}' d^3 \vec{r}$   
 $= \frac{1}{(2\pi)^3} \int \psi^+(\vec{r}', t) \psi(\vec{r}, t) [e^{-i\vec{p} \cdot (\vec{r}' - \vec{r})} + e^{i\vec{p} \cdot (\vec{r}' - \vec{r})}] d^3 \vec{p} d^3 \vec{r}' d^3 \vec{r}$

□

## 2.7 复标量场的彭加莱对称性

推论2.7.1.  $\hat{P}_a(0) = \int \psi^+(\vec{r}, t) \hat{P}_a i \dot{\psi}(\vec{r}, t) d^3 \vec{r}$ ,  $M_{ab}(n) = \int \psi^+(\vec{r}, t) \hat{M}_{ab} i \dot{\psi}(\vec{r}, t) d^3 \vec{r}$

引理2.7.1.  $[\dot{\psi}_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}^+(\vec{r}', t)] = -i \delta_{k_\zeta l_\zeta} \delta^3(\vec{r} - \vec{r}')$

定理2.7.1.  $\begin{cases} [L_{ab}, L_{cd}] = -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \\ [L_{ab}, P_c] = -i(g_{bc}P_a - g_{ac}P_b), [P_a, P_b] = 0 \end{cases}$

证明:  $[L_{ab}, L_{cd}]$   
 $= - \int d^3 \vec{r} d^3 \vec{r}' [\psi^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) i \dot{\psi}(\vec{r}, t), \psi^+(\vec{r}', t) (r'_c \partial'_d - r'_d \partial'_c) i \dot{\psi}(\vec{r}', t)]$   
 $= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' [\psi_{k_\zeta}^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) (r'_c \partial'_d - r'_d \partial'_c) \dot{\psi}_{l'_\zeta}(\vec{r}', t)]$   
 $= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}'$   
 $\{ \psi_{k_\zeta}^+(\vec{r}, t) [(r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) (r'_c \partial'_d - r'_d \partial'_c) \dot{\psi}_{l'_\zeta}(\vec{r}', t)]$   
 $+ \psi_{k'_\zeta}^+(\vec{r}', t) [\psi_{k_\zeta}^+(\vec{r}, t), (r'_c \partial'_d - r'_d \partial'_c) \dot{\psi}_{l'_\zeta}(\vec{r}', t)] (r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t) \}$   
 $= - \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}'$   
 $\{ \psi_{k_\zeta}^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) (-i) \delta_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \dot{\psi}_{l'_\zeta}(\vec{r}', t)$   
 $- \psi_{k'_\zeta}^+(\vec{r}', t) (r'_c \partial'_d - r'_d \partial'_c) (-i) \delta_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t) \}$   
 $= - \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}'$   
 $\{ \psi_{k_\zeta}^+(\vec{r}, t) (r_a \partial'_b - r_b \partial'_a) (-i) \delta_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \dot{\psi}_{l'_\zeta}(\vec{r}', t)$   
 $- \psi_{k'_\zeta}^+(\vec{r}', t) (r'_c \partial'_d - r'_d \partial'_c) (-i) \delta_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t) \}$   
 $= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}'$   
 $\{ \psi_{k_\zeta}^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) (-i) \delta_{l_\zeta k'_\zeta} (r_c \partial_d - r_d \partial_c) \dot{\psi}_{l'_\zeta}(\vec{r}', t)$   
 $- \psi_{k'_\zeta}^+(\vec{r}', t) (r_c \partial_d - r_d \partial_c) (-i) \delta_{l'_\zeta k_\zeta} (r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t) \}$   
 $= - \int \psi^+(\vec{r}, t) [-i(r_a \partial_b - r_b \partial_a), -i(r_c \partial_d - r_d \partial_c)] (-i) \dot{\psi}(\vec{r}, t) d^3 \vec{r}$   
 $= \int \psi^+(\vec{r}, t) [\hat{L}_{ab}, \hat{L}_{cd}] i \dot{\psi}(\vec{r}, t) d^3 \vec{r}$   
 $= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac})$

□

证明:  $[L_{ab}, P_c]$   
 $= - \int d^3 \vec{r} d^3 \vec{r}' [\psi^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) i \dot{\psi}(\vec{r}, t), \psi^+(\vec{r}', t) \partial'_c i \dot{\psi}(\vec{r}', t)]$   
 $= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' [\psi_{k_\zeta}^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) \partial'_c \dot{\psi}_{l'_\zeta}(\vec{r}', t)]$   
 $= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}'$

$$\begin{aligned}
& \{ \psi_{k_\zeta}^+(\vec{r}, t) [(r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] \partial'_c \dot{\psi}_{l'_\zeta}(\vec{r}', t) + \psi_{k'_\zeta}^+(\vec{r}', t) [\psi_{k_\zeta}^+(\vec{r}, t), \partial'_c \dot{\psi}_{l'_\zeta}(\vec{r}', t)] (r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t) \} \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \{ \psi_{k_\zeta}^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) (-i) \delta_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \partial'_c \dot{\psi}_{l'_\zeta}(\vec{r}', t) \\
& - \psi_{k'_\zeta}^+(\vec{r}', t) \partial'_c (-i) \delta_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t) \} \\
& = -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \{ \psi_{k_\zeta}^+(\vec{r}, t) (r_a \partial'_b - r_b \partial'_a) (-i) \delta_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \partial'_c \dot{\psi}_{l'_\zeta}(\vec{r}', t) \\
& - \psi_{k'_\zeta}^+(\vec{r}', t) \partial_c (-i) \delta_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t) \} \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} \\
& \{ \psi_{k_\zeta}^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) (-i) \delta_{l_\zeta k'_\zeta} \partial_c \dot{\psi}_{l'_\zeta}(\vec{r}, t) - \psi_{k'_\zeta}^+(\vec{r}, t) \partial_c (-i) \delta_{l'_\zeta k_\zeta} (r_a \partial_b - r_b \partial_a) \dot{\psi}_{l_\zeta}(\vec{r}, t) \} \\
& = -\int \psi^+(\vec{r}, t) [-i(r_a \partial_b - r_b \partial_a), -i \partial'_c] (-i) \dot{\psi}(\vec{r}, t) d^3 \vec{r} \\
& = \int \psi^+(\vec{r}, t) [\hat{L}_{ab}, \hat{P}_c] i \dot{\psi}(\vec{r}, t) d^3 \vec{r} \\
& = -i(g_{bc} P_a - g_{ac} P_b)
\end{aligned}$$

□

证明:  $[P_a, P_b]$ 

$$\begin{aligned}
& = -\int [\psi^+(\vec{r}, t) \partial_a i \dot{\psi}(\vec{r}, t), \psi^+(\vec{r}', t) \partial'_b i \dot{\psi}(\vec{r}', t)] d^3 \vec{r} d^3 \vec{r}' \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int [\psi_{k_\zeta}^+(\vec{r}, t) \partial_a \dot{\psi}_{l_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) \partial'_b \dot{\psi}_{l'_\zeta}(\vec{r}', t)] d^3 \vec{r} d^3 \vec{r}' \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \{ \psi_{k_\zeta}^+(\vec{r}, t) [\partial_a \dot{\psi}_{l_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] \partial'_b \dot{\psi}_{l'_\zeta}(\vec{r}', t) + \psi_{k'_\zeta}^+(\vec{r}', t) [\psi_{k_\zeta}^+(\vec{r}, t), \partial'_b \dot{\psi}_{l'_\zeta}(\vec{r}', t)] \partial_a \dot{\psi}_{l_\zeta}(\vec{r}, t) \} \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \{ \psi_{k_\zeta}^+(\vec{r}, t) (-i) \delta_{l_\zeta k'_\zeta} \partial_a \delta^3(\vec{r} - \vec{r}') \partial'_b \dot{\psi}_{l'_\zeta}(\vec{r}', t) - \psi_{k'_\zeta}^+(\vec{r}', t) (-i) \delta_{l'_\zeta k_\zeta} \partial'_b \delta^3(\vec{r}' - \vec{r}) \partial_a \dot{\psi}_{l_\zeta}(\vec{r}, t) \} \\
& = -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \{ \psi_{k_\zeta}^+(\vec{r}, t) (-i) \delta_{l_\zeta k'_\zeta} \partial'_a \delta^3(\vec{r} - \vec{r}') \partial'_b \dot{\psi}_{l'_\zeta}(\vec{r}', t) - \psi_{k'_\zeta}^+(\vec{r}', t) (-i) \delta_{l'_\zeta k_\zeta} \partial_b \delta^3(\vec{r}' - \vec{r}) \partial_a \dot{\psi}_{l_\zeta}(\vec{r}, t) \} \\
& = \int \{ \psi_{k_\zeta}^+(\vec{r}, t) (-i) \delta^{k_\zeta l'_\zeta} \partial_a \partial_b \dot{\psi}_{l'_\zeta}(\vec{r}, t) - \psi_{k'_\zeta}^+(\vec{r}, t) (-i) \delta^{k'_\zeta l_\zeta} \partial_b \partial_a \frac{\dot{\psi}_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \} d^3 \vec{r} \\
& = \int \psi^+(\vec{r}, t) (\partial_a \partial_b - \partial_b \partial_a) (-i) \dot{\psi}(\vec{r}, t) d^3 \vec{r} \\
& = -\int \psi^+(\vec{r}, t) (\partial_a \partial_b - \partial_b \partial_a) i \dot{\psi}(\vec{r}, t) d^3 \vec{r} \\
& = \int \psi^+(\vec{r}, t) [\hat{P}_a, \hat{P}_b] i \dot{\psi}(\vec{r}, t) d^3 \vec{r} = 0
\end{aligned}$$

□

### 3 中微子场协变量子化方案

#### 3.1 中微子自旋算符方程及其平面波解

定理3.1.1.  $[\frac{1}{2} \partial_a + i S_{ab} (\frac{1}{2}, \varsigma) \partial^b] \psi(x) = 0 \Leftrightarrow (\sigma, -i\varsigma)^a \partial_a \psi(x) = 0$ 

$$\text{推论3.1.1. } \begin{cases} \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \lambda(\hat{p}, -\frac{\varsigma}{2}) [a_1(\vec{p}, -\frac{\varsigma}{2}) e^{ip \cdot x} + a_2^+(\vec{p}, -\frac{\varsigma}{2}) e^{-ip \cdot x}] d^3 \vec{p} \\ a_1(\vec{p}, -\frac{\varsigma}{2}) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -\frac{\varsigma}{2}) \psi(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r} = \frac{1}{(2\pi)^{3/2}} \int \frac{i}{|\vec{p}|} \lambda^+(\hat{p}, -\frac{\varsigma}{2}) \dot{\psi}(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r} \\ a_2^+(\vec{p}, -\frac{\varsigma}{2}) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -\frac{\varsigma}{2}) \psi(\vec{r}, t) e^{ip \cdot x} d^3 \vec{r} = \frac{1}{(2\pi)^{3/2}} \int \frac{-i}{|\vec{p}|} \lambda^+(\hat{p}, -\frac{\varsigma}{2}) \dot{\psi}(\vec{r}, t) e^{ip \cdot x} d^3 \vec{r} \end{cases}$$

定义3.1.1. 投影算子:  $\hat{P}_{A_\varsigma A'_\zeta}(\frac{1}{2}, \varsigma) := \lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\varsigma}{2}), \hat{P}^2(\frac{1}{2}, \varsigma) = \hat{P}(\frac{1}{2}, \varsigma), \hat{P}^+(\frac{1}{2}, \varsigma) = \hat{P}(\frac{1}{2}, \varsigma)$ 

#### 3.2 中微子自旋算符方程平面波解的洛伦兹变换

$$\text{推论3.2.1. } \Lambda_{\vec{v}} = e^{-\varsigma \ln[\gamma_v(1+v)] \hat{v} \cdot \sigma(\frac{1}{2})} = \frac{1}{\sqrt{2(1+\gamma_v)}} [1 + \gamma_v - \gamma_v v \hat{v} \cdot \sigma] = \frac{1}{\sqrt{2(1+\gamma_v)}} \begin{bmatrix} 1 + \gamma_v(1 - v_z) & -\gamma_v v_x + i\gamma_v v_y \\ -\gamma_v v_x - i\gamma_v v_y & 1 + \gamma_v(1 + v_z) \end{bmatrix}$$

$$\text{推论3.2.2. } L_{\vec{v}} = e^{-\ln[\gamma_v(1+v)] \hat{v} \cdot L} = 1 - \gamma_v (\vec{v} \cdot L) + \frac{\gamma_v - 1}{v^2} (\vec{v} \cdot L)^2 = \gamma_v (1 - \vec{v} \cdot L) - \frac{\gamma_v - 1}{v^2} (\vec{v} \cdot R)^2$$

$$\text{推论3.2.3. } \psi'(L_{\vec{v}} x) = \frac{1}{(2\pi)^{3/2}} \int \Lambda_{\vec{v}} \lambda(\hat{p}, -\frac{\varsigma}{2}) [a_1(\vec{p}, -\frac{\varsigma}{2}) e^{iL_{\vec{v}} p \cdot L_{\vec{v}} x} + a_2^+(\vec{p}, -\frac{\varsigma}{2}) e^{-iL_{\vec{v}} p \cdot L_{\vec{v}} x}] d^3 \vec{p}$$

$$\text{推论3.2.4. } L_{\vec{v}} p = \Lambda_{\vec{v}} \lambda(\hat{p}, -\frac{\varsigma}{2}) = \lambda(L_{\vec{v}} \hat{p}, -\frac{\varsigma}{2})$$

$$\text{推论3.2.5. } \begin{bmatrix} \gamma_u \vec{u}' \\ i\gamma_u u' \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \gamma_u \vec{u} \\ i\gamma_u u \end{bmatrix}, \begin{bmatrix} \vec{p}' \\ iE' \end{bmatrix} = L_{\vec{v}} \begin{bmatrix} \vec{p} \\ iE \end{bmatrix}$$

$$\text{推论3.2.6. } \lambda(\hat{p}, \frac{1}{2}) = \frac{1}{\sqrt{2}\sqrt{1+\hat{p}_z}} \begin{bmatrix} 1 + \hat{p}_z \\ \hat{p}_x + i\hat{p}_y \end{bmatrix}, \lambda(\hat{p}, -\frac{1}{2}) = \frac{1}{\sqrt{2}\sqrt{1+\hat{p}_z}} \begin{bmatrix} -\hat{p}_x + i\hat{p}_y \\ 1 + \hat{p}_z \end{bmatrix}$$

证明:  $\Lambda_{-\vec{v}}\lambda(\hat{p}, \frac{1}{2})$

$$\begin{aligned} &= \frac{1}{\sqrt{2(1+\hat{p}_z)}} \Lambda_{-\vec{v}} \begin{bmatrix} 1 + \hat{p}_z \\ \hat{p}_x + i\hat{p}_y \end{bmatrix} = \frac{1}{\sqrt{2(1+\hat{u}_z)}} \Lambda_{-\vec{v}} \begin{bmatrix} 1 + \hat{u}_z \\ \hat{u}_x + i\hat{u}_y \end{bmatrix} \\ &= \frac{1}{\sqrt{2(1+\hat{u}_z)}} \frac{1}{\sqrt{2(1+\gamma_v)}} \begin{bmatrix} 1 + \gamma_v(1+v_z) & \gamma_v v_x - i\gamma_v v_y \\ \gamma_v v_x + i\gamma_v v_y & 1 + \gamma_v(1-v_z) \end{bmatrix} \begin{bmatrix} 1 + \hat{u}_z \\ \hat{u}_x + i\hat{u}_y \end{bmatrix} \\ &= \frac{1}{2\sqrt{(1+\hat{u}_z)(1+\gamma_v)}} \begin{bmatrix} [1 + \gamma_v(1+v_z)](1 + \hat{u}_z) + (\gamma_v v_x - i\gamma_v v_y)(\hat{u}_x + i\hat{u}_y) \\ (\gamma_v v_x + i\gamma_v v_y)(1 + \hat{u}_z) + [1 + \gamma_v(1-v_z)](\hat{u}_x + i\hat{u}_y) \end{bmatrix} \\ &= \frac{1}{2\sqrt{(1+\hat{u}_z)(1+\gamma_v)}} \begin{bmatrix} (1 + \gamma_v)(1 + \hat{u}_z) + \gamma_v[v_z + i(\vec{v} \times \hat{u})_z + \vec{v} \cdot \hat{u}] \\ (1 + \gamma_v)(\hat{u}_x + i\hat{u}_y) + \gamma_v\{(v_x + iv_y) + i[(\vec{v} \times \hat{u})_x + i(\vec{v} \times \hat{u})_y]\} \end{bmatrix} \end{aligned}$$

□

$$\text{推论3.2.7. } \hat{u}' = [\hat{u} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \hat{u})\vec{v}/v^2]/[\gamma_v(1 + \vec{v} \cdot \hat{u})]$$

$$\text{推论3.2.8. } \hat{p}' = [\hat{p} + \gamma_v \vec{v} + (\gamma_v - 1)(\vec{v} \cdot \hat{p})\vec{v}/v^2]/[\gamma_v(1 + \vec{v} \cdot \hat{p})]$$

$$\text{推论3.2.9. } 1 + \hat{u}'_z = 1 + [\hat{u}_z + \gamma_v \vec{v}_z + (\gamma_v - 1)(\vec{v} \cdot \hat{u})\vec{v}_z/v^2]/[\gamma_v(1 + \vec{v} \cdot \hat{u})]$$

$$\text{推论3.2.10. } \hat{u}'_x + i\hat{u}'_y = \{(\hat{u}_x + i\hat{u}_y) + \gamma_v(v_x + iv_y) + (\gamma_v - 1)(\vec{v} \cdot \hat{u})(v_x + iv_y)/v^2\}/[\gamma_v(1 + \vec{v} \cdot \hat{u})]$$

推论3.2.11.

$$\begin{aligned} &\{(1 + \gamma_v)(\hat{u}_x + i\hat{u}_y) + \gamma_v\{(v_x + iv_y) + i[(\vec{v} \times \hat{u})_x + i(\vec{v} \times \hat{u})_y]\}\} \\ &\{\gamma_v(1 + \vec{v} \cdot \hat{u}) + [\hat{u}_z + \gamma_v \vec{v}_z + (\gamma_v - 1)(\vec{v} \cdot \hat{u})\vec{v}_z/v^2]\} \\ &= \{(1 + \gamma_v)(\hat{u}_x + i\hat{u}_y) + \gamma_v\{(v_x + iv_y) + i[(\vec{v} \times \hat{u})_x + i(\vec{v} \times \hat{u})_y]\}\} \\ &\{\gamma_v(1 + \vec{v} \cdot \hat{u}) + [\hat{u}_z + \gamma_v \vec{v}_z + (\gamma_v - 1)(\vec{v} \cdot \hat{u})\vec{v}_z/v^2]\} \\ &= \{(1 + \gamma_v)(\hat{u}_x + i\hat{u}_y) + \gamma_v\{(v_x + iv_y) + i[(\vec{v} \times \hat{u})_x + i(\vec{v} \times \hat{u})_y]\}\} \\ &\{1 + \hat{u}_z + \gamma_v(1 + \vec{v} \cdot \hat{u}) - 1 + \gamma_v \vec{v}_z + (\gamma_v - 1)(\vec{v} \cdot \hat{u})\vec{v}_z/v^2\} \\ &= \{(1 + \gamma_v)(\hat{u}_x + i\hat{u}_y) + \gamma_v v\{(\hat{v}_x + i\hat{v}_y) + i[(\hat{v} \times \hat{u})_x + i(\hat{v} \times \hat{u})_y]\}\} \\ &\{1 + \hat{u}_z + \gamma_v(1 + \vec{v} \cdot \hat{u}) - 1 + \gamma_v v\hat{v}_z + (\gamma_v - 1)(\hat{v} \cdot \hat{u})\hat{v}_z\} \end{aligned}$$

$$\text{推论3.2.12. } \{(1 + \gamma_v)(1 + \hat{u}_z) + \gamma_v[v_z + i(\vec{v} \times \hat{u})_z + \vec{v} \cdot \hat{u}]\}$$

$$\{(\hat{u}_x + i\hat{u}_y) + \gamma_v(v_x + iv_y) + (\gamma_v - 1)(\vec{v} \cdot \hat{u})(v_x + iv_y)/v^2\}$$

$$= \{(1 + \gamma_v)(1 + \hat{u}_z) + \gamma_v v[\hat{v}_z + i(\hat{v} \times \hat{u})_z + \hat{v} \cdot \hat{u}]\}$$

$$\{(\hat{u}_x + i\hat{u}_y) + \gamma_v v(\hat{v}_x + i\hat{v}_y) + (\gamma_v - 1)(\hat{v} \cdot \hat{u})(\hat{v}_x + i\hat{v}_y)\}$$

=

推论3.2.13.

$$[(1 + \gamma_v)(\hat{u}_x + i\hat{u}_y) - \gamma_v v(\hat{u}_x + i\hat{u}_y)]\{\gamma_v(1 + v\hat{u}_z) + [\hat{u}_z + \gamma_v v + (\gamma_v - 1)\hat{u}_z]\}$$

$$= (1 + \gamma_v - \gamma_v v)(\hat{u}_x + i\hat{u}_y)[\gamma_v(1 + v\hat{u}_z) + \gamma_v(v + \hat{u}_z)]$$

$$= (1 + \gamma_v - \gamma_v v)(\hat{u}_x + i\hat{u}_y)\gamma_v(1 + v)(1 + \hat{u}_z)$$

$$= (1 + \gamma_v + \gamma_v v)(1 + \hat{u}_z)(\hat{u}_x + i\hat{u}_y)$$

$$\text{推论3.2.14. } [(1 + \gamma_v)(1 + \hat{u}_z) + \gamma_v v(1 + \hat{u}_z)](\hat{u}_x + i\hat{u}_y)$$

$$= (1 + \gamma_v + \gamma_v v)(1 + \hat{u}_z)(\hat{u}_x + i\hat{u}_y)$$

### 3.3 中微子场协变常数不变张量的性质

推论3.3.1.

$$\Gamma_{A_\zeta A'_\zeta}^a(\frac{1}{2}) := \frac{-i\zeta}{\sqrt{2}}(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a$$

$$\Gamma_{A_\zeta A'_\zeta}^\pi(\frac{1}{2}) = (\frac{1}{\sqrt{2}})^1 \delta_{A_\zeta A'_\zeta}$$

$$\Gamma_{A_\zeta A'_\zeta}^i(\frac{1}{2}) = -i\zeta(\frac{1}{\sqrt{2}})^1 2\sigma^i(\frac{1}{2})_{A_\zeta A'_\zeta}$$

引理3.3.1.  $\Gamma_{A_\zeta A'_\zeta}^a p_a = i\sqrt{2}|\vec{p}|\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}), \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) = -\frac{\zeta}{2}(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \hat{p}_a$

证明:  $\Gamma_{A_\zeta A'_\zeta}^a p_a$

$$\begin{aligned} &= (\frac{1}{\sqrt{2}})^1 i \{ -2\zeta [\sigma(\frac{1}{2}) \cdot \vec{p}]_{A_\zeta}^{B_\zeta} [\lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) + \lambda_{B_\zeta}(\hat{p}, \zeta)\lambda_{A'_\zeta}^+(\hat{p}, \zeta)] + |\vec{p}| [\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) \\ &+ \lambda_{A_\zeta}(\hat{p}, \zeta)\lambda_{A'_\zeta}^+(\hat{p}, \zeta)] \} \\ &= (\frac{1}{\sqrt{2}})^1 i \{ [|\vec{p}|\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) - |\vec{p}|\lambda_{A_\zeta}(\hat{p}, \zeta)\lambda_{A'_\zeta}^+(\hat{p}, \zeta)] + |\vec{p}| [\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) + \lambda_{A_\zeta}(\hat{p}, \zeta)\lambda_{A'_\zeta}^+(\hat{p}, \zeta)] \} \\ &= i\sqrt{2}|\vec{p}|\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) \end{aligned} \quad \square$$

推论3.3.2.  $|\vec{p}|\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) = \frac{-\zeta}{2}(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \vec{p}_a$

推论3.3.3. 投影算子:  $\hat{P}_{A_\zeta A'_\zeta}(\frac{1}{2}, \zeta) = -\frac{i}{\sqrt{2}}\Gamma_{A_\zeta A'_\zeta}^a \hat{p}_a \rightarrow -\frac{1}{\sqrt{2}}\Gamma_{A_\zeta A'_\zeta}^a \hat{\partial}_a$

### 3.4 中微子场数学上一般的协变对易规则

定理3.4.1.

$$\begin{cases} [a_\sigma(\vec{p}, -\zeta), a_{\sigma'}^+(\vec{p}', -\zeta)]_\pm = \delta_\sigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\zeta), a_{\sigma'}(\vec{p}', -\zeta)]_\pm = 0 \\ [a_\sigma^+(\vec{p}, -\zeta), a_{\sigma'}^+(\vec{p}', -\zeta)]_\pm = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{A_\zeta}(x), \psi_{A'_\zeta}^+(x')]_\pm \\ = -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a [(\delta_1 - \pm\delta_2)\Delta^{(+\zeta)}(x-x') \pm \delta_2\Delta(x-x')] \\ [\psi_{A_\zeta}(x), \psi_{B_\zeta}(x')]_\pm = 0 \\ [\psi_{A'_\zeta}^+(x), \psi_{B'_\zeta}^+(x')]_\pm = 0 \end{cases}$$

证明:  $[\psi_{A_\zeta}^{(+)}(x), \psi_{A'_\zeta}^{(+)}(x')]_\pm$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}', -\frac{\zeta}{2}) [a_1(\vec{p}, -\frac{\zeta}{2}), a_1^+(\vec{p}', -\frac{\zeta}{2})]_\pm e^{i(p \cdot x - p' \cdot x')} d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \delta_1 \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}', -\frac{\zeta}{2}) \delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x-x')} d^3\vec{p} d^3\vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \delta_1 \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) e^{ip \cdot (x-x')} d^3\vec{p} \\ &= -i \frac{1}{(2\pi)^3} \int \delta_1 \frac{1}{2|\vec{p}|} \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a p_a e^{ip \cdot (x-x')} d^3\vec{p} \\ &= -\frac{1}{(2\pi)^3} \int \delta_1 \frac{1}{2|\vec{p}|} \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a e^{ip \cdot (x-x')} d^3\vec{p} \\ &= -i\sqrt{2}\delta_1 \Gamma_{A_\zeta A'_\zeta}^a \partial_a \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} e^{ip \cdot (x-x')} d^3\vec{p} \\ &= -i\sqrt{2}\delta_1 \Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{(+)}(x-x') \end{aligned} \quad \square$$

证明:  $[\psi_{A_\zeta}^{(-)}(x), \psi_{A'_\zeta}^{(-)}(x')]_\pm$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}', -\frac{\zeta}{2}) [a_2(\vec{p}, -\frac{\zeta}{2}), a_2(\vec{p}', -\frac{\zeta}{2})]_\pm e^{-i(p \cdot x - p' \cdot x')} d^3\vec{p} d^3\vec{p}' \\ &= \pm \frac{1}{(2\pi)^3} \int \delta_2 \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}', -\frac{\zeta}{2}) \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x-x')} d^3\vec{p} d^3\vec{p}' \\ &= \pm \frac{1}{(2\pi)^3} \int \delta_2 \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2})\lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) e^{-ip \cdot (x-x')} d^3\vec{p} \\ &= -\pm i \frac{1}{(2\pi)^3} \int \delta_2 \frac{1}{2|\vec{p}|} \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a p_a e^{-ip \cdot (x-x')} d^3\vec{p} \\ &= \pm \frac{1}{(2\pi)^3} \int \delta_2 \frac{1}{2|\vec{p}|} \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a e^{-ip \cdot (x-x')} d^3\vec{p} \\ &= \pm i\sqrt{2}\delta_2 \Gamma_{A_\zeta A'_\zeta}^a \partial_a \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} e^{-ip \cdot (x-x')} d^3\vec{p} \\ &= -\pm i\sqrt{2}\delta_2 \Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{(-)}(x-x') \end{aligned} \quad \square$$

证明:  $[\psi_{A_\zeta}(x), \psi_{A'_\zeta}^+(x')]_\pm$

$$\begin{aligned} &= [\psi_{A_\zeta}^{(+)}(x), \psi_{A'_\zeta}^{(+)}(x')]_\pm + [\psi_{A_\zeta}^{(-)}(x), \psi_{A'_\zeta}^{(-)}(x')]_\pm \\ &= -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a [\delta_1 \Delta^{(+\zeta)}(x-x') \pm \delta_2 \Delta^{(-\zeta)}(x-x')] \\ &= -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a [(\delta_1 - \pm\delta_2)\Delta^{(+\zeta)}(x-x') \pm \delta_2 \Delta(x-x')] \end{aligned} \quad \square$$

从上式可知, 只有 $\delta_1 - \pm\delta_2 = 0$ 时, 才满足微观因果性, 同时只有 $\delta_1, \delta_2 \geq 0$ 时, 才满足几率非负性。所以数学上八种协变对易或反对易方案中, 物理上合理的只有一种: 即 $\delta_1 = \delta_2 = 1$ , 且满足反对易关系。其实还有两种, 即 $\delta_1 = \delta_2 = 0$ , 且满足对易或反对易关系, 就是经典情形。

### 3.5 中微子场物理的协变反对易规则

定理3.5.1.

$$\begin{cases} \{a_\sigma(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}^+(\vec{p}', -\frac{\varepsilon}{2})\} = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\sigma(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}(\vec{p}', -\frac{\varepsilon}{2})\} = 0 \\ \{a_\sigma^+(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}^+(\vec{p}', -\frac{\varepsilon}{2})\} = 0 \end{cases} \Rightarrow \begin{cases} \{\psi_{A_\zeta}(x), \psi_{A'_\zeta}^+(x')\} = -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta(x - x') \\ \{\psi_{A_\zeta}(x), \psi_{B_\zeta}(x')\} = 0 \\ \{\psi_{A'_\zeta}^+(x), \psi_{B'_\zeta}^+(x')\} = 0 \end{cases}$$

证明:  $\{\psi_{A_\zeta}(x), \psi_{A'_\zeta}^+(x')\}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda_{A'_\zeta}^+(\hat{p}', -\frac{\varepsilon}{2}) \{a_1(\vec{p}, -\frac{\varepsilon}{2}), a_1^+(\vec{p}', -\frac{\varepsilon}{2})\} e^{i(p \cdot x - p' \cdot x')} + \{a_2^+(\vec{p}, -\frac{\varepsilon}{2}), a_2(\vec{p}', -\frac{\varepsilon}{2})\} e^{-ip \cdot (x - x')} d^3 \vec{p} d^3 \vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda_{A'_\zeta}^+(\hat{p}', -\frac{\varepsilon}{2}) [\delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x - x')} + \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x - x')}] d^3 \vec{p} d^3 \vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\varepsilon}{2}) [e^{ip \cdot (x - x')} + e^{-ip \cdot (x - x')}] d^3 \vec{p} \\ &= -i \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a p_a [e^{ip \cdot (x - x')} + e^{-ip \cdot (x - x')}] d^3 \vec{p} \\ &= -\frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a [e^{ip \cdot (x - x')} - e^{-ip \cdot (x - x')}] d^3 \vec{p} \\ &= -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} [e^{ip \cdot (x - x')} - e^{-ip \cdot (x - x')}] d^3 \vec{p} \\ &= -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta(x - x') \end{aligned} \quad \square$$

定理3.5.2.

$$\begin{cases} \{a_\sigma(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}^+(\vec{p}', -\frac{\varepsilon}{2})\} = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\sigma(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}(\vec{p}', -\frac{\varepsilon}{2})\} = 0 \\ \{a_\sigma^+(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}^+(\vec{p}', -\frac{\varepsilon}{2})\} = 0 \end{cases} \Rightarrow \begin{cases} \{\psi_{A_\zeta}^{(\tau)}(x), \psi_{A'_\zeta}^{(\kappa)+}(x')\} = -i\sqrt{2}\delta^{\tau\kappa}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{(\tau)}(x - x') \\ \{\psi_{A_\zeta}^{(\tau)}(x), \psi_{B_\zeta}^{(\kappa)}(x')\} = 0 \\ \{\psi_{A'_\zeta}^{(\tau)+}(x), \psi_{B'_\zeta}^{(\kappa)+}(x')\} = 0 \end{cases}$$

证明:  $\{\psi_{A_\zeta}^{(+)}(x), \psi_{A'_\zeta}^{(+)+}(x')\}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda_{A'_\zeta}^+(\hat{p}', -\frac{\varepsilon}{2}) \{a_1(\vec{p}, -\frac{\varepsilon}{2}), a_1^+(\vec{p}', -\frac{\varepsilon}{2})\} e^{i(p \cdot x - p' \cdot x')} d^3 \vec{p} d^3 \vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\varepsilon}{2}) \delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x - x')} d^3 \vec{p} d^3 \vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\varepsilon}{2}) e^{ip \cdot (x - x')} d^3 \vec{p} \\ &= -i \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a p_a e^{ip \cdot (x - x')} d^3 \vec{p} \\ &= -\frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a e^{ip \cdot (x - x')} d^3 \vec{p} \\ &= -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} e^{ip \cdot (x - x')} d^3 \vec{p} \\ &= -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{(+)}(x - x') \end{aligned} \quad \square$$

证明:  $\{\psi_{A_\zeta}^{(-)}(x), \psi_{A'_\zeta}^{(-)+}(x')\}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda_{A'_\zeta}^+(\hat{p}', -\frac{\varepsilon}{2}) \{a_2^+(\vec{p}, -\frac{\varepsilon}{2}), a_2(\vec{p}', -\frac{\varepsilon}{2})\} e^{i(p \cdot x - p' \cdot x')} d^3 \vec{p} d^3 \vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\varepsilon}{2}) \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x - x')} d^3 \vec{p} d^3 \vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \lambda_{A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\varepsilon}{2}) e^{-ip \cdot (x - x')} d^3 \vec{p} \\ &= -i \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a p_a e^{-ip \cdot (x - x')} d^3 \vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a e^{-ip \cdot (x - x')} d^3 \vec{p} \\ &= i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} e^{-ip \cdot (x - x')} d^3 \vec{p} \\ &= -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{(-)}(x - x') \end{aligned} \quad \square$$

### 3.6 中微子场的等时反对易规则

推论3.6.1.

$$\begin{cases} \{\psi_{A_\zeta}(x), \psi_{A'_\zeta}^+(x')\} = -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta(x - x') \\ \{\psi_{A_\zeta}(x), \psi_{B_\zeta}(x')\} = 0 \\ \{\psi_{A'_\zeta}^+(x), \psi_{B'_\zeta}^+(x')\} = 0 \end{cases} \Rightarrow \begin{cases} \{\psi_{A_\zeta}(\vec{r}, t), \psi_{A'_\zeta}^+(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\psi_{A_\zeta}(\vec{r}, t), \psi_{A'_\zeta}^+(\vec{r}', t)\} = (\sigma \cdot \nabla)_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\psi_{A_\zeta}(\vec{r}, t), \psi_{B_\zeta}(\vec{r}', t)\} = 0 \\ \{\psi_{A'_\zeta}^+(\vec{r}, t), \psi_{B'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases}$$

$$\begin{aligned}
& \text{证明: } \{\psi_{A_\zeta}(x), \psi_{A'_\zeta}^+(x')\} = -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta(x-x') \\
& \Rightarrow \{\psi_{A_\zeta}(\vec{r}, t), \psi_{A'_\zeta}^+(\vec{r}', t)\} = -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta(x-x')|_{t=t'} \\
& \Leftrightarrow \{\psi_{A_\zeta}(\vec{r}, t), \psi_{A'_\zeta}^+(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r}-\vec{r}')
\end{aligned}$$

□

推论3.6.2.

$$\begin{cases} \{\psi_{A_\zeta}(\vec{r}, t), \psi_{A'_\zeta}^+(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r}-\vec{r}') \\ \{\psi_{A_\zeta}(\vec{r}, t), \psi_{B_\zeta}(\vec{r}', t)\} = 0 \\ \{\psi_{A'_\zeta}^+(\vec{r}, t), \psi_{B'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{a_\sigma(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}^+(\vec{p}', -\frac{\varepsilon}{2})\} = \delta_{\sigma\sigma'} \delta^3(\vec{p}-\vec{p}') \\ \{a_\sigma(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}(\vec{p}', -\frac{\varepsilon}{2})\} = 0 \\ \{a_\sigma^+(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}^+(\vec{p}', -\frac{\varepsilon}{2})\} = 0 \end{cases}$$

$$\begin{aligned}
& \text{证明: } \{a_1(\vec{p}, -\frac{\varepsilon}{2}), a_1^+(\vec{p}', -\frac{\varepsilon}{2})\} \\
& = \frac{1}{(2\pi)^3} \int \{\lambda^{+A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \psi_{A_\zeta}(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r}-Et)}, \lambda^{A'_\zeta}(\hat{p}', -\frac{\varepsilon}{2}) \psi_{A'_\zeta}^+(\vec{r}', t) e^{i(\vec{p}'\cdot\vec{r}'-E't)}\} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \lambda^{+A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda^{A'_\zeta}(\hat{p}', -\frac{\varepsilon}{2}) \{\psi_{A_\zeta}(\vec{r}, t), \psi_{A'_\zeta}^+(\vec{r}', t)\} e^{-i(\vec{p}\cdot\vec{r}-Et)} e^{i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \lambda^{+A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda^{A'_\zeta}(\hat{p}', -\frac{\varepsilon}{2}) \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r}-\vec{r}') e^{-i(\vec{p}\cdot\vec{r}-Et)} e^{i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \lambda^+(\hat{p}, -\frac{\varepsilon}{2}) \lambda(\hat{p}', -\frac{\varepsilon}{2}) e^{-i(\vec{p}\cdot\vec{r}-Et)} e^{i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} \\
& = \lambda^+(\hat{p}, -\frac{\varepsilon}{2}) \lambda(\hat{p}', -\frac{\varepsilon}{2}) \delta^3(\vec{p}-\vec{p}') \\
& = \delta^3(\vec{p}-\vec{p}')
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } \{a_2^+(\vec{p}, -\frac{\varepsilon}{2}), a_2(\vec{p}', -\frac{\varepsilon}{2})\} \\
& = \frac{1}{(2\pi)^3} \int \{\lambda^{+A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \psi_{A_\zeta}(\vec{r}, t) e^{i(\vec{p}\cdot\vec{r}-Et)}, \lambda^{A'_\zeta}(\hat{p}', -\frac{\varepsilon}{2}) \psi_{A'_\zeta}^+(\vec{r}', t) e^{-i(\vec{p}'\cdot\vec{r}'-E't)}\} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \lambda^{+A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda^{A'_\zeta}(\hat{p}', -\frac{\varepsilon}{2}) \{\psi_{A_\zeta}(\vec{r}, t), \psi_{A'_\zeta}^+(\vec{r}', t)\} e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \lambda^{+A_\zeta}(\hat{p}, -\frac{\varepsilon}{2}) \lambda^{A'_\zeta}(\hat{p}', -\frac{\varepsilon}{2}) \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r}-\vec{r}') e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \lambda^+(\hat{p}, -\frac{\varepsilon}{2}) \lambda(\hat{p}', -\frac{\varepsilon}{2}) e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} \\
& = \lambda^+(\hat{p}, -\frac{\varepsilon}{2}) \lambda(\hat{p}', -\frac{\varepsilon}{2}) \delta^3(\vec{p}-\vec{p}') \\
& = \delta^3(\vec{p}-\vec{p}')
\end{aligned}$$

□

### 3.7 中微子场反对易规则小结

以上几个小节的证明正好形成一个逻辑闭环，故有如下性质：

$$\text{推论3.7.1. } \begin{cases} \{a_\sigma(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}^+(\vec{p}', -\frac{\varepsilon}{2})\} = \delta_{\sigma\sigma'} \delta^3(\vec{p}-\vec{p}') \\ \{a_\sigma(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}(\vec{p}', -\frac{\varepsilon}{2})\} = 0 \\ \{a_\sigma^+(\vec{p}, -\frac{\varepsilon}{2}), a_{\sigma'}^+(\vec{p}', -\frac{\varepsilon}{2})\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')\} = \delta_{\sigma\sigma'} \delta^3(\vec{p}-\vec{p}') \\ \{a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')\} = 0 \\ \{a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')\} = 0 \end{cases}$$

⇕

⇕

$$\text{推论3.7.2. } \begin{cases} \{\psi_{A_\zeta}(x), \psi_{A'_\zeta}^+(x')\} = -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta(x-x') \\ \{\psi_{A_\zeta}(x), \psi_{B_\zeta}(x')\} = 0 \\ \{\psi_{A'_\zeta}^+(x), \psi_{B'_\zeta}^+(x')\} = 0 \end{cases} \Leftrightarrow \begin{cases} \{\psi_{A_\zeta}(\vec{r}, t), \psi_{A'_\zeta}^+(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r}-\vec{r}') \\ \{\psi_{A_\zeta}(\vec{r}, t), \psi_{B_\zeta}(\vec{r}', t)\} = 0 \\ \{\psi_{A'_\zeta}^+(\vec{r}, t), \psi_{B'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases}$$

### 3.8 中微子场的对易函数、因果函数和费曼传播子

推论3.8.1.

$$\begin{cases} \Delta_{A_\zeta A'_\zeta}(\frac{1}{2}; x) := -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta(x) \\ \Delta_{A_\zeta A'_\zeta}^{(+)}(\frac{1}{2}; x) := -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{(+)}(x) \\ \Delta_{A_\zeta A'_\zeta}^{(-)}(\frac{1}{2}; x) := -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{(-)}(x) \\ \Delta_{A_\zeta A'_\zeta}^{(l)}(\frac{1}{2}; x) := -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{(l)}(x) \end{cases}$$



推论3.8.2.

$$\begin{cases} \Delta_{A_\zeta A'_\zeta}^{(c)}(\frac{1}{2}; x) := -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{(c)}(x) - i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^\pi \delta(t)\Delta(x) = -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{(c)}(x) \\ \Delta_{A_\zeta A'_\zeta}^{ret}(\frac{1}{2}; x) := -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{ret}(x) - i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^\pi \delta(t)\Delta(x) = -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{ret}(x) \\ \Delta_{A_\zeta A'_\zeta}^{adv}(\frac{1}{2}; x) := -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{adv}(x) - i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^\pi \delta(t)\Delta(x) = -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta^{adv}(x) \\ \Delta_{FA_\zeta A'_\zeta}(\frac{1}{2}; x) := -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta_F(x) + \sqrt{2}\Gamma_{A_\zeta A'_\zeta}^\pi \delta(t)\Delta(x) = -\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta_F(x) \\ = i\Delta_{A_\zeta A'_\zeta}^{(c)}(\frac{1}{2}; x) = \frac{1}{(2\pi)^4} \int \Delta_{FA_\zeta A'_\zeta}(\frac{1}{2}; p) e^{ipx} d^4p \\ \Delta_{FA_\zeta A'_\zeta}(\frac{1}{2}; p) = \frac{-\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a p_a}{p^2 - i\varepsilon} = \frac{i\varsigma(\sigma, i\varsigma)_{A_\zeta A'_\zeta}^a p_a}{p^2 - i\varepsilon} \end{cases}$$

推论3.8.3.

$$\begin{cases} [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta(\frac{1}{2}; x) = 0 & \left\{ \begin{array}{l} [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{(c)}(\frac{1}{2}; x) = -\varsigma[\sigma(\frac{1}{2}), i\frac{1}{2}\varsigma]_a \delta(t)\Delta(\frac{1}{2}; x)|_{t=0} \\ [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{ret}(\frac{1}{2}; x) = -\varsigma[\sigma(\frac{1}{2}), i\frac{1}{2}\varsigma]_a \delta(t)\Delta(\frac{1}{2}; x)|_{t=0} \\ [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{adv}(\frac{1}{2}; x) = -\varsigma[\sigma(\frac{1}{2}), i\frac{1}{2}\varsigma]_a \delta(t)\Delta(\frac{1}{2}; x)|_{t=0} \\ [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta_F(\frac{1}{2}; x) = -i\varsigma[\sigma(\frac{1}{2}), i\frac{1}{2}\varsigma]_a \delta(t)\Delta(\frac{1}{2}; x)|_{t=0} \end{array} \right. \\ \left\{ \begin{array}{l} [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{(+)}(\frac{1}{2}; x) = 0 \\ [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{(-)}(\frac{1}{2}; x) = 0 \\ [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{(l)}(\frac{1}{2}; x) = 0 \end{array} \right. \end{cases}$$

⇕

推论3.8.4.

$$\begin{cases} [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta(\frac{1}{2}; x) = 0 & \left\{ \begin{array}{l} [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{(c)}(\frac{1}{2}; x) = -\frac{1}{\sqrt{2}}\Gamma_a \delta^4(x) \\ [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{ret}(\frac{1}{2}; x) = -\frac{1}{\sqrt{2}}\Gamma_a \delta^4(x) \\ [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{adv}(\frac{1}{2}; x) = -\frac{1}{\sqrt{2}}\Gamma_a \delta^4(x) \\ [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta_F(\frac{1}{2}; x) = -i\frac{1}{\sqrt{2}}\Gamma_a \delta^4(x) \end{array} \right. \\ \left\{ \begin{array}{l} [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{(+)}(\frac{1}{2}; x) = 0 \\ [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{(-)}(\frac{1}{2}; x) = 0 \\ [\frac{1}{2}\partial_a + iS_{ab}(\frac{1}{2}, \varsigma)\partial^b]\Delta^{(l)}(\frac{1}{2}; x) = 0 \end{array} \right. \end{cases}$$

⇕

推论3.8.5.

$$\begin{cases} (\sigma, -i\varsigma)^a \partial_a \Delta(\frac{1}{2}; x) = 0 & \left\{ \begin{array}{l} (\sigma, -i\varsigma)^a \partial_a \Delta^{(c)}(\frac{1}{2}; x) = i\varsigma \delta^4(x) \\ (\sigma, -i\varsigma)^a \partial_a \Delta^{ret}(\frac{1}{2}; x) = i\varsigma \delta^4(x) \\ (\sigma, -i\varsigma)^a \partial_a \Delta^{adv}(\frac{1}{2}; x) = i\varsigma \delta^4(x) \\ (\sigma, -i\varsigma)^a \partial_a \Delta_F(\frac{1}{2}; x) = -\varsigma \delta^4(x) \end{array} \right. \\ (\sigma, -i\varsigma)^a \partial_a \Delta^{(+)}(\frac{1}{2}; x) = 0 \\ (\sigma, -i\varsigma)^a \partial_a \Delta^{(-)}(\frac{1}{2}; x) = 0 \\ (\sigma, -i\varsigma)^a \partial_a \Delta^{(l)}(\frac{1}{2}; x) = 0 \end{cases}$$

### 3.9 中微子场能量动量算符的提取

推论3.9.1.  $H = \int |\vec{p}| [a_1^+(\vec{p}, -\frac{\varsigma}{2})a_1(\vec{p}, -\frac{\varsigma}{2}) - a_2(\vec{p}, -\frac{\varsigma}{2})a_2^+(\vec{p}, -\frac{\varsigma}{2})] d^3\vec{p}$   
 $= i\varsigma \int \psi^+(\vec{r}, t) \sigma \cdot \nabla \psi(\vec{r}, t) d^3\vec{r} = i \int \psi^+(\vec{r}, t) \partial_t \psi(\vec{r}, t) d^3\vec{r}$

证明:  $H = \int |\vec{p}| [a_1^+(\vec{p}, -\frac{\varsigma}{2})a_1(\vec{p}, -\frac{\varsigma}{2}) - a_2(\vec{p}, -\frac{\varsigma}{2})a_2^+(\vec{p}, -\frac{\varsigma}{2})] d^3\vec{p}$   
 $= \frac{1}{(2\pi)^3} \int |\vec{p}| [\lambda_m^{A'_\zeta}(\hat{p}, -\frac{\varsigma}{2}) \psi_{A'_\zeta}^+(\vec{r}', t) e^{ip \cdot x'} \lambda_m^{A_\zeta}(\hat{p}, -\frac{\varsigma}{2}) \psi_{A_\zeta}(\vec{r}, t) e^{-ip \cdot x}$   
 $- \lambda_m^{A'_\zeta}(\hat{p}, -\frac{\varsigma}{2}) \psi_{A'_\zeta}^+(\vec{r}', t) e^{-ip \cdot x'} \lambda_m^{A_\zeta}(\hat{p}, -\frac{\varsigma}{2}) \psi_{A_\zeta}(\vec{r}, t) e^{ip \cdot x}] d^3\vec{p} d^3\vec{r}' d^3\vec{r}$   
 $= \frac{1}{(2\pi)^3} \int |\vec{p}| \lambda_m^{A_\zeta}(\hat{p}, -\frac{\varsigma}{2}) \lambda_m^{A'_\zeta}(\hat{p}, -\frac{\varsigma}{2}) \psi_{A'_\zeta}^+(\vec{r}', t) \psi_{A_\zeta}(\vec{r}, t) [e^{-i\vec{p} \cdot (\vec{r} - \vec{r}')} - e^{i\vec{p} \cdot (\vec{r} - \vec{r}')}] d^3\vec{p} d^3\vec{r}' d^3\vec{r}$   
 $= \frac{1}{(2\pi)^3} \int (i\sqrt{2})^{-1} (\Gamma_a)^{A'_\zeta A_\zeta} p^a \psi_{A'_\zeta}^+(\vec{r}', t) \psi_{A_\zeta}(\vec{r}, t) [e^{-i\vec{p} \cdot (\vec{r} - \vec{r}')} - e^{i\vec{p} \cdot (\vec{r} - \vec{r}')}] d^3\vec{p} d^3\vec{r}' d^3\vec{r}$   
 $= i\varsigma \int \psi_{A'_\zeta}^+(\vec{r}, t) (\sigma \cdot \nabla)^{A'_\zeta A_\zeta} \psi_{A_\zeta}(\vec{r}, t) d^3\vec{r}$   
 $= i\varsigma \int \psi^+(\vec{r}, t) \sigma \cdot \nabla \psi(\vec{r}, t) d^3\vec{r}$   
 $= i \int \psi^+(\vec{r}, t) \partial_t \psi(\vec{r}, t) d^3\vec{r}$  □

推论3.9.2.  $\vec{P} = \int \vec{p} [a_1^+(\vec{p}, -\frac{\varsigma}{2})a_1(\vec{p}, -\frac{\varsigma}{2}) - a_2(\vec{p}, -\frac{\varsigma}{2})a_2^+(\vec{p}, -\frac{\varsigma}{2})] d^3\vec{p} = -i \int \psi^+(\vec{r}, t) \nabla \psi(\vec{r}, t) d^3\vec{r}$

证明:  $\vec{P} = \int \vec{p} [a_1^+(\vec{p}, -\frac{\varsigma}{2})a_1(\vec{p}, -\frac{\varsigma}{2}) - a_2(\vec{p}, -\frac{\varsigma}{2})a_2^+(\vec{p}, -\frac{\varsigma}{2})] d^3\vec{p}$   
 $= \frac{1}{(2\pi)^3} \int \vec{p} [\lambda_m^{A'_\zeta}(\hat{p}, -\frac{\varsigma}{2}) \psi_{A'_\zeta}^+(\vec{r}', t) e^{ip \cdot x'} \lambda_m^{A_\zeta}(\hat{p}, -\frac{\varsigma}{2}) \psi_{A_\zeta}(\vec{r}, t) e^{-ip \cdot x}$

$$\begin{aligned}
& -\lambda_m^{A'_\zeta}(\hat{p}, -\frac{\xi}{2})\psi_{A'_\zeta}^+(\vec{r}', t)e^{-ip\cdot x'}\lambda_m^{A_\zeta}(\hat{p}, -\frac{\xi}{2})\psi_{A_\zeta}(\vec{r}, t)e^{ip\cdot x}]d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3}\int\vec{p}\lambda_m^{A_\zeta}(\hat{p}, -\frac{\xi}{2})\lambda_m^{A'_\zeta}(\hat{p}, -\frac{\xi}{2})\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3}\int\frac{\hat{p}}{|\hat{p}|}(i\sqrt{2})^{-1}(\Gamma_a)^{A'_\zeta A_\zeta}p^a\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3}\frac{1}{2}\int\vec{p}\delta^{A'_\zeta A_\zeta}\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = -\frac{1}{(2\pi)^3}\int\vec{p}\delta^{A'_\zeta A_\zeta}\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = i\frac{1}{(2\pi)^3}\int\delta^{A'_\zeta A_\zeta}\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)\nabla e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = i\int\delta^{A'_\zeta A_\zeta}\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)\nabla\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}' \\
& = -i\int\psi^+(\vec{r}, t)\nabla\psi(\vec{r}, t)d^3\vec{r}
\end{aligned}$$

□

$$\text{推论3.9.3. } P^a = \int p^a[a_1^+(\vec{p}, -\frac{\xi}{2})a_1(\vec{p}, -\frac{\xi}{2}) - a_2(\vec{p}, -\frac{\xi}{2})a_2^+(\vec{p}, -\frac{\xi}{2})]d^3\vec{p} = -i\int\psi^+(\vec{r}, t)\partial^a\psi(\vec{r}, t)d^3\vec{r}$$

### 3.10 中微子场轻子数算符的提取

$$\text{推论3.10.1. } Q = \int[a_1^+(\vec{p}, -\frac{\xi}{2})a_1(\vec{p}, -\frac{\xi}{2}) + a_2(\vec{p}, -\frac{\xi}{2})a_2^+(\vec{p}, -\frac{\xi}{2})]d^3\vec{p} = \int\psi^+(\vec{r}, t)\psi(\vec{r}, t)d^3\vec{r}$$

$$\begin{aligned}
\text{证明: } Q & = \int[a_1^+(\vec{p}, -\frac{\xi}{2})a_1(\vec{p}, -\frac{\xi}{2}) + a_2(\vec{p}, -\frac{\xi}{2})a_2^+(\vec{p}, -\frac{\xi}{2})]d^3\vec{p} \\
& = \frac{1}{(2\pi)^3}\int\lambda^{A'_\zeta}(\hat{p}, -\frac{\xi}{2})\lambda^{A_\zeta}(\hat{p}, -\frac{\xi}{2})\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3}\frac{1}{i\sqrt{2}}\int(\Gamma_a)^{A'_\zeta A_\zeta}\hat{p}_a\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3}\frac{1}{i\sqrt{2}}\int(\frac{i}{\sqrt{2}})^1\delta^{A'_\zeta A_\zeta}\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \int\psi^+(\vec{r}', t)\psi(\vec{r}, t)\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}' \\
& = \int\psi^+(\vec{r}, t)\psi(\vec{r}, t)d^3\vec{r}
\end{aligned}$$

□

### 3.11 中微子场粒子数算符的提取

$$\text{推论3.11.1. } N = \int[a_1^+(\vec{p}, -\frac{\xi}{2})a_1(\vec{p}, -\frac{\xi}{2}) - a_2(\vec{p}, -\frac{\xi}{2})a_2^+(\vec{p}, -\frac{\xi}{2})]d^3\vec{p} = \int\psi^+(\vec{r}, t)\frac{i\partial_t}{\sqrt{-\nabla^2}}\psi(\vec{r}, t)d^3\vec{r}$$

$$\begin{aligned}
\text{证明: } N & = \int[a_1^+(\vec{p}, -\frac{\xi}{2})a_1(\vec{p}, -\frac{\xi}{2}) - a_2(\vec{p}, -\frac{\xi}{2})a_2^+(\vec{p}, -\frac{\xi}{2})]d^3\vec{p} \\
& = \frac{1}{(2\pi)^3}\int\lambda^{A'_\zeta}(\hat{p}, -\frac{\xi}{2})\lambda^{A_\zeta}(\hat{p}, -\frac{\xi}{2})\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3}\frac{1}{i\sqrt{2}}\int(\Gamma_a)^{A'_\zeta A_\zeta}\hat{p}_a\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3}\frac{1}{i\sqrt{2}}\int[-i\zeta(\frac{1}{\sqrt{2}})^1(\sigma^i)^{A'_\zeta A_\zeta}\hat{p}_i]\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)[e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} - e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}]d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = -\zeta\frac{1}{(2\pi)^3}\int\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)(\sigma\cdot\hat{p})^{A'_\zeta A_\zeta}e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = -i\frac{1}{(2\pi)^3}\int\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)\frac{(\sigma\cdot\nabla)^{A'_\zeta A_\zeta}}{\sqrt{-\nabla^2}}e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}d^3\vec{p}d^3\vec{r}d^3\vec{r}' \\
& = -i\zeta\int\psi_{A'_\zeta}^+(\vec{r}', t)\psi_{A_\zeta}(\vec{r}, t)\frac{(\sigma\cdot\nabla)^{A'_\zeta A_\zeta}}{\sqrt{-\nabla^2}}\delta^3(\vec{r}-\vec{r}')d^3\vec{r}d^3\vec{r}' \\
& = i\zeta\int\psi_{A'_\zeta}^+(\vec{r}, t)\frac{(\sigma\cdot\nabla)^{A'_\zeta A_\zeta}}{\sqrt{-\nabla^2}}\psi_{A_\zeta}(\vec{r}, t)d^3\vec{r} \\
& = i\zeta\int\psi^+(\vec{r}, t)\frac{\sigma\cdot\nabla}{\sqrt{-\nabla^2}}\psi(\vec{r}, t)d^3\vec{r} \\
& = \int\psi^+(\vec{r}, t)\frac{i\partial_t}{\sqrt{-\nabla^2}}\psi(\vec{r}, t)d^3\vec{r}
\end{aligned}$$

□

### 3.12 中微子场角动量算符的提取

#### 3.12.1 中微子场空间轨道角动量算符

$$\text{定理3.12.1. } L_{ij} = -i\int\psi^+(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\psi(\vec{r}, t)d^3\vec{r}$$

$$= i\int\{a_1^+(\vec{p}, -\frac{\xi}{2})(p_j\partial_{p^i} - p_i\partial_{p^j})a_1(\vec{p}, -\frac{\xi}{2}) + a_2(\vec{p}, -\frac{\xi}{2})(p_j\partial_{p^i} - p_i\partial_{p^j})a_2^+(\vec{p}, -\frac{\xi}{2})\}d^3\vec{p}$$

$$\begin{aligned}
\text{证明: } L_{ij}^{(+)} & = -i\int\psi^{(+)}(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\psi^{(+)}(\vec{r}, t)d^3\vec{r} \\
& = -i\frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}', -\frac{\xi}{2})a_1^+(\vec{p}', -\frac{\xi}{2})e^{i|\vec{p}'|t}][\lambda(\hat{p}, -\frac{\xi}{2})a_1(\vec{p}, -\frac{\xi}{2})e^{-i|\vec{p}|t}][(r_i\partial_j - r_j\partial_i)e^{i(\vec{p}-\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r} \\
& = -i\frac{1}{(2\pi)^{3/2}}\int[\lambda^+(\hat{p}', -\frac{\xi}{2})a_1^+(\vec{p}', -\frac{\xi}{2})e^{i\zeta|\vec{p}'|t}][\lambda(\hat{p}, -\frac{\xi}{2})a_1(\vec{p}, -\frac{\xi}{2})e^{-i\zeta|\vec{p}|t}][(r_i\partial_j - r_j\partial_i)e^{i\zeta(\vec{p}-\vec{p}')\cdot\vec{r}}]d^3\vec{p}d^3\vec{p}'d^3\vec{r} \\
& = -i\int[\lambda^+(\hat{p}', -\frac{\xi}{2})a_1^+(\vec{p}', -\frac{\xi}{2})e^{i\zeta|\vec{p}'|t}][\lambda(\hat{p}, -\frac{\xi}{2})a_1(\vec{p}, -\frac{\xi}{2})e^{-i\zeta|\vec{p}|t}](p_j\partial_{p^i} - p_i\partial_{p^j})e^{i\zeta(\vec{p}-\vec{p}')\cdot\vec{r}}d^3\vec{p}d^3\vec{p}' \\
& = -i\int[\lambda^+(\hat{p}', -\frac{\xi}{2})a_1^+(\vec{p}', -\frac{\xi}{2})e^{i\zeta|\vec{p}'|t}][\lambda(\hat{p}, -\frac{\xi}{2})a_1(\vec{p}, -\frac{\xi}{2})e^{-i\zeta|\vec{p}|t}](p_j\partial_{p^i} - p_i\partial_{p^j})\delta^3(\vec{p}-\vec{p}')d^3\vec{p}d^3\vec{p}' \\
& = i\int[\lambda^+(\hat{p}, -\frac{\xi}{2})a_1^+(\vec{p}, -\frac{\xi}{2})e^{i\zeta|\vec{p}|t}](p_j\partial_{p^i} - p_i\partial_{p^j})[\lambda(\hat{p}, -\frac{\xi}{2})a_1(\vec{p}, -\frac{\xi}{2})e^{-i\zeta|\vec{p}|t}]d^3\vec{p}
\end{aligned}$$



$$\begin{aligned}
&= \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_1^+(\vec{p}', -\frac{\xi}{2}) e^{i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) e^{-i\xi|\vec{p}|t}] \partial_{p^i} \delta^3(\vec{p} - \vec{p}') d^3\vec{p}' d^3\vec{p} - i\zeta t \int p^i a_1^+(\vec{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) d^3\vec{p} \\
&= - \int [\lambda^+(\hat{p}, -\frac{\xi}{2}) a_1^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] \partial_{p^i} [|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) e^{-i\xi|\vec{p}|t}] d^3\vec{p}' - i\zeta t \int p^i a_1^+(\vec{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) d^3\vec{p} \\
&= - \int a_1^+(\vec{p}, -\frac{\xi}{2}) (\partial_{p^i} |\vec{p}'| - i\zeta t p^i) a_1(\vec{p}, -\frac{\xi}{2}) d^3\vec{p}' - i\zeta t \int p^i a_1^+(\vec{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) d^3\vec{p} \\
&= - \int a_1^+(\vec{p}, -\frac{\xi}{2}) \partial_{p^i} \{|\vec{p}'| a_1(\vec{p}, -\frac{\xi}{2})\} d^3\vec{p}'
\end{aligned}$$

□

$$\begin{aligned}
\text{证明: } L_{i\pi}^{(-\zeta)} &= -i \int \psi^{(-\zeta)+}(\vec{r}, t) [r_i \partial_\pi - it \partial_i] \psi^{(-\zeta)}(\vec{r}, t) d^3\vec{r} \\
&= i \int \psi^{(-\zeta)+}(\vec{r}, t) r_i i \partial_t \psi^{(-\zeta)}(\vec{r}, t) d^3\vec{r} - it [-i \int \psi^{(-\zeta)+}(\vec{r}, t) \partial_i \psi^{(-\zeta)}(\vec{r}, t) d^3\vec{r}] \\
&= \frac{1}{(2\pi)^{3/2}} \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_2(\vec{p}', -\frac{\xi}{2}) e^{-i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] [-i\zeta r_i e^{-i\xi(\vec{p}-\vec{p}')\cdot\vec{r}}] d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\
&+ i\zeta t \int p^i a_2(\vec{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) d^3\vec{p} \\
&= \frac{1}{(2\pi)^{3/2}} \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_2(\vec{p}', -\frac{\xi}{2}) e^{-i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] [-i\zeta r_i e^{-i\xi(\vec{p}-\vec{p}')\cdot\vec{r}}] d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\
&+ i\zeta t \int p^i a_2(\vec{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) d^3\vec{p} \\
&= \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_2(\vec{p}', -\frac{\xi}{2}) e^{-i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] \partial_{p^i} e^{i\xi(\vec{p}-\vec{p}')\cdot\vec{r}} d^3\vec{p}' d^3\vec{p} + i\zeta t \int p^i a_2(\vec{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) d^3\vec{p} \\
&= \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_2(\vec{p}', -\frac{\xi}{2}) e^{-i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] \partial_{p^i} \delta^3(\vec{p} - \vec{p}') d^3\vec{p}' d^3\vec{p} + i\zeta t \int p^i a_2(\vec{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) d^3\vec{p} \\
&= - \int [\lambda^+(\hat{p}, -\frac{\xi}{2}) a_2(\vec{p}, -\frac{\xi}{2}) e^{-i\xi|\vec{p}|t}] \partial_{p^i} [|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] d^3\vec{p}' + i\zeta t \int p^i a_2(\vec{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) d^3\vec{p} \\
&= - \int a_2(\vec{p}, -\frac{\xi}{2}) (\partial_{p^i} |\vec{p}'| + i\zeta t p^i) a_2^+(\vec{p}, -\frac{\xi}{2}) d^3\vec{p}' + i\zeta t \int p^i a_2(\vec{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) d^3\vec{p} \\
&= - \int a_2(\vec{p}, -\frac{\xi}{2}) \partial_{p^i} \{|\vec{p}'| a_2^+(\vec{p}, -\frac{\xi}{2})\} d^3\vec{p}'
\end{aligned}$$

□

$$\begin{aligned}
\text{证明: } L_{i\pi}^{(+\zeta)} &= -i \int \psi^{(+\zeta)+}(\vec{r}, t) [r_i \partial_\pi - it \partial_i] \psi^{(+\zeta)}(\vec{r}, t) d^3\vec{r} \\
&= i \int \psi^{(+\zeta)+}(\vec{r}, t) r_i i \partial_t \psi^{(+\zeta)}(\vec{r}, t) d^3\vec{r} - it [-i \int \psi^{(+\zeta)+}(\vec{r}, t) \partial_i \psi^{(+\zeta)}(\vec{r}, t) d^3\vec{r}] \\
&= i \int \psi^{(+\zeta)+}(\vec{r}, t) r_i i \partial_t \psi^{(+\zeta)}(\vec{r}, t) d^3\vec{r} + 0 \\
&= \frac{1}{(2\pi)^{3/2}} \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_1^+(\vec{p}', -\frac{\xi}{2}) e^{i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] [-i\zeta r_i e^{-i\xi(\vec{p}+\vec{p}')\cdot\vec{r}}] d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\
&= \frac{1}{(2\pi)^{3/2}} \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_1^+(\vec{p}', -\frac{\xi}{2}) e^{i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] [-i\zeta r_i e^{-i\xi(\vec{p}+\vec{p}')\cdot\vec{r}}] d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\
&= \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_1^+(\vec{p}', -\frac{\xi}{2}) e^{i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] \partial_{p^i} e^{i\xi(\vec{p}+\vec{p}')\cdot\vec{r}} d^3\vec{p}' d^3\vec{p} \\
&= \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_1^+(\vec{p}', -\frac{\xi}{2}) e^{i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] \partial_{p^i} \delta^3(\vec{p} + \vec{p}') d^3\vec{p}' d^3\vec{p} \\
&= - \int [\lambda^+(-\hat{p}, -\frac{\xi}{2}) a_1^+(-\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] \partial_{p^i} [|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2}) e^{i\xi|\vec{p}|t}] d^3\vec{p}' \\
&= 0
\end{aligned}$$

□

$$\begin{aligned}
\text{证明: } L_{i\pi}^{(-\zeta)} &= -i \int \psi^{(-\zeta)+}(\vec{r}, t) [r_i \partial_\pi - it \partial_i] \psi^{(+\zeta)}(\vec{r}, t) d^3\vec{r} \\
&= i \int \psi^{(-\zeta)+}(\vec{r}, t) r_i i \partial_t \psi^{(+\zeta)}(\vec{r}, t) d^3\vec{r} - it [-i \int \psi^{(-\zeta)+}(\vec{r}, t) \partial_i \psi^{(+\zeta)}(\vec{r}, t) d^3\vec{r}] \\
&= i \int \psi^{(+\zeta)+}(\vec{r}, t) r_i i \partial_t \psi^{(-\zeta)}(\vec{r}, t) d^3\vec{r} + 0 \\
&= \frac{1}{(2\pi)^{3/2}} \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_2(\vec{p}', -\frac{\xi}{2}) e^{-i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) e^{-i\xi|\vec{p}|t}] [i\zeta r_i e^{i\xi(\vec{p}+\vec{p}')\cdot\vec{r}}] d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\
&= \frac{1}{(2\pi)^{3/2}} \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_2(\vec{p}', -\frac{\xi}{2}) e^{-i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) e^{-i\xi|\vec{p}|t}] [i\zeta r_i e^{i\xi(\vec{p}+\vec{p}')\cdot\vec{r}}] d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\
&= \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_2(\vec{p}', -\frac{\xi}{2}) e^{-i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) e^{-i\xi|\vec{p}|t}] \partial_{p^i} e^{-i\xi(\vec{p}+\vec{p}')\cdot\vec{r}} d^3\vec{p}' d^3\vec{p} \\
&= \int [\lambda^+(\hat{p}', -\frac{\xi}{2}) a_2(\vec{p}', -\frac{\xi}{2}) e^{-i\xi|\vec{p}'|t}][|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) e^{-i\xi|\vec{p}|t}] \partial_{p^i} \delta^3(\vec{p} + \vec{p}') d^3\vec{p}' d^3\vec{p} \\
&= - \int [\lambda^+(-\hat{p}, -\frac{\xi}{2}) a_2(-\vec{p}, -\frac{\xi}{2}) e^{-i\xi|\vec{p}|t}] \partial_{p^i} [|\vec{p}'|\lambda(\hat{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) e^{-i\xi|\vec{p}|t}] d^3\vec{p}' \\
&= 0
\end{aligned}$$

□

$$\begin{aligned}
\text{证明: } M_{i\pi} &= -i \int \psi^+(\vec{r}, t) [r_i \partial_\pi - it \partial_i] \psi(\vec{r}, t) d^3\vec{r} \\
&= i \int \psi^+(\vec{r}, t) r_i i \partial_t \psi(\vec{r}, t) d^3\vec{r} - it [-i \int \psi^{(+\zeta)+}(\vec{r}, t) \partial_i \psi^{(+\zeta)}(\vec{r}, t) d^3\vec{r}] \\
&= \frac{1}{(2\pi)^{3/2}} \int |\vec{p}'|\lambda^+(\hat{p}', -\frac{\xi}{2}) \lambda(\hat{p}, -\frac{\xi}{2}) [a_1^+(\vec{p}', -\frac{\xi}{2}) e^{-i\xi p' \cdot x} + a_2(\vec{p}', -\frac{\xi}{2}) e^{i\xi p' \cdot x}] i\zeta r_i [a_1(\vec{p}, -\frac{\xi}{2}) e^{i\xi p \cdot x} - a_2^+(\vec{p}, -\frac{\xi}{2}) e^{-i\xi p \cdot x}] \\
& d^3\vec{p}' d^3\vec{p} d^3\vec{r} - i\zeta t \int p^i a_1^+(\vec{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) d^3\vec{p} \\
&= L_{i\pi}^{(+\zeta)} + L_{i\pi}^{(-\zeta)} + L_{i\pi}^{(+\zeta)} + L_{i\pi}^{(+\zeta)} \\
&= - \int a_1^+(\vec{p}, -\frac{\xi}{2}) \partial_{p^i} \{|\vec{p}'| a_1(\vec{p}, -\frac{\xi}{2})\} + a_2(\vec{p}, -\frac{\xi}{2}) \partial_{p^i} \{|\vec{p}'| a_2^+(\vec{p}, -\frac{\xi}{2})\} d^3\vec{p}
\end{aligned}$$

□

### 3.12.3 中微子场自旋角动量算符

$$\begin{aligned}
\text{定理3.12.3. } S_{ab} &= \int \psi^+(\vec{r}, t) S_{ab}(\frac{1}{2}, \zeta) \psi(\vec{r}, t) d^3\vec{r} = i\sigma_{\zeta ab}^{\alpha\zeta} \int \psi^+(\vec{r}, t) \sigma_{\alpha\zeta}(\frac{1}{2}) \psi(\vec{r}, t) d^3\vec{r} \\
&= \frac{-i\zeta}{2} \sigma_{\zeta ab}^{\alpha\zeta} \int \hat{p}_{\alpha\zeta} [a_1^+(\vec{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) + a_2(\vec{p}, -\frac{\xi}{2}) a_2^+(\vec{p}, -\frac{\xi}{2})] d^3\vec{p}
\end{aligned}$$

$$\text{定理3.12.4. } \hat{s}_{\alpha_c} = \int \psi^+(\vec{r}, t) \sigma_{\alpha_c}(\frac{1}{2}) \psi(\vec{r}, t) d^3\vec{r} = -\frac{\epsilon}{2} \int \hat{p}_{\alpha_c} [a_1^+(\vec{p}, -\frac{\epsilon}{2}) a_1(\vec{p}, -\frac{\epsilon}{2}) + a_2(\vec{p}, -\frac{\epsilon}{2}) a_2^+(\vec{p}, -\frac{\epsilon}{2})] d^3\vec{p}$$

$$\text{证明: } \hat{s}_{\alpha_c}^{(+)} = \int \psi^{(+)+}(\vec{r}, t) \sigma_{\alpha_c} \psi^{(+)}(\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned} &= \frac{1}{(2\pi)^{3/2}} \int [\lambda^+(\hat{p}', -\frac{\epsilon}{2}) a_1^+(\vec{p}', -\frac{\epsilon}{2}) e^{i|\vec{p}'|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_1(\vec{p}, -\frac{\epsilon}{2}) e^{-i|\vec{p}|t}] e^{i(\vec{p}-\vec{p}')\cdot\vec{r}} d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\ &= \int [\lambda^+(\hat{p}', -\frac{\epsilon}{2}) a_1^+(\vec{p}', -\frac{\epsilon}{2}) e^{i|\vec{p}'|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_1(\vec{p}, -\frac{\epsilon}{2}) e^{-i|\vec{p}|t}] \delta^3(\vec{p}-\vec{p}') d^3\vec{p}' d^3\vec{p} \\ &= \int [\lambda^+(\hat{p}, -\frac{\epsilon}{2}) a_1^+(\vec{p}, -\frac{\epsilon}{2}) e^{i|\vec{p}|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_1(\vec{p}, -\frac{\epsilon}{2}) e^{-i|\vec{p}|t}] d^3\vec{p} \\ &= -\frac{\epsilon}{2} \int a_1^+(\vec{p}, -\frac{\epsilon}{2}) \hat{p}_{\alpha_c} a_1(\vec{p}, -\frac{\epsilon}{2}) d^3\vec{p} \quad \square \end{aligned}$$

$$\text{证明: } \hat{s}_{\alpha_c}^{(-)} = \int \psi^{(-)+}(\vec{r}, t) \sigma_{\alpha_c} \psi^{(-)}(\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned} &= \frac{1}{(2\pi)^{3/2}} \int [\lambda^+(\hat{p}', -\frac{\epsilon}{2}) a_2(\vec{p}', -\frac{\epsilon}{2}) e^{-i|\vec{p}'|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_2^+(\vec{p}, -\frac{\epsilon}{2}) e^{i|\vec{p}|t}] e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}} d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\ &= \int [\lambda^+(\hat{p}', -\frac{\epsilon}{2}) a_2(\vec{p}', -\frac{\epsilon}{2}) e^{-i|\vec{p}'|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_2^+(\vec{p}, -\frac{\epsilon}{2}) e^{i|\vec{p}|t}] \delta^3(\vec{p}-\vec{p}') d^3\vec{p}' d^3\vec{p} \\ &= \int [\lambda^+(\hat{p}, -\frac{\epsilon}{2}) a_2(\vec{p}, -\frac{\epsilon}{2}) e^{-i|\vec{p}|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_2^+(\vec{p}, -\frac{\epsilon}{2}) e^{i|\vec{p}|t}] d^3\vec{p} \\ &= -\frac{\epsilon}{2} \int a_2(\vec{p}, -\frac{\epsilon}{2}) \hat{p}_{\alpha_c} a_2^+(\vec{p}, -\frac{\epsilon}{2}) d^3\vec{p} \quad \square \end{aligned}$$

$$\text{证明: } \hat{s}_{\alpha_c}^{(+-)} = \int \psi^{(+)+}(\vec{r}, t) \sigma_{\alpha_c} \psi^{(-)}(\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned} &= \frac{1}{(2\pi)^{3/2}} \int [\lambda^+(\hat{p}', -\frac{\epsilon}{2}) a_1^+(\vec{p}', -\frac{\epsilon}{2}) e^{i|\vec{p}'|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_2^+(\vec{p}, -\frac{\epsilon}{2}) e^{i|\vec{p}|t}] e^{-i(\vec{p}+\vec{p}')\cdot\vec{r}} d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\ &= \int [\lambda^+(\hat{p}', -\frac{\epsilon}{2}) a_1^+(\vec{p}', -\frac{\epsilon}{2}) e^{i|\vec{p}'|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_2^+(\vec{p}, -\frac{\epsilon}{2}) e^{i|\vec{p}|t}] \delta^3(\vec{p}+\vec{p}') d^3\vec{p}' d^3\vec{p} \\ &= \int [\lambda^+(-\hat{p}, -\frac{\epsilon}{2}) a_1^+(-\vec{p}, -\frac{\epsilon}{2}) e^{i|\vec{p}|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_2^+(\vec{p}, -\frac{\epsilon}{2}) e^{i|\vec{p}|t}] d^3\vec{p} \\ &\neq 0 \quad \square \end{aligned}$$

$$\text{证明: } \hat{s}_{\alpha_c}^{(-+)} = \int \psi^{(-)+}(\vec{r}, t) \sigma_{\alpha_c} \psi^{(+)}(\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned} &= \frac{1}{(2\pi)^{3/2}} \int [\lambda^+(\hat{p}', -\frac{\epsilon}{2}) a_2(\vec{p}', -\frac{\epsilon}{2}) e^{-i|\vec{p}'|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_1(\vec{p}, -\frac{\epsilon}{2}) e^{-i|\vec{p}|t}] e^{i(\vec{p}+\vec{p}')\cdot\vec{r}} d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\ &= \int [\lambda^+(\hat{p}', -\frac{\epsilon}{2}) a_2(\vec{p}', -\frac{\epsilon}{2}) e^{-i|\vec{p}'|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_1(\vec{p}, -\frac{\epsilon}{2}) e^{-i|\vec{p}|t}] \delta^3(\vec{p}+\vec{p}') d^3\vec{p}' d^3\vec{p} \\ &= \int [\lambda^+(-\hat{p}, -\frac{\epsilon}{2}) a_2(-\vec{p}, -\frac{\epsilon}{2}) e^{-i|\vec{p}|t}] \sigma_{\alpha_c}(\frac{1}{2}) [\lambda(\hat{p}, -\frac{\epsilon}{2}) a_1(\vec{p}, -\frac{\epsilon}{2}) e^{-i|\vec{p}|t}] d^3\vec{p} \\ &\neq 0 \quad \square \end{aligned}$$

$$\text{证明: } \hat{s}_{\alpha_c} = \int \psi^+(\vec{r}, t) \sigma_{\alpha_c}(\frac{1}{2}) \psi(\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned} &= \frac{1}{(2\pi)^{3/2}} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} \\ &\lambda^+(\hat{p}', -\frac{\epsilon}{2}) \sigma_{\alpha_c}(\frac{1}{2}) \lambda(\hat{p}, -\frac{\epsilon}{2}) [a_1^+(\vec{p}', -\frac{\epsilon}{2}) e^{-i\vec{p}'\cdot\vec{x}} + a_2(\vec{p}', -\frac{\epsilon}{2}) e^{i\vec{p}'\cdot\vec{x}}] [a_1(\vec{p}, -\frac{\epsilon}{2}) e^{i\vec{p}\cdot\vec{x}} + a_2^+(\vec{p}, -\frac{\epsilon}{2}) e^{-i\vec{p}\cdot\vec{x}}] \\ &= \hat{s}_{\alpha_c}^{(+)} + \hat{s}_{\alpha_c}^{(-)} + \hat{s}_{\alpha_c}^{(+-)} + \hat{s}_{\alpha_c}^{(-+)} \\ &= -\frac{\epsilon}{2} \int \hat{p}_{\alpha_c} [a_1^+(\vec{p}, -\frac{\epsilon}{2}) a_1(\vec{p}, -\frac{\epsilon}{2}) + a_2(\vec{p}, -\frac{\epsilon}{2}) a_2^+(\vec{p}, -\frac{\epsilon}{2})] d^3\vec{p} \quad \square \end{aligned}$$

$$\text{证明: } [\hat{s}_{\alpha_c}, \hat{s}_{\beta_c}]$$

$$\begin{aligned} &= \int \hat{p}_{\alpha_c} \hat{p}'_{\beta_c} [[a_1^+(\vec{p}, -\frac{\epsilon}{2}) a_1(\vec{p}, -\frac{\epsilon}{2}) + a_2(\vec{p}, -\frac{\epsilon}{2}) a_2^+(\vec{p}, -\frac{\epsilon}{2})], [a_1^+(\vec{p}', -\frac{\epsilon}{2}) a_1(\vec{p}', -\frac{\epsilon}{2}) + a_2(\vec{p}', -\frac{\epsilon}{2}) a_2^+(\vec{p}', -\frac{\epsilon}{2})]] d^3\vec{p}' d^3\vec{p} \\ &\neq 0 \quad \square \end{aligned}$$

$$\text{推论3.12.2. } [\hat{s}_{\alpha_c}, \hat{s}_{\beta_c}] = i\epsilon_{\alpha_c\beta_c} \gamma_c \hat{s}_{\gamma_c}$$

$$\text{证明: } [\hat{s}_{\alpha_c}, \hat{s}_{\beta_c}] = \int d^3\vec{r}' d^3\vec{r}'' \sigma_{\alpha_c}(\frac{1}{2})^{A'_c A_c} \sigma_{\beta_c}(\frac{1}{2})^{B'_c B_c} [\psi_{A'_c}^+(\vec{r}', t) \psi_{A_c}(\vec{r}, t) \psi_{B'_c}^+(\vec{r}'', t) \psi_{B_c}(\vec{r}, t)]$$

$$= \int d^3\vec{r}' d^3\vec{r}'' \sigma_{\alpha_c}(\frac{1}{2})^{A'_c A_c} (\sigma_{\beta_c})^{B'_c B_c}$$

$$\{-[\psi_{B'_c}^+(\vec{r}'', t), \psi_{A'_c}^+(\vec{r}', t) \psi_{A_c}(\vec{r}, t)] \psi_{B_c}(\vec{r}, t) - \psi_{B'_c}^+(\vec{r}'', t) [\psi_{B_c}(\vec{r}, t), \psi_{A'_c}^+(\vec{r}', t) \psi_{A_c}(\vec{r}, t)]\}$$

$$= \int d^3\vec{r}' d^3\vec{r}'' \sigma_{\alpha_c}(\frac{1}{2})^{A'_c A_c} \sigma_{\beta_c}(\frac{1}{2})^{B'_c B_c}$$

$$\{\psi_{A'_c}^+(\vec{r}', t) \{\psi_{B'_c}^+(\vec{r}'', t), \psi_{A_c}(\vec{r}, t)\} \psi_{B_c}(\vec{r}, t) - \psi_{B'_c}^+(\vec{r}'', t) \{\psi_{B_c}(\vec{r}, t), \psi_{A'_c}^+(\vec{r}', t)\} \psi_{A_c}(\vec{r}, t)\}$$

$$= \int d^3\vec{r}' \sigma_{\alpha_c}(\frac{1}{2})^{A'_c A_c} \sigma_{\beta_c}(\frac{1}{2})^{B'_c B_c} \{\psi_{A'_c}^+(\vec{r}', t) \delta_{A_c B'_c} \psi_{B_c}(\vec{r}, t) - \psi_{B'_c}^+(\vec{r}'', t) \delta_{A'_c B_c} \psi_{A_c}(\vec{r}, t)\}$$

$$= \int d^3\vec{r}' \{\psi^+(\vec{r}, t) \sigma_{\alpha_c}(\frac{1}{2}) \sigma_{\beta_c}(\frac{1}{2}) \psi(\vec{r}', t) - \psi^+(\vec{r}', t) \sigma_{\beta_c}(\frac{1}{2}) \sigma_{\alpha_c}(\frac{1}{2}) \psi(\vec{r}, t)\}$$

$$= \int d^3\vec{r}' \psi^+(\vec{r}, t) [\sigma_{\alpha_c}(\frac{1}{2}), \sigma_{\beta_c}(\frac{1}{2})] \psi(\vec{r}', t)$$

$$= i\epsilon_{\alpha_c\beta_c} \gamma_c \hat{s}_{\gamma_c} \quad \square$$

综合以上两点，得到自由场以下有点奇怪的结论，物理意义是正反粒子必须成对产生、湮灭。

推论3.12.3.  $\hat{s}_{\alpha_c} \neq 0$

### 3.13 中微子场角动量算符小结

定义3.13.1.  $\tilde{\partial}_a := \partial_{p^a}, \tilde{\partial}_\pi \equiv \frac{1}{i|\vec{p}|}$

推论3.13.1.  $L_{ij} = -i \int \psi^+(\vec{r}, t)(r_i \partial_j - r_j \partial_i) \psi(\vec{r}, t) d^3 \vec{r}$   
 $= -i \int \{a_1^+(\vec{p}, -\frac{\xi}{2})(p_i \partial_{p_j} - p_j \partial_{p_i}) a_1(\vec{p}, -\frac{\xi}{2}) + a_2(\vec{p}, -\frac{\xi}{2})(p_i \partial_{p_j} - p_j \partial_{p_i}) a_2^+(\vec{p}, -\frac{\xi}{2})\} d^3 \vec{p}$

推论3.13.2.  $L_{i\pi} = -i \int \psi^+(\vec{r}, t)[r_i \partial_\pi - it \partial_i] \psi(\vec{r}, t) d^3 \vec{r}$   
 $= -i \int a_1^+(\vec{p}, -\frac{\xi}{2})(\frac{p_i}{i|\vec{p}|} - i|\vec{p}| \partial_{p_i}) a_1(\vec{p}, -\frac{\xi}{2}) + a_2(\vec{p}, -\frac{\xi}{2})(\frac{p_i}{i|\vec{p}|} - i|\vec{p}| \partial_{p_i}) a_2^+(\vec{p}, -\frac{\xi}{2}) d^3 \vec{p}$

推论3.13.3.  $S_{ab} = \int \psi^+(\vec{r}, t) S_{ab}(\frac{1}{2}, \varsigma) \psi(\vec{r}, t) d^3 \vec{r} = \frac{i}{2} \sigma_{\varsigma ab}^{\alpha_c} \int \psi^+(\vec{r}, t) \sigma_{\alpha_c} \psi(\vec{r}, t) d^3 \vec{r}$   
 $= -i \int [a_1^+(\vec{p}, -\frac{\xi}{2}) \frac{\xi}{2} \sigma_{\varsigma ab}^{\alpha_c} \hat{p}_{\alpha_c} a_1(\vec{p}, -\frac{\xi}{2}) + a_2(\vec{p}, -\frac{\xi}{2}) \frac{\xi}{2} \sigma_{\varsigma ab}^{\alpha_c} \hat{p}_{\alpha_c} a_2^+(\vec{p}, -\frac{\xi}{2})] d^3 \vec{p}$

推论3.13.4.  $\hat{M}_{ab} = -i(x_a \partial_b - x_b \partial_a) + \hat{S}_{ab}, \tilde{M}_{ab} = -i(p_a \tilde{\partial}_b - p_b \tilde{\partial}_a) + \frac{-i\varsigma}{2} \sigma_{\varsigma ab}^{\alpha_c} \hat{p}_{\alpha_c}$

得到以下重要定理。

定理3.13.1.  $M_{ab} = \int \psi^+(\vec{r}, t) \hat{M}_{ab} \psi(\vec{r}, t) d^3 \vec{r} = \int \{a_1^+(\vec{p}, -\frac{\xi}{2}) \tilde{M}_{ab} a_1(\vec{p}, -\frac{\xi}{2}) + a_2(\vec{p}, -\frac{\xi}{2}) \tilde{M}_{ab} a_2^+(\vec{p}, -\frac{\xi}{2})\} d^3 \vec{p}$

### 3.14 中微子场正规化能量动量算符

推论3.14.1.  $H_0 = \varsigma \int |\vec{p}| [a_1^+(\vec{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) + a_2^+(\vec{p}, -\frac{\xi}{2}) a_2(\vec{p}, -\frac{\xi}{2})] d^3 \vec{p}$   
 $= \frac{i\varsigma}{2} \int [\psi_{A'_\varsigma}^+(\vec{r}, t), (\sigma \cdot \nabla)^{A'_\varsigma A_\varsigma} \psi_{A_\varsigma}(\vec{r}, t)] d^3 \vec{r} + \frac{\xi}{2} \int \{\psi_{A'_\varsigma}^+(\vec{r}, t), \delta^{A'_\varsigma A_\varsigma} \sqrt{-\nabla^2} \psi_{A_\varsigma}(\vec{r}, t)\} d^3 \vec{r}$

证明:  $H_0 = \varsigma \int |\vec{p}| [a_1^+(\vec{p}, -\frac{\xi}{2}) a_1(\vec{p}, -\frac{\xi}{2}) + a_2^+(\vec{p}, -\frac{\xi}{2}) a_2(\vec{p}, -\frac{\xi}{2})] d^3 \vec{p}$   
 $= \frac{1}{(2\pi)^3} \varsigma \int |\vec{p}| [\lambda_m^{A'_\varsigma}(\hat{p}, -\frac{\xi}{2}) \psi_{A'_\varsigma}^+(\vec{r}', t) e^{i\varsigma \vec{p} \cdot \vec{x}'} \lambda_m^{A_\varsigma}(\hat{p}, -\frac{\xi}{2}) \psi_{A_\varsigma}(\vec{r}, t) e^{-i\varsigma \vec{p} \cdot \vec{x}}$   
 $+ \lambda_m^{A'_\varsigma}(\hat{p}, -\frac{\xi}{2}) \psi_{A_\varsigma}(\vec{r}, t) e^{i\varsigma \vec{p} \cdot \vec{x}} \lambda_m^{A'_\varsigma}(\hat{p}, -\frac{\xi}{2}) \psi_{A'_\varsigma}^+(\vec{r}', t) e^{-i\varsigma \vec{p} \cdot \vec{x}'}] d^3 \vec{p} d^3 \vec{r}' d^3 \vec{r}$   
 $= \frac{1}{(2\pi)^3} \varsigma \int |\vec{p}| \lambda_m^{A'_\varsigma}(\hat{p}, -\frac{\xi}{2}) \lambda_m^{A_\varsigma}(\hat{p}, -\frac{\xi}{2}) [\psi_{A'_\varsigma}^+(\vec{r}', t) \psi_{A_\varsigma}(\vec{r}, t) e^{-i\varsigma \vec{p} \cdot (\vec{r}' - \vec{r})} + \psi_{A_\varsigma}(\vec{r}, t) \psi_{A'_\varsigma}^+(\vec{r}', t) e^{i\varsigma \vec{p} \cdot (\vec{r}' - \vec{r})}] d^3 \vec{p} d^3 \vec{r}' d^3 \vec{r}$   
 $= \frac{1}{(2\pi)^3} \varsigma \int (i\sqrt{2})^{-1} (\Gamma_a)^{A'_\varsigma A_\varsigma} p^a [\psi_{A'_\varsigma}^+(\vec{r}', t) \psi_{A_\varsigma}(\vec{r}, t) e^{-i\varsigma \vec{p} \cdot (\vec{r}' - \vec{r})} + \psi_{A_\varsigma}(\vec{r}, t) \psi_{A'_\varsigma}^+(\vec{r}', t) e^{i\varsigma \vec{p} \cdot (\vec{r}' - \vec{r})}] d^3 \vec{p} d^3 \vec{r}' d^3 \vec{r}$   
 $= \frac{i\varsigma}{2} \int [\psi_{A'_\varsigma}^+(\vec{r}, t), (\sigma \cdot \nabla)^{A'_\varsigma A_\varsigma} \psi_{A_\varsigma}(\vec{r}, t)] d^3 \vec{r} + \frac{\xi}{2} \int \{\psi_{A'_\varsigma}^+(\vec{r}, t), \delta^{A'_\varsigma A_\varsigma} \sqrt{-\nabla^2} \psi_{A_\varsigma}(\vec{r}, t)\} d^3 \vec{r}$   
 $= i\varsigma \int \psi^+(\vec{r}, t) \sigma \cdot \nabla \psi(\vec{r}, t) d^3 \vec{r} + \frac{\xi}{2} \int \{\psi_{A'_\varsigma}^+(\vec{r}, t), \delta^{A'_\varsigma A_\varsigma} \sqrt{-\nabla^2} \psi_{A_\varsigma}(\vec{r}, t)\} d^3 \vec{r}$  □

### 3.15 中微子场的量子方程

推论3.15.1.

$$[\partial_a + iS_{ab}(\frac{1}{2}, \varsigma) \partial^b] \psi = 0 \Leftrightarrow [P_a, \psi(\vec{r}, t)] = i\partial_a \psi(\vec{r}, t); \begin{cases} \{\psi_A(\vec{r}, t), \psi_B^+(\vec{r}', t)\} = \delta_{AB} \delta^3(\vec{r} - \vec{r}') \\ \{\psi_A(\vec{r}, t), \psi_B(\vec{r}', t)\} = 0, \{\psi_A^+(\vec{r}, t), \psi_B^+(\vec{r}', t)\} = 0 \end{cases}$$

推论3.15.2.

$$\begin{cases} [A, BC] = [A, B]C + B[A, C], [A, CB] = [A, C]B + C[A, B] \\ [A, BC] = \{A, B\}C - B\{A, C\}, [A, CB] = \{A, C\}B - C\{A, B\} \\ [BC, A] = [B, A]C + B[C, A] \\ [BC, A] = -\{B, A\}C + B\{C, A\} \end{cases}$$

### 3.16 数学引理

引理3.16.1.

$$\begin{cases} [AB, A'B'] = [AB, A']B' + A'[AB, B'], [AB, B'A'] = [AB, B']A' + B'[AB, A'] \\ [AB, A'B'] = \{AB, A'\}B' - A'\{AB, B'\}, [AB, B'A'] = \{AB, B'\}A' - B'\{AB, A'\} \\ [A'B', AB] = [A', AB]B' + A'[B', AB] \\ [A'B', AB] = -\{A', AB\}B' + A'\{B', AB\} \end{cases}$$

推论3.16.1.

$$\begin{cases} [A, BC] = [A, B]C + B[A, C] & \begin{cases} [BC, A] = [B, A]C + B[C, A] \\ [BC, A] = -\{B, A\}C + B\{C, A\} \end{cases} \\ [A, BC] = \{A, B\}C - B\{A, C\} \end{cases}$$

引理3.16.2.  $[AB, A'B'] = [AB, A']B' + A'[AB, B'] = [A, A']BB' + A[B, A']B' + A'A[B, B'] + A'[A, B']B$

引理3.16.3.  $[AB, A'B'] = [AB, A']B' + A'[AB, B'] = -\{A, A'\}BB' + A\{B, A'\}B' - A'\{A, B'\}B + A'A\{B, B'\}$

引理3.16.4.  $[A, A'] = [B, B'] = 0 \Rightarrow [AB, A'B'] = A[B, A']B' + A'[A, B']B$

引理3.16.5.  $\{A, A'\} = \{B, B'\} = 0 \Rightarrow [AB, A'B'] = A\{B, A'\}B' - A'\{A, B'\}B$

### 3.17 中微子场的彭加莱对称性

引理3.17.1. 
$$\begin{cases} P_a = -i \int \psi^+(\vec{r}, t) \partial_a \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \hat{P}_a \psi(\vec{r}, t) d^3\vec{r} \\ L_{ab} = -i \int \psi^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \hat{L}_{ab} \psi(\vec{r}, t) d^3\vec{r} \\ M_{ab} = \int \psi^+(\vec{r}, t) [-i(r_a \partial_b - r_b \partial_a) + \hat{S}_{ab}] \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \hat{M}_{ab} \psi(\vec{r}, t) d^3\vec{r} \\ \tilde{M}_{ab} = -i \int \psi^+(\vec{r}, t) (r_a \sigma_b - r_b \sigma_a) \psi(\vec{r}, t) d^3\vec{r} \\ \bar{M}_{ab} = -i \int \psi^+(\vec{r}, t) (r_a \sigma_b - r_b \sigma_a) \psi(\vec{r}, t) d^3\vec{r} \end{cases}$$

定理3.17.1. 
$$\begin{cases} [L_{ab}, L_{cd}] = -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \\ [S_{ab}, S_{cd}] = -i(g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}) \\ [L_{ab}, P_c] = -i(g_{bc}P_a - g_{ac}P_b), [S_{ab}, L_{cd}] = 0, [S_{ab}, P_c] = 0, [P_a, P_b] = 0 \end{cases}$$

[ $\Rightarrow$ ] 
$$\begin{cases} [M_{ab}, M_{cd}] = -i(g_{ad}M_{bc} - g_{ac}M_{bd} + g_{bc}M_{ad} - g_{bd}M_{ac}) \\ [M_{ab}, P_c] = -i(g_{bc}P_a - g_{ac}P_b), [P_a, P_b] = 0 \end{cases}$$

证明:  $[L_{ab}, L_{cd}]$

$$\begin{aligned} &= -\int [\psi^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) \psi(\vec{r}, t), \psi^+(\vec{r}', t) (r'_c \partial'_d - r'_d \partial'_c) \psi(\vec{r}', t)] d^3\vec{r} d^3\vec{r}' \\ &= -\delta^{AB} \delta^{A'B'} \int [\psi_A^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) \psi_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t) (r'_c \partial'_d - r'_d \partial'_c) \psi_{B'}(\vec{r}', t)] d^3\vec{r} d^3\vec{r}' \\ &= -\delta^{AB} \delta^{A'B'} \int d^3\vec{r} d^3\vec{r}' \\ &\quad \{ \psi_A^+(\vec{r}, t) \{ (r_a \partial_b - r_b \partial_a) \psi_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t) \} (r'_c \partial'_d - r'_d \partial'_c) \psi_{B'}(\vec{r}', t) \\ &\quad - \psi_{A'}^+(\vec{r}', t) \{ \psi_A^+(\vec{r}, t), (r'_c \partial'_d - r'_d \partial'_c) \psi_{B'}(\vec{r}', t) \} (r_a \partial_b - r_b \partial_a) \psi_B(\vec{r}, t) \} \\ &= -\delta^{AB} \delta^{A'B'} \int d^3\vec{r} d^3\vec{r}' \\ &\quad \{ \psi_A^+(\vec{r}, t) \delta_{A'B} (r_a \partial_b - r_b \partial_a) \delta^3(\vec{r} - \vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \psi_{B'}(\vec{r}', t) - \psi_{A'}^+(\vec{r}', t) \delta_{AB'} (r'_c \partial'_d - r'_d \partial'_c) \delta^3(\vec{r} - \vec{r}') (r_a \partial_b - r_b \partial_a) \psi_B(\vec{r}, t) \} \\ &= -\int d^3\vec{r} d^3\vec{r}' \\ &\quad \{ \psi^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) \delta^3(\vec{r} - \vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \psi(\vec{r}', t) - \psi^+(\vec{r}', t) (r'_c \partial'_d - r'_d \partial'_c) \delta^3(\vec{r} - \vec{r}') (r_a \partial_b - r_b \partial_a) \psi(\vec{r}, t) \} \\ &= \int d^3\vec{r} d^3\vec{r}' \\ &\quad \{ \psi^+(\vec{r}, t) (r_a \partial'_b - r_b \partial'_a) \delta^3(\vec{r} - \vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \psi(\vec{r}', t) - \psi^+(\vec{r}', t) (r'_c \partial'_d - r'_d \partial'_c) \delta^3(\vec{r} - \vec{r}') (r_a \partial_b - r_b \partial_a) \psi(\vec{r}, t) \} \\ &= -\int d^3\vec{r} \{ \psi^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) (r_c \partial_d - r_d \partial_c) \psi(\vec{r}, t) - \psi^+(\vec{r}, t) (r'_c \partial'_d - r'_d \partial'_c) (r_a \partial_b - r_b \partial_a) \psi(\vec{r}, t) \} \\ &= \int \psi^+(\vec{r}, t) [-i(r_a \partial_b - r_b \partial_a), -i(r_c \partial_d - r_d \partial_c)] \psi(\vec{r}, t) d^3\vec{r} \\ &= \int \psi^+(\vec{r}, t) [\hat{L}_{ab}, \hat{L}_{cd}] \psi(\vec{r}, t) d^3\vec{r} \\ &= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \end{aligned}$$

□

证明:  $[L_{ab}, P_c]$

$$\begin{aligned} &= -\delta^{AB} \delta^{A'B'} \int [\psi^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) \psi(\vec{r}, t), \psi^+(\vec{r}', t) \partial'_c \psi(\vec{r}', t)] d^3\vec{r} d^3\vec{r}' \\ &= -\delta^{AB} \delta^{A'B'} \int [\psi_A^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) \psi_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t) \partial'_c \psi_{B'}(\vec{r}', t)] d^3\vec{r} d^3\vec{r}' \\ &= -\delta^{AB} \delta^{A'B'} \int d^3\vec{r} d^3\vec{r}' \\ &\quad \{ \psi_A^+(\vec{r}, t) [(r_a \partial_b - r_b \partial_a) \psi_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t)] \partial'_c \psi_{B'}(\vec{r}', t) - \psi_{A'}^+(\vec{r}', t) [\psi_A^+(\vec{r}, t), \partial'_c \psi_{B'}(\vec{r}', t)] (r_a \partial_b - r_b \partial_a) \psi_B(\vec{r}, t) \} \end{aligned}$$

$$\begin{aligned}
&= -\delta^{AB}\delta^{A'B'} \int d^3\vec{r}d^3\vec{r}' \\
&\{\psi_A^+(\vec{r}, t)\delta_{A'B'}(r_a\partial_b - r_b\partial_a)\delta^3(\vec{r} - \vec{r}')\partial'_c\psi_{B'}(\vec{r}', t) - \psi_{A'}^+(\vec{r}', t)\delta_{AB'}\partial'_c\delta^3(\vec{r} - \vec{r}')\psi_B(\vec{r}, t)\}(r_a\partial_b - r_b\partial_a)\psi_B(\vec{r}, t)\} \\
&= \int \{\psi_A^+(\vec{r}, t)\delta^{AB'}(r_a\partial'_b - r_b\partial'_a)\delta^3(\vec{r} - \vec{r}')\partial'_c\psi_{B'}(\vec{r}', t) - \psi_{A'}^+(\vec{r}', t)\delta^{A'B}\partial_c\delta^3(\vec{r} - \vec{r}')\psi_B(\vec{r}, t)\}d^3\vec{r}d^3\vec{r}' \\
&= -\int \{\psi_A^+(\vec{r}, t)\delta^{AB'}(r_a\partial_b - r_b\partial_a)\partial_c\psi_{B'}(\vec{r}, t) - \psi_{A'}^+(\vec{r}, t)\delta^{A'B}\partial_c(r_a\partial_b - r_b\partial_a)\psi_B(\vec{r}, t)\}d^3\vec{r} \\
&= \int \psi^+(\vec{r}, t)[\hat{L}_{ab}, \hat{P}_c]\psi(\vec{r}, t)d^3\vec{r} \\
&= -i(g_{bc}P_a - g_{ac}P_b) \quad \square
\end{aligned}$$

证明:  $[P_a, P_b]$

$$\begin{aligned}
&= -\int [\psi^+(\vec{r}, t)\partial_a\psi(\vec{r}, t), \psi^+(\vec{r}', t)\partial'_b\psi(\vec{r}', t)]d^3\vec{r}d^3\vec{r}' \\
&= -\delta^{AB}\delta^{A'B'} \int [\psi_A^+(\vec{r}, t)\partial_a\psi_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t)\partial'_b\psi_{B'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}' \\
&= -\delta^{AB}\delta^{A'B'} \int \{\psi_A^+(\vec{r}, t)\{\partial_a\psi_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t)\}\partial'_b\psi_{B'}(\vec{r}', t) - \psi_{A'}^+(\vec{r}', t)\{\psi_A^+(\vec{r}, t), \partial'_b\psi_{B'}(\vec{r}', t)\}\partial_a\psi_B(\vec{r}, t)\}d^3\vec{r}d^3\vec{r}' \\
&= -\delta^{AB}\delta^{A'B'} \int \{\psi_A^+(\vec{r}, t)\delta_{A'B}\partial_a\delta^3(\vec{r} - \vec{r}')\partial'_b\psi_{B'}(\vec{r}', t) - \psi_{A'}^+(\vec{r}', t)\delta_{AB'}\partial'_b\delta^3(\vec{r} - \vec{r}')\partial_a\psi_B(\vec{r}, t)\}d^3\vec{r}d^3\vec{r}' \\
&= \delta^{AB}\delta^{A'B'} \int \{\psi_A^+(\vec{r}, t)\delta_{A'B}\partial'_a\delta^3(\vec{r} - \vec{r}')\partial'_b\psi_{B'}(\vec{r}', t) - \psi_{A'}^+(\vec{r}', t)\delta_{AB'}\partial_b\delta^3(\vec{r} - \vec{r}')\partial_a\psi_B(\vec{r}, t)\}d^3\vec{r}d^3\vec{r}' \\
&= -\int \{\psi_A^+(\vec{r}, t)\delta^{AB'}\partial_a\partial_b\psi_{B'}(\vec{r}, t) - \psi_{A'}^+(\vec{r}, t)\delta^{A'B}\partial_b\partial_a\psi_B(\vec{r}, t)\}d^3\vec{r} \\
&= \int \psi^+(\vec{r}, t)[\hat{P}_a, \hat{P}_b]\psi(\vec{r}, t)d^3\vec{r} = 0 \quad \square
\end{aligned}$$

证明:  $[S_{ab}(t), S_{cd}(t)]$

$$\begin{aligned}
&= \int [\psi^{+A}(\vec{r}, t)S_{abA}^B\psi_B(\vec{r}, t), \psi^{+C}(\vec{r}, t)S_{cdC}^D\psi_D(\vec{r}, t)]d^3\vec{r}d^3\vec{r}' \\
&= \int [\psi^{+A}(\vec{r}, t)S_{abA}^B\psi_B(\vec{r}, t), \psi^{+C}(\vec{r}, t)]S_{cdC}^D\psi_D(\vec{r}, t) + \psi^{+C}(\vec{r}, t)[\psi^{+A}(\vec{r}, t)S_{abA}^B\psi_B(\vec{r}, t), S_{cdC}^D\psi_D(\vec{r}, t)]d^3\vec{r}d^3\vec{r}' \\
&= \int \psi^{+A}(\vec{r}, t)\{S_{abA}^B\psi_B(\vec{r}, t), \psi^{+C}(\vec{r}, t)\}S_{cdC}^D\psi_D(\vec{r}, t) - \psi^{+C}(\vec{r}, t)\{\psi^{+A}(\vec{r}, t), S_{cdC}^D\psi_D(\vec{r}, t)\}S_{abA}^B\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= \int \psi^{+A}(\vec{r}, t)S_{abA}^C\delta^3(\vec{r} - \vec{r}')S_{cdC}^D\psi_D(\vec{r}', t) - \psi^{+C}(\vec{r}, t)S_{cdC}^A\delta^3(\vec{r} - \vec{r}')S_{abA}^B\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= \int \psi^+(\vec{r}, t)[S_{ab}, S_{cd}]\psi(\vec{r}, t)d^3\vec{r} \\
&= -i(g_{ad}S_{bc} - g_{ac}S_{bd} + g_{bc}S_{ad} - g_{bd}S_{ac}) \quad \square
\end{aligned}$$

证明:  $[S_{ab}(t), L_{cd}]$

$$\begin{aligned}
&= -i \int [\psi^{+A}(\vec{r}, t)S_{abA}^B\psi_B(\vec{r}, t), \psi^{+C}(\vec{r}', t)(r'_c\partial'_d - r'_d\partial'_c)\psi_C(\vec{r}', t)]d^3\vec{r}d^3\vec{r}' \\
&= -i \int \{\psi^{+A}(\vec{r}, t)S_{abA}^B\psi_B(\vec{r}, t), \psi^{+C}(\vec{r}', t)\}(r'_c\partial'_d - r'_d\partial'_c)\psi_C(\vec{r}', t) \\
&\quad - \psi^{+C}(\vec{r}', t)\{\psi^{+A}(\vec{r}, t), (r'_c\partial'_d - r'_d\partial'_c)\psi_C(\vec{r}', t)\}S_{abA}^B\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= -i \int \psi^{+A}(\vec{r}, t)S_{abA}^B\delta_B^C\delta^3(\vec{r} - \vec{r}')\psi_C(\vec{r}', t) - \psi^{+C}(\vec{r}', t)\delta_C^A(r'_c\partial'_d - r'_d\partial'_c)\delta^3(\vec{r} - \vec{r}')S_{abA}^B\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= -i \int \psi^{+A}(\vec{r}, t)S_{abA}^B\delta_B^C\delta^3(\vec{r} - \vec{r}')\psi_C(\vec{r}, t) + \psi^{+C}(\vec{r}, t)\delta_C^A(r'_c\partial_d - r'_d\partial_c)\delta^3(\vec{r} - \vec{r}')S_{abA}^B\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= -i \int \psi^{+A}(\vec{r}, t)S_{abA}^B(r_c\partial_d - r_d\partial_c)\psi_B(\vec{r}, t) - \psi^{+A}(\vec{r}, t)(r_c\partial_d - r_d\partial_c)S_{abA}^B\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= \int \psi^+(\vec{r}, t)[S_{ab}, \hat{L}_{cd}]\psi(\vec{r}, t)d^3\vec{r} = 0 \quad \square
\end{aligned}$$

证明:  $[S_{ab}(t), P_c]$

$$\begin{aligned}
&= -i \int [\psi^{+A}(\vec{r}, t)S_{abA}^B\psi_B(\vec{r}, t), \psi^{+C}(\vec{r}, t)\partial'_c\psi_C(\vec{r}, t)]d^3\vec{r}d^3\vec{r}' \\
&= -i \int \{\psi^{+A}(\vec{r}, t)S_{abA}^B\psi_B(\vec{r}, t), \psi^{+C}(\vec{r}, t)\}\partial'_c\psi_C(\vec{r}, t) - \psi^{+C}(\vec{r}, t)\{\psi^{+A}(\vec{r}, t), \partial'_c\psi_C(\vec{r}, t)\}S_{abA}^B\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= -i \int \psi^{+A}(\vec{r}, t)S_{abA}^B\delta_B^C\delta^3(\vec{r} - \vec{r}')\partial'_c\psi_C(\vec{r}, t) - \psi^{+C}(\vec{r}, t)\delta_C^A\partial'_c\delta^3(\vec{r} - \vec{r}')S_{abA}^B\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= -i \int \psi^{+A}(\vec{r}, t)S_{abA}^B\delta_B^C\delta^3(\vec{r} - \vec{r}')\partial'_c\psi_C(\vec{r}, t) + \psi^{+C}(\vec{r}, t)\delta_C^A(r'_c\partial_d - r'_d\partial_c)\delta^3(\vec{r} - \vec{r}')S_{abA}^B\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= -i \int \psi^{+A}(\vec{r}, t)S_{abA}^B\partial_c\psi_B(\vec{r}, t) - \psi^{+A}(\vec{r}, t)\partial_cS_{abA}^B\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= \int \psi^+(\vec{r}, t)[S_{ab}, \hat{P}_c]\psi(\vec{r}, t)d^3\vec{r} = 0 \quad \square
\end{aligned}$$

推论3.17.1.

$$\begin{cases} \{\psi_{A_\zeta}(x), \psi_{A'_\zeta}^+(x')\} = -i\sqrt{2}\Gamma_{A_\zeta A'_\zeta}^a \partial_a \Delta(x - x') \\ \{\psi_{A_\zeta}(x), \psi_{B_\zeta}(x')\} = 0 \\ \{\psi_{A'_\zeta}^+(x), \psi_{B'_\zeta}^+(x')\} = 0 \end{cases} \Rightarrow \begin{cases} \{\psi_{A_\zeta}(\vec{r}, t), \psi_{A'_\zeta}^+(\vec{r}', t)\} = \delta_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\dot{\psi}_{A_\zeta}(\vec{r}, t), \psi_{A'_\zeta}^+(\vec{r}', t)\} = (\sigma \cdot \nabla)_{A_\zeta A'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\psi_{A_\zeta}(\vec{r}, t), \psi_{B_\zeta}(\vec{r}', t)\} = 0 \\ \{\psi_{A'_\zeta}^+(\vec{r}, t), \psi_{B'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases}$$



引理3.17.2.  $\{A, A'\} = \{B, B'\} = 0 \Rightarrow [AB, A'B'] = A\{B, A'\}B' - A'\{A, B'\}B$

证明:  $[\tilde{M}_{ab}, \tilde{M}_{cd}]$

$$\begin{aligned}
&= - \int [\psi^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)\dot{\psi}(\vec{r}, t), \psi^+(\vec{r}', t)(r'_c\sigma_d - r'_d\sigma_c)\dot{\psi}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}' \\
&= - \int [\psi_A^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)^{AB}\dot{\psi}_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t)(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}' \\
&= - \int \psi_A^+(\vec{r}, t)\{(r_a\sigma_b - r_b\sigma_a)^{AB}\dot{\psi}_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t)\}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}', t) \\
&\quad - \psi_{A'}^+(\vec{r}', t)\{\psi_A^+(\vec{r}, t), (r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}', t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}\dot{\psi}_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= - \int \psi_A^+(\vec{r}, t)\{\dot{\psi}_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}', t) \\
&\quad - \psi_{A'}^+(\vec{r}', t)\{\psi_A^+(\vec{r}, t), \dot{\psi}_{B'}(\vec{r}', t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= - \int \psi_A^+(\vec{r}, t)(\sigma \cdot \nabla)_{BA'}\delta^3(\vec{r} - \vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}', t) \\
&\quad - \psi_{A'}^+(\vec{r}', t)(\sigma \cdot \nabla')_{B'A}\delta^3(\vec{r} - \vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= - \int -\psi_A^+(\vec{r}, t)(\sigma \cdot \nabla')_{BA'}\delta^3(\vec{r} - \vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}', t) \\
&\quad + \psi_{A'}^+(\vec{r}', t)(\sigma \cdot \nabla)_{B'A}\delta^3(\vec{r} - \vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= - \int \psi_A^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)^{AB}(\sigma \cdot \nabla)_{BA'}[(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\dot{\psi}_{B'}(\vec{r}', t)] \\
&\quad - \psi_{A'}^+(\vec{r}', t)(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}(\sigma \cdot \nabla)_{B'A}[(r_a\sigma_b - r_b\sigma_a)^{AB}\dot{\psi}_B(\vec{r}, t)]d^3\vec{r} \\
&= - \int \{\psi^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)(\sigma \cdot \nabla)[(r'_c\sigma_d - r'_d\sigma_c)\dot{\psi}(\vec{r}', t)] - \psi^+(\vec{r}, t)(r'_c\sigma_d - r'_d\sigma_c)(\sigma \cdot \nabla)[(r_a\sigma_b - r_b\sigma_a)\dot{\psi}(\vec{r}, t)]\}d^3\vec{r} \\
&= - \int \{\psi^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)[\sigma_c, \sigma_d]\dot{\psi}(\vec{r}, t) - \psi^+(\vec{r}, t)(r'_c\sigma_d - r'_d\sigma_c)[\sigma_a, \sigma_b]\dot{\psi}(\vec{r}, t)\}d^3\vec{r} \\
&\quad - \int \{\psi^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)[(r'_c\sigma_d - r'_d\sigma_c)(\sigma \cdot \nabla)\dot{\psi}(\vec{r}, t)] - \psi^+(\vec{r}, t)(r'_c\sigma_d - r'_d\sigma_c)[(r_a\sigma_b - r_b\sigma_a)(\sigma \cdot \nabla)\dot{\psi}(\vec{r}, t)]\}d^3\vec{r} \\
&= - \int \{\psi^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)[\sigma_c, \sigma_d]\dot{\psi}(\vec{r}, t) - \psi^+(\vec{r}, t)(r'_c\sigma_d - r'_d\sigma_c)[\sigma_a, \sigma_b]\dot{\psi}(\vec{r}, t)\}d^3\vec{r} \\
&\quad - \int \{\psi^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)(r'_c\sigma_d - r'_d\sigma_c)(\sigma \cdot \nabla)\dot{\psi}(\vec{r}, t) - \psi^+(\vec{r}, t)(r'_c\sigma_d - r'_d\sigma_c)(r_a\sigma_b - r_b\sigma_a)(\sigma \cdot \nabla)\dot{\psi}(\vec{r}, t)\}d^3\vec{r} \\
&= - \int \{\psi^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)[\sigma_c, \sigma_d]\dot{\psi}(\vec{r}, t) - \psi^+(\vec{r}, t)(r'_c\sigma_d - r'_d\sigma_c)[\sigma_a, \sigma_b]\dot{\psi}(\vec{r}, t)\}d^3\vec{r} \\
&\quad - \int \psi^+(\vec{r}, t)[(r_a\sigma_b - r_b\sigma_a), (r'_c\sigma_d - r'_d\sigma_c)](\sigma \cdot \nabla)\dot{\psi}(\vec{r}, t)d^3\vec{r} \\
&= ??? \quad \square
\end{aligned}$$

证明:  $[\bar{M}_{ab}, \bar{M}_{cd}]$

$$\begin{aligned}
&= - \int [\psi^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)\psi(\vec{r}, t), \psi^+(\vec{r}', t)(r'_c\sigma_d - r'_d\sigma_c)\psi(\vec{r}', t)]d^3\vec{r}d^3\vec{r}' \\
&= - \int [\psi_A^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)^{AB}\psi_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t)(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}', t)]d^3\vec{r}d^3\vec{r}' \\
&= - \int \psi_A^+(\vec{r}, t)\{(r_a\sigma_b - r_b\sigma_a)^{AB}\psi_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t)\}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}', t) \\
&\quad - \psi_{A'}^+(\vec{r}', t)\{\psi_A^+(\vec{r}, t), (r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}', t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= - \int \psi_A^+(\vec{r}, t)\{\psi_B(\vec{r}, t), \psi_{A'}^+(\vec{r}', t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}', t) \\
&\quad - \psi_{A'}^+(\vec{r}', t)\{\psi_A^+(\vec{r}, t), \psi_{B'}(\vec{r}', t)\}(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= - \int \psi_A^+(\vec{r}, t)\delta_{BA'}\delta^3(\vec{r} - \vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}', t) \\
&\quad - \psi_{A'}^+(\vec{r}', t)\delta_{B'A}\delta^3(\vec{r} - \vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= - \int \psi_A^+(\vec{r}, t)\delta_{BA'}\delta^3(\vec{r} - \vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}', t) \\
&\quad - \psi_{A'}^+(\vec{r}', t)\delta_{B'A}\delta^3(\vec{r} - \vec{r}')(r_a\sigma_b - r_b\sigma_a)^{AB}(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_B(\vec{r}, t)d^3\vec{r}d^3\vec{r}' \\
&= - \int \psi_A^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)^{AB}\delta_{BA'}[(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\psi_{B'}(\vec{r}', t)] \\
&\quad - \psi_{A'}^+(\vec{r}', t)(r'_c\sigma_d - r'_d\sigma_c)^{A'B'}\delta_{B'A}[(r_a\sigma_b - r_b\sigma_a)^{AB}\psi_B(\vec{r}, t)]d^3\vec{r} \\
&= - \int \{\psi^+(\vec{r}, t)(r_a\sigma_b - r_b\sigma_a)[(r'_c\sigma_d - r'_d\sigma_c)\psi(\vec{r}', t)] - \psi^+(\vec{r}, t)(r'_c\sigma_d - r'_d\sigma_c)[(r_a\sigma_b - r_b\sigma_a)\psi(\vec{r}, t)]\}d^3\vec{r} \\
&= - \int \psi^+(\vec{r}, t)[(r_a\sigma_b - r_b\sigma_a), (r'_c\sigma_d - r'_d\sigma_c)]\psi(\vec{r}, t)d^3\vec{r} \quad \square
\end{aligned}$$

## 4 光子旋量场协变量子化方案

### 4.1 光子旋量自旋算符方程及其平面波解

定理4.1.1.  $[\partial_a + iS_{ab}(1, \varsigma)\partial^b]\psi(x) = 0$

$$\text{推论4.1.1.} \quad \begin{cases} \psi(\vec{r}, t) := \frac{1}{(2\pi)^{3/2}} \int \sqrt{|\vec{p}|} \lambda(\hat{p}, -\varsigma) [a_1(\vec{p}, -\varsigma) e^{ip \cdot x} + a_2^+(\vec{p}, -\varsigma) e^{-ip \cdot x}] d^3 \vec{p} \\ \sqrt{|\vec{p}|} a_1(\vec{p}, -\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -\varsigma) \psi(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r} = \frac{1}{(2\pi)^{3/2}} \int \frac{i}{|\vec{p}|} \lambda^+(\hat{p}, -\varsigma) \dot{\psi}(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r} \\ \sqrt{|\vec{p}|} a_2^+(\vec{p}, -\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -\varsigma) \psi(\vec{r}, t) e^{ip \cdot x} d^3 \vec{r} = \frac{1}{(2\pi)^{3/2}} \int \frac{-i}{|\vec{p}|} \lambda^+(\hat{p}, -\varsigma) \dot{\psi}(\vec{r}, t) e^{ip \cdot x} d^3 \vec{r} \end{cases}$$

定义4.1.1. 投影算子:  $\hat{P}_{k_\varsigma k'_\varsigma}(1, \varsigma) := \lambda_{k_\varsigma}(\hat{p}, -\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}, -\varsigma)$ ,  $\hat{P}^2(1, \varsigma) = \hat{P}(1, \varsigma)$ ,  $\hat{P}^+(1, \varsigma) = \hat{P}(1, \varsigma)$

## 4.2 光子旋量场协变常数不变张量的性质

推论4.2.1.

$$\begin{aligned} \Gamma_{k_\varsigma k'_\varsigma}^{\pi\pi}(1) &= \left(\frac{1}{\sqrt{2}}\right)^2 \delta_{k_\varsigma k'_\varsigma} \\ \Gamma_{k_\varsigma k'_\varsigma}^{i\pi}(1) &= -i\varsigma \left(\frac{1}{\sqrt{2}}\right)^2 \sigma^i(1)_{k_\varsigma k'_\varsigma} \\ \Gamma_{k_\varsigma k'_\varsigma}^{ij}(1) &= -\left(\frac{1}{\sqrt{2}}\right)^2 [\sigma^i(1) \sigma^j(1) - \delta^{ij}]_{k_\varsigma k'_\varsigma} = -\left(\frac{1}{\sqrt{2}}\right)^2 2 \frac{1}{2!} [\sigma^i(1) \sigma^j(1) - \frac{1}{2} \delta^{ij}]_{k_\varsigma k'_\varsigma} \end{aligned}$$

引理4.2.1.  $\Gamma_{k_\varsigma k'_\varsigma}^{ab} p_a p_b = -2|\vec{p}|^2 \lambda_{k_\varsigma}(\hat{p}, -\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}, -\varsigma)$

证明:  $\Gamma_{k_\varsigma k'_\varsigma}^{ab} p_a p_b$

$$\begin{aligned} &= C_2^2 \Gamma_{k_\varsigma k'_\varsigma}^{\pi\pi}(1) p_\pi^2 + C_2^1 \Gamma_{k_\varsigma k'_\varsigma}^{i\pi}(1) p_i p_\pi + C_2^0 \Gamma_{k_\varsigma k'_\varsigma}^{ij}(1) p_i p_j \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 \{ -|\vec{p}|^2 + 2|\vec{p}| \varsigma [\sigma^i(1) \cdot \vec{p}] - 2[\sigma^i(1) \cdot \vec{p}]^2 + |\vec{p}|^2 \}_{k_\varsigma k'_\varsigma} \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 |\vec{p}|^2 \{ 2\varsigma [\sigma^i(1) \cdot \hat{p}] - 2[\sigma^i(1) \cdot \hat{p}]^2 \}_{k_\varsigma k'_\varsigma} \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 |\vec{p}|^2 \{ 2\varsigma [\sigma^i(1) \cdot \hat{p}] - 2[\sigma^i(1) \cdot \hat{p}]^2 \} \sum_{h=1}^{-1} \lambda(\hat{p}, h) \lambda^+(\hat{p}, h) \}_{k_\varsigma k'_\varsigma} \\ &= -2|\vec{p}|^2 \lambda_{k_\varsigma}(\hat{p}, -\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}, -\varsigma) \end{aligned}$$

□

推论4.2.2. 投影算子:  $\hat{P}_{k_\varsigma k'_\varsigma}(1, \varsigma) = -\Gamma_{k_\varsigma k'_\varsigma}^{ab} \hat{p}_a \hat{p}_b \rightarrow \Gamma_{k_\varsigma k'_\varsigma}^{ab} \hat{\partial}_a \hat{\partial}_b$

## 4.3 光子旋量场数学上一般的协变对易规则

定理4.3.1.

$$\begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)]_{\pm} = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)]_{\pm} = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)]_{\pm} = 0 \end{cases} \Rightarrow \begin{cases} [\Psi_{k_\varsigma}(x), \Psi_{k'_\varsigma}^+(x')]_{\pm} \\ = i\Gamma_{k_\varsigma k'_\varsigma}^{ab} \partial_a \partial_b [\delta_1 \Delta(x - x') - (\delta_1 \pm \delta_2) \Delta^{(-)}(x - x')] \\ [\Psi_{k_\varsigma}(x), \Psi_{\beta_\varsigma}(x')]_{\pm} = 0 \\ [\Psi_{k'_\varsigma}^+(x), \Psi_{\beta'_\varsigma}^+(x')]_{\pm} = 0 \end{cases}$$

证明:  $[\Psi_{k_\varsigma}^{(+)}(x), \Psi_{k'_\varsigma}^{(+)}(x')]_{\pm}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}', -\varsigma) \sqrt{|\vec{p}| |\vec{p}'|} [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)]_{\pm} e^{ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}', -\varsigma) |\vec{p}| \delta_1 \delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\ &= \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}', -\varsigma) \delta_1 |\vec{p}| e^{ip \cdot (x-x')} d^3 \vec{p} \\ &= \frac{-\delta_1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \Gamma_{k_\varsigma k'_\varsigma}^{ab} p_a p_b e^{ip \cdot (x-x')} d^3 \vec{p} \\ &= i\delta_1 \Gamma_{k_\varsigma k'_\varsigma}^{ab} \partial_a \partial_b \Delta^{(+)}(x - x') \end{aligned}$$

□

证明:  $[\Psi_{k_\varsigma}^{(-)}(x), \Psi_{k'_\varsigma}^{(-)}(x')]_{\pm}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}', -\varsigma) \sqrt{|\vec{p}| |\vec{p}'|} [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)]_{\pm} e^{-ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\ &= \pm \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}', -\varsigma) |\vec{p}| \delta_2 \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\ &= \pm \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}', -\varsigma) \delta_2 |\vec{p}| e^{-ip \cdot (x-x')} d^3 \vec{p} \\ &= \pm \frac{-\delta_2}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \Gamma_{k_\varsigma k'_\varsigma}^{ab} p_a p_b e^{-ip \cdot (x-x')} d^3 \vec{p} \\ &= -\pm i\delta_2 \Gamma_{k_\varsigma k'_\varsigma}^{ab} \partial_a \partial_b \Delta^{(-)}(x - x') \end{aligned}$$

□

证明:  $[\Psi_{k_\varsigma}(x), \Psi_{k'_\varsigma}^+(x')]_{\pm}$

$$\begin{aligned} &= [\Psi_{k_\varsigma}^{(+)}(x), \Psi_{k'_\varsigma}^{(+)}(x')]_{\pm} + [\Psi_{k_\varsigma}^{(-)}(x), \Psi_{k'_\varsigma}^{(-)}(x')]_{\pm} \\ &= i\delta_1 \Gamma_{k_\varsigma k'_\varsigma}^{ab} \partial_a \partial_b \Delta^{(+)}(x - x') - \pm i\delta_2 \Gamma_{k_\varsigma k'_\varsigma}^{ab} \partial_a \partial_b \Delta^{(-)}(x - x') \end{aligned}$$

$$\begin{aligned}
&= i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b [\delta_1 \Delta^{(+)}(x-x') - \pm \delta_2 \Delta^{(-)}(x-x')] \\
&= i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b [(\delta_1 \pm \delta_2) \Delta^{(+)}(x-x') - \pm \delta_2 \Delta^{(-)}(x-x')] \\
&= i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b [\delta_1 \Delta(x-x') - (\delta_1 \pm \delta_2) \Delta^{(-)}(x-x')] \quad \square
\end{aligned}$$

从上式可知，只有  $\delta_1 \pm \delta_2 = 0$  时，才满足微观因果性，同时只有  $\delta_1, \delta_2 \geq 0$  时，才满足几率非负性。所以数学上八种协变对易或反对易方案中，物理上合理的只有一种：即  $\delta_1 = \delta_2 = 1$ ，且满足对易关系。其实还有两种，即  $\delta_1 = \delta_2 = 0$ ，且满足对易或反对易关系，就是经典情形。

#### 4.4 光子旋量场的协变对易规则

从上节可知，有物理意义的对易规则如下：（为了相互印证，重新作了证明）

$$\text{定理4.4.1.} \quad \begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \end{cases}$$

证明：  $\{\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')\}$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' \\
&\lambda_{k_\zeta}(\hat{p}, -\varsigma) \lambda_{k'_\zeta}^+(\hat{p}', -\varsigma) |\vec{p}|^{1/2} |\vec{p}'|^{1/2} \{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)] e^{ip \cdot (x-x')} + [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)] e^{-ip \cdot (x-x')} \} \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}| \lambda_{k_\zeta}(\hat{p}, -\varsigma) \lambda_{k'_\zeta}^+(\hat{p}', -\varsigma) [\delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x-x')} - \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x-x')}] d^3\vec{p} d^3\vec{p}' \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}| \lambda_{k_\zeta}(\hat{p}, -\varsigma) \lambda_{k'_\zeta}^+(\hat{p}', -\varsigma) [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3\vec{p} \\
&= -\frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \Gamma_{k_\zeta k'_\zeta}^{ab} p_a p_b [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3\vec{p} \\
&= \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3\vec{p} \\
&= i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3\vec{p} \\
&= i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \quad \square
\end{aligned}$$

#### 4.5 光子旋量场的等时对易规则

$$\text{推论4.5.1.} \quad \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] = i\zeta[\sigma(1) \cdot \nabla]_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0 \\ [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)] = 0 \end{cases}$$

$$\text{推论4.5.2.} \quad \begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] = i\zeta[\sigma(1) \cdot \nabla]_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0 \\ [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)] = 0 \end{cases} \Rightarrow \begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = 0 \end{cases}$$

证明：  $[a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)]$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int [\lambda^{+k_\zeta}(\hat{p}, -\varsigma) \Psi_{k_\zeta}(\vec{r}, t) e^{-i(\vec{p} \cdot \vec{r} - Et)}, \lambda^{k'_\zeta}(\vec{p}', -\varsigma) \Psi_{k'_\zeta}^+(\vec{r}', t) e^{i(\vec{p}' \cdot \vec{r}' - E't)}] d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda^{+k_\zeta}(\hat{p}, -\varsigma) \lambda^{k'_\zeta}(\vec{p}', -\varsigma) [\Psi_{k_\zeta}(\vec{r}, t), \Psi_{k'_\zeta}^+(\vec{r}', t)] e^{-i(\vec{p} \cdot \vec{r} - Et)} e^{i(\vec{p}' \cdot \vec{r}' - E't)} d^3\vec{r} d^3\vec{r}' \\
&= i\zeta \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda^{+k_\zeta}(\hat{p}, -\varsigma) \lambda^{k'_\zeta}(\vec{p}', -\varsigma) [\sigma(1) \cdot \nabla]_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') e^{-i(\vec{p} \cdot \vec{r} - Et)} e^{i(\vec{p}' \cdot \vec{r}' - E't)} d^3\vec{r} d^3\vec{r}' \\
&= i\zeta \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda^{+k_\zeta}(\hat{p}, -\varsigma) \lambda^{k'_\zeta}(\vec{p}', -\varsigma) [\sigma(1) \cdot \vec{p}]_{k_\zeta k'_\zeta} i e^{-i(\vec{p} \cdot \vec{r} - Et)} e^{i(\vec{p}' \cdot \vec{r}' - E't)} d^3\vec{r} \\
&= -\zeta \frac{1}{|\vec{p}|} \lambda^{+k_\zeta}(\hat{p}, -\varsigma) \lambda^{k'_\zeta}(\vec{p}', -\varsigma) [\sigma(1) \cdot \vec{p}]_{k_\zeta k'_\zeta} \delta^3(\vec{p} - \vec{p}') \\
&= -\zeta \lambda^+(\hat{p}, -\varsigma) \frac{\sigma(1) \cdot \vec{p}}{|\vec{p}|} \lambda(\vec{p}', -\varsigma) \delta^3(\vec{p} - \vec{p}') \\
&= \lambda^+(\hat{p}, -\varsigma) \lambda(\vec{p}', -\varsigma) \delta^3(\vec{p} - \vec{p}') \\
&= \delta^3(\vec{p} - \vec{p}') \quad \square
\end{aligned}$$

证明：  $[a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)]$

$$= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int [\lambda^{+k_\zeta}(\hat{p}, -\varsigma) \Psi_{k_\zeta}(\vec{r}, t) e^{i(\vec{p} \cdot \vec{r} - Et)}, \lambda^{k'_\zeta}(\vec{p}', -\varsigma) \Psi_{k'_\zeta}^+(\vec{r}', t) e^{-i(\vec{p}' \cdot \vec{r}' - E't)}] d^3\vec{r} d^3\vec{r}'$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda^{+k_\zeta}(\hat{p}, -\varsigma) \lambda^{k'_\zeta}(\vec{p}', -\varsigma) [\Psi_{k_\zeta}(\vec{r}, t), \Psi_{k'_\zeta}^+(\vec{r}', t)] e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\
&= i\varsigma \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda^{+k_\zeta}(\hat{p}, -\varsigma) \lambda^{k'_\zeta}(\vec{p}', -\varsigma) [\sigma(1) \cdot \nabla]_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\
&= i\varsigma \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}||\vec{p}'|}} \int \lambda^{+k_\zeta}(\hat{p}, -\varsigma) \lambda^{k'_\zeta}(\vec{p}', -\varsigma) [\sigma(1) \cdot \vec{p}]_{k_\zeta k'_\zeta} (-i) e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r}' \\
&= \varsigma \frac{1}{|\vec{p}|} \lambda^{+k_\zeta}(\hat{p}, -\varsigma) \lambda^{k'_\zeta}(\vec{p}', -\varsigma) [\sigma(1) \cdot \vec{p}]_{k_\zeta k'_\zeta} \delta^3(\vec{p} - \vec{p}') \\
&= \varsigma \lambda^+(\hat{p}, -\varsigma) \frac{\sigma(1)\cdot\vec{p}}{|\vec{p}|} \lambda(\vec{p}', -\varsigma) \delta^3(\vec{p} - \vec{p}') \\
&= -\lambda^+(\hat{p}, -\varsigma) \lambda(\vec{p}', -\varsigma) \delta^3(\vec{p} - \vec{p}') \\
&= -\delta^3(\vec{p} - \vec{p}') \quad \square
\end{aligned}$$

#### 4.6 光子旋量场的对易规则小结

以上几个小节的证明正好形成一个逻辑闭环，故有如下性质：

$$\begin{aligned}
\text{推论4.6.1.} \quad & \left\{ \begin{array}{l} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')] = 0 \end{array} \right. \\
& \qquad \qquad \qquad \Updownarrow \qquad \qquad \qquad \qquad \qquad \Updownarrow \\
\text{推论4.6.2.} \quad & \left\{ \begin{array}{l} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] = i\varsigma [\sigma(1) \cdot \nabla]_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0 \\ [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)] = 0 \end{array} \right.
\end{aligned}$$

$$\text{推论4.6.3. } \sigma_{-\varsigma} = S_{em}(\varsigma)(\sigma \otimes I)S_{em}^+(\varsigma), \sigma_{+\varsigma} = S_{em}(\varsigma)(I \otimes \sigma)S_{em}^+(\varsigma), \gamma = S_m(1)\sigma(1)S_m^-(1)$$

#### 4.7 电磁场多种旋量形式的等价对易规则

定理4.7.1.

$$\left\{ \begin{array}{l} [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')] = i\sigma^{\alpha_\zeta \alpha'_\zeta ab} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha_\zeta}(x), \Psi_{\beta_\zeta}(x')] = 0 \\ [\Psi_{\alpha'_\zeta}^+(x), \Psi_{\beta'_\zeta}^+(x')] = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} [\Psi_{A_\zeta B_\zeta}(x), \Psi_{A'_\zeta B'_\zeta}^+(x')] \\ = -\frac{i}{2} (\sigma, i\varsigma)^a_{A_\zeta A'_\zeta} (\sigma, i\varsigma)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{A_\zeta B_\zeta}(x), \Psi_{C_\zeta D_\zeta}(x')] = 0 \\ [\Psi_{A'_\zeta B'_\zeta}^+(x), \Psi_{C'_\zeta D'_\zeta}^+(x')] = 0 \end{array} \right.$$

证明：  $[\Psi_{A_\zeta B_\zeta}(x), \Psi_{A'_\zeta B'_\zeta}^+(x')]$

$$\begin{aligned}
&= [\frac{i\varsigma}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \Psi_{\alpha_\zeta}(x), \frac{-i\varsigma}{\sqrt{2}} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \Psi_{\alpha'_\zeta}(x')] \\
&= \frac{1}{2} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}(x')] \\
&= \frac{1}{2} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\
&= \frac{i}{2} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \frac{-i\varsigma}{\sqrt{2}} (\sigma, i\varsigma)^a_{C_\zeta C'_\zeta} \frac{-i\varsigma}{\sqrt{2}} (\sigma, i\varsigma)^b_{D_\zeta D'_\zeta} \frac{-i\varsigma}{\sqrt{2}} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} \frac{i\varsigma}{\sqrt{2}} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} \partial_a \partial_b \Delta(x - x') \\
&= -\frac{i}{8} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} (\sigma, i\varsigma)^a_{C_\zeta C'_\zeta} (\sigma, i\varsigma)^b_{D_\zeta D'_\zeta} \partial_a \partial_b \Delta(x - x') \\
&= -\frac{i}{8} \delta_{\{A_\zeta B_\zeta\}}^{C_\zeta D_\zeta} \delta_{\{A'_\zeta B'_\zeta\}}^{C'_\zeta D'_\zeta} (\sigma, i\varsigma)^a_{C_\zeta C'_\zeta} (\sigma, i\varsigma)^b_{D_\zeta D'_\zeta} \partial_a \partial_b \Delta(x - x') \\
&= -\frac{i}{8} (\sigma, i\varsigma)^a_{\{A_\zeta(A'_\zeta} (\sigma, i\varsigma)^b_{B_\zeta\} B'_\zeta)} \partial_a \partial_b \Delta(x - x') \\
&= -\frac{i}{2} (\sigma, i\varsigma)^a_{A_\zeta A'_\zeta} (\sigma, i\varsigma)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \quad \square
\end{aligned}$$

证明：  $[\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')]$

$$\begin{aligned}
&= [\frac{i\varsigma}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \Psi_{A_\zeta B_\zeta}(x), \frac{-i\varsigma}{\sqrt{2}} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \Psi_{A'_\zeta B'_\zeta}^+(x')] \\
&= \frac{1}{2} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} [\Psi_{A_\zeta B_\zeta}(x), \Psi_{A'_\zeta B'_\zeta}^+(x')] \\
&= -\frac{i}{4} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} (\sigma, i\varsigma)^a_{A_\zeta A'_\zeta} (\sigma, i\varsigma)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \\
&= i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \quad \square
\end{aligned}$$

定理4.7.2.

$$\begin{cases} [\Psi_{A_\zeta B_\zeta}(x), \Psi_{A'_\zeta B'_\zeta}^+(x')] \\ = -\frac{i}{2}(\sigma, i\zeta)^a_{A_\zeta A'_\zeta}(\sigma, i\zeta)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{A_\zeta B_\zeta}(x), \Psi_{C_\zeta D_\zeta}(x')] = 0 \\ [\Psi_{A'_\zeta B'_\zeta}^+(x), \Psi_{C'_\zeta D'_\zeta}^+(x')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \end{cases}$$

证明:  $[\Psi_{k_\zeta}(x), \Psi_{k'_\zeta}^+(x')]$

$$\begin{aligned} &= [\Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1) \Psi_{A_\zeta B_\zeta}(x), \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta}(1) \Psi_{A'_\zeta B'_\zeta}^+(x')] \\ &= \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta}(1) [\Psi_{A_\zeta B_\zeta}(x), \Psi_{A'_\zeta B'_\zeta}^+(x')] \\ &= -\frac{i}{2} \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta}(1) (\sigma, i\zeta)^a_{A_\zeta A'_\zeta} (\sigma, i\zeta)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \\ &= i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \end{aligned} \quad \square$$

证明:  $[\Psi_{A_\zeta B_\zeta}(x), \Psi_{A'_\zeta B'_\zeta}^+(x')]$

$$\begin{aligned} &= [\Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1) \psi_{k_\zeta}(x), \Gamma_{A'_\zeta B'_\zeta}^{k'_\zeta}(1) \psi_{k'_\zeta}^+(x')] \\ &= \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1) \Gamma_{A'_\zeta B'_\zeta}^{k'_\zeta}(1) [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] \\ &= \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1) \Gamma_{A'_\zeta B'_\zeta}^{k'_\zeta}(1) i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ &= -\frac{i}{2} \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1) \Gamma_{A'_\zeta B'_\zeta}^{k'_\zeta}(1) \Gamma_{C_\zeta D_\zeta}^{C'_\zeta D'_\zeta}(1) \Gamma_{C'_\zeta D'_\zeta}^{C_\zeta D_\zeta}(1) (\sigma, i\zeta)^a_{C_\zeta C'_\zeta} (\sigma, i\zeta)^b_{D_\zeta D'_\zeta} \partial_a \partial_b \Delta(x - x') \\ &= -\frac{i}{8} \delta_{A_\zeta}^{[C_\zeta} \delta_{B_\zeta}^{D_\zeta]} \delta_{A'_\zeta}^{[C'_\zeta} \delta_{B'_\zeta}^{D'_\zeta]} (\sigma, i\zeta)^a_{C_\zeta C'_\zeta} (\sigma, i\zeta)^b_{D_\zeta D'_\zeta} \partial_a \partial_b \Delta(x - x') \\ &= -\frac{i}{8} (\sigma, i\zeta)^a_{\{A_\zeta(A'_\zeta} (\sigma, i\zeta)^b_{B_\zeta)B'_\zeta\}} \partial_a \partial_b \Delta(x - x') \\ &= -\frac{i}{2} (\sigma, i\zeta)^a_{A_\zeta A'_\zeta} (\sigma, i\zeta)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \end{aligned} \quad \square$$

引理4.7.1.  $\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} = \Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Gamma_{\alpha'_\zeta}^{k'_\zeta}(1) \Gamma_{k_\zeta k'_\zeta}^{ab}, \Gamma_{k_\zeta k'_\zeta}^{ab} = \Gamma_{k_\zeta}^{\alpha_\zeta}(1) \Gamma_{k'_\zeta}^{\alpha'_\zeta}(1) \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab}$

定理4.7.3.

$$\begin{cases} [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha_\zeta}(x), \Psi_{\beta_\zeta}(x')] = 0 \\ [\Psi_{\alpha'_\zeta}^+(x), \Psi_{\beta'_\zeta}^+(x')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \end{cases}$$

证明:  $[\Psi_{k_\zeta}(x), \Psi_{k'_\zeta}^+(x')]$

$$\begin{aligned} &= [\Gamma_{k_\zeta}^{\alpha_\zeta}(1) \Psi_{\alpha_\zeta}(x), \Gamma_{k'_\zeta}^{\alpha'_\zeta}(1) \Psi_{\alpha'_\zeta}^+(x')] \\ &= \Gamma_{k_\zeta}^{\alpha_\zeta}(1) \Gamma_{k'_\zeta}^{\alpha'_\zeta}(1) [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')] \\ &= \Gamma_{k_\zeta}^{\alpha_\zeta}(1) \Gamma_{k'_\zeta}^{\alpha'_\zeta}(1) i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ &= i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \end{aligned} \quad \square$$

证明:  $[\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')]$

$$\begin{aligned} &= [\Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Psi_{k_\zeta}(x), \Gamma_{\alpha'_\zeta}^{k'_\zeta}(1) \Psi_{k'_\zeta}^+(x')] \\ &= \Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Gamma_{\alpha'_\zeta}^{k'_\zeta}(1) [\Psi_{k_\zeta}(x), \Psi_{k'_\zeta}^+(x')] \\ &= \Gamma_{\alpha_\zeta}^{k_\zeta}(1) \Gamma_{\alpha'_\zeta}^{k'_\zeta}(1) i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ &= i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \end{aligned} \quad \square$$

## 4.8 光子旋量场的对易函数、因果函数和费曼传播子

推论4.8.1.

$$\begin{cases} \Delta_{k_\zeta k'_\zeta}(1; x) := \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{(+)}(1; x) := \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta^{(+)}(x) \\ \Delta_{k_\zeta k'_\zeta}^{(-)}(1; x) := \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta^{(-)}(x) \\ \Delta_{k_\zeta k'_\zeta}^{(l)}(1; x) := \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta^{(l)}(x) \end{cases}$$

推论4.8.2.

$$\begin{cases} \Delta_{k_\zeta k'_\zeta}^{(c)}(1; x) := \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta^{(c)}(x) + [\Gamma_{k_\zeta k'_\zeta}^{\pi\pi} \delta'(t) + 2i\Gamma_{k_\zeta k'_\zeta}^{i\pi} \delta(t) \partial_i] \Delta(x) = \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta^{(c)}(x) + \Gamma_{k_\zeta k'_\zeta}^{\pi\pi} \delta^4(x) \\ \Delta_{k_\zeta k'_\zeta}^{ret}(1; x) := \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta^{ret}(x) + [\Gamma_{k_\zeta k'_\zeta}^{\pi\pi} \delta'(t) + 2i\Gamma_{k_\zeta k'_\zeta}^{i\pi} \delta(t) \partial_i] \Delta(x) = \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta^{(c)}(x) + \Gamma_{k_\zeta k'_\zeta}^{\pi\pi} \delta^4(x) \\ \Delta_{k_\zeta k'_\zeta}^{adv}(1; x) := \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta^{adv}(x) + [\Gamma_{k_\zeta k'_\zeta}^{\pi\pi} \delta'(t) + 2i\Gamma_{k_\zeta k'_\zeta}^{i\pi} \delta(t) \partial_i] \Delta(x) = \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta^{(c)}(x) + \Gamma_{k_\zeta k'_\zeta}^{\pi\pi} \delta^4(x) \\ \Delta_{Fk_\zeta k'_\zeta}(1; x) := \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta_F(x) + i[\Gamma_{k_\zeta k'_\zeta}^{\pi\pi} \delta'(t) + 2i\Gamma_{k_\zeta k'_\zeta}^{i\pi} \delta(t) \partial_i] \Delta(x) = \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta^{(c)}(x) + i\Gamma_{k_\zeta k'_\zeta}^{\pi\pi} \delta^4(x) \\ = i\Delta_{k_\zeta k'_\zeta}^{(c)}(1; x) \\ \Delta_{Fk_\zeta k'_\zeta}(1; p) = \frac{i\Gamma_{k_\zeta k'_\zeta}^{ab} p_a p_b}{p^2 - i\varepsilon} + \dots \end{cases}$$

推论4.8.3.

$$\begin{cases} [\partial_a + iS_{ab}(1, \varsigma) \partial^b] \Delta(1; x) = 0 \\ [\partial_a + iS_{ab}(1, \varsigma) \partial^b] \Delta^{(+)}(1; x) = 0 \\ [\partial_a + iS_{ab}(1, \varsigma) \partial^b] \Delta^{(-)}(1; x) = 0 \\ [\partial_a + iS_{ab}(1, \varsigma) \partial^b] \Delta^{(l)}(1; x) = 0 \end{cases} \quad \begin{cases} [\partial^a + iS^{ab}(1, \varsigma) \partial_b] \Delta^{(c)}(1; x) = -\varsigma[\sigma(1), i\varsigma]_a \delta(t) \Delta(1; x)|_{t=0} \\ [\partial^a + iS^{ab}(1, \varsigma) \partial_b] \Delta^{ret}(1; x) = -\varsigma[\sigma(1), i\varsigma]_a \delta(t) \Delta(1; x)|_{t=0} \\ [\partial^a + iS^{ab}(1, \varsigma) \partial_b] \Delta^{adv}(1; x) = -\varsigma[\sigma(1), i\varsigma]_a \delta(t) \Delta(1; x)|_{t=0} \\ [\partial^a + iS^{ab}(1, \varsigma) \partial_b] \Delta_F(1; x) = -i\varsigma[\sigma(1), i\varsigma]_a \delta(t) \Delta(1; x)|_{t=0} \end{cases}$$

[⇕] [⇕]

推论4.8.4.

$$\begin{cases} (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta(1; x) = 0 \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{(+)}(1; x) = 0 \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{(-)}(1; x) = 0 \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{(l)}(1; x) = 0 \end{cases} \quad \begin{cases} (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{(c)}(1; x) = -\varsigma \delta(t) N(1) \Delta(1; x)|_{t=0} \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{ret}(1; x) = -\varsigma \delta(t) N(1) \Delta(1; x)|_{t=0} \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{adv}(1; x) = -\varsigma \delta(t) N(1) \Delta(1; x)|_{t=0} \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta_F(1; x) = -i\varsigma \delta(t) N(1) \Delta(1; x)|_{t=0} \end{cases}$$

[⇕] [⇕]

推论4.8.5.

$$\begin{cases} (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta(1; x) \bar{N}(1) = 0 \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{(+)}(1; x) \bar{N}(1) = 0 \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{(-)}(1; x) \bar{N}(1) = 0 \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{(l)}(1; x) \bar{N}(1) = 0 \end{cases} \quad \begin{cases} (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{(c)}(1; x) \bar{N}(1) = -\varsigma \delta(t) N(1) \Delta(1; x)|_{t=0} \bar{N}(1) \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{ret}(1; x) \bar{N}(1) = -\varsigma \delta(t) N(1) \Delta(1; x)|_{t=0} \bar{N}(1) \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta^{adv}(1; x) \bar{N}(1) = -\varsigma \delta(t) N(1) \Delta(1; x)|_{t=0} \bar{N}(1) \\ (\sigma \otimes I, -i\varsigma)_a \partial^a N(1) \Delta_F(1; x) \bar{N}(1) = -i\varsigma \delta(t) N(1) \Delta(1; x)|_{t=0} \bar{N}(1) \end{cases}$$

[⇕] [⇕]

推论4.8.6.

$$\begin{cases} [\sigma(1), -i\varsigma]_a \partial^a \Delta(1; x) = 0 \\ [\sigma(1), -i\varsigma]_a \partial^a \Delta^{(+)}(1; x) = 0 \\ [\sigma(1), -i\varsigma]_a \partial^a \Delta^{(-)}(1; x) = 0 \\ [\sigma(1), -i\varsigma]_a \partial^a \Delta^{(l)}(1; x) = 0 \end{cases} \quad \begin{cases} [\sigma(1), -i\varsigma]_a \partial^a \Delta^{(c)}(1; x) = -\varsigma \delta(t) \Delta(1; x)|_{t=0} \\ [\sigma(1), -i\varsigma]_a \partial^a \Delta^{ret}(1; x) = -\varsigma \delta(t) \Delta(1; x)|_{t=0} \\ [\sigma(1), -i\varsigma]_a \partial^a \Delta^{adv}(1; x) = -\varsigma \delta(t) \Delta(1; x)|_{t=0} \\ [\sigma(1), -i\varsigma]_a \partial^a \Delta_F(1; x) = -i\varsigma \delta(t) \Delta(1; x)|_{t=0} \end{cases}$$

## 4.9 光子旋量场的量子方程

推论4.9.1.

$$[\partial_a + iS_{ab}(1, \varsigma) \partial^b] \psi = 0 \Leftrightarrow [P_a, \psi(\vec{r}, t)] = i\partial_a \psi(\vec{r}, t); \quad \begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^\dagger(\vec{r}', t)] = i\varsigma \sigma^i(1)_{k_\zeta k'_\zeta} \partial_i \delta^3(\vec{r} - \vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0, [\psi_{k'_\zeta}^\dagger(\vec{r}, t), \psi_{l'_\zeta}^\dagger(\vec{r}', t)] = 0 \end{cases}$$

## 4.10 光子旋量场的彭加莱对称性

推论4.10.1.

$$\begin{cases} \Gamma_{abc \cdots}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \partial_\pi \Delta(x-x')|_{t=t'} = i \sum_{l=0}^{[s]} (-1)^l C_{2s}^{2l} \Gamma_{ij \cdots \pi \cdots \pi}^{2s-2l, 2l}(s) \overbrace{\partial_i \partial_j \cdots}^{2s-2l} \nabla^{2l} \delta^3(\vec{r}-\vec{r}') \\ \Gamma_{abc \cdots}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \cdots}^{2s} \partial_\pi \Delta(x-x')|_{t=t'} = i \sum_{l=0}^{[s]} (-1)^l C_{2s}^{2l} \Gamma_{ij \cdots \pi \cdots \pi}^{2s-2l, 2l}(s) \overbrace{\hat{\partial}_i \hat{\partial}_j \cdots}^{2s-2l} \delta^3(\vec{r}-\vec{r}') \end{cases}$$

推论4.10.2.

$$\begin{aligned} \Gamma_{k_\zeta k'_\zeta}^{\pi\pi}(1) &= \left(\frac{1}{\sqrt{2}}\right)^2 \delta_{k_\zeta k'_\zeta} \\ \Gamma_{k_\zeta k'_\zeta}^{i\pi}(1) &= -i\zeta \left(\frac{1}{\sqrt{2}}\right)^2 \sigma^i(1)_{k_\zeta k'_\zeta} \\ \Gamma_{k_\zeta k'_\zeta}^{ij}(1) &= -\left(\frac{1}{\sqrt{2}}\right)^2 [\sigma^i(1)\sigma^j(1) - \delta^{ij}]_{k_\zeta k'_\zeta} = -\left(\frac{1}{\sqrt{2}}\right)^2 2\frac{1}{2!} [\sigma^i(1)\sigma^j(1) - \frac{1}{2}\delta^{ij}]_{k_\zeta k'_\zeta} \end{aligned}$$

推论4.10.3.  $\Gamma^{ab}(1)\partial_a\partial_b\partial_\pi\Delta(x-x')|_{t=t'} = i\{\Gamma^{ij}(1)\partial_i\partial_j\delta^3(\vec{r}-\vec{r}') - \Gamma^{\pi\pi}(1)\nabla^2\delta^3(\vec{r}-\vec{r}')\} = -i[\sigma(1) \cdot \nabla]^2\delta^3(\vec{r}-\vec{r}')$ 

推论4.10.4.

$$\begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = -\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b |\partial_\pi \Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \end{cases} \Rightarrow \begin{cases} \left[\frac{\psi_{k_\zeta}(\vec{r}, t)}{\sqrt{(-\nabla^2)}}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{\sqrt{(-\nabla'^2)}}\right] = -i[\sigma(1) \cdot \hat{\nabla}]^2 \delta^3(\vec{r}-\vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0 \\ [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)] = 0 \end{cases}$$

推论4.10.5.

$$\begin{aligned} \hat{P}_a(n) &= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{P}_a \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} = -i \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{P}_a \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \\ M_{ab}(n) &= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{M}_{ab} \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} = -i \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{M}_{ab} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \end{aligned}$$

$$\text{定理4.10.1.} \begin{cases} [L_{ab}, L_{cd}] = -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \\ [L_{ab}, P_c] = -i(g_{bc}P_a - g_{ac}P_b), [P_a, P_b] = 0 \end{cases}$$

证明:  $[L_{ab}, L_{cd}]$ 

$$\begin{aligned} &= -\int d^3\vec{r} d^3\vec{r}' \left[ \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} (r_a \partial_b - r_b \partial_a) \frac{i\psi(\vec{r}, t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}', t)}{\sqrt{-\nabla'^2}} (r'_c \partial'_d - r'_d \partial'_c) \frac{i\psi(\vec{r}', t)}{\sqrt{-\nabla'^2}} \right] \\ &= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} d^3\vec{r}' \left[ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{\sqrt{-\nabla'^2}} (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \right] \\ &= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} d^3\vec{r}' \\ &\quad \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}} [(r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{\sqrt{-\nabla'^2}}] (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \right. \\ &\quad \left. + \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{\sqrt{-\nabla'^2}} \left[ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}}, (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \right] (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} \\ &= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} d^3\vec{r}' \\ &\quad \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}} (r_a \partial_b - r_b \partial_a) \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l_\zeta k'_\zeta} \delta^3(\vec{r}-\vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \right. \\ &\quad \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{\sqrt{-\nabla'^2}} (r'_c \partial'_d - r'_d \partial'_c) \{-i[\sigma(1) \cdot \hat{\nabla}']^2\}_{l'_\zeta k_\zeta} \delta^3(\vec{r}'-\vec{r}) (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} \\ &= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} d^3\vec{r}' \\ &\quad \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}} (r_a \partial'_b - r_b \partial'_a) \{-i[\sigma(1) \cdot \hat{\nabla}']^2\}_{l_\zeta k'_\zeta} \delta^3(\vec{r}-\vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \right. \\ &\quad \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{\sqrt{-\nabla'^2}} (r'_c \partial_d - r'_d \partial_c) \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l'_\zeta k_\zeta} \delta^3(\vec{r}'-\vec{r}) (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} \\ &= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} \\ &\quad \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}} (r_a \partial_b - r_b \partial_a) \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l_\zeta k'_\zeta} (r_c \partial_d - r_d \partial_c) \frac{\psi_{l'_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right. \\ &\quad \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}} (r_c \partial_d - r_d \partial_c) \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l'_\zeta k_\zeta} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} \\ &= -\int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} [-i(r_a \partial_b - r_b \partial_a), -i(r_c \partial_d - r_d \partial_c)] \{-i[\sigma(1) \cdot \hat{\nabla}]^2\} \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3\vec{r} \\ &= \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} [\hat{L}_{ab}, \hat{L}_{cd}] \frac{i\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3\vec{r} \\ &= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \end{aligned}$$

□

证明:  $[L_{ab}, P_c]$

$$\begin{aligned}
&= - \int d^3\vec{r}d^3\vec{r}' \left[ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} (r_a\partial_b - r_b\partial_a) \frac{i\psi(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}} \partial'_c \frac{i\psi(\vec{r}',t)}{\sqrt{-\nabla'^2}} \right] \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r}d^3\vec{r}' \left[ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} (r_a\partial_b - r_b\partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}} \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}} \right] \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r}d^3\vec{r}' \\
&\quad \left\{ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} [(r_a\partial_b - r_b\partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}}] \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}} + \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}} [\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}, \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}}] (r_a\partial_b - r_b\partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}} \right\} \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r}d^3\vec{r}' \\
&\quad \left\{ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} (r_a\partial_b - r_b\partial_a) \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}} \right. \\
&\quad \left. - \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}} \partial'_c \{-i[\sigma(1) \cdot \hat{\nabla}']^2\}_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a\partial_b - r_b\partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}} \right\} \\
&= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r}d^3\vec{r}' \\
&\quad \left\{ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} (r_a\partial'_b - r_b\partial'_a) \{-i[\sigma(1) \cdot \hat{\nabla}']^2\}_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}} \right. \\
&\quad \left. - \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}} \partial'_c \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a\partial_b - r_b\partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}} \right\} \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r}' \\
&\quad \left\{ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} (r_a\partial_b - r_b\partial_a) \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l_\zeta k'_\zeta} \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}} - \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} \partial'_c \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l'_\zeta k_\zeta} (r_a\partial_b - r_b\partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}} \right\} \\
&= - \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} [-i(r_a\partial_b - r_b\partial_a), -i\partial'_c] \{-i[\sigma(1) \cdot \hat{\nabla}]^2\} \frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}} d^3\vec{r}' \\
&= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} [\hat{L}_{ab}, \hat{P}_c] \frac{i\psi(\vec{r},t)}{\sqrt{-\nabla^2}} d^3\vec{r}' \\
&= -i(g_{bc}P_a - g_{ac}P_b)
\end{aligned}$$

□

证明:  $[P_a, P_b]$

$$\begin{aligned}
&= - \int \left[ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} \partial_a \frac{i\psi(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}} \partial'_b \frac{i\psi(\vec{r}',t)}{\sqrt{-\nabla'^2}} \right] d^3\vec{r}d^3\vec{r}' \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int \left[ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} \partial_a \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}} \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}} \right] d^3\vec{r}d^3\vec{r}' \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r}d^3\vec{r}' \left\{ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} [\partial_a \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}}, \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}}] \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}} + \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}} [\frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}}, \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}}] \partial_a \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}} \right\} \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r}d^3\vec{r}' \\
&\quad \left\{ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l_\zeta k'_\zeta} \partial_a \delta^3(\vec{r} - \vec{r}') \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}} - \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}} \{-i[\sigma(1) \cdot \hat{\nabla}']^2\}_{l'_\zeta k_\zeta} \partial'_b \delta^3(\vec{r}' - \vec{r}) \partial_a \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}} \right\} \\
&= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r}d^3\vec{r}' \\
&\quad \left\{ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l_\zeta k'_\zeta} \partial'_a \delta^3(\vec{r} - \vec{r}') \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}',t)}{\sqrt{-\nabla'^2}} - \frac{\psi^+(\vec{r}',t)}{\sqrt{-\nabla'^2}} \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{l'_\zeta k_\zeta} \partial_b \delta^3(\vec{r}' - \vec{r}) \partial_a \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}} \right\} \\
&= \int \left\{ \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{k_\zeta l'_\zeta} \partial_a \partial_b \frac{\psi_{l'_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}} - \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} \{-i[\sigma(1) \cdot \hat{\nabla}]^2\}_{k'_\zeta l_\zeta} \partial_b \partial_a \frac{\psi_{l_\zeta}(\vec{r},t)}{\sqrt{-\nabla^2}} \right\} d^3\vec{r}' \\
&= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} (\partial_a \partial_b - \partial_b \partial_a) \{-i[\sigma(1) \cdot \hat{\nabla}]^2\} \frac{\psi(\vec{r},t)}{\sqrt{-\nabla^2}} d^3\vec{r}' \\
&= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} (\partial_a \partial_b - \partial_b \partial_a) \frac{-i\psi(\vec{r},t)}{\sqrt{-\nabla^2}} d^3\vec{r}' \\
&= \int \frac{\psi^+(\vec{r},t)}{\sqrt{-\nabla^2}} [\hat{P}_a, \hat{P}_b] \frac{i\psi(\vec{r},t)}{\sqrt{-\nabla^2}} d^3\vec{r}' = 0
\end{aligned}$$

□

#### 4.11 光子自旋的彭加莱对称性

$$\text{定理4.11.1.} \quad \begin{cases} \nabla \cdot \vec{E} = -\rho, \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = -\vec{J} + \partial_t \vec{E} \\ \vec{A} = \frac{\nabla \times \vec{B}}{-\nabla^2}, \vec{\phi} = \frac{\nabla \cdot \vec{E}}{-\nabla^2} \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J} + \partial_t \nabla \vec{\phi} \\ \nabla^2 \vec{\phi} = \rho, \nabla \cdot \vec{A} = 0 \\ \vec{E} = -\partial_t \vec{A} - \nabla \vec{\phi}, \vec{B} = \nabla \times \vec{A} \end{cases}$$

$$\text{推论4.11.1.} \quad \begin{cases} [\partial_a + iS_{ab}(\gamma, \zeta) \partial^b] \Psi = -i\sigma_{\zeta ab}^{[\beta_\zeta]} J^b \\ \vec{A} = \frac{-i \nabla \times (\Psi - \Psi^*)}{\sqrt{2}}, i\vec{\phi} = \frac{-i \nabla \cdot (\Psi + \Psi^*)}{\sqrt{2}} \\ F_{ab} = \frac{i}{2} (\sigma_{-ab}^{\alpha'} \psi_{\alpha'} + \sigma_{+ab}^{\alpha} \psi_{\alpha}) \end{cases} \Leftrightarrow \begin{cases} \nabla^2 \vec{A} - \partial_t^2 \vec{A} = \vec{J} + \partial_t \nabla \vec{\phi} \\ \nabla^2 \vec{\phi} = \rho, \nabla \cdot \vec{A} = 0 \\ \sqrt{2} \Psi = -\partial_t \vec{A} - \nabla \vec{\phi} - i\zeta \nabla \times \vec{A} \end{cases}$$

定义4.11.1. 电磁场复矢量  $\psi_{\alpha_\zeta} := \frac{i}{2} \sigma_{\zeta \alpha_\zeta}^{ab} F_{ab} = i\zeta(E - i\zeta B)_{\alpha_\zeta} = (i\zeta E + B)_{\alpha_\zeta}$

定义4.11.2.  $\psi_\alpha = i(E - iB)_\alpha, \psi_\alpha^* = \psi_{\alpha'} = -i(E + iB)_{\alpha'}$



$SO(4)$ 群生成元矩阵的正分支:

$$\sigma_+ = R + L = \left\{ \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix} \right\} \quad (23.1a)$$

$SO(4)$ 群生成元矩阵的负分支:

$$\sigma_- = R - L = \left\{ \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \right\} \quad (23.2a)$$

$$\begin{aligned} \text{定理4.11.2. } \Sigma_{ij\pi} &= F_{i\pi}A_j - F_{j\pi}A_i = -i(E_iA_j - E_jA_i) \\ &= \frac{i}{2}(\sigma_{-i\pi}^{\alpha} \psi_{\alpha}^* + \sigma_{+i\pi}^{\alpha} \psi_{\alpha}) \frac{-i}{\sqrt{2}} \varepsilon_{jlm} \frac{\partial^l(\Psi - \Psi^*)^m}{\nabla^2} - \frac{i}{2}(\sigma_{-j\pi}^{\alpha} \psi_{\alpha}^* + \sigma_{+j\pi}^{\alpha} \psi_{\alpha}) \frac{-i}{\sqrt{2}} \varepsilon_{ilm} \frac{\partial^l(\Psi - \Psi^*)^m}{\nabla^2} \\ &= \frac{1}{2\sqrt{2}} [(-i\psi_i^* + i\psi_i)\varepsilon_{jlm} - (-i\psi_j^* + i\psi_j)\varepsilon_{ilm}] \frac{\partial^l(\Psi - \Psi^*)^m}{\nabla^2} \\ &= \frac{1}{4}[(\psi_i - \psi_i^*)\varepsilon_{jlm} - (\psi_j - \psi_j^*)\varepsilon_{ilm}] \frac{\partial^l(\psi + \psi^*)^m}{\nabla^2} \\ &= i(E_i\varepsilon_{jlm} - E_j\varepsilon_{ilm}) \frac{\partial^l B^m}{\nabla^2} \end{aligned}$$

$$\begin{aligned} \text{定理4.11.3. } \varepsilon^{kij}\Sigma_{ij\pi} &= \varepsilon^{kij}i(E_i\varepsilon_{jlm} - E_j\varepsilon_{ilm}) \frac{\partial^l B^m}{\nabla^2} = -2i[\frac{\vec{E}}{\sqrt{-\nabla^2}} \cdot \partial^k \frac{\vec{B}}{\sqrt{-\nabla^2}} - (\frac{\vec{E}}{\sqrt{-\nabla^2}} \cdot \nabla) \frac{B^k}{\sqrt{-\nabla^2}}] \\ \varepsilon_{\alpha\beta\gamma\zeta}\varepsilon^{\gamma\zeta\rho\sigma} &= \delta_{\alpha\zeta\rho\sigma}\delta_{\beta\zeta\sigma\zeta} - \delta_{\alpha\zeta\sigma\zeta}\delta_{\beta\zeta\rho\zeta} \end{aligned}$$

$$\text{定理4.11.4. } L_{ij\pi} = x_i F_{k\pi} \partial_j A^k - x_j F_{k\pi} \partial_i A^k = -iE_k(x_i \partial_j - x_j \partial_i) A^k = -iE_k(x_i \partial_j - x_j \partial_i) \varepsilon^{klm} \frac{\partial_l B_m}{\nabla^2}$$

$$\text{定理4.11.5. } \varepsilon^{kij}L_{ij\pi} = -iE_n(x_i \partial_j - x_j \partial_i) \varepsilon^{kij} \varepsilon^{nlm} \frac{\partial_l B_m}{\nabla^2}$$

$$\text{定理4.11.6. } \Sigma_{i\pi\pi} = F_{i\pi}A_{\pi} - F_{\pi\pi}A_i = E_i\phi$$

$$\text{定理4.11.7. } L_{i\pi\pi} = x_i F_{k\pi} \partial_{\pi} A^k - x_{\pi} F_{k\pi} \partial_i A^k - \frac{1}{2}x_i \vec{E}^2 + \frac{1}{2}x_i \vec{B}^2 = -iE_k(x_i \partial_{\pi} - x_{\pi} \partial_i) A^k - \frac{1}{2}x_i \vec{E}^2 + \frac{1}{2}x_i \vec{B}^2$$

## 4.12 光子角动量的彭加莱对称性

$$\begin{aligned} \text{定理4.12.1. } \Psi &:= \frac{1}{\sqrt{2}}(\vec{E} - i\zeta\vec{B}) \\ &= -\zeta \int \Psi^+(\vec{r}, t)(r_i\gamma_j - r_j\gamma_i)\Psi(\vec{r}, t)d^3\vec{r} \\ i\Sigma_{ij\pi} &= [(E_iA_j - E_jA_i) + E_k(x_i\partial_j - x_j\partial_i)A^k] \\ &= [(E_iA_j - E_jA_i) + (x_i\partial_j - x_j\partial_i)(\vec{E} \cdot \vec{A})_{\vec{E}}] \\ &= [(E_iA_j - E_jA_i) + x_i(\vec{E} \times \vec{B})_j + x_i(\vec{E} \cdot \nabla)\vec{A}_j - x_j(\vec{E} \times \vec{B})_i - x_j(\vec{E} \cdot \nabla)\vec{A}_i] \\ &= [x_i(\vec{E} \times \vec{B})_j - x_j(\vec{E} \times \vec{B})_i] \\ &= i\zeta[x_i(\Psi^+ \times \Psi)_j - x_j(\Psi^+ \times \Psi)_i] \\ &= -\zeta\Psi^+(x_i\gamma_j - x_j\gamma_i)\Psi \end{aligned}$$

$$\begin{aligned} \text{定理4.12.2. } M_{i\pi\pi} &= E_i\phi + \vec{E} \cdot (-x_i\partial_t - t\partial_i)\vec{A} - \frac{1}{2}x_i\vec{E}^2 + \frac{1}{2}x_i\vec{B}^2 \\ &= E_i\phi + x_i\vec{E} \cdot (\vec{E} + \nabla\phi) - t\partial_i(\vec{E} \cdot \vec{A})_{\vec{E}} - \frac{1}{2}x_i\vec{E}^2 + \frac{1}{2}x_i\vec{B}^2 \\ &= -t\partial_i(\vec{E} \cdot \vec{A})_{\vec{E}} + \frac{1}{2}x_i(\vec{E}^2 + \vec{B}^2) \\ &= -t(\vec{E} \times \vec{B})_i + \frac{1}{2}x_i(\vec{E}^2 + \vec{B}^2) \\ &= -i\zeta t(\Psi^+ \times \Psi)_i + x_i\Psi^+\Psi \\ &= -i\zeta\pi\Psi^+\gamma_i\Psi + x_i\Psi^+\Psi \\ iM_{i\pi\pi} &= -\zeta\Psi^+[x_i(-i\zeta) - \pi\gamma_i]\Psi \\ iM_{ab\pi} &= -\zeta\Psi^+(x_a\gamma_b - x_b\gamma_a)\Psi, \gamma_a = (\gamma, -i\zeta) \end{aligned}$$

$$\text{定理4.12.3. } M_{ij} = -\zeta \int \Psi^+(\vec{r}, t)(x_i\gamma_j - x_j\gamma_i)\Psi(\vec{r}, t)d^3\vec{r}$$

$$\begin{aligned} \text{证明: } [M_{ab}(\vec{r}, t), M_{a'b'}(\vec{r}', t)] &= \int d^3\vec{r}d^3\vec{r}' [\Psi^+(\vec{r}, t)(r_a\gamma_b - r_b\gamma_a)\Psi(\vec{r}, t), \Psi^+(\vec{r}', t)(r'_{a'}\gamma_{b'} - r'_{b'}\gamma_{a'})\Psi(\vec{r}', t)] \\ &= \int d^3\vec{r}d^3\vec{r}' [\Psi_{\alpha\zeta}^+(\vec{r}, t)(r_a\gamma_b - r_b\gamma_a)^{\alpha\zeta\alpha\zeta}\Psi_{\alpha\zeta}(\vec{r}, t), \Psi_{\beta\zeta}^+(\vec{r}', t)(r'_{a'}\gamma_{b'} - r'_{b'}\gamma_{a'})^{\beta\zeta\beta\zeta}\Psi_{\beta\zeta}(\vec{r}', t)] \\ &= \int d^3\vec{r}d^3\vec{r}' (r_a\gamma_b - r_b\gamma_a)^{\alpha\zeta\alpha\zeta}(r'_{a'}\gamma_{b'} - r'_{b'}\gamma_{a'})^{\beta\zeta\beta\zeta} [\Psi_{\alpha\zeta}^+(\vec{r}, t)\Psi_{\alpha\zeta}(\vec{r}, t), \Psi_{\beta\zeta}^+(\vec{r}', t)\Psi_{\beta\zeta}(\vec{r}', t)] \\ &= \int d^3\vec{r}d^3\vec{r}' (r_a\gamma_b - r_b\gamma_a)^{\alpha\zeta\alpha\zeta}(r'_{a'}\gamma_{b'} - r'_{b'}\gamma_{a'})^{\beta\zeta\beta\zeta} \end{aligned}$$



$$\begin{aligned}
&= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\zeta\alpha_\zeta}(\gamma, -i\zeta)_{a'}^{\beta'_\zeta\beta_\zeta} [\Psi_{\alpha'_\zeta}^+(\vec{r}', t)\Psi_{\alpha_\zeta}(\vec{r}, t), \Psi_{\beta'_\zeta}^+(\vec{r}', t)\Psi_{\beta_\zeta}(\vec{r}, t)] \\
&= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\zeta\alpha_\zeta}(\gamma, -i\zeta)_{a'}^{\beta'_\zeta\beta_\zeta} \\
&\{\Psi_{\alpha'_\zeta}^+(\vec{r}', t)[\Psi_{\alpha_\zeta}(\vec{r}, t), \Psi_{\beta'_\zeta}^+(\vec{r}', t)]\Psi_{\beta_\zeta}(\vec{r}, t) - \Psi_{\beta'_\zeta}^+(\vec{r}', t)[\Psi_{\beta_\zeta}(\vec{r}, t), \Psi_{\alpha'_\zeta}^+(\vec{r}', t)]\Psi_{\alpha_\zeta}(\vec{r}, t)\} \\
&= \int d^3\vec{r}d^3\vec{r}'(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\zeta\alpha_\zeta}(\gamma, -i\zeta)_{a'}^{\beta'_\zeta\beta_\zeta} \\
&\{\Psi_{\alpha'_\zeta}^+(\vec{r}', t)i\zeta[\gamma \cdot \nabla]_{\alpha_\zeta\beta'_\zeta}\delta^3(\vec{r} - \vec{r}')\Psi_{\beta_\zeta}(\vec{r}, t) - \Psi_{\beta'_\zeta}^+(\vec{r}', t)i\zeta[\gamma \cdot \nabla']_{\beta_\zeta\alpha'_\zeta}\delta^3(\vec{r}' - \vec{r})\Psi_{\alpha_\zeta}(\vec{r}, t)\} \\
&= i\zeta \int d^3\vec{r}\{ (r_i\gamma_j - r_j\gamma_i)^{\alpha'_\zeta\alpha_\zeta}\Psi_{\alpha'_\zeta}^+(\vec{r}, t)[\gamma \cdot \nabla]_{\alpha_\zeta\beta'_\zeta}[(\gamma, -i\zeta)_{a'}^{\beta'_\zeta\beta_\zeta}\Psi_{\beta_\zeta}(\vec{r}, t)] \\
&- [(\gamma, -i\zeta)_{a'}^{\beta'_\zeta\beta_\zeta}\Psi_{\beta'_\zeta}^+(\vec{r}, t)[\gamma \cdot \nabla]_{\beta_\zeta\alpha'_\zeta}(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\zeta\alpha_\zeta}\Psi_{\alpha_\zeta}(\vec{r}, t)]\} \\
&= i\zeta \int d^3\vec{r}\{ -[(\gamma, -i\zeta)_{a'}^{\beta'_\zeta\beta_\zeta}\Psi_{\beta'_\zeta}^+(\vec{r}, t)i\varepsilon_{ijk}(\gamma^k)_{\beta_\zeta\alpha'_\zeta}\Psi_{\alpha_\zeta}(\vec{r}, t)] \\
&+ \{(r_i\gamma_j - r_j\gamma_i)^{\alpha'_\zeta\alpha_\zeta}[(\gamma, -i\zeta)_{a'}^{\beta'_\zeta\beta_\zeta}\Psi_{\alpha'_\zeta}^+(\vec{r}, t)(\gamma^{k'})_{\alpha_\zeta\beta'_\zeta}\partial_{k'}\Psi_{\beta_\zeta}(\vec{r}, t)] \\
&- (r_i\gamma_j - r_j\gamma_i)^{\alpha'_\zeta\alpha_\zeta}[(\gamma, -i\zeta)_{a'}^{\beta'_\zeta\beta_\zeta}\Psi_{\beta'_\zeta}^+(\vec{r}, t)(\gamma^k)_{\beta_\zeta\alpha'_\zeta}\partial_k\Psi_{\alpha_\zeta}(\vec{r}, t)]\} \\
&= i\zeta \int d^3\vec{r}\{ -i\varepsilon_{ijk}\Psi^+(\vec{r}, t)(\gamma, -i\zeta)_{a'}\gamma^k\Psi(\vec{r}, t) \\
&+ \Psi^+(\vec{r}, t)(r_i\gamma_j - r_j\gamma_i)\gamma_{k'}(\gamma, -i\zeta)_{a'}\partial^{k'}\Psi(\vec{r}, t) - \Psi^+(\vec{r}, t)(\gamma, -i\zeta)_{a'}\gamma_k(r_i\gamma_j - r_j\gamma_i)\partial^k\Psi(\vec{r}, t)\} \\
&= i\zeta \int d^3\vec{r}\{\Psi^+(\vec{r}, t)(r_i\gamma_j - r_j\gamma_i)\gamma_k(\gamma, -i\zeta)_{a'}\partial^k\Psi(\vec{r}, t) - \Psi^+(\vec{r}, t)(\gamma, -i\zeta)_{a'}\gamma_k\partial^k[(r_i\gamma_j - r_j\gamma_i)\Psi(\vec{r}, t)]\} \quad \square
\end{aligned}$$

证明:

$$= \int d^3\vec{r}\Psi^+(\vec{r}, t)\{(r_i\gamma_j - r_j\gamma_i)[\gamma \cdot \nabla][(r_i\gamma_{j'} - r_{j'}\gamma_{i'}) - [(r_i\gamma_{j'} - r_{j'}\gamma_{i'})[\gamma \cdot \nabla](r_i\gamma_j - r_j\gamma_i)]\}\Psi(\vec{r}, t) \quad \square$$

## 5 电磁场复场强协变量子化新方案

本节换成电磁表象再次对光子协变量子化方案重新进行了一次完整的描述，方便后面章节使用。

### 5.1 电磁场方程各种等价形式 [23, 25]

$$\text{定义5.1.1. } \Psi_{\alpha_\zeta} := \frac{-i\zeta}{\sqrt{2}}\psi_{\alpha_\zeta} = \frac{-i\zeta}{\sqrt{2}}\frac{i}{2}\sigma_{\zeta\alpha_\zeta}^{ab}F_{ab} = \frac{-i\zeta}{\sqrt{2}}i\zeta(E - i\zeta B)_{\alpha_\zeta}$$

$$\text{定义5.1.2. } \Psi := \frac{1}{\sqrt{2}}(\vec{E} - i\zeta\vec{B}) = \frac{1}{\sqrt{2}}(\vec{E} - i\zeta\nabla \times \vec{A}), \Psi_i = \frac{1}{\sqrt{2}}(E_i - i\zeta\varepsilon_i{}^{jk}\partial_j A_k), p \cdot x := \vec{p} \cdot \vec{r} - Et$$

定理5.1.1.

$$\begin{cases} \partial^a F_{ab} = 0 \\ \partial^a * F_{ab} = 0 \end{cases} \Leftrightarrow \begin{cases} \nabla \cdot \vec{E} = 0, \nabla \times \vec{E} = -\partial_t \vec{B} \\ \nabla \cdot \vec{B} = 0, \nabla \times \vec{B} = \partial_t \vec{E} \end{cases} \Leftrightarrow \begin{cases} (\gamma, -i\zeta)^a \partial_a \Psi = 0 \\ \nabla \cdot \Psi = 0 \end{cases} \Leftrightarrow \begin{cases} [\partial_a + iS_{ab}(\gamma, \zeta)\partial^b]\Psi = 0 \\ S_{ab}(\gamma, \zeta) = i\sigma_{\zeta ab}^{\alpha_\zeta}\gamma_{\alpha_\zeta}(s) \end{cases}$$

### 5.2 电磁场复场强自旋方程及其平面波解

$$\text{定理5.2.1. } [\partial_a + iS_{ab}(\gamma, \zeta)\partial^b]\Psi(x) = 0$$

$$\text{推论5.2.1. } \begin{cases} \Psi(\vec{r}, t) := \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \sqrt{|\vec{p}|}\lambda_m(\hat{p}, -\zeta)[a_1(\vec{p}, -\zeta)e^{i\zeta p \cdot x} + a_2^+(\vec{p}, -\zeta)e^{-i\zeta p \cdot x}]d^3\vec{p} \\ \sqrt{|\vec{p}|}a_1(\vec{p}, -\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda_m^+(\hat{p}, -\zeta)\Psi(\vec{r}, t)e^{-i\zeta p \cdot x}d^3\vec{r} \\ \sqrt{|\vec{p}|}a_2^+(\vec{p}, -\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda_m^+(\hat{p}, -\zeta)\Psi(\vec{r}, t)e^{i\zeta p \cdot x}d^3\vec{r} \end{cases}$$

$$\text{推论5.2.2. } (\gamma, -i\zeta)^a \partial_a \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \sum_{h=1}^{-1} \sqrt{|\vec{p}|}\lambda_m(\hat{p}, h)[a_1(\vec{p}, -\zeta)e^{i\zeta p \cdot x} + a_2^+(\vec{p}, -\zeta)e^{-i\zeta p \cdot x}]d^3\vec{p} = 0$$

$$\int_{\vec{p} \neq 0} \sum_{h=1}^{-1} \sqrt{|\vec{p}|}(\gamma, -i\zeta)^a p_a \lambda_m(\hat{p}, h)[a_1(\vec{p}, -\zeta)e^{i\zeta p \cdot x} - a_2^+(\vec{p}, -\zeta)e^{-i\zeta p \cdot x}]d^3\vec{p} = 0$$

### 5.3 电磁场常数不变张量 $\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}$ 的性质

从常数不变张量分析可知:

推论5.3.1.

$$\sigma_{\alpha_\zeta\alpha'_\zeta}^{\pi\pi} = \frac{1}{2}\delta_{\alpha_\zeta\alpha'_\zeta}$$

$$\sigma_{\alpha_\zeta\alpha'_\zeta}^{kk} = \sigma_{\alpha_\zeta\alpha'_\zeta}^{\pi k} = -\frac{\zeta}{2}\varepsilon^k{}_{\alpha_\zeta\alpha'_\zeta}$$

$$\sigma_{\alpha_\zeta\alpha'_\zeta}^{kl} = \frac{1}{2}(\delta_{\alpha_\zeta}^k \delta_{\alpha'_\zeta}^l + \delta_{\alpha'_\zeta}^k \delta_{\alpha_\zeta}^l - \delta^{kl}\delta_{\alpha_\zeta\alpha'_\zeta})$$

推论5.3.2.  $\sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b = \partial_{\alpha_s} \partial_{\alpha'_s} - \frac{1}{2} \delta_{\alpha_s \alpha'_s} (\nabla^2 + \partial_t^2) + i \zeta \varepsilon^k_{\alpha_s \alpha'_s} \partial_k \partial_t$

证明:  $\sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b = \sigma_{\alpha_s \alpha'_s}^{kl} \partial_k \partial_l + 2 \sigma_{\alpha_s \alpha'_s}^{k\pi} \partial_k \partial_\pi + \sigma_{\alpha_s \alpha'_s}^{\pi\pi} \partial_\pi \partial_\pi$   
 $= \partial_{\alpha_s} \partial_{\alpha'_s} - \frac{1}{2} \delta_{\alpha_s \alpha'_s} (\nabla^2 - \partial_\pi^2) - \zeta \varepsilon^k_{\alpha_s \alpha'_s} \partial_k \partial_\pi$   
 $= \partial_{\alpha_s} \partial_{\alpha'_s} - \frac{1}{2} \delta_{\alpha_s \alpha'_s} (\nabla^2 + \partial_t^2) + i \zeta \varepsilon^k_{\alpha_s \alpha'_s} \partial_k \partial_t$  □

推论5.3.3.  $\sigma_{\alpha_s \alpha'_s}^{ab} p_a p_b = p_{\alpha_s} p_{\alpha'_s} - \delta_{\alpha_s \alpha'_s} |\vec{p}|^2 - i \zeta \varepsilon^k_{\alpha_s \alpha'_s} p_k |\vec{p}|$

推论5.3.4.  $\sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b \Delta(x) = (\partial_{\alpha_s} \partial_{\alpha'_s} - \delta_{\alpha_s \alpha'_s} \nabla^2 + i \zeta \varepsilon^k_{\alpha_s \alpha'_s} \partial_k \partial_t) \Delta(x) = -\sigma_{\alpha_s \alpha'_s}^{ab} p_a p_b \Delta(x)$

推论5.3.5.

$$\begin{cases} \sigma_{\{\alpha_s \alpha'_s\}}^{ab} \partial_a \partial_b = 2 \partial_{\alpha_s} \partial_{\alpha'_s} - \delta_{\alpha_s \alpha'_s} (\nabla^2 + \partial_t^2) \\ \sigma_{[\alpha_s \alpha'_s]}^{ab} \partial_a \partial_b = 2 i \zeta \varepsilon^k_{\alpha_s \alpha'_s} \partial_k \partial_t = -2 \zeta (\gamma \cdot \nabla)_{\alpha_s \alpha'_s} \partial_t \end{cases} \quad \begin{cases} \sigma_{\{\alpha_s \alpha'_s\}}^{ab} \partial_a \partial_b \Delta(x) = 2 (\partial_{\alpha_s} \partial_{\alpha'_s} - \delta_{\alpha_s \alpha'_s} \nabla^2) \Delta(x) \\ \sigma_{[\alpha_s \alpha'_s]}^{ab} \partial_a \partial_b \Delta(x) = 2 i \zeta \varepsilon^k_{\alpha_s \alpha'_s} \partial_k \partial_t \Delta(x) \end{cases}$$

推论5.3.6.

$$\begin{cases} \sigma_{\{\alpha_s \alpha'_s\}}^{ab} p_a p_b = 2 (p_{\alpha_s} p_{\alpha'_s} - \delta_{\alpha_s \alpha'_s} |\vec{p}|^2) \\ \sigma_{[\alpha_s \alpha'_s]}^{ab} p_a p_b = -2 i \zeta \varepsilon^k_{\alpha_s \alpha'_s} p_k |\vec{p}| = 2 \zeta \gamma^k_{\alpha_s \alpha'_s} p_k |\vec{p}| \end{cases} \quad \begin{cases} \sigma_{\{\alpha_s \alpha'_s\}}^{ab} \hat{p}_a \hat{p}_b = 2 (\hat{p}_{\alpha_s} \hat{p}_{\alpha'_s} - \delta_{\alpha_s \alpha'_s}) \\ \sigma_{[\alpha_s \alpha'_s]}^{ab} \hat{p}_a \hat{p}_b = -2 i \zeta \varepsilon^k_{\alpha_s \alpha'_s} \hat{p}_k = 2 \zeta \gamma^k_{\alpha_s \alpha'_s} \hat{p}_k \end{cases}$$

引理5.3.1.  $\sigma_{\alpha_s \alpha'_s}^{ab} p_a p_b = -2 |\vec{p}|^2 \lambda_{m \alpha_s}(\hat{p}, -\zeta) \lambda_{m \alpha'_s}^+(\hat{p}, -\zeta)$

证明:  $\sigma_{\alpha_s \alpha'_s}^{ab} p_a p_b$

$$\begin{aligned} &= p_{\alpha_s} p_{\alpha'_s} + \zeta \gamma^k_{\alpha_s \alpha'_s} p_k |\vec{p}| - \delta_{\alpha_s \alpha'_s} |\vec{p}|^2 \\ &= p_{\alpha_s} p_{\alpha'_s} + \zeta |\vec{p}| \gamma^k_{\alpha_s \alpha'_s} p_k \delta_{\beta_s \alpha'_s} - \delta_{\alpha_s \alpha'_s} |\vec{p}|^2 \\ &= \lambda_{m \alpha_s}(\hat{p}, 0) \lambda_{m \alpha'_s}^+(\hat{p}, 0) |\vec{p}|^2 + \zeta |\vec{p}| \gamma^k_{\alpha_s \alpha'_s} p_k \sum_{h=1}^{-1} \lambda_{m \beta_s}(\hat{p}, h) \lambda_{m \alpha'_s}^+(\hat{p}, h) - \delta_{\alpha_s \alpha'_s} |\vec{p}|^2 \\ &= \lambda_{m \alpha_s}(\hat{p}, 0) \lambda_{m \alpha'_s}^+(\hat{p}, 0) |\vec{p}|^2 + \zeta |\vec{p}| [\zeta |\vec{p}| \lambda_{m \beta_s}(\hat{p}, \zeta) \lambda_{m \alpha'_s}^+(\hat{p}, \zeta) - \zeta |\vec{p}| \lambda_{m \beta_s}(\hat{p}, -\zeta) \lambda_{m \alpha'_s}^+(\hat{p}, -\zeta)] - \delta_{\alpha_s \alpha'_s} |\vec{p}|^2 \\ &= |\vec{p}|^2 \sum_{h=1}^{-1} \lambda_{m \alpha_s}(\hat{p}, h) \lambda_{m \alpha'_s}^+(\hat{p}, h) - \delta_{\alpha_s \alpha'_s} |\vec{p}|^2 - 2 |\vec{p}|^2 \lambda_{m \alpha_s}(\hat{p}, -\zeta) \lambda_{m \alpha'_s}^+(\hat{p}, -\zeta) \\ &= -2 |\vec{p}|^2 \lambda_{m \alpha_s}(\hat{p}, -\zeta) \lambda_{m \alpha'_s}^+(\hat{p}, -\zeta) \end{aligned}$$
 □

以上引理把常数不变张量分析与螺旋度分析联系起来。

$$\text{推论5.3.7.} \quad \begin{cases} (\sigma^{ab} \hat{p}_a \hat{p}_b)^n = (-2)^{n-1} \sigma^{ab} \hat{p}_a \hat{p}_b \\ (\frac{\sigma^{ab} \partial_a \partial_b}{\nabla^2})^n = (-2)^{n-1} \frac{\sigma^{ab} \partial_a \partial_b}{\nabla^2} \end{cases} \quad \begin{cases} (\hat{p}^T \hat{p} - 1)^n = (-1)^{n-1} (\hat{p}^T \hat{p} - 1) \\ (\frac{\nabla^T \nabla}{\nabla^2} - 1)^n = (-2)^{n-1} (\frac{\nabla^T \nabla}{\nabla^2} - 1) \end{cases}$$

$$\text{推论5.3.8.} \quad \begin{cases} (\zeta \gamma \cdot \hat{p})^{2n} = -(\hat{p}^T \hat{p} - 1) \\ (\frac{-i \zeta \gamma \cdot \nabla}{\sqrt{-\nabla^2}})^{2n} = -(\frac{\nabla^T \nabla}{\nabla^2} - 1) \end{cases} \quad \begin{cases} (\zeta \gamma \cdot \hat{p})^{2n-1} = (\zeta \gamma \cdot \hat{p}) \\ (\frac{-i \zeta \gamma \cdot \nabla}{\sqrt{-\nabla^2}})^{2n-1} = (\frac{-i \zeta \gamma \cdot \nabla}{\sqrt{-\nabla^2}}) \end{cases}$$

$$\text{推论5.3.9.} \quad \begin{cases} (\hat{p}^T \hat{p} - 1) (\zeta \gamma \cdot \hat{p}) = (\zeta \gamma \cdot \hat{p}) (\hat{p}^T \hat{p} - 1) = -(\zeta \gamma \cdot \hat{p}) \\ (\frac{\nabla^T \nabla}{\nabla^2} - 1) (\frac{-i \zeta \gamma \cdot \nabla}{\sqrt{-\nabla^2}}) = (\frac{-i \zeta \gamma \cdot \nabla}{\sqrt{-\nabla^2}}) (\frac{\nabla^T \nabla}{\nabla^2} - 1) = -(\frac{-i \zeta \gamma \cdot \nabla}{\sqrt{-\nabla^2}}) \end{cases}$$

## 5.4 电磁场数学上一般的协变对易规则

定理5.4.1.

$$\begin{cases} [a_\sigma(\vec{p}, -\zeta), a_{\sigma'}^+(\vec{p}', -\zeta)]_{\pm} = \zeta^0 \delta_\sigma \delta_{\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\zeta), a_{\sigma'}(\vec{p}', -\zeta)]_{\pm} = 0 \\ [a_\sigma^+(\vec{p}, -\zeta), a_{\sigma'}^+(\vec{p}', -\zeta)]_{\pm} = 0 \end{cases} \Rightarrow \begin{cases} [\Psi_{\alpha_s}(x), \Psi_{\alpha'_s}^+(x')]_{\pm} \\ = i \zeta^0 \sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b [\delta_1 \Delta^{(+\zeta)}(x - x') - \pm \delta_2 \Delta^{(-\zeta)}(x - x')] \\ [\Psi_{\alpha_s}(x), \Psi_{\beta_s}(x')]_{\pm} = 0 \\ [\Psi_{\alpha'_s}^+(x), \Psi_{\beta'_s}^+(x')]_{\pm} = 0 \end{cases}$$

证明:  $[\Psi_{\alpha_s}^{(+\zeta)}(x), \Psi_{\alpha'_s}^{(+\zeta)+}(x')]_{\pm}$

$$= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \lambda_{m \alpha_s}(\hat{p}, -\zeta) \lambda_{m \alpha'_s}^+(\vec{p}', -\zeta) \sqrt{|\vec{p}| |\vec{p}'|} [a_1(\vec{p}, -\zeta), a_1^+(\vec{p}', -\zeta)]_{\pm} e^{i \zeta \vec{p} \cdot (x - x')} d^3 \vec{p} d^3 \vec{p}'$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\vec{p}', -\varsigma) |\vec{p}'| \varsigma^0 \delta_1 \delta^3(\vec{p} - \vec{p}') e^{i\varsigma\vec{p}\cdot(x-x')} d^3\vec{p} d^3\vec{p}' \\
&= \frac{1}{(2\pi)^3} \int \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\hat{p}, -\varsigma) \varsigma^0 \delta_1 |\vec{p}'| e^{i\varsigma\vec{p}\cdot(x-x')} d^3\vec{p}' \\
&= \frac{\varsigma^0 \delta_1}{(2\pi)^3} \int \frac{1}{2|\vec{p}'|} \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} p_a p_b e^{i\varsigma\vec{p}\cdot(x-x')} d^3\vec{p}' \\
&= i\varsigma^0 \delta_1 \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta^{(+\varsigma)}(x - x') \quad \square
\end{aligned}$$

$$\begin{aligned}
&\text{证明: } [\Psi_{\alpha_\zeta}^{(-\varsigma)}(x), \Psi_{\alpha'_\zeta}^{(-\varsigma)+}(x')]_{\pm} \\
&= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\vec{p}', -\varsigma) \sqrt{|\vec{p}'| |\vec{p}'|} [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)]_{\pm} e^{-i\varsigma\vec{p}\cdot(x-x')} d^3\vec{p} d^3\vec{p}' \\
&= \pm \frac{1}{(2\pi)^3} \int \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\vec{p}', -\varsigma) |\vec{p}'| \varsigma^0 \delta_2 \delta^3(\vec{p} - \vec{p}') e^{-i\varsigma\vec{p}\cdot(x-x')} d^3\vec{p} d^3\vec{p}' \\
&= \pm \frac{1}{(2\pi)^3} \int \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\hat{p}, -\varsigma) \varsigma^0 \delta_2 |\vec{p}'| e^{-i\varsigma\vec{p}\cdot(x-x')} d^3\vec{p}' \\
&= \pm \frac{\varsigma^0 \delta_2}{(2\pi)^3} \int \frac{1}{2|\vec{p}'|} \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} p_a p_b e^{-i\varsigma\vec{p}\cdot(x-x')} d^3\vec{p}' \\
&= -\pm i\varsigma^0 \delta_2 \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta^{(-\varsigma)}(x - x') \quad \square
\end{aligned}$$

$$\begin{aligned}
&\text{证明: } [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')]_{\pm} \\
&= [\Psi_{\alpha_\zeta}^{(+\varsigma)}(x), \Psi_{\alpha'_\zeta}^{(+\varsigma)+}(x')]_{\pm} + [\Psi_{\alpha_\zeta}^{(-\varsigma)}(x), \Psi_{\alpha'_\zeta}^{(-\varsigma)+}(x')]_{\pm} \\
&= i\varsigma^0 \delta_1 \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta^{(+\varsigma)}(x - x') - \pm i\varsigma^0 \delta_2 \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta^{(-\varsigma)}(x - x') \\
&= i\varsigma^0 \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b [\delta_1 \Delta^{(+\varsigma)}(x - x') - \pm \delta_2 \Delta^{(-\varsigma)}(x - x')] \\
&= i\varsigma^0 \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b [(\delta_1 \pm \delta_2) \Delta^{(+\varsigma)}(x - x') - \pm \delta_2 \Delta(x - x')] \quad \square
\end{aligned}$$

从上式可知，只有 $\delta_1 \pm \delta_2 = 0$ 时，才满足微观因果性，同时只有 $\delta_1, \delta_2 \geq 0$ 时，才满足几率非负性。所以数学上八种协变对易或反对易方案中，物理上合理的只有一种：即 $\delta_1 = \delta_2 = 1$ ，且满足对易关系。其实还有两种，即 $\delta_1 = \delta_2 = 0$ ，且满足对易或反对易关系，就是经典情形。

## 5.5 电磁场物理的协变对易规则

从上节可知，有物理意义的对易规则如下：（为了相互印证，重新作了证明）

定理5.5.1.

$$\begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = \varsigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = 0 \end{cases} \Rightarrow \begin{cases} [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha_\zeta}(x), \Psi_{\beta_\zeta}(x')] = 0 \\ [\Psi_{\alpha'_\zeta}^+(x), \Psi_{\beta'_\zeta}^+(x')] = 0 \end{cases}$$

证明:  $[\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')]$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} d^3\vec{p} d^3\vec{p}' \\
&\lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\vec{p}', -\varsigma) \sqrt{|\vec{p}'| |\vec{p}'|} \{ [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)] e^{i\varsigma\vec{p}\cdot(x-x')} + [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)] e^{-i\varsigma\vec{p}\cdot(x-x')} \} \\
&= \frac{1}{(2\pi)^3} \int \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\vec{p}', -\varsigma) |\vec{p}'| [\varsigma \delta^3(\vec{p} - \vec{p}') e^{i\varsigma\vec{p}\cdot(x-x')} - \varsigma \delta^3(\vec{p} - \vec{p}') e^{-i\varsigma\vec{p}\cdot(x-x')}] d^3\vec{p} d^3\vec{p}' \\
&= \frac{1}{(2\pi)^3} \int \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\hat{p}, -\varsigma) \varsigma |\vec{p}'| [e^{i\varsigma\vec{p}\cdot(x-x')} - e^{-i\varsigma\vec{p}\cdot(x-x')}] d^3\vec{p}' \\
&= \frac{\varsigma}{(2\pi)^3} \int \frac{1}{2|\vec{p}'|} \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} p_a p_b [e^{i\varsigma\vec{p}\cdot(x-x')} - e^{-i\varsigma\vec{p}\cdot(x-x')}] d^3\vec{p}' \\
&= i\varsigma \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta[\varsigma(x - x')] \\
&= i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \quad \square
\end{aligned}$$

定理5.5.2.

$$\begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = \varsigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = 0 \end{cases} \Rightarrow \begin{cases} [\Psi_{\alpha_\zeta}^{(\tau)}(x), \Psi_{\alpha'_\zeta}^{(\kappa)+}(x')] = i\delta^{\tau\kappa} \sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta^{(\tau)}(x - x') \\ [\Psi_{\alpha_\zeta}^{(\tau)}(x), \Psi_{\beta_\zeta}^{(\kappa)}(x')] = 0 \\ [\Psi_{\alpha'_\zeta}^{(\tau)+}(x), \Psi_{\beta'_\zeta}^{(\kappa)+}(x')] = 0 \end{cases}$$

证明:  $[\Psi_{\alpha_\zeta}^{(+\varsigma)}(x), \Psi_{\alpha'_\zeta}^{(+\varsigma)+}(x')]$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\vec{p}', -\varsigma) \sqrt{|\vec{p}'| |\vec{p}'|} [a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)] e^{i\varsigma\vec{p}\cdot(x-x')} d^3\vec{p} d^3\vec{p}' \\
&= \frac{1}{(2\pi)^3} \int \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\vec{p}', -\varsigma) |\vec{p}'| \varsigma \delta^3(\vec{p} - \vec{p}') e^{i\varsigma\vec{p}\cdot(x-x')} d^3\vec{p} d^3\vec{p}'
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\hat{p}, -\varsigma) \varsigma |\vec{p}| e^{i\varsigma \vec{p} \cdot (x-x')} d^3 \vec{p} \\
&= \frac{-\varsigma}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} p_a p_b e^{i\varsigma \vec{p} \cdot (x-x')} d^3 \vec{p} \\
&= i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta^{(+\varsigma)}(x-x')
\end{aligned}$$

□

证明:  $[\Psi_{\alpha_\zeta}^{(-\varsigma)}(x), \Psi_{\alpha'_\zeta}^{(-\varsigma)+}(x')]$ 

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\vec{p}', -\varsigma) \sqrt{|\vec{p}| |\vec{p}'|} [a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)] e^{-i\varsigma \vec{p} \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= -\frac{1}{(2\pi)^3} \int \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\vec{p}', -\varsigma) |\vec{p}| \varsigma \delta^3(\vec{p} - \vec{p}') e^{-i\varsigma \vec{p} \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= -\frac{1}{(2\pi)^3} \int \lambda_{m\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_{m\alpha'_\zeta}^+(\hat{p}, -\varsigma) \varsigma |\vec{p}| e^{-i\varsigma \vec{p} \cdot (x-x')} d^3 \vec{p} \\
&= \frac{\varsigma}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} p_a p_b e^{-i\varsigma \vec{p} \cdot (x-x')} d^3 \vec{p} \\
&= i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta^{(-\varsigma)}(x-x')
\end{aligned}$$

□

## 5.6 电磁场的等时对易规则

$$\text{推论5.6.1.} \quad \begin{cases} [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\Psi_{\alpha_\zeta}(x), \Psi_{\beta_\zeta}(x')] = 0 \\ [\Psi_{\alpha'_\zeta}^+(x), \Psi_{\beta'_\zeta}^+(x')] = 0 \end{cases} \Rightarrow \begin{cases} [\Psi_{\alpha_\zeta}(\vec{r}, t), \Psi_{\alpha'_\zeta}^+(\vec{r}', t)] = \varsigma \varepsilon^k{}_{\alpha_\zeta \alpha'_\zeta} \partial_k \delta(\vec{r} - \vec{r}') \\ [\Psi_{\alpha_\zeta}(\vec{r}, t), \Psi_{\beta_\zeta}(\vec{r}', t)] = 0 \\ [\Psi_{\alpha'_\zeta}^+(\vec{r}, t), \Psi_{\beta'_\zeta}^+(\vec{r}', t)] = 0 \end{cases}$$

证明:  $[\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x')$  $\Rightarrow [\Psi_{\alpha_\zeta}(\vec{r}, t), \Psi_{\alpha'_\zeta}^+(\vec{r}', t)] = 2i\sigma_{\alpha_\zeta \alpha'_\zeta}^{k\pi} \partial_k \partial_\pi \Delta(x-x')|_{t=t'}$  $\Leftrightarrow [\Psi_{\alpha_\zeta}(\vec{r}, t), \Psi_{\alpha'_\zeta}^+(\vec{r}', t)] = \varsigma \varepsilon^k{}_{\alpha_\zeta \alpha'_\zeta} \partial_k \delta(\vec{r} - \vec{r}')$ 

□

$$\text{推论5.6.2.} \quad \begin{cases} [\Psi_{\alpha_\zeta}(\vec{r}, t), \Psi_{\alpha'_\zeta}^+(\vec{r}', t)] = \varsigma \varepsilon^k{}_{\alpha_\zeta \alpha'_\zeta} \partial_k \delta(\vec{r} - \vec{r}') \\ [\Psi_{\alpha_\zeta}(\vec{r}, t), \Psi_{\beta_\zeta}(\vec{r}', t)] = 0 \\ [\Psi_{\alpha'_\zeta}^+(\vec{r}, t), \Psi_{\beta'_\zeta}^+(\vec{r}', t)] = 0 \end{cases} \Rightarrow \begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = \varsigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = 0 \end{cases}$$

证明:  $[a_1(\vec{p}, -\varsigma), a_1^+(\vec{p}', -\varsigma)]$ 

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}| |\vec{p}'|}} \int [\lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)}, \lambda_m^{\alpha'_\zeta}(\vec{p}', -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{i\varsigma(\vec{p}' \cdot \vec{r}' - E't)}] d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}| |\vec{p}'|}} \int \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_m^{\alpha'_\zeta}(\vec{p}', -\varsigma) [\Psi_{\alpha_\zeta}(\vec{r}, t), \Psi_{\alpha'_\zeta}^+(\vec{r}', t)] e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r} d^3 \vec{r}' \\
&= i\varsigma \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}| |\vec{p}'|}} \int \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_m^{\alpha'_\zeta}(\vec{p}', -\varsigma) \gamma^k{}_{\alpha_\zeta \alpha'_\zeta} \partial_k \delta^3(\vec{r} - \vec{r}') e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r} d^3 \vec{r}' \\
&= i\varsigma \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}| |\vec{p}'|}} \int \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_m^{\alpha'_\zeta}(\vec{p}', -\varsigma) \gamma^k{}_{\alpha_\zeta \alpha'_\zeta} i\varsigma p_k e^{-i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r}' \\
&= -\frac{1}{|\vec{p}|} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_m^{\alpha'_\zeta}(\hat{p}', h') \gamma^k{}_{\alpha_\zeta \alpha'_\zeta} p_k \delta^3(\vec{p} - \vec{p}') \\
&= -\lambda_m^+(\hat{p}, -\varsigma) \frac{\gamma^k{}_{\alpha_\zeta \alpha'_\zeta} p_k}{|\vec{p}|} \lambda_m(\hat{p}', -\varsigma) \delta^3(\vec{p} - \vec{p}') \\
&= \varsigma \lambda_m^+(\hat{p}, -\varsigma) \lambda_m(\hat{p}', -\varsigma) \delta^3(\vec{p} - \vec{p}') \\
&= \varsigma \delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

证明:  $[a_2^+(\vec{p}, -\varsigma), a_2(\vec{p}', -\varsigma)]$ 

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}| |\vec{p}'|}} \int [\lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)}, \lambda_m^{\alpha'_\zeta}(\vec{p}', -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{-i\varsigma(\vec{p}' \cdot \vec{r}' - E't)}] d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}| |\vec{p}'|}} \int \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_m^{\alpha'_\zeta}(\vec{p}', -\varsigma) [\Psi_{\alpha_\zeta}(\vec{r}, t), \Psi_{\alpha'_\zeta}^+(\vec{r}', t)] e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{-i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r} d^3 \vec{r}' \\
&= i\varsigma \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}| |\vec{p}'|}} \int \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_m^{\alpha'_\zeta}(\vec{p}', -\varsigma) \gamma^k{}_{\alpha_\zeta \alpha'_\zeta} \partial_k \delta^3(\vec{r} - \vec{r}') e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{-i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r} d^3 \vec{r}' \\
&= i\varsigma \frac{1}{(2\pi)^3} \frac{1}{\sqrt{|\vec{p}| |\vec{p}'|}} \int \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_m^{\alpha'_\zeta}(\vec{p}', -\varsigma) \gamma^k{}_{\alpha_\zeta \alpha'_\zeta} (-i\varsigma p_k) e^{i\varsigma(\vec{p} \cdot \vec{r} - Et)} e^{-i\varsigma(\vec{p}' \cdot \vec{r}' - E't)} d^3 \vec{r}' \\
&= \frac{1}{|\vec{p}|} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \lambda_m^{\alpha'_\zeta}(\hat{p}', h') \gamma^k{}_{\alpha_\zeta \alpha'_\zeta} p_k \delta^3(\vec{p} - \vec{p}') \\
&= \lambda_m^+(\hat{p}, -\varsigma) \frac{\gamma^k{}_{\alpha_\zeta \alpha'_\zeta} p_k}{|\vec{p}|} \lambda_m(\hat{p}', h') \delta^3(\vec{p} - \vec{p}') \\
&= -\varsigma \lambda_m^+(\hat{p}, -\varsigma) \lambda_m(\hat{p}', h') \delta^3(\vec{p} - \vec{p}') \\
&= -\varsigma \delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

## 5.7 电磁场对易规则小结

以上几个小节的证明正好形成一个逻辑闭环，故有如下性质：

$$\begin{aligned} \text{推论5.7.1.} \quad & \begin{cases} [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = \varsigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -\varsigma), a_{\sigma'}(\vec{p}', -\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -\varsigma), a_{\sigma'}^+(\vec{p}', -\varsigma)] = 0 \end{cases} \Leftrightarrow \begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')] = 0 \end{cases} \\ & \Updownarrow \qquad \qquad \qquad \Updownarrow \\ \text{推论5.7.2.} \quad & \begin{cases} [\Psi_{\alpha_\varsigma}(x), \Psi_{\alpha'_\varsigma}^+(x')] = i\sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha_\varsigma}(x), \Psi_{\beta_\varsigma}(x')] = 0 \\ [\Psi_{\alpha'_\varsigma}^+(x), \Psi_{\beta'_\varsigma}^+(x')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_{\alpha_\varsigma}(\vec{r}, t), \Psi_{\alpha'_\varsigma}^+(\vec{r}', t)] = \varsigma \varepsilon^k \alpha_\varsigma \alpha'_\varsigma \partial_k \delta(\vec{r} - \vec{r}') \\ [\Psi_{\alpha_\varsigma}(\vec{r}, t), \Psi_{\beta_\varsigma}(\vec{r}', t)] = 0 \\ [\Psi_{\alpha'_\varsigma}^+(\vec{r}, t), \Psi_{\beta'_\varsigma}^+(\vec{r}', t)] = 0 \end{cases} \end{aligned}$$

## 5.8 电磁场的对易函数、因果函数和费曼传播子(好像跟波戈留波夫的差一个负号)

定义5.8.1.

$$\begin{cases} [\varphi(x), \varphi(x')] = i\Delta(x - x'), \varphi^+(x) = \varphi(x) \\ \Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) = \Delta^{ret}(x) - \Delta^{adv}(x) \\ \Delta^{(+)}(x) = i[\Delta^{(-)}(x) - \Delta^{(+)}(x)] \\ N_m(1) = \begin{bmatrix} I_3 \\ 0 \end{bmatrix}, \bar{N}_m(1) = [I_3, 0] \end{cases} \begin{cases} \Delta^{(c)}(x) = \theta(t)\Delta^{(+)}(x) - \theta(-t)\Delta^{(-)}(x) \\ \Delta^{ret}(x) = \theta(t)\Delta(x) = \Delta^{(c)}(x) + \Delta^{(-)}(x) \\ \Delta^{adv}(x) = -\theta(-t)\Delta(x) = \Delta^{(c)}(x) - \Delta^{(+)}(x) \\ \Delta_F(x) = \langle T\varphi(x)\varphi(x') \rangle_0 = i\Delta^{(c)}(x - x') \end{cases}$$

定义5.8.2.  $\tilde{\Delta}^{(l)}(x) := \begin{bmatrix} \Delta^{(l)}(x) \\ 0 \end{bmatrix}$ ,  $\Delta^{(c)}(x - x') := i\langle T\varphi(x)\varphi(x') \rangle_0$ ,  $\vec{\partial} := (\partial_x, \partial_y, \partial_z)$

推论5.8.1.

$$\begin{cases} \Delta_{\alpha_\varsigma\alpha'_\varsigma}(\gamma; x) := \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab} \partial_a \partial_b \Delta(x) \\ \Delta_{\alpha_\varsigma\alpha'_\varsigma}^{(+)}(\gamma; x) := \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab} \partial_a \partial_b \Delta^{(+)}(x) \\ \Delta_{\alpha_\varsigma\alpha'_\varsigma}^{(-)}(\gamma; x) := \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab} \partial_a \partial_b \Delta^{(-)}(x) \\ \Delta_{\alpha_\varsigma\alpha'_\varsigma}^{(l)}(\gamma; x) := \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab} \partial_a \partial_b \Delta^{(l)}(x) \end{cases} \begin{cases} \Delta_{\alpha_\varsigma\alpha'_\varsigma}^{(c)}(\gamma; x) := \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab} \partial_a \partial_b \Delta^{(c)}(x) + \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{\pi\pi} \delta^4(x) \\ \Delta_{\alpha_\varsigma\alpha'_\varsigma}^{ret}(\gamma; x) := \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab} \partial_a \partial_b \Delta^{ret}(x) + \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{\pi\pi} \delta^4(x) \\ \Delta_{\alpha_\varsigma\alpha'_\varsigma}^{adv}(\gamma; x) := \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab} \partial_a \partial_b \Delta^{adv}(x) + \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{\pi\pi} \delta^4(x) \\ \Delta_{F\alpha_\varsigma\alpha'_\varsigma}(\gamma; x) := \sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab} \partial_a \partial_b \Delta_F(x) + i\sigma_{\alpha_\varsigma\alpha'_\varsigma}^{\pi\pi} \delta^4(x) = i\Delta_{\alpha_\varsigma\alpha'_\varsigma}^{(c)}(\gamma; x) \\ \Delta_{F\alpha_\varsigma\alpha'_\varsigma}(\gamma; x) = \frac{i\sigma_{\alpha_\varsigma\alpha'_\varsigma}^{ab} p_a p_b}{p^2 - i\varepsilon} + \dots \end{cases}$$

推论5.8.2.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]\Delta(\gamma; x) = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]\Delta^{(+)}(\gamma; x) = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]\Delta^{(-)}(\gamma; x) = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]\Delta^{(l)}(\gamma; x) = 0 \end{cases} \begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]\Delta^{(c)}(\gamma; x) = -\varsigma(\gamma, i\varsigma)_a \delta(t)\Delta(\gamma; x)|_{t=0} \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]\Delta^{ret}(\gamma; x) = -\varsigma(\gamma, i\varsigma)_a \delta(t)\Delta(\gamma; x)|_{t=0} \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]\Delta^{adv}(\gamma; x) = -\varsigma(\gamma, i\varsigma)_a \delta(t)\Delta(\gamma; x)|_{t=0} \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]\Delta_F(\gamma; x) = -i\varsigma(\gamma, i\varsigma)_a \delta(t)\Delta(\gamma; x)|_{t=0} \end{cases}$$

[ $\Updownarrow$ ] [ $\Updownarrow$ ]

推论5.8.3.

$$\begin{cases} (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1)\Delta(\gamma; x) = 0 \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1)\Delta^{(+)}(\gamma; x) = 0 \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1)\Delta^{(-)}(\gamma; x) = 0 \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1)\Delta^{(l)}(\gamma; x) = 0 \end{cases} \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1)\Delta^{(c)}(\gamma; x) = -\varsigma\delta(t)N_m(1)\Delta(\gamma; x)|_{t=0} \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1)\Delta^{ret}(\gamma; x) = -\varsigma\delta(t)N_m(1)\Delta(\gamma; x)|_{t=0} \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1)\Delta^{adv}(\gamma; x) = -\varsigma\delta(t)N_m(1)\Delta(\gamma; x)|_{t=0} \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1)\Delta_F(\gamma; x) = -i\varsigma\delta(t)N_m(1)\Delta(\gamma; x)|_{t=0} \end{cases}$$

[ $\Updownarrow$ ] [ $\Updownarrow$ ]

推论5.8.4.

$$\begin{cases} (\sigma_{-\zeta}, -i\varsigma)_a \partial^a N_m(1) \Delta(\gamma; x) \bar{N}_m(1) = 0 \\ (\sigma_{-\zeta}, -i\varsigma)_a \partial^a N_m(1) \Delta^{(+)}(\gamma; x) \bar{N}_m(1) = 0 \\ (\sigma_{-\zeta}, -i\varsigma)_a \partial^a N_m(1) \Delta^{(-)}(\gamma; x) \bar{N}_m(1) = 0 \\ (\sigma_{-\zeta}, -i\varsigma)_a \partial^a N_m(1) \Delta^{(l)}(\gamma; x) \bar{N}_m(1) = 0 \end{cases} \quad \begin{cases} (\sigma_{-\zeta}, -i\varsigma)_a \partial^a N_m(1) \Delta^{(c)}(\gamma; x) \bar{N}_m(1) = -\varsigma \delta(t) N_m(1) \Delta(\gamma; x)|_{t=0} \bar{N}_m(1) \\ (\sigma_{-\zeta}, -i\varsigma)_a \partial^a N_m(1) \Delta^{ret}(\gamma; x) \bar{N}_m(1) = -\varsigma \delta(t) N_m(1) \Delta(\gamma; x)|_{t=0} \bar{N}_m(1) \\ (\sigma_{-\zeta}, -i\varsigma)_a \partial^a N_m(1) \Delta^{adv}(\gamma; x) \bar{N}_m(1) = -\varsigma \delta(t) N_m(1) \Delta(\gamma; x)|_{t=0} \bar{N}_m(1) \\ (\sigma_{-\zeta}, -i\varsigma)_a \partial^a N_m(1) \Delta_F(\gamma; x) \bar{N}_m(1) = -i\varsigma \delta(t) N_m(1) \Delta(\gamma; x)|_{t=0} \bar{N}_m(1) \end{cases}$$

[↓] [↓]

推论5.8.5.

$$\begin{cases} (\gamma, -i\varsigma)_a \partial^a \Delta(\gamma; x) = 0 \\ (\gamma, -i\varsigma)_a \partial^a \Delta^{(+)}(\gamma; x) = 0 \\ (\gamma, -i\varsigma)_a \partial^a \Delta^{(-)}(\gamma; x) = 0 \\ (\gamma, -i\varsigma)_a \partial^a \Delta^{(l)}(\gamma; x) = 0 \end{cases} \quad \begin{cases} (\gamma, -i\varsigma)_a \partial^a \Delta^{(c)}(\gamma; x) = -\varsigma \delta(t) \Delta(\gamma; x)|_{t=0} \\ (\gamma, -i\varsigma)_a \partial^a \Delta^{ret}(\gamma; x) = -\varsigma \delta(t) \Delta(\gamma; x)|_{t=0} \\ (\gamma, -i\varsigma)_a \partial^a \Delta^{adv}(\gamma; x) = -\varsigma \delta(t) \Delta(\gamma; x)|_{t=0} \\ (\gamma, -i\varsigma)_a \partial^a \Delta_F(\gamma; x) = -i\varsigma \delta(t) \Delta(\gamma; x)|_{t=0} \end{cases}$$

## 5.9 电磁场能量动量算符的提取

推论5.9.1.  $H = \int_{\vec{p} \neq 0} |\vec{p}| [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) + a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)] d^3 \vec{p} = \int \Psi^+(\vec{r}, t) \Psi(\vec{r}, t) d^3 \vec{r}$

证明:  $H = \int_{\vec{p} \neq 0} |\vec{p}| [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) + a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)] d^3 \vec{p}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} [\lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{i\varsigma \vec{p} \cdot \vec{x}'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{-i\varsigma \vec{p} \cdot \vec{x}} \\ &+ \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{-i\varsigma \vec{p} \cdot \vec{x}'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{i\varsigma \vec{p} \cdot \vec{x}}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} p^a p^b \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} (\hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta} + \varsigma \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k - \delta^{\alpha'_\zeta \alpha_\zeta}) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} (\delta^{\alpha'_\zeta \alpha_\zeta} - \hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta}) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \int [\delta^{\alpha'_\zeta \alpha_\zeta} - \frac{\partial^{\alpha'_\zeta \alpha_\zeta}}{\nabla^2}] \delta^3(\vec{r} - \vec{r}') \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{r} d^3 \vec{r}' \\ &= \int \Psi_{\alpha'_\zeta}^+(\vec{r}, t) [\delta^{\alpha'_\zeta \alpha_\zeta} - \frac{\partial^{\alpha'_\zeta \alpha_\zeta}}{\nabla^2}] \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{r} \text{ (利用了运动方程: } \nabla \cdot \Psi(\vec{r}, t) = 0) \\ &= \int \Psi^+(\vec{r}, t) \Psi(\vec{r}, t) d^3 \vec{r} \end{aligned}$$

□

推论5.9.2.  $\vec{P} = \int_{\vec{p} \neq 0} \vec{p} [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) + a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)] d^3 \vec{p} = -\varsigma \int \Psi^+(\vec{r}, t) \gamma \Psi(\vec{r}, t) d^3 \vec{r}$

证明:  $\vec{P} = \int_{\vec{p} \neq 0} \vec{p} [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) + a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)] d^3 \vec{p}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \vec{p} [\lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{i\varsigma \vec{p} \cdot \vec{x}'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{-i\varsigma \vec{p} \cdot \vec{x}} \\ &+ \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{-i\varsigma \vec{p} \cdot \vec{x}'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{i\varsigma \vec{p} \cdot \vec{x}}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \vec{p} \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \hat{p} \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} p^a p^b \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \hat{p} (\hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta} + \varsigma \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k - \delta^{\alpha'_\zeta \alpha_\zeta}) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= -\frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \hat{p} \varsigma \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \varsigma \nabla(\gamma \cdot \nabla)^{\alpha'_\zeta \alpha_\zeta} \frac{1}{|\vec{p}|^2} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \int \Psi_{\alpha'_\zeta}^+(\vec{r}, t) \Psi_{\alpha_\zeta}(\vec{r}, t) \varsigma \frac{\nabla(\gamma \cdot \nabla)^{\alpha'_\zeta \alpha_\zeta}}{-\nabla^2} \delta^3(\vec{r} - \vec{r}') d^3 \vec{r} d^3 \vec{r}' \end{aligned}$$



$$= \varsigma \int \Psi^+(\vec{r}, t) \frac{\nabla(\gamma \cdot \nabla)}{-\nabla^2} \Psi(\vec{r}, t) d^3 \vec{r}$$

$$= -\varsigma \int \Psi^+(\vec{r}, t) \gamma \Psi(\vec{r}, t) d^3 \vec{r}$$

□

推论5.9.3.  $P^a = -\varsigma \int \Psi^+(\vec{r}, t) (\gamma, -i\varsigma)^a \Psi(\vec{r}, t) d^3 \vec{r}$

## 5.10 电磁场类电荷算符的提取

推论5.10.1.  $Q = \varsigma \int_{\vec{p} \neq 0} [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)] d^3 \vec{p} = i\varsigma \int \Psi^+(\vec{r}, t) \frac{\gamma \cdot \nabla}{-\nabla^2} \Psi(\vec{r}, t) d^3 \vec{r}$

证明:  $Q = \varsigma \int_{\vec{p} \neq 0} [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)] d^3 \vec{p}$

$$= \varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} [\lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{i\varsigma p \cdot x'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{-i\varsigma p \cdot x}$$

$$- \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{-i\varsigma p \cdot x'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{i\varsigma p \cdot x}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} - e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \varsigma \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} p^a p^b \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} - e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \varsigma \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} (\hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta} + \varsigma \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k - \delta^{\alpha'_\zeta \alpha_\zeta}) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} - e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \varsigma \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \varsigma \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} - e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k \Psi_{\alpha_\zeta}(\vec{r}, t) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= -i\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \frac{(\gamma \cdot \nabla)^{\alpha'_\zeta \alpha_\zeta}}{-\nabla^2} \delta^3(\vec{r} - \vec{r}') d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= i\varsigma \int \Psi^+(\vec{r}, t) \frac{\gamma \cdot \nabla}{-\nabla^2} \Psi(\vec{r}, t) d^3 \vec{r} = \int \Psi^+(\vec{r}, t) \frac{i\partial_t}{-\nabla^2} \Psi(\vec{r}, t) d^3 \vec{r}$$

□

## 5.11 电磁场粒子数算符的提取

推论5.11.1.  $N = \int_{\vec{p} \neq 0} [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) + a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)] d^3 \vec{p} = \int \Psi^+(\vec{r}, t) \frac{1}{\sqrt{-\nabla^2}} \Psi(\vec{r}, t) d^3 \vec{r}$

证明:  $N = \int_{\vec{p} \neq 0} [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) + a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)] d^3 \vec{p}$

$$= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} [\lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{i\varsigma p \cdot x'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{-i\varsigma p \cdot x}$$

$$+ \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{-i\varsigma p \cdot x'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{i\varsigma p \cdot x}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} p^a p^b \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} (\hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta} + \varsigma \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k - \delta^{\alpha'_\zeta \alpha_\zeta}) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} (\delta^{\alpha'_\zeta \alpha_\zeta} - \hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta}) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \int \frac{1}{\sqrt{-\nabla^2}} [\delta^{\alpha'_\zeta \alpha_\zeta} - \frac{\partial^{\alpha'_\zeta} \partial^{\alpha_\zeta}}{\nabla^2}] \delta^3(\vec{r} - \vec{r}') \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{r} d^3 \vec{r}'$$

$$= \int \Psi_{\alpha'_\zeta}^+(\vec{r}, t) \frac{1}{\sqrt{-\nabla^2}} [\delta^{\alpha'_\zeta \alpha_\zeta} - \frac{\partial^{\alpha'_\zeta} \partial^{\alpha_\zeta}}{\nabla^2}] \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{r} \text{ (利用了运动方程: } \nabla \cdot \Psi(\vec{r}, t) = 0)$$

$$= \int \Psi^+(\vec{r}, t) \frac{1}{\sqrt{-\nabla^2}} \Psi(\vec{r}, t) d^3 \vec{r}$$

□

## 5.12 电磁场能量动量正规化算符

推论5.12.1.  $H_0 = \int_{\vec{p} \neq 0} |\vec{p}| [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) + a_2^+(\vec{p}, -\varsigma) a_2(\vec{p}, -\varsigma)] d^3 \vec{p}$

$$= \int \Psi^+(\vec{r}, t) \Psi(\vec{r}, t) d^3 \vec{r} - \frac{i}{2} \int [\Psi^+(\vec{r}, t) (\frac{\gamma \cdot \nabla}{\sqrt{-\nabla^2}}) \Psi(\vec{r}, t) + \Psi^T(\vec{r}, t) (\frac{\gamma \cdot \nabla}{\sqrt{-\nabla^2}}) \Psi^*(\vec{r}, t)] d^3 \vec{r}$$

证明:  $H_0 = \int_{\vec{p} \neq 0} |\vec{p}| [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) + a_2^+(\vec{p}, -\varsigma) a_2(\vec{p}, -\varsigma)] d^3 \vec{p}$

$$= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} |\vec{p}| [\lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{i\varsigma p \cdot x'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{-i\varsigma p \cdot x}$$

$$\begin{aligned}
& + \lambda_m^{\alpha\prime\alpha\prime}(\hat{p}, -\varsigma) \Psi_{\alpha\prime}(\vec{r}, t) e^{i\varsigma \vec{p} \cdot \vec{x}} \lambda_m^{\alpha\prime\alpha\prime}(\hat{p}, -\varsigma) \Psi_{\alpha\prime}^+(\vec{r}', t) e^{-i\varsigma \vec{p} \cdot \vec{x}'} ] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \lambda_m^{\alpha\prime\alpha\prime}(\hat{p}, -\varsigma) \lambda_m^{\alpha\prime\alpha\prime}(\hat{p}, -\varsigma) [\Psi_{\alpha\prime}^+(\vec{r}', t) \Psi_{\alpha\prime}(\vec{r}, t) e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + \Psi_{\alpha\prime}(\vec{r}, t) \Psi_{\alpha\prime}^+(\vec{r}', t) e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} ] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = \frac{-1}{2(2\pi)^3} \int_{\vec{p} \neq 0} (\hat{p}^{\alpha\prime\alpha\prime} \hat{p}^{\alpha\prime\alpha\prime} + \varsigma \gamma_k^{\alpha\prime\alpha\prime} \hat{p}^k - \delta^{\alpha\prime\alpha\prime}) [\Psi_{\alpha\prime}^+(\vec{r}', t) \Psi_{\alpha\prime}(\vec{r}, t) e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + \Psi_{\alpha\prime}(\vec{r}, t) \Psi_{\alpha\prime}^+(\vec{r}', t) e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} ] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = \int \Psi^+(\vec{r}, t) \Psi(\vec{r}, t) d^3 \vec{r} \\
& + \frac{-1}{2(2\pi)^3} \int_{\vec{p} \neq 0} \varsigma \gamma_k^{\alpha\prime\alpha\prime} \hat{p}^k [\Psi_{\alpha\prime}^+(\vec{r}', t) \Psi_{\alpha\prime}(\vec{r}, t) e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + \Psi_{\alpha\prime}(\vec{r}, t) \Psi_{\alpha\prime}^+(\vec{r}', t) e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} ] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = \int \Psi^+(\vec{r}, t) \Psi(\vec{r}, t) d^3 \vec{r} + \frac{-1}{2} \int i [\Psi_{\alpha\prime}^+(\vec{r}, t) (\frac{\gamma \cdot \nabla}{\sqrt{-\nabla^2}})^{\alpha\prime\alpha\prime} \Psi_{\alpha\prime}(\vec{r}, t) - i [(\frac{\gamma \cdot \nabla}{\sqrt{-\nabla^2}})^{\alpha\prime\alpha\prime} \Psi_{\alpha\prime}(\vec{r}, t) \Psi_{\alpha\prime}^+(\vec{r}, t) d^3 \vec{r} \\
& = \int \Psi^+(\vec{r}, t) \Psi(\vec{r}, t) d^3 \vec{r} - \frac{i}{2} \int [\Psi_{\alpha\prime}^+(\vec{r}, t) (\frac{\gamma \cdot \nabla}{\sqrt{-\nabla^2}})^{\alpha\prime\alpha\prime} \Psi_{\alpha\prime}(\vec{r}, t) + [\Psi_{\alpha\prime}(\vec{r}, t) (\frac{\gamma \cdot \nabla}{\sqrt{-\nabla^2}})^{\alpha\prime\alpha\prime} \Psi_{\alpha\prime}^+(\vec{r}, t) d^3 \vec{r} \\
& = \int \Psi^+(\vec{r}, t) \Psi(\vec{r}, t) d^3 \vec{r} - \frac{i}{2} \int [\Psi^+(\vec{r}, t) (\frac{\gamma \cdot \nabla}{\sqrt{-\nabla^2}}) \Psi(\vec{r}, t) + \Psi^T(\vec{r}, t) (\frac{\gamma \cdot \nabla}{\sqrt{-\nabla^2}}) \Psi^*(\vec{r}, t)] d^3 \vec{r} \\
& = H - H_g
\end{aligned}$$

□

推论5.12.2.  $H = H_0 + H_g$

$$H_g = \frac{i}{2} \int [\Psi^+(\vec{r}, t) (\frac{\gamma \cdot \nabla}{\sqrt{-\nabla^2}}) \Psi(\vec{r}, t) + \Psi^T(\vec{r}, t) (\frac{\gamma \cdot \nabla}{\sqrt{-\nabla^2}}) \Psi^*(\vec{r}, t)] d^3 \vec{r} = \int_{\vec{p} \neq 0} |\vec{p}| [a_2(\vec{p}, -\varsigma), a_2^\dagger(\vec{p}, -\varsigma)] d^3 \vec{p}$$

## 5.13 电磁场角动量算符的提取

### 5.13.1 电磁场空间轨道角动量算符的提取

引理5.13.1.  $\lambda_m^+(\hat{p}, -\varsigma) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma) = ?0$ ,  $\lambda_m^+(-\hat{p}, -\varsigma) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma) = ?0$

$$\begin{aligned}
& \text{推论5.13.1. } M_{ij}(1, \varsigma) = -\varsigma \int \Psi^+(\vec{r}, t) (r_i \gamma_j - r_j \gamma_i) \Psi(\vec{r}, t) d^3 \vec{r} \\
& = -i\varsigma \int_{\vec{p} \neq 0} a_1^+(\vec{p}, -\varsigma) (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_2^\dagger(\vec{p}, -\varsigma) d^3 \vec{p} \\
& + i \int_{\vec{p} \neq 0} |\vec{p}| [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) a_2^\dagger(\vec{p}, -\varsigma)] \lambda_m^+(\hat{p}, -\varsigma) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma) d^3 \vec{p}
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } M_{ij}(1, \varsigma) = -\varsigma \int \Psi^+(\vec{r}, t) (r_i \gamma_j - r_j \gamma_i) \Psi(\vec{r}, t) d^3 \vec{r} \\
& = \frac{-\varsigma}{(2\pi)^3} \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_1^+(\vec{p}', -\varsigma) e^{-i\varsigma \vec{p}' \cdot \vec{r}} e^{i\varsigma |\vec{p}'| t} \lambda_m^+(\hat{p}', -\varsigma)] (r_i \gamma_j - r_j \gamma_i) [\lambda_m(\hat{p}, -\varsigma) \sqrt{|\vec{p}|} a_1(\vec{p}, -\varsigma) e^{i\varsigma \vec{p} \cdot \vec{r}} e^{-i\varsigma |\vec{p}| t}] d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \\
& - \frac{\varsigma}{(2\pi)^3} \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_2(\vec{p}', -\varsigma) e^{i\varsigma \vec{p}' \cdot \vec{r}} e^{-i\varsigma |\vec{p}'| t} \lambda_m^+(\hat{p}', -\varsigma)] (r_i \gamma_j - r_j \gamma_i) [\lambda_m(\hat{p}, -\varsigma) \sqrt{|\vec{p}|} a_2^\dagger(\vec{p}, -\varsigma) e^{-i\varsigma \vec{p} \cdot \vec{r}} e^{i\varsigma |\vec{p}| t}] d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \\
& = \frac{-\varsigma}{(2\pi)^3} \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_1^+(\vec{p}', -\varsigma) e^{i\varsigma |\vec{p}'| t} \lambda_m^+(\hat{p}', -\varsigma)] [-i\varsigma (\tilde{\partial}_i \gamma_j - \tilde{\partial}_j \gamma_i) e^{i\varsigma (\vec{p} - \vec{p}') \cdot \vec{r}}] [\lambda_m(\hat{p}, -\varsigma) \sqrt{|\vec{p}|} a_1(\vec{p}, -\varsigma) e^{-i\varsigma |\vec{p}| t}] d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \\
& - \frac{\varsigma}{(2\pi)^3} \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_2(\vec{p}', -\varsigma) e^{-i\varsigma |\vec{p}'| t} \lambda_m^+(\hat{p}', -\varsigma)] [i\varsigma (\tilde{\partial}_i \gamma_j - \tilde{\partial}_j \gamma_i) e^{-i\varsigma (\vec{p} - \vec{p}') \cdot \vec{r}}] [\lambda_m(\hat{p}, -\varsigma) \sqrt{|\vec{p}|} a_2^\dagger(\vec{p}, -\varsigma) e^{i\varsigma |\vec{p}| t}] d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \\
& = i \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_1^+(\vec{p}', -\varsigma) e^{i\varsigma |\vec{p}'| t} \lambda_m^+(\hat{p}', -\varsigma)] [(\tilde{\partial}_i \gamma_j - \tilde{\partial}_j \gamma_i) \delta^3(\vec{p} - \vec{p}')] [\lambda_m(\hat{p}, -\varsigma) \sqrt{|\vec{p}|} a_1(\vec{p}, -\varsigma) e^{-i\varsigma |\vec{p}| t}] d^3 \vec{p} d^3 \vec{p}' \\
& - i \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_2(\vec{p}', -\varsigma) e^{-i\varsigma |\vec{p}'| t} \lambda_m^+(\hat{p}', -\varsigma)] [(\tilde{\partial}_i \gamma_j - \tilde{\partial}_j \gamma_i) \delta^3(\vec{p} - \vec{p}')] [\lambda_m(\hat{p}, -\varsigma) \sqrt{|\vec{p}|} a_2^\dagger(\vec{p}, -\varsigma) e^{i\varsigma |\vec{p}| t}] d^3 \vec{p} d^3 \vec{p}' \\
& = -i \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_1^+(\vec{p}, -\varsigma) e^{i\varsigma |\vec{p}| t} \lambda_m^+(\hat{p}, -\varsigma)] \{(\tilde{\partial}_i \gamma_j - \tilde{\partial}_j \gamma_i) [\lambda_m(\hat{p}, -\varsigma) \sqrt{|\vec{p}|} a_1(\vec{p}, -\varsigma) e^{-i\varsigma |\vec{p}| t}]\} d^3 \vec{p} \\
& + i \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_2(\vec{p}, -\varsigma) e^{-i\varsigma |\vec{p}| t} \lambda_m^+(\hat{p}, -\varsigma)] \{(\tilde{\partial}_i \gamma_j - \tilde{\partial}_j \gamma_i) [\lambda_m(\hat{p}, -\varsigma) \sqrt{|\vec{p}|} a_2^\dagger(\vec{p}, -\varsigma) e^{i\varsigma |\vec{p}| t}]\} d^3 \vec{p} \\
& = -i \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_1^+(\vec{p}, -\varsigma) e^{i\varsigma |\vec{p}| t} \lambda_m^+(\hat{p}, -\varsigma)] \{(\gamma_j \tilde{\partial}_i - \gamma_i \tilde{\partial}_j) [\lambda_m(\hat{p}, -\varsigma) \sqrt{|\vec{p}|} a_1(\vec{p}, -\varsigma) e^{-i\varsigma |\vec{p}| t}]\} d^3 \vec{p} \\
& + i \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_2(\vec{p}, -\varsigma) e^{-i\varsigma |\vec{p}| t} \lambda_m^+(\hat{p}, -\varsigma)] \{(\gamma_j \tilde{\partial}_i - \gamma_i \tilde{\partial}_j) [\lambda_m(\hat{p}, -\varsigma) \sqrt{|\vec{p}|} a_2^\dagger(\vec{p}, -\varsigma) e^{i\varsigma |\vec{p}| t}]\} d^3 \vec{p} \\
& = i\varsigma \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_1^+(\vec{p}, -\varsigma) e^{i\varsigma |\vec{p}| t} \lambda_m^+(\hat{p}, -\varsigma)] \{(\hat{p}_j \tilde{\partial}_i - \hat{p}_i \tilde{\partial}_j) [\sqrt{|\vec{p}|} a_1(\vec{p}, -\varsigma) e^{-i\varsigma |\vec{p}| t}]\} d^3 \vec{p} \\
& + i \int_{\vec{p} \neq 0} [|\vec{p}| a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) \lambda_m^+(\hat{p}, -\varsigma)] \{(\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma)\} d^3 \vec{p} \\
& - i\varsigma \int_{\vec{p} \neq 0} [\sqrt{|\vec{p}|} a_2(\vec{p}, -\varsigma) e^{-i\varsigma |\vec{p}| t} \lambda_m^+(\hat{p}, -\varsigma)] \{(\hat{p}_j \tilde{\partial}_i - \hat{p}_i \tilde{\partial}_j) [\sqrt{|\vec{p}|} a_2^\dagger(\vec{p}, -\varsigma) e^{i\varsigma |\vec{p}| t}]\} d^3 \vec{p} \\
& - i \int_{\vec{p} \neq 0} [|\vec{p}| a_2(\vec{p}, -\varsigma) a_2^\dagger(\vec{p}, -\varsigma) \lambda_m^+(\hat{p}, -\varsigma)] \{(\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma)\} d^3 \vec{p} \\
& = -i\varsigma \int_{\vec{p} \neq 0} a_1^+(\vec{p}, -\varsigma) (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_1(\vec{p}, -\varsigma) d^3 \vec{p} + i \int_{\vec{p} \neq 0} |\vec{p}| a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) \lambda_m^+(\hat{p}, -\varsigma) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma) d^3 \vec{p}
\end{aligned}$$

$$\begin{aligned}
& + i\varsigma \int_{\vec{p} \neq 0} a_2(\vec{p}, -\varsigma)(p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_2^+(\vec{p}, -\varsigma) d^3 \vec{p} - i \int_{\vec{p} \neq 0} |\vec{p}| a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma) \lambda_m^+(\hat{p}, -\varsigma) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma) d^3 \vec{p} \\
& = -i\varsigma \int_{\vec{p} \neq 0} a_1^+(\vec{p}, -\varsigma)(p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma)(p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_2^+(\vec{p}, -\varsigma) d^3 \vec{p} \\
& + i \int_{\vec{p} \neq 0} |\vec{p}| [a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)] \lambda_m^+(\hat{p}, -\varsigma) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma) d^3 \vec{p} \quad \square
\end{aligned}$$

尝试反推：

$$\begin{aligned}
\text{证明: } M_{ij}(1, \varsigma) & = \varsigma \int_{\vec{p} \neq 0} \{a_1^+(\vec{p}, -\varsigma) \tilde{M}_{ij}(1, \varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) \tilde{M}_{ij}(1, \varsigma) a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} \\
& = -i\varsigma \int_{\vec{p} \neq 0} \{a_1^+(\vec{p}, -\varsigma)(p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma)(p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} \\
& = -i\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{\sqrt{|\vec{p}|}} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) e^{i\varsigma p \cdot x'} (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) \left[ \frac{1}{\sqrt{|\vec{p}|}} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) e^{-i\varsigma p \cdot x} \right] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = -i\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{\sqrt{|\vec{p}|}} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) e^{i\varsigma p \cdot x'} e^{-i\varsigma p \cdot x} (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) \left[ \frac{1}{\sqrt{|\vec{p}|}} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \right] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& - i\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{\sqrt{|\vec{p}|}} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) e^{i\varsigma p \cdot x'} (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) \left[ \frac{1}{\sqrt{|\vec{p}|}} e^{-i\varsigma p \cdot x} \right] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = -i\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& - i\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} [-i\varsigma(p_i r_j - p_j r_i)] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = \frac{i\varsigma}{2(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) (\sigma)_{\alpha'_\zeta \alpha_\zeta}^{ab} \hat{p}_a \hat{p}_b [-i\varsigma(p_i r_j - p_j r_i)] e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = \frac{i\varsigma}{2(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) (\sigma)_{\alpha'_\zeta \alpha_\zeta}^{ab} \hat{p}_a \hat{p}_b [-i\varsigma(p_i r_j - p_j r_i)] [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = \frac{i\varsigma}{2(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) (\hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta} + \varsigma \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k - \delta^{\alpha'_\zeta \alpha_\zeta}) [-i\varsigma(p_i r_j - p_j r_i)] [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = -i\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{1}{|\vec{p}|} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k [-i\varsigma(r_i p_j - r_j p_i)] e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = \varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \frac{(\gamma \cdot \nabla)^{\alpha'_\zeta \alpha_\zeta}}{-\nabla^2} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = \varsigma \int \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \frac{(\gamma \cdot \nabla)^{\alpha'_\zeta \alpha_\zeta}}{-\nabla^2} \delta^3(\vec{r} - \vec{r}') d^3 \vec{r} d^3 \vec{r}' \\
& = \varsigma \int \Psi^+(\vec{r}, t) \frac{\gamma \cdot \nabla}{-\nabla^2} (\partial_j r_i - \partial_i r_j) \Psi(\vec{r}, t) d^3 \vec{r} \\
& = -\varsigma \int \Psi^+(\vec{r}, t) (r_i \gamma_j - r_j \gamma_i) \Psi(\vec{r}, t) d^3 \vec{r} \quad \square
\end{aligned}$$

### 5.13.2 电磁场时间轨道角动量算符的提取

$$\begin{aligned}
\text{推论5.13.2. } L_{i\pi}(1, \varsigma) & = \varsigma \int_{\vec{p} \neq 0} \{a_1^+(\vec{p}, -\varsigma) \tilde{M}_{i\pi}(1, \varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) \tilde{M}_{i\pi}(1, \varsigma) a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} \\
& = \int \Psi^+(\vec{r}, t) (i r_i + i \varsigma t \gamma_i) \Psi(\vec{r}, t) d^3 \vec{r}
\end{aligned}$$

$$\begin{aligned}
\text{证明: } L_{i\pi}(1, \varsigma) & = \varsigma \int_{\vec{p} \neq 0} \{a_1^+(\vec{p}, -\varsigma) \tilde{M}_{i\pi}(1, \varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) \tilde{M}_{i\pi}(1, \varsigma) a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} \\
& = -i\varsigma \int_{\vec{p} \neq 0} \{a_1^+(\vec{p}, -\varsigma)(p_i \tilde{\partial}_\pi - p_\pi \tilde{\partial}_i) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma)(p_i \tilde{\partial}_\pi - p_\pi \tilde{\partial}_i) a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} \\
& = -\varsigma \int_{\vec{p} \neq 0} \hat{p}_i \{a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} - \varsigma \int_{\vec{p} \neq 0} |\vec{p}| \{a_1^+(\vec{p}, -\varsigma) \tilde{\partial}_i a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) \tilde{\partial}_i a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} \\
& = -\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \sqrt{|\vec{p}|} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) e^{i\varsigma p \cdot x'} \tilde{\partial}_i \left[ \frac{1}{\sqrt{|\vec{p}|}} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) (e^{-i\varsigma p \cdot x}) \right] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& - \varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \sqrt{|\vec{p}|} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) e^{-i\varsigma p \cdot x'} \tilde{\partial}_i \left[ \frac{1}{\sqrt{|\vec{p}|}} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) (-e^{i\varsigma p \cdot x}) \right] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' - \hat{s}_i(1, \varsigma) \\
& = -\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) e^{i\varsigma p \cdot x'} \tilde{\partial}_i [\lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) (e^{-i\varsigma p \cdot x})] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& - \varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) e^{-i\varsigma p \cdot x'} \tilde{\partial}_i [\lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) (-e^{i\varsigma p \cdot x})] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& = -\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) e^{i\varsigma p \cdot x'} \tilde{\partial}_i (e^{-i\varsigma p \cdot x}) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
& - \varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) e^{-i\varsigma p \cdot x'} \tilde{\partial}_i (-e^{i\varsigma p \cdot x}) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'
\end{aligned}$$

$$\begin{aligned}
&= -\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] \tilde{\partial}_i(-i\varsigma \vec{p} \cdot \vec{x}) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= i \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] (r_i - t\hat{p}_i) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= -i/2 \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) (\sigma)_{\alpha'_\zeta \alpha_\zeta}^{ab} \hat{p}_a \hat{p}_b [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] (r_i - t\hat{p}_i) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= -i/2 \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) (\hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta} + \varsigma \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k - \delta^{\alpha'_\zeta \alpha_\zeta}) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} + e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] (r_i - t\hat{p}_i) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= -i \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) r_i \Psi_{\alpha_\zeta}(\vec{r}, t) (\hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta} - \delta^{\alpha'_\zeta \alpha_\zeta}) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&+ i\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) t \Psi_{\alpha_\zeta}(\vec{r}, t) (\gamma \cdot \hat{p})^{\alpha'_\zeta \alpha_\zeta} \hat{p}_i e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= -i \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) r_i \Psi_{\alpha_\zeta}(\vec{r}, t) \left( \frac{-\partial^{\alpha'_\zeta} \partial^{\alpha_\zeta}}{-\nabla^2} - \delta^{\alpha'_\zeta \alpha_\zeta} \right) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&+ i\varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) t \Psi_{\alpha_\zeta}(\vec{r}, t) \frac{(-i\gamma \cdot \nabla)^{\alpha'_\zeta \alpha_\zeta} (-i\partial_i)}{-\nabla^2} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= -i \int \Psi_{\alpha'_\zeta}^+(\vec{r}', t) r_i \Psi_{\alpha_\zeta}(\vec{r}, t) \left( \frac{\partial^{\alpha'_\zeta} \partial^{\alpha_\zeta}}{\nabla^2} - \delta^{\alpha'_\zeta \alpha_\zeta} \right) \delta^3(\vec{r} - \vec{r}') d^3 \vec{r} d^3 \vec{r}' \\
&+ i\varsigma \int \Psi_{\alpha'_\zeta}^+(\vec{r}', t) t \Psi_{\alpha_\zeta}(\vec{r}, t) \frac{(\gamma \cdot \nabla)^{\alpha'_\zeta \alpha_\zeta} (\partial_i)}{\nabla^2} \delta^3(\vec{r} - \vec{r}') d^3 \vec{r} d^3 \vec{r}' \\
&= -i \int \Psi_{\alpha'_\zeta}^+(\vec{r}, t) \left( \frac{\partial^{\alpha'_\zeta} \partial^{\alpha_\zeta}}{\nabla^2} - \delta^{\alpha'_\zeta \alpha_\zeta} \right) r_i \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{r} + i\varsigma \int \Psi_{\alpha'_\zeta}^+(\vec{r}, t) \frac{(\gamma \cdot \nabla)^{\alpha'_\zeta \alpha_\zeta} (\partial_i)}{\nabla^2} t \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{r} \\
&= i \int \Psi^+(\vec{r}, t) r_i \Psi(\vec{r}, t) d^3 \vec{r} + i\varsigma \int \Psi^+(\vec{r}, t) t \gamma_i \Psi(\vec{r}, t) d^3 \vec{r} \\
&= \int \Psi^+(\vec{r}, t) (ir_i + i\varsigma t \gamma_i) \Psi(\vec{r}, t) d^3 \vec{r}
\end{aligned}$$

□

### 5.13.3 电磁场轨道角动量算符的提取

$$\begin{aligned}
\text{推论5.13.3. } M_{ab}(1, \varsigma) &= \varsigma \int_{\vec{p} \neq 0} \{a_1^+(\vec{p}, -\varsigma) \tilde{M}_{ab}(1, \varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) \tilde{M}_{ab}(1, \varsigma) a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} \\
&= \int \Psi^+(\vec{r}, t) [r_a \gamma_b(1, \varsigma) - r_b \gamma_a(1, \varsigma)] \Psi(\vec{r}, t) d^3 \vec{r}
\end{aligned}$$

### 5.13.4 电磁场自旋角动量算符的提取

$$\text{推论5.13.4. } \hat{s}(1, \varsigma) = \varsigma \int_{\vec{p} \neq 0} \hat{p} \{a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} = \int \Psi^+(\vec{r}, t) \frac{-i\nabla}{-\nabla^2} \Psi(\vec{r}, t) d^3 \vec{r}$$

$$\begin{aligned}
\text{证明: } \hat{s}(1, \varsigma) &= \varsigma \int_{\vec{p} \neq 0} \hat{p} \{a_1^+(\vec{p}, -\varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} \\
&= \varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{\hat{p}}{|\vec{p}|} [\lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{i\varsigma \vec{p} \cdot \vec{x}'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{-i\varsigma \vec{p} \cdot \vec{x}} \\
&- \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) e^{-i\varsigma \vec{p} \cdot \vec{x}'} \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha_\zeta}(\vec{r}, t) e^{i\varsigma \vec{p} \cdot \vec{x}}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= \varsigma \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{\hat{p}}{|\vec{p}|} \lambda_m^{\alpha'_\zeta}(\hat{p}, -\varsigma) \lambda_m^{+\alpha_\zeta}(\hat{p}, -\varsigma) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} - e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= \varsigma \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \frac{\hat{p}}{|\vec{p}|} \sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} p^a p^b \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} - e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= \varsigma \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \frac{\hat{p}}{|\vec{p}|} (\hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta} + \varsigma \gamma_k^{\alpha'_\zeta \alpha_\zeta} \hat{p}^k - \delta^{\alpha'_\zeta \alpha_\zeta}) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} - e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= \varsigma \frac{1}{(2\pi)^3} \frac{-1}{2} \int_{\vec{p} \neq 0} \frac{\hat{p}}{|\vec{p}|} (\hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta} - \delta^{\alpha'_\zeta \alpha_\zeta}) \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) [e^{-i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')} - e^{i\varsigma \vec{p} \cdot (\vec{r} - \vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{\hat{p}}{|\vec{p}|} (\hat{p}^{\alpha'_\zeta} \hat{p}^{\alpha_\zeta} - \delta^{\alpha'_\zeta \alpha_\zeta}) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \frac{-i\nabla}{-\nabla^2} \left( \frac{-\partial^{\alpha'_\zeta} \partial^{\alpha_\zeta}}{-\nabla^2} - \delta^{\alpha'_\zeta \alpha_\zeta} \right) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= i \int \frac{\nabla}{\nabla^2} \left( \frac{\partial^{\alpha'_\zeta} \partial^{\alpha_\zeta}}{\nabla^2} - \delta^{\alpha'_\zeta \alpha_\zeta} \right) \delta^3(\vec{r} - \vec{r}') \Psi_{\alpha'_\zeta}^+(\vec{r}', t) \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{r} d^3 \vec{r}' \\
&= -i \int \Psi_{\alpha'_\zeta}^+(\vec{r}, t) \frac{\nabla}{\nabla^2} \left( \frac{\partial^{\alpha'_\zeta} \partial^{\alpha_\zeta}}{\nabla^2} - \delta^{\alpha'_\zeta \alpha_\zeta} \right) \Psi_{\alpha_\zeta}(\vec{r}, t) d^3 \vec{r} \\
&= \int \Psi^+(\vec{r}, t) \frac{-i\nabla}{-\nabla^2} \Psi(\vec{r}, t) d^3 \vec{r}
\end{aligned}$$

□

## 5.14 电磁场角动量算符的小结

$$\begin{aligned}
\text{推论5.14.1. } L_{ab}(1, \varsigma) &= \varsigma \int_{\vec{p} \neq 0} \{a_1^+(\vec{p}, -\varsigma) \tilde{M}_{ab}(1, \varsigma) a_1(\vec{p}, -\varsigma) - a_2(\vec{p}, -\varsigma) \tilde{M}_{ab}(1, \varsigma) a_2^+(\vec{p}, -\varsigma)\} d^3 \vec{p} \\
&= \int \Psi^+(\vec{r}, t) [r_a \gamma_b(1, \varsigma) - r_b \gamma_a(1, \varsigma) - \frac{\sigma_{ab}^{\alpha'_\zeta \alpha_\zeta} \partial_{\alpha_\zeta}}{-\nabla^2}] \Psi(\vec{r}, t) d^3 \vec{r}
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } [\vec{r} \times (\vec{E} \times \vec{B}), \vec{r}' \times (\vec{E}' \times \vec{B}')] \\
&= [E_i(\vec{r} \cdot \vec{B}) - (\vec{r} \cdot \vec{E})B_i, E'_j(\vec{r}' \cdot \vec{B}') - (\vec{r}' \cdot \vec{E}')B'_j] \\
&= [E_i(\vec{r} \cdot \vec{B}), E'_j(\vec{r}' \cdot \vec{B}')] - [E_i(\vec{r} \cdot \vec{B}), (\vec{r}' \cdot \vec{E}')B'_j] - [(\vec{r}' \cdot \vec{E}')B_i, E'_j(\vec{r}' \cdot \vec{B}')] + [(\vec{r}' \cdot \vec{E}')B_i, (\vec{r}' \cdot \vec{E}')B'_j] \\
&= E_i[(\vec{r} \cdot \vec{B}), E'_j(\vec{r}' \cdot \vec{B}')] + E'_j[E_i, (\vec{r}' \cdot \vec{B}')](\vec{r} \cdot \vec{B}) \\
&\quad - E_i[(\vec{r} \cdot \vec{B}), (\vec{r}' \cdot \vec{E}')B'_j] - (\vec{r}' \cdot \vec{E}')[E_i, B'_j](\vec{r} \cdot \vec{B}) \\
&\quad - (\vec{r}' \cdot \vec{E}')[B_i, E'_j](\vec{r}' \cdot \vec{B}') - E'_j[(\vec{r}' \cdot \vec{E}'), (\vec{r}' \cdot \vec{B}')]B_i \\
&\quad + (\vec{r}' \cdot \vec{E}')[B_i, (\vec{r}' \cdot \vec{E}')B'_j] + (\vec{r}' \cdot \vec{E}')[(\vec{r}' \cdot \vec{E}'), B'_j]B_i \\
&= r^k r'^l \{ E_i [B_k, E'_j] B'_l + E'_j [E_i, B'_l] B_k \\
&\quad - E_i [B_k, E'_l] B'_j - E'_l [E_i, B'_j] B_k \\
&\quad - E_k [B_i, E'_j] B'_l - E'_j [E_k, B'_l] B_i \\
&\quad + E_k [B_i, E'_l] B'_j + E'_l [E_k, B'_j] B_i \} \\
&= r^k r'^l \\
&\quad \{ E_i B'_l [i\varepsilon_{kj}{}^m \partial_m \delta(x-x')] + E'_j B_k [-i\varepsilon_{il}{}^m \partial_m \delta(x-x')] \\
&\quad - E_i B'_j [i\varepsilon_{kl}{}^m \partial_m \delta(x-x')] - E'_l B_k [-i\varepsilon_{ij}{}^m \partial_m \delta(x-x')] \\
&\quad - E_k B'_l [i\varepsilon_{ij}{}^m \partial_m \delta(x-x')] - E'_j B_i [-i\varepsilon_{kl}{}^m \partial_m \delta(x-x')] \\
&\quad + E_k B'_j [i\varepsilon_{il}{}^m \partial_m \delta(x-x')] + E'_l B_i [-i\varepsilon_{kj}{}^m \partial_m \delta(x-x')] \} \\
&= -r^k r'^l \\
&\quad \{ \partial_m E_i B'_l [i\varepsilon_{kj}{}^m \delta(x-x')] + \partial'_m E'_j B_k [i\varepsilon_{il}{}^m \delta(x-x')] - \partial_m E_i B'_j [i\varepsilon_{kl}{}^m \delta(x-x')] - \partial'_m E'_l B_k [i\varepsilon_{ij}{}^m \delta(x-x')] \\
&\quad - \partial_m E_k B'_l [i\varepsilon_{ij}{}^m \delta(x-x')] - \partial'_m E'_j B_i [i\varepsilon_{kl}{}^m \delta(x-x')] + \partial_m E_k B'_j [i\varepsilon_{il}{}^m \delta(x-x')] + \partial'_m E'_l B_i [i\varepsilon_{kj}{}^m \delta(x-x')] \} \\
&\quad + r^k E'_l B_k [i\varepsilon_{ij}{}^l \delta(x-x')] + r^l E_k B'_l [i\varepsilon_{ij}{}^k \delta(x-x')] - r^l E_k B'_j [i\varepsilon_{il}{}^k \delta(x-x')] - r^k E'_l B_i [i\varepsilon_{kj}{}^l \delta(x-x')] \\
&= -i r^k r'^l \delta(x-x') \\
&\quad \{ \partial_m E_i B'_l \varepsilon_{kj}{}^m + \partial'_m E'_j B_k \varepsilon_{il}{}^m - \partial_m E_i B'_j \varepsilon_{kl}{}^m - \partial'_m E'_l B_k \varepsilon_{ij}{}^m \\
&\quad - \partial_m E_k B'_l \varepsilon_{ij}{}^m - \partial'_m E'_j B_i \varepsilon_{kl}{}^m + \partial_m E_k B'_j \varepsilon_{il}{}^m + \partial'_m E'_l B_i \varepsilon_{kj}{}^m \} \\
&\quad + r^k E'_l B_k [i\varepsilon_{ij}{}^l \delta(x-x')] + r^l E_k B'_l [i\varepsilon_{ij}{}^k \delta(x-x')] - r^l E_k B'_j [i\varepsilon_{il}{}^k \delta(x-x')] - r^k E'_l B_i [i\varepsilon_{kj}{}^l \delta(x-x')] \\
&= -i \delta(x-x') \\
&\quad \{ \varepsilon_{kj}{}^m r^k \partial_m E_i r^l B_l + \varepsilon_{il}{}^m r^l \partial_m E_j r^k B_k - \varepsilon_{kl}{}^m r^k r^l \partial_m E_i B_j - \varepsilon_{ij}{}^m r^l \partial_m E_l r^k B_k \\
&\quad - \varepsilon_{ij}{}^m r^k \partial_m E_k r^l B_l - \varepsilon_{kl}{}^m r^k r^l \partial_m E_j B_i + \varepsilon_{il}{}^m r^k r^l \partial_m E_k B_j + \varepsilon_{kj}{}^m r^k r^l \partial_m E_l B_i \\
&\quad - \varepsilon_{ij}{}^l E_l r^k B_k - \varepsilon_{ij}{}^k E_k r^l B_l + \varepsilon_{il}{}^k r^l E_k B_j + \varepsilon_{kj}{}^l r^k E_l B_i \} \\
&= -i \delta(x-x') \\
&\quad \{ \varepsilon_{kj}{}^m r^k \partial_m E_i (\vec{r} \cdot \vec{B}) + \varepsilon_{il}{}^m r^l \partial_m E_j (\vec{r}' \cdot \vec{B}') - \varepsilon_{kl}{}^m r^k r^l \partial_m E_i B_j - \varepsilon_{ij}{}^m r^l \partial_m E_l (\vec{r} \cdot \vec{B}) \\
&\quad - \varepsilon_{ij}{}^m r^k \partial_m E_k (\vec{r} \cdot \vec{B}) - \varepsilon_{kl}{}^m r^k r^l \partial_m E_j B_i + \varepsilon_{il}{}^m r^k r^l \partial_m E_k B_j + \varepsilon_{kj}{}^m r^k r^l \partial_m E_l B_i \\
&\quad - \varepsilon_{ij}{}^l E_l (\vec{r} \cdot \vec{B}) - \varepsilon_{ij}{}^k E_k (\vec{r}' \cdot \vec{B}') + \varepsilon_{il}{}^k r^l E_k B_j + \varepsilon_{kj}{}^l r^k E_l B_i \} \\
&= -i \delta(x-x') \\
&\quad \{ -\varepsilon_{jk}{}^l r^k \partial_l E_i (\vec{r} \cdot \vec{B}) + \varepsilon_{ik}{}^l r^k \partial_l E_j (\vec{r}' \cdot \vec{B}') - 2\varepsilon_{ij}{}^k r^l \partial_k E_l (\vec{r} \cdot \vec{B}) - 2\varepsilon_{ij}{}^k E_k (\vec{r}' \cdot \vec{B}') \\
&\quad + \varepsilon_{ik}{}^l r^k r^m \partial_l E_m B_j - \varepsilon_{jk}{}^l r^k r^m \partial_l E_m B_i + \varepsilon_{ik}{}^l r^k E_l B_j - \varepsilon_{jk}{}^l r^k E_l B_i \}
\end{aligned}$$

□

定理5.14.1.

$$\begin{cases} [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\Psi_{\alpha_\zeta}(x), \Psi_{\beta_\zeta}(x')] = 0, [\Psi_{\alpha'_\zeta}^+(x), \Psi_{\beta'_\zeta}^+(x')] = 0 \\ \Psi(x) = \frac{1}{\sqrt{2}} [\vec{E}(x) - i\zeta \vec{B}(x)] \end{cases} \Leftrightarrow \begin{cases} [E_i(x), E_j(x')] = -i(\delta_{ij} \nabla^2 - \partial_i \partial_j) \Delta(x-x') \\ [B_i(x), B_j(x')] = -i(\delta_{ij} \nabla^2 - \partial_i \partial_j) \Delta(x-x') \\ [E_i(x), B_j(x')] = i\varepsilon_{ij}{}^k \partial_k \partial_t \Delta(x-x') \\ [B_i(x), E_j(x')] = -i\varepsilon_{ij}{}^k \partial_k \partial_t \Delta(x-x') \end{cases}$$

### 5.15 对易和反对易公式

$$\text{推论5.15.1. } \begin{cases} [A, BC] = [A, B]C + B[A, C] \\ [BC, A] = [B, A]C + B[C, A] \end{cases}$$

$$\begin{cases} [AB, A'B'] = [AB, A']B' + A'[AB, B'] = [A, A']BB' + A[B, A']B' + A'[A, B']B + A'A[B, B'] \\ [AB, A'B'] = A[B, A'B'] + [A, A'B']B = AA'[B, B'] + A[B, A']B' + A'[A, B']B + [A, A']B'B \end{cases}$$

$$\text{推论5.15.2. } \begin{cases} [A, BC] = [A, B]C + B[A, C], [A, CB] = [A, C]B + C[A, B] \\ [A, BC] = \{A, B\}C - B\{A, C\}, [A, CB] = \{A, C\}B - C\{A, B\} \end{cases}$$

$$\text{推论5.15.3. } \begin{cases} [A, \{B, C\}] = \{[A, B], C\} + \{B, [A, C]\} \\ [A, [B, C]] = \{\{A, B\}, C\} - \{B, \{A, C\}\} \end{cases}$$

## 6 自由无耦合Yang-Mills场协变量子化新方案

### 6.1 自由无耦合Yang-Mills场方程各种等价形式<sup>[23, 25]</sup>

$$\text{定义6.1.1. } \Psi_{\alpha_\zeta}^\rho := \frac{-i\zeta}{\sqrt{2}}\psi_{\alpha_\zeta}^\rho = \frac{-i\zeta}{\sqrt{2}}\frac{i}{2}\sigma_{\zeta\alpha_\zeta}^{ab}F_{ab}^\rho = \frac{-i\zeta}{\sqrt{2}}i\zeta(E^\rho - i\zeta B^\rho)_{\alpha_\zeta}$$

$$\text{定义6.1.2. } \Psi^\rho := \frac{1}{\sqrt{2}}(\vec{E}^\rho - i\zeta\vec{B}^\rho) = \frac{1}{\sqrt{2}}(\vec{E}^\rho - i\zeta\nabla \times \vec{A}^\rho), \Psi_i^\rho = \frac{1}{\sqrt{2}}(E_i^\rho - i\zeta\varepsilon_i^{jk}\partial_j A_k^\rho), p \cdot x := \vec{p} \cdot \vec{r} - Et$$

定理6.1.1.

$$\begin{cases} \partial^a F_{ab}^\rho = 0 \\ \partial^a * F_{ab}^\rho = 0 \end{cases} \Leftrightarrow \begin{cases} \nabla \cdot \vec{E}^\rho = 0, \nabla \times \vec{E}^\rho = -\partial_t \vec{B}^\rho \\ \nabla \cdot \vec{B}^\rho = 0, \nabla \times \vec{B}^\rho = \partial_t \vec{E}^\rho \end{cases} \Leftrightarrow \begin{cases} (\gamma, -i\zeta)^a \partial_a \Psi^\rho = 0 \\ \nabla \cdot \Psi^\rho = 0 \end{cases} \Leftrightarrow \begin{cases} [\partial_a + iS_{ab}(\gamma, \zeta)\partial^b]\Psi^\rho = 0 \\ S_{ab}(\gamma, \zeta) = i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta}(s) \end{cases}$$

### 6.2 自由无耦合Yang-Mills场复场强自旋方程及其平面波解

$$\text{定理6.2.1. } [\partial_a + iS_{ab}(\gamma, \zeta)\partial^b]\Psi^\rho(x) = 0$$

$$\text{推论6.2.1. } \begin{cases} \Psi^\rho(\vec{r}, t) := \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \sqrt{|\vec{p}|} \lambda_m(\hat{p}, -\zeta) [a_1^\rho(\vec{p}, -\zeta) e^{i\zeta p \cdot x} + a_2^{\rho+}(\vec{p}, -\zeta) e^{-i\zeta p \cdot x}] d^3 \vec{p} \\ \sqrt{|\vec{p}|} a_1^\rho(\vec{p}, -\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda_m^+(\hat{p}, -\zeta) \Psi^\rho(\vec{r}, t) e^{-i\zeta p \cdot x} d^3 \vec{r} \\ \sqrt{|\vec{p}|} a_2^{\rho+}(\vec{p}, -\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda_m^+(\hat{p}, -\zeta) \Psi^\rho(\vec{r}, t) e^{i\zeta p \cdot x} d^3 \vec{r} \end{cases}$$

### 6.3 自由无耦合Yang-Mills场对易规则

$$\text{推论6.3.1. } \begin{cases} [a_\sigma^\rho(\vec{p}, -\zeta), a_{\sigma'}^{\rho'+}(\vec{p}', -\zeta)] = \zeta \delta^{\rho\rho'} \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma^\rho(\vec{p}, -\zeta), a_{\sigma'}^\tau(\vec{p}', -\zeta)] = 0 \\ [a_{\sigma'}^{\rho'+}(\vec{p}', -\zeta), a_{\sigma'}^{\tau'+}(\vec{p}', -\zeta)] = 0 \end{cases} \Leftrightarrow \begin{cases} [a_\sigma^\rho(\vec{p}), a_{\sigma'}^{\rho'+}(\vec{p}')] = \delta^{\rho\rho'} \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma^\rho(\vec{p}), a_{\sigma'}^\tau(\vec{p}')] = 0 \\ [a_{\sigma'}^{\rho'+}(\vec{p}'), a_{\sigma'}^{\tau'+}(\vec{p}')] = 0 \end{cases}$$

⇕

⇕

$$\text{推论6.3.2. } \begin{cases} [\Psi_{\alpha_\zeta}^\rho(x), \Psi_{\alpha'_\zeta}^{\rho'+}(x')] = i\delta^{\rho\rho'} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha_\zeta}^\rho(x), \Psi_{\beta_\zeta}^\tau(x')] = 0 \\ [\Psi_{\alpha'_\zeta}^{\rho'+}(x), \Psi_{\beta'_\zeta}^{\tau'+}(x')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_{\alpha_\zeta}^\rho(\vec{r}, t), \Psi_{\alpha'_\zeta}^{\rho'+}(\vec{r}', t)] = \zeta \delta^{\rho\rho'} \varepsilon^k{}_{\alpha_\zeta \alpha'_\zeta} \partial_k \delta(\vec{r} - \vec{r}') \\ [\Psi_{\alpha_\zeta}^\rho(\vec{r}, t), \Psi_{\beta_\zeta}^\tau(\vec{r}', t)] = 0 \\ [\Psi_{\alpha'_\zeta}^{\rho'+}(\vec{r}, t), \Psi_{\beta'_\zeta}^{\tau'+}(\vec{r}', t)] = 0 \end{cases}$$

## 6.4 自由无耦合Yang-Mills场的对易函数、因果函数和费曼传播子

推论6.4.1.

$$\begin{cases} \Delta_{\alpha_s \alpha'_s}^{\rho \rho'}(\gamma; x) := \delta^{\rho \rho'} \sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b \Delta(x) \\ \Delta_{\alpha_s \alpha'_s}^{\rho \rho'(+)}(\gamma; x) := \delta^{\rho \rho'} \sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b \Delta^{(+)}(x) \\ \Delta_{\alpha_s \alpha'_s}^{\rho \rho'(-)}(\gamma; x) := \delta^{\rho \rho'} \sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b \Delta^{(-)}(x) \\ \Delta_{\alpha_s \alpha'_s}^{\rho \rho'(l)}(\gamma; x) := \delta^{\rho \rho'} \sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b \Delta^{(l)}(x) \end{cases} \begin{cases} \Delta_{\alpha_s \alpha'_s}^{\rho \rho'(c)}(\gamma; x) := \delta^{\rho \rho'} [\sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b \Delta^{(c)}(x) + \sigma_{\alpha_s \alpha'_s}^{\pi \pi} \delta^4(x)] \\ \Delta_{\alpha_s \alpha'_s}^{\rho \rho' ret}(\gamma; x) := \delta^{\rho \rho'} [\sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b \Delta^{ret}(x) + \sigma_{\alpha_s \alpha'_s}^{\pi \pi} \delta^4(x)] \\ \Delta_{\alpha_s \alpha'_s}^{\rho \rho' adv}(\gamma; x) := \delta^{\rho \rho'} [\sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b \Delta^{adv}(x) + \sigma_{\alpha_s \alpha'_s}^{\pi \pi} \delta^4(x)] \\ \Delta_{F \alpha_s \alpha'_s}^{\rho \rho'}(\gamma; x) := \delta^{\rho \rho'} [\sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b \Delta_F(x) + i \sigma_{\alpha_s \alpha'_s}^{\pi \pi} \delta^4(x)] = i \Delta_{\alpha_s \alpha'_s}^{\rho \rho'(c)}(\gamma; x) \\ \Delta_{F \alpha_s \alpha'_s}(\gamma; x) = \frac{i \delta^{\rho \rho'} \sigma_{\alpha_s \alpha'_s}^{ab} p_a p_b}{p^2 - i\varepsilon} + \dots \end{cases}$$

推论6.4.2.

$$\begin{cases} [\partial_a + i S_{ab}(\gamma, \varsigma) \partial^b] \Delta^{\rho \rho'}(\gamma; x) = 0 \\ [\partial_a + i S_{ab}(\gamma, \varsigma) \partial^b] \Delta^{\rho \rho'(+)}(\gamma; x) = 0 \\ [\partial_a + i S_{ab}(\gamma, \varsigma) \partial^b] \Delta^{\rho \rho'(-)}(\gamma; x) = 0 \\ [\partial_a + i S_{ab}(\gamma, \varsigma) \partial^b] \Delta^{\rho \rho'(l)}(\gamma; x) = 0 \end{cases} \begin{cases} [\partial_a + i S_{ab}(\gamma, \varsigma) \partial^b] \Delta^{\rho \rho'(c)}(\gamma; x) = -\varsigma(\gamma, i\varsigma)_a \delta(t) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \\ [\partial_a + i S_{ab}(\gamma, \varsigma) \partial^b] \Delta^{\rho \rho' ret}(\gamma; x) = -\varsigma(\gamma, i\varsigma)_a \delta(t) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \\ [\partial_a + i S_{ab}(\gamma, \varsigma) \partial^b] \Delta^{\rho \rho' adv}(\gamma; x) = -\varsigma(\gamma, i\varsigma)_a \delta(t) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \\ [\partial_a + i S_{ab}(\gamma, \varsigma) \partial^b] \Delta_F^{\rho \rho'}(\gamma; x) = -i\varsigma(\gamma, i\varsigma)_a \delta(t) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \end{cases}$$

[⇕] [⇕]

推论6.4.3.

$$\begin{cases} (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho'}(\gamma; x) = 0 \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho'(+)}(\gamma; x) = 0 \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho'(-)}(\gamma; x) = 0 \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho'(l)}(\gamma; x) = 0 \end{cases} \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho'(c)}(\gamma; x) = -\varsigma \delta(t) N_m(1) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho' ret}(\gamma; x) = -\varsigma \delta(t) N_m(1) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho' adv}(\gamma; x) = -\varsigma \delta(t) N_m(1) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta_F(\gamma; x) = -i\varsigma \delta(t) N_m(1) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \end{cases}$$

[⇕] [⇕]

推论6.4.4.

$$\begin{cases} (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho'}(\gamma; x) \bar{N}_m(1) = 0 \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho'(+)}(\gamma; x) \bar{N}_m(1) = 0 \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho'(-)}(\gamma; x) \bar{N}_m(1) = 0 \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho'(l)}(\gamma; x) \bar{N}_m(1) = 0 \end{cases} \begin{cases} (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho'(c)}(\gamma; x) \bar{N}_m(1) = -\varsigma \delta(t) N_m(1) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \bar{N}_m(1) \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho' ret}(\gamma; x) \bar{N}_m(1) = -\varsigma \delta(t) N_m(1) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \bar{N}_m(1) \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta^{\rho \rho' adv}(\gamma; x) \bar{N}_m(1) = -\varsigma \delta(t) N_m(1) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \bar{N}_m(1) \\ (\sigma_{-\varsigma}, -i\varsigma)_a \partial^a N_m(1) \Delta_F(\gamma; x) \bar{N}_m(1) = -i\varsigma \delta(t) N_m(1) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \bar{N}_m(1) \end{cases}$$

[⇕] [⇕]

推论6.4.5.

$$\begin{cases} (\gamma, -i\varsigma)_a \partial^a \Delta^{\rho \rho'}(\gamma; x) = 0 \\ (\gamma, -i\varsigma)_a \partial^a \Delta^{\rho \rho'(+)}(\gamma; x) = 0 \\ (\gamma, -i\varsigma)_a \partial^a \Delta^{\rho \rho'(-)}(\gamma; x) = 0 \\ (\gamma, -i\varsigma)_a \partial^a \Delta^{\rho \rho'(l)}(\gamma; x) = 0 \end{cases} \begin{cases} (\gamma, -i\varsigma)_a \partial^a \Delta^{\rho \rho'(c)}(\gamma; x) = -\varsigma \delta(t) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \\ (\gamma, -i\varsigma)_a \partial^a \Delta^{\rho \rho' ret}(\gamma; x) = -\varsigma \delta(t) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \\ (\gamma, -i\varsigma)_a \partial^a \Delta^{\rho \rho' adv}(\gamma; x) = -\varsigma \delta(t) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \\ (\gamma, -i\varsigma)_a \partial^a \Delta_F(\gamma; x) = -i\varsigma \delta(t) \Delta^{\rho \rho'}(\gamma; x)|_{t=0} \end{cases}$$

## 6.5 自由无耦合Yang-Mills场在辐射规范下势对易关系与复场强协变对易关系的等价性

引理6.5.1.

$$\begin{cases} \nabla^2 \tilde{A}^\rho - \partial_t^2 \tilde{A}^\rho = \tilde{J}^\rho + \partial_t \nabla \tilde{\phi}^\rho, \nabla^2 \tilde{\phi}^\rho = \rho^\rho \\ \sqrt{2} \Psi^\rho = -\partial_t \tilde{A}^\rho - \nabla \tilde{\phi}^\rho - i\varsigma \nabla \times \tilde{A}^\rho, \nabla \cdot \tilde{A}^\rho = 0 \end{cases} \Leftrightarrow \begin{cases} [\partial_a + i S_{ab}(\gamma, \varsigma) \partial^b] \Psi^\rho = -i \sigma_{\varsigma ab}^{[\beta \varsigma]} J^{b\rho} \\ \tilde{A}^\rho = \frac{-i\varsigma}{\sqrt{2}} \frac{\nabla \times (\Psi^\rho - \Psi^{+\rho})}{\nabla^2}, \tilde{\phi}^\rho = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi^\rho + \Psi^{+\rho})}{\nabla^2} \end{cases}$$

引理6.5.2.

$$\begin{cases} [\tilde{A}_i^\rho(x), \tilde{A}_j^\tau(x')] = i \delta^{\rho\tau} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x - x') \\ [\tilde{A}_i^\rho(x), \tilde{\phi}^\tau(x')] = 0, [\tilde{\phi}^\rho(x), \tilde{\phi}^\tau(x')] = 0 \\ \sqrt{2} \Psi^\rho = -\partial_t \tilde{A}^\rho - \nabla \tilde{\phi}^\rho - i\varsigma \nabla \times \tilde{A}^\rho, \nabla \cdot \tilde{A}^\rho = 0 \end{cases} \Rightarrow \begin{cases} [\Psi_{\alpha_s}^\rho(x), \Psi_{\alpha'_s}^{\rho'+}(x')] = i \delta^{\rho \rho'} \sigma_{\alpha_s \alpha'_s}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\Psi_{\alpha_s}^\rho(x), \Psi_{\beta_s}^\tau(x')] = 0, [\Psi_{\alpha'_s}^{\rho'+}(x), \Psi_{\beta'_s}^{\tau'+}(x')] = 0 \\ \tilde{A}^\rho = \frac{-i\varsigma}{\sqrt{2}} \frac{\nabla \times (\Psi^\rho - \Psi^{+\rho})}{\nabla^2}, \tilde{\phi}^\rho = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi^\rho + \Psi^{+\rho})}{\nabla^2} \end{cases}$$

引理6.5.3.

$$\begin{cases} [\Psi_{\alpha'_\zeta}^\rho(x), \Psi_{\alpha'_\zeta}^{\rho'+}(x')] = i\delta^{\rho\rho'} \sigma_{\alpha'_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\Psi_{\alpha'_\zeta}^\rho(x), \Psi_{\beta'_\zeta}^{\tau'+}(x')] = 0, [\Psi_{\alpha'_\zeta}^{\rho'+}(x), \Psi_{\beta'_\zeta}^{\tau'+}(x')] = 0 \\ \tilde{A}^\rho = \frac{-i\zeta}{\sqrt{2}} \frac{\nabla \times (\Psi^\rho - \Psi^{+\rho})}{\nabla^2}, \tilde{\phi}^\rho = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi^\rho + \Psi^{+\rho})}{\nabla^2} \end{cases} \Rightarrow \begin{cases} [\tilde{A}_i^\rho(x), \tilde{A}_j^\tau(x')] = i\delta^{\rho\tau} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x-x') \\ [\tilde{A}_i^\rho(x), \tilde{\phi}^\tau(x')] = 0, [\tilde{\phi}^\rho(x), \tilde{\phi}^\tau(x')] = 0 \end{cases}$$

定理6.5.1.

$$\begin{cases} [\tilde{A}_i^\rho(x), \tilde{A}_j^\tau(x')] = i\delta^{\rho\tau} (\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}) \Delta(x-x') \\ [\tilde{A}_i^\rho(x), \tilde{\phi}^\tau(x')] = 0, [\tilde{\phi}^\rho(x), \tilde{\phi}^\tau(x')] = 0 \\ \nabla^2 \tilde{A}^\rho - \partial_t^2 \tilde{A}^\rho = \tilde{J}^\rho + \partial_t \nabla \tilde{\phi}^\rho, \nabla^2 \tilde{\phi}^\rho = \rho^\rho \\ \sqrt{2} \Psi^\rho = -\partial_t \tilde{A}^\rho - \nabla \tilde{\phi}^\rho - i\zeta \nabla \times \tilde{A}^\rho, \nabla \cdot \tilde{A}^\rho = 0 \end{cases} \Leftrightarrow \begin{cases} [\Psi_{\alpha'_\zeta}^\rho(x), \Psi_{\alpha'_\zeta}^{\rho'+}(x')] = i\delta^{\rho\rho'} \sigma_{\alpha'_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\Psi_{\alpha'_\zeta}^\rho(x), \Psi_{\beta'_\zeta}^{\tau'+}(x')] = 0, [\Psi_{\alpha'_\zeta}^{\rho'+}(x), \Psi_{\beta'_\zeta}^{\tau'+}(x')] = 0 \\ [\partial_a + iS_{ab}(\gamma, \zeta) \partial^b] \Psi^\rho = -i\sigma_{\zeta ab}^{[\beta\zeta]} J^{b\rho} \\ \tilde{A}^\rho = \frac{-i\zeta}{\sqrt{2}} \frac{\nabla \times (\Psi^\rho - \Psi^{+\rho})}{\nabla^2}, \tilde{\phi}^\rho = -\frac{1}{\sqrt{2}} \frac{\nabla \cdot (\Psi^\rho + \Psi^{+\rho})}{\nabla^2} \end{cases}$$

## 6.6 在 $\lambda$ -规范下导出自由无耦合Yang-Mills场的协变对易规则

推论6.6.1.

$$\begin{cases} [A_a^\rho(x), A_b^\tau(x')] = i\delta^{\rho\tau} (\delta_{ab} - \frac{\lambda-1}{\lambda} \frac{\partial_a \partial_b}{\square+i\epsilon}) \Delta(x-x') \\ \phi = -iA_0, \sqrt{2} \Psi^\rho = -\partial_t \tilde{A}^\rho - \nabla \phi^\rho - i\zeta \nabla \times \tilde{A}^\rho \end{cases} \Rightarrow \begin{cases} [\Psi_{\alpha'_\zeta}^\rho(x), \Psi_{\alpha'_\zeta}^{\rho'+}(x')] = i\delta^{\rho\rho'} \sigma_{\alpha'_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\Psi_{\alpha'_\zeta}^\rho(x), \Psi_{\beta'_\zeta}^{\tau'+}(x')] = 0, [\Psi_{\alpha'_\zeta}^{\rho'+}(x), \Psi_{\beta'_\zeta}^{\tau'+}(x')] = 0 \\ [\Psi_i^\rho(x), \phi^\tau(x')] = [\Psi_i^{\rho'+}(x), \phi^{\tau'+}(x')] = \frac{i}{\sqrt{2}} \delta^{\rho\tau} \partial_i \Delta(x-x') \\ [\phi^\rho(x), \phi^\tau(x')] = -i\delta^{\rho\tau} (1 + \frac{\lambda-1}{\lambda} \frac{\nabla^2}{\square+i\epsilon}) \Delta(x-x') \end{cases}$$

## 7 引力微子场协变量子化方案

### 7.1 引力微子自旋算符方程及其平面波解

定理7.1.1.  $[\frac{3}{2} \partial_a + iS_{ab}(\frac{3}{2}, \zeta) \partial^b] \psi(x) = 0$ 

$$\begin{cases} \psi(\vec{r}, t) := \frac{1}{(2\pi)^{3/2}} \int |\vec{p}| \lambda(\hat{p}, -\frac{3}{2}\zeta) [a_1(\vec{p}, -\frac{3}{2}\zeta) e^{ip \cdot x} + a_2^+(\vec{p}, -\frac{3}{2}\zeta) e^{-ip \cdot x}] d^3 \vec{p} \\ |\vec{p}| a_1(\vec{p}, -\frac{3}{2}\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -\frac{3}{2}\zeta) \psi(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r} \\ |\vec{p}| a_2^+(\vec{p}, -\frac{3}{2}\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -\frac{3}{2}\zeta) \psi(\vec{r}, t) e^{ip \cdot x} d^3 \vec{r} \end{cases}$$

定义7.1.1. 投影算子:  $\hat{P}_{k_\zeta k'_\zeta}(\frac{3}{2}, \zeta) := \lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -\frac{3}{2}\zeta)$ ,  $\hat{P}^2(\frac{3}{2}, \zeta) = \hat{P}(\frac{3}{2}, \zeta)$ ,  $\hat{P}^+(\frac{3}{2}, \zeta) = \hat{P}(\frac{3}{2}, \zeta)$ 推论7.1.2.  $H_2 = \int |\vec{p}| [a_1^+(\vec{p}, -\frac{3}{2}\zeta) a_1(\vec{p}, -\frac{3}{2}\zeta) - a_2(\vec{p}, -\frac{3}{2}\zeta) a_2^+(\vec{p}, -\frac{3}{2}\zeta)] d^3 \vec{p} = \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{i\partial_t}{-\nabla^2} \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r}$ 

$$\begin{aligned} \text{证明: } H_2 &= \int |\vec{p}| [a_1^+(\vec{p}, -\frac{3}{2}\zeta) a_1(\vec{p}, -\frac{3}{2}\zeta) - a_2(\vec{p}, -\frac{3}{2}\zeta) a_2^+(\vec{p}, -\frac{3}{2}\zeta)] d^3 \vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|} [\lambda^{k'_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \psi_{k'_\zeta}^+(\vec{r}, t) e^{ip \cdot x'} \lambda^{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \psi_{k_\zeta}(\vec{r}, t) e^{-ip \cdot x} \\ &\quad - \lambda^{k'_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \psi_{k'_\zeta}^+(\vec{r}, t) e^{-ip \cdot x'} \lambda^{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \psi_{k_\zeta}(\vec{r}, t) e^{ip \cdot x}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|} \lambda^{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda^{k'_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \psi_{k'_\zeta}^+(\vec{r}, t) \psi_{k_\zeta}(\vec{r}, t) [e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')} - e^{i\vec{p} \cdot (\vec{r}-\vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int (-2\sqrt{2}i)^{-1} \frac{1}{|\vec{p}|^4} \Gamma_{k_\zeta k'_\zeta}^{abc} p_a p_b p_c \psi_{k'_\zeta}^+(\vec{r}, t) \psi_{k_\zeta}(\vec{r}, t) [e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')} - e^{i\vec{p} \cdot (\vec{r}-\vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int (-2\sqrt{2}i)^{-1} \frac{1}{|\vec{p}|^4} (\frac{1}{\sqrt{2}})^3 \frac{i}{6} \{ [3|\vec{p}|^3 - 2\zeta |\vec{p}|^2 [\sigma(\frac{3}{2}) \cdot \vec{p}] - 12|\vec{p}| [\sigma(\frac{3}{2}) \cdot \vec{p}]^2 + 8\zeta [\sigma(\frac{3}{2}) \cdot \vec{p}]^3] \} \psi_{k'_\zeta}^+(\vec{r}, t) \psi_{k_\zeta}(\vec{r}, t) \\ &\quad [e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')} - e^{i\vec{p} \cdot (\vec{r}-\vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int \frac{-1}{48} \frac{1}{|\vec{p}|^4} \{ [3|\vec{p}|^3 - 2\zeta |\vec{p}|^2 [\sigma(\frac{3}{2}) \cdot \vec{p}] - 12|\vec{p}| [\sigma(\frac{3}{2}) \cdot \vec{p}]^2 + 8\zeta [\sigma(\frac{3}{2}) \cdot \vec{p}]^3] \} \psi_{k'_\zeta}^+(\vec{r}, t) \psi_{k_\zeta}(\vec{r}, t) \\ &\quad [e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')} - e^{i\vec{p} \cdot (\vec{r}-\vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int \frac{-1}{48} \frac{1}{|\vec{p}|^4} \{ [-2\zeta |\vec{p}|^2 [\sigma(\frac{3}{2}) \cdot \vec{p}] + 8\zeta [\sigma(\frac{3}{2}) \cdot \vec{p}]^3] \} \psi_{k'_\zeta}^+(\vec{r}, t) \psi_{k_\zeta}(\vec{r}, t) [e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')} - e^{i\vec{p} \cdot (\vec{r}-\vec{r}')}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int \frac{i\zeta}{24} \psi_{k'_\zeta}^+(\vec{r}, t) \psi_{k_\zeta}(\vec{r}, t) \{ [\frac{1}{|\vec{p}|^2} [\sigma(\frac{3}{2}) \cdot \nabla] + \frac{1}{|\vec{p}|^4} 4[\sigma(\frac{3}{2}) \cdot \nabla]^3] [e^{-i\vec{p} \cdot (\vec{r}-\vec{r}')} + e^{i\vec{p} \cdot (\vec{r}-\vec{r}')}] \} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{-i\zeta}{12} \int \psi_{k'_\zeta}^+(\vec{r}, t) \psi_{k_\zeta}(\vec{r}, t) \{ \frac{\sigma(\frac{3}{2}) \cdot \nabla}{\nabla^2} - 4 \frac{[\sigma(\frac{3}{2}) \cdot \nabla]^3}{\nabla^4} \} \delta^3(\vec{r}-\vec{r}') d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{i\zeta}{12} \int \psi_{k'_\zeta}^+(\vec{r}, t) \{ \frac{\sigma(\frac{3}{2}) \cdot \nabla}{\nabla^2} - 4 \frac{[\sigma(\frac{3}{2}) \cdot \nabla]^3}{\nabla^4} \} \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r} \\ &= \frac{-i\zeta}{3/2} \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{\sigma(\frac{3}{2}) \cdot \nabla}{\nabla^2} \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r} \\ &= \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{i\partial_t}{-\nabla^2} \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r} \end{aligned}$$

□



## 7.2 引力微子场协变常数不变张量的性质

推论7.2.1.

$$\begin{aligned}\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi}\left(\frac{3}{2}\right) &= \left(\frac{1}{\sqrt{2}}\right)^3 \delta_{k_\zeta k'_\zeta} \\ \Gamma_{k_\zeta k'_\zeta}^{i\pi\pi}\left(\frac{3}{2}\right) &= -i\zeta\left(\frac{1}{\sqrt{2}}\right)^3 \frac{2}{3} \sigma^i\left(\frac{3}{2}\right)_{k_\zeta k'_\zeta} \\ \Gamma_{k_\zeta k'_\zeta}^{ij\pi}\left(\frac{3}{2}\right) &= -\left(\frac{1}{\sqrt{2}}\right)^3 \frac{1}{3} [\sigma^i\left(\frac{3}{2}\right)\sigma^j\left(\frac{3}{2}\right) - \frac{3}{2}\delta^{ij}]_{k_\zeta k'_\zeta} = -\left(\frac{1}{\sqrt{2}}\right)^3 \frac{2}{3} \frac{1}{2!} [\sigma^i\left(\frac{3}{2}\right)\sigma^j\left(\frac{3}{2}\right) - \frac{3}{4}\delta^{ij}]_{k_\zeta k'_\zeta} \\ \Gamma_{k_\zeta k'_\zeta}^{ijk}\left(\frac{3}{2}\right) &= \left(\frac{1}{\sqrt{2}}\right)^3 \frac{2i\zeta}{3} \left\{ \sigma^i\left(\frac{3}{2}\right)[\sigma^j\left(\frac{3}{2}\right)\sigma^k\left(\frac{3}{2}\right)] - \left[\frac{1}{2}\sigma^i\left(\frac{3}{2}\right)\delta^{jk} + \frac{3}{2}\delta^{ij}\sigma^k\left(\frac{3}{2}\right)\right] \right\}_{k_\zeta k'_\zeta} \\ &= \left(\frac{1}{\sqrt{2}}\right)^3 \frac{4i\zeta}{3} \frac{1}{3!} [\sigma^i\left(\frac{3}{2}\right)\sigma^j\left(\frac{3}{2}\right)\sigma^k\left(\frac{3}{2}\right) - \frac{7}{4}\delta^{ij}\sigma^k\left(\frac{3}{2}\right)]_{k_\zeta k'_\zeta}\end{aligned}$$

$$\text{推论7.2.2. } \Gamma^{abc}\left(\frac{3}{2}\right)\partial_a\partial_b\partial_c\Delta(x-x')|_{t=t'} = \frac{i}{4\sqrt{2}}\{\nabla^2 - 4[\sigma\left(\frac{3}{2}\right)\cdot\nabla]^2\}\delta^3(\vec{r}-\vec{r}')$$

证明:  $\Gamma^{abc}\left(\frac{3}{2}\right)\partial_a\partial_b\partial_c\Delta(x-x')|_{t=t'}$

$$\begin{aligned}&= i\sum_{l=0}^1(-1)^l C_3^{2l+1} \Gamma_{i^j\cdots\pi\cdots\pi}^{2-2l\ 2l+1}\left(\frac{3}{2}\right) \partial_i\partial_j\cdots\nabla^{2l}\delta^3(\vec{r}-\vec{r}') \\ &= i[C_3^1\Gamma^{ij\pi}\left(\frac{3}{2}\right)\partial_i\partial_j - C_3^3\Gamma^{\pi\pi\pi}\left(\frac{3}{2}\right)\nabla^2]\delta^3(\vec{r}-\vec{r}') \\ &= i\left\{-\left(\frac{1}{\sqrt{2}}\right)^3[\sigma^i\left(\frac{3}{2}\right)\sigma^j\left(\frac{3}{2}\right) - \frac{3}{2}\delta^{ij}]\partial_i\partial_j - \left(\frac{1}{\sqrt{2}}\right)^3\nabla^2\right\}\delta^3(\vec{r}-\vec{r}') \\ &= -i\left(\frac{1}{\sqrt{2}}\right)^3\left\{[\sigma^i\left(\frac{3}{2}\right)\sigma^j\left(\frac{3}{2}\right) - \frac{3}{2}\delta^{ij}]\partial_i\partial_j + \nabla^2\right\}\delta^3(\vec{r}-\vec{r}') \\ &= i\frac{1}{4\sqrt{2}}\{\nabla^2 - 4[\sigma\left(\frac{3}{2}\right)\cdot\nabla]^2\}\delta^3(\vec{r}-\vec{r}') \\ &= \frac{i}{4\sqrt{2}}\{\nabla^2 - 9[\frac{2}{3}\sigma\left(\frac{3}{2}\right)\cdot\nabla]^2\}\delta^3(\vec{r}-\vec{r}')\end{aligned}$$

□

$$\text{推论7.2.3. } \Gamma^{abc}\left(\frac{3}{2}\right)\partial_a\partial_b\partial_c\Delta(x-x')|_{t=t'}\partial_\pi\Delta(x-x')|_{t=t'} = \frac{\zeta}{4\sqrt{2}}\{\nabla^2 - 9[\frac{2}{3}\sigma\left(\frac{3}{2}\right)\cdot\nabla]^2\}[\frac{2}{3}\sigma\left(\frac{3}{2}\right)\cdot\nabla]\delta^3(\vec{r}-\vec{r}')$$

证明:  $\Gamma^{abc}\left(\frac{3}{2}\right)\partial_a\partial_b\partial_c\Delta(x-x')|_{t=t'}\partial_\pi\Delta(x-x')|_{t=t'}$

$$\begin{aligned}&= i\sum_{l=0}^1(-1)^l C_3^{2l} \Gamma_{i^j\cdots\pi\cdots\pi}^{3-2l\ 2l}\left(\frac{3}{2}\right) \partial_i\partial_j\cdots\nabla^{2l}\delta^3(\vec{r}-\vec{r}') \\ &= i[C_3^0\Gamma^{ijk}\left(\frac{3}{2}\right)\partial_i\partial_j\partial_k - C_3^2\Gamma^{i\pi\pi}\left(\frac{3}{2}\right)\partial_i\nabla^2]\delta^3(\vec{r}-\vec{r}') \\ &= i\left[\left(\frac{1}{\sqrt{2}}\right)^3\frac{4i\zeta}{3}\frac{1}{3!}[\sigma^i\left(\frac{3}{2}\right)\sigma^j\left(\frac{3}{2}\right)\sigma^k\left(\frac{3}{2}\right) - \frac{7}{4}\delta^{ij}\sigma^k\left(\frac{3}{2}\right)]\partial_i\partial_j\partial_k + 2i\zeta\left(\frac{1}{\sqrt{2}}\right)^3\sigma^i\left(\frac{3}{2}\right)\partial_i\nabla^2\right]\delta^3(\vec{r}-\vec{r}') \\ &= i\left(\frac{1}{\sqrt{2}}\right)^3 2i\zeta\left\{\frac{2}{3}[\sigma\left(\frac{3}{2}\right)\cdot\nabla]^3 - \frac{7}{4}[\sigma\left(\frac{3}{2}\right)\cdot\nabla]\nabla^2\right\} + [\sigma\left(\frac{3}{2}\right)\cdot\nabla]\nabla^2\delta^3(\vec{r}-\vec{r}') \\ &= -i\left(\frac{1}{\sqrt{2}}\right)^3 \frac{4i\zeta}{3}\{\nabla^2 - 4[\sigma\left(\frac{3}{2}\right)\cdot\nabla]^2\}[\sigma\left(\frac{3}{2}\right)\cdot\nabla]\delta^3(\vec{r}-\vec{r}') \\ &= \frac{\zeta}{4\sqrt{2}}\{\nabla^2 - 9[\frac{2}{3}\sigma\left(\frac{3}{2}\right)\cdot\nabla]^2\}[\frac{2}{3}\sigma\left(\frac{3}{2}\right)\cdot\nabla]\delta^3(\vec{r}-\vec{r}')\end{aligned}$$

□

$$\text{引理7.2.1. } \Gamma_{k_\zeta k'_\zeta}^{abc}p_a p_b p_c = -2\sqrt{2}i|\vec{p}|^3\lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta)\lambda_{k'_\zeta}^+(\hat{p}, -\frac{3}{2}\zeta)$$

证明:  $\Gamma_{k_\zeta k'_\zeta}^{abc}p_a p_b p_c$

$$\begin{aligned}&\succ = C_3^3\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi}(1)p_\pi^3 + C_3^2\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi}(1)p_i p_\pi^2 + C_3^1\Gamma_{k_\zeta k'_\zeta}^{ij\pi}(1)p_i p_j p_\pi + C_3^0\Gamma_{k_\zeta k'_\zeta}^{ijk}(1)p_i p_j p_k \\ &= \left(\frac{1}{\sqrt{2}}\right)^3[-i|\vec{p}|^3 + 2i\zeta|\vec{p}|^2\sigma\left(\frac{3}{2}\right)\cdot\vec{p} - 2i|\vec{p}|[\sigma\left(\frac{3}{2}\right)\cdot\vec{p}]^2 + i\frac{3}{2}|\vec{p}|^3 + \frac{4i\zeta}{3}\{[\sigma\left(\frac{3}{2}\right)\cdot\vec{p}]^3 - \frac{7}{4}|\vec{p}|^2[\sigma\left(\frac{3}{2}\right)\cdot\vec{p}]\}] \\ &= \left(\frac{1}{\sqrt{2}}\right)^3 \frac{i}{6}[3|\vec{p}|^3 - 2\zeta|\vec{p}|^2[\sigma\left(\frac{3}{2}\right)\cdot\vec{p}] - 12|\vec{p}|[\sigma\left(\frac{3}{2}\right)\cdot\vec{p}]^2 + 8\zeta[\sigma\left(\frac{3}{2}\right)\cdot\vec{p}]^3 \\ &= \left(\frac{1}{\sqrt{2}}\right)^3 \frac{i}{6}|\vec{p}|^3[3 - 2\zeta[\sigma\left(\frac{3}{2}\right)\cdot\hat{p}] - 12[\sigma\left(\frac{3}{2}\right)\cdot\hat{p}]^2 + 8\zeta[\sigma\left(\frac{3}{2}\right)\cdot\hat{p}]^3 \\ &= \left\{\left(\frac{1}{\sqrt{2}}\right)^3 \frac{i}{6}|\vec{p}|^3[3 - 2\zeta[\sigma\left(\frac{3}{2}\right)\cdot\hat{p}] - 12[\sigma\left(\frac{3}{2}\right)\cdot\hat{p}]^2 + 8\zeta[\sigma\left(\frac{3}{2}\right)\cdot\hat{p}]^3\right\} \sum_{h=3/2}^{-3/2} \lambda(\hat{p}, h)\lambda^+(\hat{p}, h) \\ &= \prec -2\sqrt{2}i|\vec{p}|^3\lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta)\lambda_{k'_\zeta}^+(\hat{p}, -\frac{3}{2}\zeta)\end{aligned}$$

□

$$\text{推论7.2.4. 投影算子: } \hat{F}_{k_\zeta k'_\zeta}\left(\frac{3}{2}, \zeta\right) = \frac{i}{2\sqrt{2}}\Gamma_{k_\zeta k'_\zeta}^{abc}\hat{p}_a\hat{p}_b\hat{p}_c \rightarrow -\frac{1}{2\sqrt{2}}\Gamma_{k_\zeta k'_\zeta}^{abc}\hat{\partial}_a\hat{\partial}_b\hat{\partial}_c$$

## 7.3 引力微子场数学上一般的协变对易规则

定理7.3.1.

$$\begin{cases} [a_\sigma(\vec{p}, -\frac{3}{2}\zeta), a_{\sigma'}^+(\vec{p}', -\frac{3}{2}\zeta)]_\pm \\ = \delta_\sigma\delta_{\sigma'}\delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}, -\frac{3}{2}\zeta), a_{\sigma'}(\vec{p}', -\frac{3}{2}\zeta)]_\pm = 0 \\ [a_\sigma^+(\vec{p}, -\frac{3}{2}\zeta), a_{\sigma'}^+(\vec{p}', -\frac{3}{2}\zeta)]_\pm = 0 \end{cases} \Rightarrow \begin{cases} [\Psi_{k_\zeta}(x), \Psi_{k'_\zeta}^+(x')]_\pm \\ = -\frac{i}{\sqrt{2}}\Gamma_{k_\zeta k'_\zeta}^{abc}\partial_a\partial_b\partial_c[(\delta_1 - \pm\delta_2)\Delta^{(+)}(x-x') \pm \delta_2\Delta(x-x')] \\ [\Psi_{k_\zeta}(x), \Psi_{l_\zeta}(x')]_\pm = 0 \\ [\Psi_{k'_\zeta}^+(x), \Psi_{l'_\zeta}^+(x')]_\pm = 0 \end{cases}$$

$$\begin{aligned}
& \text{证明: } [\Psi_{k_\zeta}^{(+)}(x), \Psi_{k'_\zeta}^{(+)+}(x')]_{\pm} \\
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda_{k'_\zeta}^+(\vec{p}', -\frac{3}{2}\zeta) |\vec{p}| |\vec{p}'| [a_1(\vec{p}, -\frac{3}{2}\zeta), a_1^+(\vec{p}', -\frac{3}{2}\zeta)]_{\pm} e^{ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda_{k'_\zeta}^+(\vec{p}', -\frac{3}{2}\zeta) |\vec{p}|^2 \delta_1 \delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -\frac{3}{2}\zeta) \delta_1 |\vec{p}|^2 e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= \frac{\delta_1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} p_a p_b p_c e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= \frac{\delta_1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= -\frac{i}{\sqrt{2}} \delta_1 \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta^{(+)}(x-x') \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } [\Psi_{k_\zeta}^{(-)}(x), \Psi_{k'_\zeta}^{(-)+}(x')]_{\pm} \\
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda_{k'_\zeta}^+(\vec{p}', -\frac{3}{2}\zeta) |\vec{p}| |\vec{p}'| [a_2(\vec{p}, -\frac{3}{2}\zeta), a_2^+(\vec{p}', -\frac{3}{2}\zeta)]_{\pm} e^{-ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \pm \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda_{k'_\zeta}^+(\vec{p}', -\frac{3}{2}\zeta) |\vec{p}|^2 \delta_2 \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \pm \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -\frac{3}{2}\zeta) \delta_2 |\vec{p}|^2 e^{-ip \cdot (x-x')} d^3 \vec{p} \\
&= \pm \frac{\delta_2}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} p_a p_b p_c e^{-ip \cdot (x-x')} d^3 \vec{p} \\
&= \pm \frac{\delta_2}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c e^{-ip \cdot (x-x')} d^3 \vec{p} \\
&= -\pm \frac{i}{\sqrt{2}} \delta_2 \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta^{(-)}(x-x') \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } [\Psi_{k_\zeta}(x), \Psi_{k'_\zeta}^+(x')]_{\pm} \\
&= [\Psi_{k_\zeta}^{(+)}(x), \Psi_{k'_\zeta}^{(+)+}(x')]_{\pm} + [\Psi_{k_\zeta}^{(-)}(x), \Psi_{k'_\zeta}^{(-)+}(x')]_{\pm} \\
&= -\frac{i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c [\delta_1 \Delta^{(+)}(x-x') \pm \delta_2 \Delta^{(-)}(x-x')] \\
&= -\frac{i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c [(\delta_1 - \pm \delta_2) \Delta^{(+)}(x-x') \pm \delta_2 \Delta^{(-)}(x-x')] \quad \square
\end{aligned}$$

从上式可知，只有 $\delta_1 - \pm \delta_2 = 0$ 时，才满足微观因果性，同时只有 $\delta_1, \delta_2 \geq 0$ 时，才满足几率非负性。所以数学上八种协变对易或反对易方案中，物理上合理的只有一种：即 $\delta_1 = \delta_2 = 1$ ，且满足反对易关系。其实还有两种，即 $\delta_1 = \delta_2 = 0$ ，且满足对易或反对易关系，就是经典情形。

#### 7.4 引力微子场物理的协变反对易规则

$$\text{定理7.4.1. } \begin{cases} \{a_\sigma(\vec{p}, -\frac{3}{2}\zeta), a_{\sigma'}^+(\vec{p}', -\frac{3}{2}\zeta)\} = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\sigma(\vec{p}, -\frac{3}{2}\zeta), a_{\sigma'}(\vec{p}', -\frac{3}{2}\zeta)\} = 0 \\ \{a_\sigma^+(\vec{p}, -\frac{3}{2}\zeta), a_{\sigma'}^+(\vec{p}', -\frac{3}{2}\zeta)\} = 0 \end{cases} \Rightarrow \begin{cases} \{\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')\} = \frac{-i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta(x-x') \\ \{\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')\} = 0 \\ \{\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')\} = 0 \end{cases}$$

$$\begin{aligned}
& \text{证明: } \{\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')\} \\
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda_{k'_\zeta}^+(\hat{p}', -\frac{3}{2}\zeta) |\vec{p}| |\vec{p}'| \\
&\{ \{a_1(\vec{p}, -\frac{3}{2}\zeta), a_1^+(\vec{p}', -\frac{3}{2}\zeta)\} e^{ip \cdot (x-x')} + \{a_2^+(\vec{p}, -\frac{3}{2}\zeta), a_2(\vec{p}', -\frac{3}{2}\zeta)\} e^{-ip \cdot (x-x')} \} d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^2 \lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda_{k'_\zeta}^+(\hat{p}', -\frac{3}{2}\zeta) [\delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x-x')} + \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x-x')}] d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^2 \lambda_{k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -\frac{3}{2}\zeta) [e^{ip \cdot (x-x')} + e^{-ip \cdot (x-x')}] d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} p_a p_b p_c [e^{ip \cdot (x-x')} + e^{-ip \cdot (x-x')}] d^3 \vec{p} \\
&= i \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3 \vec{p} \\
&= \frac{-i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3 \vec{p} \\
&= \frac{-i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta(x-x') \quad \square
\end{aligned}$$

#### 7.5 引力微子场的等时反对易规则

推论7.5.1.

$$\begin{cases} \{\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')\} = \frac{-i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta(x-x') \\ \{\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')\} = 0 \\ \{\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')\} = 0 \end{cases} \Rightarrow \begin{cases} \{\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)\} \\ = \frac{1}{8} \{ \nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2 \}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)\} = 0, \{\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases}$$

$$\text{性质7.5.1.} \quad \begin{cases} \Delta^*(x) = \Delta(x), \Delta(-x) = -\Delta(x), (\nabla^2 - \partial_t^2)\Delta(x) = 0 \\ \partial_t \Delta(x)|_{t=0} = -\delta^3(\vec{r}), \partial_k \partial_t \Delta(x)|_{t=0} = \partial_t \partial_k \Delta(x)|_{t=0} = -\partial_k \delta^3(\vec{r}) \\ \partial_k \Delta(x)|_{t=0} = 0, \partial_k \partial_l \Delta(x)|_{t=0} = 0, \partial_t^2 \Delta(x)|_{t=0} = 0 \end{cases}$$

$$\begin{aligned} \text{证明: } & \{\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)\} = \frac{-i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta[(x-x')]|_{t=t'} \\ & = C_3^1 \frac{-i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{ij\pi} \partial_i \partial_j \partial_\pi \Delta[(x-x')]|_{t=t'} + \frac{-i}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi} \partial_\pi \partial_\pi \partial_\pi \Delta[(x-x')]|_{t=t'} \\ & = \frac{1}{8} \{\nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2\}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \end{aligned}$$

推论7.5.2.

$$\begin{cases} \{\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)\} \\ = \frac{1}{8} \{\nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2\}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)\} = 0, \{\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)\} = 0 \end{cases} \Rightarrow \begin{cases} \{a_\sigma(\vec{p}, -\frac{3}{2}\zeta), a_\sigma^+(\vec{p}', -\frac{3}{2}\zeta)\} = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\sigma(\vec{p}, -\frac{3}{2}\zeta), a_{\sigma'}(\vec{p}', -\frac{3}{2}\zeta)\} = 0 \\ \{a_\sigma^+(\vec{p}, -\frac{3}{2}\zeta), a_{\sigma'}^+(\vec{p}', -\frac{3}{2}\zeta)\} = 0 \end{cases}$$

$$\begin{aligned} \text{证明: } & \{a_1(\vec{p}, -\frac{3}{2}\zeta), a_1^+(\vec{p}', -\frac{3}{2}\zeta)\} \\ & = \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}||\vec{p}'|} \int \{\lambda^{+k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \Psi_{k_\zeta}(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r}-Et)}, \lambda^{k'_\zeta}(\vec{p}', -\frac{3}{2}\zeta) \Psi_{k'_\zeta}^+(\vec{r}', t) e^{i(\vec{p}'\cdot\vec{r}'-E't)}\} d^3\vec{r} d^3\vec{r}' \\ & = \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}||\vec{p}'|} \int \lambda^{+k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda^{k'_\zeta}(\vec{p}', -\frac{3}{2}\zeta) [\Psi_{k_\zeta}(\vec{r}, t), \Psi_{k'_\zeta}^+(\vec{r}', t)] e^{-i(\vec{p}\cdot\vec{r}-Et)} e^{i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\ & = \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}||\vec{p}'|} \int \lambda^{+k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda^{k'_\zeta}(\vec{p}', -\frac{3}{2}\zeta) \frac{1}{8} \{\nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2\}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') e^{-i(\vec{p}\cdot\vec{r}-Et)} e^{i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\ & = \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}||\vec{p}'|} \int \lambda^{+k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda^{k'_\zeta}(\vec{p}', -\frac{3}{2}\zeta) \frac{-1}{8} \{\vec{p}^2 - 4[\sigma(\frac{3}{2}) \cdot \vec{p}]^2\}_{k_\zeta k'_\zeta} e^{-i(\vec{p}\cdot\vec{r}-Et)} e^{i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} \\ & = \frac{1}{|\vec{p}||\vec{p}'|} \lambda^{+k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda^{k'_\zeta}(\vec{p}', -\frac{3}{2}\zeta) \frac{-1}{8} \{\vec{p}^2 - 4[\sigma(\frac{3}{2}) \cdot \vec{p}]^2\}_{k_\zeta k'_\zeta} \delta^3(\vec{p} - \vec{p}') \\ & = \lambda^+(\hat{p}, -\frac{3}{2}\zeta) \frac{-1}{8} \{\vec{p}^2 - 4[\sigma(\frac{3}{2}) \cdot \vec{p}]^2\} \lambda(\hat{p}, -\frac{3}{2}\zeta) \delta^3(\vec{p} - \vec{p}') \\ & = \lambda^+(\hat{p}, -\frac{3}{2}\zeta) \lambda(\hat{p}, -\frac{3}{2}\zeta) \delta^3(\vec{p} - \vec{p}') \\ & = \delta^3(\vec{p} - \vec{p}') \end{aligned}$$

$$\begin{aligned} \text{证明: } & \{a_2^+(\vec{p}, -\frac{3}{2}\zeta), a_2(\vec{p}', -\frac{3}{2}\zeta)\} \\ & = \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}||\vec{p}'|} \int \{\lambda^{+k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \Psi_{k_\zeta}(\vec{r}, t) e^{i(\vec{p}\cdot\vec{r}-Et)}, \lambda^{k'_\zeta}(\vec{p}', -\frac{3}{2}\zeta) \Psi_{k'_\zeta}^+(\vec{r}', t) e^{-i(\vec{p}'\cdot\vec{r}'-E't)}\} d^3\vec{r} d^3\vec{r}' \\ & = \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}||\vec{p}'|} \int \lambda^{+k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda^{k'_\zeta}(\vec{p}', -\frac{3}{2}\zeta) [\Psi_{k_\zeta}(\vec{r}, t), \Psi_{k'_\zeta}^+(\vec{r}', t)] e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\ & = \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}||\vec{p}'|} \int \lambda^{+k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda^{k'_\zeta}(\vec{p}', -\frac{3}{2}\zeta) \frac{1}{8} \{\nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2\}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\ & = \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}||\vec{p}'|} \int \lambda^{+k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda^{k'_\zeta}(\vec{p}', -\frac{3}{2}\zeta) \frac{-1}{8} \{\vec{p}^2 - 4[\sigma(\frac{3}{2}) \cdot \vec{p}]^2\}_{k_\zeta k'_\zeta} e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} \\ & = \frac{1}{|\vec{p}||\vec{p}'|} \lambda^{+k_\zeta}(\hat{p}, -\frac{3}{2}\zeta) \lambda^{k'_\zeta}(\vec{p}', -\frac{3}{2}\zeta) \frac{-1}{8} \{\vec{p}^2 - 4[\sigma(\frac{3}{2}) \cdot \vec{p}]^2\}_{k_\zeta k'_\zeta} \delta^3(\vec{p} - \vec{p}') \\ & = \lambda^+(\hat{p}, -\frac{3}{2}\zeta) \frac{-1}{8} \{\vec{p}^2 - 4[\sigma(\frac{3}{2}) \cdot \vec{p}]^2\} \lambda(\hat{p}, -\frac{3}{2}\zeta) \delta^3(\vec{p} - \vec{p}') \\ & = \lambda^+(\hat{p}, -\frac{3}{2}\zeta) \lambda(\hat{p}, -\frac{3}{2}\zeta) \delta^3(\vec{p} - \vec{p}') \\ & = \delta^3(\vec{p} - \vec{p}') \end{aligned}$$

## 7.6 引力微子场的对易函数、因果函数和费曼传播子

推论7.6.1.

$$\begin{cases} \Delta_{k_\zeta k'_\zeta}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{(+)}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta^{(+)}(x) \\ \Delta_{k_\zeta k'_\zeta}^{(-)}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta^{(-)}(x) \\ \Delta_{k_\zeta k'_\zeta}^{(l)}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta^{(l)}(x) \end{cases}$$

推论7.6.2.

$$\begin{cases} \Delta_{k_\zeta k'_\zeta}^{(c)}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta^{(c)}(x) + \frac{i}{\sqrt{2}} [\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi} \delta''(t) + 3i \Gamma_{k_\zeta k'_\zeta}^{i\pi\pi} \delta'(t) \partial_i - 3 \Gamma_{k_\zeta k'_\zeta}^{ij\pi} \delta(t) \partial_i \partial_j] \Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{ret}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{ijk} \partial_i \partial_j \partial_k \Delta^{ret}(x) + \frac{i}{\sqrt{2}} [\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi} \delta''(t) + 3i \Gamma_{k_\zeta k'_\zeta}^{i\pi\pi} \delta'(t) \partial_i - 3 \Gamma_{k_\zeta k'_\zeta}^{ij\pi} \delta(t) \partial_i \partial_j] \Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{adv}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta^{adv}(x) + \frac{i}{\sqrt{2}} [\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi} \delta''(t) + 3i \Gamma_{k_\zeta k'_\zeta}^{i\pi\pi} \delta'(t) \partial_i - 3 \Gamma_{k_\zeta k'_\zeta}^{ij\pi} \delta(t) \partial_i \partial_j] \Delta(x) \\ \Delta_{F k_\zeta k'_\zeta}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta_F(x) + \frac{-1}{\sqrt{2}} [\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi} \delta''(t) + 3i \Gamma_{k_\zeta k'_\zeta}^{i\pi\pi} \delta'(t) \partial_i - 3 \Gamma_{k_\zeta k'_\zeta}^{ij\pi} \delta(t) \partial_i \partial_j] \Delta(x) \\ = i \Delta_{k_\zeta k'_\zeta}^{(c)}(\frac{3}{2}; x) \end{cases}$$

## 推论7.6.3.

$$\begin{cases} \Delta_{k_\zeta k'_\zeta}^{(c)}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta^{(c)}(x) + \frac{i}{\sqrt{2}} [2\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi} \partial_t \delta^4(x) + 3i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi} \partial_i \delta^4(x)] \\ \Delta_{k_\zeta k'_\zeta}^{ret}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{ijk} \partial_i \partial_j \partial_k \Delta^{ret}(x) + \frac{i}{\sqrt{2}} [2\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi} \partial_t \delta^4(x) + 3i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi} \partial_i \delta^4(x)] \\ \Delta_{k_\zeta k'_\zeta}^{adv}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta^{adv}(x) + \frac{i}{\sqrt{2}} [2\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi} \partial_t \delta^4(x) + 3i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi} \partial_i \delta^4(x)] \\ \Delta_{F k_\zeta k'_\zeta}(\frac{3}{2}; x) := \frac{-1}{\sqrt{2}} \Gamma_{k_\zeta k'_\zeta}^{abc} \partial_a \partial_b \partial_c \Delta_F(x) + \frac{-i}{\sqrt{2}} [2\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi} \partial_t \delta^4(x) + 3i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi} \partial_i \delta^4(x)] \\ = i\Delta_{k_\zeta k'_\zeta}^{(c)}(\frac{3}{2}; x) \\ \Delta_{F k_\zeta k'_\zeta}(\frac{3}{2}; p) = \frac{1}{\sqrt{2}} \frac{\Gamma_{k_\zeta k'_\zeta}^{abc} p_a p_b p_c}{p^2 - i\varepsilon} + \dots \end{cases}$$

## 推论7.6.4.

$$\begin{cases} [s\partial_a + iS_{ab}(\frac{3}{2}, \varsigma)\partial^b] \Delta(\frac{3}{2}; x) = 0 & [s\partial_a + iS_{ab}(\frac{3}{2}, \varsigma)\partial^b] \Delta^{(c)}(\frac{3}{2}; x) = -\varsigma[\sigma(\frac{3}{2}), i\frac{3}{2}\varsigma]_a \delta(t) \Delta(\frac{3}{2}; x)|_{t=0} \\ [s\partial_a + iS_{ab}(\frac{3}{2}, \varsigma)\partial^b] \Delta^{(+)}(\frac{3}{2}; x) = 0 & [s\partial_a + iS_{ab}(\frac{3}{2}, \varsigma)\partial^b] \Delta^{ret}(\frac{3}{2}; x) = -\varsigma[\sigma(\frac{3}{2}), i\frac{3}{2}\varsigma]_a \delta(t) \Delta(\frac{3}{2}; x)|_{t=0} \\ [s\partial_a + iS_{ab}(\frac{3}{2}, \varsigma)\partial^b] \Delta^{(-)}(\frac{3}{2}; x) = 0 & [s\partial_a + iS_{ab}(\frac{3}{2}, \varsigma)\partial^b] \Delta^{adv}(\frac{3}{2}; x) = -\varsigma[\sigma(\frac{3}{2}), i\frac{3}{2}\varsigma]_a \delta(t) \Delta(\frac{3}{2}; x)|_{t=0} \\ [s\partial_a + iS_{ab}(\frac{3}{2}, \varsigma)\partial^b] \Delta^{(l)}(\frac{3}{2}; x) = 0 & [s\partial_a + iS_{ab}(\frac{3}{2}, \varsigma)\partial^b] \Delta_F(\frac{3}{2}; x) = -i\varsigma[\sigma(\frac{3}{2}), i\frac{3}{2}\varsigma]_a \delta(t) \Delta(\frac{3}{2}; x)|_{t=0} \end{cases}$$

## 7.7 引力微子的量子方程

## 推论7.7.1.

$$\begin{cases} \Delta_{k_\zeta k'_\zeta}^{(c)}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} \overbrace{(\partial_a \partial_b \partial_c \dots)}^{2s} \Delta^{(c)}(x) + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \Gamma_{k_\zeta k'_\zeta}^{ab \dots \pi \dots} \overbrace{(\partial_t^{2s-1-n} \delta(t))}^{n} \overbrace{(\partial_a \partial_b \dots)}^n \Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{ret}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} \overbrace{(\partial_a \partial_b \partial_c \dots)}^{2s} \Delta^{ret}(x) + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \Gamma_{k_\zeta k'_\zeta}^{ab \dots \pi \dots} \overbrace{(\partial_t^{2s-1-n} \delta(t))}^{n} \overbrace{(\partial_a \partial_b \dots)}^n \Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{adv}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} \overbrace{(\partial_a \partial_b \partial_c \dots)}^{2s} \Delta^{adv}(x) + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \Gamma_{k_\zeta k'_\zeta}^{ab \dots \pi \dots} \overbrace{(\partial_t^{2s-1-n} \delta(t))}^{n} \overbrace{(\partial_a \partial_b \dots)}^n \Delta(x) \\ \Delta_{F k_\zeta k'_\zeta}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} \overbrace{(\partial_a \partial_b \partial_c \dots)}^{2s} \Delta_F(x) + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \Gamma_{k_\zeta k'_\zeta}^{ab \dots \pi \dots} \overbrace{(\partial_t^{2s-1-n} \delta(t))}^{n} \overbrace{(\partial_a \partial_b \dots)}^n \Delta(x) \\ = i\Delta_{k_\zeta k'_\zeta}^{(c)}(s; x) \end{cases}$$

$$\text{推论7.7.2. } [\frac{3}{2}\partial_a + iS_{ab}(\frac{3}{2}, \varsigma)\partial^b] \psi(x) = 0 \Rightarrow \begin{cases} \dot{\psi}(\vec{r}, t) = -i[\psi(\vec{r}, t), H] \\ \nabla\psi(\vec{r}, t) = i[\psi(\vec{r}, t), \vec{P}] \\ \partial_a \psi(\vec{r}, t) = i[\psi(\vec{r}, t), P_a] \end{cases}$$

## 定理7.7.1.

$$\begin{cases} \{\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)\} = \frac{1}{8} \{\nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2\}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ \{\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)\} = 0, \{\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)\} = 0 \\ H = \frac{-i\varsigma}{3/2} \int \psi^+(\vec{r}, t) \frac{\sigma(\frac{3}{2}) \cdot \nabla}{\nabla^2} \psi(\vec{r}, t) d^3\vec{r}, \vec{P} = \int \psi^+(\vec{r}, t) \frac{-i\nabla}{\nabla^2} \psi(\vec{r}, t) d^3\vec{r} \\ \Rightarrow \begin{cases} [\psi(\vec{r}, t), H] = \frac{-i\varsigma}{3/2} \frac{1}{8} \{\nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2\} \frac{\sigma(\frac{3}{2}) \cdot \nabla}{\nabla^2} \psi(\vec{r}, t) \\ [\psi(\vec{r}, t), \vec{P}] = \frac{1}{8} \{\nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2\} \frac{-i\nabla}{\nabla^2} \psi(\vec{r}, t) \end{cases} \end{cases}$$

证明:  $[\psi(\vec{r}, t), H]$ 

$$\begin{aligned} &= \frac{-i\varsigma}{3/2} \delta^{k'_\zeta k_\zeta} \int d^3\vec{r}' [\psi_{j_\zeta}(\vec{r}', t), \psi_{k'_\zeta}^+(\vec{r}', t) \frac{\sigma(\frac{3}{2}) \cdot \nabla'}{\nabla'^2} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \frac{-i\varsigma}{3/2} \delta^{k'_\zeta k_\zeta} \int d^3\vec{r}' [\psi_{j_\zeta}(\vec{r}', t), \psi_{k'_\zeta}^+(\vec{r}', t)] \frac{\sigma(\frac{3}{2}) \cdot \nabla'}{\nabla'^2} \psi_{k_\zeta}(\vec{r}', t) \\ &= \frac{-i\varsigma}{3/2} \delta^{k'_\zeta k_\zeta} \int d^3\vec{r}' \frac{1}{8} \{\nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2\}_{j_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \frac{\sigma(\frac{3}{2}) \cdot \nabla'}{\nabla'^2} \psi_{k_\zeta}(\vec{r}', t) \\ &= \frac{-i\varsigma}{3/2} \frac{1}{8} \delta^{k'_\zeta k_\zeta} \{\nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2\}_{k_\zeta k'_\zeta} \frac{\sigma(\frac{3}{2}) \cdot \nabla}{\nabla^2} \psi_{k_\zeta}(\vec{r}, t) \\ &= \frac{-i\varsigma}{3/2} \frac{1}{8} \{\nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2\} \frac{\sigma(\frac{3}{2}) \cdot \nabla}{\nabla^2} \psi(\vec{r}, t) \end{aligned}$$

□

证明:  $[\psi(\vec{r}, t), P]$

$$\begin{aligned} &= \delta^{k'_\zeta k_\zeta} \int d^3\vec{r}' [\psi_{j_\zeta}(\vec{r}', t), \psi_{k'_\zeta}^+(\vec{r}', t) \frac{-i\nabla}{\sqrt{-\nabla^2}} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \delta^{k'_\zeta k_\zeta} \int d^3\vec{r}' [\psi_{j_\zeta}(\vec{r}', t), \psi_{k'_\zeta}^+(\vec{r}', t) \frac{-i\nabla}{\sqrt{-\nabla^2}} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \delta^{k'_\zeta k_\zeta} \int d^3\vec{r}' \frac{1}{8} \{ \nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2 \}_{j_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \frac{-i\nabla}{\sqrt{-\nabla^2}} \psi_{k_\zeta}(\vec{r}', t) \\ &= \frac{1}{8} \delta^{k'_\zeta k_\zeta} \{ \nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2 \}_{k_\zeta k'_\zeta} \frac{-i\nabla}{\sqrt{-\nabla^2}} \psi_{k_\zeta}(\vec{r}, t) \\ &= \frac{1}{8} \{ \nabla^2 - 4[\sigma(\frac{3}{2}) \cdot \nabla]^2 \} \frac{-i\nabla}{\sqrt{-\nabla^2}} \psi(\vec{r}, t) \end{aligned} \quad \square$$

推论7.7.3.

$$\begin{cases} \dot{\psi}(\vec{r}, t) = -i[\psi(\vec{r}, t), H] \\ \nabla\psi(\vec{r}, t) = i[\psi(\vec{r}, t), \vec{P}] \end{cases} \Leftrightarrow \begin{cases} \dot{\psi}(\vec{r}, t) = \frac{-i}{12} \{ \sigma(\frac{3}{2}) \cdot \nabla - \frac{4}{\sqrt{2}} [\sigma(\frac{3}{2}) \cdot \nabla]^3 \} \psi(\vec{r}, t) \\ \nabla\psi(\vec{r}, t) = -\frac{1}{8} \{ 1 - \frac{4}{\sqrt{2}} [\sigma(\frac{3}{2}) \cdot \nabla]^2 \} \nabla\psi(\vec{r}, t) \end{cases} \Leftrightarrow \begin{cases} \partial^a \partial_a \psi(\vec{r}, t) = 0 \\ [\sigma(\frac{3}{2}), -\frac{3}{2} i\zeta]^a \partial_a \psi(\vec{r}, t) = 0 \end{cases}$$

推论7.7.4.

$$\begin{cases} (\frac{3}{2})^2 \nabla\psi = \frac{3}{2} \sigma(\frac{3}{2}) \cdot \nabla \sigma(\frac{3}{2}) \psi - \frac{1}{2} \sigma(\frac{3}{2}) [\sigma(\frac{3}{2}) \cdot \nabla] \psi \\ [\sigma(\frac{3}{2}), -\frac{3}{2} i\zeta]^a \partial_a \psi(\vec{r}, t) = 0 \end{cases} \Rightarrow \begin{cases} \partial^a \partial_a \psi(\vec{r}, t) = 0 \\ [\sigma(2), -\frac{3}{2} i\zeta]^a \partial_a \psi(\vec{r}, t) = 0 \end{cases}$$

$$\text{推论7.7.5. } \frac{6}{2} \begin{bmatrix} 3\partial_z & \sqrt{3}\partial_- & 0 & 0 \\ \sqrt{3}\partial_+ & \partial_z & 2\partial_- & 0 \\ 0 & 2\partial_+ & -\partial_z & \sqrt{3}\partial_- \\ 0 & 0 & \sqrt{3}\partial_+ & -3\partial_z \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} = \frac{6}{4} \begin{bmatrix} 9\partial_z & \sqrt{3}\partial_- & 0 & 0 \\ 3\sqrt{3}\partial_+ & \partial_z & -2\partial_- & 0 \\ 0 & 2\partial_+ & \partial_z & -3\sqrt{3}\partial_- \\ 0 & 0 & -\sqrt{3}\partial_+ & 9\partial_z \end{bmatrix}$$

$$\text{推论7.7.6. } -\frac{2}{2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3\partial_z & \sqrt{3}\partial_- & 0 & 0 \\ \sqrt{3}\partial_+ & \partial_z & 2\partial_- & 0 \\ 0 & 2\partial_+ & -\partial_z & \sqrt{3}\partial_- \\ 0 & 0 & \sqrt{3}\partial_+ & -3\partial_z \end{bmatrix} = -\frac{2}{4} \begin{bmatrix} 9\partial_z & 3\sqrt{3}\partial_- & 0 & 0 \\ \sqrt{3}\partial_+ & \partial_z & 2\partial_- & 0 \\ 0 & -2\partial_+ & \partial_z & -\sqrt{3}\partial_- \\ 0 & 0 & -3\sqrt{3}\partial_+ & 9\partial_z \end{bmatrix}$$

$$\sigma(\frac{3}{2}) = \left( \frac{1}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}, \frac{i}{2} \begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \right) \quad (23.3)$$

## 7.8 引力微子的彭加莱对称性

$$\text{引理7.8.1. } \nabla^2(r_i \partial_j - r_j \partial_i) = (r_i \partial_j - r_j \partial_i) \nabla^2$$

$$\text{引理7.8.2. } [\sigma(s) \cdot \nabla](r_i \partial_j - r_j \partial_i) = (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla] + [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i]$$

$$\text{引理7.8.3. } [\sigma(s) \cdot \nabla]^2(r_i \partial_j - r_j \partial_i) = (r_i \partial_j - r_j \partial_i)$$

证明:  $[\sigma(s) \cdot \nabla]^2(r_i \partial_j - r_j \partial_i)$

$$\begin{aligned} &= [\sigma(s) \cdot \nabla] \{ (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla] + [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i] \} \\ &= (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla]^2 + [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i] [\sigma(s) \cdot \nabla] + [\sigma(s) \cdot \nabla] [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i] \end{aligned} \quad \square$$

$$\text{推论7.8.1. } \frac{-i}{\sqrt{2}} \Gamma^{abc} (\frac{3}{2}) \partial_a \partial_b \partial_c \Delta(x - x')|_{t=t'} = \frac{1}{8} \{ \{ \nabla^2 - 9[\frac{2}{3} \sigma(\frac{3}{2}) \cdot \nabla]^2 \} \} \delta^3(\vec{r} - \vec{r}')$$

$$\text{推论7.8.2. } \frac{-i}{\sqrt{2}} \Gamma^{abc} (\frac{3}{2}) \partial_a \partial_b \partial_c \partial_\pi \Delta(x - x')|_{t=t'} = \frac{1}{8} \{ \nabla^2 - 9[\frac{2}{3} \sigma(\frac{3}{2}) \cdot \nabla]^2 \} [-i\zeta \frac{2}{3} \sigma(\frac{3}{2}) \cdot \nabla] \delta^3(\vec{r} - \vec{r}')$$

## 7.9 引力微子场的彭加莱对称性

$$\text{引理7.9.1. } \begin{cases} P_a = -i \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \partial_a \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3\vec{r} = \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \hat{P}_a \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3\vec{r} \\ L_{ab} = -i \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} (r_a \partial_b - r_b \partial_a) \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3\vec{r} = \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \hat{L}_{ab} \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3\vec{r} \\ M_{ab} = \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} [-i(r_a \partial_b - r_b \partial_a) + \hat{S}_{ab}] \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3\vec{r} = \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \hat{M}_{ab} \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3\vec{r} \end{cases}$$

$$\text{推论7.9.1. } \begin{cases} \{ \frac{\psi_{k_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{\sqrt{-\nabla'^2}} \} = \frac{1}{8} \{ 9[\frac{2}{3} \sigma(\frac{3}{2}) \cdot \hat{\nabla}]^2 - 1 \}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}'), \hat{\nabla} := \frac{-i\nabla}{\sqrt{-\nabla^2}} \\ \{ \frac{\psi_{k_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}}, \frac{\psi_{l_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \} = 0, \{ \frac{\psi_{k'_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}}, \frac{\psi_{l'_\zeta}^+(\vec{r}', t)}{\sqrt{-\nabla'^2}} \} = 0 \end{cases}$$

$$\text{定理7.9.1. } \begin{cases} [L_{ab}, L_{cd}] = -i(g_{ad} L_{bc} - g_{ac} L_{bd} + g_{bc} L_{ad} - g_{bd} L_{ac}) \\ [L_{ab}, P_c] = -i(g_{bc} P_a - g_{ac} P_b), [P_a, P_b] = 0 \end{cases}$$



$$\begin{aligned}
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \frac{1}{8} \left\{ 9 \left[ \frac{2}{3} \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right]^2 - 1 \right\} l_\zeta k'_\zeta \partial_a \delta^3(\vec{r} - \vec{r}') \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{\sqrt{-\nabla'^2}} \frac{1}{8} \left\{ 9 \left[ \frac{2}{3} \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla}' \right]^2 - 1 \right\} l'_\zeta k_\zeta \partial'_b \delta^3(\vec{r}' - \vec{r}) \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r}' d^3 \vec{r} \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \frac{1}{8} \left\{ 9 \left[ \frac{2}{3} \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right]^2 - 1 \right\} l_\zeta k'_\zeta \partial'_a \delta^3(\vec{r} - \vec{r}') \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{\sqrt{-\nabla'^2}} \frac{1}{8} \left\{ 9 \left[ \frac{2}{3} \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla}' \right]^2 - 1 \right\} l'_\zeta k_\zeta \partial_b \delta^3(\vec{r}' - \vec{r}) \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} \\
& = - \int \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \frac{1}{8} \left\{ 9 \left[ \frac{2}{3} \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right]^2 - 1 \right\} k_\zeta l'_\zeta \partial_a \partial_b \frac{\psi_{l'_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} - \frac{\psi_{k'_\zeta}^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \frac{1}{8} \left\{ 9 \left[ \frac{2}{3} \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right]^2 - 1 \right\} k'_\zeta l_\zeta \partial_b \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} d^3 \vec{r}' \\
& = - \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} (\partial_a \partial_b - \partial_b \partial_a) \frac{1}{8} \left\{ 9 \left[ \frac{2}{3} \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right]^2 - 1 \right\} \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3 \vec{r}' \\
& = - \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} (\partial_a \partial_b - \partial_b \partial_a) \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3 \vec{r}' \\
& = \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} [\hat{P}_a, \hat{P}_b] \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3 \vec{r}' = 0
\end{aligned}$$

□

$$\text{引理 7.9.2. } \left[ \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right]^2 \sigma \left( \frac{3}{2} \right) \psi = \left\{ 2 \left[ \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right] \hat{\nabla} + \frac{1}{4} \sigma \left( \frac{3}{2} \right) \right\} \psi, S_{ab} \left( \frac{3}{2} \right) = i \sigma_{\zeta ab}^{\alpha \zeta} \sigma_{\alpha \zeta} \left( \frac{3}{2} \right)$$

$$\text{引理 7.9.3. } \left[ \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right]^2 S_{ab} \left( \frac{3}{2} \right) \psi = \left\{ 2 \left[ \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right] i \sigma_{\zeta ab}^{\alpha \zeta} \hat{\nabla}_{\alpha \zeta} + \frac{1}{4} S_{ab} \left( \frac{3}{2} \right) \right\} \psi$$

$$\text{引理 7.9.4. } \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \sigma_{\alpha \zeta} \left( \frac{3}{2} \right) \left[ \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right] \hat{\nabla}_{\beta \zeta} \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3 \vec{r}' = \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \sigma_{\alpha \zeta} \left( \frac{3}{2} \right) \sigma_{\beta \zeta} \left( \frac{3}{2} \right) \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3 \vec{r}'$$

证明:  $[S_{ab}(t), S_{cd}(t)]$ 

$$\begin{aligned}
& = \int \left[ \frac{\psi^{+k_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{abk_\zeta} l_\zeta \left( \frac{3}{2}, \zeta \right) \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}}, \frac{\psi^{+m_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} S_{cdm_\zeta} n_\zeta \left( \frac{3}{2}, \zeta \right) \frac{\psi_{n_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \right] d^3 \vec{r}' d^3 \vec{r} \\
& = \int \left\{ \frac{\psi^{+k_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \left\{ S_{abk_\zeta} l_\zeta \left( \frac{3}{2}, \zeta \right) \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}}, \frac{\psi^{+m_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \right\} S_{cdm_\zeta} n_\zeta \left( \frac{3}{2}, \zeta \right) \frac{\psi_{n_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \right. \\
& \quad \left. - \frac{\psi^{+m_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \left\{ \frac{\psi^{+k_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}}, S_{abk_\zeta} l_\zeta \left( \frac{3}{2}, \zeta \right) \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} S_{cdm_\zeta} n_\zeta \left( \frac{3}{2}, \zeta \right) \right\} d^3 \vec{r}' d^3 \vec{r} \\
& = \int \left\{ \frac{\psi^{+k_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{abk_\zeta} l_\zeta \left( \frac{3}{2}, \zeta \right) S_{cdm_\zeta} n_\zeta \left( \frac{3}{2}, \zeta \right) \frac{1}{8} \left\{ -1 + 4 \frac{[\sigma \left( \frac{3}{2} \right) \cdot \nabla']^2}{\nabla'^2} \right\} l_\zeta m_\zeta \delta^3(\vec{r}' - \vec{r}) \frac{\psi_{n_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} \right. \\
& \quad \left. - \frac{\psi^{+m_\zeta}(\vec{r}', t)}{\sqrt{-\nabla'^2}} S_{cdm_\zeta} n_\zeta \left( \frac{3}{2}, \zeta \right) S_{abk_\zeta} l_\zeta \left( \frac{3}{2}, \zeta \right) \frac{1}{8} \left\{ -1 + 4 \frac{[\sigma \left( \frac{3}{2} \right) \cdot \nabla]^2}{\nabla^2} \right\} n_\zeta k_\zeta \delta^3(\vec{r} - \vec{r}') \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} d^3 \vec{r}' d^3 \vec{r} \\
& = \int \left\{ \frac{\psi^{+k_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{abk_\zeta} l_\zeta \left( \frac{3}{2}, \zeta \right) \frac{1}{8} \left\{ -1 + 4 \frac{[\sigma \left( \frac{3}{2} \right) \cdot \nabla]^2}{\nabla^2} \right\} l_\zeta m_\zeta S_{cdm_\zeta} n_\zeta \left( \frac{3}{2}, \zeta \right) \frac{\psi_{n_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right. \\
& \quad \left. - \frac{\psi^{+m_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{cdm_\zeta} n_\zeta \left( \frac{3}{2}, \zeta \right) \frac{1}{8} \left\{ -1 + 4 \frac{[\sigma \left( \frac{3}{2} \right) \cdot \nabla]^2}{\nabla^2} \right\} n_\zeta k_\zeta S_{abk_\zeta} l_\zeta \left( \frac{3}{2}, \zeta \right) \frac{\psi_{l_\zeta}(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} d^3 \vec{r}' \\
& = \int \left\{ \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{ab} \left( \frac{3}{2}, \zeta \right) \frac{1}{8} \left\{ -1 + 4 \frac{[\sigma \left( \frac{3}{2} \right) \cdot \nabla]^2}{\nabla^2} \right\} S_{cd} \left( \frac{3}{2}, \zeta \right) \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} - \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{cd} \left( \frac{3}{2}, \zeta \right) \frac{1}{8} \left\{ -1 + 4 \frac{[\sigma \left( \frac{3}{2} \right) \cdot \nabla]^2}{\nabla^2} \right\} S_{ab} \left( \frac{3}{2}, \zeta \right) \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} d^3 \vec{r}' \\
& = \int \left\{ \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{ab} \left( \frac{3}{2}, \zeta \right) \frac{1}{8} \left\{ -1 + 4 \left[ \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right]^2 \right\} S_{cd} \left( \frac{3}{2}, \zeta \right) \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} - \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{cd} \left( \frac{3}{2}, \zeta \right) \frac{1}{8} \left\{ -1 + 4 \left[ \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right]^2 \right\} S_{ab} \left( \frac{3}{2}, \zeta \right) \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} d^3 \vec{r}' \\
& = \int \left\{ \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{ab} \left( \frac{3}{2}, \zeta \right) \left[ \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right] i \sigma_{\zeta ab}^{\alpha \zeta} \hat{\nabla}_{\alpha \zeta} \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} - \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{cd} \left( \frac{3}{2}, \zeta \right) \left[ \sigma \left( \frac{3}{2} \right) \cdot \hat{\nabla} \right] i \sigma_{\zeta cd}^{\alpha \zeta} \hat{\nabla}_{\alpha \zeta} \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} d^3 \vec{r}' \\
& ? = \int \left\{ \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{ab} \left( \frac{3}{2}, \zeta \right) i \sigma_{\zeta ab}^{\alpha \zeta} \sigma_{\alpha \zeta} \left( \frac{3}{2} \right) \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} - \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} S_{cd} \left( \frac{3}{2}, \zeta \right) i \sigma_{\zeta cd}^{\alpha \zeta} \sigma_{\alpha \zeta} \left( \frac{3}{2} \right) \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} \right\} d^3 \vec{r}' \\
& = \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} [S_{ab} \left( \frac{3}{2}, \zeta \right), S_{cd} \left( \frac{3}{2}, \zeta \right)] \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3 \vec{r}'
\end{aligned}$$

□

$$\text{证明: } \left[ \sigma(s) \cdot \hat{\nabla} \right]^n \sigma(s) \psi = \left\{ s [s^n - (s-1)^n] (\zeta \hat{\partial}_t)^{n-1} \hat{\nabla} + (s-1)^n (\zeta \hat{\partial}_t)^n \sigma(s) \right\} \psi$$

$$\left[ \sigma(s) \cdot \hat{\nabla} \right]^n \sigma(s) \psi = \left\{ s [s^n - (s-1)^n] \left[ \frac{1}{s} \sigma(s) \cdot \hat{\nabla} \right]^{n-1} \hat{\nabla} + (s-1)^n \sigma(s) \left[ \frac{1}{s} \sigma(s) \cdot \hat{\nabla} \right]^n \right\} \psi$$

□

## 8 自由无耦合引力子场协变量子化方案

### 8.1 引力子自旋算符方程及其平面波解

定理 8.1.1.  $[2\partial_a + iS_{ab}(2, \zeta) \partial^b] \psi(x) = 0$ 

$$\text{推论 8.1.1. } \begin{cases} \psi(\vec{r}, t) := \frac{1}{(2\pi)^{3/2}} \int |\vec{p}|^{3/2} \lambda(\hat{p}, -2\zeta) [a_1(\vec{p}, -2\zeta) e^{i\vec{p} \cdot \vec{x}} + a_2^+(\vec{p}, -2\zeta) e^{-i\vec{p} \cdot \vec{x}}] d^3 \vec{p} \\ \vec{p}^{3/2} a_1(\vec{p}, -2\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -2\zeta) \psi(\vec{r}, t) e^{-i\vec{p} \cdot \vec{x}} d^3 \vec{r} = \frac{1}{(2\pi)^{3/2}} \int \frac{i}{|\vec{p}|} \lambda^+(\hat{p}, -2\zeta) \dot{\psi}(\vec{r}, t) e^{-i\vec{p} \cdot \vec{x}} d^3 \vec{r} \\ |\vec{p}|^{3/2} a_2^+(\vec{p}, -2\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -2\zeta) \psi(\vec{r}, t) e^{i\vec{p} \cdot \vec{x}} d^3 \vec{r} = \frac{1}{(2\pi)^{3/2}} \int \frac{-i}{|\vec{p}|} \lambda^+(\hat{p}, -2\zeta) \dot{\psi}(\vec{r}, t) e^{i\vec{p} \cdot \vec{x}} d^3 \vec{r} \end{cases}$$

定义 8.1.1. 投影算子:  $\hat{P}_{k_\zeta k'_\zeta}(2, \zeta) := \lambda_{k_\zeta}(\hat{p}, -2\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -2\zeta)$ ,  $\hat{P}^2(2, \zeta) = \hat{P}(2, \zeta)$ ,  $\hat{P}^+(2, \zeta) = \hat{P}(2, \zeta)$ 

$$\text{推论 8.1.2. } H_2 = \int |\vec{p}| [a_1^+(\vec{p}, -2\zeta) a_1(\vec{p}, -2\zeta) + a_2(\vec{p}, -2\zeta) a_2^+(\vec{p}, -2\zeta)] d^3 \vec{p} = \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{1}{-\nabla^2} \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r}$$

$$\begin{aligned}
& \text{证明: } H_2 = \int |\vec{p}| [a_1^+(\vec{p}, -2\zeta) a_1(\vec{p}, -2\zeta) + a_2(\vec{p}, -2\zeta) a_2^+(\vec{p}, -2\zeta)] d^3 \vec{p} \\
& = \frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|^2} [\lambda^{k'_\zeta}(\hat{p}, -2\zeta) \psi_{k'_\zeta}^+(\vec{r}, t) e^{i\vec{p} \cdot \vec{x}} \lambda^{+k_\zeta}(\hat{p}, -2\zeta) \psi_{k_\zeta}(\vec{r}, t) e^{-i\vec{p} \cdot \vec{x}}
\end{aligned}$$

$$\begin{aligned}
& + \lambda_{k'_\zeta}^+ (\hat{p}, -2\zeta) \psi_{k'_\zeta}^+ (\vec{r}', t) e^{-i\vec{p}\cdot\vec{x}'} \lambda^{+k_\zeta} (\hat{p}, -2\zeta) \psi_{k_\zeta} (\vec{r}, t) e^{i\vec{p}\cdot\vec{x}}] d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|^2} \lambda^{+k_\zeta} (\hat{p}, -2\zeta) \lambda_{k'_\zeta}^+ (\hat{p}, -2\zeta) \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) [e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \frac{1}{4|\vec{p}|^6} 4|\vec{p}|^4 \lambda_{k_\zeta} (\hat{p}, -2\zeta) \lambda_{k'_\zeta}^+ (\hat{p}, -2\zeta) \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) [e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \Gamma_{k_\zeta k'_\zeta}^{abcd} p_a p_b p_c p_d \frac{1}{|\vec{p}|^6} [e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \left\{ \left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{3} |\vec{p}|^4 \{0 + 4\zeta[\sigma(2) \cdot \hat{p}] - 2[\sigma(2) \cdot \hat{p}]^2 - 4\zeta[\sigma(2) \cdot \hat{p}]^3 + 2[\sigma(2) \cdot \hat{p}]^4\} \frac{1}{|\vec{p}|^6} [e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}] \right\} d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{24} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \left\{ -[\sigma(2) \cdot \hat{p}]^2 + [\sigma(2) \cdot \hat{p}]^4 \right\} \frac{1}{|\vec{p}|^2} [e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{12} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \left\{ -[\sigma(2) \cdot \hat{p}]^2 + [\sigma(2) \cdot \hat{p}]^4 \right\} \frac{1}{|\vec{p}|^2} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{12} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \left\{ \frac{1}{|\vec{p}|^4} [\sigma(2) \cdot i\vec{p}]^2 + \frac{1}{|\vec{p}|^6} [\sigma(2) \cdot i\vec{p}]^4 \right\} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{12} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \left\{ \frac{[\sigma(2) \cdot \nabla]^2}{\nabla^4} - \frac{[\sigma(2) \cdot \nabla]^4}{\nabla^6} \right\} \delta^3(\vec{r} - \vec{r}') d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{12} \int \psi_{k'_\zeta}^+ (\vec{r}, t) \left\{ \frac{[\sigma(2) \cdot \nabla]^2}{\nabla^4} - \frac{[\sigma(2) \cdot \nabla]^4}{\nabla^6} \right\} \psi_{k_\zeta} (\vec{r}, t) d^3\vec{r} \\
& = \int \psi_{k'_\zeta}^+ (\vec{r}, t) \frac{1}{\nabla^2} \psi_{k_\zeta} (\vec{r}, t) d^3\vec{r} \quad \square
\end{aligned}$$

$$\text{推论8.1.3. } P_2 = \int \vec{p} [a_1^+(\vec{p}, -2\zeta) a_1(\vec{p}, -2\zeta) + a_2(\vec{p}, -2\zeta) a_2^+(\vec{p}, -2\zeta)] d^3\vec{p} = \frac{\nabla^2}{2} \int \psi_{k'_\zeta}^+ (\vec{r}, t) \frac{\sigma(2)}{\nabla^2} \psi_{k_\zeta} (\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned}
\text{证明: } P_2 & = \int \vec{p} [a_1^+(\vec{p}, -2\zeta) a_1(\vec{p}, -2\zeta) + a_2(\vec{p}, -2\zeta) a_2^+(\vec{p}, -2\zeta)] d^3\vec{p} \\
& = \frac{1}{(2\pi)^3} \int \frac{\hat{p}}{|\vec{p}|^2} [\lambda_{k'_\zeta}^+ (\hat{p}, -2\zeta) \psi_{k'_\zeta}^+ (\vec{r}', t) e^{i\vec{p}\cdot\vec{x}'} \lambda^{+k_\zeta} (\hat{p}, -2\zeta) \psi_{k_\zeta} (\vec{r}, t) e^{-i\vec{p}\cdot\vec{x}} \\
& + \lambda_{k'_\zeta}^+ (\hat{p}, -2\zeta) \psi_{k'_\zeta}^+ (\vec{r}', t) e^{-i\vec{p}\cdot\vec{x}'} \lambda^{+k_\zeta} (\hat{p}, -2\zeta) \psi_{k_\zeta} (\vec{r}, t) e^{i\vec{p}\cdot\vec{x}}] d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \frac{\hat{p}}{|\vec{p}|^2} \lambda^{+k_\zeta} (\hat{p}, -2\zeta) \lambda_{k'_\zeta}^+ (\hat{p}, -2\zeta) \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) [e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \frac{\hat{p}}{4|\vec{p}|^6} 4|\vec{p}|^4 \lambda_{k_\zeta} (\hat{p}, -2\zeta) \lambda_{k'_\zeta}^+ (\hat{p}, -2\zeta) \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) [e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \Gamma_{k_\zeta k'_\zeta}^{abcd} p_a p_b p_c p_d \frac{\hat{p}}{|\vec{p}|^6} [e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{4} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \left\{ \left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{3} |\vec{p}|^4 \{0 + 4\zeta[\sigma(2) \cdot \hat{p}] - 2[\sigma(2) \cdot \hat{p}]^2 - 4\zeta[\sigma(2) \cdot \hat{p}]^3 + 2[\sigma(2) \cdot \hat{p}]^4\} \frac{\hat{p}}{|\vec{p}|^6} [e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}] \right\} d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{12} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \left\{ \zeta[\sigma(2) \cdot \hat{p}] - \zeta[\sigma(2) \cdot \hat{p}]^3 \right\} \frac{\hat{p}}{|\vec{p}|^2} [e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{\zeta}{6} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \left\{ [\sigma(2) \cdot \hat{p}] - [\sigma(2) \cdot \hat{p}]^3 \right\} \frac{\hat{p}}{|\vec{p}|^2} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{\zeta}{6} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+ (\vec{r}', t) \psi_{k_\zeta} (\vec{r}, t) \left\{ -[\sigma(2) \cdot i\vec{p}] \frac{i\vec{p}}{|\vec{p}|^4} - [\sigma(2) \cdot i\vec{p}]^3 \frac{i\vec{p}}{|\vec{p}|^6} \right\} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{p} d^3\vec{r} d^3\vec{r}' \\
& = \frac{\zeta}{6} \int \psi_{k'_\zeta}^+ (\vec{r}, t) \psi_{k_\zeta} (\vec{r}, t) \left\{ -[\sigma(2) \cdot \nabla] \frac{\nabla}{\nabla^4} + [\sigma(2) \cdot \nabla]^3 \frac{\nabla}{\nabla^6} \right\} \delta^3(\vec{r} - \vec{r}') d^3\vec{r} d^3\vec{r}' \\
& = \frac{\zeta}{6} \int \psi_{k'_\zeta}^+ (\vec{r}, t) \left\{ -[\sigma(2) \cdot \nabla] \frac{\nabla}{\nabla^4} + [\sigma(2) \cdot \nabla]^3 \frac{\nabla}{\nabla^6} \right\} \psi_{k_\zeta} (\vec{r}, t) d^3\vec{r} \\
& = \frac{\zeta}{2} \int \psi_{k'_\zeta}^+ (\vec{r}, t) [\sigma(2) \cdot \nabla] \frac{\nabla}{\nabla^4} \psi_{k_\zeta} (\vec{r}, t) d^3\vec{r} \\
& = \frac{\nabla^2}{2} \int \psi_{k'_\zeta}^+ (\vec{r}, t) \frac{\sigma(2)}{\nabla^2} \psi_{k_\zeta} (\vec{r}, t) d^3\vec{r} \quad \square
\end{aligned}$$

$$\text{推论8.1.4. } P_a = \int p_a [a_1^+(\vec{p}, -2\zeta) a_1(\vec{p}, -2\zeta) + a_2(\vec{p}, -2\zeta) a_2^+(\vec{p}, -2\zeta)] d^3\vec{p} = \frac{\nabla^2}{2} \int \psi_{k'_\zeta}^+ (\vec{r}, t) \frac{[\sigma(2) \cdot \nabla] a}{\nabla^2} \psi_{k_\zeta} (\vec{r}, t) d^3\vec{r}$$

$$\text{推论8.1.5. } P_2 = \int \vec{p} [a_1^+(\vec{p}, -2\zeta) a_1(\vec{p}, -2\zeta) + a_2(\vec{p}, -2\zeta) a_2^+(\vec{p}, -2\zeta)] d^3\vec{p} = \frac{\zeta}{2} \int \psi_{k'_\zeta}^+ (\vec{r}, t) [\sigma(2) \cdot \nabla] \frac{\nabla}{\nabla^4} \psi_{k_\zeta} (\vec{r}, t) d^3\vec{r}$$

## 8.2 引力子场协变常数不变张量的性质

### 推论8.2.1.

$$\begin{aligned}
\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}(2) & = \left(\frac{1}{\sqrt{2}}\right)^4 \delta_{k_\zeta k'_\zeta} \\
\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}(2) & = -i\zeta \left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{2} \sigma^i(2)_{k_\zeta k'_\zeta} \\
\Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}(2) & = -\left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{6} [\sigma^i(2)\sigma^j(2) - 2\delta^{ij}]_{k_\zeta k'_\zeta} = -\left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{3} \frac{1}{2!} [\sigma^i(2)\sigma^j(2) - \delta^{ij}]_{k_\zeta k'_\zeta} \\
\Gamma_{k_\zeta k'_\zeta}^{ijk\pi}(2) & = \left(\frac{1}{\sqrt{2}}\right)^4 \frac{i\zeta}{6} \{ \sigma^i(2)[\sigma^j(2)]\sigma^k(2) - [\sigma^i(2)\delta^{jk} + 2\delta^{ij}\sigma^k(2)] \}_{k_\zeta k'_\zeta} \\
& = \left(\frac{1}{\sqrt{2}}\right)^4 \frac{i\zeta}{3} \frac{1}{3!} \{ \sigma^i(2)\sigma^j(2)\sigma^k(2) - \frac{5}{2} \sigma^i(2)\delta^{jk} \}_{k_\zeta k'_\zeta} \\
\Gamma_{k_\zeta k'_\zeta}^{ijkl}(2) & = \left(\frac{1}{\sqrt{2}}\right)^4 \frac{2}{3} \frac{1}{4!} [\sigma^i(2)\sigma^j(2)\sigma^k(2)\sigma^l(2) - 4\sigma^i(2)\sigma^j(2)\delta^{kl}]_{k_\zeta k'_\zeta} + \frac{3}{2} \delta^{ij}\delta^{kl}]_{k_\zeta k'_\zeta}
\end{aligned}$$

$$\text{引理8.2.1. } \Gamma_{k_\zeta k'_\zeta}^{abcd} p_a p_b p_c p_d = 4|\vec{p}|^4 \lambda_{k_\zeta} (\hat{p}, -2\zeta) \lambda_{k'_\zeta}^+ (\hat{p}, -2\zeta)$$

$$\begin{aligned}
\text{证明: } \Gamma_{k_\zeta k'_\zeta}^{abcd} p_a p_b p_c p_d & = C_4^4 \Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}(1) p_\pi^4 + C_4^3 \Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}(1) p_i p_\pi^3 + C_4^2 \Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}(1) p_i p_j p_\pi^2 + C_4^1 \Gamma_{k_\zeta k'_\zeta}^{ijk\pi}(1) p_i p_j p_k p_\pi + C_4^0 \Gamma_{k_\zeta k'_\zeta}^{ijkl}(1) p_i p_j p_k p_l
\end{aligned}$$



$$\begin{aligned}
&= \left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{3} \{3|\vec{p}|^4 - 6\varsigma|\vec{p}|^3[\sigma(2) \cdot \vec{p}] + 6|\vec{p}|^2\{[\sigma(2) \cdot \vec{p}]^2 - |\vec{p}|^2\} - 4\varsigma|\vec{p}|\{[\sigma(2) \cdot \vec{p}]^3 - \frac{5}{2}|\vec{p}|^2[\sigma(2) \cdot \vec{p}]\}\} \\
&+ 2\{[\sigma(2) \cdot \vec{p}]^4 - 4|\vec{p}|^2[\sigma(2) \cdot \vec{p}]^2 + \frac{3}{2}|\vec{p}|^4\}\}_{k_\varsigma k'_\varsigma} \\
&= \left\{ \left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{3} |\vec{p}|^4 \{0 + 4\varsigma[\sigma(2) \cdot \vec{p}] - 2[\sigma(2) \cdot \vec{p}]^2 - 4\varsigma[\sigma(2) \cdot \vec{p}]^3 + 2[\sigma(2) \cdot \vec{p}]^4\} \sum_{h=2}^{-2} \lambda(\hat{p}, h) \lambda^+(\hat{p}, h) \right\}_{k_\varsigma k'_\varsigma} \\
&= 4|\vec{p}|^4 \lambda_{k_\varsigma}(\hat{p}, -2\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}, -2\varsigma)
\end{aligned}$$

□

推论8.2.2. 投影算子:  $\hat{P}_{k_\varsigma k'_\varsigma}(2, \varsigma) = \frac{1}{4} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \hat{p}_a \hat{p}_b \hat{p}_c \hat{p}_d \rightarrow \frac{1}{4} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \hat{\partial}_d$

### 8.3 引力子场数学上一般的协变对易规则

定理8.3.1.

$$\begin{cases} [a_\sigma(\vec{p}, -2\varsigma), a_{\sigma'}^+(\vec{p}', -2\varsigma)]_\pm \\ = \delta_\sigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -2\varsigma), a_{\sigma'}(\vec{p}', -2\varsigma)]_\pm = 0 \\ [a_\sigma^+(\vec{p}, -2\varsigma), a_{\sigma'}^+(\vec{p}', -2\varsigma)]_\pm = 0 \end{cases} \Rightarrow \begin{cases} [\Psi_{k_\varsigma}(x), \Psi_{k'_\varsigma}^+(x')]_\pm \\ = \frac{i}{2} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \partial_a \partial_b \partial_c \partial_d [\delta_1 \Delta(x-x') - (\delta_1 \pm \delta_2) \Delta^{(-)}(x-x')] \\ [\Psi_{k_\varsigma}(x), \Psi_{\beta_\varsigma}(x')]_\pm = 0 \\ [\Psi_{k'_\varsigma}^+(x), \Psi_{\beta'_\varsigma}^+(x')]_\pm = 0 \end{cases}$$

证明:  $[\Psi_{k_\varsigma}^{(+)}(x), \Psi_{k'_\varsigma}^{(+)}(x')]_\pm$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -2\varsigma) \lambda_{k'_\varsigma}^+(\vec{p}', -2\varsigma) |\vec{p}|^3/2 |\vec{p}'|^3/2 [a_1(\vec{p}, -2\varsigma), a_1^+(\vec{p}', -2\varsigma)]_\pm e^{ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -2\varsigma) \lambda_{k'_\varsigma}^+(\vec{p}', -2\varsigma) |\vec{p}|^3 \delta_1 \delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -2\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}, -2\varsigma) \delta_1 |\vec{p}|^3 e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= \frac{\delta_1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{1}{2} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} p_a p_b p_c p_d e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= \frac{\delta_1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{1}{2} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \partial_a \partial_b \partial_c \partial_d e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= \frac{i}{2} \delta_1 \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \partial_a \partial_b \partial_c \partial_d \Delta^{(+)}(x-x')
\end{aligned}$$

□

证明:  $[\Psi_{k_\varsigma}^{(-)}(x), \Psi_{k'_\varsigma}^{(-)}(x')]_\pm$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -2\varsigma) \lambda_{k'_\varsigma}^+(\vec{p}', -2\varsigma) |\vec{p}|^3/2 |\vec{p}'|^3/2 [a_2^+(\vec{p}, -2\varsigma), a_2(\vec{p}', -2\varsigma)]_\pm e^{-ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \pm \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -2\varsigma) \lambda_{k'_\varsigma}^+(\vec{p}', -2\varsigma) |\vec{p}|^3 \delta_2 \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \pm \frac{1}{(2\pi)^3} \int \lambda_{k_\varsigma}(\hat{p}, -2\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}, -2\varsigma) \delta_2 |\vec{p}|^3 e^{-ip \cdot (x-x')} d^3 \vec{p} \\
&= \pm \frac{\delta_2}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{1}{2} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} p_a p_b p_c p_d e^{-ip \cdot (x-x')} d^3 \vec{p} \\
&= \pm \frac{\delta_2}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{1}{2} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \partial_a \partial_b \partial_c \partial_d e^{-ip \cdot (x-x')} d^3 \vec{p} \\
&= -\pm \frac{i}{2} \delta_2 \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \partial_a \partial_b \partial_c \partial_d \Delta^{(-)}(x-x')
\end{aligned}$$

□

证明:  $[\Psi_{k_\varsigma}(x), \Psi_{k'_\varsigma}^+(x')]_\pm$

$$\begin{aligned}
&= [\Psi_{k_\varsigma}^{(+)}(x), \Psi_{k'_\varsigma}^{(+)}(x')]_\pm + [\Psi_{k_\varsigma}^{(-)}(x), \Psi_{k'_\varsigma}^{(-)}(x')]_\pm \\
&= \frac{i}{2} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \partial_a \partial_b \partial_c \partial_d [\delta_1 \Delta^{(+)}(x-x') - \pm \delta_2 \Delta^{(-)}(x-x')] \\
&= \frac{i}{2} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \partial_a \partial_b \partial_c \partial_d [(\delta_1 \pm \delta_2) \Delta^{(+)}(x-x') - \pm \delta_2 \Delta(x-x')] \\
&= \frac{i}{2} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \partial_a \partial_b \partial_c \partial_d [\delta_1 \Delta(x-x') - (\delta_1 \pm \delta_2) \Delta^{(-)}(x-x')]
\end{aligned}$$

□

从上式可知, 只有  $\delta_1 \pm \delta_2 = 0$  时, 才满足微观因果性, 同时只有  $\delta_1, \delta_2 \geq 0$  时, 才满足几率非负性。所以数学上八种协变对易或反对易方案中, 物理上合理的只有一种: 即  $\delta_1 = \delta_2 = 1$ , 且满足对易关系。其实还有两种, 即  $\delta_1 = \delta_2 = 0$ , 且满足对易或反对易关系, 就是经典情形。

### 8.4 引力子场物理的协变对易规则

从上节可知, 有物理意义的对易规则如下: (为了相互印证, 重新作了证明)

$$\text{定理8.4.1. } \begin{cases} [a_\sigma(\vec{p}, -2\varsigma), a_{\sigma'}^+(\vec{p}', -2\varsigma)] = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -2\varsigma), a_{\sigma'}(\vec{p}', -2\varsigma)] = 0 \\ [a_\sigma^+(\vec{p}, -2\varsigma), a_{\sigma'}^+(\vec{p}', -2\varsigma)] = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{k_\varsigma}(x), \psi_{k'_\varsigma}^+(x')] = \frac{i}{2} \Gamma_{k_\varsigma k'_\varsigma}^{abcd} \partial_a \partial_b \partial_c \partial_d \Delta(x-x') \\ [\psi_{k_\varsigma}(x), \psi_{l_\varsigma}(x')] = 0 \\ [\psi_{k'_\varsigma}^+(x), \psi_{l'_\varsigma}^+(x')] = 0 \end{cases}$$

$$\begin{aligned}
& \text{证明: } \{\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')\} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' \lambda_{k_\zeta}(\hat{p}, -2\zeta) \lambda_{k'_\zeta}^+(\hat{p}', -2\zeta) |\vec{p}|^{3/2} |\vec{p}'|^{3/2} \\
& \{ [a_1(\vec{p}, -2\zeta), a_1^+(\vec{p}', -2\zeta)] e^{ip \cdot (x-x')} + [a_2^+(\vec{p}, -2\zeta), a_2(\vec{p}', -2\zeta)] e^{-ip \cdot (x-x')} \} \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^3 \lambda_{k_\zeta}(\hat{p}, -2\zeta) \lambda_{k'_\zeta}^+(\hat{p}', -2\zeta) [\delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x-x')} - \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x-x')}] d^3\vec{p} d^3\vec{p}' \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^3 \lambda_{k_\zeta}(\hat{p}, -2\zeta) \lambda_{k'_\zeta}^+(\hat{p}', -2\zeta) [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3\vec{p} \\
&= \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{1}{2} \Gamma_{k_\zeta k'_\zeta}^{abcd} p_a p_b p_c p_d [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3\vec{p} \\
&= \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{1}{2} \Gamma_{k_\zeta k'_\zeta}^{abcd} \partial_a \partial_b \partial_c \partial_d [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3\vec{p} \\
&= \frac{i}{2} \Gamma_{k_\zeta k'_\zeta}^{abcd} \partial_a \partial_b \partial_c \partial_d \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3\vec{p} \\
&= \frac{i}{2} \Gamma_{k_\zeta k'_\zeta}^{abcd} \partial_a \partial_b \partial_c \partial_d \Delta(x-x')
\end{aligned}$$

□

## 8.5 引力子场的等时对易规则

推论8.5.1.

$$\begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = \frac{i}{2} \Gamma_{k_\zeta k'_\zeta}^{abcd} \partial_a \partial_b \partial_c \partial_d \Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] \\ = \frac{1}{6} i \zeta \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0, [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)] = 0 \end{cases}$$

$$\begin{aligned}
& \text{证明: } [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] = \frac{i}{2} \Gamma_{k_\zeta k'_\zeta}^{abcd} \partial_a \partial_b \partial_c \partial_d \Delta(x-x')|_{t=t'} \\
&= C \frac{1}{4} \frac{i}{2} \Gamma_{k_\zeta k'_\zeta}^{ijkl} \partial_i \partial_j \partial_k \partial_l \Delta(x-x')|_{t=t'} + C \frac{3}{4} \frac{i}{2} \Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi} \partial_i \partial_\pi \partial_\pi \partial_\pi \Delta(x-x')|_{t=t'} \\
&= \frac{1}{6} i \zeta \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}')
\end{aligned}$$

□

推论8.5.2.

$$\begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] \\ = \frac{1}{6} i \zeta \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0, [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)] = 0 \end{cases} \Rightarrow \begin{cases} [a_\sigma(\vec{p}, -2\zeta), a_\sigma^+(\vec{p}', -2\zeta)] = \zeta \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -2\zeta), a_{\sigma'}^+(\vec{p}', -2\zeta)] = 0 \\ [a_\sigma^+(\vec{p}, -2\zeta), a_{\sigma'}^+(\vec{p}', -2\zeta)] = 0 \end{cases}$$

证明:  $[a_1(\vec{p}, -2\zeta), a_1^+(\vec{p}', -2\zeta)]$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}|^{3/2} |\vec{p}'|^{3/2}} \int [\lambda^{+k_\zeta}(\hat{p}, -2\zeta) \Psi_{k_\zeta}(\vec{r}, t) e^{-i(\vec{p} \cdot \vec{r} - Et)}, \lambda^{k'_\zeta}(\vec{p}', -2\zeta) \Psi_{k'_\zeta}^+(\vec{r}', t) e^{i(\vec{p}' \cdot \vec{r}' - E't)}] d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}|^{3/2} |\vec{p}'|^{3/2}} \int \lambda^{+k_\zeta}(\hat{p}, -2\zeta) \lambda^{k'_\zeta}(\vec{p}', -2\zeta) [\Psi_{k_\zeta}(\vec{r}, t), \Psi_{k'_\zeta}^+(\vec{r}', t)] e^{-i(\vec{p} \cdot \vec{r} - Et)} e^{i(\vec{p}' \cdot \vec{r}' - E't)} d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}|^{3/2} |\vec{p}'|^{3/2}} \\
& \int \lambda^{+k_\zeta}(\hat{p}, -2\zeta) \lambda^{k'_\zeta}(\vec{p}', -2\zeta) \frac{1}{6} i \zeta \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') e^{-i(\vec{p} \cdot \vec{r} - Et)} e^{i(\vec{p}' \cdot \vec{r}' - E't)} d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}|^{3/2} |\vec{p}'|^{3/2}} \int \lambda^{+k_\zeta}(\hat{p}, -2\zeta) \lambda^{k'_\zeta}(\vec{p}', -2\zeta) \frac{1}{6} \{ [\sigma(2) \cdot \vec{p}] \vec{p}^2 - [\sigma(2) \cdot \vec{p}]^3 \}_{k_\zeta k'_\zeta} e^{-i(\vec{p} \cdot \vec{r} - Et)} e^{i(\vec{p}' \cdot \vec{r}' - E't)} d^3\vec{r} \\
&= \zeta \frac{1}{|\vec{p}|^{3/2} |\vec{p}'|^{3/2}} \lambda^{+k_\zeta}(\hat{p}, -2\zeta) \lambda^{k'_\zeta}(\vec{p}', -2\zeta) \frac{1}{6} \{ [\sigma(2) \cdot \vec{p}] \vec{p}^2 - [\sigma(2) \cdot \vec{p}]^3 \}_{k_\zeta k'_\zeta} \delta^3(\vec{p} - \vec{p}') \\
&= \zeta \lambda^+(\hat{p}, -2\zeta) \frac{1}{6} \{ [\sigma(2) \cdot \vec{p}] - [\sigma(2) \cdot \hat{p}]^3 \} \lambda(\hat{p}, -2\zeta) \delta^3(\vec{p} - \vec{p}') \\
&= \lambda^+(\hat{p}, -2\zeta) \lambda(\hat{p}, -2\zeta) \delta^3(\vec{p} - \vec{p}') \\
&= \delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

证明:  $[a_2^+(\vec{p}, -2\zeta), a_2(\vec{p}', -2\zeta)]$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}|^{3/2} |\vec{p}'|^{3/2}} \int [\lambda^{+k_\zeta}(\hat{p}, -2\zeta) \Psi_{k_\zeta}(\vec{r}, t) e^{i(\vec{p} \cdot \vec{r} - Et)}, \lambda^{k'_\zeta}(\vec{p}', -2\zeta) \Psi_{k'_\zeta}^+(\vec{r}', t) e^{-i(\vec{p}' \cdot \vec{r}' - E't)}] d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}|^{3/2} |\vec{p}'|^{3/2}} \int \lambda^{+k_\zeta}(\hat{p}, -2\zeta) \lambda^{k'_\zeta}(\vec{p}', -2\zeta) [\Psi_{k_\zeta}(\vec{r}, t), \Psi_{k'_\zeta}^+(\vec{r}', t)] e^{i(\vec{p} \cdot \vec{r} - Et)} e^{-i(\vec{p}' \cdot \vec{r}' - E't)} d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}|^{3/2} |\vec{p}'|^{3/2}} \\
& \int \lambda^{+k_\zeta}(\hat{p}, -2\zeta) \lambda^{k'_\zeta}(\vec{p}', -2\zeta) \frac{1}{6} i \zeta \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') e^{i(\vec{p} \cdot \vec{r} - Et)} e^{-i(\vec{p}' \cdot \vec{r}' - E't)} d^3\vec{r} d^3\vec{r}' \\
&= \zeta \frac{1}{(2\pi)^3} \frac{1}{|\vec{p}|^{3/2} |\vec{p}'|^{3/2}} \int \lambda^{+k_\zeta}(\hat{p}, -2\zeta) \lambda^{k'_\zeta}(\vec{p}', -2\zeta) \frac{-1}{6} \{ [\sigma(2) \cdot \vec{p}] \vec{p}^2 - [\sigma(2) \cdot \vec{p}]^3 \}_{k_\zeta k'_\zeta} e^{i(\vec{p} \cdot \vec{r} - Et)} e^{-i(\vec{p}' \cdot \vec{r}' - E't)} d^3\vec{r} \\
&= \zeta \frac{1}{|\vec{p}|^{3/2} |\vec{p}'|^{3/2}} \lambda^{+k_\zeta}(\hat{p}, -2\zeta) \lambda^{k'_\zeta}(\vec{p}', -2\zeta) \frac{-1}{6} \{ [\sigma(2) \cdot \vec{p}] \vec{p}^2 - [\sigma(2) \cdot \vec{p}]^3 \}_{k_\zeta k'_\zeta} \delta^3(\vec{p} - \vec{p}') \\
&= \zeta \lambda^+(\hat{p}, -2\zeta) \frac{-1}{6} \{ [\sigma(2) \cdot \vec{p}] - [\sigma(2) \cdot \hat{p}]^3 \} \lambda(\hat{p}, -2\zeta) \delta^3(\vec{p} - \vec{p}') \\
&= -\lambda^+(\hat{p}, -2\zeta) \lambda(\hat{p}, -2\zeta) \delta^3(\vec{p} - \vec{p}') \\
&= -\delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

## 8.6 引力子场对易规则小结

以上几个小节的证明正好形成一个逻辑闭环，故有如下性质：

推论8.6.1.

$$\begin{cases} [a_\sigma(\vec{p}, -2\zeta), a_{\sigma'}^+(\vec{p}', -2\zeta)] = \delta_{\sigma\sigma'}\delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -2\zeta), a_{\sigma'}(\vec{p}', -2\zeta)] = 0 \\ [a_\sigma^+(\vec{p}, -2\zeta), a_{\sigma'}^+(\vec{p}', -2\zeta)] = 0 \end{cases} \Leftrightarrow \begin{cases} [a_\sigma(\vec{p}), a_{\sigma'}^+(\vec{p}')] = \delta_{\sigma\sigma'}\delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}), a_{\sigma'}(\vec{p}')] = 0 \\ [a_\sigma^+(\vec{p}), a_{\sigma'}^+(\vec{p}')] = 0 \end{cases}$$

$$\Downarrow \qquad \qquad \qquad \Downarrow$$

推论8.6.2.

$$\begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = \frac{i}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] \\ = \frac{1}{6}i\zeta\{[\sigma(2) \cdot \nabla]\nabla^2 - [\sigma(2) \cdot \nabla]^3\}_{k_\zeta k'_\zeta}\delta^3(\vec{r} - \vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0, [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)] = 0 \end{cases}$$

## 8.7 引力子场的对易函数、因果函数和费曼传播子

推论8.7.1.

$$\begin{cases} \Delta_{k_\zeta k'_\zeta}(2; x) := \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{(+)}(2; x) := \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta^{(+)}(x) \\ \Delta_{k_\zeta k'_\zeta}^{(-)}(2; x) := \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta^{(-)}(x) \\ \Delta_{k_\zeta k'_\zeta}^{(l)}(2; x) := \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta^{(l)}(x) \end{cases}$$

推论8.7.2.

$$\begin{cases} \Delta_{k_\zeta k'_\zeta}^{(c)}(2; x) \\ := \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta^{(c)}(x) - \frac{1}{2}[\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}\delta'''(t) + 4i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}\delta''(t)\partial_i - 6\Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}\delta'(t)\partial_i\partial_j - 4i\Gamma_{k_\zeta k'_\zeta}^{ijk\pi}\delta(t)\partial_i\partial_j\partial_k]\Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{ret}(2; x) \\ := \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta^{ret}(x) - \frac{1}{2}[\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}\delta'''(t) + 4i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}\delta''(t)\partial_i - 6\Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}\delta'(t)\partial_i\partial_j - 4i\Gamma_{k_\zeta k'_\zeta}^{ijk\pi}\delta(t)\partial_i\partial_j\partial_k]\Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{adv}(2; x) \\ := \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta^{adv}(x) - \frac{1}{2}[\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}\delta'''(t) + 4i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}\delta''(t)\partial_i - 6\Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}\delta'(t)\partial_i\partial_j - 4i\Gamma_{k_\zeta k'_\zeta}^{ijk\pi}\delta(t)\partial_i\partial_j\partial_k]\Delta(x) \\ \Delta_{Fk_\zeta k'_\zeta}(2; x) = i\Delta_{k_\zeta k'_\zeta}^{(c)}(2; x) \\ := \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta_F(x) - \frac{i}{2}[\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}\delta'''(t) + 4i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}\delta''(t)\partial_i - 6\Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}\delta'(t)\partial_i\partial_j - 4i\Gamma_{k_\zeta k'_\zeta}^{ijk\pi}\delta(t)\partial_i\partial_j\partial_k]\Delta(x) \end{cases}$$

推论8.7.3.

$$\begin{cases} \Delta_{k_\zeta k'_\zeta}^{(c)}(2; x) = \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta^{(c)}(x) - \frac{1}{2}[\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}(3\partial_t^2 + \nabla^2) + 8i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}\partial_i\partial_t - 6\Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}\partial_i\partial_j]\delta^4(x) \\ \Delta_{k_\zeta k'_\zeta}^{ret}(2; x) = \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta^{ret}(x) - \frac{1}{2}[\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}(3\partial_t^2 + \nabla^2) + 8i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}\partial_i\partial_t - 6\Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}\partial_i\partial_j]\delta^4(x) \\ \Delta_{k_\zeta k'_\zeta}^{adv}(2; x) = \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta^{adv}(x) - \frac{1}{2}[\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}(3\partial_t^2 + \nabla^2) + 8i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}\partial_i\partial_t - 6\Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}\partial_i\partial_j]\delta^4(x) \\ \Delta_{Fk_\zeta k'_\zeta}(2; x) = i\Delta_{k_\zeta k'_\zeta}^{(c)}(2; x) = \frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\Delta_F(x) - \frac{i}{2}[\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}(3\partial_t^2 + \nabla^2) + 8i\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}\partial_i\partial_t - 6\Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}\partial_i\partial_j]\delta^4(x) \\ \Delta_{Fk_\zeta k'_\zeta}(2; p) = \frac{-i}{2}\frac{\Gamma_{k_\zeta k'_\zeta}^{abcd}p_a p_b p_c p_d}{p^2 - i\epsilon} + \dots \end{cases}$$

推论8.7.4.

$$\begin{cases} [s\partial_a + iS_{ab}(2, \zeta)\partial^b]\Delta(2; x) = 0 \\ [s\partial_a + iS_{ab}(2, \zeta)\partial^b]\Delta^{(+)}(2; x) = 0 \\ [s\partial_a + iS_{ab}(2, \zeta)\partial^b]\Delta^{(-)}(2; x) = 0 \\ [s\partial_a + iS_{ab}(2, \zeta)\partial^b]\Delta^{(l)}(2; x) = 0 \end{cases} \begin{cases} [s\partial_a + iS_{ab}(2, \zeta)\partial^b]\Delta^{(c)}(2; x) = -\zeta[\sigma(2), i2\zeta]_a\delta(t)\Delta(2; x)|_{t=0} \\ [s\partial_a + iS_{ab}(2, \zeta)\partial^b]\Delta^{ret}(2; x) = -\zeta[\sigma(2), i2\zeta]_a\delta(t)\Delta(2; x)|_{t=0} \\ [s\partial_a + iS_{ab}(2, \zeta)\partial^b]\Delta^{adv}(2; x) = -\zeta[\sigma(2), i2\zeta]_a\delta(t)\Delta(2; x)|_{t=0} \\ [s\partial_a + iS_{ab}(2, \zeta)\partial^b]\Delta_F(2; x) = -i\zeta[\sigma(2), i2\zeta]_a\delta(t)\Delta(2; x)|_{t=0} \end{cases}$$

## 8.8 引力子场的量子方程

定理8.8.1.

$$H = \frac{1}{2} \int \{ \psi_{k'_\zeta}^+(\vec{r}, t), \Gamma(\nabla) \psi_{k_\zeta}(\vec{r}, t) \} d^3 \vec{r}$$

定理8.8.2.  $[\psi_{j_\zeta}(\vec{r}, t), \int d^3 \vec{r}' \psi_{k'_\zeta}^+(\vec{r}', t) \Gamma(\nabla')^{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}', t)] = \frac{1}{2} [\psi_{j_\zeta}(\vec{r}, t), \int d^3 \vec{r}' \{ \psi_{k'_\zeta}^+(\vec{r}', t), \Gamma(\nabla')^{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}', t) \}]$

$$\begin{aligned} \text{证明: } & \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) \Gamma(\nabla')^{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) \Gamma(\nabla')^{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \int d^3 \vec{r}' \Gamma(\nabla')^{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}', t) [\psi_{j_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] \\ &= \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}, t), \Gamma(\nabla')^{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}', t) \psi_{k'_\zeta}^+(\vec{r}', t)] \end{aligned}$$

□

## 8.9 对易和反对易公式

$$\text{推论8.9.1. } \begin{cases} [A, BC] = [A, B]C + B[A, C], [A, CB] = [A, C]B + C[A, B] \\ [A, \{B, C\}] = \{A, B\}C - B\{A, C\}, [A, \{C, B\}] = \{A, C\}B - C\{A, B\} \end{cases}$$

$$\text{推论8.9.2. } \begin{cases} [A, \{B, C\}] = \{[A, B], C\} + \{B, [A, C]\} \\ [A, [B, C]] = \{ \{A, B\}, C \} - \{B, \{A, C\} \} \end{cases}$$

定理8.9.1.

$$\begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] = \frac{1}{6} i \zeta \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0, [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)] = 0 \\ H = \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{[\frac{1}{2} \sigma(2) \cdot \nabla]^2}{-\nabla^4} \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r} = \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{1}{-\nabla^2} \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r}, \vec{P} = \frac{\zeta}{2} \int \psi_{k'_\zeta}^+(\vec{r}, t) [\sigma(2) \cdot \nabla] \frac{\nabla}{\nabla^4} \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r} \end{cases}$$

$$\Rightarrow \begin{cases} [\psi(\vec{r}, t), H] = \frac{i}{6} \zeta \{ -[\sigma(2) \cdot \nabla] + \frac{1}{\nabla^2} [\sigma(2) \cdot \nabla]^3 \} \psi(\vec{r}, t) \\ [\psi(\vec{r}, t), \vec{P}] = \frac{i}{12} \{ \frac{[\sigma(2) \cdot \nabla]^2}{\nabla^2} - \frac{[\sigma(2) \cdot \nabla]^4}{\nabla^4} \} \nabla \psi(\vec{r}, t) \end{cases}$$

证明:  $[\psi(\vec{r}, t), H]$

$$\begin{aligned} &= \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) \{ \frac{[\frac{1}{2} \sigma(2) \cdot \nabla]^2}{-\nabla^4} \}_{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) \{ \frac{[\frac{1}{2} \sigma(2) \cdot \nabla]^2}{-\nabla^4} \}_{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \int d^3 \vec{r}' \frac{1}{6} i \zeta \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{j_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \{ \frac{[\frac{1}{2} \sigma(2) \cdot \nabla]^2}{-\nabla^4} \}_{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}', t) \\ &= \frac{i}{6} \zeta \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{j_\zeta k'_\zeta} \{ \frac{[\frac{1}{2} \sigma(2) \cdot \nabla]^2}{-\nabla^4} \}_{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}, t) \\ &= \frac{i}{24} \zeta \{ -\frac{1}{\nabla^2} [\sigma(2) \cdot \nabla]^3 + \frac{1}{\nabla^4} [\sigma(2) \cdot \nabla]^5 \} \psi(\vec{r}, t) \\ &= \frac{i}{6} \zeta \{ -[\sigma(2) \cdot \nabla] + \frac{1}{\nabla^2} [\sigma(2) \cdot \nabla]^3 \} \psi(\vec{r}, t) \end{aligned}$$

□

证明:  $[\psi(\vec{r}, t), H]$

$$\begin{aligned} &= \delta^{k'_\zeta k_\zeta} \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) \frac{1}{-\nabla'^2} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \delta^{k'_\zeta k_\zeta} \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) \frac{1}{-\nabla'^2} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \delta^{k'_\zeta k_\zeta} \int d^3 \vec{r}' \frac{1}{6} i \zeta \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{j_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \frac{1}{-\nabla'^2} \psi_{k_\zeta}(\vec{r}', t) \\ &= \frac{i}{6} \zeta \delta^{k'_\zeta k_\zeta} \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{j_\zeta k'_\zeta} \frac{1}{-\nabla^2} \psi_{k_\zeta}(\vec{r}, t) \\ &= \frac{i}{6} \zeta \{ -[\sigma(2) \cdot \nabla] + \frac{1}{\nabla^2} [\sigma(2) \cdot \nabla]^3 \} \psi(\vec{r}, t) \end{aligned}$$

□

证明:  $[\psi(\vec{r}, t), \vec{P}]$

$$\begin{aligned} &= \frac{\zeta}{2} \delta^{k'_\zeta k_\zeta} \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) [\sigma(2) \cdot \nabla'] \frac{\nabla'}{\nabla'^4} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \frac{\zeta}{2} \delta^{k'_\zeta k_\zeta} \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t) [\sigma(2) \cdot \nabla'] \frac{\nabla'}{\nabla'^4} \psi_{k_\zeta}(\vec{r}', t)] \\ &= \frac{\zeta}{2} \delta^{k'_\zeta k_\zeta} \int d^3 \vec{r}' \frac{1}{6} i \zeta \{ [\sigma(2) \cdot \nabla'] \nabla'^2 - [\sigma(2) \cdot \nabla']^3 \}_{j_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') [\sigma(2) \cdot \nabla'] \frac{\nabla'}{\nabla'^4} \psi_{k_\zeta}(\vec{r}, t) \\ &= \frac{i}{12} \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^4 \} \frac{\nabla}{\nabla^4} \psi(\vec{r}, t) \\ &= \frac{i}{12} \{ \frac{[\sigma(2) \cdot \nabla]^2}{\nabla^2} - \frac{[\sigma(2) \cdot \nabla]^4}{\nabla^4} \} \nabla \psi(\vec{r}, t) \end{aligned}$$

□

推论8.9.3.

$$\begin{cases} \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^1 \sigma(s)\} = [\sigma^2(s) - 1][\sigma(s) \cdot \hat{p}] \\ \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^2 \sigma(s)\} = [\sigma^2(s) - 3][\sigma(s) \cdot \hat{p}]^2 + \sigma^2(s) \\ \sigma(s) \cdot \{[\sigma(s) \cdot \hat{p}]^3 \sigma(s)\} \\ = [\sigma^2(s) - 6][\sigma(s) \cdot \hat{p}]^3 + [3\sigma^2(s) - 1]\sigma(s) \cdot \hat{p} \end{cases} \Rightarrow \begin{cases} \sigma(2) \cdot \{[\sigma(2) \cdot \hat{p}]^1 \sigma(2)\} = 5[\sigma(2) \cdot \hat{p}] \\ \sigma(2) \cdot \{[\sigma(2) \cdot \hat{p}]^2 \sigma(2)\} = 3[\sigma(2) \cdot \hat{p}]^2 + 6 \\ \sigma(2) \cdot \{[\sigma(2) \cdot \hat{p}]^3 \sigma(2)\} = 17[\sigma(2) \cdot \hat{p}] \end{cases}$$

推论8.9.4.

$$\begin{cases} \nabla \psi(\vec{r}, t) = i[\psi(\vec{r}, t), P] \\ \dot{\psi}(\vec{r}, t) = -i[\psi(\vec{r}, t), H] \end{cases} \Leftrightarrow \begin{cases} \nabla \psi(\vec{r}, t) = -\frac{1}{12} \left\{ \frac{[\sigma(2) \cdot \nabla]^2}{\nabla^2} - \frac{[\sigma(2) \cdot \nabla]^4}{\nabla^4} \right\} \nabla \psi(\vec{r}, t) \\ \dot{\psi}(\vec{r}, t) = -\frac{1}{6} \zeta \left\{ [\sigma(2) \cdot \nabla] - \frac{[\sigma(2) \cdot \nabla]^3}{\nabla^2} \right\} \psi(\vec{r}, t) \end{cases} \Leftrightarrow \begin{cases} \partial^a \partial_a \psi(\vec{r}, t) = 0 \\ [\sigma(2), -2i\zeta]^a \partial_a \psi(\vec{r}, t) = 0 \end{cases}$$

$$\text{推论8.9.5. } \begin{cases} \nabla \psi(\vec{r}, t) = -\frac{1}{12} \left\{ \frac{[\sigma(2) \cdot \nabla]^2}{\nabla^2} - \frac{[\sigma(2) \cdot \nabla]^4}{\nabla^4} \right\} \nabla \psi(\vec{r}, t) \\ \psi(\vec{r}, t) = \int \lambda(\hat{p}, -2\zeta) [a_1(\vec{p}) e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + a_2^+(\vec{p}) e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] d^3 \vec{p} \end{cases}$$

$$\Leftrightarrow \begin{cases} \nabla \psi(\vec{r}, t) = -\frac{1}{12} \left\{ [\sigma(2) \cdot \nabla] - \frac{1}{\nabla^2} [\sigma(2) \cdot \nabla]^3 \right\} \sigma(2) \psi(\vec{r}, t) \\ \psi(\vec{r}, t) = \int \lambda(\hat{p}, -2\zeta) [a_1(\vec{p}) e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + a_2^+(\vec{p}) e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] d^3 \vec{p} \end{cases}$$

$$\text{推论8.9.6. } [2\partial_a + iS_{ab}(2, \zeta) \partial^b] \psi(x) = 0 \Rightarrow \begin{cases} \partial^a \partial_a \psi(\vec{r}, t) = 0 \\ [\sigma(2), -2i\zeta]^a \partial_a \psi(\vec{r}, t) = 0 \end{cases}$$

$$\text{推论8.9.7. } [2\partial_a + iS_{ab}(2, \zeta) \partial^b] \psi(x) = 0 \Rightarrow \partial_a \psi(\vec{r}, t) = i[\psi(\vec{r}, t), P_a]$$

## 8.10 引力子场的第二种量子方程

定理8.10.1.

$$\begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)] = \frac{1}{6} i\zeta \{ [\sigma(2) \cdot \nabla] \nabla^2 - [\sigma(2) \cdot \nabla]^3 \}_{k_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0, [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)] = 0 \\ H = \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{1}{\nabla^2} \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r}, \vec{P} = \frac{-\zeta}{2} \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{\sigma(2)}{\nabla^2} \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r} \end{cases}$$

$$\Rightarrow \begin{cases} [\psi(\vec{r}, t), H] = \frac{i}{6} \zeta \{ -[\sigma(2) \cdot \nabla] + \frac{1}{\nabla^2} [\sigma(2) \cdot \nabla]^3 \} \psi(\vec{r}, t) \\ [\psi(\vec{r}, t), \vec{P}] = \frac{i}{12} \{ [\sigma(2) \cdot \nabla] - \frac{1}{\nabla^2} [\sigma(2) \cdot \nabla]^3 \} \sigma(2) \psi(\vec{r}, t) \end{cases}$$

证明:  $[\psi(\vec{r}, t), P]$

$$\begin{aligned} &= \frac{\zeta}{2} \sigma(2)^{k'_\zeta k_\zeta} \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}', t), \psi_{k'_\zeta}^+(\vec{r}', t)] \frac{1}{\nabla'^2} \psi_{k_\zeta}(\vec{r}', t) \\ &= \frac{\zeta}{2} \sigma(2)^{k'_\zeta k_\zeta} \int d^3 \vec{r}' [\psi_{j_\zeta}(\vec{r}', t), \psi_{k'_\zeta}^+(\vec{r}', t)] \frac{1}{\nabla'^2} \psi_{k_\zeta}(\vec{r}', t) \\ &= \frac{\zeta}{2} \sigma(2)^{k'_\zeta k_\zeta} \int d^3 \vec{r}' \frac{1}{6} i\zeta \{ [\sigma(2) \cdot \nabla'] \nabla'^2 - [\sigma(2) \cdot \nabla']^3 \}_{j_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \frac{1}{\nabla'^2} \psi_{k_\zeta}(\vec{r}', t) \\ &= \frac{\zeta}{2} \sigma(2)^{k'_\zeta k_\zeta} \frac{1}{6} i\zeta \{ [\sigma(2) \cdot \nabla] \nabla'^2 - [\sigma(2) \cdot \nabla]^3 \}_{j_\zeta k'_\zeta} \frac{1}{\nabla^2} \psi_{k_\zeta}(\vec{r}, t) \\ &= \frac{i}{12} \{ [\sigma(2) \cdot \nabla] - \frac{1}{\nabla^2} [\sigma(2) \cdot \nabla]^3 \} \sigma(2) \psi(\vec{r}, t) \\ &? = -i \nabla \psi(\vec{r}, t) \end{aligned}$$

□

$$\text{性质8.10.1. } i\sigma(s) \times \nabla = \sigma(s) \cdot \nabla \sigma(s) - \sigma(s) [\sigma(s) \cdot \nabla], \sigma(s) \cdot \nabla \sigma(s) = i\sigma(s) \times \nabla + \sigma(s) [\sigma(s) \cdot \nabla]$$

$$\begin{aligned} \text{推论8.10.1. } & \{ [\sigma(2) \cdot \nabla] - \frac{[\sigma(2) \cdot \nabla]^3}{\nabla^2} \} \sigma(2) \\ &= i\sigma(s) \times \nabla + \sigma(s) [\sigma(s) \cdot \nabla] - \frac{[\sigma(2) \cdot \nabla]^2}{\nabla^2} \{ i\sigma(s) \times \nabla + \sigma(s) [\sigma(s) \cdot \nabla] \} \\ &= i\sigma(s) \times \nabla + \sigma(s) [\sigma(s) \cdot \nabla] - \frac{[\sigma(2) \cdot \nabla]}{\nabla^2} i \{ i\sigma(s) \times \nabla + \sigma(s) [\sigma(s) \cdot \nabla] \} \times \nabla + \frac{[\sigma(2) \cdot \nabla]}{\nabla^2} \{ i\sigma(s) \times \nabla + \sigma(s) [\sigma(s) \cdot \nabla] \} [\sigma(s) \cdot \nabla] \end{aligned}$$

性质8.10.2.

$$\sigma(2) \cdot \hat{\nabla} \equiv -\frac{1}{12} \sigma_\alpha(2) \{ [\sigma(2) \cdot \hat{\nabla}] - [\sigma(2) \cdot \hat{\nabla}]^3 \} \sigma^\alpha(2), [\sigma(2) \cdot \hat{\nabla}]^5 \equiv -4[\sigma(2) \cdot \hat{\nabla}] + 5[\sigma(2) \cdot \hat{\nabla}]^3, \hat{\nabla} := \frac{-i\nabla}{\sqrt{-\nabla^2}}, \hat{\nabla}^2 = 1$$

推论8.10.2.

$$\begin{cases} \dot{\psi}(\vec{r}, t) = \frac{1}{6} \zeta \{ -[\sigma(2) \cdot \nabla] + \frac{1}{\nabla^2} [\sigma(2) \cdot \nabla]^3 \} \psi(\vec{r}, t) \\ \nabla \psi(\vec{r}, t) = -\frac{1}{12} \{ [\sigma(2) \cdot \nabla] - \frac{1}{\nabla^2} [\sigma(2) \cdot \nabla]^3 \} \sigma(2) \psi(\vec{r}, t) \end{cases} \Rightarrow \partial_t^2 \psi(\vec{r}, t) = \nabla^2 \psi(\vec{r}, t)$$

证明:

$$\begin{aligned}
& \left\{ \begin{aligned} \dot{\psi}(\vec{r}, t) &= \frac{1}{6}\zeta\{-[\sigma(2) \cdot \nabla] + \frac{1}{\sqrt{2}}[\sigma(2) \cdot \nabla]^3\}\psi(\vec{r}, t) \\ \nabla\psi(\vec{r}, t) &= -\frac{1}{12}\{[\sigma(2) \cdot \nabla] - \frac{1}{\sqrt{2}}[\sigma(2) \cdot \nabla]^3\}\sigma(2)\psi(\vec{r}, t) \end{aligned} \right. \\
& \Rightarrow \partial_t^2\psi(\vec{r}, t) = \frac{1}{36}\{[\sigma(2) \cdot \nabla]^2 - 2\frac{1}{\sqrt{2}}[\sigma(2) \cdot \nabla]^4 + \frac{1}{\sqrt{4}}[\sigma(2) \cdot \nabla]^6\}\psi(\vec{r}, t) \\
& = \frac{1}{36}\{[\sigma(2) \cdot \nabla]^2 - 2\frac{1}{\sqrt{2}}[\sigma(2) \cdot \nabla]^4 - 4[\sigma(2) \cdot \nabla]^2 + 5\frac{1}{\sqrt{2}}[\sigma(2) \cdot \nabla]^4\}\psi(\vec{r}, t) \\
& = -\frac{1}{12}\{[\sigma(2) \cdot \nabla]^2 - \frac{1}{\sqrt{2}}[\sigma(2) \cdot \nabla]^4\}\psi(\vec{r}, t) \\
& = \nabla^2\psi(\vec{r}, t)
\end{aligned}$$

□

推论8.10.3.

$$\left\{ \begin{aligned} \dot{\psi}(\vec{r}, t) &= \frac{1}{6}\zeta\{-[\sigma(2) \cdot \nabla] + \frac{1}{\sqrt{2}}[\sigma(2) \cdot \nabla]^3\}\psi(\vec{r}, t) \\ \nabla\psi(\vec{r}, t) &= -\frac{1}{12}\{[\sigma(2) \cdot \nabla] - \frac{1}{\sqrt{2}}[\sigma(2) \cdot \nabla]^3\}\sigma(2)\psi(\vec{r}, t) \end{aligned} \right. \quad ! \Rightarrow [\sigma(2), -2i\zeta]^a \partial_a \psi(\vec{r}, t) = 0$$

## 8.11 引力子场的彭加莱对称性

推论8.11.1.

$$\left\{ \begin{aligned} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s} \partial_\pi \Delta(x-x')|_{t=t'} &= i \sum_{l=0}^{[s]} (-1)^l C_{2s}^{2l} \Gamma_{ij \dots \pi \dots \pi}^{\overbrace{2s-2l} \quad \overbrace{2l} \quad \overbrace{2s-2l}}(s) \overbrace{\partial_i \partial_j \dots}^{2s-2l} \nabla^{2l} \delta^3(\vec{r} - \vec{r}') \\ \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \partial_\pi \Delta(x-x')|_{t=t'} &= i \sum_{l=0}^{[s]} (-1)^l C_{2s}^{2l} \Gamma_{ij \dots \pi \dots \pi}^{\overbrace{2s-2l} \quad \overbrace{2l} \quad \overbrace{2s-2l}}(s) \overbrace{\hat{\partial}_i \hat{\partial}_j \dots}^{2s-2l} \delta^3(\vec{r} - \vec{r}') \end{aligned} \right.$$

推论8.11.2.

$$\begin{aligned}
\Gamma_{k_\zeta k'_\zeta}^{\pi\pi\pi\pi}(2) &= \left(\frac{1}{\sqrt{2}}\right)^4 \delta_{k_\zeta k'_\zeta} \\
\Gamma_{k_\zeta k'_\zeta}^{i\pi\pi\pi}(2) &= -i\zeta \left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{2} \sigma^i(2)_{k_\zeta k'_\zeta} \\
\Gamma_{k_\zeta k'_\zeta}^{ij\pi\pi}(2) &= -\left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{6} [\sigma^{\{i}(2)\sigma^{j\}}(2) - 2\delta^{ij}]_{k_\zeta k'_\zeta} = -\left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{3} \frac{1}{2!} [\sigma^{\{i}(2)\sigma^{j\}}(2) - \delta^{\{ij\}}]_{k_\zeta k'_\zeta} \\
\Gamma_{k_\zeta k'_\zeta}^{ijk\pi}(2) &= \left(\frac{1}{\sqrt{2}}\right)^4 \frac{i\zeta}{6} \{\sigma^{\{i}(2)\sigma^j(2)\sigma^k\}}(2) - [\sigma^i(2)\delta^{jk} + 2\delta^{\{ij}\sigma^k\}}(2)\}_{k_\zeta k'_\zeta} \\
&= \left(\frac{1}{\sqrt{2}}\right)^4 \frac{i\zeta}{3} \frac{1}{3!} \{\sigma^{\{i}(2)\sigma^j(2)\sigma^k\}}(2) - \frac{5}{2}\sigma^{\{i}(2)\delta^{jk}\}}(2)\}_{k_\zeta k'_\zeta} \\
\Gamma_{k_\zeta k'_\zeta}^{ijkl}(2) &= \left(\frac{1}{\sqrt{2}}\right)^4 \frac{2}{3} \frac{1}{4!} [\sigma^{\{i}(2)\sigma^j(2)\sigma^k(2)\sigma^l\}}(2) - 4\sigma^{\{i}(2)\sigma^j(2)\delta^{kl}\}} + \frac{3}{2}\delta^{\{ij}\delta^{kl}\}]_{k_\zeta k'_\zeta}
\end{aligned}$$

推论8.11.3.  $\Gamma^{abcd}(2)\partial_a\partial_b\partial_c\partial_d\partial_\pi\Delta(x-x')|_{t=t'}$ 

$$\begin{aligned}
& = i \sum_{l=0}^2 (-1)^l C_4^{2l} \Gamma_{ij \dots \pi \dots \pi}^{\overbrace{4-2l} \quad \overbrace{2l} \quad \overbrace{4-2l}}(2) \overbrace{\partial_i \partial_j \dots}^{4-2l} \nabla^{2l} \delta^3(\vec{r} - \vec{r}') \\
& = i\{\Gamma^{ijkl}(2)\partial_i\partial_j\partial_k\partial_l\delta^3(\vec{r} - \vec{r}') - 6\Gamma^{ij\pi\pi}(2)\partial_i\partial_j\nabla^2\delta^3(\vec{r} - \vec{r}') + \Gamma^{\pi\pi\pi\pi}(2)\nabla^4\delta^3(\vec{r} - \vec{r}')\} \\
& = i\left\{\left(\frac{1}{\sqrt{2}}\right)^4 \frac{2}{3} \frac{1}{4!} [\sigma^{\{i}(2)\sigma^j(2)\sigma^k(2)\sigma^l\}}(2) - 4\sigma^{\{i}(2)\sigma^j(2)\delta^{kl}\}} + \frac{3}{2}\delta^{\{ij}\delta^{kl}\}]\partial_i\partial_j\partial_k\partial_l\delta^3(\vec{r} - \vec{r}') + 6\left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{3} \frac{1}{2!} [\sigma^{\{i}(2)\sigma^j\}}(2) - \delta^{\{ij\}}]\partial_i\partial_j\nabla^2\delta^3(\vec{r} - \vec{r}') + \left(\frac{1}{\sqrt{2}}\right)^4 \nabla^4\delta^3(\vec{r} - \vec{r}')\right\} \\
& = i\left\{\frac{1}{6}\{[\sigma(2) \cdot \nabla]^4 - 4[\sigma(2) \cdot \nabla]^2\nabla^2 + \frac{3}{2}\nabla^4\}\delta^3(\vec{r} - \vec{r}') + \frac{1}{2}\{[\sigma(2) \cdot \nabla]^2\nabla^2 - \nabla^4\}\delta^3(\vec{r} - \vec{r}') + \frac{1}{4}\nabla^4\delta^3(\vec{r} - \vec{r}')\}\right\} \\
& = \frac{i}{6}\{[\sigma(2) \cdot \nabla]^4 - [\sigma(2) \cdot \nabla]^2\nabla^2\}\delta^3(\vec{r} - \vec{r}')
\end{aligned}$$

推论8.11.4.

$$\left\{ \begin{aligned} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] &= -\frac{1}{2}\Gamma_{k_\zeta k'_\zeta}^{abcd}\partial_a\partial_b\partial_c\partial_d\partial_\pi\Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] &= 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] &= 0 \end{aligned} \right. \Rightarrow \left\{ \begin{aligned} & \left[\frac{\psi_{k_\zeta}(\vec{r}, t)}{-\nabla^2}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{-\nabla'^2}\right] \\ & = \frac{i}{12}\{[\sigma(2) \cdot \hat{\nabla}]^2 - [\sigma(2) \cdot \hat{\nabla}]^4\}\delta^3(\vec{r} - \vec{r}') \\ & [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)] = 0, [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)] = 0 \end{aligned} \right.$$

推论8.11.5.  $\hat{P}_a(2) = \int \frac{\psi^+(\vec{r}, t)}{-\nabla^2} \hat{P}_a \frac{i\psi(\vec{r}, t)}{-\nabla^2} d^3\vec{r}$ ,  $M_{ab}(2) = \int \frac{\psi^+(\vec{r}, t)}{-\nabla^2} \hat{M}_{ab} \frac{i\psi(\vec{r}, t)}{-\nabla^2} d^3\vec{r}$ 

$$\text{定理8.11.1. } \left\{ \begin{aligned} [L_{ab}, L_{cd}] &= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \\ [L_{ab}, P_c] &= -i(g_{bc}P_a - g_{ac}P_b), [P_a, P_b] = 0 \end{aligned} \right.$$

证明:  $[L_{ab}, L_{cd}]$ 

$$= -\int d^3\vec{r} d^3\vec{r}' \left[ \frac{\psi^+(\vec{r}, t)}{-\nabla^2} (r_a \partial_b - r_b \partial_a) \frac{i\psi(\vec{r}, t)}{-\nabla^2}, \frac{\psi^+(\vec{r}', t)}{-\nabla'^2} (r'_c \partial'_d - r'_d \partial'_c) \frac{i\psi(\vec{r}', t)}{-\nabla'^2} \right]$$



$$\begin{aligned}
& \vec{r}) \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{-\nabla^2} \} \\
& = -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r}' d^3 \vec{r}'' \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{-\nabla^2} \frac{i}{12} \{ [\sigma(2) \cdot \hat{\nabla}]^2 - [\sigma(2) \cdot \hat{\nabla}]^4 \} \right\}_{l_\zeta k'_\zeta} \partial'_a \delta^3(\vec{r} - \vec{r}'') \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{-\nabla'^2} - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{-\nabla'^2} \frac{i}{12} \{ [\sigma(2) \cdot \hat{\nabla}]^2 - [\sigma(2) \cdot \hat{\nabla}]^4 \} \right\}_{l'_\zeta k_\zeta} \partial_b \delta^3(\vec{r} - \\
& \vec{r}'') \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{-\nabla^2} \} \\
& = \int \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{-\nabla^2} \frac{i}{12} \{ [\sigma(2) \cdot \hat{\nabla}]^2 - [\sigma(2) \cdot \hat{\nabla}]^4 \} \right\}_{k_\zeta l'_\zeta} \partial_a \partial_b \frac{\psi_{l'_\zeta}(\vec{r}, t)}{-\nabla^2} - \frac{\psi_{k'_\zeta}^+(\vec{r}, t)}{-\nabla^2} \frac{i}{12} \{ [\sigma(2) \cdot \hat{\nabla}]^2 - [\sigma(2) \cdot \hat{\nabla}]^4 \} \right\}_{k'_\zeta l_\zeta} \partial_b \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{-\nabla^2} \} d^3 \vec{r} \\
& = \int \frac{\psi^+(\vec{r}, t)}{-\nabla^2} (\partial_a \partial_b - \partial_b \partial_a) \frac{i}{12} \{ [\sigma(2) \cdot \hat{\nabla}]^2 - [\sigma(2) \cdot \hat{\nabla}]^4 \} \frac{\psi(\vec{r}, t)}{-\nabla^2} d^3 \vec{r} \\
& = \int \frac{\psi^+(\vec{r}, t)}{-\nabla^2} (\partial_a \partial_b - \partial_b \partial_a) \frac{-i \psi(\vec{r}, t)}{-\nabla^2} d^3 \vec{r} \\
& = \int \frac{\psi^+(\vec{r}, t)}{-\nabla^2} [\hat{P}_a, \hat{P}_b] \frac{i \psi(\vec{r}, t)}{-\nabla^2} d^3 \vec{r} = 0 \quad \square
\end{aligned}$$



## 第二十四章 s-自旋方程的协变量子化方案

自我评述：在本章终于按统一的方式对所有无质量自旋粒子建立了相应的量子场论。无需知道哈密顿量，就可以按统一的新程式对各种自旋粒子进行了量子化，给出了统一的量子化对易规则和能量动量算符形式，给出了部分量子彭加莱代数。但角动量算符只取得部分成功，没有彻底解决，仍需努力。

### 1 自旋波函数的傅里叶变换性质(无需满足自旋方程)

#### 1.1 坐标-动量空间的一阶对应性质

$$\text{性质1.1.1.1.} \quad \begin{cases} \int \psi^+(\vec{r}, t)\psi(\vec{r}, t)d^3\vec{r} = \int \psi^+(\vec{p}, t)\psi(\vec{p}, t)d^3\vec{p} = \int [a_1^+(\vec{p})a_1(\vec{p}) + a_2(\vec{p})a_2^+(\vec{p})]d^3\vec{p} \\ \int \psi^+(\vec{r}, t)\sigma(s)\psi(\vec{r}, t)d^3\vec{r} = \int \psi^+(\vec{p}, t)\sigma(s)\psi(\vec{p}, t)d^3\vec{p} \end{cases}$$

$$\text{性质1.1.2.} \quad \begin{cases} \int \psi^+(\vec{r}, t)\hat{\nabla}\psi(\vec{r}, t)d^3\vec{r} = \int \psi^+(\vec{p}, t)\hat{p}\psi(\vec{p}, t)d^3\vec{p} \\ \int \psi^+(\vec{r}, t)\vec{r}\psi(\vec{r}, t)d^3\vec{r} = \int \psi^+(\vec{p}, t)(i\tilde{\nabla})\psi(\vec{p}, t)d^3\vec{p} \\ \int \psi^+(\vec{r}, t)[\sigma(s) \cdot \hat{\nabla}]\psi(\vec{r}, t)d^3\vec{r} = \int \psi^+(\vec{p}, t)[\sigma(s) \cdot \hat{p}]\psi(\vec{p}, t)d^3\vec{p} \end{cases}$$

$$\text{性质1.1.3.} \quad \begin{cases} \int \psi^+(\vec{r}, t)[\sigma(s) \cdot \hat{\nabla}]\hat{\nabla}\psi(\vec{r}, t)d^3\vec{r} = \int \psi^+(\vec{p}, t)[\sigma(s) \cdot \hat{p}]\hat{p}\psi(\vec{p}, t)d^3\vec{p} \\ \int \psi^+(\vec{r}, t)\sigma(s)[\sigma(s) \cdot \hat{\nabla}]\psi(\vec{r}, t)d^3\vec{r} = \int \psi^+(\vec{p}, t)\sigma(s)[\sigma(s) \cdot \hat{p}]\psi(\vec{p}, t)d^3\vec{p} \\ \int \psi^+(\vec{r}, t)[\sigma(s) \cdot \hat{\nabla}]\sigma(s)\psi(\vec{r}, t)d^3\vec{r} = \int \psi^+(\vec{p}, t)[\sigma(s) \cdot \hat{p}]\sigma(s)\psi(\vec{p}, t)d^3\vec{p} \end{cases}$$

$$\text{性质1.1.4.} \quad \begin{cases} \int \psi^+(\vec{r}, t)r_i\sigma_j(s)\psi(\vec{r}, t)d^3\vec{r} = \int \psi^+(\vec{p}, t)\sigma_j(s)(i\tilde{\partial}_i)\psi(\vec{p}, t)d^3\vec{p} \\ \int \psi^+(\vec{r}, t)\sigma_i(s)\partial_j\psi(\vec{r}, t)d^3\vec{r} = i \int \psi^+(\vec{p}, t)\sigma_i(s)p_j\psi(\vec{p}, t)d^3\vec{p} \end{cases}$$

$$\text{性质1.1.5.} \quad \begin{cases} \int \psi^+(\vec{r}, t)[r_i\sigma_j(s) - r_j\sigma_i(s)]\psi(\vec{r}, t)d^3\vec{r} = -i \int \psi^+(\vec{p}, t)[\sigma_i(s)\tilde{\partial}_j - \sigma_j(s)\tilde{\partial}_i]\psi(\vec{p}, t)d^3\vec{p} \\ \int \psi^+(\vec{r}, t)[\sigma_i(s)\partial_j - \sigma_j(s)\partial_i]\psi(\vec{r}, t)d^3\vec{r} = i \int \psi^+(\vec{p}, t)[\sigma_i(s)p_j - \sigma_j(s)p_i]\psi(\vec{p}, t)d^3\vec{p} \end{cases}$$

#### 1.2 坐标-动量空间的二阶对应性质

$$\text{性质1.2.1.} \quad \begin{cases} \int \psi^+(\vec{r}, t)r_i\partial_j\psi(\vec{r}, t)d^3\vec{r} = \int d^3\vec{p}\psi^+(\vec{p}, t)(-\delta_{ij} - p_j\tilde{\partial}_i)\psi(\vec{p}, t) \\ \int \psi^+(\vec{r}, t)(\delta_{ij} + r_i\partial_j)\psi(\vec{r}, t)d^3\vec{r} = \int d^3\vec{p}\psi^+(\vec{p}, t)(-p_j\tilde{\partial}_i)\psi(\vec{p}, t) \\ \int \psi^+(\vec{r}, t)(r_i\partial_j - r_j\partial_i)\psi(\vec{r}, t)d^3\vec{r} = \int \psi^+(\vec{p}, t)(p_i\tilde{\partial}_j - p_j\tilde{\partial}_i)\psi(\vec{p}, t)d^3\vec{p} \end{cases}$$

$$\begin{aligned} \text{证明:} & \int \psi^+(\vec{r}, t)r_i\partial_j\psi(\vec{r}, t)d^3\vec{r} \\ &= \frac{1}{(2\pi)^3} \int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}', t)e^{-i\vec{p}'\cdot\vec{r}}\psi(\vec{p}, t)r_i\partial_j e^{i\vec{p}\cdot\vec{r}} \\ &= \frac{1}{(2\pi)^3} \int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}', t)e^{-i\vec{p}'\cdot\vec{r}}\psi(\vec{p}, t)ip_j(-i\tilde{\partial}_i)e^{i\vec{p}\cdot\vec{r}} \\ &= \frac{1}{(2\pi)^3} \int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}', t)e^{-i\vec{p}'\cdot\vec{r}}\psi(\vec{p}, t)p_j\tilde{\partial}_i e^{i\vec{p}\cdot\vec{r}} \\ &= -\frac{1}{(2\pi)^3} \int d^3\vec{p}d^3\vec{p}'d^3\vec{r}\psi^+(\vec{p}', t)\tilde{\partial}_i[\psi(\vec{p}, t)p_j]e^{i(\vec{p}-\vec{p}')\cdot\vec{r}} \\ &= -\int d^3\vec{p}d^3\vec{p}'\psi^+(\vec{p}', t)\tilde{\partial}_i[\psi(\vec{p}, t)p_j]\delta^3(\vec{p}-\vec{p}') \\ &= -\int d^3\vec{p}\psi^+(\vec{p}, t)\tilde{\partial}_i[\psi(\vec{p}, t)p_j] \\ &= \int \psi^+(\vec{p}, t)(-\delta_{ij} - p_j\tilde{\partial}_i)\psi(\vec{p}, t)d^3\vec{p} \end{aligned}$$

□

#### 性质1.2.2.

$$\begin{cases} \int \psi^+(\vec{r}, t)r_i\partial_j[\sigma(s) \cdot \nabla]\psi(\vec{r}, t)d^3\vec{r} = -i\{\int \psi^+(\vec{p}, t)\delta_{ij}[\sigma(s) \cdot \vec{p}]\psi(\vec{p}, t)d^3\vec{p} + \int \psi^+(\vec{p}, t)p_j\tilde{\partial}_i\{\sigma(s) \cdot \vec{p}\}\psi(\vec{p}, t)d^3\vec{p}\} \\ = -i\{\int \psi^+(\vec{p}, t)\delta_{ij}[\sigma(s) \cdot \vec{p}]\psi(\vec{p}, t)d^3\vec{p} + \int \psi^+(\vec{p}, t)[\sigma(s) \cdot \vec{p}]p_j\tilde{\partial}_i\psi(\vec{p}, t)d^3\vec{p} + \int \psi^+(\vec{p}, t)p_j\sigma_i(s)\psi(\vec{p}, t)d^3\vec{p}\} \end{cases}$$

$$\begin{aligned}
& \text{证明: } \int \psi^+(\vec{r}, t) r_i \partial_j [\sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \psi^+(\vec{p}', t) e^{-i\vec{p}' \cdot \vec{r}} \psi(\vec{p}, t) r_i \partial_j [\sigma(s) \cdot \nabla] e^{i\vec{p} \cdot \vec{r}} \\
&= \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \psi^+(\vec{p}', t) e^{-i\vec{p}' \cdot \vec{r}} \psi(\vec{p}, t) i p_j [i\sigma(s) \cdot \vec{p}] (-i\tilde{\partial}_i) e^{i\vec{p} \cdot \vec{r}} \\
&= \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \psi^+(\vec{p}', t) e^{-i\vec{p}' \cdot \vec{r}} \psi(\vec{p}, t) p_j [i\sigma(s) \cdot \vec{p}] \tilde{\partial}_i e^{i\vec{p} \cdot \vec{r}} \\
&= -\frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' d^3 \vec{r} \psi^+(\vec{p}', t) \tilde{\partial}_i \{ \psi(\vec{p}, t) p_j [i\sigma(s) \cdot \vec{p}] \} e^{i(\vec{p} - \vec{p}') \cdot \vec{r}} \\
&= -\int d^3 \vec{p} d^3 \vec{p}' \psi^+(\vec{p}', t) \tilde{\partial}_i \{ \psi(\vec{p}, t) p_j [i\sigma(s) \cdot \vec{p}] \} \delta^3(\vec{p} - \vec{p}') \\
&= -\int d^3 \vec{p} \psi^+(\vec{p}, t) \tilde{\partial}_i \{ \psi(\vec{p}, t) p_j [i\sigma(s) \cdot \vec{p}] \} \\
&= \int \psi^+(\vec{p}, t) (-\delta_{ij} - p_j \tilde{\partial}_i) \{ [i\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) \} d^3 \vec{p} \\
&= -i \{ \int \psi^+(\vec{p}, t) \delta_{ij} [\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) d^3 \vec{p} + \int \psi^+(\vec{p}, t) [\sigma(s) \cdot \vec{p}] p_j \tilde{\partial}_i \psi(\vec{p}, t) d^3 \vec{p} + \int \psi^+(\vec{p}, t) p_j \sigma_i(s) \psi(\vec{p}, t) d^3 \vec{p} \} \quad \square
\end{aligned}$$

性质1.2.3.

$$\begin{aligned}
& \int \psi^+(\vec{r}, t) (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3 \vec{r} = i \int \psi^+(\vec{p}, t) (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) \{ [\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) \} d^3 \vec{p} \\
&= i \{ \int \psi^+(\vec{p}, t) [\sigma(s) \cdot \vec{p}] (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) \psi(\vec{p}, t) d^3 \vec{p} + \int \psi^+(\vec{p}, t) [p_i \sigma_j(s) - p_j \sigma_i(s)] \psi(\vec{p}, t) d^3 \vec{p} \}
\end{aligned}$$

性质1.2.4.

$$\int \psi^+(\vec{r}, t) [\sigma(s) \cdot \nabla] \{ r_i \partial_j \psi(\vec{r}, t) \} d^3 \vec{r} = -i \{ \int \psi^+(\vec{p}, t) \delta_{ij} [\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) d^3 \vec{p} + \int \psi^+(\vec{p}, t) [\sigma(s) \cdot \vec{p}] p_j \tilde{\partial}_i \psi(\vec{p}, t) d^3 \vec{p} \}$$

证明:

$$\begin{aligned}
& \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \nabla] \{ r_i \partial_j \psi(\vec{r}, t) \} d^3 \vec{r} \\
&= \int \psi^+(\vec{r}, t) \sigma_i(s) \partial_j \psi(\vec{r}, t) d^3 \vec{r} + \int \psi^+(\vec{r}, t) r_i \partial_j [\sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3 \vec{r} \\
&= i \int \psi^+(\vec{p}, t) \sigma_i(s) p_j \psi(\vec{p}, t) d^3 \vec{p} - i \{ \int \psi^+(\vec{p}, t) \delta_{ij} [\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) d^3 \vec{p} + \int \psi^+(\vec{p}, t) p_j \tilde{\partial}_i \{ [\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) \} d^3 \vec{p} \} \\
&= -i \{ \int \psi^+(\vec{p}, t) \delta_{ij} [\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) d^3 \vec{p} + \int \psi^+(\vec{p}, t) [\sigma(s) \cdot \vec{p}] p_j \tilde{\partial}_i \psi(\vec{p}, t) d^3 \vec{p} \} \quad \square
\end{aligned}$$

$$\text{性质1.2.5. } \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \nabla] [(r_i \partial_j - r_j \partial_i) \psi(\vec{r}, t)] d^3 \vec{r} = i \int \psi^+(\vec{p}, t) [\sigma(s) \cdot \vec{p}] (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) \psi(\vec{p}, t) d^3 \vec{p}$$

### 1.3 坐标空间和动量空间的高阶对应性质

## 2 坐标空间中的自旋方程

### 2.1 s-自旋方程及其平面波解

定理2.1.1.  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x) = 0$

$$\text{推论2.1.1. } \begin{cases} \psi(\vec{r}, t) := \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma) e^{i\vec{p} \cdot \vec{r}} + a_2^+(\vec{p}, -s\varsigma) e^{-i\vec{p} \cdot \vec{r}}] d^3 \vec{p} \\ |\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -s\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -s\varsigma) \psi(\vec{r}, t) e^{-i\vec{p} \cdot \vec{r}} d^3 \vec{r} = \frac{1}{(2\pi)^{3/2}} \int \frac{i}{|\vec{p}|} \lambda^+(\hat{p}, -s\varsigma) \dot{\psi}(\vec{r}, t) e^{-i\vec{p} \cdot \vec{r}} d^3 \vec{r} \\ |\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}, -s\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -s\varsigma) \psi(\vec{r}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{r} = \frac{1}{(2\pi)^{3/2}} \int \frac{-i}{|\vec{p}|} \lambda^+(\hat{p}, -s\varsigma) \dot{\psi}(\vec{r}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{r} \end{cases}$$

定义2.1.1. 投影算子:  $\hat{P}_{k_\varsigma k'_\varsigma}(s, \varsigma) := \lambda_{k_\varsigma}(\hat{p}, -s\varsigma) \lambda_{k'_\varsigma}^+(\hat{p}, -s\varsigma)$ ,  $\hat{P}^2(s, \varsigma) = \hat{P}(s, \varsigma)$ ,  $\hat{P}^+(s, \varsigma) = \hat{P}(s, \varsigma)$

定义2.1.2.  $A(\vec{r}, t) := \frac{\partial_t}{\nabla^2} \psi(\vec{r}, t) \Leftrightarrow \psi(\vec{r}, t) = \partial_t A(\vec{r}, t)$

### 2.2 动量空间中自旋方程的平面波解

$$\text{推论2.2.1. } \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \psi(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}, \psi(\vec{p}, t) = |\vec{p}|^{(s-\frac{1}{2})} [a_1(\vec{p}, -s\varsigma) \lambda(\hat{p}, -s\varsigma) e^{-i|\vec{p}|t} + a_2^+(-\vec{p}, -s\varsigma) \lambda(-\hat{p}, -s\varsigma) e^{i|\vec{p}|t}]$$

$$\begin{aligned}
& \text{证明: } \psi(\vec{r}, t) := \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma) e^{i\vec{p} \cdot \vec{r}} + a_2^+(\vec{p}, -s\varsigma) e^{-i\vec{p} \cdot \vec{r}}] d^3 \vec{p} \\
&\Leftrightarrow \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma) e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + a_2^+(\vec{p}, -s\varsigma) e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] d^3 \vec{p} \\
&\Leftrightarrow \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma) e^{-i|\vec{p}|t} e^{i\vec{p} \cdot \vec{r}} + a_2^+(\vec{p}, -s\varsigma) e^{i|\vec{p}|t} e^{-i\vec{p} \cdot \vec{r}}] d^3 \vec{p} \\
&\Leftrightarrow \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} [a_1(\vec{p}, -s\varsigma) \lambda(\hat{p}, -s\varsigma) e^{-i|\vec{p}|t} + a_2^+(-\vec{p}, -s\varsigma) \lambda(-\hat{p}, -s\varsigma) e^{i|\vec{p}|t}] e^{i\vec{p} \cdot \vec{r}} d^3 \vec{p}
\end{aligned}$$

$$\begin{aligned} \Leftrightarrow \psi(\vec{r}, t) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \psi(\vec{p}, t) e^{i\vec{p}\cdot\vec{r}} d^3\vec{p}, \psi(\vec{p}, t) = |\vec{p}|^{s-\frac{1}{2}} [a_1(\vec{p}, -s\zeta) \lambda(\hat{p}, -s\zeta) e^{-i|\vec{p}|t} + a_2^\dagger(-\vec{p}, -s\zeta) \lambda(-\hat{p}, -s\zeta) e^{i|\vec{p}|t}] \\ \Leftrightarrow \psi(\vec{p}, t) &= \frac{1}{(2\pi)^{3/2}} \int \psi(\vec{r}, t) e^{-i\vec{p}\cdot\vec{r}} d^3\vec{r} \quad \square \end{aligned}$$

### 2.3 几个重要引理

引理2.3.1.  $s\nabla\psi(\vec{r}, t) = [i\sigma(s) \times \nabla + \zeta\sigma(s)\partial_t]\psi(\vec{r}, t) \Rightarrow [\frac{1}{s}\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = \zeta\partial_t\psi(\vec{r}, t)$

证明:  $s\nabla\psi(\vec{r}, t) = [i\sigma(s) \times \nabla + \zeta\sigma(s)\partial_t]\psi(\vec{r}, t)$

$$\Rightarrow s\sigma(s) \cdot \nabla\psi(\vec{r}, t) = \sigma(s) \cdot [i\sigma(s) \times \nabla + \zeta\sigma(s)\partial_t]\psi(\vec{r}, t)$$

$$\Leftrightarrow s[\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = -[\sigma(s) \cdot \nabla]\psi(\vec{r}, t) + \zeta\sigma^2(s)\partial_t\psi(\vec{r}, t)$$

$$\Leftrightarrow (s+1)[\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = \zeta s(s+1)\partial_t\psi(\vec{r}, t)$$

$$\Leftrightarrow [\frac{1}{s}\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = \zeta\partial_t\psi(\vec{r}, t) \quad \square$$

引理2.3.2.  $s\nabla\psi(\vec{r}, t) = [\sigma(s) \cdot \nabla - \zeta(s-1)\partial_t]\sigma(s)\psi(\vec{r}, t) \stackrel{s \neq 1}{\Leftrightarrow} [\frac{1}{s}\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = \zeta\partial_t\psi(\vec{r}, t)$

证明:  $s\nabla\psi(\vec{r}, t) = [\sigma(s) \cdot \nabla - \zeta(s-1)\partial_t]\sigma(s)\psi(\vec{r}, t)$

$$\Rightarrow s\sigma(s) \cdot \nabla\psi(\vec{r}, t) = \sigma(s) \cdot [\sigma(s) \cdot \nabla - \zeta(s-1)\partial_t]\sigma(s)\psi(\vec{r}, t)$$

$$\Leftrightarrow s[\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = \sigma(s) \cdot [\sigma(s) \cdot \nabla]\sigma(s)\psi(\vec{r}, t) - \zeta(s-1)\sigma^2(s)\partial_t\psi(\vec{r}, t)$$

$$\Leftrightarrow s[\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = [\sigma^2(s) - 1][\sigma(s) \cdot \nabla]\psi(\vec{r}, t) - \zeta(s-1)\sigma^2(s)\partial_t\psi(\vec{r}, t)$$

$$\Leftrightarrow (s+1)[\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = \sigma^2(s)[\sigma(s) \cdot \nabla]\psi(\vec{r}, t) - \zeta(s-1)\sigma^2(s)\partial_t\psi(\vec{r}, t)$$

$$\Leftrightarrow (s-1)[\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = \zeta(s-1)s\partial_t\psi(\vec{r}, t)$$

$$\Leftrightarrow \stackrel{s \neq 1}{\Rightarrow} [\frac{1}{s}\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = \zeta\partial_t\psi(\vec{r}, t) \quad \square$$

推论2.3.1.  $\begin{cases} s\nabla\psi(\vec{r}, t) = \{[\sigma(s) \cdot \nabla, \sigma(s)] + \zeta\sigma(s)\partial_t\}\psi(\vec{r}, t) \stackrel{s \neq 1}{\Leftrightarrow} s\nabla\psi(\vec{r}, t) = [\sigma(s) \cdot \nabla - \zeta(s-1)\partial_t]\sigma(s)\psi(\vec{r}, t) \\ s\nabla\psi(\vec{r}, t) = \{[\sigma(s) \cdot \nabla, \sigma(s)] + \zeta\sigma(s)\partial_t\}\psi(\vec{r}, t) \stackrel{s=1}{\Rightarrow} s\nabla\psi(\vec{r}, t) = [\sigma(s) \cdot \nabla]\sigma(s)\psi(\vec{r}, t) \end{cases}$

推论2.3.2.  $\nabla\psi(\vec{r}, t) = \{[\sigma(s) \cdot \nabla, \sigma(s)] + \zeta\sigma(s)\partial_t\}\psi(\vec{r}, t) \stackrel{s=1}{\Rightarrow} \begin{cases} \nabla\psi(\vec{r}, t) = [\sigma(s) \cdot \nabla]\sigma(s)\psi(\vec{r}, t) \\ [\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = \zeta\partial_t\psi(\vec{r}, t) \end{cases}$

推论2.3.3.  $[\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\psi(\vec{r}, t) = [\sigma(s) \cdot \hat{\nabla}]^{n-1} \hat{\nabla}\psi(\vec{r}, t), s=1$

引理2.3.3.  $\begin{cases} s^2\nabla\psi(\vec{r}, t) = \{is\sigma(s) \times \nabla + \sigma(s)[\sigma(s) \cdot \nabla]\}\psi(\vec{r}, t) \Rightarrow \nabla^2\psi(\vec{r}, t) = [\frac{1}{s}\sigma(s) \cdot \nabla]^2\psi(\vec{r}, t) \\ \updownarrow \\ s^2\nabla\psi(\vec{r}, t) = \{s\sigma(s) \cdot \nabla\sigma(s) - (s-1)\sigma(s)[\sigma(s) \cdot \nabla]\}\psi(\vec{r}, t) \Rightarrow \nabla^2\psi(\vec{r}, t) = [\frac{1}{s}\sigma(s) \cdot \nabla]^2\psi(\vec{r}, t) \end{cases}$

### 2.4 s-自旋方程的多种等价形式(证明略)

定理2.4.1.

$$[s\partial_a + iS_{ab}(s, \zeta)\partial^b]\psi(x) = 0 \quad [\Leftrightarrow] \quad \sigma(s) \cdot \nabla\psi = s\zeta\partial_t\psi, O(s) \cdot \nabla\psi(x) = 0$$

$$[\Updownarrow] \quad [\Updownarrow]$$

$$s\nabla\psi(x) = [i\sigma(s) \times \nabla + \zeta\sigma(s)\partial_t]\psi(x) \quad [\Leftrightarrow] \quad \begin{cases} \sigma(s) \cdot \nabla\psi(x) = s\zeta\partial_t\psi(x) \\ s^2\nabla\psi(x) = is\sigma(s) \times \nabla + \sigma(s)[\sigma(s) \cdot \nabla]\psi(x) \end{cases}$$

$$[\Updownarrow] \quad [\Updownarrow]$$

$$s\nabla\psi(x) = \{[\sigma(s) \cdot \nabla, \sigma(s)] + \zeta\sigma(s)\partial_t\}\psi(x) [\Leftrightarrow] \begin{cases} \sigma(s) \cdot \nabla\psi(x) = s\zeta\partial_t\psi(x) \\ s^2\nabla\psi(x) = \{s\sigma(s) \cdot \nabla\sigma(s) - (s-1)\sigma(s)[\sigma(s) \cdot \nabla]\}\psi(x) \end{cases}$$

## 2.5 约束方程的多种等价形式

推论2.5.1.  $O(1) = \frac{1}{\sqrt{2}} \left\{ \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}, i \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2} & 0 \end{bmatrix} \right\}$

推论2.5.2.  $O(1)S_m^+(1) = \left\{ \begin{bmatrix} i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i \end{bmatrix} \right\}$

推论2.5.3.  $S_m(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & -i \\ -1 & 0 & -1 \\ 0 & -i\sqrt{2} & 0 \end{bmatrix}, S_m^+(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 & 0 \\ 0 & 0 & i\sqrt{2} \\ i & -1 & 0 \end{bmatrix}, S_m(1)S_m^+(1) = S_m^+(1)S_m(1) = I_3$

推论2.5.4.  $[\gamma \cdot \nabla] \gamma \Psi = \nabla \Psi \Leftrightarrow O(1)S_m^+(1) \cdot \nabla \Psi = 0 \Leftrightarrow \nabla \cdot \Psi = 0$

推论2.5.5.  $O_x(s) = -\sqrt{s(s-\frac{1}{2})} [\bar{N}_{1\zeta}(s-\frac{1}{2})\bar{N}_{1\zeta}(s) - \bar{N}_{2\zeta}(s-\frac{1}{2})\bar{N}_{2\zeta}(s)]$   
 $= \frac{1}{2} \begin{bmatrix} -\sqrt{2s \cdot (2s-1)} & 0 & \sqrt{2 \cdot 1} & 0 & 0 & 0 \\ 0 & -\sqrt{(2s-1) \cdot (2s-2)} & 0 & \sqrt{3 \cdot 2} & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & -\sqrt{2 \cdot 1} & 0 & \sqrt{2s \cdot (2s-1)} \end{bmatrix}$

推论2.5.6.  $O_y(s) = -i\sqrt{s(s-\frac{1}{2})} [\bar{N}_{1\zeta}(s-\frac{1}{2})\bar{N}_{1\zeta}(s) + \bar{N}_{2\zeta}(s-\frac{1}{2})\bar{N}_{2\zeta}(s)]$   
 $= \frac{i}{2} \begin{bmatrix} -\sqrt{2s \cdot (2s-1)} & 0 & -\sqrt{2 \cdot 1} & 0 & 0 & 0 \\ 0 & -\sqrt{(2s-1) \cdot (2s-2)} & 0 & -\sqrt{3 \cdot 2} & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & -\sqrt{2 \cdot 1} & 0 & -\sqrt{2s \cdot (2s-1)} \end{bmatrix}$

推论2.5.7.  $O_z(s) = \sqrt{s(s-\frac{1}{2})} [\bar{N}_{1\zeta}(s-\frac{1}{2})\bar{N}_{2\zeta}(s) + \bar{N}_{2\zeta}(s-\frac{1}{2})\bar{N}_{1\zeta}(s)]$   
 $= \begin{bmatrix} 0 & \sqrt{1 \cdot (2s-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2 \cdot (2s-2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{(2s-1) \cdot 1} & 0 \end{bmatrix}, \bar{N}_{1\zeta}(s-\frac{1}{2})\bar{N}_{2\zeta}(s) = \bar{N}_{2\zeta}(s-\frac{1}{2})\bar{N}_{1\zeta}(s)$

$$\sigma(s) = \left( \frac{1}{2} \begin{bmatrix} 0 & A_1 & 0 & 0 & 0 \\ A_1 & 0 & A_2 & 0 & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & A_{2s} \\ 0 & 0 & 0 & A_{2s} & 0 \end{bmatrix}, \frac{i}{2} \begin{bmatrix} 0 & -A_1 & 0 & 0 & 0 \\ A_1 & 0 & -A_2 & 0 & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -A_{2s} \\ 0 & 0 & 0 & A_{2s} & 0 \end{bmatrix}, \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix} \right) \quad (24.1a)$$

$$A_n = \sqrt{n} \cdot \sqrt{2s+1-n}, n = 1, 2, \dots, 2s; \sigma(s) \prec \sigma_{\alpha_\zeta k_\zeta}^{l_\zeta}(s) \simeq \sigma_{\alpha'_\zeta k'_\zeta}^{l'_\zeta}(s) \quad (24.1b)$$

$$\sigma(s) \times \sigma(s) = i\sigma(s), \sigma^2(s) = s(s+1), \sigma^+(s) = \sigma(s), s = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots \quad (24.1c)$$

## 2.6 约束方程的重要推论

推论2.6.1.  $O(s) \cdot \nabla \psi = 0 \Leftrightarrow$

$$\frac{1}{2} \begin{bmatrix} -\sqrt{2s \cdot (2s-1)}(\partial_x + i\partial_y) & 2\sqrt{1 \cdot (2s-1)}\partial_z & \sqrt{2 \cdot 1}(\partial_x - i\partial_y) & 0 & 0 & 0 \\ 0 & -\sqrt{(2s-1) \cdot (2s-2)}(\partial_x + i\partial_y) & 2\sqrt{2 \cdot (2s-2)}\partial_z & \sqrt{3 \cdot 2}(\partial_x - i\partial_y) & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & -\sqrt{2 \cdot 1}(\partial_x + i\partial_y) & 2\sqrt{(2s-1) \cdot 1}\partial_z & \sqrt{2s \cdot (2s-1)}(\partial_x - i\partial_y) \end{bmatrix} \psi = 0$$

推论2.6.2.  $\{s^2 \hat{\nabla} + s[\sigma(s), \sigma(s) \cdot \hat{\nabla}] - \sigma(s)[\sigma(s) \cdot \hat{\nabla}]\} \psi = 0 \Leftrightarrow O(s) \cdot \nabla \psi = 0$

证明:  $\{s^2 \partial_y + s[\sigma_y(s), \sigma(s) \cdot \hat{\nabla}] - \sigma_y(s)[\sigma(s) \cdot \hat{\nabla}]\} \psi = 0$

$$\Leftrightarrow \{s^2 \partial_y + is[\sigma_x(s)\partial_z - \sigma_z(s)\partial_x] - [\sigma_y^2(s)\partial_y + \sigma_y(s)\sigma_x(s)\partial_x + \sigma_y(s)\sigma_z(s)\partial_z]\} \psi = 0$$

$$\Leftrightarrow \{[s^2 - \sigma_y^2(s)]\partial_y + [is\sigma_x(s) - \sigma_y(s)\sigma_z(s)]\partial_z - [is\sigma_z(s) + \sigma_y(s)\sigma_x(s)]\partial_x\} \psi = 0$$

$$\Leftrightarrow \left\{ [s^2 - \frac{1}{4}] \begin{bmatrix} A_1^2 & 0 & -A_1 A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2 + A_1^2 & 0 & -A_2 A_3 & 0 & 0 & 0 \\ -A_1 A_2 & 0 & A_3^2 + A_2^2 & 0 & \dots & 0 & 0 \\ 0 & -A_2 A_3 & 0 & \dots & 0 & -A_{2s-1} A_{2s} & 0 \\ 0 & 0 & \dots & 0 & A_{2s-1}^2 + A_{2s-2}^2 & 0 & -A_{2s-1} A_{2s} \\ 0 & 0 & 0 & -A_{2s-2} A_{2s-1} & 0 & A_{2s}^2 + A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & -A_{2s-1} A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \right\} \partial_y$$

$$+ [is \frac{1}{2} \begin{bmatrix} 0 & A_1 & 0 & 0 & 0 \\ A_1 & 0 & A_2 & 0 & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & A_{2s} \\ 0 & 0 & 0 & A_{2s} & 0 \end{bmatrix} - \frac{i}{2} \begin{bmatrix} 0 & -(s-1)A_1 & 0 & 0 & 0 \\ sA_1 & 0 & -(s-2)A_2 & 0 & 0 \\ 0 & (s-1)A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & sA_{2s} \\ 0 & 0 & 0 & -(s-1)A_{2s} & 0 \end{bmatrix}] \partial_z$$

$$\begin{aligned}
& - \left[ i s \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix} + \frac{i}{4} \begin{bmatrix} -A_1^2 & 0 & -A_1 A_2 & 0 & 0 & 0 & 0 \\ 0 & A_1^2 - A_2^2 & 0 & -A_2 A_3 & 0 & 0 & 0 \\ A_1 A_2 & 0 & A_2^2 - A_3^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2 A_3 & 0 & \cdots & 0 & -A_{2s-2} A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-2}^2 - A_{2s-1}^2 & 0 & -A_{2s-1} A_{2s} \\ 0 & 0 & 0 & A_{2s-2} A_{2s-1} & 0 & A_{2s-1}^2 - A_{2s}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1} A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \right] \partial_x \psi = 0 \\
& \Leftrightarrow \left\{ \left[ s^2 - \frac{1}{4} \begin{bmatrix} A_1^2 & 0 & -A_1 A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2 + A_1^2 & 0 & -A_2 A_3 & 0 & 0 & 0 \\ -A_1 A_2 & 0 & A_3^2 + A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & -A_2 A_3 & 0 & \cdots & 0 & -A_{2s-1} A_{2s} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2 + A_{2s-2}^2 & 0 & -A_{2s-1} A_{2s} \\ 0 & 0 & 0 & -A_{2s-2} A_{2s-1} & 0 & A_{2s}^2 + A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & -A_{2s-1} A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \right] \partial_y \right. \\
& \left. - \left[ i s \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix} + \frac{i}{4} \begin{bmatrix} -A_1^2 & 0 & -A_1 A_2 & 0 & 0 & 0 & 0 \\ 0 & A_1^2 - A_2^2 & 0 & -A_2 A_3 & 0 & 0 & 0 \\ A_1 A_2 & 0 & A_2^2 - A_3^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2 A_3 & 0 & \cdots & 0 & -A_{2s-2} A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-2}^2 - A_{2s-1}^2 & 0 & -A_{2s-1} A_{2s} \\ 0 & 0 & 0 & A_{2s-2} A_{2s-1} & 0 & A_{2s-1}^2 - A_{2s}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1} A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \right] \partial_x \right. \\
& \left. - \frac{i}{2} \begin{bmatrix} 0 & -(2s-1)A_1 & 0 & 0 & 0 \\ 0A_1 & 0 & -(2s-2)A_2 & 0 & 0 \\ 0 & -1A_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0A_{2s} \\ 0 & 0 & 0 & -(2s-1)A_{2s} & 0 \end{bmatrix} \partial_z \right\} \psi = 0 \quad \square
\end{aligned}$$

$$\text{证明: } \{s^2 \partial_x + s[\sigma_x(s), \sigma(s) \cdot \hat{\nabla}] - \sigma_x(s)[\sigma(s) \cdot \hat{\nabla}]\} \psi = 0$$

$$\Leftrightarrow \{s^2 \partial_x + i s[\sigma_z(s) \partial_y - \sigma_y(s) \partial_z] - [\sigma_x^2(s) \partial_x + \sigma_x(s) \sigma_y(s) \partial_y + \sigma_x(s) \sigma_z(s) \partial_z]\} \psi = 0$$

$$\Leftrightarrow \{[s^2 - \sigma_x^2(s)] \partial_x + [i s \sigma_z(s) - \sigma_x(s) \sigma_y(s)] \partial_y - [i s \sigma_y(s) + \sigma_x(s) \sigma_z(s)] \partial_z\} \psi = 0$$

$$\begin{aligned}
& \Leftrightarrow \left\{ \left[ s^2 - \frac{1}{4} \begin{bmatrix} A_1^2 & 0 & A_1 A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2 + A_1^2 & 0 & A_2 A_3 & 0 & 0 & 0 \\ A_1 A_2 & 0 & A_3^2 + A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2 A_3 & 0 & \cdots & 0 & A_{2s-2} A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2 + A_{2s-2}^2 & 0 & A_{2s-1} A_{2s} \\ 0 & 0 & 0 & A_{2s-2} A_{2s-1} & 0 & A_{2s}^2 + A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1} A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \right] \partial_x \right. \\
& \left. + \left[ i s \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix} - \frac{i}{4} \begin{bmatrix} A_1^2 & 0 & -A_1 A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2 - A_1^2 & 0 & -A_2 A_3 & 0 & 0 & 0 \\ A_1 A_2 & 0 & A_3^2 - A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2 A_3 & 0 & \cdots & 0 & -A_{2s-2} A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2 - A_{2s-2}^2 & 0 & -A_{2s-1} A_{2s} \\ 0 & 0 & 0 & A_{2s-2} A_{2s-1} & 0 & A_{2s}^2 - A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1} A_{2s} & 0 & -A_{2s}^2 \end{bmatrix} \right] \partial_y \right. \\
& \left. - \left[ i s \frac{i}{2} \begin{bmatrix} 0 & -A_1 & 0 & 0 & 0 \\ A_1 & 0 & -A_2 & 0 & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -A_{2s} \\ 0 & 0 & 0 & A_{2s} & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & (s-1)A_1 & 0 & 0 & 0 \\ sA_1 & 0 & (s-2)A_2 & 0 & 0 \\ 0 & (s-1)A_2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -sA_{2s} \\ 0 & 0 & 0 & -(s-1)A_{2s} & 0 \end{bmatrix} \right] \partial_z \right\} \psi = 0 \\
& \Leftrightarrow \left\{ \left[ s^2 - \frac{1}{4} \begin{bmatrix} A_1^2 & 0 & A_1 A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2 + A_1^2 & 0 & A_2 A_3 & 0 & 0 & 0 \\ A_1 A_2 & 0 & A_3^2 + A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2 A_3 & 0 & \cdots & 0 & A_{2s-2} A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2 + A_{2s-2}^2 & 0 & A_{2s-1} A_{2s} \\ 0 & 0 & 0 & A_{2s-2} A_{2s-1} & 0 & A_{2s}^2 + A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1} A_{2s} & 0 & A_{2s}^2 \end{bmatrix} \right] \partial_x \right. \\
& \left. + \left[ i s \begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -(s-1) & 0 \\ 0 & 0 & 0 & 0 & -s \end{bmatrix} - \frac{i}{4} \begin{bmatrix} A_1^2 & 0 & -A_1 A_2 & 0 & 0 & 0 & 0 \\ 0 & A_2^2 - A_1^2 & 0 & -A_2 A_3 & 0 & 0 & 0 \\ A_1 A_2 & 0 & A_3^2 - A_2^2 & 0 & \cdots & 0 & 0 \\ 0 & A_2 A_3 & 0 & \cdots & 0 & -A_{2s-2} A_{2s-1} & 0 \\ 0 & 0 & \cdots & 0 & A_{2s-1}^2 - A_{2s-2}^2 & 0 & -A_{2s-1} A_{2s} \\ 0 & 0 & 0 & A_{2s-2} A_{2s-1} & 0 & A_{2s}^2 - A_{2s-1}^2 & 0 \\ 0 & 0 & 0 & 0 & A_{2s-1} A_{2s} & 0 & -A_{2s}^2 \end{bmatrix} \right] \partial_y \right. \\
& \left. - \frac{1}{2} \begin{bmatrix} 0 & (2s-1)A_1 & 0 & 0 & 0 \\ 0A_1 & 0 & (2s-2)A_2 & 0 & 0 \\ 0 & -1A_2 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0A_{2s} \\ 0 & 0 & 0 & -(2s-1)A_{2s} & 0 \end{bmatrix} \partial_z \right\} \psi = 0 \quad \square
\end{aligned}$$

$$\text{推论2.6.3. } \begin{bmatrix} -(2s-1)A_1(\partial_x + i\partial_y) & 1(2s-1)2 & 1A_2(\partial_x - i\partial_y) & 0 & 0 \\ 0 & -(2s-2)A_2(\partial_x + i\partial_y) & 2(2s-2)2 & \cdots & 0 \\ 0 & 0 & \cdots & 1(2s-1)2, (2s-1)A_{2s}(\partial_x - i\partial_y) & 0 \end{bmatrix} \psi = 0$$

$$\text{证明: } \{s^2 \partial_z + s[\sigma_z(s), \sigma(s) \cdot \hat{\nabla}] - \sigma_z(s)[\sigma(s) \cdot \hat{\nabla}]\} \psi = 0, A_n = \sqrt{n} \cdot \sqrt{2s+1-n}, n = 1, 2, \dots, 2s;$$

$$\Leftrightarrow \{s^2 \partial_z + i s[\sigma_y(s) \partial_x - \sigma_x(s) \partial_y] - [\sigma_z^2(s) \partial_z + \sigma_z(s) \sigma_x(s) \partial_x + \sigma_z(s) \sigma_y(s) \partial_y]\} \psi = 0$$

$$\Leftrightarrow \{[s^2 - \sigma_z^2(s)]\partial_z + [is\sigma_y(s) - \sigma_z(s)\sigma_x(s)]\partial_x - [is\sigma_x(s) + \sigma_z(s)\sigma_y(s)]\partial_y\}\psi = 0$$

$$\Leftrightarrow \{[s^2 - \sigma_z^2(s)]\partial_z$$

$$+ [-\frac{1}{2} \begin{bmatrix} 0 & -sA_1 & 0 & 0 & 0 \\ sA_1 & 0 & -sA_2 & 0 & 0 \\ 0 & sA_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -sA_{2s} \\ 0 & 0 & 0 & sA_{2s} & 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & sA_1 & 0 & 0 & 0 \\ (s-1)A_1 & 0 & (s-1)A_2 & 0 & 0 \\ 0 & (s-2)A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -(s-1)A_{2s} \\ 0 & 0 & 0 & 0 & -sA_{2s} \end{bmatrix}] \partial_x$$

$$- [\frac{i}{2} \begin{bmatrix} 0 & sA_1 & 0 & 0 & 0 \\ sA_1 & 0 & sA_2 & 0 & 0 \\ 0 & sA_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & sA_{2s} \\ 0 & 0 & 0 & sA_{2s} & 0 \end{bmatrix} + \frac{i}{2} \begin{bmatrix} 0 & -sA_1 & 0 & 0 & 0 \\ (s-1)A_1 & 0 & -(s-1)A_2 & 0 & 0 \\ 0 & (s-2)A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & (s-1)A_{2s} \\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix}] \partial_y\}\psi = 0$$

$$\Leftrightarrow \{[s^2 - \begin{bmatrix} s^2 & 0 & 0 & 0 & 0 \\ 0 & (s-1)^2 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & (s-1)^2 & 0 \\ 0 & 0 & 0 & 0 & s^2 \end{bmatrix}] \partial_z - \frac{1}{2} \begin{bmatrix} 0 & -0A_1 & 0 & 0 & 0 \\ (2s-1)A_1 & 0 & -1A_2 & 0 & 0 \\ 0 & (2s-2)A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -(2s-1)A_{2s} \\ 0 & 0 & 0 & 0A_{2s} & 0 \end{bmatrix}] \partial_x - \frac{i}{2} \begin{bmatrix} 0 & 0A_1 & 0 & 0 & 0 \\ (2s-1)A_1 & 0 & 1A_2 & 0 & 0 \\ 0 & (2s-2)A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & (2s-1)A_{2s} \\ 0 & 0 & 0 & 0 & 0A_{2s} \end{bmatrix}] \partial_y\}$$

$$\Leftrightarrow \left\{ \begin{bmatrix} 0(2s) & 0 & 0 & 0 & 0 \\ 0 & 1(2s-1) & 0 & 0 & 0 \\ 0 & 0 & 2(2s-2) & 0 & 0 \\ 0 & 0 & 0 & 1(2s-1) & 0 \\ 0 & 0 & 0 & 0 & 0(2s) \end{bmatrix} \partial_z - \frac{1}{2} \begin{bmatrix} (2s-1)A_1 & -0A_1 & 0 & 0 & 0 \\ 0 & (2s-2)A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -(2s-1)A_{2s} \\ 0 & 0 & 0 & 0A_{2s} & 0 \end{bmatrix} \partial_x - \frac{i}{2} \begin{bmatrix} (2s-1)A_1 & 0A_1 & 0 & 0 & 0 \\ 0 & (2s-2)A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & (2s-1)A_{2s} \\ 0 & 0 & 0 & 0 & 0A_{2s} \end{bmatrix} \partial_y \right\} \psi = 0$$

$$\Leftrightarrow \begin{bmatrix} 0(2s)2 & 0A_1(\partial_x - i\partial_y) & 0 & 0 & 0 \\ -(2s-1)A_1(\partial_x + i\partial_y) & 1(2s-1)2 & 1A_2(\partial_x - i\partial_y) & 0 & 0 \\ 0 & -(2s-2)A_2(\partial_x + i\partial_y) & 2(2s-2)2 & \dots & 0 \\ 0 & 0 & \dots & 1(2s-1)2 & (2s-1)A_{2s}(\partial_x - i\partial_y) \\ 0 & 0 & 0 & -0A_{2s}(\partial_x + i\partial_y) & 0(2s)2 \end{bmatrix} \psi = 0$$

$$\Leftrightarrow \begin{bmatrix} -(2s-1)A_1(\partial_x + i\partial_y) & 1(2s-1)2 & 1A_2(\partial_x - i\partial_y) & 0 & 0 \\ 0 & -(2s-2)A_2(\partial_x + i\partial_y) & 2(2s-2)2 & \dots & 0 \\ 0 & 0 & \dots & 1(2s-1)2 & (2s-1)A_{2s}(\partial_x - i\partial_y) \end{bmatrix} \psi = 0 \quad \square$$

推论2.6.4.

$$\sigma_z(s)\sigma_x(s) = \frac{1}{2} \begin{bmatrix} 0 & sA_1 & 0 & 0 & 0 \\ (s-1)A_1 & 0 & (s-1)A_2 & 0 & 0 \\ 0 & (s-2)A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -(s-1)A_{2s} \\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix}$$

$$\sigma_z(s)\sigma_y(s) = \frac{i}{2} \begin{bmatrix} 0 & -sA_1 & 0 & 0 & 0 \\ (s-1)A_1 & 0 & -(s-1)A_2 & 0 & 0 \\ 0 & (s-2)A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & (s-1)A_{2s} \\ 0 & 0 & 0 & -sA_{2s} & 0 \end{bmatrix}$$

## 2.7 自旋方程的重要推论

$$\text{定理2.7.1. } [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(\vec{r}, t) = 0 \Leftrightarrow \begin{cases} [\frac{1}{s}\sigma(s) \cdot \nabla]\psi(\vec{r}, t) = \varsigma\partial_t\psi(\vec{r}, t) \\ [\sigma(s) \cdot \hat{\nabla}]\sigma(s)\psi(\vec{r}, t) = [s\hat{\nabla} + \varsigma(s-1)\hat{\partial}_t\sigma(s)]\psi(\vec{r}, t) \\ \hat{\nabla} := \frac{-i\nabla}{\sqrt{-\nabla^2}}, \hat{\nabla}^2 = 1, \hat{\partial}_t := \frac{-i\partial_t}{\sqrt{-\nabla^2}} \simeq -1 \end{cases}$$

$$\Leftrightarrow \begin{cases} [\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\psi(\vec{r}, t) = \{s[s^n - (s-1)^n](\varsigma\hat{\partial}_t)^{n-1}\hat{\nabla} + (s-1)^n(\varsigma\hat{\partial}_t)^n\sigma(s)\}\psi(\vec{r}, t) \\ [\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\psi(\vec{r}, t) = \{s[s^n - (s-1)^n][\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^{n-1}\hat{\nabla} + (s-1)^n\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^n\}\psi(\vec{r}, t) \\ [\frac{1}{s}\sigma(s) \cdot \nabla]^n \psi(\vec{r}, t) = \varsigma^n \partial_t^n \psi(\vec{r}, t), \nabla^{2n}\psi(\vec{r}, t) = [\frac{1}{s}\sigma(s) \cdot \nabla]^{2n}\psi(\vec{r}, t) = \partial_t^{2n}\psi(\vec{r}, t), n \geq 1 \end{cases}$$

证明:

$$[\sigma(s) \cdot \hat{\nabla}]\sigma(s)\psi(\vec{r}, t) = [e_1\hat{\nabla} + d_1\sigma(s)]\psi(\vec{r}, t), e_1 = s, d_1 = \varsigma(s-1)\hat{\partial}_t$$

..

$$[\sigma(s) \cdot \hat{\nabla}]^{n-1}\sigma(s)\psi(\vec{r}, t) = [e_{n-1}\hat{\nabla} + d_{n-1}\sigma(s)]\psi(\vec{r}, t)$$

$$[\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\psi(\vec{r}, t) = [e_n\hat{\nabla} + d_n\sigma(s)]\psi(\vec{r}, t)$$

..

$$[\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\psi(\vec{r}, t)$$

$$= [\sigma(s) \cdot \hat{\nabla}][e_{n-1}\hat{\nabla} + d_{n-1}\sigma(s)]\psi(\vec{r}, t) = [(-s\varsigma e_{n-1} + d_1^{n-1}e_1)\hat{\nabla} + d_{n-1}d_1\sigma(s)]\psi(\vec{r}, t)$$

$$\begin{cases} e_n = e_{n-1}s\varsigma\hat{\partial}_t + e_1d_1^{n-1} \\ d_n = d_{n-1}d_1 \\ e_1 = s, d_1 = \varsigma(s-1)\hat{\partial}_t \end{cases} \Leftrightarrow \begin{cases} e_n = s[s^n - (s-1)^n](\varsigma\hat{\partial}_t)^{n-1} \\ d_n = d_1^n = (s-1)^n(\varsigma\hat{\partial}_t)^n \end{cases}$$

$$[\sigma(s) \cdot \hat{\nabla}]^n \sigma(s) \psi(\vec{r}, t) = \{s[s^n - (s-1)^n](\zeta \hat{\partial}_t)^{n-1} \hat{\nabla} + (s-1)^n (\zeta \hat{\partial}_t)^n \sigma(s)\} \psi(\vec{r}, t)$$

$$[\sigma(s) \cdot \hat{\nabla}]^n \sigma(s) \psi(\vec{r}, t) = \{s[s^n - (s-1)^n][\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^{n-1} \hat{\nabla} + (s-1)^n \sigma(s)[\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^n\} \psi(\vec{r}, t), n \geq 1 \quad \square$$

推论2.7.1.  $[s\partial_a + iS_{ab}(s, \zeta)\partial^b]\psi(\vec{r}, t) = 0$

$$[\Rightarrow] \begin{cases} \sigma^\alpha(s)[\sigma(s) \cdot \hat{\nabla}]^n \sigma_\alpha(s) \psi = s[s^{n+1} + (s-1)^n](\zeta \hat{\partial}_t)^n(\vec{r}, t) \\ \sigma^\alpha(s)[\sigma(s) \cdot \hat{\nabla}]^n \sigma_\alpha(s) \psi(\vec{r}, t) = s[s^{n+1} + (s-1)^n][\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^n \psi(\vec{r}, t) \end{cases}$$

定理2.7.2.  $[\sigma(s) \cdot \hat{\nabla}]\sigma(s)\psi(\vec{r}, t) = \{s\hat{\nabla} + (s-1)\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]\}\psi(\vec{r}, t)$

$$[\Leftrightarrow] \begin{cases} [\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\psi(\vec{r}, t) = \{s[s^n - (s-1)^n][\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^{n-1} \hat{\nabla} + (s-1)^n \sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^n\} \psi(\vec{r}, t) \\ [\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^{2n} \psi(\vec{r}, t) = \psi(\vec{r}, t), n \geq 1 \end{cases}$$

证明:

$$[\sigma(s) \cdot \hat{\nabla}]\sigma(s)\psi(\vec{r}, t) = [e_1 \hat{\nabla} + d_1 \sigma(s)]\psi(\vec{r}, t), e_1 = s, d_1 = (s-1)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]$$

..

$$[\sigma(s) \cdot \hat{\nabla}]^{n-1} \sigma(s)\psi(\vec{r}, t) = [e_{n-1} \hat{\nabla} + d_{n-1} \sigma(s)]\psi(\vec{r}, t)$$

$$[\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\psi(\vec{r}, t) = [e_n \hat{\nabla} + d_n \sigma(s)]\psi(\vec{r}, t)$$

..

$$[\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\psi(\vec{r}, t)$$

$$= [\sigma(s) \cdot \hat{\nabla}][e_{n-1} \hat{\nabla} + d_{n-1} \sigma(s)]\psi(\vec{r}, t) = [(e_{n-1} s[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}] + e_1 d_1^{n-1}) \hat{\nabla} + d_{n-1} d_1 \sigma(s)]\psi(\vec{r}, t)$$

$$\begin{cases} e_n = e_{n-1} s[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}] + e_1 d_1^{n-1} \\ d_n = d_{n-1} d_1 \\ e_1 = s, d_1 = (s-1)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}] \end{cases} \Leftrightarrow \begin{cases} e_n = s[s^n - (s-1)^n][\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^{n-1} \\ d_n = d_1^n = (s-1)^n [\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^n \end{cases}$$

$$[\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\psi(\vec{r}, t) = \{s[s^n - (s-1)^n][\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^{n-1} \hat{\nabla} + (s-1)^n \sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^n\} \psi(\vec{r}, t), n \geq 1 \quad \square$$

推论2.7.2.  $[\sigma(s) \cdot \hat{\nabla}]\sigma(s)\psi(\vec{r}, t) = \{s\hat{\nabla} + (s-1)\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]\}\psi(\vec{r}, t)$

$$[\Rightarrow] \sigma^\alpha(s)[\sigma(s) \cdot \hat{\nabla}]^n \sigma_\alpha(s)\psi(\vec{r}, t) = s[s^{n+1} + (s-1)^n][\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^n \psi(\vec{r}, t), n \geq 1$$

推论2.7.3.  $[\sigma(s) \cdot \hat{\nabla}]\sigma(s)\psi(\vec{r}, t) = \{s\hat{\nabla} + (s-1)\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]\}\psi(\vec{r}, t)$

$$[\Rightarrow] \begin{cases} \sigma(s) \times \{[\sigma(s) \cdot \hat{\nabla}]\sigma(s)\}\psi(\vec{r}, t) = i\{-s^2[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]\sigma(s) + (s^2 + s - 1)\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]\}\psi(\vec{r}, t) \\ \sigma(s) \cdot \{\sigma(s) \times [\sigma(s) \cdot \hat{\nabla}]\sigma(s)\}\psi(\vec{r}, t) = i[s^2 + s - 1][\sigma(s) \cdot \hat{\nabla}]\psi(\vec{r}, t) \end{cases}$$

推论2.7.4.  $[\sigma(s) \cdot \hat{\nabla}]\sigma(s)\psi(\vec{r}, t) = \{s\hat{\nabla} + (s-1)\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]\}\psi(\vec{r}, t)$

$$[\Rightarrow] \begin{cases} \sigma(s) \times \{[\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\}\psi(\vec{r}, t) \\ = i\{-s^2[s^n - (s-1)^n][\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^{n-1} + [s^{n+2} - (s+1)(s-1)^{n+1}]\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^n\}\psi(\vec{r}, t) \\ \sigma(s) \cdot \{\sigma(s) \times [\sigma(s) \cdot \hat{\nabla}]^n \sigma(s)\}\psi(\vec{r}, t) = is[s^{n+1} + (s-1)^n][\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^n \psi(\vec{r}, t), n \geq 1 \end{cases}$$

### 3 动量空间中的自旋方程

#### 3.1 动量空间中s-自旋方程的多种等价形式

定理3.1.1.

$$[s\partial_a + iS_{ab}(s, \zeta)\partial^b]\psi(x) = 0 \quad [\Leftrightarrow] \quad \sigma(s) \cdot \nabla \psi = s\zeta \partial_t \psi, O(s) \cdot \nabla \psi(x) = 0$$

$[\Downarrow]$

$[\Downarrow]$

$$s\nabla \psi(x) = [i\sigma(s) \times \nabla + \zeta \sigma(s) \partial_t] \psi(x) \quad [\Leftrightarrow] \quad \begin{cases} \sigma(s) \cdot \nabla \psi(x) = s\zeta \partial_t \psi(x) \\ s^2 \nabla \psi(x) = is\sigma(s) \times \nabla + \sigma(s)[\sigma(s) \cdot \nabla] \psi(x) \end{cases}$$

$[\Downarrow]$

$[\Downarrow]$

$$s\nabla \psi(x) = \{[\sigma(s) \cdot \nabla, \sigma(s)] + \zeta \sigma(s) \partial_t\} \psi(x) [\Leftrightarrow] \begin{cases} \sigma(s) \cdot \nabla \psi(x) = s\zeta \partial_t \psi(x) \\ s^2 \nabla \psi(x) = \{s[\sigma(s) \cdot \nabla] \sigma(s) - (s-1)\sigma(s)[\sigma(s) \cdot \nabla]\} \psi(x) \end{cases}$$

[⇔]

[⇔]

定理3.1.2.

$$[s(\vec{p}, -\partial_t)_a + iS_{ab}(s, \varsigma)(\vec{p}, -\partial_t)^b]\psi(\vec{p}, t) = 0 \quad [\Leftrightarrow] \quad \frac{1}{s}\sigma(s) \cdot \vec{p}\psi(\vec{p}, t) = -i\varsigma\partial_t\psi, O(s) \cdot \vec{p}\psi(\vec{p}, t) = 0$$

[⇔]

[⇔]

$$[s\vec{p} - i\sigma(s) \times \vec{p}]\psi(\vec{p}, t) = -\sigma(s)i\varsigma\partial_t\psi(\vec{p}, t) \quad [\Leftrightarrow] \quad \begin{cases} \frac{1}{s}\sigma(s) \cdot \vec{p}\psi(\vec{p}, t) = -i\varsigma\partial_t\psi(\vec{p}, t) \\ \{s^2\vec{p} - is\sigma(s) \times \vec{p} - \sigma(s)[\sigma(s) \cdot \vec{p}]\}\psi(\vec{p}, t) = 0 \end{cases}$$

[⇔]

[⇔]

$$\{s\vec{p} - [\sigma(s) \cdot \vec{p}, \sigma(s)]\}\psi(\vec{p}, t) = -i\varsigma\sigma(s)\partial_t\psi(\vec{p}, t) \quad [\Leftrightarrow] \quad \begin{cases} \frac{1}{s}\sigma(s) \cdot \vec{p}\psi(\vec{p}, t) = -i\varsigma\partial_t\psi(\vec{p}, t) \\ \{s^2\vec{p} + (s-1)\sigma(s)[\sigma(s) \cdot \vec{p}] - s[\sigma(s) \cdot \vec{p}]\sigma(s)\}\psi(\vec{p}, t) = 0 \end{cases}$$

$$\text{推论3.1.1.} \quad \begin{cases} \sigma(s) \cdot \nabla\psi(x) = s\varsigma\partial_t\psi(x) \\ \nabla\psi(x) = [\sigma(s) \cdot \nabla]\sigma(s)\psi(x) \end{cases} \quad [\Leftrightarrow] \quad \begin{cases} [\sigma(s) \cdot \vec{p}]\psi(\vec{p}, t) = -i\varsigma\partial_t\psi(\vec{p}, t) \\ \vec{p}\psi(\vec{p}, t) = [\sigma(s) \cdot \vec{p}]\sigma(s)\psi(\vec{p}, t) \end{cases} ; s = 1$$

### 3.2 动量空间中s-自旋方程的平面波解

$$\text{推论3.2.1.} \quad \psi(\vec{p}, t) = |\vec{p}|^{(s-\frac{1}{2})}[a_1(\vec{p}, -s\varsigma)\lambda(\hat{p}, -s\varsigma)e^{-i|\vec{p}t} + a_2^+(\vec{p}, -s\varsigma)\lambda(-\hat{p}, -s\varsigma)e^{i|\vec{p}t}]$$

### 3.3 动量空间中平面波解的性质

#### 3.3.1 重要性质一

$$\text{推论3.3.1.} \quad \begin{cases} [\frac{1}{s}\sigma(s) \cdot \hat{p}]\sigma(s)\psi(\vec{p}, t) = \{\hat{p} + (1 - \frac{1}{s})\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{p}]\}\psi(\vec{p}, t) \\ [\Rightarrow][\frac{1}{s}\sigma(s) \cdot \hat{p}]^2\sigma(s)\psi(\vec{p}, t) = \{s[1 - (1 - \frac{1}{s})^2][\frac{1}{s}\sigma(s) \cdot \hat{p}]\hat{p} + (1 - \frac{1}{s})^2\sigma(s)\}\psi(\vec{p}, t) \end{cases}$$

$$\text{推论3.3.2.} \quad \begin{cases} [\frac{1}{s}\sigma(s) \cdot \hat{p}]\sigma(s)\psi(\vec{p}, t) = \{\hat{p} + (1 - \frac{1}{s})\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{p}]\}\psi(\vec{p}, t) \\ [\Leftrightarrow] \begin{cases} [\frac{1}{s}\sigma(s) \cdot \hat{p}]^n\sigma(s)\psi(\vec{p}, t) = \{s[1 - (1 - \frac{1}{s})^n][\frac{1}{s}\sigma(s) \cdot \hat{p}]^{n-1}\hat{p} + (1 - \frac{1}{s})^n\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{p}]^n\}\psi(\vec{p}, t) \\ [\frac{1}{s}\sigma(s) \cdot \hat{p}]^{2n}\psi(\vec{p}, t) = \psi(\vec{p}, t) \end{cases} \\ [\Leftrightarrow] \begin{cases} [\frac{1}{s}\sigma(s) \cdot \hat{p}]^{2k+1}\sigma(s)\psi(\vec{p}, t) = \{s[1 - (1 - \frac{1}{s})^{2k+1}]\hat{p} + (1 - \frac{1}{s})^{2k+1}\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{p}]\}\psi(\vec{p}, t) \\ [\frac{1}{s}\sigma(s) \cdot \hat{p}]^{2k}\sigma(s)\psi(\vec{p}, t) = \{s[1 - (1 - \frac{1}{s})^{2k}][\frac{1}{s}\sigma(s) \cdot \hat{p}]\hat{p} + (1 - \frac{1}{s})^{2k}\sigma(s)\}\psi(\vec{p}, t) \\ [\frac{1}{s}\sigma(s) \cdot \hat{p}]^{2k}\psi(\vec{p}, t) = \psi(\vec{p}, t) \end{cases} \end{cases}$$

推论3.3.3.

$$\begin{cases} [\sigma(1) \cdot \hat{p}]^{2k+1}\sigma(1)\psi(\vec{p}, t) = [\sigma(1) \cdot \hat{p}]\sigma(1)\psi(\vec{p}, t) = \hat{p}\psi(\vec{p}, t) \\ [\sigma(1) \cdot \hat{p}]^{2k+2}\sigma(1)\psi(\vec{p}, t) = [\sigma(1) \cdot \hat{p}]^2\sigma(1)\psi(\vec{p}, t) = [\sigma(1) \cdot \hat{p}]\hat{p}\psi(\vec{p}, t) \end{cases}$$

#### 3.3.2 重要性质二

定理3.3.1.

$$\begin{cases} \psi^+(\vec{p}, t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{p}]\psi(\vec{p}, t) = \psi^+(\vec{p}, t)[\frac{1}{s}\sigma(s) \cdot \hat{p}]\frac{1}{s}\sigma(s)\psi(\vec{p}, t) = \psi^+(\vec{p}, t)\hat{p}\psi(\vec{p}, t), s \geq 1 \\ \psi^+(\vec{p}, t)\{\sigma(\frac{1}{2})[\sigma(\frac{1}{2}) \cdot \hat{p}] + [\sigma(\frac{1}{2}) \cdot \hat{p}]\sigma(\frac{1}{2})\}\psi(\vec{p}, t) = \frac{1}{2}\psi^+(\vec{p}, t)\hat{p}\psi(\vec{p}, t) \end{cases}$$

证明:  $\psi^+(\vec{p}, t)\{s^2\hat{p} + (s-1)\sigma(s)[\sigma(s) \cdot \hat{p}] - s[\sigma(s) \cdot \hat{p}]\sigma(s)\}\psi(\vec{p}, t) = 0$ 

$$\Leftrightarrow \begin{cases} \psi^+(\vec{p}, t)\{s^2\hat{p} + (s-1)\sigma(s)[\sigma(s) \cdot \hat{p}] - s[\sigma(s) \cdot \hat{p}]\sigma(s)\}\psi(\vec{p}, t) = 0 \\ \psi^+(\vec{p}, t)\{s^2\hat{p} + (s-1)[\sigma(s) \cdot \hat{p}]\sigma(s) - s\sigma(s)[\sigma(s) \cdot \hat{p}]\}\psi(\vec{p}, t) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \psi^+(\vec{p}, t)\frac{1}{s}\sigma(s)[\frac{1}{s}\sigma(s) \cdot \hat{p}]\psi(\vec{p}, t) = \psi^+(\vec{p}, t)[\frac{1}{s}\sigma(s) \cdot \hat{p}]\frac{1}{s}\sigma(s)\psi(\vec{p}, t) = \psi^+(\vec{p}, t)\hat{p}\psi(\vec{p}, t), s \geq 1 \\ \psi^+(\vec{p}, t)\{\sigma(\frac{1}{2})[\sigma(\frac{1}{2}) \cdot \hat{p}] + [\sigma(\frac{1}{2}) \cdot \hat{p}]\sigma(\frac{1}{2})\}\psi(\vec{p}, t) = \frac{1}{2}\psi^+(\vec{p}, t)\hat{p}\psi(\vec{p}, t) \end{cases} \quad \square$$

$$\text{定理3.3.2.} \quad [\frac{1}{s}\sigma(s) \cdot \vec{p}]^2\psi(\vec{p}, t) = \vec{p}^2\psi(\vec{p}, t) = -\partial_t^2\psi(\vec{p}, t)$$



**推论3.3.4.**

$$\begin{cases} \int \psi^+(\vec{r}, t) \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}] \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}] \frac{1}{s} \sigma(s) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \hat{\nabla} \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \\ \int \psi^+(\vec{r}, t) \{ \sigma(\frac{1}{2}) [\sigma(\frac{1}{2}) \cdot \hat{\nabla}] + [\sigma(\frac{1}{2}) \cdot \hat{\nabla}] \sigma(\frac{1}{2}) \} \psi(\vec{r}, t) d^3\vec{r} = \int \frac{1}{2} \psi^+(\vec{r}, t) \hat{\nabla} \psi(\vec{r}, t) d^3\vec{r} \end{cases}$$

**3.3.3 重要性质三**

$$\text{引理3.3.1. } \psi^+(\vec{p}, t) \sigma(s) \psi(\vec{p}, t) = (-s\zeta) |\vec{p}|^{(2s-1)} [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) - a_2(-\vec{p}, -s\zeta) a_2^+(-\vec{p}, -s\zeta)] \hat{p}, s \geq 1$$

证明:  $\psi^+(\vec{p}, t) \sigma(s) \psi(\vec{p}, t), s \geq 1$

$$\begin{aligned} &= |\vec{p}|^{(s-\frac{1}{2})} [a_1^+(\vec{p}, -s\zeta) \lambda^+(\hat{p}, -s\zeta) e^{i|\vec{p}|t} + a_2(-\vec{p}, -s\zeta) \lambda^+(-\hat{p}, -s\zeta) e^{-i|\vec{p}|t}] \sigma(s) |\vec{p}|^{(s-\frac{1}{2})} [a_1(\vec{p}, -s\zeta) \lambda(\hat{p}, -s\zeta) e^{-i|\vec{p}|t} + a_2^+(-\vec{p}, -s\zeta) \lambda(-\hat{p}, -s\zeta) e^{i|\vec{p}|t}] \\ &= |\vec{p}|^{(2s-1)} [a_1^+(\vec{p}, -s\zeta) \lambda^+(\hat{p}, -s\zeta) e^{i|\vec{p}|t} + a_2(-\vec{p}, -s\zeta) \lambda^+(-\hat{p}, -s\zeta) e^{-i|\vec{p}|t}] \sigma(s) [a_1(\vec{p}, -s\zeta) \lambda(\hat{p}, -s\zeta) e^{-i|\vec{p}|t} + a_2^+(-\vec{p}, -s\zeta) \lambda(-\hat{p}, -s\zeta) e^{i|\vec{p}|t}] \\ &= (-s\zeta) |\vec{p}|^{(2s-1)} [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) - a_2(-\vec{p}, -s\zeta) a_2^+(-\vec{p}, -s\zeta)] \hat{p}, s \geq 1 \quad \square \end{aligned}$$

$$\text{引理3.3.2. } \psi^+(\vec{p}, t) \sigma(s) [\sigma(s) \cdot \hat{p}] \psi(\vec{p}, t) = s^2 |\vec{p}|^{(2s-1)} [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + a_2(-\vec{p}, -s\zeta) a_2^+(-\vec{p}, -s\zeta)] \hat{p}, s \geq 1$$

证明:  $\psi^+(\vec{p}, t) \sigma(s) [\sigma(s) \cdot \hat{p}] \psi(\vec{p}, t), s \geq 1$

$$\begin{aligned} &= |\vec{p}|^{(s-\frac{1}{2})} [a_1^+(\vec{p}, -s\zeta) \lambda^+(\hat{p}, -s\zeta) e^{i|\vec{p}|t} + a_2(-\vec{p}, -s\zeta) \lambda^+(-\hat{p}, -s\zeta) e^{-i|\vec{p}|t}] \sigma(s) [\sigma(s) \cdot \hat{p}] |\vec{p}|^{(s-\frac{1}{2})} [a_1(\vec{p}, -s\zeta) \lambda(\hat{p}, -s\zeta) e^{-i|\vec{p}|t} + a_2^+(-\vec{p}, -s\zeta) \lambda(-\hat{p}, -s\zeta) e^{i|\vec{p}|t}] \\ &= |\vec{p}|^{(2s-1)} [a_1^+(\vec{p}, -s\zeta) \lambda^+(\hat{p}, -s\zeta) e^{i|\vec{p}|t} + a_2(-\vec{p}, -s\zeta) \lambda^+(-\hat{p}, -s\zeta) e^{-i|\vec{p}|t}] \sigma(s) [-s\zeta a_1(\vec{p}, -s\zeta) \lambda(\hat{p}, -s\zeta) e^{-i|\vec{p}|t} + s\zeta a_2^+(-\vec{p}, -s\zeta) \lambda(-\hat{p}, -s\zeta) e^{i|\vec{p}|t}] \\ &= s^2 |\vec{p}|^{(2s-1)} [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + a_2(-\vec{p}, -s\zeta) a_2^+(-\vec{p}, -s\zeta)] \hat{p}, s \geq 1 \quad \square \end{aligned}$$

**定理3.3.3.**

$$\begin{cases} \psi^+(\vec{p}, t) \sigma(s) \psi(\vec{p}, t) = \psi^+(\vec{p}, t) [\sigma(s) \cdot \hat{p}] \psi(\vec{p}, t), \psi^+(\vec{p}, t) \sigma(s) \times \hat{p} \psi(\vec{p}, t) = 0, s \geq 1 \\ \psi^+(\vec{p}, t) \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{p}] \psi(\vec{p}, t) = \psi^+(\vec{p}, t) [\frac{1}{s} \sigma(s) \cdot \hat{p}] \frac{1}{s} \sigma(s) \psi(\vec{p}, t) = \psi^+(\vec{p}, t) \hat{p} \psi(\vec{p}, t), s \geq 1 \\ \psi^+(\vec{p}, t) \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{p}]^k \psi(\vec{p}, t) = \psi^+(\vec{p}, t) [\frac{1}{s} \sigma(s) \cdot \hat{p}]^k \frac{1}{s} \sigma(s) \psi(\vec{p}, t), s \geq 1 \end{cases}$$

**推论3.3.5.**

$$\begin{cases} \int \psi^+(\vec{r}, t) \sigma(s) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \hat{\nabla}] \hat{\nabla} \psi(\vec{r}, t) d^3\vec{r}, \int \psi^+(\vec{r}, t) [\sigma(s) \times \hat{\nabla}] \psi(\vec{r}, t) d^3\vec{r} = 0, s \geq 1 \\ \int \psi^+(\vec{r}, t) \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}] \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}] \frac{1}{s} \sigma(s) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \hat{\nabla} \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \\ \int \psi^+(\vec{r}, t) \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^k \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^k \frac{1}{s} \sigma(s) \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \\ \int \psi^+(\vec{r}, t) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^i \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^j \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^j \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^i \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \end{cases}$$

**推论3.3.6.**

$$\begin{cases} \psi^+(\vec{p}, t) [\frac{1}{s} \sigma(s) \cdot \hat{p}] \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{p}] \psi(\vec{p}, t) = \psi^+(\vec{p}, t) \frac{1}{s} \sigma(s) \psi(\vec{p}, t), s \geq 1 \\ \int \psi^+(\vec{r}, t) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}] \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}] \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \frac{1}{s} \sigma(s) \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \\ \int \psi^+(\vec{r}, t) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^i \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^j \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^{i+j} \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \end{cases}$$

**4 自旋和角动量算符的各种性质(满足自旋方程)****4.1 自旋波函数的一般性质**

$$\text{定义4.1.1. } \Gamma(n; m, l) := (\sqrt{-\nabla^2})^n \partial_\pi^m \overbrace{\partial_i \partial_j \cdots}^l, n \in Z, m, l \in N$$

$$\text{推论4.1.1. } \int \psi^+(\vec{r}, t) \Gamma(n; m, l) \dot{\psi}(\vec{r}, t) d^3\vec{r} = - \int \dot{\psi}^+(\vec{r}, t) \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r}$$

$$\text{推论4.1.2. } \begin{cases} \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \hat{\nabla}] \hat{\nabla} \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \sigma(s) \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \\ \int \psi^+(\vec{r}, t) \sigma(s) \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} = s\zeta \int \psi^+(\vec{r}, t) \frac{\nabla^2}{\zeta^2} \Gamma(n; m, l) \dot{\psi}(\vec{r}, t) d^3\vec{r}, s \geq 1 \end{cases}$$

$$\text{推论4.1.3. } \begin{cases} \int \psi^+(\vec{r}, t) \frac{1}{s} \sigma(s) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}] \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \hat{\nabla} \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \\ \int \psi^+(\vec{r}, t) \sigma(s) \Gamma(n; m, l) \dot{\psi}(\vec{r}, t) d^3\vec{r} = s\zeta \int \psi^+(\vec{r}, t) \nabla \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \end{cases}$$

$$\text{性质4.1.1.} \quad \begin{cases} \int \psi^+(\vec{r}, t) [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i] \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} = 0, s \geq 1 \\ \int \psi^+(\vec{r}, t) [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i] \dot{\psi}(\vec{r}, t) d^3\vec{r} = 0, s \geq 1 \end{cases}$$

推论4.1.4.

$$\begin{cases} \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \hat{\nabla}]^j \sigma(s) [\sigma(s) \cdot \hat{\nabla}]^k \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \hat{\nabla}]^k \sigma(s) [\sigma(s) \cdot \hat{\nabla}]^j \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \\ \int \psi^+ [\sigma(s) \cdot \hat{\nabla}]^j \sigma(s) [\sigma(s) \cdot \hat{\nabla}]^k \Gamma(n; m, l) \psi d^3\vec{r} = s_c \int \psi^+ [\sigma(s) \cdot \hat{\nabla}]^{j+k} \frac{\nabla}{\nabla^2} \Gamma(n; m, l) \dot{\psi} d^3\vec{r}, s \geq 1 \\ \int \psi^+ [\sigma(s) \cdot \hat{\nabla}]^j \sigma(s) [\sigma(s) \cdot \hat{\nabla}]^k \Gamma(n; m, l) \dot{\psi} d^3\vec{r} = s_c \int \psi^+ [\sigma(s) \cdot \hat{\nabla}]^{j+k} \nabla \Gamma(n; m, l) \psi d^3\vec{r}, s \geq 1 \end{cases}$$

## 4.2 角动量算符的性质一

$$\text{引理4.2.1.} \quad \nabla^2 (r_i \partial_j - r_j \partial_i) = (r_i \partial_j - r_j \partial_i) \nabla^2$$

$$\text{引理4.2.2.} \quad [\sigma(s) \cdot \nabla] (r_i \partial_j - r_j \partial_i) = (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla] + [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i]$$

$$\begin{aligned} \text{引理4.2.3.} \quad & [\sigma(s) \cdot \nabla]^2 (r_i \partial_j - r_j \partial_i) \\ &= (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla]^2 + [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i] [\sigma(s) \cdot \nabla] + [\sigma(s) \cdot \nabla] [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i] \end{aligned}$$

$$\begin{aligned} \text{证明:} \quad & [\sigma(s) \cdot \nabla]^2 (r_i \partial_j - r_j \partial_i) \\ &= [\sigma(s) \cdot \nabla] \{ (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla] + [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i] \} \\ &= (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla]^2 + [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i] [\sigma(s) \cdot \nabla] + [\sigma(s) \cdot \nabla] [\sigma_i(s) \partial_j - \sigma_j(s) \partial_i] \end{aligned} \quad \square$$

性质4.2.1.

$$\begin{cases} s \geq 1, n \in Z, l, m \in N, \\ \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \nabla] (r_i \partial_j - r_j \partial_i) \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla] \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} \\ \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \nabla] (r_i \partial_j - r_j \partial_i) \dot{\psi}(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla] \dot{\psi}(\vec{r}, t) d^3\vec{r} \end{cases}$$

性质4.2.2.

$$\begin{cases} s \geq 1, n \in Z, l, m \in N, \\ \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \nabla]^2 (r_i \partial_j - r_j \partial_i) \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla]^2 \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} \\ \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \nabla]^2 (r_i \partial_j - r_j \partial_i) \dot{\psi}(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla]^2 \dot{\psi}(\vec{r}, t) d^3\vec{r} \end{cases}$$

推论4.2.1.

$$\begin{cases} s \geq 1, n \in Z, l, m \in N, \\ \int \psi^+(\vec{r}, t) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^2 (r_i \partial_j - r_j \partial_i) \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \Gamma(n; m, l) \psi(\vec{r}, t) d^3\vec{r} \\ \int \psi^+(\vec{r}, t) [\frac{1}{s} \sigma(s) \cdot \hat{\nabla}]^2 (r_i \partial_j - r_j \partial_i) \dot{\psi}(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \dot{\psi}(\vec{r}, t) d^3\vec{r} \end{cases}$$

## 4.3 角动量算符的性质二

$$\text{推论4.3.1.} \quad \begin{cases} \int \psi^+(\vec{r}, t) (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3\vec{r} = i \int \psi^+(\vec{p}, t) (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) \{ [\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) \} d^3\vec{p}, s \geq \frac{1}{2} \\ \int \psi^+(\vec{r}, t) (r_i \partial_j - r_j \partial_i) [\sigma(s) \cdot \nabla] \dot{\psi}(\vec{r}, t) d^3\vec{r} = i \int \psi^+(\vec{p}, t) [\sigma(s) \cdot \vec{p}] (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) \dot{\psi}(\vec{p}, t) d^3\vec{p}, s \geq 1 \end{cases}$$

$$\text{推论4.3.2.} \quad \begin{cases} \int \psi^+(\vec{r}, t) (r_i \hat{\partial}_j - r_j \hat{\partial}_i) [\sigma(s) \cdot \hat{\nabla}] \psi(\vec{r}, t) d^3\vec{r} = i \int \psi^+(\vec{p}, t) (\hat{p}_i \tilde{\partial}_j - \hat{p}_j \tilde{\partial}_i) \{ [\sigma(s) \cdot \hat{p}] \psi(\vec{p}, t) \} d^3\vec{p}, s \geq \frac{1}{2} \\ \int \psi^+(\vec{r}, t) (r_i \hat{\partial}_j - r_j \hat{\partial}_i) [\sigma(s) \cdot \hat{\nabla}] \dot{\psi}(\vec{r}, t) d^3\vec{r} = i \int \psi^+(\vec{p}, t) [\sigma(s) \cdot \hat{p}] (\hat{p}_i \tilde{\partial}_j - \hat{p}_j \tilde{\partial}_i) \dot{\psi}(\vec{p}, t) d^3\vec{p}, s \geq 1 \end{cases}$$

$$\text{推论4.3.3.} \quad \begin{cases} \int \psi^+(\vec{r}, t) [r_i \sigma_j(s) - r_j \sigma_i(s)] \psi(\vec{r}, t) d^3\vec{r} = -i \int \psi^+(\vec{p}, t) [\sigma_i(s) \tilde{\partial}_j - \sigma_j(s) \tilde{\partial}_i] \psi(\vec{p}, t) d^3\vec{p} \\ \int \psi^+(\vec{r}, t) (r_i \partial_j - r_j \partial_i) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{p}, t) (p_i \tilde{\partial}_j - p_j \tilde{\partial}_i) \psi(\vec{p}, t) d^3\vec{p} \end{cases}$$

$$\begin{aligned} \text{推论4.3.4.} \quad & -i \int \psi^+(\vec{r}, t) [r_i \partial_j - r_j \partial_i - \sigma_{ij}^k \sigma_k(s)] \psi(\vec{r}, t) d^3\vec{r} \\ &= -i \int \psi^+(\vec{r}, t) [r_i \partial_j - r_j \partial_i + i \varepsilon_{ij}^k \sigma_k(s)] \psi(\vec{r}, t) d^3\vec{r} \\ &= -i \int \psi^+(\vec{r}, t) \{ r_i \partial_j - r_j \partial_i + [\sigma_i(s), \sigma_j(s)] \} \psi(\vec{r}, t) d^3\vec{r} \\ &= -i \int \psi^+(\vec{p}, t) \{ p_i \tilde{\partial}_j - p_j \tilde{\partial}_i + [\sigma_i(s), \sigma_j(s)] \} \psi(\vec{p}, t) d^3\vec{p} \end{aligned}$$

## 4.4 角动量算符的性质???

推论4.4.1.  $\int \psi^+(\vec{r}, t) r_i \sigma_j(s) [\sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3\vec{r} = - \int \psi^+(\vec{p}, t) \{ \sigma_j(s) \sigma_i(s) + \sigma_j(s) [\sigma(s) \cdot \vec{p}] \tilde{\partial}_i \} \psi(\vec{p}, t) d^3\vec{p}$

证明:  $\int \psi^+(\vec{r}, t) r_i \sigma_j(s) [\sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3\vec{r}$   
 $= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' d^3\vec{r} \psi^+(\vec{p}', t) e^{-i\vec{p}' \cdot \vec{r}} r_i \sigma_j(s) [\sigma(s) \cdot \nabla] [\psi(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}}]$   
 $= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' d^3\vec{r} \psi^+(\vec{p}', t) e^{-i\vec{p}' \cdot \vec{r}} \sigma_j(s) [\sigma(s) \cdot i\vec{p}] \psi(\vec{p}, t) (-i\tilde{\partial}_i) e^{i\vec{p} \cdot \vec{r}}$   
 $= - \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' d^3\vec{r} \psi^+(\vec{p}', t) \tilde{\partial}_i \{ \sigma_j(s) [\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) \} e^{i(\vec{p} - \vec{p}') \cdot \vec{r}}$   
 $= - \int d^3\vec{p} d^3\vec{p}' \psi^+(\vec{p}', t) \tilde{\partial}_i \{ \sigma_j(s) [\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) \} \delta^3(\vec{p} - \vec{p}')$   
 $= - \int \psi^+(\vec{p}, t) \sigma_j(s) \tilde{\partial}_i \{ \sigma(s) \cdot \vec{p} \} \psi(\vec{p}, t) d^3\vec{p}$   
 $= - \int \psi^+(\vec{p}, t) \{ \sigma_j(s) \sigma_i(s) + \sigma_j(s) [\sigma(s) \cdot \vec{p}] \tilde{\partial}_i \} \psi(\vec{p}, t) d^3\vec{p}$  □

推论4.4.2.  $\int \psi^+(\vec{r}, t) [\sigma(s) \cdot \nabla] \{ r_i \sigma_j(s) \psi(\vec{r}, t) \} d^3\vec{r} = - \int d^3\vec{p} \psi^+(\vec{p}, t) [\sigma(s) \cdot \vec{p}] \sigma_j(s) \tilde{\partial}_i \psi(\vec{p}, t)$

证明:  $\int \psi^+(\vec{r}, t) [\sigma(s) \cdot \nabla] \{ r_i \sigma_j(s) \psi(\vec{r}, t) \} d^3\vec{r}$   
 $= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' d^3\vec{r} \psi^+(\vec{p}', t) e^{-i\vec{p}' \cdot \vec{r}} [\sigma(s) \cdot \nabla] [r_i \sigma_j(s) \psi(\vec{p}, t) e^{i\vec{p} \cdot \vec{r}}]$   
 $= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' d^3\vec{r} \psi^+(\vec{p}', t) e^{-i\vec{p}' \cdot \vec{r}} [\sigma(s) \cdot \nabla] [r_i e^{i\vec{p} \cdot \vec{r}}] \sigma_j(s) \psi(\vec{p}, t)$   
 $= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' d^3\vec{r} \psi^+(\vec{p}', t) e^{-i\vec{p}' \cdot \vec{r}} \{ [\sigma(s) \cdot \nabla] r_i [e^{i\vec{p} \cdot \vec{r}}] + r_i [\sigma(s) \cdot \nabla] e^{i\vec{p} \cdot \vec{r}} \} \sigma_j(s) \psi(\vec{p}, t)$   
 $= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' d^3\vec{r} \psi^+(\vec{p}', t) e^{-i\vec{p}' \cdot \vec{r}} \{ [\sigma_i(s) \sigma_j(s) e^{i\vec{p} \cdot \vec{r}}] + [\sigma(s) \cdot \vec{p}] [\sigma_j(s) \tilde{\partial}_i e^{i\vec{p} \cdot \vec{r}}] \} \psi(\vec{p}, t)$   
 $= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' d^3\vec{r} \psi^+(\vec{p}', t) \{ \sigma_i(s) \sigma_j(s) \psi(\vec{p}, t) - \tilde{\partial}_i [\sigma(s) \cdot \vec{p} \sigma_j(s) \psi(\vec{p}, t)] \} e^{i(\vec{p} - \vec{p}') \cdot \vec{r}}$   
 $= \int d^3\vec{p} \psi^+(\vec{p}, t) \{ [\sigma_i(s) \sigma_j(s)] \psi(\vec{p}, t) - \tilde{\partial}_i [\sigma(s) \cdot \vec{p} \sigma_j(s) \psi(\vec{p}, t)] \}$   
 $= - \int \psi^+(\vec{p}, t) [\sigma(s) \cdot \vec{p}] \sigma_j(s) \tilde{\partial}_i \psi(\vec{p}, t) d^3\vec{p}$  □

推论4.4.3.

$$\begin{cases} \int \psi^+(\vec{r}, t) [r_i \sigma_j(s) - r_j \sigma_i(s)] [\sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{p}, t) [\sigma_i(s) \tilde{\partial}_j - \sigma_j(s) \tilde{\partial}_i] \{ [\sigma(s) \cdot \vec{p}] \psi(\vec{p}, t) \} d^3\vec{p} \\ \int \psi^+(\vec{r}, t) [\sigma(s) \cdot \nabla] \{ [r_i \sigma_j(s) - r_j \sigma_i(s)] \psi(\vec{r}, t) \} d^3\vec{r} = \int \psi^+(\vec{p}, t) [\sigma(s) \cdot \vec{p}] [\sigma_i(s) \tilde{\partial}_j - \sigma_j(s) \tilde{\partial}_i] \psi(\vec{p}, t) d^3\vec{p} \end{cases}$$

引理4.4.1.  $\begin{cases} \Psi(\vec{p}, t) = |\vec{p}|^{\frac{1}{2}} [a_1(\vec{p}, -\varsigma) \lambda_m(\hat{p}, -\varsigma) e^{-i|\vec{p}|t} + a_2^+(\vec{p}, -\varsigma) \lambda_m(-\hat{p}, -\varsigma) e^{i|\vec{p}|t}] \\ (\gamma \cdot \vec{p}) \Psi(\vec{p}, t) = -\varsigma |\vec{p}|^{\frac{1}{2}} [a_1(\vec{p}, -\varsigma) \lambda_m(\hat{p}, -\varsigma) e^{-i|\vec{p}|t} - a_2^+(\vec{p}, -\varsigma) \lambda_m(-\hat{p}, -\varsigma) e^{i|\vec{p}|t}] \end{cases}$

推论4.4.4.  $\lambda_m^+(\hat{p}, -\varsigma) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma) = 0, \lambda_m^+(\hat{p}, -\varsigma) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(-\hat{p}, -\varsigma) = 0$

定理4.4.1.  $\int d^3\vec{r} \{ \Psi^+(\vec{r}, t) (r_i \gamma_j - r_j \gamma_i) (\gamma \cdot \nabla) \Psi(\vec{r}, t) - \Psi^+(\vec{r}, t) (\gamma \cdot \nabla) [(r_i \gamma_j - r_j \gamma_i) \Psi(\vec{r}, t)] \} = 0$

证明:  $\int d^3\vec{r} \{ \Psi^+(\vec{r}, t) (r_i \gamma_j - r_j \gamma_i) (\gamma \cdot \nabla) \Psi(\vec{r}, t) - \Psi^+(\vec{r}, t) (\gamma \cdot \nabla) [(r_i \gamma_j - r_j \gamma_i) \Psi(\vec{r}, t)] \}$   
 $= \int d^3\vec{r} \{ \Psi^+(\vec{p}, t) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) [(\gamma \cdot \vec{p}) \Psi(\vec{p}, t)] - \Psi^+(\vec{p}, t) (\gamma \cdot \vec{p}) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \Psi(\vec{p}, t) \}$   
 $= \int d^3\vec{r} \{ \Psi^+(\vec{p}, t) (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) [(\gamma \cdot \vec{p}) \Psi(\vec{p}, t)] - [(\gamma \cdot \vec{p}) \Psi(\vec{p}, t)]^+ (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \Psi(\vec{p}, t) \}$   
 $= -\varsigma \int d^3\vec{r} \{ |\vec{p}|^{\frac{1}{2}} [a_1^+(\vec{p}, -\varsigma) \lambda_m^+(\hat{p}, -\varsigma) e^{i|\vec{p}|t} + a_2(-\vec{p}, -\varsigma) \lambda_m^+(-\hat{p}, -\varsigma) e^{-i|\vec{p}|t}]$   
 $(\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \{ |\vec{p}|^{\frac{1}{2}} [a_1(\vec{p}, -\varsigma) \lambda_m(\hat{p}, -\varsigma) e^{-i|\vec{p}|t} - a_2^+(-\vec{p}, -\varsigma) \lambda_m(-\hat{p}, -\varsigma) e^{i|\vec{p}|t}] \}$   
 $- \{ |\vec{p}|^{\frac{1}{2}} [a_1^+(\vec{p}, -\varsigma) \lambda_m^+(\hat{p}, -\varsigma) e^{i|\vec{p}|t} - a_2(-\vec{p}, -\varsigma) \lambda_m^+(-\hat{p}, -\varsigma) e^{-i|\vec{p}|t}] \}$   
 $(\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \{ |\vec{p}|^{\frac{1}{2}} [a_1(\vec{p}, -\varsigma) \lambda_m(\hat{p}, -\varsigma) e^{-i|\vec{p}|t} + a_2^+(-\vec{p}, -\varsigma) \lambda_m(-\hat{p}, -\varsigma) e^{i|\vec{p}|t}] \} \}$   
 $= 2\varsigma \int d^3\vec{r} |\vec{p}|^{\frac{1}{2}} [a_1^+(\vec{p}, -\varsigma) \lambda_m^+(\hat{p}, -\varsigma) e^{i|\vec{p}|t}] (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \{ |\vec{p}|^{\frac{1}{2}} [a_2^+(-\vec{p}, -\varsigma) \lambda_m(-\hat{p}, -\varsigma) e^{i|\vec{p}|t}] \}$   
 $- 2\varsigma \int d^3\vec{r} |\vec{p}|^{\frac{1}{2}} [a_2(-\vec{p}, -\varsigma) \lambda_m^+(-\hat{p}, -\varsigma) e^{-i|\vec{p}|t}] (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \{ |\vec{p}|^{\frac{1}{2}} [a_1(\vec{p}, -\varsigma) \lambda_m(\hat{p}, -\varsigma) e^{-i|\vec{p}|t}] \}$   
 $= 2\varsigma \int d^3\vec{r} |\vec{p}|^{\frac{1}{2}} [a_1^+(\vec{p}, -\varsigma) \lambda_m^+(\hat{p}, -\varsigma) e^{i|\vec{p}|t}] (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \{ |\vec{p}|^{\frac{1}{2}} [a_2^+(-\vec{p}, -\varsigma) \lambda_m(-\hat{p}, -\varsigma) e^{i|\vec{p}|t}] \}$   
 $- 2\varsigma \int d^3\vec{r} |\vec{p}|^{\frac{1}{2}} [a_2(-\vec{p}, -\varsigma) \lambda_m^+(-\hat{p}, -\varsigma) e^{-i|\vec{p}|t}] (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \{ |\vec{p}|^{\frac{1}{2}} [a_1(\vec{p}, -\varsigma) \lambda_m(\hat{p}, -\varsigma) e^{-i|\vec{p}|t}] \}$   
 $= 2\varsigma \int d^3\vec{r} |\vec{p}|^{\frac{1}{2}} [a_1^+(\vec{p}, -\varsigma) \lambda_m^+(\hat{p}, -\varsigma) e^{i|\vec{p}|t}] (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(-\hat{p}, -\varsigma) \{ |\vec{p}|^{\frac{1}{2}} [a_2^+(-\vec{p}, -\varsigma) e^{i|\vec{p}|t}] \}$   
 $- 2\varsigma \int d^3\vec{r} |\vec{p}|^{\frac{1}{2}} [a_2(-\vec{p}, -\varsigma) \lambda_m^+(-\hat{p}, -\varsigma) e^{-i|\vec{p}|t}] (\gamma_i \tilde{\partial}_j - \gamma_j \tilde{\partial}_i) \lambda_m(\hat{p}, -\varsigma) \{ |\vec{p}|^{\frac{1}{2}} [a_1(\vec{p}, -\varsigma) e^{-i|\vec{p}|t}] \}$   
 $+ 2\varsigma \int d^3\vec{r} |\vec{p}|^{\frac{1}{2}} [a_1^+(\vec{p}, -\varsigma) \lambda_m^+(\hat{p}, -\varsigma) e^{i|\vec{p}|t}] [\gamma_i \lambda_m(-\hat{p}, -\varsigma) \tilde{\partial}_j - \gamma_j \lambda_m(-\hat{p}, -\varsigma) \tilde{\partial}_i] \{ |\vec{p}|^{\frac{1}{2}} [a_2^+(-\vec{p}, -\varsigma) e^{i|\vec{p}|t}] \}$

$$-2\zeta \int d^3\vec{r} |\vec{p}|^{\frac{1}{2}} [a_2(-\vec{p}, -\zeta) \lambda_m^+(-\hat{p}, -\zeta) e^{-i|\vec{p}|t}] [\gamma_i \lambda_m(\hat{p}, -\zeta) \tilde{\partial}_j - \gamma_j \lambda_m(\hat{p}, -\zeta) \tilde{\partial}_i] \{ |\vec{p}|^{\frac{1}{2}} [a_1(\vec{p}, -\zeta) e^{-i|\vec{p}|t}] \} \\ = 0 - 0 + 0 - 0 = 0 \quad \square$$

$$\text{推论4.4.5. } \int d^3\vec{r} \{ \psi^+(\vec{r}, t) [r_i \sigma_j(1) - r_j \sigma_i(1)] [\sigma(1) \cdot \nabla] \psi(\vec{r}, t) - \psi^+(\vec{r}, t) [\sigma(1) \cdot \nabla] \{ [r_i \sigma_j(1) - r_j \sigma_i(1)] \psi(\vec{r}, t) \} \} = 0$$

#### 4.5 角动量算符的性质三

$$\text{推论4.5.1. } \psi^+(\vec{p}, t) \sigma_i(s) [\sigma(s) \cdot \vec{p}] \tilde{\partial}_j \psi(\vec{p}, t) = \psi^+(\vec{p}, t) p_i \tilde{\partial}_j \psi(\vec{p}, t)$$

$$\text{引理4.5.1. } \begin{cases} P_a = -i \int \psi^+(\vec{r}, t) \partial_a \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \hat{P}_a \psi(\vec{r}, t) d^3\vec{r} \\ L_{ab} = -i \int \psi^+(\vec{r}, t) (r_a \partial_b - r_b \partial_a) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \hat{L}_{ab} \psi(\vec{r}, t) d^3\vec{r} \\ M_{ab} = \int \psi^+(\vec{r}, t) [-i(r_a \partial_b - r_b \partial_a) + i\sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\alpha\zeta}(s)] \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \hat{M}_{ab} \psi(\vec{r}, t) d^3\vec{r} \\ \tilde{M}_{ab} = -i \int \psi^+(\vec{r}, t) (r_a \sigma_b - r_b \sigma_a) \dot{\psi}(\vec{r}, t) d^3\vec{r} \\ \bar{M}_{ab} = -i \int \psi^+(\vec{r}, t) (r_a \sigma_b - r_b \sigma_a) \psi(\vec{r}, t) d^3\vec{r} \end{cases}$$

$$\text{定理4.5.1. } S_{ab} = \int \psi^+(\vec{r}, t) S_{ab}(\frac{1}{2}, \zeta) \psi(\vec{r}, t) d^3\vec{r} = i\sigma_{\zeta ab}^{\alpha\zeta} \int \psi^+(\vec{r}, t) \sigma_{\alpha\zeta}(\frac{1}{2}) \psi(\vec{r}, t) d^3\vec{r} \\ = \frac{-i\zeta}{2} \sigma_{\zeta ab}^{\alpha\zeta} \int \hat{p}_{\alpha\zeta} [a_1^+(\vec{p}, -\frac{\zeta}{2}) a_1(\vec{p}, -\frac{\zeta}{2}) + a_2(\vec{p}, -\frac{\zeta}{2}) a_2^+(\vec{p}, -\frac{\zeta}{2})] d^3\vec{p}$$

#### 4.6 自旋算符的性质一

$$\text{推论4.6.1. } \begin{cases} \int \psi^+(\vec{r}, t) \sigma(s) \psi(\vec{r}, t) d^3\vec{r} = s\zeta \int \psi^+(\vec{r}, t) \frac{\nabla}{\zeta^2} \dot{\psi}(\vec{r}, t) d^3\vec{r}, s \geq 1 \\ \int \psi^+(\vec{r}, t) \sigma(s) \dot{\psi}(\vec{r}, t) d^3\vec{r} = s\zeta \int \psi^+(\vec{r}, t) \nabla \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \end{cases}$$

$$\text{推论4.6.2. } \begin{cases} \int \psi^+(\vec{r}, t) S_{ab}(s, \zeta) \psi(\vec{r}, t) d^3\vec{r} = i s \zeta \sigma_{\zeta ab}^{\alpha\zeta} \int \psi^+(\vec{r}, t) \frac{\nabla_{\alpha\zeta}}{\zeta^2} \dot{\psi}(\vec{r}, t) d^3\vec{r}, s \geq 1 \\ \int \psi^+(\vec{r}, t) S_{ab}(s, \zeta) \dot{\psi}(\vec{r}, t) d^3\vec{r} = i s \zeta \sigma_{\zeta ab}^{\alpha\zeta} \int \psi^+(\vec{r}, t) \nabla_{\alpha\zeta} \psi(\vec{r}, t) d^3\vec{r}, s \geq 1 \end{cases}$$

$$\text{推论4.6.3. } \begin{cases} \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha\zeta}(n + \frac{1}{2}) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} = -(n + \frac{1}{2}) \zeta \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\alpha\zeta} \frac{\dot{\psi}(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n+1}}, n + \frac{1}{2} \geq 1 \\ \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha\zeta}(n) \frac{i\dot{\psi}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} = n\zeta \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \nabla_{\alpha\zeta} \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, n \geq 1 \end{cases}$$

$$\text{推论4.6.4. } \begin{cases} \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} S_{ab}(n + \frac{1}{2}, \zeta) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} = -(n + \frac{1}{2}) i \zeta \sigma_{\zeta ab}^{\alpha\zeta} \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\alpha\zeta} \frac{\dot{\psi}(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n+1}}, n + \frac{1}{2} \geq 1 \\ \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} S_{ab}(n, \zeta) \frac{i\dot{\psi}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} = n i \zeta \sigma_{\zeta ab}^{\alpha\zeta} \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \nabla_{\alpha\zeta} \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, n \geq 1 \end{cases}$$

#### 4.7 自旋算符的性质二

$$\text{推论4.7.1. } \int \psi^+(\vec{r}, t) \sigma_i(s) [\sigma(s) \cdot \nabla] \sigma_j(s) \psi(\vec{r}, t) d^3\vec{r} = \int \psi^+(\vec{r}, t) \sigma_i(s) \sigma_j(s) [\sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3\vec{r}$$

$$\text{推论4.7.2. } \int \psi^+(\vec{r}, t) \sigma_i(s) \sigma_j(s) \psi(\vec{r}, t) d^3\vec{r} = ???$$

$$\text{推论4.7.3. } [\sigma(s) \cdot \nabla] \sigma(s) \psi = \{ s \nabla + (s-1) \sigma(s) [\frac{1}{s} \sigma(s) \cdot \nabla] \} \psi$$

$$\text{证明: } \int \psi^+(\vec{r}, t) \sigma_i(s) [\sigma(s) \cdot \nabla] \sigma_j(s) \psi(\vec{r}, t) d^3\vec{r} \\ = \int \psi^+(\vec{r}, t) \sigma_i(s) \{ s \partial_j + (s-1) \sigma_j(s) [\frac{1}{s} \sigma(s) \cdot \nabla] \} \psi(\vec{r}, t) d^3\vec{r} \\ = \int \psi^+(\vec{r}, t) s \sigma_i(s) \partial_j \psi(\vec{r}, t) + \psi^+(\vec{r}, t) (s-1) \sigma_i(s) \sigma_j(s) [\frac{1}{s} \sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3\vec{r} \\ = \int \psi^+(\vec{r}, t) s [\sigma(s) \cdot \nabla] \hat{\partial}_i \hat{\partial}_j \psi(\vec{r}, t) + \psi^+(\vec{r}, t) (s-1) [s^2 \hat{\partial}_i \hat{\partial}_j + \frac{s}{2} (\delta_{ij} - \hat{\partial}_i \hat{\partial}_j + i \zeta \varepsilon_{ij}^k \hat{\partial}_k \hat{\partial}_t)] [\frac{1}{s} \sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3\vec{r} \\ = \int \psi^+(\vec{r}, t) [s^2 \hat{\partial}_i \hat{\partial}_j + \frac{s-1}{2} (\delta_{ij} - \hat{\partial}_i \hat{\partial}_j + i \zeta \varepsilon_{ij}^k \hat{\partial}_k \hat{\partial}_t)] [\sigma(s) \cdot \nabla] \psi(\vec{r}, t) d^3\vec{r} \quad \square$$

$$\text{推论4.7.4. } [\sigma(s) \cdot \nabla]^2 \sigma(s) \psi = \{ (2s-1) [\sigma(s) \cdot \nabla] \nabla + (1 - \frac{1}{s})^2 \sigma(s) [\sigma(s) \cdot \nabla]^2 \} \psi$$

$$\text{证明: } \int \psi^+(\vec{r}, t) \sigma_i(s) [\sigma(s) \cdot \nabla]^2 \sigma_j(s) \psi(\vec{r}, t) d^3\vec{r} \\ = \int \psi^+(\vec{r}, t) \sigma_i(s) \{ (2s-1) [\sigma(s) \cdot \nabla] \partial_j + (1 - \frac{1}{s})^2 \sigma_j(s) [\sigma(s) \cdot \nabla]^2 \} \psi(\vec{r}, t) d^3\vec{r} \\ = \int \psi^+(\vec{r}, t) \{ (2s-1) \sigma_i(s) [\sigma(s) \cdot \nabla] \partial_j + (s-1)^2 \sigma_i(s) \sigma_j(s) \nabla^2 \} \psi(\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned}
&= \int \psi^+(\vec{r}, t) \{ (2s-1)s^2 \partial_i \partial_j + (s-1)^2 \sigma_i(s) \sigma_j(s) \nabla^2 \} \psi(\vec{r}, t) d^3 \vec{r} \\
&= \int \psi^+(\vec{r}, t) \{ (2s-1)s^2 \partial_i \partial_j + (s-1)^2 [s^2 \partial_i \partial_j + \frac{s}{2} (\delta_{ij} \nabla^2 - \partial_i \partial_j + i \zeta \varepsilon_{ij}^k \partial_k \partial_t)] \} \psi(\vec{r}, t) d^3 \vec{r} \\
&= \int \psi^+(\vec{r}, t) \{ s^4 \partial_i \partial_j + \frac{s}{2} (s-1)^2 (\delta_{ij} \nabla^2 - \partial_i \partial_j + i \zeta \varepsilon_{ij}^k \partial_k \partial_t) \} \psi(\vec{r}, t) d^3 \vec{r} \quad \square
\end{aligned}$$

$$\text{证明: } \int \psi^+(\vec{r}, t) \sigma_{[i}(s) [\sigma(s) \cdot \nabla]^2 \sigma_{j]}(s) \psi(\vec{r}, t) d^3 \vec{r} = \int \psi^+(\vec{r}, t) s (s-1)^2 i \zeta \varepsilon_{ij}^k \partial_k \partial_t \psi(\vec{r}, t) d^3 \vec{r} \quad \square$$

$$\text{证明: } \int \psi^+(\vec{r}, t) \sigma_{[i}(s) [\sigma(s) \cdot \hat{\nabla}]^2 \sigma_{j]}(s) \psi(\vec{r}, t) d^3 \vec{r} = (s-1)^2 i \varepsilon_{ij}^k \int \psi^+(\vec{r}, t) \sigma_k(s) \psi(\vec{r}, t) d^3 \vec{r} \quad \square$$

$$\text{证明: } \int \psi^+(\vec{r}, t) \sigma_i(s) \sigma_j(s) \psi(\vec{r}, t) d^3 \vec{r} = \int \psi^+(\vec{r}, t) [s^2 \hat{\partial}_i \hat{\partial}_j + \frac{s}{2} (\delta_{ij} \hat{\nabla}^2 - \hat{\partial}_i \hat{\partial}_j + i \zeta \varepsilon_{ij}^k \hat{\partial}_k \hat{\partial}_t)] \psi(\vec{r}, t) d^3 \vec{r}, s \neq 1 \quad \square$$

$$\text{证明: } \int \psi^+(\vec{r}, t) \sigma_{[i}(s) \sigma_{j]}(s) \psi(\vec{r}, t) d^3 \vec{r} = i \zeta s \int \psi^+(\vec{r}, t) \varepsilon_{ij}^k \hat{\partial}_k \hat{\partial}_t \psi(\vec{r}, t) d^3 \vec{r} = i \varepsilon_{ij}^k \int \psi^+(\vec{r}, t) \sigma_k(s) \psi(\vec{r}, t) d^3 \vec{r}, s \neq 1 \quad \square$$

$$\text{推论4.7.5. } \psi(\vec{p}, t) = |\vec{p}|^{(s-\frac{1}{2})} [a_1(\vec{p}, -s\zeta) \lambda(\hat{p}, -s\zeta) e^{-i|\vec{p}|t} + a_2^+(\vec{p}, -s\zeta) \lambda(-\hat{p}, -s\zeta) e^{i|\vec{p}|t}]$$

## 4.8 自旋算符的性质三

$$\text{证明: } \int \frac{\psi^+(\vec{r}, t)}{\sqrt{-\nabla^2}} \sigma_{[i}(\frac{3}{2}, \zeta) \frac{1}{8} \{-1 + 4[\sigma(\frac{3}{2}) \cdot \hat{\nabla}]^2\} \sigma_{j]}(\frac{3}{2}, \zeta) \frac{\psi(\vec{r}, t)}{\sqrt{-\nabla^2}} d^3 \vec{r} \quad \square$$

猜想4.8.1.

$$\begin{cases} P_a(n + \frac{1}{2}) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{P}_a \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\ M_{ab}(n + \frac{1}{2}) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{M}_{ab} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \end{cases} \quad \begin{cases} P_a(n) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{P}_a \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\ M_{ab}(n) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{M}_{ab} \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \end{cases}$$

猜想4.8.2.

$$\begin{cases} \hat{s}(n + \frac{1}{2}) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \sigma(n + \frac{1}{2}) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} = \zeta(n + \frac{1}{2}) \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n+1}} d^3 \vec{r} \\ \hat{s}(n) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \sigma_\alpha(n) \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} = i \zeta n \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \nabla \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \end{cases}$$

## 5 常数不变张量 $\Gamma_{k_\zeta k'_\zeta}^{abc \dots}(s)$ 的数学分析

### 5.1 s-自旋场协变常数不变张量 $\Gamma_{k_\zeta k'_\zeta}^{abc \dots}(s)$ 的性质

$$\begin{aligned} \text{性质5.1.1. } & \Gamma_{k_\zeta k'_\zeta}^{\pi \pi \pi \dots}(s) = (\frac{-i\zeta}{\sqrt{2}})^{2s} \overbrace{(\zeta)_{A_\zeta A'_\zeta} (\zeta)_{B_\zeta B'_\zeta} (\zeta)_{C_\zeta C'_\zeta} \dots}^{2s} \cdot \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s) \\ &= (\frac{1}{\sqrt{2}})^{2s} \delta_{k_\zeta k'_\zeta} \end{aligned}$$

$$\begin{aligned} \text{性质5.1.2. } & \Gamma_{k_\zeta k'_\zeta}^{\pi \pi \pi \dots}(s) = (\frac{-i\zeta}{\sqrt{2}})^{2s} \overbrace{(\sigma)_{A_\zeta A'_\zeta}^i (\zeta)_{B_\zeta B'_\zeta} (\zeta)_{C_\zeta C'_\zeta} \dots}^{2s} \cdot \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s) \\ &= -i\zeta (\frac{1}{\sqrt{2}})^{2s} \frac{1}{s} \sigma^i(s)_{k_\zeta k'_\zeta} \end{aligned}$$

$$\begin{aligned} \text{性质5.1.3. } & \Gamma_{k_\zeta k'_\zeta}^{ij \pi \dots}(s) = (\frac{-i\zeta}{\sqrt{2}})^{2s} \overbrace{(\sigma)_{A_\zeta A'_\zeta}^i (\sigma)_{B_\zeta B'_\zeta}^j (\zeta)_{C_\zeta C'_\zeta} \dots}^{2s} \cdot \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s) \\ &= -(\frac{1}{\sqrt{2}})^{2s} \frac{1}{s(s-\frac{1}{2})} \frac{1}{2!} [\sigma^{\{ij}(s) \sigma^{\}j}(s) - \frac{s}{2} \delta^{\{ij\}}]_{k_\zeta k'_\zeta} \end{aligned}$$

$$\begin{aligned} \text{性质5.1.4. } & \Gamma_{k_\zeta k'_\zeta}^{ijk \pi \dots}(s) = (\frac{-i\zeta}{\sqrt{2}})^{2s} \overbrace{(\sigma)_{A_\zeta A'_\zeta}^i (\sigma)_{B_\zeta B'_\zeta}^j (\sigma)_{C_\zeta C'_\zeta}^k (\zeta)_{D_\zeta D'_\zeta} \dots}^{2s} \cdot \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta D'_\zeta \dots}(s) \\ &= (\frac{1}{\sqrt{2}})^{2s} \frac{i\zeta}{s(s-\frac{1}{2})(s-1)} \frac{1}{3!} [\sigma^{\{ij}(s) \sigma^j(s) \sigma^k\}(s) + \frac{1-3s}{2} \delta^{\{ij} \sigma^k\}(s)]_{k_\zeta k'_\zeta} \end{aligned}$$

$$\begin{aligned} \text{性质5.1.5. } & \Gamma_{k_\zeta k'_\zeta}^{ijkl \pi \dots}(s) = (\frac{-i\zeta}{\sqrt{2}})^{2s} \overbrace{(\sigma)_{A_\zeta A'_\zeta}^i (\sigma)_{B_\zeta B'_\zeta}^j (\sigma)_{C_\zeta C'_\zeta}^k (\sigma)_{D_\zeta D'_\zeta}^l \dots}^{2s} \cdot \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots}(s) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta D'_\zeta \dots}(s) \\ &= (\frac{1}{\sqrt{2}})^{2s} \frac{1}{s(s-\frac{1}{2})(s-1)(s-\frac{3}{2})} \frac{1}{4!} [\sigma^{\{ij}(s) \sigma^j(s) \sigma^k(s) \sigma^l\}(s) + (2-3s) \sigma^{\{ij}(s) \sigma^j(s) \delta^{kl}\} + \frac{3}{4} s(s-1) \delta^{\{ij} \delta^{kl}\}]_{k_\zeta k'_\zeta} \end{aligned}$$

## 5.2 重要关系

引理5.2.1.

$$\begin{cases} (\sigma \cdot \nabla)^{A_\zeta A_\zeta} \Gamma_{A_\zeta B_\zeta}^{k_\zeta} \dots (s) \psi_{k_\zeta}(\vec{r}, t) = i\zeta \partial_\pi \delta^{A_\zeta A_\zeta} \Gamma_{A_\zeta B_\zeta}^{k_\zeta} \dots (s) \psi_{k_\zeta}(\vec{r}, t), [\sigma(s) \cdot \nabla]^{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}, t) = i s \zeta \partial_\pi \delta^{k'_\zeta k_\zeta} \psi_{k_\zeta}(\vec{r}, t) \\ (\sigma \cdot \nabla)_{A_\zeta A'_\zeta} \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta} \dots (s) \psi^{k'_\zeta}(\vec{r}, t) = -i\zeta \partial_\pi \delta^{A_\zeta A'_\zeta} \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta} \dots (s) \psi^{k'_\zeta}(\vec{r}, t), [\sigma(s) \cdot \nabla]_{k_\zeta k'_\zeta} \psi^{k'_\zeta}(\vec{r}, t) = -i s \zeta \partial_\pi \delta_{k_\zeta k'_\zeta} \psi^{k'_\zeta}(\vec{r}, t) \end{cases}$$

$$\text{定理5.2.1. } \Gamma_{k_\zeta k'_\zeta}^{ij \dots} (s) \underbrace{\partial_i \partial_j \dots}_n \psi(\vec{r}, t) = 2^{-s} \delta_{k_\zeta k'_\zeta} (\partial_\pi)^n \psi^{k'_\zeta}(\vec{r}, t)$$

$$\text{证明: } \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \underbrace{\pi \dots \pi}_{2s-n} (s) \underbrace{\partial_i \partial_j \dots}_n \psi^{k'_\zeta}(\vec{r}, t)$$

$$\begin{aligned} &= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma)_{A_\zeta A'_\zeta}^i (\sigma)_{B_\zeta B'_\zeta}^j \dots}^n \overbrace{(i\zeta)_{P_\zeta P'_\zeta} (i\zeta)_{Q_\zeta Q'_\zeta} \dots}^{2s-n} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta \dots P_\zeta Q_\zeta} \dots}^{2s} (s) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta \dots P'_\zeta Q'_\zeta} \dots}^{2s} (s) \underbrace{\partial_i \partial_j \dots}_n \psi^{k'_\zeta}(\vec{r}, t) \\ &= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma \cdot \nabla)_{A_\zeta A'_\zeta} (\sigma \cdot \nabla)_{B_\zeta B'_\zeta} \dots}^n \overbrace{(i\zeta)_{P_\zeta P'_\zeta} (i\zeta)_{Q_\zeta Q'_\zeta} \dots}^{2s-n} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta \dots P_\zeta Q_\zeta} \dots}^{2s} (s) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta \dots P'_\zeta Q'_\zeta} \dots}^{2s} (s) \psi^{k'_\zeta}(\vec{r}, t) \\ &= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(-i\zeta \partial_\pi)_{A_\zeta A'_\zeta} (-i\zeta \partial_\pi)_{B_\zeta B'_\zeta} \dots}^n \overbrace{(i\zeta)_{P_\zeta P'_\zeta} (i\zeta)_{Q_\zeta Q'_\zeta} \dots}^{2s-n} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta \dots P_\zeta Q_\zeta} \dots}^{2s} (s) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta \dots P'_\zeta Q'_\zeta} \dots}^{2s} (s) \psi^{k'_\zeta}(\vec{r}, t) \\ &= 2^{-s} \delta_{k_\zeta k'_\zeta} (-\partial_\pi)^n \psi(\vec{r}, t) \end{aligned}$$

□

$$\text{推论5.2.1. } \begin{cases} \lambda_{k_\zeta}(\hat{p}, -s\zeta) = \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta} \dots (s) \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \dots \\ \Gamma_{A_\zeta B_\zeta C_\zeta}^{k_\zeta} \dots (s) \lambda_{k_\zeta}(\hat{p}, -s\zeta) = \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \dots \end{cases}$$

$$\text{定理5.2.2. } \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \underbrace{\pi \dots \pi}_{2s-n} (s) \hat{p}_i \hat{p}_j \dots \lambda^{k'_\zeta}(\hat{p}, -s\zeta) = \frac{i^n}{2^s} \lambda_{k_\zeta}(\hat{p}, -s\zeta), \Gamma_{ij \dots}^{k'_\zeta k_\zeta} \underbrace{\pi \dots \pi}_{2s-n} (s) \hat{p}^i \hat{p}^j \dots \lambda_{k_\zeta}(\hat{p}, s\zeta) = \frac{i^n}{2^s} \lambda^{k'_\zeta}(\hat{p}, s\zeta)$$

$$\text{证明: } \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \underbrace{\pi \dots \pi}_{2s-n} (s) \hat{p}_i \hat{p}_j \dots \lambda^{k'_\zeta}(\hat{p}, -s\zeta)$$

$$\begin{aligned} &= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma)_{A_\zeta A'_\zeta}^i (\sigma)_{B_\zeta B'_\zeta}^j \dots}^n \overbrace{(i\zeta)_{P_\zeta P'_\zeta} (i\zeta)_{Q_\zeta Q'_\zeta} \dots}^{2s-n} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta \dots P_\zeta Q_\zeta} \dots}^{2s} (s) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta \dots P'_\zeta Q'_\zeta} \dots}^{2s} (s) \hat{p}_i \hat{p}_j \dots \lambda^{k'_\zeta}(\hat{p}, -s\zeta) \\ &= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta \dots P_\zeta Q_\zeta} \dots}^{2s} (s) \\ &= \overbrace{(\sigma \cdot \hat{p})_{A_\zeta A'_\zeta} (\sigma \cdot \hat{p})_{B_\zeta B'_\zeta} \dots}^n \overbrace{(i\zeta)_{P_\zeta P'_\zeta} (i\zeta)_{Q_\zeta Q'_\zeta} \dots}^{2s-n} \overbrace{\lambda^{A'_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{B'_\zeta}(\hat{p}, -\frac{\zeta}{2}) \dots \lambda^{P'_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{Q'_\zeta}(\hat{p}, -\frac{\zeta}{2}) \dots}^{2s} \\ &= \frac{i^n}{2^s} \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta \dots P_\zeta Q_\zeta} \dots}^{2s} (s) \overbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \dots \lambda_{P_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{Q_\zeta}(\hat{p}, -\frac{\zeta}{2}) \dots}^{2s} \\ &= \frac{i^n}{2^s} \lambda_{k_\zeta}(\hat{p}, -s\zeta) \end{aligned}$$

□

$$\text{定理5.2.3. } \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \underbrace{\pi \dots \pi}_{2s-n} (s) \hat{p}_i \hat{p}_j \dots \lambda^{k'_\zeta}(\hat{p}, s\zeta) = \frac{(-i)^n}{2^s} \lambda_{k_\zeta}(\hat{p}, s\zeta), \Gamma_{ij \dots}^{k'_\zeta k_\zeta} \underbrace{\pi \dots \pi}_{2s-n} (s) \hat{p}^i \hat{p}^j \dots \lambda_{k_\zeta}(\hat{p}, -s\zeta) = \frac{(-i)^n}{2^s} \lambda^{k'_\zeta}(\hat{p}, -s\zeta)$$

$$\text{定理5.2.4. } \Gamma_{ij \dots}^{k'_\zeta k_\zeta} \underbrace{\pi \dots \pi}_{2s-n} (s) \hat{p}^i \hat{p}^j \dots \lambda_{k_\zeta}(\hat{p}, s\zeta) = \frac{i^n}{2^s} \lambda^{k'_\zeta}(\hat{p}, s\zeta)$$

$$\text{定理5.2.5. } \begin{cases} \Gamma_{k'_\zeta k_\zeta}^{ij \dots} (s) \underbrace{\partial_i \partial_j \dots}_n \psi^{k_\zeta}(\vec{r}, t) = 2^{-s} \delta_{k'_\zeta k_\zeta} \partial_\pi^n \psi(\vec{r}, t) \\ \Gamma_{k'_\zeta k_\zeta}^{abc \dots} (s) := \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma, i\zeta)_{A'_\zeta A_\zeta}^a (\sigma, i\zeta)_{B'_\zeta B_\zeta}^b (\sigma, i\zeta)_{C'_\zeta C_\zeta}^c \dots}^{2s} \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta} \dots}^{2s} (s) \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta} \dots}^{2s} (s) \end{cases}$$

$$\begin{aligned}
& \text{证明: } \Gamma_{k'_c k'_\zeta}^{ij \cdots \pi \cdots \pi} (s) \underbrace{\partial_i \partial_j \cdots}_{n} \psi^{k_\zeta}(\vec{r}, t) \\
&= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma)_{A'_\zeta A'_\zeta}^i (\sigma)_{B'_\zeta B'_\zeta}^j \cdots}^n \overbrace{(i\zeta)_{P'_\zeta P'_\zeta} (i\zeta)_{Q'_\zeta Q'_\zeta} \cdots}^{2s-n} \overbrace{\Gamma_{k'_c}^{A'_\zeta B'_\zeta \cdots P'_\zeta Q'_\zeta} \cdots}^{2s} (s) \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta \cdots P_\zeta Q_\zeta} \cdots}^{2s} (s) \underbrace{\partial_i \partial_j \cdots}_{n} \psi^{k_\zeta}(\vec{r}, t) \\
&= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma \cdot \nabla)_{A'_\zeta A'_\zeta} (\sigma \cdot \nabla)_{B'_\zeta B'_\zeta} \cdots}^n \overbrace{(i\zeta)_{P'_\zeta P'_\zeta} (i\zeta)_{Q'_\zeta Q'_\zeta} \cdots}^{2s-n} \overbrace{\Gamma_{k'_c}^{A'_\zeta B'_\zeta \cdots P'_\zeta Q'_\zeta} \cdots}^{2s} (s) \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta \cdots P_\zeta Q_\zeta} \cdots}^{2s} (s) \psi^{k_\zeta}(\vec{r}, t) \\
&= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(i\zeta \partial_\pi)_{A'_\zeta A'_\zeta} (i\zeta \partial_\pi)_{B'_\zeta B'_\zeta} \cdots}^n \overbrace{(i\zeta)_{P'_\zeta P'_\zeta} (i\zeta)_{Q'_\zeta Q'_\zeta} \cdots}^{2s-n} \overbrace{\Gamma_{k'_c}^{A'_\zeta B'_\zeta \cdots P'_\zeta Q'_\zeta} \cdots}^{2s} (s) \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta \cdots P_\zeta Q_\zeta} \cdots}^{2s} (s) \psi^{k_\zeta}(\vec{r}, t) \\
&= 2^{-s} \delta_{k'_c k'_\zeta} (\partial_\pi)^n \psi^{k_\zeta}(\vec{r}, t)
\end{aligned}$$

□

### 5.3 s-自旋场协变常数不变张量 $\Gamma_{k'_c k'_\zeta}^{abc \cdots} (s)$ 的重要定理

$$\text{引理5.3.1. } \Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_a \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_b \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_c}^{2s} \cdots = (i\sqrt{2})^{2s} \lambda_{k_\zeta} \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, -s\zeta \right) \lambda_{k'_\zeta}^+ \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, -s\zeta \right)$$

$$\begin{aligned}
& \text{证明: } \Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_a \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_b \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_c}^{2s} \cdots \\
&= \left(\frac{-i\zeta}{\sqrt{2}}\right)^{2s} \overbrace{(\sigma, i\zeta)_{A'_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B'_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C'_\zeta C'_\zeta}^c \cdots}^{2s} \overbrace{\Gamma_{k'_c}^{A'_\zeta B'_\zeta C'_\zeta} \cdots}^{2s} (s) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta} \cdots}^{2s} (s) \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_a \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_b \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_c}^{2s} \cdots \\
&= \begin{cases} (i\sqrt{2})^{2s} \overbrace{\Gamma_{k'_c}^{1'_\zeta 1'_\zeta 1'_\zeta} \cdots}^{2s} (s) \overbrace{\Gamma_{k'_\zeta}^{1'_\zeta 1'_\zeta 1'_\zeta} \cdots}^{2s} (s) = (i\sqrt{2})^{2s} \lambda_{k_\zeta} \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, s \right) \lambda_{k'_\zeta}^+ \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, s \right), \zeta = -1 \\ (i\sqrt{2})^{2s} \overbrace{\Gamma_{k'_c}^{2'_\zeta 2'_\zeta 2'_\zeta} \cdots}^{2s} (s) \overbrace{\Gamma_{k'_\zeta}^{2'_\zeta 2'_\zeta 2'_\zeta} \cdots}^{2s} (s) = (i\sqrt{2})^{2s} \lambda_{k_\zeta} \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, -s \right) \lambda_{k'_\zeta}^+ \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, -s \right), \zeta = 1 \end{cases} \\
&= (i\sqrt{2})^{2s} \lambda_{k_\zeta} \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, -s\zeta \right) \lambda_{k'_\zeta}^+ \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, -s\zeta \right)
\end{aligned}$$

□

$$\text{定理5.3.1. } \Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdots}^{2s} = (i\sqrt{2})^{2s} \lambda_{k_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -s\zeta)$$

以上  $s = \frac{1}{2}, 1, \frac{3}{2}, 2$  已被证明,  $s > 2$  还属于猜想, 下面用常数不变张量分析法来统一证明它:

$$\begin{aligned}
& \text{证明: } \Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_a \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_b \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}_c}^{2s} \cdots = (i\sqrt{2})^{2s} \lambda_{k_\zeta} \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, -s\zeta \right) \lambda_{k'_\zeta}^+ \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, -s\zeta \right) \\
&\Leftrightarrow \overbrace{\Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) [\exp\{-i \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\}]_a^{\bar{a}} \exp\{-i \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\}]_b^{\bar{b}} \exp\{-i \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\}]_c^{\bar{c}} \cdots}^{2s} \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdots}^{2s} \\
&= (i\sqrt{2})^{2s} \exp\{-i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} |_{k_\zeta}^{\bar{k}_\zeta} \lambda_{\bar{k}_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -s\zeta) \exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} |_{k'_\zeta}^{\bar{k}'_\zeta} \\
&\Leftrightarrow [\exp\{i \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\}]_a^{\bar{a}} \exp\{i \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\}]_b^{\bar{b}} \exp\{i \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\}]_c^{\bar{c}} \cdots \\
&\exp\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\} |_{\bar{k}_\zeta}^{k_\zeta} \overbrace{\Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) \exp\{-i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z\}}^{2s} |_{k'_\zeta}^{\bar{k}'_\zeta} \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdots}^{2s} = (i\sqrt{2})^{2s} \lambda_{\bar{k}_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -s\zeta) \\
&\Leftrightarrow \overbrace{\Gamma_{k'_c k'_\zeta}^{\bar{a} \bar{b} \bar{c} \cdots} (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdots}^{2s}}^{2s} = (i\sqrt{2})^{2s} \lambda_{\bar{k}_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -s\zeta) \\
&\Leftrightarrow \overbrace{\Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdots}^{2s}}^{2s} = (i\sqrt{2})^{2s} \lambda_{k_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -s\zeta)
\end{aligned}$$

□

$$\text{推论5.3.1. 投影算子: } \hat{P}_{k'_c k'_\zeta}(s, \zeta) = (i\sqrt{2})^{-2s} \overbrace{\Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdots}^{2s}}^{2s} \overbrace{\Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdots}^{2s}}^{2s} = (i\sqrt{2})^{2s} \hat{P}_{k'_c k'_\zeta}(s, \zeta)$$

$$\text{推论5.3.2. } \overbrace{\Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdots}^{2s}}^{2s} \succ \overbrace{\Gamma_{k'_c k'_\zeta}^{abc \cdots} (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdots}^{2s}}^{2s} = (i\sqrt{2})^{2s} \lambda(\hat{p}, -s\zeta) \lambda^+(\hat{p}, -s\zeta), s \geq 0$$

$$\text{推论5.3.3. } \overbrace{\Gamma^{abc\dots}(s)}^{2s} \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \dots}^{2s} \lambda(\hat{p}, -s\zeta) = (i\sqrt{2})^{2s} \lambda(\hat{p}, -s\zeta)$$

$$\text{推论5.3.4. } \begin{cases} \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \lambda_{k_\zeta}(\hat{p}, -s\zeta) = 1, \lambda^{+k_\zeta}(-\hat{p}, -s\zeta) \lambda_{k_\zeta}(\hat{p}, -s\zeta) = 0 \\ \lambda_{k_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -s\zeta) = (i\sqrt{2})^{-2s} \overbrace{\Gamma^{abc\dots}(s)}^{2s} \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \dots}^{2s} \end{cases}$$

#### 5.4 算符 $\hat{p}_a, \hat{\partial}_a$ 和 $\Gamma_{\pm}^{abc\dots}(s), \Gamma_{\pm}^{abc\dots}(s)$ 定义的重新汇总

$$\text{定义5.4.1. } \hat{p}_a := \frac{p_a}{|\vec{p}|} = (\hat{p}, i); \hat{p} = \frac{\vec{p}}{|\vec{p}|}, \hat{p}_\pi = \frac{p_\pi}{|\vec{p}|} = i; \hat{p}^2 = 1, \hat{p}_\pi^2 = i^2$$

$$\text{定义5.4.2. } \hat{\partial}_a := \frac{\partial_a}{i\sqrt{-\nabla^2}} = \frac{-i\partial_a}{\sqrt{-\nabla^2}} = \frac{(-i\nabla, -\partial_t)}{\sqrt{-\nabla^2}}; \hat{\nabla} = \frac{\nabla}{i\sqrt{-\nabla^2}} = \frac{-i\nabla}{\sqrt{-\nabla^2}}; \hat{\nabla}^2 = 1, \hat{\nabla}_\pi^2 = i^2$$

$$\text{推论5.4.1. } p_a \simeq -i\partial_a, |\vec{p}| \simeq \sqrt{-\nabla^2}, \hat{p}_a \simeq \hat{\partial}_a, p_a = |\vec{p}| \hat{p}_a, \partial_a = (i\sqrt{-\nabla^2}) \hat{\partial}_a$$

$$\text{定义5.4.3. } odd := -, even := +$$

$$\text{定义5.4.4. } \begin{cases} \overbrace{\Gamma^{abc\dots}(s)}^{2s} = 1 \cdot \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l} \overbrace{\pi\dots\pi}^{2l}, 1 \cdot \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l-1} \overbrace{\pi\dots\pi}^{2l+1}, l = 0, \dots, 2s \\ \overbrace{\Gamma_{+}^{abc\dots}(s)}^{2s} := 1 \cdot \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l} \overbrace{\pi\dots\pi}^{2l}, 0 \cdot \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l-1} \overbrace{\pi\dots\pi}^{2l+1}, l = 0, \dots, 2s \\ \overbrace{\Gamma_{-}^{abc\dots}(s)}^{2s} := 0 \cdot \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l} \overbrace{\pi\dots\pi}^{2l}, 1 \cdot \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l-1} \overbrace{\pi\dots\pi}^{2l+1}, l = 0, \dots, 2s \end{cases}$$

$$\text{推论5.4.2. } \overbrace{\Gamma^{abc\dots}(s)}^{2s} = \overbrace{\Gamma_{+}^{abc\dots}(s)}^{2s} + \overbrace{\Gamma_{-}^{abc\dots}(s)}^{2s}$$

#### 5.5 算符 $\Gamma_{\pm}^{abc\dots}(s) \partial_a \partial_b \partial_c \dots$ 和 $\Gamma_{\pm}^{abc\dots}(s) \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots$ 在质壳条件下的性质

$$\text{推论5.5.1. } \partial^a \partial_a \psi = 0 \Rightarrow \begin{cases} \overbrace{\Gamma_{+}^{abc\dots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \dots}^{2s} \psi = \sum_{l=0}^{[s]} C_{2s}^{2n} \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l} \overbrace{\pi\dots\pi}^{2l} \overbrace{\partial_i \partial_j \dots}^{2s-2l} (\sqrt{-\nabla^2})^{2l} \psi \\ \overbrace{\Gamma_{-}^{abc\dots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \dots}^{2s} \psi = \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2n+1} \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l-1} \overbrace{\pi\dots\pi}^{2l+1} \overbrace{\partial_i \partial_j \dots}^{2s-2n-1} (\sqrt{-\nabla^2})^{2l} \partial_\pi \psi \end{cases}$$

$$\text{推论5.5.2. } \partial^a \partial_a \psi = 0 \Rightarrow \begin{cases} \overbrace{\Gamma_{+}^{abc\dots}(s)}^{2s} \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \psi = \sum_{n=0}^{[s]} (-1)^l C_{2s}^{2l} \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l} \overbrace{\pi\dots\pi}^{2l} \overbrace{\hat{\partial}_i \hat{\partial}_j \dots}^{2s-2l} \psi \\ \overbrace{\Gamma_{-}^{abc\dots}(s)}^{2s} \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \psi = \sum_{n=0}^{[s-\frac{1}{2}]} (-1)^l C_{2s}^{2l+1} \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l-1} \overbrace{\pi\dots\pi}^{2l+1} \overbrace{\hat{\partial}_i \hat{\partial}_j \dots}^{2s-2l-1} \hat{\partial}_\pi \psi \end{cases}$$

$$\text{推论5.5.3. } \partial^a \partial_a \psi = 0 \Rightarrow \begin{cases} \overbrace{\Gamma_{+}^{abc\dots}(s)}^{2s} \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \psi = \overbrace{\Gamma_{+}^{abc\dots}(s)}^{2s} \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s, \hat{\partial}_\pi \rightarrow i} \psi \\ \overbrace{\Gamma_{-}^{abc\dots}(s)}^{2s} \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \psi = -i \overbrace{\Gamma_{-}^{abc\dots}(s)}^{2s} \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s, \hat{\partial}_\pi \rightarrow i} \hat{\partial}_\pi \psi \end{cases}$$

#### 5.6 算符 $\Gamma^{abc\dots}(s) \partial_a \partial_b \partial_c \dots \Delta(x-x')|_{t=t'}$ 的性质

$$\text{性质5.6.1. } \begin{cases} \overbrace{\Gamma_{-}^{abc\dots}(s)}^{2s} \overbrace{p_a p_b p_c \dots}^{2s} := \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l-1} \overbrace{\pi\dots\pi}^{2l+1} \overbrace{p_i p_j \dots}^{2s-2l-1} p_\pi^{2l+1} \\ \overbrace{\Gamma_{-}^{abc\dots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \dots}^{2s} := \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l-1} \overbrace{\pi\dots\pi}^{2l+1} \overbrace{\partial_i \partial_j \dots}^{2s-2l-1} \partial_\pi^{2l+1} \end{cases}$$

推论5.6.1.

$$\begin{cases} \overbrace{\Gamma^{abc\dots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \dots}^{2s} \Delta(x-x')|_{t=t'} = i \sum_{l=0}^{[s-\frac{1}{2}]} (-1)^l C_{2s}^{2l+1} \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l-1} \overbrace{\pi\dots\pi}^{2l+1} \overbrace{\partial_i \partial_j \dots}^{2s-2l-1} \nabla^{2l} \delta^3(\vec{r} - \vec{r}') \\ \overbrace{\Gamma^{abc\dots}(s)}^{2s} \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \Delta(x-x')|_{t=t'} = \frac{1}{\sqrt{-\nabla^2}} \sum_{l=0}^{[s-\frac{1}{2}]} (-1)^l C_{2s}^{2l+1} \overbrace{\Gamma^{ij\dots\pi\dots\pi}(s)}^{2s-2l-1} \overbrace{\pi\dots\pi}^{2l+1} \overbrace{\hat{\partial}_i \hat{\partial}_j \dots}^{2s-2l-1} \delta^3(\vec{r} - \vec{r}') \end{cases}$$



$$\begin{cases} \overbrace{\Gamma^{abc\dots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s} \Delta(x-x')|_{t=t'} = (i\sqrt{-\nabla^2})^{2s-1} \overbrace{\Gamma_-^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \\ \overbrace{\Gamma^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \Delta(x-x')|_{t=t'} = \frac{1}{i\sqrt{-\nabla^2}} \overbrace{\Gamma_-^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \end{cases}$$

### 5.7 算符 $\overbrace{\Gamma^{abc\dots}}^{2s}(s) \partial_a \partial_b \partial_c \dots | \partial_\pi \Delta(x-x')|_{t=t'}$ 的性质

推论5.7.1.

$$\begin{cases} \overbrace{\Gamma^{abc\dots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s} \partial_\pi \Delta(x-x')|_{t=t'} = i \sum_{l=0}^{[s]} (-1)^l C_{2s}^{2l} \overbrace{\Gamma^{ij\dots}}^{2s-2l} \overbrace{\pi \dots \pi}^{2l} (s) \overbrace{\partial_i \partial_j \dots}^{2s-2l} \nabla^{2l} \delta^3(\vec{r} - \vec{r}') \\ \overbrace{\Gamma^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \partial_\pi \Delta(x-x')|_{t=t'} = i \sum_{l=0}^{[s]} (-1)^l C_{2s}^{2l} \overbrace{\Gamma^{ij\dots}}^{2s-2l} \overbrace{\pi \dots \pi}^{2l} (s) \overbrace{\hat{\partial}_i \hat{\partial}_j \dots}^{2s-2l} \delta^3(\vec{r} - \vec{r}') \\ \overbrace{\Gamma^{abc\dots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s} \partial_\pi \Delta(x-x')|_{t=t'} = i(i\sqrt{-\nabla^2})^{2s} \overbrace{\Gamma_+^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \\ \overbrace{\Gamma^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \partial_\pi \Delta(x-x')|_{t=t'} = i \overbrace{\Gamma_+^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \end{cases}$$

### 5.8 几个重要定理

定理5.8.1.

$$\begin{cases} \overbrace{\Gamma^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \hat{\partial}_\pi \Delta(x-x')|_{t=t'} = i \overbrace{\Gamma_+^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \\ \overbrace{\Gamma^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s} \Delta(x-x')|_{t=t'} = \frac{1}{i\sqrt{-\nabla^2}} \overbrace{\Gamma_-^{abc\dots}}^{2s}(s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2s, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \end{cases}$$

定理5.8.2.

$$\begin{cases} \overbrace{\Gamma^{abc\dots}}^{2n}(n) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2n} \hat{\partial}_\pi \Delta(x-x')|_{t=t'} = i \overbrace{\Gamma_+^{abc\dots}}^{2n}(n) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2n, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \\ \overbrace{\Gamma^{abc\dots}}^{2n+1}(n + \frac{1}{2}) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2n+1} \Delta(x-x')|_{t=t'} = \frac{1}{i\sqrt{-\nabla^2}} \overbrace{\Gamma_-^{abc\dots}}^{2n+1}(n + \frac{1}{2}) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \end{cases}$$

猜想5.8.1.

$$\overbrace{\Gamma_+^{ab\dots}}^{2s}(s) \overbrace{\hat{p}_a \hat{p}_b \dots}^{2s} \lambda(\hat{p}, -s_\zeta) = \overbrace{\Gamma_-^{ab\dots}}^{2s}(s) \overbrace{\hat{p}_a \hat{p}_b \dots}^{2s} \lambda(\hat{p}, -s_\zeta) = \frac{1}{2} \overbrace{\Gamma^{ab\dots}}^{2s}(s) \overbrace{\hat{p}_a \hat{p}_b \dots}^{2s} \lambda(\hat{p}, -s_\zeta) = \frac{(i\sqrt{2})^{2s}}{2} \lambda(\hat{p}, -s_\zeta)$$

此猜想对低自旋情形已验证正确, 对一般情形还需要严格加以证明。

推论5.8.1.

$$\begin{cases} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc\dots}}^{2n}(n) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2n} \psi(\vec{r}, t; n) = (-2)^n \psi(\vec{r}, t; n) \\ \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc\dots}}^{2n+1}(n + \frac{1}{2}) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2n+1} \hat{\partial}_\pi \psi(\vec{r}, t; n + \frac{1}{2}) = -(-2)^n \sqrt{2} \psi(\vec{r}, t; n + \frac{1}{2}) \end{cases}$$

$$\text{证明: } \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc\dots}}^{2n}(n) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2n} \psi(\vec{r}, t; n)$$

$$\begin{aligned} &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(n-\frac{1}{2})} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc\dots}}^{2n}(n) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \dots}^{2n} \lambda(\hat{p}, -n_\zeta) [a_1(\vec{p}, -n_\zeta) e^{ip \cdot x} + (-1)^{2n} a_2^+(\vec{p}, -n_\zeta) e^{-ip \cdot x}] d^3 \vec{p} \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(n-\frac{1}{2})} (i\sqrt{2})^{2n} \lambda(\hat{p}, -s_\zeta) \lambda(\hat{p}, -n_\zeta) [a_1(\vec{p}, -n_\zeta) e^{ip \cdot x} + a_2^+(\vec{p}, -n_\zeta) e^{-ip \cdot x}] d^3 \vec{p} \\ &= (-2)^n \psi(\vec{r}, t; n) \end{aligned}$$

$$\begin{aligned} &\text{证明: } \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc\dots}}^{2n+1}(n + \frac{1}{2}) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots}^{2n+1} \hat{\partial}_\pi \psi(\vec{r}, t; n + \frac{1}{2}) \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} d^3 \vec{p} \end{aligned}$$

$$\begin{aligned}
& |\vec{p}|^n \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc}}^{2n} \cdot \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c}^{2n+1} \cdot \hat{p}_\pi \lambda(\hat{p}, -(n + \frac{1}{2})\zeta) [a_1(\vec{p}, -(n + \frac{1}{2})\zeta) e^{ip \cdot x} - (-1)^{2n+1} a_2^+(\vec{p}, -(n + \frac{1}{2})\zeta) e^{-ip \cdot x}] \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^n i(i\sqrt{2})^{2n+1} \lambda(\hat{p}, -(n + \frac{1}{2})\zeta) \lambda(\hat{p}, -(n + \frac{1}{2})\zeta) [a_1(\vec{p}, -(n + \frac{1}{2})\zeta) e^{ip \cdot x} + a_2^+(\vec{p}, -(n + \frac{1}{2})\zeta) e^{-ip \cdot x}] d^3 \vec{p} \\
&= i(i\sqrt{2})^{2n+1} \psi(\vec{r}, t; n + \frac{1}{2})
\end{aligned}$$

□

## 6 s-自旋场的对易规则

### 6.1 对易和反对易公式

$$\text{推论6.1.1.} \quad \begin{cases} [A, BC] = [A, B]C + B[A, C], [A, CB] = [A, C]B + C[A, B] \\ [A, BC] = \{A, B\}C - B\{A, C\}, [A, CB] = \{A, C\}B - C\{A, B\} \end{cases}$$

$$\text{推论6.1.2.} \quad \begin{cases} [A, \{B, C\}] = \{[A, B], C\} + \{B, [A, C]\} \\ [A, [B, C]] = \{\{A, B\}, C\} - \{B, \{A, C\}\} \end{cases}$$

### 6.2 s-自旋场数学上一般的协变对易规则

$$\text{定理6.2.1.} \quad \begin{cases} [a_\sigma(\vec{p}, -s\zeta), a_{\sigma'}^+(\vec{p}', -s\zeta)]_\pm = \delta_\sigma \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -s\zeta), a_{\sigma'}^+(\vec{p}', -s\zeta)]_\pm = 0, [a_\sigma^+(\vec{p}, -s\zeta), a_{\sigma'}^+(\vec{p}', -s\zeta)]_\pm = 0 \end{cases}$$

$$\Rightarrow \begin{cases} [\Psi_{k_\zeta}(x), \Psi_{k'_\zeta}^+(x')]_\pm \\ = i(-\sqrt{2})^{-2(s-1)} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc}}^{2s} \cdot (s) \overbrace{\partial_a \partial_b \partial_c}^{2s} \cdot \{\delta_1 \pm (-1)^{2s} \delta_2\} \Delta^{(+)}(x - x') \pm (-1)^{2s+1} \delta_2 \Delta(x - x') \\ [\Psi_{k_\zeta}(x), \Psi_{\beta_\zeta}(x')]_\pm = 0, [\Psi_{k'_\zeta}^+(x), \Psi_{\beta'_\zeta}^+(x')]_\pm = 0 \end{cases}$$

$$\begin{aligned}
& \text{证明: } [\Psi_{k_\zeta}^{(+)}(x), \Psi_{k'_\zeta}^{(+)}(x')]_\pm \\
&= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \lambda_{k_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\vec{p}', -s\zeta) |\vec{p}|^{(2s-1)/2} |\vec{p}'|^{(2s-1)/2} [a_1(\vec{p}, -s\zeta), a_1^+(\vec{p}', -s\zeta)]_\pm e^{ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\vec{p}', -s\zeta) |\vec{p}|^{2s-1} \delta_1 \delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -s\zeta) \delta_1 |\vec{p}|^{2s-1} e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int (i\sqrt{2})^{-2s} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc}}^{2s} \cdot (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c}^{2s} \cdot \delta_1 |\vec{p}|^{2s-1} e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= -(i\sqrt{2})^{-2(s-1)} \frac{\delta_1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc}}^{2s} \cdot (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c}^{2s} \cdot e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= i^{-2s} (\sqrt{2})^{-2(s-1)} \frac{\delta_1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} i^{-2s} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc}}^{2s} \cdot (s) \overbrace{\partial_a \partial_b \partial_c}^{2s} \cdot e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= (-\sqrt{2})^{-2(s-1)} \frac{\delta_1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc}}^{2s} \cdot (s) \overbrace{\partial_a \partial_b \partial_c}^{2s} \cdot e^{ip \cdot (x-x')} d^3 \vec{p} \\
&= i(-\sqrt{2})^{-2(s-1)} \delta_1 \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc}}^{2s} \cdot (s) \overbrace{\partial_a \partial_b \partial_c}^{2s} \cdot \Delta^{(+)}(x - x')
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } [\Psi_{k_\zeta}^{(-)}(x), \Psi_{k'_\zeta}^{(-)}(x')]_\pm \\
&= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \lambda_{k_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\vec{p}', -s\zeta) |\vec{p}|^{(2s-1)/2} |\vec{p}'|^{(2s-1)/2} [a_2^+(\vec{p}, -s\zeta), a_2(\vec{p}', -s\zeta)]_\pm e^{-ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \pm \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\vec{p}', -s\zeta) |\vec{p}|^{2s-1} \delta_2 \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x-x')} d^3 \vec{p} d^3 \vec{p}' \\
&= \pm \frac{1}{(2\pi)^3} \int \lambda_{k_\zeta}(\hat{p}, -s\zeta) \lambda_{k'_\zeta}^+(\hat{p}, -s\zeta) \delta_2 |\vec{p}|^{2s-1} e^{-ip \cdot (x-x')} d^3 \vec{p} \\
&= \pm \frac{1}{(2\pi)^3} \int (i\sqrt{2})^{-2s} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc}}^{2s} \cdot (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c}^{2s} \cdot \delta_2 |\vec{p}|^{2s-1} e^{-ip \cdot (x-x')} d^3 \vec{p} \\
&= -\pm (i\sqrt{2})^{-2(s-1)} \frac{\delta_2}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc}}^{2s} \cdot (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c}^{2s} \cdot e^{-ip \cdot (x-x')} d^3 \vec{p} \\
&= \pm i^{-2s} (\sqrt{2})^{-2(s-1)} \frac{\delta_2}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} (-i)^{-2s} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc}}^{2s} \cdot (s) \overbrace{\partial_a \partial_b \partial_c}^{2s} \cdot e^{-ip \cdot (x-x')} d^3 \vec{p}
\end{aligned}$$

$$\begin{aligned}
&= \pm(\sqrt{2})^{-2(s-1)} \frac{\delta_2}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} e^{-ip \cdot (x-x')} d^3 \vec{p} \\
&= -\pm i(\sqrt{2})^{-2(s-1)} \delta_2 \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(-)}(x-x')
\end{aligned}$$

□

$$\begin{aligned}
&\text{证明: } [\Psi_{k_\zeta}(x), \Psi_{k'_\zeta}^+(x')]_{\pm} \\
&= [\Psi_{k_\zeta}^{(+)}(x), \Psi_{k'_\zeta}^{(+)}(x')]_{\pm} + [\Psi_{k_\zeta}^{(-)}(x), \Psi_{k'_\zeta}^{(-)}(x')]_{\pm} \\
&= i(-\sqrt{2})^{-2(s-1)} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} [\delta_1 \Delta^{(+)}(x-x') \pm (-1)^{2s+1} \delta_2 \Delta^{(-)}(x-x')] \\
&= i(-\sqrt{2})^{-2(s-1)} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \{[\delta_1 \pm (-1)^{2s} \delta_2] \Delta^{(+)}(x-x') \pm (-1)^{2s+1} \delta_2 \Delta^{(-)}(x-x')\}
\end{aligned}$$

□

从上式可知，只有 $\delta_1 \pm (-1)^{-2s} \delta_2 = 0$ 时，才满足微观因果性，同时只有 $\delta_1, \delta_2 \geq 0$ 时，才满足几率非负性。所以数学上八种协变对易或反对易方案中，物理上合理的只有一种：即 $\delta_1 = \delta_2 = 1$  (如果不是1可以归一化)，且对于玻色子满足对易关系；对于费米子满足反对易关系。其实还有两种，即 $\delta_1 = \delta_2 = 0$ ，且满足对易或反对易关系，就是经典情形。

### 6.3 s-自旋场物理的协变对易规则

$$\text{定义6.3.1. } \Delta_{k_\zeta k'_\zeta}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x)$$

$$\text{推论6.3.1. } \Delta(s, \varsigma; x) = \frac{(i\varsigma)^{2s}}{2^{2s-1}} \overbrace{\bar{\Gamma}(s)}^{2s} (\sigma, i\varsigma)^a \otimes (\sigma, i\varsigma)^b \otimes (\sigma, i\varsigma)^c \otimes \cdots \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x) \Gamma(s)$$

$$\text{推论6.3.2. } \Delta(s, \varsigma; x) = \left(\frac{i\varsigma}{2}\right) \bar{N}(s) [(\sigma, i\varsigma)^a \otimes \partial_a \Delta(s - \frac{1}{2}, \varsigma; x)] N(s)$$

$$\begin{aligned}
&\text{证明: } \Delta(s, \varsigma; x) = \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s, \varsigma)}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x) \\
&= \frac{(-1)^{2s}}{2^{s-1}} \left(\frac{-i\varsigma}{\sqrt{2}}\right) \bar{N}(s) [(\sigma, i\varsigma)^a \otimes \overbrace{\Gamma^{bc \cdots}(s - \frac{1}{2}, \varsigma)}^{2s-1}] N(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x) \\
&= \left(\frac{i\varsigma}{2}\right) \bar{N}(s) [(\sigma, i\varsigma)^a \otimes \partial_a \Delta(s - \frac{1}{2}, \varsigma; x)] N(s) \\
&= \left(\frac{i\varsigma}{2}\right) \bar{N}(s) [(\sigma, i\varsigma)^a \otimes \partial_a \Delta(s - \frac{1}{2}, \varsigma; x)] N(s)
\end{aligned}$$

□

定理6.3.1.

$$\begin{cases} [a_\sigma(\vec{p}, -s\varsigma), a_{\sigma'}^+(\vec{p}', -s\varsigma)]_{-2s+1} = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -s\varsigma), a_{\sigma'}(\vec{p}', -s\varsigma)]_{-2s+1} = 0 \\ [a_\sigma^+(\vec{p}, -s\varsigma), a_{\sigma'}^+(\vec{p}', -s\varsigma)]_{-2s+1} = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')]_{-2s+1}, s \geq 0 \\ = i \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x-x'), \Gamma(0) := 1 \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')]_{-2s+1} = 0, [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')]_{-2s+1} = 0 \end{cases}$$

证明:  $\{\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')\}$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int_{\vec{p} \neq 0} \lambda_{k_\zeta}(\hat{p}, -s\varsigma) \lambda_{k'_\zeta}^+(\hat{p}', -s\varsigma) |\vec{p}|^{(2s-1)/2} |\vec{p}'|^{(2s-1)/2} \\
&\{[a_1(\vec{p}, -s\varsigma), a_1^+(\vec{p}', -s\varsigma)]_{-2s+1} e^{ip \cdot (x-x')} + [a_2^+(\vec{p}, -s\varsigma), a_2(\vec{p}', -s\varsigma)]_{-2s+1} e^{-ip \cdot (x-x')}\} d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^{2s-1} \lambda_{k_\zeta}(\hat{p}, -s\varsigma) \lambda_{k'_\zeta}^+(\hat{p}', -s\varsigma) [\delta^3(\vec{p} - \vec{p}') e^{ip \cdot (x-x')} + (-1)^{2s+1} \delta^3(\vec{p} - \vec{p}') e^{-ip \cdot (x-x')}] d^3 \vec{p} d^3 \vec{p}' \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^{2s-1} \lambda_{k_\zeta}(\hat{p}, -s\varsigma) \lambda_{k'_\zeta}^+(\hat{p}', -s\varsigma) [e^{ip \cdot (x-x')} + (-1)^{2s+1} e^{-ip \cdot (x-x')}] d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{-1}{(i\sqrt{2})^{2(s-1)}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s)}^{2s} \overbrace{p_a p_b p_c \cdots}^{2s} [e^{ip \cdot (x-x')} + (-1)^{2s+1} e^{-ip \cdot (x-x')}] d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} \frac{(-i)^{2(s-1)}}{(i\sqrt{2})^{2(s-1)}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3 \vec{p} \\
&= \frac{i}{(-\sqrt{2})^{2(s-1)}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}|} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3 \vec{p} \\
&= \frac{i}{(-\sqrt{2})^{2(s-1)}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s)}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x-x') \\
&= i \Delta_{k_\zeta k'_\zeta}(s; x-x')
\end{aligned}$$

□

## 6.4 s-自旋场的等时对易规则

推论6.4.1.

$$\Delta_{k_\zeta k'_\zeta}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \cdots} (s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x), \Delta_{k_\zeta k'_\zeta}(s; x)|_{t=0} = \frac{(-1)^{2s}}{2^{s-1}} (i\sqrt{-\nabla^2})^{2s-1} \Gamma_{-}^{abc \cdots} (s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \cdots}^{2s, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r})$$

推论6.4.2.

$$\begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')]_{-2s+1}, s \geq 0 \\ = i \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \cdots} (s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')]_{-2s+1} = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')]_{-2s+1} = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)]_{-2s+1}, s > 0 \\ = i \frac{(-1)^{2s}}{2^{s-1}} (i\sqrt{-\nabla^2})^{2s-1} \Gamma_{-}^{abc \cdots} (s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \cdots}^{2s, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r}-\vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)]_{-2s+1} = 0 \\ [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)]_{-2s+1} = 0 \end{cases}$$

推论6.4.3.

$$\begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')]_{-2s+1}, s \geq 0 \\ = i \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \cdots} (s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} i \partial_\pi \Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')]_{-2s+1} = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')]_{-2s+1} = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)]_{-2s+1}, s > 0 \\ = i \frac{(-1)^{2s+1}}{2^{s-1}} (i\sqrt{-\nabla^2})^{2s} \Gamma_{+}^{abc \cdots} (s) \overbrace{\hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \cdots}^{2s, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r}-\vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)]_{-2s+1} = 0 \\ [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)]_{-2s+1} = 0 \end{cases}$$

推论6.4.4.

$$\begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')]_{-2s+1}, s \geq 0 \\ = i \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \cdots} (s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')]_{-2s+1} = 0 \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')]_{-2s+1} = 0 \end{cases} \Rightarrow \begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)]_{-2s+1} = \frac{(-1)^{2s+1}}{2^{s-1}}, s > 0 \\ \sum_{n=0}^{[s-\frac{1}{2}]} (-1)^n C_{2s}^{2n+1} \Gamma_{k_\zeta k'_\zeta}^{ij \cdots} \overbrace{\pi \cdots \pi}^{2n+1} (s) \overbrace{\partial_i \partial_j \cdots}^{2s-2n-1} \nabla^{2n} \delta^3(\vec{r}-\vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)]_{-2s+1} = 0 \\ [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)]_{-2s+1} = 0 \end{cases}$$

推论6.4.5.

$$\begin{cases} [\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^+(\vec{r}', t)]_{-2s+1} = \frac{(-1)^{2s+1}}{2^{s-1}}, s > 0 \\ \sum_{n=0}^{[s-\frac{1}{2}]} (-1)^n C_{2s}^{2n+1} \Gamma_{k_\zeta k'_\zeta}^{ij \cdots} \overbrace{\pi \cdots \pi}^{2n+1} (s) \overbrace{\partial_i \partial_j \cdots}^{2s-2n-1} \nabla^{2n} \delta^3(\vec{r}-\vec{r}') \\ [\psi_{k_\zeta}(\vec{r}, t), \psi_{l_\zeta}(\vec{r}', t)]_{-2s+1} = 0 \\ [\psi_{k'_\zeta}^+(\vec{r}, t), \psi_{l'_\zeta}^+(\vec{r}', t)]_{-2s+1} = 0 \end{cases} \Rightarrow \begin{cases} [a_\sigma(\vec{p}, -s\zeta), a_{\sigma'}^+(\vec{p}', -s\zeta)]_{-2s+1} \\ = \delta_{\sigma\sigma'} \delta^3(\vec{p}-\vec{p}') \\ [a_\sigma(\vec{p}, -s\zeta), a_{\sigma'}(\vec{p}', -s\zeta)]_{-2s+1} = 0 \\ [a_\sigma^+(\vec{p}, -s\zeta), a_{\sigma'}^+(\vec{p}', -s\zeta)]_{-2s+1} = 0 \end{cases}$$

证明:  $[a_1(\vec{p}, -s\zeta), a_1^+(\vec{p}', -s\zeta)]_{-2s+1}$ 

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \frac{1}{(|\vec{p}||\vec{p}'|)^{(2s-1)/2}} \int [\lambda^{+k_\zeta}(\hat{p}, -s\zeta) \Psi_{k_\zeta}(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r}-Et)}, \lambda^{k'_\zeta}(\vec{p}', -s\zeta) \Psi_{k'_\zeta}^+(\vec{r}', t) e^{i(\vec{p}'\cdot\vec{r}'-E't)}]_{-2s+1} d^3\vec{r} d^3\vec{r}' \\ &= \frac{1}{(2\pi)^3} \frac{1}{(|\vec{p}||\vec{p}'|)^{(2s-1)/2}} \int \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \lambda^{k'_\zeta}(\vec{p}', -s\zeta) [\Psi_{k_\zeta}(\vec{r}, t), \Psi_{k'_\zeta}^+(\vec{r}', t)]_{-2s+1} e^{-i(\vec{p}\cdot\vec{r}-Et)} e^{i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\ &= \frac{1}{(2\pi)^3} \frac{1}{(|\vec{p}||\vec{p}'|)^{(2s-1)/2}} \int d^3\vec{r} d^3\vec{r}' \frac{(-1)^{2s+1}}{2^{s-1}} \\ &\quad \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \lambda^{k'_\zeta}(\vec{p}', -s\zeta) \sum_{n=0}^{[s-\frac{1}{2}]} (-1)^n C_{2s}^{2n+1} \Gamma_{k_\zeta k'_\zeta}^{ij \cdots} \overbrace{\pi \cdots \pi}^{2n+1} (s) \overbrace{\partial_i \partial_j \cdots}^{2s-2n-1} \nabla^{2n} \delta^3(\vec{r}-\vec{r}') e^{-i(\vec{p}\cdot\vec{r}-Et)} e^{i(\vec{p}'\cdot\vec{r}'-E't)} \\ &= \frac{1}{(2\pi)^3} \frac{1}{(|\vec{p}||\vec{p}'|)^{(2s-1)/2}} \int d^3\vec{r}' \frac{(-1)^{2s+1}}{2^{s-1}} \\ &\quad i^{2s-1} \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \lambda^{k'_\zeta}(\vec{p}', -s\zeta) \sum_{n=0}^{[s-\frac{1}{2}]} (-1)^n C_{2s}^{2n+1} \Gamma_{k_\zeta k'_\zeta}^{ij \cdots} \overbrace{\pi \cdots \pi}^{2n+1} (s) \overbrace{p_i p_j \cdots}^{2s-2n-1} \vec{p}^{2n} e^{-i(\vec{p}\cdot\vec{r}-Et)} e^{i(\vec{p}'\cdot\vec{r}'-E't)} \\ &= \frac{(-i)^{2s-1}}{2^{s-1}} \lambda^+(\hat{p}, -s\zeta) \sum_{n=0}^{[s-\frac{1}{2}]} (-1)^n C_{2s}^{2n+1} \Gamma_{k_\zeta k'_\zeta}^{ij \cdots} \overbrace{\pi \cdots \pi}^{2n+1} (s) \overbrace{\vec{p}_i \vec{p}_j \cdots}^{2s-2n-1} \lambda(\vec{p}', -s\zeta) \delta^3(\vec{p}-\vec{p}') \\ &= \frac{(-i)^{2s}}{2^{s-1}} \lambda^+(\hat{p}, -s\zeta) \Gamma_{-}^{abc \cdots} (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \cdots}^{2s} \lambda(\vec{p}', -s\zeta) \delta^3(\vec{p}-\vec{p}') \end{aligned}$$

$$\begin{aligned}
&= \frac{(-i)^{2s}}{2^{s-1}} \frac{1}{2} (i\sqrt{2})^{2s} \lambda^+(\hat{p}, -s\zeta) \lambda(\hat{p}, -s\zeta) \delta^3(\vec{p} - \vec{p}') \\
&= \lambda^+(\hat{p}, -s\zeta) \lambda(\hat{p}, -s\zeta) \delta^3(\vec{p} - \vec{p}') \\
&= \delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

**证明:**  $[a_2^+(\vec{p}, -s\zeta), a_2(\vec{p}', -s\zeta)]_{-2s+1}$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \frac{1}{(|\vec{p}||\vec{p}'|)^{(2s-1)/2}} \int [\lambda^{+k_\zeta}(\hat{p}, -s\zeta) \Psi_{k_\zeta}(\vec{r}, t) e^{i(\vec{p}\cdot\vec{r}-Et)}, \lambda^{k'_\zeta}(\vec{p}', -s\zeta) \Psi_{k'_\zeta}^+(\vec{r}', t) e^{-i(\vec{p}'\cdot\vec{r}'-E't)}]_{-2s+1} d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{(|\vec{p}||\vec{p}'|)^{(2s-1)/2}} \int \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \lambda^{k'_\zeta}(\vec{p}', -s\zeta) [\Psi_{k_\zeta}(\vec{r}, t), \Psi_{k'_\zeta}^+(\vec{r}', t)]_{-2s+1} e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{(2\pi)^3} \frac{1}{(|\vec{p}||\vec{p}'|)^{(2s-1)/2}} \int d^3\vec{r} d^3\vec{r}' \frac{(-1)^{2s+1}}{2^{s-1}} \\
&\lambda^{+k_\zeta}(\hat{p}, -s\zeta) \lambda^{k'_\zeta}(\vec{p}', -s\zeta) \sum_{n=0}^{[s-\frac{1}{2}]} (-1)^n C_{2s}^{2n+1} \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \overbrace{\pi \dots \pi}^{2s-2n-1} (s) \overbrace{\partial_i \partial_j \dots}^{2s-2n-1} \nabla^{2n} \delta^3(\vec{r} - \vec{r}') e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}'-E't)} \\
&= \frac{1}{(2\pi)^3} \frac{1}{(|\vec{p}||\vec{p}'|)^{(2s-1)/2}} \int d^3\vec{r} \frac{(-1)^{2s+1}}{2^{s-1}} i \\
&(-i)^{2s-1} \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \lambda^{k'_\zeta}(\vec{p}', -s\zeta) \sum_{n=0}^{[s-\frac{1}{2}]} (-1)^n C_{2s}^{2n+1} \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \overbrace{\pi \dots \pi}^{2s-2n-1} (s) \overbrace{\hat{p}_i \hat{p}_j \dots}^{2s-2n-1} \vec{p}^{2n} e^{i(\vec{p}\cdot\vec{r}-Et)} e^{-i(\vec{p}'\cdot\vec{r}-E't)} \\
&= \frac{(i)^{2s-1}}{2^{s-1}} \lambda^+(\hat{p}, -s\zeta) \sum_{n=0}^{[s-\frac{1}{2}]} (-1)^n C_{2s}^{2n+1} \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \overbrace{\pi \dots \pi}^{2s-2n-1} (s) \overbrace{\hat{p}_i \hat{p}_j \dots}^{2s-2n-1} \lambda(\vec{p}', -s\zeta) \delta^3(\vec{p} - \vec{p}') \\
&= -\frac{(i)^{2s}}{2^{s-1}} \zeta \lambda^+(\hat{p}, -s\zeta) \overbrace{\Gamma_{-}^{abc \dots}}^{2s} (s) \overbrace{\hat{p}_a \hat{p}_b \hat{p}_c \dots}^{2s} \lambda(\vec{p}', -s\zeta) \delta^3(\vec{p} - \vec{p}') \\
&= -\frac{(i)^{2s}}{2^{s-1}} \frac{1}{2} (i\sqrt{2})^{2s} \lambda^+(\hat{p}, -s\zeta) \lambda(\hat{p}, -s\zeta) \delta^3(\vec{p} - \vec{p}') \\
&= (-1)^{2s+1} \lambda^+(\hat{p}, -s\zeta) \lambda(\hat{p}, -s\zeta) \delta^3(\vec{p} - \vec{p}') \\
&= (-1)^{2s+1} \delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

## 6.5 s-自旋场的对易函数、因果函数和费曼传播子

**引理6.5.1.**  $[\theta(t), \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s}] \Delta^{(+)}(x) - [\theta(-t), \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s}] \Delta^{(-)}(x)$

$$= \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \overbrace{\pi \dots \pi}^{2s-n} (s) [\partial_t^{2s-1-n} \delta(t)] \overbrace{\partial_i \partial_j \dots}^n \Delta(x)$$

**证明:**  $[\theta(t), \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s}]$

$$\begin{aligned}
&= \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) [\theta(t), \overbrace{\partial_a \partial_b \partial_c \dots}^{2s}] \\
&= \frac{(-1)^{2s-1}}{2^{s-1}} \sum_{n=0}^{2s-1} C_{2s}^n \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \overbrace{\pi \dots \pi}^{2s-n} (s) [\partial_\pi^{2s-n} \theta(t)] \overbrace{\partial_i \partial_j \dots}^n \\
&= \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \overbrace{\pi \dots \pi}^{2s-n} (s) [\partial_t^{2s-n} \theta(t)] \overbrace{\partial_i \partial_j \dots}^n
\end{aligned}$$

□

**证明:**  $[\theta(-t), \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s}]$

$$\begin{aligned}
&= \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) [\theta(-t), \overbrace{\partial_a \partial_b \partial_c \dots}^{2s}] \\
&= \frac{(-1)^{2s-1}}{2^{s-1}} \sum_{n=0}^{2s-1} C_{2s}^n \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \overbrace{\pi \dots \pi}^{2s-n} (s) [\partial_\pi^{2s-n} \theta(-t)] \overbrace{\partial_i \partial_j \dots}^n \\
&= \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \Gamma_{k_\zeta k'_\zeta}^{ij \dots} \overbrace{\pi \dots \pi}^{2s-n} (s) [\partial_t^{2s-n} \theta(-t)] \overbrace{\partial_i \partial_j \dots}^n
\end{aligned}$$

□

**证明:**  $[\theta(t), \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s}] \Delta^{(+)}(x) - [\theta(-t), \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s}] \Delta^{(-)}(x)$

$$= [\theta(t), \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s}] \Delta(x) - [\theta(t) + \theta(-t), \frac{(-1)^{2s}}{2^{s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s}] \Delta^{(-)}(x)$$

$$\begin{aligned}
&= [\theta(t), \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s}] \Delta(x) \\
&= \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^m \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij \cdots}}^n \overbrace{\pi \cdots}^{2s-n}(s) [\partial_t^{2s-n} \theta(t)] \overbrace{\partial_i \partial_j \cdots}^n \Delta(x) \\
&= \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^m \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij \cdots}}^n \overbrace{\pi \cdots}^{2s-n}(s) [\partial_t^{2s-1-n} \delta(t)] \overbrace{\partial_i \partial_j \cdots}^n \Delta(x)
\end{aligned}$$

□

引理6.5.2.  $[sp_a + iS_{ab}(s, \varsigma)p^b] \lambda(\hat{p}, -s\varsigma) = 0$

$$\begin{aligned}
&\text{证明: } [sp_a + iS_{ab}(s, \varsigma)p^b] \lambda(\hat{p}, -s\varsigma) \\
&= |\vec{p}| \left\{ s \left[ \exp\left\{ i \frac{(R \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z \right\} \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} \right]_a + iS_{ab}(s, \varsigma) \left[ \exp\left\{ i \frac{(R \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z \right\} \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} \right]^b \right\} \\
&\exp\left\{ i \frac{[\sigma(2) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z \right\} \lambda \left( \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}, -s\varsigma \right) \\
&= \exp\left\{ i \frac{(R \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z \right\} |a^c \exp\left\{ i \frac{[\sigma(2) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z \right\} |\vec{p}| \left[ s \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}_c + iS_{cd}(s, \varsigma) \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}^d \right] \lambda \left( \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix}, -s\varsigma \right) \\
&= \exp\left\{ i \frac{(R \times \hat{p})_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z \right\} |a^c \exp\left\{ i \frac{[\sigma(2) \times \hat{p}]_z}{\sqrt{1-\hat{p}_z^2}} \arccos \hat{p}_z \right\} |\vec{p}| \cdot 0 \\
&= 0
\end{aligned}$$

□

引理6.5.3.  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]_{j_\zeta}^{k_\zeta} \Delta_{k_\zeta k'_\zeta}(s; x) = 0, [s\partial_a + iS_{ab}(s, \varsigma)\partial^b] \Delta(s; x) = 0$

$$\begin{aligned}
&\text{证明: } [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]_{j_\zeta}^{k_\zeta} \Delta_{k_\zeta k'_\zeta}(s; x) \\
&= \left( \frac{-1}{\sqrt{2}} \right)^{2(s-1)} [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]_{j_\zeta}^{k_\zeta} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x) \\
&= \left( \frac{-1}{\sqrt{2}} \right)^{2(s-1)} i^{2s+1} \frac{-i}{(2\pi)^3} \int [sp_a + iS_{ab}(s, \varsigma)p^b]_{j_\zeta}^{k_\zeta} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{p_a p_b p_c \cdots}^{2s} \frac{1}{2|\vec{p}|} [e^{ip \cdot x} - (-1)^{2s+1} e^{-ip \cdot x}] d^3 \vec{p} \\
&= \left( -\frac{1}{2} \right)^{2s-1} \frac{-1}{(2\pi)^3} \int [sp_a + iS_{ab}(s, \varsigma)p^b]_{j_\zeta}^{k_\zeta} \lambda_{k_\zeta}(\hat{p}, -s\varsigma) \lambda_{k'_\zeta}^+(\hat{p}, -s\varsigma) \overbrace{p_a p_b p_c \cdots}^{2s} \frac{1}{2|\vec{p}|^{2s+1}} [e^{ip \cdot x} - (-1)^{2s+1} e^{-ip \cdot x}] d^3 \vec{p} \\
&= \left( -\frac{1}{2} \right)^{2s-1} \frac{-1}{(2\pi)^3} \int 0 \cdot \lambda_{k'_\zeta}^+(\hat{p}, -s\varsigma) \overbrace{p_a p_b p_c \cdots}^{2s} \frac{1}{2|\vec{p}|^{2s+1}} [e^{ip \cdot x} - (-1)^{2s+1} e^{-ip \cdot x}] d^3 \vec{p} \\
&= 0
\end{aligned}$$

□

定义6.5.1.

$$\begin{cases} \Delta(x) = \Delta^{(+)}(x) + \Delta^{(-)}(x) = \Delta^{ret}(x) - \Delta^{adv}(x) \\ \Delta^{(l)}(x) = i[\Delta^{(-)}(x) - \Delta^{(+)}(x)] \\ \Delta_F(x) = \langle T\varphi(x)\varphi(x') \rangle_0 = i\Delta^{(c)}(x - x') \end{cases} \quad \begin{cases} \Delta^{(c)}(x) = \theta(t)\Delta^{(+)}(x) - \theta(-t)\Delta^{(-)}(x) \\ \Delta^{ret}(x) = \theta(t)\Delta(x) = \Delta^{(c)}(x) + \Delta^{(-)}(x) \\ \Delta^{adv}(x) = -\theta(-t)\Delta(x) = \Delta^{(c)}(x) - \Delta^{(+)}(x) \end{cases}$$

推论6.5.1.

$$\begin{cases} \Delta_{k_\zeta k'_\zeta}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{(+)}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(+)}(x) \\ \Delta_{k_\zeta k'_\zeta}^{(-)}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(-)}(x) \\ \Delta_{k_\zeta k'_\zeta}^{(l)}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(l)}(x) \end{cases}$$

推论6.5.2.

$$\begin{cases}
\Delta_{k_\zeta k'_\zeta}^{(c)}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(c)}(x) + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij \cdots \pi \cdots}}^{n \quad 2s-n}(s) [\partial_t^{2s-1-n} \delta(t)] \overbrace{\partial_i \partial_j \cdots}^n \Delta(x) \\
\Delta_{k_\zeta k'_\zeta}^{ret}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{ret}(x) + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij \cdots \pi \cdots}}^{n \quad 2s-n}(s) [\partial_t^{2s-1-n} \delta(t)] \overbrace{\partial_i \partial_j \cdots}^n \Delta(x) \\
\Delta_{k_\zeta k'_\zeta}^{adv}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{adv}(x) + \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij \cdots \pi \cdots}}^{n \quad 2s-n}(s) [\partial_t^{2s-1-n} \delta(t)] \overbrace{\partial_i \partial_j \cdots}^n \Delta(x) \\
\Delta_{F k_\zeta k'_\zeta}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta_F(x) + \frac{i^{2s-1}}{2^{s-1}} \sum_{n=0}^{2s-1} i^n C_{2s}^n \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij \cdots \pi \cdots}}^{n \quad 2s-n}(s) [\partial_t^{2s-1-n} \delta(t)] \overbrace{\partial_i \partial_j \cdots}^n \Delta(x) \\
= i \Delta_{k_\zeta k'_\zeta}^{(c)}(s; x)
\end{cases}$$

推论6.5.3.

$$\begin{cases}
\Delta_{k_\zeta k'_\zeta}^{(c)}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{(c)}(x) \\
+ \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{2s-2-[(2s-2-n)/2]} i^n C_{2s}^n C_{2s-1-n}^{2l+1} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij \cdots \pi \cdots}}^{n \quad 2s-n}(s) \overbrace{\partial_i \partial_j \cdots}^n \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\
\Delta_{k_\zeta k'_\zeta}^{ret}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{ret}(x) \\
+ \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{2s-2-[(2s-2-n)/2]} i^n C_{2s}^n C_{2s-1-n}^{2l+1} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij \cdots \pi \cdots}}^{n \quad 2s-n}(s) \overbrace{\partial_i \partial_j \cdots}^n \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\
\Delta_{k_\zeta k'_\zeta}^{adv}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta^{adv}(x) \\
+ \frac{i^{2s-2}}{2^{s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{2s-2-[(2s-2-n)/2]} i^n C_{2s}^n C_{2s-1-n}^{2l+1} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij \cdots \pi \cdots}}^{n \quad 2s-n}(s) \overbrace{\partial_i \partial_j \cdots}^n \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\
\Delta_{F k_\zeta k'_\zeta}(s; x) = i \Delta_{k_\zeta k'_\zeta}^{(c)}(s; x) := \frac{(-1)^{2s}}{2^{s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta_F(x) \\
+ \frac{i^{2s-1}}{2^{s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{2s-2-[(2s-2-n)/2]} i^n C_{2s}^n C_{2s-1-n}^{2l+1} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{ij \cdots \pi \cdots}}^{n \quad 2s-n}(s) \overbrace{\partial_i \partial_j \cdots}^n \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\
\Delta_{F k_\zeta k'_\zeta}(s; p) = \frac{(-i)^{2s+1}}{2^{s-1}} \frac{\overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{p_a p_b p_c \cdots}^{2s}}{p^2 - i\varepsilon} + \cdots
\end{cases}$$

引理6.5.4.  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\theta(t) = -\varsigma[\sigma(s), i s \varsigma]_a \delta(t)$

证明:  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\theta(t)$

$$= [-i s \delta_{a4} + S_{a4}(s, \varsigma)]\delta(t) = [-i s \delta_{a4} - \sigma_a(s)]\delta(t) = -\varsigma[\sigma(s), i s \varsigma]_a \delta(t) \quad \square$$

引理6.5.5.  $\frac{1}{\sqrt{-\nabla^2}} \delta^3(\vec{r}) = 2\Delta^{(+)}(x)|_{t=0} = 2\Delta^{(-)}(x)|_{t=0}$

引理6.5.6.  $[sD_a + iS_{ab}(s, \varsigma)D^b]\psi(s, \varsigma) = -\sqrt{2}\varsigma s \bar{Z}_a(s, \varsigma) \tilde{J}(s) \Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a \tilde{\psi}(s, \varsigma) = i \tilde{J}(s, \varsigma)$

推论6.5.4.

$$\begin{cases}
[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\Delta(s; x) = 0 \\
[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\Delta^{(+)}(s; x) = 0 \\
[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\Delta^{(-)}(s; x) = 0 \\
[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\Delta^{(l)}(s; x) = 0
\end{cases}
\begin{cases}
[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\Delta^{(c)}(s; x) = -\varsigma[\sigma(s), i s \varsigma]_a \delta(t) \Delta(s; x)|_{t=0} \\
= -\sqrt{2}\varsigma s [\frac{-i\varsigma}{\sqrt{2}} \bar{N}(s)(\sigma, i\varsigma)_a] i \varsigma \delta(t) N(s) \Delta(s; x)|_{t=0} \\
[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\Delta^{ret}(s; x) = -\varsigma[\sigma(s), i s \varsigma]_a \delta(t) \Delta(s; x)|_{t=0} \\
[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\Delta^{adv}(s; x) = -\varsigma[\sigma(s), i s \varsigma]_a \delta(t) \Delta(s; x)|_{t=0} \\
[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\Delta_F(s; x) = -i\varsigma[\sigma(s), i s \varsigma]_a \delta(t) \Delta(s; x)|_{t=0}
\end{cases}$$

(\Downarrow)

(\Downarrow)

推论6.5.5.

$$\begin{cases} (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta(s; x) = 0 \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{(+)}(s; x) = 0 \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{(-)}(s; x) = 0 \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{(l)}(s; x) = 0 \end{cases} \quad \begin{cases} (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{(c)}(s; x) = -\varsigma \delta(t) \Gamma(s) \Delta(s; x)|_{t=0} \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{ret}(s; x) = -\varsigma \delta(t) \Gamma(s) \Delta(s; x)|_{t=0} \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{adv}(s; x) = -\varsigma \delta(t) \Gamma(s) \Delta(s; x)|_{t=0} \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta_F(s; x) = -i\varsigma \delta(t) \Gamma(s) \Delta(s; x)|_{t=0} \end{cases}$$

[\Uparrow] \qquad \qquad \qquad [\Downarrow]

推论6.5.6.

$$\begin{cases} (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta(s; x) = 0 \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{(+)}(s; x) = 0 \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{(-)}(s; x) = 0 \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{(l)}(s; x) = 0 \end{cases} \quad \begin{cases} (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{(c)}(s; x) = -\varsigma \delta(t) N(s) \Delta(s; x)|_{t=0} \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{ret}(s; x) = -\varsigma \delta(t) N(s) \Delta(s; x)|_{t=0} \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{adv}(s; x) = -\varsigma \delta(t) N(s) \Delta(s; x)|_{t=0} \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta_F(s; x) = -i\varsigma \delta(t) N(s) \Delta(s; x)|_{t=0} \end{cases}$$

[\Uparrow] \qquad \qquad \qquad [\Downarrow]

推论6.5.7.

$$\begin{cases} (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta(s; x) \bar{\Gamma}(s) = 0 \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{(+)}(s; x) \bar{\Gamma}(s) = 0 \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{(-)}(s; x) \bar{\Gamma}(s) = 0 \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{(l)}(s; x) \bar{\Gamma}(s) = 0 \end{cases} \quad \begin{cases} (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{(c)}(s; x) \bar{\Gamma}(s) = -\varsigma \delta(t) \Gamma(s) \Delta(s; x)|_{t=0} \bar{\Gamma}(s) \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{ret}(s; x) \bar{\Gamma}(s) = -\varsigma \delta(t) \Gamma(s) \Delta(s; x)|_{t=0} \bar{\Gamma}(s) \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta^{adv}(s; x) \bar{\Gamma}(s) = -\varsigma \delta(t) \Gamma(s) \Delta(s; x)|_{t=0} \bar{\Gamma}(s) \\ (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a \partial^a \Gamma(s) \Delta_F(s; x) \bar{\Gamma}(s) = -i\varsigma \delta(t) \Gamma(s) \Delta(s; x)|_{t=0} \bar{\Gamma}(s) \end{cases}$$

[\Uparrow] \qquad \qquad \qquad [\Downarrow]

推论6.5.8.

$$\begin{cases} (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta(s; x) \bar{N}(s) = 0 \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{(+)}(s; x) \bar{N}(s) = 0 \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{(-)}(s; x) \bar{N}(s) = 0 \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{(l)}(s; x) \bar{N}(s) = 0 \end{cases} \quad \begin{cases} (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{(c)}(s; x) \bar{N}(s) = -\varsigma \delta(t) N(s) \Delta(s; x)|_{t=0} \bar{N}(s) \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{ret}(s; x) \bar{N}(s) = -\varsigma \delta(t) N(s) \Delta(s; x)|_{t=0} \bar{N}(s) \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta^{adv}(s; x) \bar{N}(s) = -\varsigma \delta(t) N(s) \Delta(s; x)|_{t=0} \bar{N}(s) \\ (\sigma \otimes I_{2s}, -i\varsigma)_a \partial^a N(s) \Delta_F(s; x) \bar{N}(s) = -i\varsigma \delta(t) N(s) \Delta(s; x)|_{t=0} \bar{N}(s) \end{cases}$$

[\Downarrow] \qquad \qquad \qquad [\Downarrow]

推论6.5.9.

$$\begin{cases} [\sigma(s), -i\varsigma]_a \partial^a \Delta(s; x) = 0 \\ [\sigma(s), -i\varsigma]_a \partial^a \Delta^{(+)}(s; x) = 0 \\ [\sigma(s), -i\varsigma]_a \partial^a \Delta^{(-)}(s; x) = 0 \\ [\sigma(s), -i\varsigma]_a \partial^a \Delta^{(l)}(s; x) = 0 \end{cases} \quad \begin{cases} [\sigma(s), -i\varsigma]_a \partial^a \Delta^{(c)}(s; x) = -\varsigma \delta(t) \Delta(s; x)|_{t=0} \\ [\sigma(s), -i\varsigma]_a \partial^a \Delta^{ret}(s; x) = -\varsigma \delta(t) \Delta(s; x)|_{t=0} \\ [\sigma(s), -i\varsigma]_a \partial^a \Delta^{adv}(s; x) = -\varsigma \delta(t) \Delta(s; x)|_{t=0} \\ [\sigma(s), -i\varsigma]_a \partial^a \Delta_F(s; x) = -i\varsigma \delta(t) \Delta(s; x)|_{t=0} \end{cases}$$

## 6.6 s-自旋场能量动量算符的提取

$$\text{推论6.6.1.} \quad \begin{cases} \psi(\vec{r}, t) := \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -\varsigma) [a_1(\vec{p}, -\varsigma) e^{ip \cdot x} + a_2^+(\vec{p}, -\varsigma) e^{-ip \cdot x}] d^3 \vec{p} \\ |\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -\varsigma) \psi(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r} \\ |\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}, -\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -\varsigma) \psi(\vec{r}, t) e^{ip \cdot x} d^3 \vec{r} \end{cases}$$

引理6.6.1.

$$\Gamma_{\underset{2s-n}{k_\varsigma k'_\varsigma}}^{i_j \dots \pi \dots \pi} (s) \underbrace{\partial_i \partial_j \dots}_{2s-n} \psi(\vec{r}, t) = \left(\frac{1}{\sqrt{2}}\right)^{2s} \delta_{k_\varsigma k'_\varsigma} \partial_\pi^{2s-n} \psi(\vec{r}, t), \Gamma_{\underset{2s-n}{i_j \dots \pi \dots \pi}}^{k'_\varsigma k_\varsigma} (s) \overbrace{\partial^i \partial^j \dots}^{2s-n} \psi(\vec{r}, t) = \left(\frac{1}{\sqrt{2}}\right)^{2s} \delta^{k'_\varsigma k_\varsigma} \partial_\pi^{2s-n} \psi(\vec{r}, t)$$



$$\text{引理6.6.2. } \Gamma_{k_\zeta k'_\zeta}^{ij \dots \pi \dots \pi} (s) \underbrace{\partial_i \partial_j \dots \partial_\pi}_{2s-n} \psi(\vec{r}, t) = \left(\frac{1}{\sqrt{2}}\right)^{2s} \delta_{k_\zeta k'_\zeta} \partial_\pi^{2s} \psi(\vec{r}, t)$$

$$\text{性质6.6.1. } \begin{cases} \Gamma_{abc \dots}^{2s} (s) \overbrace{p_a p_b p_c \dots}^{2s} = \sum_{n=0}^{2s} C_{2s}^n \Gamma_{ij \dots \pi \dots \pi}^{2s-n \ n} (s) \overbrace{p_i p_j \dots p_\pi}^{2s-n} p_\pi^n \\ \Gamma_{abc \dots}^{2s} (s) \overbrace{\partial_a \partial_b \partial_c \dots}^{2s} = \sum_{n=0}^{2s} C_{2s}^n \Gamma_{ij \dots \pi \dots \pi}^{2s-n \ n} (s) \overbrace{\partial_i \partial_j \dots \partial_\pi}^{2s-n} \end{cases}$$

定理6.6.1.

$$H(s) = \int |\vec{p}| [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] d^3 \vec{p} = \int \psi^+(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3 \vec{r}$$

$$\text{证明: } H(s) = \int |\vec{p}| [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] d^3 \vec{p}$$

$$= \frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|^{2s-2}} [\lambda^{k'_\zeta}(\hat{p}, -s\zeta) \psi_{k'_\zeta}^+(\vec{r}', t) e^{ip \cdot x'} \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \psi_{k_\zeta}(\vec{r}, t) e^{-ip \cdot x}$$

$$+ (-1)^{2s} \lambda^{k'_\zeta}(\hat{p}, -s\zeta) \psi_{k'_\zeta}^+(\vec{r}', t) e^{-ip \cdot x'} \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \psi_{k_\zeta}(\vec{r}, t) e^{ip \cdot x}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|^{2s-2}} \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \lambda^{k'_\zeta}(\hat{p}, -s\zeta) \psi_{k_\zeta}(\vec{r}, t) \psi_{k'_\zeta}^+(\vec{r}', t) [e^{-i\vec{p} \cdot (\vec{r} - \vec{r}')} + (-1)^{2s} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} ] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= (i\sqrt{2})^{-2s} \frac{1}{(2\pi)^3} \int \frac{1}{|\vec{p}|^{2s-2}} \psi_{k'_\zeta}^+(\vec{r}', t) \Gamma_{abc \dots}^{k'_\zeta k_\zeta} (s) \underbrace{\hat{p}^a \hat{p}^b \hat{p}^c \dots}_{2s} \psi_{k_\zeta}(\vec{r}, t) [e^{-i\vec{p} \cdot (\vec{r} - \vec{r}')} + (-1)^{2s} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} ] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= (-i\sqrt{2})^{-2s} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+(\vec{r}', t) \psi_{k_\zeta}(\vec{r}, t) \frac{1}{|\vec{p}|^{4s-2}} \Gamma_{abc \dots}^{k'_\zeta k_\zeta} (s) \underbrace{(p^a p^b p^c \dots + p^{+a} p^{+b} p^{+c} \dots)}_{2s} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= (-i\sqrt{2})^{-2s} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+(\vec{r}', t) \psi_{k_\zeta}(\vec{r}, t) \frac{1}{|\vec{p}|^{4s-2}} \sum_{n=0}^{2s} C_{2s}^n \Gamma_{ij \dots \pi \dots \pi}^{k'_\zeta k_\zeta} (s) \underbrace{(p^i p^j \dots p_\pi^n + p^i p^j \dots p_\pi^{+n})}_{2s-n} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= (\sqrt{2})^{-2s} \int \psi_{k'_\zeta}^+(\vec{r}', t) \psi_{k_\zeta}(\vec{r}, t) \frac{1}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n (\sqrt{-\nabla^2})^n \Gamma_{ij \dots \pi \dots \pi}^{k'_\zeta k_\zeta} (s) \underbrace{\partial^i \partial^j \dots}_{2s-n} [1 + (-1)^n] \delta^3(\vec{r} - \vec{r}') d^3 \vec{r} d^3 \vec{r}'$$

$$= (-\sqrt{2})^{-2s} \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{1}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n (\sqrt{-\nabla^2})^n \Gamma_{ij \dots \pi \dots \pi}^{k'_\zeta k_\zeta} (s) \underbrace{\partial^i \partial^j \dots}_{2s-n} [1 + (-1)^n] \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r}$$

$$= \left(\frac{-1}{\sqrt{2}}\right)^{2s} (\sqrt{2})^{-2s} \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{1}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n (\sqrt{-\nabla^2})^n \delta^{k'_\zeta k_\zeta} \partial_\pi^{2s-n} [1 + (-1)^n] \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r}$$

$$= \left(\frac{-1}{\sqrt{2}}\right)^{2s} (\sqrt{2})^{-2s} \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{1}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n \partial_\pi^n \delta^{k'_\zeta k_\zeta} \partial_\pi^{2s-n} [1 + (-1)^n] \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r}$$

$$= \frac{1}{(-2)^{2s}} \int \psi^{+k_\zeta}(\vec{r}, t) \frac{1}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n (-i\partial_t)^n (-i\partial_t)^{2s-n} [1 + (-1)^n] \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r}$$

$$= \frac{1}{(-2)^{2s}} \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{(-i\partial_t)^{2s}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n [1 + (-1)^n] \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r}$$

$$= \frac{1}{(-2)^{2s}} \int \psi^{+k_\zeta}(\vec{r}, t) \frac{(-i\partial_t)^{2s}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n [1 + (-1)^n] \psi_{k_\zeta}(\vec{r}, t) d^3 \vec{r}$$

$$= \int \psi^+(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3 \vec{r} \quad \square$$

定理6.6.2.

$$P(s) = \int \vec{p} [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] d^3 \vec{p} = \int \psi^+(\vec{r}, t) \frac{-i\nabla(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3 \vec{r}$$

$$\text{证明: } P(s) = \int \vec{p} [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] d^3 \vec{p}$$

$$= \frac{1}{(2\pi)^3} \int \frac{\vec{p}}{|\vec{p}|^{2s-2}} [\lambda^{k'_\zeta}(\hat{p}, -s\zeta) \psi_{k'_\zeta}^+(\vec{r}', t) e^{ip \cdot x'} \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \psi_{k_\zeta}(\vec{r}, t) e^{-ip \cdot x}$$

$$+ (-1)^{2s} \lambda^{k'_\zeta}(\hat{p}, -s\zeta) \psi_{k'_\zeta}^+(\vec{r}', t) e^{-ip \cdot x'} \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \psi_{k_\zeta}(\vec{r}, t) e^{ip \cdot x}] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= \frac{1}{(2\pi)^3} \int \frac{\vec{p}}{|\vec{p}|^{2s-2}} \lambda^{+k_\zeta}(\hat{p}, -s\zeta) \lambda^{k'_\zeta}(\hat{p}, -s\zeta) \psi_{k_\zeta}(\vec{r}, t) \psi_{k'_\zeta}^+(\vec{r}', t) [e^{-i\vec{p} \cdot (\vec{r} - \vec{r}')} + (-1)^{2s} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} ] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= (i\sqrt{2})^{-2s} \frac{1}{(2\pi)^3} \int \frac{\vec{p}}{|\vec{p}|^{2s-2}} \psi_{k'_\zeta}^+(\vec{r}', t) (\Gamma_{abc \dots}^{k'_\zeta k_\zeta} (s) \underbrace{\hat{p}^a \hat{p}^b \hat{p}^c \dots}_{2s} \psi_{k_\zeta}(\vec{r}, t) [e^{-i\vec{p} \cdot (\vec{r} - \vec{r}')} + (-1)^{2s} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} ] d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$= (-i\sqrt{2})^{-2s} \frac{1}{(2\pi)^3} \int \psi_{k'_\zeta}^+(\vec{r}', t) \psi_{k_\zeta}(\vec{r}, t) \frac{\vec{p}}{|\vec{p}|^{4s-2}} \Gamma_{abc \dots}^{k'_\zeta k_\zeta} (s) \underbrace{(p^a p^b p^c \dots - p^{+a} p^{+b} p^{+c} \dots)}_{2s} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}'$$

$$\begin{aligned}
&= (-i\sqrt{2})^{-2s} \frac{1}{(2\pi)^3} \int \psi_{k'_c}^+(\vec{r}', t) \psi_{k_c}(\vec{r}, t) \frac{\hat{p}}{|\vec{p}|^{4s-2}} \sum_{n=0}^{2s} C_{2s}^n \Gamma_{ij \dots \pi \dots \pi}^{k'_c k_c} (s) \overbrace{(p^i p^j \dots p_\pi^n - p^i p^j \dots p_\pi^{+n})}^{2s-n} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{p} d^3 \vec{r} d^3 \vec{r}' \\
&= (\sqrt{2})^{-2s} \int \psi_{k'_c}^+(\vec{r}', t) \psi_{k_c}(\vec{r}, t) \frac{\hat{V}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n (\sqrt{-\nabla^2})^n \Gamma_{ij \dots \pi \dots \pi}^{k'_c k_c} (s) \overbrace{\partial^i \partial^j \dots}^{2s-n} [(-1)^n - 1] \delta^3(\vec{r} - \vec{r}') d^3 \vec{r} d^3 \vec{r}' \\
&= (-\sqrt{2})^{-2s} \int \psi_{k'_c}^+(\vec{r}', t) \frac{\hat{V}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n (\sqrt{-\nabla^2})^n \Gamma_{ij \dots \pi \dots \pi}^{k'_c k_c} (s) \overbrace{\partial^i \partial^j \dots}^{2s-n} [(-1)^n - 1] \psi_{k_c}(\vec{r}, t) d^3 \vec{r}' \\
&= \left(\frac{1}{\sqrt{2}}\right)^{2s} (-\sqrt{2})^{-2s} \int \psi_{k'_c}^+(\vec{r}, t) \frac{\hat{V}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n (\sqrt{-\nabla^2})^n \delta^{k'_c k_c} \partial_\pi^{2s-n} [(-1)^n - 1] \psi_{k_c}(\vec{r}, t) d^3 \vec{r}' \\
&= \frac{1}{(-2)^{2s}} \int \psi^{+k_c}(\vec{r}, t) \frac{\hat{V}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n \sqrt{-\nabla^2} (-i\partial_t)^{n-1} (-i\partial_t)^{2s-n} [(-1)^n - 1] \psi_{k_c}(\vec{r}, t) d^3 \vec{r}' \\
&= \frac{1}{(-2)^{2s}} \int \psi_{k'_c}^+(\vec{r}, t) \frac{-i\nabla(-i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n [(-1)^n - 1] \psi_{k_c}(\vec{r}, t) d^3 \vec{r}' \\
&= \frac{1}{(-2)^{2s}} \int \psi^{+k_c}(\vec{r}, t) \frac{-i\nabla(-i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \sum_{n=0}^{2s} C_{2s}^n [(-1)^n - 1] \psi_{k_c}(\vec{r}, t) d^3 \vec{r}' \\
&= \int \psi^+(\vec{r}, t) \frac{-i\nabla(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3 \vec{r}' \quad \square
\end{aligned}$$

定理6.6.3.

$$P_u(s) = \int p_u [a_1^+(\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s} a_2(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] d^3 \vec{p} = \int \psi^+(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3 \vec{r}'$$

## 6.7 s-自旋场方程的各种物理算符

$$\text{推论6.7.1.} \quad \begin{cases} \psi(\vec{r}, t) := \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -s_\zeta) [a_1(\vec{p}, -s_\zeta) e^{i\vec{p} \cdot \vec{x}} + a_2^+(\vec{p}, -s_\zeta) e^{-i\vec{p} \cdot \vec{x}}] d^3 \vec{p} \\ |\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -s_\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -s_\zeta) \psi(\vec{r}, t) e^{-i\vec{p} \cdot \vec{x}} d^3 \vec{r}' \\ |\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}, -s_\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -s_\zeta) \psi(\vec{r}, t) e^{i\vec{p} \cdot \vec{x}} d^3 \vec{r}' \end{cases}$$

定理6.7.1.

$$P_u(s) = \int \psi^+(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3 \vec{r}' = \int p_u [a^+(\vec{p}, -s_\zeta) a(\vec{p}, -s_\zeta) + (-1)^{2s} b(\vec{p}, -s_\zeta) b^+(\vec{p}, -s_\zeta)] d^3 \vec{p}$$

$$\begin{aligned}
\text{证明: } P_u(s) &= \int \psi^+(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r}' |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^+(\hat{p}', -s_\zeta) \lambda(\hat{p}, -s_\zeta) \frac{p_u}{|\vec{p}|^{2s-1}} \\
&\quad [a_1^+(\vec{p}', -s_\zeta) e^{-i(\vec{p}' \cdot \vec{r}' - |\vec{p}'|t)} + a_2(\vec{p}', -s_\zeta) e^{i(\vec{p}' \cdot \vec{r}' - |\vec{p}'|t)}] [a_1(\vec{p}, -s_\zeta) e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + (-1)^{2s} a_2^+(\vec{p}, -s_\zeta) e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] \\
&= \int |\vec{p}'|^{2s-1} \lambda^+(\hat{p}', -s_\zeta) \lambda(\hat{p}, -s_\zeta) \frac{p_u}{|\vec{p}|^{2s-1}} \{ [a_1^+(\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s} a_2(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] \delta^3(\vec{p}' - \vec{p}) \\
&\quad + [(-1)^{2s} a_1^+(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta) e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) e^{2i|\vec{p}|t}] \delta^3(\vec{p}' + \vec{p}) \} d^3 \vec{p}' d^3 \vec{p} \\
&= \int p_u [a_1^+(\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s} a_2(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] d^3 \vec{p} \quad \square
\end{aligned}$$

$$\text{定理6.7.2. } Q(s) = \int \psi^+(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3 \vec{r}' = \int [a^+(\vec{p}, -s_\zeta) a(\vec{p}, -s_\zeta) + (-1)^{2s-1} b(\vec{p}, -s_\zeta) b^+(\vec{p}, -s_\zeta)] d^3 \vec{p}$$

$$\begin{aligned}
\text{证明: } Q(s) &= \int \psi^+(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r}' |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^+(\hat{p}', -s_\zeta) \lambda(\hat{p}, -s_\zeta) \frac{1}{|\vec{p}|^{2s-1}} \\
&\quad [a_1^+(\vec{p}', -s_\zeta) e^{-i(\vec{p}' \cdot \vec{r}' - |\vec{p}'|t)} + a_2(\vec{p}', -s_\zeta) e^{i(\vec{p}' \cdot \vec{r}' - |\vec{p}'|t)}] [a_1(\vec{p}, -s_\zeta) e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + (-1)^{2s-1} a_2^+(\vec{p}, -s_\zeta) e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] \\
&= \int |\vec{p}'|^{2s-1} \lambda^+(\hat{p}', -s_\zeta) \lambda(\hat{p}, -s_\zeta) \frac{1}{|\vec{p}|^{2s-1}} \{ [a_1^+(\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s-1} a_2(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] \delta^3(\vec{p}' - \vec{p}) \\
&\quad + [(-1)^{2s-1} a_1^+(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta) e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) e^{2i|\vec{p}|t}] \delta^3(\vec{p}' + \vec{p}) \} d^3 \vec{p}' d^3 \vec{p} \\
&= \int [a_1^+(\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s-1} a_2(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] d^3 \vec{p} \quad \square
\end{aligned}$$

$$\text{定理6.7.3. } N(s) = \int \psi^+(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \psi(\vec{r}, t) d^3 \vec{r}' = \int [a^+(\vec{p}, -s_\zeta) a(\vec{p}, -s_\zeta) + (-1)^{2s} b(\vec{p}, -s_\zeta) b^+(\vec{p}, -s_\zeta)] d^3 \vec{p}$$

$$\begin{aligned}
\text{证明: } N(s) &= \int \psi^+(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \psi(\vec{r}, t) d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r}' |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^+(\hat{p}', -s_\zeta) \lambda(\hat{p}, -s_\zeta) \frac{1}{|\vec{p}|^{2s-1}}
\end{aligned}$$

$$\begin{aligned}
& [a_1^+(\vec{p}', -s\zeta)e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)} + a_2(\vec{p}', -s\zeta)e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p}, -s\zeta)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)} + (-1)^{2s}a_2^+(\vec{p}, -s\zeta)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}] \\
& = \int \vec{p}'^{2s-1}\lambda^+(\hat{p}', -s\zeta)\lambda(\hat{p}, -s\zeta)\frac{1}{|\vec{p}'|^{2s-1}}\{[a_1^+(\vec{p}, -s\zeta)a_1(\vec{p}, -s\zeta) + (-1)^{2s}a_2(\vec{p}, -s\zeta)a_2^+(\vec{p}, -s\zeta)]\delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s}a_1^+(\vec{p}, -s\zeta)a_2^+(\vec{p}, -s\zeta)e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s\zeta)a_1(\vec{p}, -s\zeta)e^{2i|\vec{p}|t}]\delta^3(\vec{p}' + \vec{p})\}d^3\vec{p}'d^3\vec{p} \\
& = \int [a_1^+(\vec{p}, -s\zeta)a_1(\vec{p}, -s\zeta) + (-1)^{2s}a_2(\vec{p}, -s\zeta)a_2^+(\vec{p}, -s\zeta)]d^3\vec{p}
\end{aligned}$$

□

$$\text{定理6.7.4. } \vec{S}(s) = \int \psi^+(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3\vec{r} = \int \hat{p}[a^+(\vec{p}, -s\zeta)a(\vec{p}, -s\zeta) + (-1)^{2s-1}b(\vec{p}, -s\zeta)b^+(\vec{p}, -s\zeta)]d^3\vec{p}$$

$$\text{证明: } \vec{S}(s) = \int \psi^+(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned}
& = \frac{1}{(2\pi)^3} \int d^3\vec{p}'d^3\vec{p}d^3\vec{r}|\vec{p}'|^{s-\frac{1}{2}}|\vec{p}|^{s-\frac{1}{2}}\lambda^+(\hat{p}', -s\zeta)\lambda(\hat{p}, -s\zeta)\frac{\hat{p}}{|\vec{p}|^{2s-1}} \\
& [a_1^+(\vec{p}', -s\zeta)e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)} + a_2(\vec{p}', -s\zeta)e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p}, -s\zeta)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)} + (-1)^{2s}a_2^+(\vec{p}, -s\zeta)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}] \\
& = \int \vec{p}'^{2s-1}\lambda^+(\hat{p}', -s\zeta)\lambda(\hat{p}, -s\zeta)\frac{\hat{p}}{|\vec{p}|^{2s-1}}\{[a_1^+(\vec{p}, -s\zeta)a_1(\vec{p}, -s\zeta) + (-1)^{2s}a_2(\vec{p}, -s\zeta)a_2^+(\vec{p}, -s\zeta)]\delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s}a_1^+(\vec{p}, -s\zeta)a_2^+(\vec{p}, -s\zeta)e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s\zeta)a_1(\vec{p}, -s\zeta)e^{2i|\vec{p}|t}]\delta^3(\vec{p}' + \vec{p})\}d^3\vec{p}'d^3\vec{p} \\
& = \int \hat{p}[a_1^+(\vec{p}, -s\zeta)a_1(\vec{p}, -s\zeta) + (-1)^{2s}a_2(\vec{p}, -s\zeta)a_2^+(\vec{p}, -s\zeta)]d^3\vec{p}
\end{aligned}$$

□

$$\text{定理6.7.5. } \vec{M}(s) = \int \psi^+(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \psi(\vec{r}, t) d^3\vec{r} = \int \hat{p}[a^+(\vec{p}, -s\zeta)a(\vec{p}, -s\zeta) + (-1)^{2s}b(\vec{p}, -s\zeta)b^+(\vec{p}, -s\zeta)]d^3\vec{p}$$

$$\text{证明: } \vec{M}(s) = \int \psi^+(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \psi(\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned}
& = \frac{1}{(2\pi)^3} \int d^3\vec{p}'d^3\vec{p}d^3\vec{r}|\vec{p}'|^{s-\frac{1}{2}}|\vec{p}|^{s-\frac{1}{2}}\lambda^+(\hat{p}', -s\zeta)\lambda(\hat{p}, -s\zeta)\frac{\hat{p}}{|\vec{p}|^{2s-1}} \\
& [a_1^+(\vec{p}', -s\zeta)e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)} + a_2(\vec{p}', -s\zeta)e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}][a_1(\vec{p}, -s\zeta)e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)} + (-1)^{2s-1}a_2^+(\vec{p}, -s\zeta)e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}] \\
& = \int \vec{p}'^{2s-1}\lambda^+(\hat{p}', -s\zeta)\lambda(\hat{p}, -s\zeta)\frac{\hat{p}}{|\vec{p}|^{2s-1}}\{[a_1^+(\vec{p}, -s\zeta)a_1(\vec{p}, -s\zeta) + (-1)^{2s-1}a_2(\vec{p}, -s\zeta)a_2^+(\vec{p}, -s\zeta)]\delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s-1}a_1^+(\vec{p}, -s\zeta)a_2^+(\vec{p}, -s\zeta)e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s\zeta)a_1(\vec{p}, -s\zeta)e^{2i|\vec{p}|t}]\delta^3(\vec{p}' + \vec{p})\}d^3\vec{p}'d^3\vec{p} \\
& = \int \hat{p}[a_1^+(\vec{p}, -s\zeta)a_1(\vec{p}, -s\zeta) + (-1)^{2s-1}a_2(\vec{p}, -s\zeta)a_2^+(\vec{p}, -s\zeta)]d^3\vec{p}
\end{aligned}$$

□

## 6.8 s-自旋场能量动量算符的小结

$$\text{定理6.8.1. } [s\partial_a + iS_{ab}(s, \zeta)\partial^b] \frac{\psi(x)}{(\sqrt{-\nabla^2})^{|s|}} = 0$$

$$\text{定理6.8.2. } P_a(s) = \int \psi^+(\vec{r}, t) \frac{-i\partial_a(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi(\vec{r}, t) d^3\vec{r}$$

$$\text{定理6.8.3. } P_a(n) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \partial_a \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r}, P_a(n + \frac{1}{2}) = -i \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \partial_a \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r}$$

$$\text{定理6.8.4. } \begin{cases} M_{ab}(n) = i \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [-i(x_a\partial_b - x_b\partial_a) + S_{ab}(n, \zeta)] \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \\ M_{ab}(n + \frac{1}{2}) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [-i(x_a\partial_b - x_b\partial_a) + S_{ab}(n + \frac{1}{2}, \zeta)] \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \end{cases}$$

## 6.9 s-自旋场合理的哈密顿型能量动量算符

定理6.9.1.

$$\begin{cases} \hat{H}(\frac{1}{2}) = \frac{i\zeta}{1/2} \int \psi^+(\vec{r}, t) \sigma(\frac{1}{2}) \cdot \nabla \psi(\vec{r}, t) d^3\vec{r} \\ \hat{H}(1) = \int \psi^+(\vec{r}, t) \frac{[\sigma(1)\cdot\nabla]^2}{\nabla^2} \psi(\vec{r}, t) d^3\vec{r} \\ \hat{H}(\frac{3}{2}) = \frac{-i\zeta}{3/2} \int \psi^+(\vec{r}, t) \frac{\sigma(\frac{3}{2})\cdot\nabla}{\nabla^2} \psi(\vec{r}, t) d^3\vec{r} \\ \hat{H}(2) = (\frac{-i\zeta}{2})^2 \int \psi^+(\vec{r}, t) \frac{[\sigma(2)\cdot\nabla]^2}{\nabla^4} \psi(\vec{r}, t) d^3\vec{r} \end{cases} \quad \begin{cases} \hat{P}(\frac{1}{2}) = - \int \psi^+(\vec{r}, t) i\nabla \psi(\vec{r}, t) d^3\vec{r} \\ \hat{P}(1) = i\zeta \int \psi^+(\vec{r}, t) \frac{[\sigma(1)\cdot\nabla]i\nabla}{\nabla^2} \psi(\vec{r}, t) d^3\vec{r} \\ \hat{P}(\frac{3}{2}) = \int \psi^+(\vec{r}, t) \frac{i\nabla}{\nabla^2} \psi(\vec{r}, t) d^3\vec{r} \\ \hat{P}(2) = (\frac{-i\zeta}{2}) \int \psi^+(\vec{r}, t) \frac{[\sigma(2)\cdot\nabla]i\nabla}{\nabla^4} \psi(\vec{r}, t) d^3\vec{r} \end{cases}$$

定理6.9.2.

$$\begin{cases} \hat{H}(n + \frac{1}{2}) = \int \psi^+(\vec{r}, t) \frac{i\zeta \sigma(n + \frac{1}{2}) \cdot \nabla}{(\sqrt{-\nabla^2})^{2n}} \psi(\vec{r}, t) d^3\vec{r} \\ \hat{P}(n + \frac{1}{2}) = \int \psi^+(\vec{r}, t) \frac{-i\nabla}{(\sqrt{-\nabla^2})^{2n}} \psi(\vec{r}, t) d^3\vec{r} \end{cases} \quad \begin{cases} \hat{H}(n) = \int \psi^+(\vec{r}, t) \frac{[\frac{i\zeta}{n} \sigma(n) \cdot \nabla]^2}{(\sqrt{-\nabla^2})^{2n}} \psi(\vec{r}, t) d^3\vec{r} \\ \hat{P}(n) = \int \psi^+(\vec{r}, t) \frac{-i\nabla [\frac{i\zeta}{n} \sigma(n) \cdot \nabla]}{(\sqrt{-\nabla^2})^{2n}} \psi(\vec{r}, t) d^3\vec{r} \end{cases}$$

## 6.10 导出的能量动量算符和角动量算符

$$\text{定义6.10.1. } \begin{cases} \hat{M}_{ab}(s, \varsigma) = x_a \hat{P}_b - x_b \hat{P}_a + i\sigma_{sab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s) \delta(s - \frac{1}{2}), \hat{P}_a = -i\partial_a \\ \Gamma_{ab}(s, \varsigma) = x_a \Gamma_b(s, \varsigma) - x_b \Gamma_a(s, \varsigma), \Gamma_a(s, \varsigma) := -\varsigma[\frac{1}{s}\sigma(s), -i\varsigma]_a \end{cases}$$

推论6.10.1.

$$\begin{cases} P_a(n + \frac{1}{2}) = \int \psi^+(\vec{r}, t) \frac{-i\partial_a}{(\sqrt{-\nabla^2})^{2n}} \psi(\vec{r}, t) d^3\vec{r} \\ P_a(n + \frac{1}{2}) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{P}_a \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \\ M_{ab}(n + \frac{1}{2}) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{M}_{ab} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \end{cases} \quad \begin{cases} P_a(n) = \int \psi^+(\vec{r}, t) \frac{-i\partial_a [\frac{1}{n}\sigma(n) \cdot \nabla]}{(\sqrt{-\nabla^2})^{2n}} \psi(\vec{r}, t) d^3\vec{r} \\ \hat{P}_a(n) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{P}_a \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \\ M_{ab}(n) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{M}_{ab} \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \end{cases}$$

定理6.10.1.

$$\begin{cases} H(1) = \int \psi_{k'_\zeta}^+(\vec{r}, t) \psi_{k_\zeta}(\vec{r}, t) d^3\vec{r} \\ H(2) = -\int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{1}{\nabla^2} \psi_{k_\zeta}(\vec{r}, t) d^3\vec{r} \end{cases} \quad \begin{cases} P(1) = -\varsigma \int \psi_{k'_\zeta}^+(\vec{r}, t) \sigma(1) \psi_{k_\zeta}(\vec{r}, t) d^3\vec{r} \\ P(2) = (\frac{\varsigma}{2}) \int \psi_{k'_\zeta}^+(\vec{r}, t) \frac{\sigma(2)}{\nabla^2} \psi_{k_\zeta}(\vec{r}, t) d^3\vec{r} \end{cases}$$

推论6.10.2.

$$\begin{cases} P_a(n - \frac{1}{2}) = \int \psi^+(\vec{r}, t) \frac{-i\partial_a}{(\sqrt{-\nabla^2})^{2(n-1)}} \psi(\vec{r}, t) d^3\vec{r} \\ P_a(n - \frac{1}{2}) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n-1}} \hat{P}_a \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n-1}} d^3\vec{r} \\ M_{ab}(n - \frac{1}{2}) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n-1}} \hat{M}_{ab} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n-1}} d^3\vec{r} \end{cases} \quad \begin{cases} P_a(n) = \int \psi^+(\vec{r}, t) \frac{-\varsigma[\frac{1}{n}\sigma(n), -i\varsigma]_a}{(\sqrt{-\nabla^2})^{2(n-1)}} \psi(\vec{r}, t) d^3\vec{r} \\ P_a(n) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n-1}} \Gamma_a \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n-1}} d^3\vec{r} \\ M_{ab}(n) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n-1}} \Gamma_{ab} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n-1}} d^3\vec{r} \end{cases}$$

## 6.11 s-自旋场量子方程的提取

$$\text{定理6.11.1. } [\psi(\vec{r}, t), H(s)] = \frac{(-1)^{2s}}{4^{s-1}} \sqrt{-\nabla^2} [\Gamma_{-}^{abc\dots}(s) \hat{\partial}_{\pi \rightarrow i} \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots] [\Gamma_{+}^{abc\dots}(s) \hat{\partial}_{\pi \rightarrow i} \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots] \psi(\vec{r}, t)$$

$$\begin{aligned} \text{证明: } [\psi(\vec{r}, t), H(s)] &= [\psi(\vec{r}, t), \frac{i^{-2s}}{2^{s-1}} \int \frac{\psi^+(\vec{r}', t)}{(\sqrt{-\nabla^2})^{s-1}} \Gamma_{+}^{abc\dots}(s) \hat{\partial}'_a \hat{\partial}'_b \hat{\partial}'_c \dots \frac{\psi(\vec{r}', t)}{(\sqrt{-\nabla^2})^{s-1}} d^3\vec{r}'] \\ &= \frac{i^{-2s}}{2^{s-1}} \int [\psi(\vec{r}, t), \frac{\psi^+(\vec{r}', t)}{(\sqrt{-\nabla^2})^{s-1}}]_{-2s+1} \Gamma_{+}^{abc\dots}(s) \hat{\partial}'_a \hat{\partial}'_b \hat{\partial}'_c \dots \frac{\psi(\vec{r}', t)}{(\sqrt{-\nabla^2})^{s-1}} d^3\vec{r}' \\ &= \frac{i^{-2s}}{2^{s-1}} \int i \frac{(-1)^{2s}}{2^{s-1}} (i\sqrt{-\nabla^2})^{2s-1} [\Gamma_{-}^{abc\dots}(s) \hat{\partial}_{\pi \rightarrow i} \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots] \delta^3(\vec{r} - \vec{r}') [\Gamma_{+}^{abc\dots}(s) \hat{\partial}'_a \hat{\partial}'_b \hat{\partial}'_c \dots] \frac{\psi(\vec{r}', t)}{(\sqrt{-\nabla^2})^{2(s-1)}} d^3\vec{r}' \\ &= \frac{(-1)^{2s}}{4^{s-1}} \sqrt{-\nabla^2} [\Gamma_{-}^{abc\dots}(s) \hat{\partial}_{\pi \rightarrow i} \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots] [\Gamma_{+}^{abc\dots}(s) \hat{\partial}_{\pi \rightarrow i} \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots] \psi(\vec{r}, t) \quad \square \end{aligned}$$

$$\text{定理6.11.2. } [\psi(\vec{r}, t), \vec{P}(s)] = \frac{(-1)^{2s}}{4^{s-1}} (-i\nabla) [\Gamma_{-}^{abc\dots}(s) \hat{\partial}_{\pi \rightarrow i} \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots] [\Gamma_{-}^{abc\dots}(s) \hat{\partial}_{\pi \rightarrow i} \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots] \psi(\vec{r}, t)$$

定理6.11.3.

$$\begin{aligned} ???[\psi(\vec{r}, t), P(s)] &= \frac{(-1)^{2s}}{4^{s-1}} \sqrt{-\nabla^2} \{ [\Gamma_{-}^{abc\dots}(s) \hat{\partial}_{\pi \rightarrow i} \hat{\nabla}, i\Gamma_{+}^{abc\dots}(s) \hat{\partial}_{\pi \rightarrow i} \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots] [\Gamma_{-}^{abc\dots}(s) \hat{\partial}_{\pi \rightarrow i} \hat{\partial}_a \hat{\partial}_b \hat{\partial}_c \dots] \psi(\vec{r}, t) \\ &= \frac{(-1)^{2s}}{4^{s-1}} \sqrt{-\nabla^2} \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} d^3\vec{p} |\vec{p}|^{(s-\frac{1}{2})} \\ &\quad \{ \{ [\Gamma_{-}^{ab\dots}(s) \hat{\nabla}, i\Gamma_{+}^{ab\dots}(s) \hat{\nabla}] (\zeta\hat{p}, i)_a (\zeta\hat{p}, i)_b \dots \} [\Gamma_{-}^{ab\dots}(s) (\zeta\hat{p}, i)_a (\zeta\hat{p}, i)_b \dots] \lambda(\hat{p}, -s\varsigma) a_1(\vec{p}, -s\varsigma) e^{ip \cdot x} \\ &\quad + [\Gamma_{-}^{ab\dots}(s) \hat{\nabla}, i\Gamma_{+}^{ab\dots}(s) \hat{\nabla}] (-\zeta\hat{p}, i)_a (-\zeta\hat{p}, i)_b \dots \} [\Gamma_{-}^{ab\dots}(s) (-\zeta\hat{p}, i)_a (-\zeta\hat{p}, i)_b \dots] \lambda(\hat{p}, -s\varsigma) a_2^+(\vec{p}, -s\varsigma) e^{-ip \cdot x} \} \\ &= \frac{(-1)^{2s}}{4^{s-1}} \sqrt{-\nabla^2} \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \{ [\Gamma_{-}^{abc\dots}(s) \hat{\nabla}, i\Gamma_{+}^{abc\dots}(s) \hat{\nabla}] \hat{p}_a \hat{p}_b \hat{p}_c \dots \} \frac{2^{s-1}}{i^{-2s}} \lambda(\hat{p}, -s\varsigma) \\ &\quad \vec{p}|^{(s-\frac{1}{2})} [a_1(\vec{p}, -s\varsigma) e^{ip \cdot x} + a_2^+(\vec{p}, -s\varsigma) e^{-ip \cdot x}] d^3\vec{p} \\ &= \frac{(-1)^{2s}}{4^{s-1}} \sqrt{-\nabla^2} \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} (\hat{\nabla}, i) (\frac{2^{s-1}}{i^{-2s}})^2 \lambda(\hat{p}, -s\varsigma) |\vec{p}|^{(s-\frac{1}{2})} [a_1(\vec{p}, -s\varsigma) e^{ip \cdot x} + a_2^+(\vec{p}, -s\varsigma) e^{-ip \cdot x}] d^3\vec{p} \\ &= \sqrt{-\nabla^2} (\hat{\nabla}, i) \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma) e^{ip \cdot x} + a_2^+(\vec{p}, -s\varsigma) e^{-ip \cdot x}] d^3\vec{p} \end{aligned}$$

$$= \sqrt{-\nabla^2}(\hat{\nabla}, i)\psi$$

$$= (-i\nabla, i\sqrt{-\nabla^2})\psi$$

## 6.12 对易和反对易公式

$$\text{推论6.12.1.} \quad \begin{cases} [A, BC] = [A, B]C + B[A, C], [A, CB] = [A, C]B + C[A, B] \\ [A, BC] = \{A, B\}C - B\{A, C\}, [A, CB] = \{A, C\}B - C\{A, B\} \end{cases}$$

$$\text{推论6.12.2.} \quad \begin{cases} [A, \{B, C\}] = \{[A, B], C\} + \{B, [A, C]\} \\ [A, [B, C]] = \{\{A, B\}, C\} - \{B, \{A, C\}\} \end{cases}$$

## 6.13 猜测：s-自旋场的能量动量算符和角动量算符的可能错误表述

推论6.13.1.

$$\begin{cases} \hat{M}_{ab}(s, \varsigma) = -i(x_a \partial_b - x_b \partial_a) + i\sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s) \\ \gamma_{ab}(s, \varsigma) = x_a \gamma_b(s, \varsigma) - x_b \gamma_a(s, \varsigma) + \frac{\sigma_{\varsigma ab}^{\alpha\varsigma} \partial_{\alpha\varsigma}}{(\sqrt{-\nabla^2})^{2s}}, \gamma_a(s, \varsigma) := -\varsigma[\frac{1}{s}\sigma(s), -i\varsigma]_a \\ \tilde{M}_{ab}(s, \varsigma) = -i(p_a \tilde{\partial}_b - p_b \tilde{\partial}_a) - i s \varsigma \sigma_{\varsigma ab}^{\alpha\varsigma} \hat{p}_{\alpha\varsigma}, \tilde{\partial}_\pi \equiv \frac{1}{i|\vec{p}|} \\ \tilde{M}_{ab}(s, \varsigma)? =? -i(p_a \tilde{\partial}_b - p_b \tilde{\partial}_a) - i s \varsigma \sigma_{\varsigma ab}^{\alpha\varsigma} \hat{p}_{\alpha\varsigma}, \tilde{\partial}_\pi \equiv \frac{1}{i|\vec{p}|} \end{cases}$$

推论6.13.2.

$$\begin{cases} P_a(s, \varsigma) = \int_{\vec{p} \neq 0} \{a_1^+(\vec{p}, -s\varsigma) p_a a_1(\vec{p}, -s\varsigma) + (-1)^{2s} a_2(\vec{p}, -s\varsigma) p_a a_2^+(\vec{p}, -s\varsigma)\} d^3 \vec{p} \\ M_{ab}(s, \varsigma) = \int_{\vec{p} \neq 0} \{a_1^+(\vec{p}, -s\varsigma) \tilde{M}_{ab}(s, \varsigma) a_1(\vec{p}, -s\varsigma) + (-1)^{2s+1} a_2(\vec{p}, -s\varsigma) \tilde{M}_{ab}(s, \varsigma) a_2^+(\vec{p}, -s\varsigma)\} d^3 \vec{p} \end{cases}$$

## 6.14 s-自旋场的量子方程

$$\text{推论6.14.1.} \quad [2\partial_a + iS_{ab}(2, \varsigma)\partial^b]\psi(x) = 0 \Rightarrow \begin{cases} \dot{\psi}(\vec{r}, t) = -i[\psi(\vec{r}, t), H] \\ \nabla\psi(\vec{r}, t) = i[\psi(\vec{r}, t), \vec{P}] \\ \partial_a\psi(\vec{r}, t) = i[\psi(\vec{r}, t), P_a] \end{cases}$$

$$\text{推论6.14.2.} \quad [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x) = 0[\text{波场}] \Leftrightarrow \begin{cases} \partial^a \partial_a \psi(\vec{r}, t) = 0 \\ [\sigma(s), -i s \varsigma]^a \partial_a \psi(\vec{r}, t) = 0 \end{cases} \Leftrightarrow \partial_a \psi(\vec{r}, t) = i[\psi(\vec{r}, t), P_a]$$

$$\text{推论6.14.3.} \quad \begin{cases} s^2 \vec{p} \lambda(\hat{p}, -s\varsigma) = s\sigma(s) \cdot \vec{p} \sigma(s) \lambda(\hat{p}, -s\varsigma) - (s-1)\sigma(s)[\sigma(s) \cdot \vec{p}] \lambda(\hat{p}, -s\varsigma) \\ [\sigma(s) \cdot \vec{p} + s\varsigma p] \sigma(s) \lambda(\hat{p}, -s\varsigma) = (s\vec{p} + \varsigma p \sigma(s)) \lambda(\hat{p}, -s\varsigma) \\ [\sigma(s) \cdot \vec{p} + s\varsigma p] \lambda(\hat{p}, -s\varsigma) = 0 \end{cases}$$

推论6.14.4.  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x) = 0 \Rightarrow$

$$\begin{cases} [\sigma(s) \cdot \hat{\nabla}]^n \sigma(s) \psi = \{(\varsigma \hat{\partial}_t)^{n-1} s[s^n - (s-1)^n] \hat{\nabla} + (\varsigma \hat{\partial}_t)^n (s-1)^n \sigma(s)\} \psi \\ [\sigma(s) \cdot \hat{\nabla}]^n \sigma(s) \psi = \{[\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^{n-1} s[s^n - (s-1)^n] \hat{\nabla} + (s-1)^n \sigma(s) [\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^n\} \psi \\ \sigma(s) \cdot [\sigma(s) \cdot \hat{\nabla}]^n \sigma(s) \psi = [s^{n+2} + s(s-1)^n] [\frac{1}{s}\sigma(s) \cdot \hat{\nabla}]^n \psi \end{cases}$$

## 6.15 s-自旋场的彭加莱对易代数

$$\text{定义6.15.1.} \quad \begin{cases} \hat{M}_{ab}(s, \varsigma) = x_a \hat{P}_b - x_b \hat{P}_a + i\sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s), \hat{P}_a = -i\partial_a \\ \Gamma_{ab}(s, \varsigma) = x_a \Gamma_b(s, \varsigma) - x_b \Gamma_a(s, \varsigma), \Gamma_a(s, \varsigma) := -\varsigma[\frac{1}{s}\sigma(s), -i\varsigma]_a \end{cases}$$

猜想6.15.1.

$$\begin{cases} P_a(n + \frac{1}{2}) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{P}_a \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\ M_{ab}(n + \frac{1}{2}) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{M}_{ab} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \end{cases} \quad \begin{cases} P_a(n) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{P}_a \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\ M_{ab}(n) = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \hat{M}_{ab} \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \end{cases}$$

$$\begin{aligned}
\text{证明: } [P_a(x), P_b(x')] &= [\int \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \hat{P}_a \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r}, \int \frac{\psi^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} \hat{P}_b \frac{\psi(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} d^3\vec{r}'] \\
&= - \int \frac{1}{\sqrt{2n}\sqrt{\nabla'^2n}} [\psi^+(\vec{r},t) \partial_a \psi(\vec{r},t), \psi^+(\vec{r}',t') \partial'_b \psi(\vec{r}',t')] d^3\vec{r} d^3\vec{r}' \\
&= - \int \frac{\delta^{k_\zeta k'_\zeta} \delta^{l_\zeta l'_\zeta}}{\sqrt{2n}\sqrt{\nabla'^2n}} [\psi_{k'_\zeta}^+(\vec{r},t) \partial_a \psi_{k_\zeta}(\vec{r},t), \psi_{l'_\zeta}^+(\vec{r}',t') \partial'_b \psi_{l_\zeta}(\vec{r}',t')] d^3\vec{r} d^3\vec{r}' \\
&= - \int d^3\vec{r} d^3\vec{r}' \frac{\delta^{k_\zeta k'_\zeta} \delta^{l_\zeta l'_\zeta}}{\sqrt{2n}\sqrt{\nabla'^2n}} \\
&\quad \{ [\psi_{k'_\zeta}^+(\vec{r},t) \partial_a \psi_{k_\zeta}(\vec{r},t), \psi_{l'_\zeta}^+(\vec{r}',t') \partial'_b \psi_{l_\zeta}(\vec{r}',t')] + \psi_{l'_\zeta}^+(\vec{r}',t') [\psi_{k'_\zeta}^+(\vec{r},t) \partial_a \psi_{k_\zeta}(\vec{r},t), \partial'_b \psi_{l_\zeta}(\vec{r}',t')] \} \\
&= - \int d^3\vec{r} d^3\vec{r}' \frac{\delta^{k_\zeta k'_\zeta} \delta^{l_\zeta l'_\zeta}}{\sqrt{2n}\sqrt{\nabla'^2n}} \\
&\quad \{ \psi_{k'_\zeta}^+(\vec{r},t) \{ \partial_a \psi_{k_\zeta}(\vec{r},t), \psi_{l'_\zeta}^+(\vec{r}',t') \} \partial'_b \psi_{l_\zeta}(\vec{r}',t') - \psi_{l'_\zeta}^+(\vec{r}',t') \{ \psi_{k'_\zeta}^+(\vec{r},t), \partial'_b \psi_{l_\zeta}(\vec{r}',t') \} \partial_a \psi_{k_\zeta}(\vec{r},t) \} \\
&= - \int d^3\vec{r} d^3\vec{r}' \frac{\delta^{k_\zeta k'_\zeta} \delta^{l_\zeta l'_\zeta}}{\sqrt{2n}\sqrt{\nabla'^2n}} \frac{-i}{2^{2n}} \\
&\quad \{ \psi_{k'_\zeta}^+(\vec{r},t) \partial_a \Gamma_{k_\zeta l'_\zeta}^{cd} \cdot (n + \frac{1}{2}) \overbrace{\partial_c \partial_d}^{2n+1} \cdot \Delta(x-x') \partial'_b \psi_{l_\zeta}(\vec{r}',t') - \psi_{l'_\zeta}^+(\vec{r}',t') \partial'_b \Gamma_{l_\zeta k'_\zeta}^{cd} \cdot (n + \frac{1}{2}) \overbrace{\partial'_c \partial'_d}^{2n+1} \cdot \Delta(x'-x) \partial_a \psi_{k_\zeta}(\vec{r},t) \} \\
&= \int d^3\vec{r} d^3\vec{r}' \frac{\delta^{k_\zeta k'_\zeta} \delta^{l_\zeta l'_\zeta}}{2^{2n}\sqrt{4n}} \\
&\quad \{ \psi_{k'_\zeta}^+(\vec{r},t) \partial'_b \psi_{l_\zeta}(\vec{r}',t') \Gamma_{k_\zeta l'_\zeta}^{cd} \cdot (n + \frac{1}{2}) \partial_a \overbrace{\partial_c \partial_d}^{2n+1} \cdot \Delta(x-x') + \psi_{l'_\zeta}^+(\vec{r}',t') \partial_a \psi_{k_\zeta}(\vec{r},t) \Gamma_{l_\zeta k'_\zeta}^{cd} \cdot (n + \frac{1}{2}) \partial_b \overbrace{\partial'_c \partial'_d}^{2n+1} \cdot \Delta(x-x') \} \\
&= 0
\end{aligned}$$

□

## 7 s-自旋粒子的彭加莱对称性

### 7.1 玻色子的彭加莱对称性

$$\text{引理 7.1.1. } \left\{ \begin{aligned} & [\frac{\psi_{k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n}] = \frac{i}{(-2)^{n-1}} \Gamma_+^{ef} \cdot (n) \overbrace{\hat{\partial}_e \hat{\partial}_f}^{2n, \hat{\partial}_\pi \rightarrow i} \cdot \delta^3(\vec{r}-\vec{r}') \\ & [\frac{\psi_{k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n}] = \frac{i}{(-2)^{n-1}} \frac{-1}{i\sqrt{-\nabla^2}} \Gamma_-^{ef} \cdot (n) \overbrace{\hat{\partial}_e \hat{\partial}_f}^{2n, \hat{\partial}_\pi \rightarrow i} \cdot \delta^3(\vec{r}-\vec{r}'), n > 0 \end{aligned} \right.$$

$$\text{定理 7.1.1. } \left\{ \begin{aligned} & [L_{ab}, L_{cd}] = -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \\ & [L_{ab}, P_c] = -i(g_{bc}P_a - g_{ac}P_b), [P_a, P_b] = 0 \end{aligned} \right.$$

证明:  $[L_{ab}, L_{cd}]$

$$\begin{aligned}
&= - \int d^3\vec{r} d^3\vec{r}' \left[ \frac{\psi^+(\vec{r},t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} (r'_c \partial'_d - r'_d \partial'_c) \frac{i\psi(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} \right] \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} d^3\vec{r}' \left[ \frac{\psi_{k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} \right] \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} d^3\vec{r}' \\
&\quad \left\{ \frac{\psi_{k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} [(r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n}] (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} \right. \\
&\quad \left. + \frac{\psi_{k'_\zeta}^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} [\frac{\psi_{k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n}, (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n}] (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \right\} \\
&= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} d^3\vec{r}' \\
&\quad \left\{ \frac{\psi_{k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i}{(-2)^{n-1}} \{ \Gamma_+^{ef} \cdot (n) \overbrace{\hat{\partial}_e \hat{\partial}_f}^{2n, \hat{\partial}_\pi \rightarrow i} \}_{l_\zeta k'_\zeta} \delta^3(\vec{r}-\vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} \right. \\
&\quad - \frac{\psi_{k'_\zeta}^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} (r'_c \partial'_d - r'_d \partial'_c) \frac{i}{(-2)^{n-1}} \{ \Gamma_+^{ef} \cdot (n) \overbrace{\hat{\partial}_e \hat{\partial}_f}^{2n, \hat{\partial}_\pi \rightarrow i} \}_{l'_\zeta k_\zeta} \delta^3(\vec{r}'-\vec{r}) (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \} \\
&= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} d^3\vec{r}' \\
&\quad \left\{ \frac{\psi_{k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} (r_a \partial'_b - r_b \partial'_a) \frac{i}{(-2)^{n-1}} \{ \Gamma_+^{ef} \cdot (n) \overbrace{\hat{\partial}_e \hat{\partial}_f}^{2n, \hat{\partial}_\pi \rightarrow i} \}_{l_\zeta k'_\zeta} \delta^3(\vec{r}-\vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} \right. \\
&\quad - \frac{\psi_{k'_\zeta}^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} (r'_c \partial'_d - r'_d \partial'_c) \frac{i}{(-2)^{n-1}} \{ \Gamma_+^{ef} \cdot (n) \overbrace{\hat{\partial}_e \hat{\partial}_f}^{2n, \hat{\partial}_\pi \rightarrow i} \}_{l'_\zeta k_\zeta} \delta^3(\vec{r}'-\vec{r}) (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \} \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} d^3\vec{r}' \\
&\quad \left\{ \frac{\psi_{k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i}{(-2)^{n-1}} \{ \Gamma_+^{ef} \cdot (n) \overbrace{\hat{\partial}_e \hat{\partial}_f}^{2n, \hat{\partial}_\pi \rightarrow i} \}_{l_\zeta k'_\zeta} (r_c \partial_d - r_d \partial_c) \frac{\psi_{l'_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \right. \\
&\quad \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} (r'_c \partial'_d - r'_d \partial'_c) \frac{i}{(-2)^{n-1}} \{ \Gamma_+^{ef} \cdot (n) \overbrace{\hat{\partial}_e \hat{\partial}_f}^{2n, \hat{\partial}_\pi \rightarrow i} \}_{l'_\zeta k_\zeta} (r_c \partial_d - r_d \partial_c) \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \right\} \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3\vec{r} d^3\vec{r}' \\
&\quad \left\{ \frac{\psi_{k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i}{(-2)^{n-1}} \{ \Gamma_+^{ef} \cdot (n) \overbrace{\hat{\partial}_e \hat{\partial}_f}^{2n, \hat{\partial}_\pi \rightarrow i} \}_{l_\zeta k'_\zeta} (r_c \partial_d - r_d \partial_c) \frac{\psi_{l'_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \right. \\
&\quad \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}',t')}{(\sqrt{-\nabla'^2})^n} (r'_c \partial'_d - r'_d \partial'_c) \frac{i}{(-2)^{n-1}} \{ \Gamma_+^{ef} \cdot (n) \overbrace{\hat{\partial}_e \hat{\partial}_f}^{2n, \hat{\partial}_\pi \rightarrow i} \}_{l'_\zeta k_\zeta} (r_c \partial_d - r_d \partial_c) \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \right\}
\end{aligned}$$

$$\begin{aligned}
& - \frac{\psi_{k'_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_c \partial_d - r_d \partial_c) \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots\}_{l'_\zeta k'_\zeta} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \} \\
& = - \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [-i(r_a \partial_b - r_b \partial_a), -i(r_c \partial_d - r_d \partial_c)] \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots\} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
& = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [\hat{L}_{ab}, \hat{L}_{cd}] \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
& = -i(g_{ad} L_{bc} - g_{ac} L_{bd} + g_{bc} L_{ad} - g_{bd} L_{ac})
\end{aligned}$$

□

证明:  $[L_{ab}, P_c]$ 

$$\begin{aligned}
& = - \int d^3 \vec{r} d^3 \vec{r}' \left[ \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_c \frac{i\psi(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \left[ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [(r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n}] \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} + \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \left[ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots\}_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right. \\
& \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_c \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}'_e \hat{\partial}'_f \dots\}_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial'_b - r_b \partial'_a) \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}'_e \hat{\partial}'_f \dots\}_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right. \\
& \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_c \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots\}_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r}' \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots\}_{l_\zeta k'_\zeta} \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_c \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots\}_{l'_\zeta k_\zeta} \right. \\
& \left. (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = - \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [-i(r_a \partial_b - r_b \partial_a), -i\partial'_c] \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots\} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
& = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [\hat{L}_{ab}, \hat{P}_c] \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
& = -i(g_{bc} P_a - g_{ac} P_b)
\end{aligned}$$

□

证明:  $[P_a, P_b]$ 

$$\begin{aligned}
& = - \int \left[ \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \partial_a \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_b \frac{i\psi(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] d^3 \vec{r} d^3 \vec{r}' \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int \left[ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] d^3 \vec{r} d^3 \vec{r}' \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \left[ \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} + \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \left[ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots\}_{l_\zeta k'_\zeta} \partial_a \delta^3(\vec{r} - \vec{r}') \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}'_e \hat{\partial}'_f \dots\}_{l'_\zeta k_\zeta} \right. \\
& \left. \partial'_b \delta^3(\vec{r}' - \vec{r}) \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}'_e \hat{\partial}'_f \dots\}_{l_\zeta k'_\zeta} \partial'_a \delta^3(\vec{r} - \vec{r}') \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \frac{i}{(-2)^{n-1}} \{\Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots\}_{l'_\zeta k_\zeta} \right. \\
& \left. \partial_b \delta^3(\vec{r}' - \vec{r}) \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{i}{(-2)^{n-1}} \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \left\{ \Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots \right\}^{k_\zeta l'_\zeta} \partial_a \partial_b \frac{\psi_{l'_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} - \frac{\psi_{k'_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \left\{ \Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots \right\}^{k'_\zeta l_\zeta} \partial_b \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} d^3 \vec{r} \\
&= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (\partial_a \partial_b - \partial_b \partial_a) \frac{i}{(-2)^{n-1}} \left\{ \Gamma_+^{ef \dots} (n) \hat{\partial}_e \hat{\partial}_f \dots \right\} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
&= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (\partial_a \partial_b - \partial_b \partial_a) \frac{-i \psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
&= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [\hat{P}_a, \hat{P}_b] \frac{i \psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} = 0
\end{aligned}$$

□

## 7.2 费米子的彭加莱对称性

$$\text{引理 7.2.1. } \left\{ \frac{\psi_{k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} = \frac{i}{(-2)^{n-1} \sqrt{2}} \Gamma_-^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \delta^3(\vec{r} - \vec{r}')$$

$$\text{定理 7.2.1. } \begin{cases} [L_{ab}, L_{cd}] = -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac}) \\ [L_{ab}, P_c] = -i(g_{bc}P_a - g_{ac}P_b), [P_a, P_b] = 0 \end{cases}$$

证明:  $[L_{ab}, L_{cd}]$

$$\begin{aligned}
&= - \int d^3 \vec{r} d^3 \vec{r}' \left[ \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] \\
&= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \left[ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] \\
&= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
&\quad \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \left\{ (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right. \\
&\quad \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
&= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
&\quad \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_-^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right. \\
&\quad \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} (r'_c \partial'_d - r'_d \partial'_c) \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_-^{ef \dots} (n + \frac{1}{2}) \hat{\partial}'_e \hat{\partial}'_f \dots \right\}_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
&= \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
&\quad \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial'_b - r_b \partial'_a) \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_-^{ef \dots} (n + \frac{1}{2}) \hat{\partial}'_e \hat{\partial}'_f \dots \right\}_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') (r'_c \partial'_d - r'_d \partial'_c) \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right. \\
&\quad \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} (r'_c \partial_d - r'_d \partial_c) \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_-^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
&= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} \\
&\quad \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_-^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l_\zeta k'_\zeta} (r_c \partial_d - r_d \partial_c) \frac{\psi_{l'_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right. \\
&\quad \left. - \frac{\psi_{k'_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_c \partial_d - r_d \partial_c) \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_-^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l'_\zeta k_\zeta} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
&= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [-i(r_a \partial_b - r_b \partial_a), -i(r_c \partial_d - r_d \partial_c)] \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_-^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \right\} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
&= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [\hat{L}_{ab}, \hat{L}_{cd}] \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
&= -i(g_{ad}L_{bc} - g_{ac}L_{bd} + g_{bc}L_{ad} - g_{bd}L_{ac})
\end{aligned}$$

□

证明:  $[L_{ab}, P_c]$

$$\begin{aligned}
&= - \int d^3 \vec{r} d^3 \vec{r}' \left[ \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_c \frac{\psi(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] \\
&= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \left[ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] \\
&= -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}'
\end{aligned}$$



$$\begin{aligned}
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \left\{ (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right. \\
& - \left. \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_c \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial'_b - r_b \partial'_a) \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right. \\
& - \left. \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_c \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l'_\zeta k_\zeta} \delta^3(\vec{r}' - \vec{r}) (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r}' \\
& \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (r_a \partial_b - r_b \partial_a) \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l_\zeta k'_\zeta} \delta^3(\vec{r} - \vec{r}') \partial'_c \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right. \\
& - \left. \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_c \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l'_\zeta k_\zeta} (r_a \partial_b - r_b \partial_a) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [-i(r_a \partial_b - r_b \partial_a), -i \partial'_c] \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l_\zeta k'_\zeta} \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} d^3 \vec{r}' \\
& = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [\hat{L}_{ab}, \hat{P}_c] \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
& = -i(g_{bc} P_a - g_{ac} P_b) \quad \square
\end{aligned}$$

证明:  $[P_a, P_b]$

$$\begin{aligned}
& = -\int \left[ \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \partial_a \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_b \frac{\psi(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] d^3 \vec{r} d^3 \vec{r}' \\
& = -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int \left[ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] d^3 \vec{r} d^3 \vec{r}' \\
& = -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \\
& \int d^3 \vec{r} d^3 \vec{r}' \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \left\{ \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} - \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = -\delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l_\zeta k'_\zeta} \partial_a \delta^3(\vec{r} - \vec{r}') \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right. \\
& - \left. \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l'_\zeta k_\zeta} \partial'_b \delta^3(\vec{r}' - \vec{r}) \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = \delta^{k_\zeta l_\zeta} \delta^{k'_\zeta l'_\zeta} \int d^3 \vec{r} d^3 \vec{r}' \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l_\zeta k'_\zeta} \partial'_a \delta^3(\vec{r} - \vec{r}') \partial'_b \frac{\psi_{l'_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right. \\
& - \left. \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l'_\zeta k_\zeta} \partial_b \delta^3(\vec{r}' - \vec{r}) \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} \\
& = -\int \left\{ \frac{\psi_{k_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l_\zeta k'_\zeta} \partial_a \partial_b \frac{\psi_{l'_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right. \\
& - \left. \frac{\psi_{k'_\zeta}^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l'_\zeta k_\zeta} \partial_b \partial_a \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \right\} d^3 \vec{r} \\
& = -\int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (\partial_a \partial_b - \partial_b \partial_a) \frac{i}{(-2)^{n-1} \sqrt{2}} \left\{ \Gamma_{-}^{ef \dots} \left( n + \frac{1}{2} \right) \hat{\partial}_e \hat{\partial}_f \dots \right\}_{l_\zeta k'_\zeta} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
& = -\int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} (\partial_a \partial_b - \partial_b \partial_a) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
& = \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [\hat{P}_a, \hat{P}_b] \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} = 0 \quad \square
\end{aligned}$$

## 7.3 费米子自旋的彭加莱对称性

$$\text{引理7.3.1.} \quad \left\{ \begin{aligned} \left\{ \frac{\psi_{k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} &= \frac{i}{(-2)^{n-1}\sqrt{2}} \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \\ \left\{ \frac{\dot{\psi}_{k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\dot{\psi}_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} &= \frac{i}{(-2)^{n-1}\sqrt{2}} (-i\sqrt{-\nabla^2}) \Gamma_+^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \\ \left\{ \frac{\psi_{k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n+1}}, \frac{\psi_{k'_\zeta}^+(\vec{r}', t)}{(\sqrt{-\nabla'^2})^{n+1}} \right\} &= \frac{i}{(-2)^{n-1}\sqrt{2}} \frac{1}{i\sqrt{-\nabla^2}} \Gamma_+^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \delta^3(\vec{r} - \vec{r}') \end{aligned} \right.$$

$$\text{推论7.3.1.} \quad \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\zeta} (n + \frac{1}{2}) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} = -\zeta (n + \frac{1}{2}) \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\alpha_\zeta} \frac{\dot{\psi}(\vec{r}, t)}{(\sqrt{-\nabla^2})^{n+1}}, n + \frac{1}{2} \geq 1$$

$$\text{推论7.3.2.} \quad \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\zeta} (n) \frac{i\dot{\psi}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} = n\zeta \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \nabla_{\alpha_\zeta} \frac{i\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, n \geq 1$$

证明:  $[S_{ab}(t), S_{cd}(t)]$

$$\begin{aligned} &= \int \left[ \frac{\psi^{+k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} S_{abk_\zeta} l_\zeta (n + \frac{1}{2}, \zeta) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^{+m_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} S_{cdm_\zeta} n_\zeta (n + \frac{1}{2}, \zeta) \frac{\psi_{n_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] d^3\vec{r} d^3\vec{r}' \\ &= \int \frac{\psi^{+k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \left\{ S_{abk_\zeta} l_\zeta (n + \frac{1}{2}, \zeta) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^{+m_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} S_{cdm_\zeta} n_\zeta (n + \frac{1}{2}, \zeta) \frac{\psi_{n_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \\ &\quad - \frac{\psi^{+m_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \left\{ \frac{\psi^{+k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, S_{cdm_\zeta} n_\zeta (n + \frac{1}{2}, \zeta) \frac{\psi_{n_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} S_{abk_\zeta} l_\zeta (n + \frac{1}{2}, \zeta) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} d^3\vec{r}' \\ &= \int \frac{\psi^{+k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} S_{abk_\zeta} l_\zeta (n + \frac{1}{2}, \zeta) S_{cdm_\zeta} n_\zeta (n + \frac{1}{2}, \zeta) \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}'_e \hat{\partial}'_f \cdots}^{2n+1, \hat{\partial}'_\pi \rightarrow i} \right\} l_\zeta m_\zeta \delta^3(\vec{r} - \vec{r}') \frac{\psi_{n_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \\ &\quad - \frac{\psi^{+m_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} S_{cdm_\zeta} n_\zeta (n + \frac{1}{2}, \zeta) S_{abk_\zeta} l_\zeta (n + \frac{1}{2}, \zeta) \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \right\} n_\zeta k_\zeta \delta^3(\vec{r} - \vec{r}') \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} d^3\vec{r}' \\ &= \int \frac{\psi^{+k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} S_{abk_\zeta} l_\zeta (n + \frac{1}{2}, \zeta) \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \right\} l_\zeta m_\zeta S_{cdm_\zeta} n_\zeta (n + \frac{1}{2}, \zeta) \frac{\psi_{n_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \\ &\quad - \frac{\psi^{+m_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} S_{cdm_\zeta} n_\zeta (n + \frac{1}{2}, \zeta) \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \right\} n_\zeta k_\zeta S_{abk_\zeta} l_\zeta (n + \frac{1}{2}, \zeta) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \\ &= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} S_{ab} (n + \frac{1}{2}, \zeta) \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \right\} S_{cd} (n + \frac{1}{2}, \zeta) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \\ &\quad - \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} S_{cd} (n + \frac{1}{2}, \zeta) \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \right\} S_{ab} (n + \frac{1}{2}, \zeta) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \\ &= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} S_{ab} (n + \frac{1}{2}, \zeta) \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \right\} S_{cd} (n + \frac{1}{2}, \zeta) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \\ &\quad - \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} S_{cd} (n + \frac{1}{2}, \zeta) \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \right\} S_{ab} (n + \frac{1}{2}, \zeta) \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \\ ??? &= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [S_{ab} (n + \frac{1}{2}, \zeta), S_{cd} (n + \frac{1}{2}, \zeta)] \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \right\} \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \\ &= \int \frac{\psi^+(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} [S_{ab} (n + \frac{1}{2}, \zeta), S_{cd} (n + \frac{1}{2}, \zeta)] \frac{\psi(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} \\ &= -i[g_{ad}S_{bc} (n + \frac{1}{2}, \zeta) - g_{ac}S_{bd} (n + \frac{1}{2}, \zeta) + g_{bc}S_{ad} (n + \frac{1}{2}, \zeta) - g_{bd}S_{ac} (n + \frac{1}{2}, \zeta)] \quad \square \end{aligned}$$

证明:  $[\sigma_{\alpha_\zeta}(t), \sigma_{\beta_\zeta}(t)]$

$$\begin{aligned} &= \int \left[ \frac{\psi^{+k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\zeta k_\zeta} l_\zeta (n + \frac{1}{2}) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^{+m_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \sigma_{\beta_\zeta m_\zeta} n_\zeta (n + \frac{1}{2}) \frac{\psi_{n_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right] d^3\vec{r} d^3\vec{r}' \\ &= \int \frac{\psi^{+k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \left\{ \sigma_{\alpha_\zeta k_\zeta} l_\zeta (n + \frac{1}{2}) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \frac{\psi^{+m_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} \sigma_{\beta_\zeta m_\zeta} n_\zeta (n + \frac{1}{2}) \frac{\psi_{n_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \\ &\quad - \frac{\psi^{+m_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \left\{ \frac{\psi^{+k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n}, \sigma_{\beta_\zeta m_\zeta} n_\zeta (n + \frac{1}{2}) \frac{\psi_{n_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \right\} \sigma_{\alpha_\zeta k_\zeta} l_\zeta (n + \frac{1}{2}) \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} d^3\vec{r}' \\ &= \int \frac{\psi^{+k_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\zeta k_\zeta} l_\zeta (n + \frac{1}{2}) \sigma_{\beta_\zeta m_\zeta} n_\zeta (n + \frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}'_e \hat{\partial}'_f \cdots}^{2n+1, \hat{\partial}'_\pi \rightarrow i} \right\} l_\zeta m_\zeta \delta^3(\vec{r} - \vec{r}') \frac{\psi_{n_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \\ &\quad - \frac{\psi^{+m_\zeta}(\vec{r}', t)}{(\sqrt{-\nabla'^2})^n} \sigma_{\beta_\zeta m_\zeta} n_\zeta (n + \frac{1}{2}) \sigma_{\alpha_\zeta k_\zeta} l_\zeta (n + \frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \left\{ \Gamma_-^{ef \cdots} (n + \frac{1}{2}) \overbrace{\hat{\partial}_e \hat{\partial}_f \cdots}^{2n+1, \hat{\partial}_\pi \rightarrow i} \right\} n_\zeta k_\zeta \delta^3(\vec{r} - \vec{r}') \frac{\psi_{l_\zeta}(\vec{r}, t)}{(\sqrt{-\nabla^2})^n} d^3\vec{r} d^3\vec{r}' \end{aligned}$$

$$\begin{aligned}
&= \int \frac{\psi^{+k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\zeta k_\zeta} l_\zeta (n + \frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \Gamma_{-}^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \}_{l_\zeta}^{2n+1, \hat{\partial}_\pi \rightarrow i} m_\zeta \sigma_{\beta_\zeta m_\zeta} n_\zeta (n + \frac{1}{2}) \frac{\psi_{n_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \\
&- \frac{\psi^{+m_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\beta_\zeta m_\zeta} n_\zeta (n + \frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \Gamma_{-}^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \}_{n_\zeta}^{2n+1, \hat{\partial}_\pi \rightarrow i} k_\zeta \sigma_{\alpha_\zeta k_\zeta} l_\zeta (n + \frac{1}{2}) \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
&= \int \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\zeta} (n + \frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \Gamma_{-}^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \}_{\sigma_{\beta_\zeta}}^{2n+1, \hat{\partial}_\pi \rightarrow i} \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \\
&- \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\beta_\zeta} (n + \frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \Gamma_{-}^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \}_{\sigma_{\alpha_\zeta}}^{2n+1, \hat{\partial}_\pi \rightarrow i} \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
&= \int \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\alpha_\zeta} (n + \frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \Gamma_{-}^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \}_{\sigma_{\beta_\zeta}}^{2n+1, \hat{\partial}_\pi \rightarrow i} \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \\
&- \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} \sigma_{\beta_\zeta} (n + \frac{1}{2}) \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \Gamma_{-}^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \}_{\sigma_{\alpha_\zeta}}^{2n+1, \hat{\partial}_\pi \rightarrow i} \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} \\
&??? = \int \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} [\sigma_{\alpha_\zeta} (n + \frac{1}{2}), \sigma_{\beta_\zeta} (n + \frac{1}{2})] \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \Gamma_{-}^{ef \dots} (n + \frac{1}{2}) \hat{\partial}_e \hat{\partial}_f \dots \}_{\psi(\vec{r},t)}^{2n+1, \hat{\partial}_\pi \rightarrow i} d^3 \vec{r} \\
&= \int \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^n} [\sigma_{\alpha_\zeta} (n + \frac{1}{2}), \sigma_{\beta_\zeta} (n + \frac{1}{2})] \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^n} d^3 \vec{r} = i \varepsilon_{\alpha_\zeta \beta_\zeta} \gamma_\zeta \sigma_{\gamma_\zeta} (t) \quad \square
\end{aligned}$$

证明:  $[\sigma_{\alpha_\zeta}(t), \sigma_{\beta_\zeta}(t)](n + \frac{1}{2})^{-2}$

$$\begin{aligned}
&= \int [ \frac{\psi^{+k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\alpha_\zeta} \delta_{k_\zeta} l_\zeta \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}, \frac{\psi^{+m_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\beta_\zeta} \delta_{m_\zeta} n_\zeta \frac{\psi_{n_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} ] d^3 \vec{r} d^3 \vec{r}' \\
&= \int \frac{\psi^{+k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \{ \nabla_{\alpha_\zeta} \delta_{k_\zeta} l_\zeta \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}, \frac{\psi^{+m_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\beta_\zeta} \delta_{m_\zeta} n_\zeta \frac{\psi_{n_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \} \\
&- \frac{\psi^{+m_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \{ \frac{\psi^{+k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}}, \nabla_{\beta_\zeta} \delta_{m_\zeta} n_\zeta \frac{\psi_{n_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \} \nabla_{\alpha_\zeta} \delta_{k_\zeta} l_\zeta \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} d^3 \vec{r} d^3 \vec{r}' \\
&= \int \frac{\psi^{+k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\alpha_\zeta} \delta_{k_\zeta} l_\zeta \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \Gamma_{+}^{ef \dots} \hat{\partial}_e \hat{\partial}_f \dots \}_{l_\zeta}^{2n+1, \hat{\partial}_\pi \rightarrow i} m_\zeta \delta^3(\vec{r} - \vec{r}') \nabla_{\beta_\zeta} \delta_{m_\zeta} n_\zeta \frac{\psi_{n_\zeta}(\vec{r}',t)}{(\sqrt{-\nabla^2})^{n+1}} \\
&- \frac{\psi^{+m_\zeta}(\vec{r}',t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\beta_\zeta} \delta_{m_\zeta} n_\zeta \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \Gamma_{+}^{ef \dots} \hat{\partial}_e \hat{\partial}_f \dots \}_{n_\zeta}^{2n+1, \hat{\partial}_\pi \rightarrow i} k_\zeta \delta^3(\vec{r} - \vec{r}') \nabla_{\alpha_\zeta} \delta_{k_\zeta} l_\zeta \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} d^3 \vec{r} d^3 \vec{r}' \\
&= \int \frac{\psi^{+k_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\alpha_\zeta} \delta_{k_\zeta} l_\zeta \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \Gamma_{+}^{ef \dots} \hat{\partial}_e \hat{\partial}_f \dots \}_{l_\zeta}^{2n+1, \hat{\partial}_\pi \rightarrow i} m_\zeta \nabla_{\beta_\zeta} \delta_{m_\zeta} n_\zeta \frac{\psi_{n_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \\
&- \frac{\psi^{+m_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\beta_\zeta} \delta_{m_\zeta} n_\zeta \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \Gamma_{+}^{ef \dots} \hat{\partial}_e \hat{\partial}_f \dots \}_{n_\zeta}^{2n+1, \hat{\partial}_\pi \rightarrow i} k_\zeta \nabla_{\alpha_\zeta} \delta_{k_\zeta} l_\zeta \frac{\psi_{l_\zeta}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} d^3 \vec{r} \\
&= \int \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\alpha_\zeta} \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \Gamma_{+}^{ef \dots} \hat{\partial}_e \hat{\partial}_f \dots \}_{\nabla_{\beta_\zeta}}^{2n+1, \hat{\partial}_\pi \rightarrow i} \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \\
&- \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\beta_\zeta} \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \Gamma_{+}^{ef \dots} \hat{\partial}_e \hat{\partial}_f \dots \}_{\nabla_{\alpha_\zeta}}^{2n+1, \hat{\partial}_\pi \rightarrow i} \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} d^3 \vec{r} \\
&= \int \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\alpha_\zeta} \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \Gamma_{+}^{ef \dots} \hat{\partial}_e \hat{\partial}_f \dots \}_{\nabla_{\beta_\zeta}}^{2n+1, \hat{\partial}_\pi \rightarrow i} \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \\
&- \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} \nabla_{\beta_\zeta} \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \Gamma_{+}^{ef \dots} \hat{\partial}_e \hat{\partial}_f \dots \}_{\nabla_{\alpha_\zeta}}^{2n+1, \hat{\partial}_\pi \rightarrow i} \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} d^3 \vec{r} \\
&= \int \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} [\nabla_{\alpha_\zeta}, \nabla_{\beta_\zeta}] \frac{i}{(-2)^{n-1}\sqrt{2}} \{ \frac{1}{i\sqrt{-\nabla^2}} \Gamma_{+}^{ef \dots} \hat{\partial}_e \hat{\partial}_f \dots \}_{\psi(\vec{r},t)}^{2n+1, \hat{\partial}_\pi \rightarrow i} d^3 \vec{r} \\
&= \int \frac{\psi^{+}(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} [\nabla_{\alpha_\zeta}, \nabla_{\beta_\zeta}] \frac{\psi(\vec{r},t)}{(\sqrt{-\nabla^2})^{n+1}} d^3 \vec{r} = 0? = i \varepsilon_{\alpha_\zeta \beta_\zeta} \gamma_\zeta \sigma_{\gamma_\zeta} (t) \quad \square
\end{aligned}$$

推论7.3.3.  $\lambda^+(\hat{p}, -s_\zeta) \sigma_i(s) [\sigma(s) \cdot \hat{p}]^n \sigma_j(s) \lambda(\hat{p}, -s_\zeta)$

$$\begin{aligned}
&= \lambda^+(\hat{p}, -s_\zeta) \sigma_i(s) \{ (-\zeta)^{n-1} s [s^n - (s-1)^n] \hat{p}_j + (-\zeta)^n (s-1)^n \sigma_j(s) \} \lambda(\hat{p}, -s_\zeta) \\
&= (-\zeta)^n s^2 [s^n - (s-1)^n] \hat{p}_i \hat{p}_j + (-\zeta)^n (s-1)^n \lambda^+(\hat{p}, -s_\zeta) \sigma_i(s) \sigma_j(s) \lambda(\hat{p}, -s_\zeta)
\end{aligned}$$

$$\begin{aligned}
&= (-\varsigma)^n s^2 [s^n - (s-1)^n] \hat{p}_i \hat{p}_j + (-\varsigma)^n (s-1)^n [s^2 \hat{p}_i \hat{p}_j + \frac{\varsigma}{2} (\delta_{ij} - \hat{p}_i \hat{p}_j - i\varsigma \varepsilon_{ij}^k \hat{p}_k)] \\
&= (-\varsigma)^n s^2 s^n \hat{p}_i \hat{p}_j + (-\varsigma)^n (s-1)^n [\frac{\varsigma}{2} (\delta_{ij} - \hat{p}_i \hat{p}_j - i\varsigma \varepsilon_{ij}^k \hat{p}_k)]
\end{aligned}$$

## 8 Penrose全对称方程的协变量子化

自我评述：由于Penrose全对称方程与自旋方程完全等价，所以Penrose全对称方程的协变量子化原则上也已成功完成，只需从自旋方程等价转换过来即可。但直接从Penrose全对称方程出发可以提供一种全新的解法，对有质量粒子的协变量子化有启示作用。由于详细结论已经由自旋方程方法全部得到，所以下面只给出Penrose全对称方程解法的精髓，不再求全，是自旋方程方法的补充。

### 8.1 Penrose全对称方程<sup>[1,2]</sup>及其平面波解

定理8.1.1.

$$[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x) = 0 \Leftrightarrow (\sigma, -i\varsigma)_a^{A'_\varsigma A_\varsigma} \partial^a \underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \dots}(x)}_{2s} = 0, \underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \dots}(x)}_{2s} = \underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}(s)}_{2s} \psi_{k_\varsigma}(x)$$

推论8.1.1.

$$\begin{cases}
\underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \dots}(x)}_{2s} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}(s)}_{2s} \lambda_{k_\varsigma}(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma) e^{ip \cdot x} + a_2^+(\vec{p}, -s\varsigma) e^{-ip \cdot x}] d^3 \vec{p} \\
|\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -s\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^{+k_\varsigma}(\hat{p}, -s\varsigma) \underbrace{\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma \dots}(s)}_{2s} \underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \dots}(x)}_{2s} e^{-ip \cdot x} d^3 \vec{r} \\
|\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}, -s\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^{+k_\varsigma}(\hat{p}, -s\varsigma) \underbrace{\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma \dots}(s)}_{2s} \underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \dots}(x)}_{2s} e^{ip \cdot x} d^3 \vec{r}
\end{cases}$$

推论8.1.2.

$$\begin{cases}
\lambda_{k_\varsigma}(\hat{p}, -s\varsigma) = \underbrace{\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma \dots}(s)}_{2s} \underbrace{\lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda_{B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda_{C_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \dots}_{2s} \\
\underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}(s)}_{2s} \lambda_{k_\varsigma}(\hat{p}, -s\varsigma) = \underbrace{\lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda_{B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda_{C_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \dots}_{2s}
\end{cases}$$

推论8.1.3.

$$\begin{cases}
\underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \dots}(x)}_{2s} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \underbrace{\lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda_{B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda_{C_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \dots}_{2s} [a_1(\vec{p}, -s\varsigma) e^{ip \cdot x} + a_2^+(\vec{p}, -s\varsigma) e^{-ip \cdot x}] d^3 \vec{p} \\
|\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -s\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^{+A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda^{+B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda^{+C_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \dots \underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \dots}(x)}_{2s} e^{-ip \cdot x} d^3 \vec{r} \\
|\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}, -s\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^{+A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda^{+B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda^{+C_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \dots \underbrace{\psi_{A_\varsigma B_\varsigma C_\varsigma \dots}(x)}_{2s} e^{ip \cdot x} d^3 \vec{r}
\end{cases}$$

### 8.2 Penrose全对称方程的自旋基及其性质

定义8.2.1.  $\lambda_{A_\varsigma B_\varsigma \dots}(\hat{p}, -s\varsigma) := \underbrace{\lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda_{B_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \dots}_{2s}$

推论8.2.1.  $\begin{cases} \lambda^{+A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) = 1, \lambda^{+A_\varsigma}(-\hat{p}, -\frac{\varsigma}{2}) \lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) = 1 \\ \lambda_{A_\varsigma}(\hat{p}, -\frac{\varsigma}{2}) \lambda_{A'_\varsigma}^+(\hat{p}, -\frac{\varsigma}{2}) = -\frac{\varsigma}{2} (\sigma, i\varsigma)_{A_\varsigma A'_\varsigma}^a \hat{p}_a \end{cases}$

推论8.2.2.  $\begin{cases} \lambda^{+A_\varsigma B_\varsigma \dots}(\hat{p}, -s\varsigma) \lambda_{A_\varsigma B_\varsigma \dots}(\hat{p}, -s\varsigma) = 1, \lambda^{+A_\varsigma B_\varsigma \dots}(-\hat{p}, -s\varsigma) \lambda_{A_\varsigma B_\varsigma \dots}(\hat{p}, -s\varsigma) = 0 \\ \lambda_{A_\varsigma B_\varsigma \dots}(\hat{p}, -s\varsigma) \lambda_{A'_\varsigma B'_\varsigma \dots}^+(\hat{p}, -s\varsigma) = (-\frac{\varsigma}{2})^{2s} \frac{1}{[(2s)!]^2} \underbrace{(\sigma, i\varsigma)_{A_\varsigma A'_\varsigma}^a (\sigma, i\varsigma)_{B_\varsigma B'_\varsigma}^b \dots}_{2s} \underbrace{\hat{p}_a \hat{p}_b \dots}_{2s} \end{cases}$

### 8.3 Penrose全对称方程的各种物理算符

定理8.3.1.  $P_u(s)$

$$= \int \psi^{+\overbrace{A_\varsigma B_\varsigma \dots}^{2s}}(\vec{r}, t) \frac{-i\partial_u (i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \underbrace{\psi_{A_\varsigma B_\varsigma \dots}(\vec{r}, t)}_{2s} d^3 \vec{r} \int p_u [a_1^+(\vec{p}, -s\varsigma) a_1(\vec{p}, -s\varsigma) + (-1)^{2s} a_2(\vec{p}, -s\varsigma) a_2^+(\vec{p}, -s\varsigma)] d^3 \vec{p}$$

$$\begin{aligned}
\text{证明: } P_u(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta}^{2s}}(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi_{\overbrace{A_\zeta B_\zeta}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \overbrace{\lambda^{+A_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}^{2s} \frac{p_u}{|\vec{p}|^{2s-1}} \\
& [a_1^+(\vec{p}', -s\zeta) e^{-i(\vec{p}' \cdot \vec{r} - |\vec{p}'|t)} + a_2(\vec{p}', -s\zeta) e^{i(\vec{p}' \cdot \vec{r} - |\vec{p}'|t)}] [a_1(\vec{p}, -s\zeta) e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + (-1)^{2s} a_2^+(\vec{p}, -s\zeta) e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] \\
&= \int |\vec{p}|^{2s-1} \overbrace{\lambda^{+A_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}^{2s} \frac{p_u}{|\vec{p}|^{2s-1}} \\
& \{ [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] \delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s} a_1^+(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta) e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) e^{2i|\vec{p}|t}] \delta^3(\vec{p}' + \vec{p}) \} d^3\vec{p}' d^3\vec{p} \\
&= \int \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}^{2s} \\
& p_u [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] d^3\vec{p} \\
&= \int p_u [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] d^3\vec{p} \quad \square
\end{aligned}$$

$$\begin{aligned}
\text{定理8.3.2. } Q(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi_{\overbrace{A_\zeta B_\zeta}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \int [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s-1} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] d^3\vec{p}
\end{aligned}$$

$$\begin{aligned}
\text{证明: } Q(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi_{\overbrace{A_\zeta B_\zeta}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \overbrace{\lambda^{+A_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}^{2s} \frac{1}{|\vec{p}|^{2s-1}} \\
& [a_1^+(\vec{p}', -s\zeta) e^{-i(\vec{p}' \cdot \vec{r} - |\vec{p}'|t)} + a_2(\vec{p}', -s\zeta) e^{i(\vec{p}' \cdot \vec{r} - |\vec{p}'|t)}] [a_1(\vec{p}, -s\zeta) e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + (-1)^{2s-1} a_2^+(\vec{p}, -s\zeta) e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] \\
&= \int |\vec{p}|^{2s-1} \overbrace{\lambda^{+A_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}^{2s} \frac{1}{|\vec{p}|^{2s-1}} \\
& \{ [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s-1} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] \delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s-1} a_1^+(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta) e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) e^{2i|\vec{p}|t}] \delta^3(\vec{p}' + \vec{p}) \} d^3\vec{p}' d^3\vec{p} \\
&= \int [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s-1} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] d^3\vec{p} \quad \square
\end{aligned}$$

$$\begin{aligned}
\text{定理8.3.3. } N(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \psi_{\overbrace{A_\zeta B_\zeta}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \int [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] d^3\vec{p}
\end{aligned}$$

$$\begin{aligned}
\text{证明: } N(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \psi_{\overbrace{A_\zeta B_\zeta}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \overbrace{\lambda^{+A_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}^{2s} \frac{1}{|\vec{p}|^{2s-1}} \\
& [a_1^+(\vec{p}', -s\zeta) e^{-i(\vec{p}' \cdot \vec{r} - |\vec{p}'|t)} + a_2(\vec{p}', -s\zeta) e^{i(\vec{p}' \cdot \vec{r} - |\vec{p}'|t)}] [a_1(\vec{p}, -s\zeta) e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + (-1)^{2s} a_2^+(\vec{p}, -s\zeta) e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] \\
&= \int |\vec{p}|^{2s-1} \overbrace{\lambda^{+A_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}^{2s} \frac{1}{|\vec{p}|^{2s-1}} \\
& \{ [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] \delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s} a_1^+(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta) e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) e^{2i|\vec{p}|t}] \delta^3(\vec{p}' + \vec{p}) \} d^3\vec{p}' d^3\vec{p} \\
&= \int [a_1^+(\vec{p}, -s\zeta) a_1(\vec{p}, -s\zeta) + (-1)^{2s} a_2(\vec{p}, -s\zeta) a_2^+(\vec{p}, -s\zeta)] d^3\vec{p} \quad \square
\end{aligned}$$

$$\begin{aligned} \text{定理8.3.4. } \vec{S}(s) &= \int \psi^+ \overbrace{A_\zeta B_\zeta \cdots}^{2s}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \underbrace{\psi_{A_\zeta B_\zeta \cdots}}_{2s}(\vec{r}, t) d^3\vec{r} \\ &= \int \hat{p} [a_1^+(\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s} a_2(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] d^3\vec{p} \end{aligned}$$

$$\begin{aligned} \text{证明: } \vec{S}(s) &= \int \psi^+ \overbrace{A_\zeta B_\zeta \cdots}^{2s}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \underbrace{\psi_{A_\zeta B_\zeta \cdots}}_{2s}(\vec{r}, t) d^3\vec{r} \\ &= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \overbrace{\lambda^{+A_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots}_{2s} \underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}_{2s} \frac{\hat{p}}{|\vec{p}|^{2s-1}} \\ &\quad [a_1^+(\vec{p}', -s_\zeta) e^{-i(\vec{p}' \cdot \vec{r} - |\vec{p}'|t)} + a_2(\vec{p}', -s_\zeta) e^{i(\vec{p}' \cdot \vec{r} - |\vec{p}'|t)}] [a_1(\vec{p}, -s_\zeta) e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + (-1)^{2s} a_2^+(\vec{p}, -s_\zeta) e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] \\ &= \int |\vec{p}|^{2s-1} \overbrace{\lambda^{+A_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots}_{2s} \underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}_{2s} \frac{\hat{p}}{|\vec{p}|^{2s-1}} \\ &\quad \{ [a_1^+(\vec{p}', -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s} a_2(\vec{p}', -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] \delta^3(\vec{p}' - \vec{p}) \\ &\quad + [(-1)^{2s} a_1^+(\vec{p}', -s_\zeta) a_2^+(\vec{p}, -s_\zeta) e^{-2i|\vec{p}|t} + a_2(-\vec{p}', -s_\zeta) a_1(\vec{p}, -s_\zeta) e^{2i|\vec{p}|t}] \delta^3(\vec{p}' + \vec{p}) \} d^3\vec{p}' d^3\vec{p} \\ &= \int \hat{p} [a_1^+(\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s} a_2(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] d^3\vec{p} \quad \square \end{aligned}$$

$$\begin{aligned} \text{定理8.3.5. } \vec{M}(s) &= \int \psi^+ \overbrace{A_\zeta B_\zeta \cdots}^{2s}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \underbrace{\psi_{A_\zeta B_\zeta \cdots}}_{2s}(\vec{r}, t) d^3\vec{r} \\ &= \int \hat{p} [a_1^+(\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s-1} a_2(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] d^3\vec{p} \end{aligned}$$

$$\begin{aligned} \text{证明: } \vec{M}(s) &= \int \psi^+ \overbrace{A_\zeta B_\zeta \cdots}^{2s}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \underbrace{\psi_{A_\zeta B_\zeta \cdots}}_{2s}(\vec{r}, t) d^3\vec{r} \\ &= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \overbrace{\lambda^{+A_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots}_{2s} \underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}_{2s} \frac{\hat{p}}{|\vec{p}|^{2s-1}} \\ &\quad [a_1^+(\vec{p}', -s_\zeta) e^{-i(\vec{p}' \cdot \vec{r} - |\vec{p}'|t)} + a_2(\vec{p}', -s_\zeta) e^{i(\vec{p}' \cdot \vec{r} - |\vec{p}'|t)}] [a_1(\vec{p}, -s_\zeta) e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + (-1)^{2s-1} a_2^+(\vec{p}, -s_\zeta) e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] \\ &= \int |\vec{p}|^{2s-1} \overbrace{\lambda^{+A_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots}_{2s} \underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}_{2s} \frac{\hat{p}}{|\vec{p}|^{2s-1}} \\ &\quad \{ [a_1^+(\vec{p}', -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s-1} a_2(\vec{p}', -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] \delta^3(\vec{p}' - \vec{p}) \\ &\quad + [(-1)^{2s-1} a_1^+(\vec{p}', -s_\zeta) a_2^+(\vec{p}, -s_\zeta) e^{-2i|\vec{p}|t} + a_2(-\vec{p}', -s_\zeta) a_1(\vec{p}, -s_\zeta) e^{2i|\vec{p}|t}] \delta^3(\vec{p}' + \vec{p}) \} d^3\vec{p}' d^3\vec{p} \\ &= \int \hat{p} [a_1^+(\vec{p}, -s_\zeta) a_1(\vec{p}, -s_\zeta) + (-1)^{2s-1} a_2(\vec{p}, -s_\zeta) a_2^+(\vec{p}, -s_\zeta)] d^3\vec{p} \quad \square \end{aligned}$$

## 8.4 Penrose全对称方程的协变对易规则

定理8.4.1.

$$\begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}(x')]_{-2s+1} = i \frac{(-1)^{2s}}{2^{2s-1}} \overbrace{\Gamma_{k_\zeta k'_\zeta}^{abc \cdots}}^{2s}(s) \overbrace{\partial_a \partial_b \partial_c \cdots}_{2s} \Delta(x-x'), \Gamma(0) := 1 & \Leftrightarrow \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')]_{-2s+1} = 0, [\psi_{k'_\zeta}(x), \psi_{l'_\zeta}(x')]_{-2s+1} = 0, s \geq 0 \\ \left\{ \begin{aligned} [\psi_{\underbrace{A_\zeta B_\zeta C_\zeta \cdots}_{2s}}(x), \psi_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \cdots}_{2s}}(x')]_{-2s+1} &= i \frac{(i\zeta)^{2s}}{2^{2s-1} [(2s)!]^2} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c \cdots}_{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}_{2s} \Delta(x-x') \\ [\psi_{\underbrace{A_\zeta B_\zeta C_\zeta \cdots}_{2s}}(x), \psi_{\underbrace{E_\zeta F_\zeta G_\zeta \cdots}_{2s}}(x')]_{-2s+1} &= 0, [\psi_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \cdots}_{2s}}(x), \psi_{\underbrace{E'_\zeta F'_\zeta G'_\zeta \cdots}_{2s}}(x')]_{-2s+1} = 0, s \geq 0 \end{aligned} \right. \end{cases}$$

引理8.4.1.

$$\begin{cases} \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) = -\frac{\zeta}{2} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \hat{p}_a \\ \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) = \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a p_a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b p_b = (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a p_a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b p_b, p^a p_a = 0 \\ (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a p_a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b p_b = (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a p_a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b p_b, p^a p_a = 0 \end{cases}$$

引理8.4.2.  $\begin{cases} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \partial_a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_b = (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \partial_a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_b, \partial^a \partial_a = 0 \\ (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \partial_a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_b = (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \partial_a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_b, \partial^a \partial_a = 0 \end{cases}$

直接验证便可以证明以上两个引理。

推论8.4.1.  $\frac{1}{[(2s)!]^2} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c \cdots}^{2s} \overbrace{p_a p_b p_c \cdots}^{2s} = \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c \cdots}^{2s} \overbrace{p_a p_b p_c \cdots}^{2s}$

推论8.4.2.  $\frac{1}{[(2s)!]^2} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots}^{2s} \overbrace{\partial_a \partial_b \cdots}^{2s} \Delta(x-x') = \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots}^{2s} \overbrace{\partial_a \partial_b \cdots}^{2s} \Delta(x-x')$

推论8.4.3.

$$\begin{cases} [\psi_{A_\zeta B_\zeta C_\zeta \cdots} (x), \psi_{A'_\zeta B'_\zeta C'_\zeta \cdots}^+ (x')]_{-2s+1} = i \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c \cdots}^{2s} \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x-x') \\ [\psi_{A_\zeta B_\zeta C_\zeta \cdots} (x), \psi_{E_\zeta F_\zeta G_\zeta \cdots} (x')]_{-2s+1} = 0, [\psi_{A'_\zeta B'_\zeta C'_\zeta \cdots}^+ (x), \psi_{E'_\zeta F'_\zeta G'_\zeta \cdots}^+ (x')]_{-2s+1} = 0, s \geq 0 \end{cases}$$

推论8.4.4.  $\psi_{\alpha_\zeta \beta_\zeta} = \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \frac{i\zeta}{\sqrt{2}} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \psi_{A_\zeta B_\zeta C_\zeta D_\zeta} = -\frac{1}{2} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \psi_{A_\zeta B_\zeta C_\zeta D_\zeta}, [\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta}]^* = \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta}$   
 $\Psi_{\alpha_\zeta} := \frac{-i\zeta}{\sqrt{2}} \frac{i}{2} \sigma_{\alpha_\zeta}^{ab} F_{ab}$

推论8.4.5.  $[\psi_{\alpha_\zeta \beta_\zeta}, \psi_{\alpha'_\zeta \beta'_\zeta}^+] = \frac{i}{2} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\beta_\zeta \beta'_\zeta}^{cd} \partial_a \partial_b \partial_c \partial_d \Delta(x-x')$

证明:  $[\psi_{\alpha_\zeta \beta_\zeta}, \psi_{\alpha'_\zeta \beta'_\zeta}^+]$

$$\begin{aligned} &= \frac{1}{4} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \sigma_{\beta'_\zeta}^{C'_\zeta D'_\zeta} [\psi_{A_\zeta B_\zeta C_\zeta D_\zeta}, \psi_{A'_\zeta B'_\zeta C'_\zeta D'_\zeta}^+] \\ &= \frac{i}{32} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \sigma_{\beta'_\zeta}^{C'_\zeta D'_\zeta} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c (\sigma, i\zeta)_{D_\zeta D'_\zeta}^d \partial_a \partial_b \partial_c \partial_d \Delta(x-x') \\ &= \frac{i}{2} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\beta_\zeta \beta'_\zeta}^{cd} \partial_a \partial_b \partial_c \partial_d \Delta(x-x') \end{aligned}$$

□

推论8.4.6.  $[\underbrace{\psi_{\alpha_\zeta \beta_\zeta \cdots}}_{2n}, \underbrace{\psi_{\alpha'_\zeta \beta'_\zeta \cdots}^+}_{2n}] = i \frac{(-1)^n}{2^{n-1}} \underbrace{\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\beta_\zeta \beta'_\zeta}^{cd} \cdots}_n \underbrace{\partial_a \partial_b \partial_c \partial_d \cdots}_n \Delta(x-x')$

推论8.4.7.  $\psi_{\alpha_\zeta \beta_\zeta} = \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta}, [\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta}]^* = \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta}$

推论8.4.8.  $[\psi_{\alpha_\zeta}, \psi_{\alpha'_\zeta}^+] = \frac{i}{2} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x')$

证明:  $[\psi_{\alpha_\zeta}, \psi_{\alpha'_\zeta}^+]$

$$\begin{aligned} &= -\frac{1}{2} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} [\psi_{A_\zeta B_\zeta}, \psi_{A'_\zeta B'_\zeta}^+] \\ &= \frac{i}{4} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_a \partial_b \Delta(x-x') \\ &= -i \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \end{aligned}$$

□

## 8.5 光子Penrose全对称方程的协变量子化

定理8.5.1.

$$[\partial_a + iS_{ab}(1, \zeta) \partial^b] \psi(x) = 0 \Leftrightarrow (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \psi_{A_\zeta B_\zeta}(x) = 0, \psi_{A_\zeta B_\zeta}(x) = \Gamma_{A_\zeta B_\zeta}^{k_\zeta} \psi_{k_\zeta}(x)$$

推论8.5.1.

$$\begin{cases} \psi_{A_\zeta B_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{\frac{1}{2}} \Gamma_{A_\zeta B_\zeta}^{k_\zeta} \lambda_{k_\zeta}(\hat{p}, -\zeta) [a_1(\vec{p}, -\zeta) e^{ip \cdot x} + a_2^+(\vec{p}, -\zeta) e^{-ip \cdot x}] d^3 \vec{p} \\ |\vec{p}|^{\frac{1}{2}} a_1(\vec{p}, -\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda^{+k_\zeta}(\hat{p}, -\zeta) \Gamma_{k_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta}(x) e^{-ip \cdot x} d^3 \vec{r} \\ |\vec{p}|^{\frac{1}{2}} a_2^+(\vec{p}, -\zeta) = \frac{1}{(2\pi)^{3/2}} \int \lambda^{+k_\zeta}(\hat{p}, -\zeta) \Gamma_{k_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta}(x) e^{ip \cdot x} d^3 \vec{r} \end{cases}$$

定理8.5.2.

$$\begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0, [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] \\ = -\frac{i}{8}(\sigma, i\zeta)_{A_\zeta(A'_\zeta(\sigma, i\zeta)_{B_\zeta}^b}^{a} B'_\zeta) \partial_a \partial_b \Delta(x-x') \\ [\psi_{A_\zeta B_\zeta}(x), \psi_{C_\zeta D_\zeta}(x')] = 0, [\psi_{A'_\zeta B'_\zeta}^+(x), \psi_{C'_\zeta D'_\zeta}^+(x')] = 0 \end{cases}$$

$$\text{定理8.5.3. } H(1) = \int \psi^+(\vec{r}, t) \frac{[\sigma(1) \cdot \nabla]^2}{\nabla^2} \psi(\vec{r}, t) d^3 \vec{r} = \int \psi_{A'_\zeta B'_\zeta}^+(\vec{r}, t) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta} \frac{[\sigma(1) \cdot \nabla]^2 |k'_\zeta|^{k_\zeta}}{\nabla^2} \Gamma_{k_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta}(\vec{r}, t) d^3 \vec{r}$$

## 8.6 光子Penrose一般方程的协变对易规则

$$\text{猜想8.6.1. } [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] = -\frac{i}{2}(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_a \partial_b \Delta(x-x') + ik \varepsilon_{A_\zeta B_\zeta} \varepsilon_{A'_\zeta B'_\zeta} \Delta(x-x') \\ \Leftrightarrow [\Psi_{\alpha_\zeta}(x), \Psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x-x'), [\phi(x), \phi^+(x')] = i\Delta(x-x'), [\Psi_{\alpha_\zeta}(x), \phi^+(x')] = 0$$

自我评述: 以上都是从自旋方程的已知结论等价转换过来的。下面将直接从Penrose全对称方程出发重新求解, 提供一种新的解法。

## 8.7 Penrose全对称方程平面波的直接解法

$$\text{定理8.7.1. } (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \psi_{A_\zeta B_\zeta C_\zeta \dots}(x) = 0, \psi_{A_\zeta B_\zeta C_\zeta \dots}(x) = \frac{1}{(2s)!} \psi_{\{A_\zeta B_\zeta C_\zeta \dots\}}(x)$$

$$\Leftrightarrow \psi_{A_\zeta B_\zeta C_\zeta \dots}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \dots}_{2s} \cdot [a_1(\vec{p}, -s\zeta) e^{ip \cdot x} + a_2^+(\vec{p}, -s\zeta) e^{-ip \cdot x}] d^3 \vec{p}$$

$$\text{证明: } (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \psi_{A_\zeta B_\zeta C_\zeta \dots}(x) = 0, \psi_{A_\zeta B_\zeta C_\zeta \dots}(x) = \frac{1}{(2s)!} \psi_{\{A_\zeta B_\zeta C_\zeta \dots\}}(x)$$

$$\Rightarrow \psi_{A_\zeta B_\zeta C_\zeta \dots}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) [a_{B_\zeta C_\zeta \dots}(\vec{p}) e^{ip \cdot x} + b_{B_\zeta C_\zeta \dots}^+(\vec{p}) e^{-ip \cdot x}] d^3 \vec{p}$$

$$\Rightarrow \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) a_{B_\zeta C_\zeta \dots}(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \int \psi_{A_\zeta B_\zeta C_\zeta \dots}(x) e^{-ip \cdot x} d^3 \vec{r} = \frac{1}{(2s)!} \lambda_{\{A_\zeta(\hat{p}, -\frac{\zeta}{2}) a_{B_\zeta C_\zeta \dots}\}}(\vec{p}, -s\zeta)$$

$$\Rightarrow \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) a_{B_\zeta C_\zeta \dots}(\vec{p}) = \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) a_{A_\zeta C_\zeta \dots}(\vec{p})$$

$$\Rightarrow \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) [a_{C_\zeta D_\zeta \dots}(\vec{p}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) + a'_{C_\zeta D_\zeta \dots}(\vec{p}) \lambda_{B_\zeta}(\hat{p}, \frac{\zeta}{2})]$$

$$= \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) [a_{C_\zeta D_\zeta \dots}(\vec{p}) \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) + a'_{C_\zeta D_\zeta \dots}(\vec{p}) \lambda_{A_\zeta}(\hat{p}, \frac{\zeta}{2})]$$

$$\Rightarrow \lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, \frac{\zeta}{2}) \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) [a_{C_\zeta D_\zeta \dots}(\vec{p}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) + a'_{C_\zeta D_\zeta \dots}(\vec{p}) \lambda_{B_\zeta}(\hat{p}, \frac{\zeta}{2})]$$

$$= \lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, \frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) [a_{C_\zeta D_\zeta \dots}(\vec{p}) \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) + a'_{C_\zeta D_\zeta \dots}(\vec{p}) \lambda_{A_\zeta}(\hat{p}, \frac{\zeta}{2})]$$

$$\Rightarrow a'_{B_\zeta C_\zeta \dots}(\vec{p}) = 0, \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) a_{B_\zeta C_\zeta \dots}(\vec{p}) = \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) a_{C_\zeta D_\zeta \dots}(\vec{p}) = \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2}) a_{B_\zeta D_\zeta \dots}(\vec{p})$$

$\Rightarrow \dots$

$$\Rightarrow \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) a_{B_\zeta C_\zeta \dots}(\vec{p}) = a(\vec{p}, -s\zeta) \underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2})}_{2s} \dots$$

$$\text{同理: } \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) b_{B_\zeta C_\zeta \dots}^+(\vec{p}) = b^+(\vec{p}, -s\zeta) \underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2})}_{2s} \dots$$

$$\Rightarrow \psi_{A_\zeta B_\zeta C_\zeta \dots}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2})}_{2s} \cdot [a_1(\vec{p}, -s\zeta) e^{ip \cdot x} + a_2^+(\vec{p}, -s\zeta) e^{-ip \cdot x}] d^3 \vec{p}$$

$$|\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -s\zeta) = a(\vec{p}, -s\zeta), |\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}, -s\zeta) = b^+(\vec{p}, -s\zeta)$$

$$\Rightarrow (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \psi_{A_\zeta B_\zeta C_\zeta \dots}(x) = 0, \psi_{A_\zeta B_\zeta C_\zeta \dots}(x) = \frac{1}{(2s)!} \psi_{\{A_\zeta B_\zeta C_\zeta \dots\}}(x) \quad \square$$

推论8.7.1.



$$\begin{cases} |\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -s\zeta) = \frac{1}{(2\pi)^{3/2}} \int \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+C_\zeta}(\hat{p}, -\frac{\zeta}{2})}^{2s} \cdot \underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s} e^{-ip \cdot x} d^3 \vec{r} \\ |\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}, -s\zeta) = \frac{1}{(2\pi)^{3/2}} \int \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+C_\zeta}(\hat{p}, -\frac{\zeta}{2})}^{2s} \cdot \underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s} e^{ip \cdot x} d^3 \vec{r} \end{cases}$$

## 8.8 从Penrose全对称方程平面波解重新证明协变对易关系

定理8.8.1.

$$\begin{cases} [a_\sigma(\vec{p}, -s\zeta), a_{\sigma'}^+(\vec{p}', -s\zeta)]_{-2s+1} = \delta_{\sigma\sigma'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -s\zeta), a_{\sigma'}(\vec{p}', -s\zeta)]_{-2s+1} = 0, [a_\sigma^+(\vec{p}, -s\zeta), a_{\sigma'}^+(\vec{p}', -s\zeta)]_{-2s+1} = 0 \end{cases} \Leftrightarrow \begin{cases} [\underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s}, \underbrace{\psi_{A'_\zeta B'_\zeta C'_\zeta} \dots(x')^+}_{2s}]_{-2s+1} = i \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c}^{2s} \cdot \overbrace{\partial_a \partial_b \partial_c \dots}_{2s} \Delta(x - x') \\ [\underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s}, \underbrace{\psi_{E_\zeta F_\zeta G_\zeta} \dots(x')^+}_{2s}]_{-2s+1} = 0, [\underbrace{\psi_{A'_\zeta B'_\zeta C'_\zeta} \dots(x)}_{2s}, \underbrace{\psi_{E'_\zeta F'_\zeta G'_\zeta} \dots(x')^+}_{2s}]_{-2s+1} = 0, s \geq 0 \end{cases}$$

证明:  $[\underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s}, \underbrace{\psi_{A'_\zeta B'_\zeta C'_\zeta} \dots(x')^+}_{2s}]_{-2s+1} = \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' (|\vec{p}| |\vec{p}'|)^{(s-\frac{1}{2})}$

$$\underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2})}_{2s} \cdot \underbrace{\lambda_{A'_\zeta}^+(\hat{p}', -\frac{\zeta}{2}) \lambda_{B'_\zeta}^+(\hat{p}', -\frac{\zeta}{2}) \lambda_{C'_\zeta}^+(\hat{p}', -\frac{\zeta}{2})}_{2s} \cdot$$

$$[[a_1(\vec{p}, -s\zeta) e^{ip \cdot x} + a_2^+(\vec{p}, -s\zeta) e^{-ip \cdot x}], [a_1^+(\vec{p}', -s\zeta) e^{-ip' \cdot x'} + a_2(\vec{p}', -s\zeta) e^{ip' \cdot x'}]]_{-2s+1}$$

$$\Rightarrow [\underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s}, \underbrace{\psi_{A'_\zeta B'_\zeta C'_\zeta} \dots(x')^+}_{2s}]_{-2s+1} = \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' (|\vec{p}| |\vec{p}'|)^{(s-\frac{1}{2})}$$

$$[\underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}', -\frac{\zeta}{2})}_{2s} [\underbrace{\lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B'_\zeta}^+(\hat{p}', -\frac{\zeta}{2})}_{2s}] [\underbrace{\lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C'_\zeta}^+(\hat{p}', -\frac{\zeta}{2})}_{2s}]] \cdot$$

$$\{[a_1(\vec{p}, -s\zeta), a_1^+(\vec{p}', -s\zeta)]_{-2s+1} e^{i(p \cdot x - p' \cdot x')} + [a_2^+(\vec{p}, -s\zeta), a_2(\vec{p}', -s\zeta)]_{-2s+1} e^{-i(p \cdot x - p' \cdot x')}\}$$

$$\Rightarrow [\underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s}, \underbrace{\psi_{A'_\zeta B'_\zeta C'_\zeta} \dots(x')^+}_{2s}]_{-2s+1} = \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' (|\vec{p}| |\vec{p}'|)^{(s-\frac{1}{2})}$$

$$[\underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}', -\frac{\zeta}{2})}_{2s} [\underbrace{\lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B'_\zeta}^+(\hat{p}', -\frac{\zeta}{2})}_{2s}] [\underbrace{\lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C'_\zeta}^+(\hat{p}', -\frac{\zeta}{2})}_{2s}]] \cdot$$

$$\delta^3(\vec{p} - \vec{p}') [e^{i(p \cdot x - p' \cdot x')} + (-1)^{2s+1} e^{-i(p \cdot x - p' \cdot x')}]$$

$$\Rightarrow [\underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s}, \underbrace{\psi_{A'_\zeta B'_\zeta C'_\zeta} \dots(x')^+}_{2s}]_{-2s+1}$$

$$= \frac{1}{(2\pi)^3} \int d^3 \vec{p} [\underbrace{|\vec{p}| \lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2})}_{2s} [\underbrace{|\vec{p}| \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B'_\zeta}^+(\hat{p}, -\frac{\zeta}{2})}_{2s}] [\underbrace{|\vec{p}| \lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C'_\zeta}^+(\hat{p}, -\frac{\zeta}{2})}_{2s}]] \cdot$$

$$|\vec{p}|^{-1} [e^{ip \cdot (x-x')} + (-1)^{2s+1} e^{-ip \cdot (x-x')}]$$

$$\Rightarrow [\underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s}, \underbrace{\psi_{A'_\zeta B'_\zeta C'_\zeta} \dots(x')^+}_{2s}]_{-2s+1}$$

$$= \frac{1}{(2\pi)^3} \int d^3 \vec{p} (-\frac{\zeta}{2})^{2s} \underbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a p_a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b p_b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c p_c}_{2s} \cdot |\vec{p}|^{-1} [e^{ip \cdot (x-x')} + (-1)^{2s+1} e^{-ip \cdot (x-x')}]$$

$$\Rightarrow [\underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s}, \underbrace{\psi_{A'_\zeta B'_\zeta C'_\zeta} \dots(x')^+}_{2s}]_{-2s+1}$$

$$= \frac{1}{(2\pi)^3} 2i \frac{(i\zeta)^{2s}}{2^{2s-1}} \underbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \partial_a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c \partial_c}_{2s} \cdot \int \frac{-i}{|\vec{p}|} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3 \vec{p}$$

$$\Rightarrow [\underbrace{\psi_{A_\zeta B_\zeta C_\zeta} \dots(x)}_{2s}, \underbrace{\psi_{A'_\zeta B'_\zeta C'_\zeta} \dots(x')^+}_{2s}]_{-2s+1} = i \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b (\sigma, i\zeta)_{C_\zeta C'_\zeta}^c}^{2s} \cdot \overbrace{\partial_a \partial_b \partial_c \dots}_{2s} \Delta(x - x')$$

□

证明:  $[a_1(\vec{p}, -s\zeta), a_1^+(\vec{p}', -s\zeta)]$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^{-(2s-1)} \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots \lambda^{A'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{B'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{C'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots}^{2s} \\
&\quad \underbrace{[\psi_{A_\zeta B_\zeta C_\zeta \cdots}(x), \psi_{A'_\zeta B'_\zeta C'_\zeta \cdots}^+(x')]}_{2s} e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^{-(2s-1)} \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots \lambda^{A'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{B'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{C'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots}^{2s} \\
&\quad i \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b(\sigma, i\zeta)_{C_\zeta C'_\zeta}^c \cdots \partial_a \partial_b \partial_c \cdots \Delta(x-x')}_{2s} e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int |\vec{p}|^{-(2s-1)} \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots \lambda^{A'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{B'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{C'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots}^{2s} \\
&\quad i \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b(\sigma, i\zeta)_{C_\zeta C'_\zeta}^c \cdots \{\partial_a \partial_b \partial_c \cdots \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}_0|} [e^{ip_0 \cdot (x-x')} - e^{-ip_0 \cdot (x-x')}] d^3 \vec{p}_0\}}_{2s} e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
&= [\frac{1}{(2\pi)^3}]^2 \int d^3 \vec{p}_0 d^3 \vec{r} d^3 \vec{r}' |\vec{p}|^{-(2s-1)} \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots \lambda^{A'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{B'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{C'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots}^{2s} \\
&\quad \underbrace{(-\frac{\zeta}{2})^{2s} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a p_{0a}(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b p_{0b}(\sigma, i\zeta)_{C_\zeta C'_\zeta}^c p_{0c} \cdots \frac{1}{|\vec{p}_0|} [e^{i(p_0-p) \cdot x} e^{-i(p_0-p') \cdot x'} + (-1)^{2s+1} e^{-i(p_0+p) \cdot x} e^{i(p_0+p') \cdot x'}]}_{2s} \\
&= [\frac{1}{(2\pi)^3}]^2 \int |\vec{p}|^{-(2s-1)} \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots \lambda^{A'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{B'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{C'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots}^{2s} \\
&\quad \underbrace{\lambda_{A_\zeta}(\hat{p}_0, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}_0, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}_0, -\frac{\zeta}{2}) \cdots \lambda_{A'_\zeta}^+(\hat{p}_0, -\frac{\zeta}{2}) \lambda_{B'_\zeta}^+(\hat{p}_0, -\frac{\zeta}{2}) \lambda_{C'_\zeta}^+(\hat{p}_0, -\frac{\zeta}{2}) \cdots}_{2s} \\
&\quad \frac{1}{|\vec{p}_0|} [e^{i(p_0-p) \cdot x} e^{-i(p_0-p') \cdot x'} + (-1)^{2s+1} e^{-i(p_0+p) \cdot x} e^{i(p_0+p') \cdot x'}] d^3 \vec{p}_0 d^3 \vec{r} d^3 \vec{r}' \\
&= \int |\vec{p}|^{-(2s-1)} \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots \lambda^{A'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{B'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \lambda^{C'_\zeta}(\hat{p}', -\frac{\zeta}{2}) \cdots}^{2s} \\
&\quad \underbrace{\lambda_{A_\zeta}(\hat{p}_0, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}_0, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}_0, -\frac{\zeta}{2}) \cdots \lambda_{A'_\zeta}^+(\hat{p}_0, -\frac{\zeta}{2}) \lambda_{B'_\zeta}^+(\hat{p}_0, -\frac{\zeta}{2}) \lambda_{C'_\zeta}^+(\hat{p}_0, -\frac{\zeta}{2}) \cdots}_{2s} \\
&\quad \frac{1}{|\vec{p}_0|} [\delta^3(\vec{p}_0 - \vec{p}) \delta^3(\vec{p}_0 - \vec{p}') + (-1)^{2s+1} e^{2iE_0(t-t')} \delta^3(\vec{p}_0 + \vec{p}) \delta^3(\vec{p}_0 + \vec{p}')] d^3 \vec{p}_0 \\
&= \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots \lambda^{A'_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{B'_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{C'_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}_{2s} \\
&\quad \underbrace{\lambda_{A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots \lambda_{A'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) \lambda_{B'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) \lambda_{C'_\zeta}^+(\hat{p}, -\frac{\zeta}{2}) \cdots}_{2s} \delta^3(\vec{p} - \vec{p}') \\
&+ (-1)^{2s+1} e^{2iE_0(t-t')} \overbrace{\lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+C_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots \lambda^{A'_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{B'_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda^{C'_\zeta}(\hat{p}, -\frac{\zeta}{2}) \cdots}_{2s} \\
&\quad \underbrace{\lambda_{A_\zeta}(-\hat{p}, -\frac{\zeta}{2}) \lambda_{B_\zeta}(-\hat{p}, -\frac{\zeta}{2}) \lambda_{C_\zeta}(-\hat{p}, -\frac{\zeta}{2}) \cdots \lambda_{A'_\zeta}^+(-\hat{p}, -\frac{\zeta}{2}) \lambda_{B'_\zeta}^+(-\hat{p}, -\frac{\zeta}{2}) \lambda_{C'_\zeta}^+(-\hat{p}, -\frac{\zeta}{2}) \cdots}_{2s} \delta^3(\vec{p} - \vec{p}') \\
&= \delta^3(\vec{p} - \vec{p}') + 0 = \delta^3(\vec{p} - \vec{p}') \quad \square
\end{aligned}$$

自我评述：以上的证法不再基于等时对易规则，直接基于协变对易规则，看似更难了，其实是更简单了，因为不要求出复杂的等时对易规则(参看下一节)，即使求出也难于使用，而协变对易规则本身已知且很有规律，也可以分解为自旋基的乘积，整个证明过程基本上只依赖于自旋基的性质，没有复杂的计算。这个证明方法可以推广到其他所有类似情况，从而简化所有类似的证明。其他的几个对易括号也可按同样的方法求出，不再列出。

## 8.9 Penrose全对称方程的等时量子化规则

$$\begin{aligned}
&\text{定理8.9.1. } [\psi_{A_\zeta B_\zeta \cdots}(x), \psi_{A'_\zeta B'_\zeta \cdots}^+(x')]_{-2s+1} = i \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots \Delta(x-x')}^{2s} \\
&\Rightarrow [\psi_{A_\zeta B_\zeta \cdots E_\zeta F_\zeta \cdots Z_\zeta}(\vec{r}, t), \psi_{A'_\zeta B'_\zeta \cdots E'_\zeta F'_\zeta \cdots Z'_\zeta}^+(\vec{r}', t)]_{-2s+1} \\
&= -\frac{(i\zeta)^{2s+1}}{2^{2s-1}} \sum_{k=0}^{[s-\frac{1}{2}]} \frac{(2s)!}{(2s-2k-1)!(2k)!} \overbrace{(\sigma \cdot \nabla)_{A_\zeta A'_\zeta} (\sigma \cdot \nabla)_{B_\zeta B'_\zeta} \cdots \delta_{E_\zeta E'_\zeta} \delta_{F_\zeta F'_\zeta} \cdots \nabla^2 \delta_{Z_\zeta Z'_\zeta}}^{2s-2k-1} \delta^3(\vec{r} - \vec{r}')
\end{aligned}$$

## 8.10 Penrose全对称方程的对易函数、因果函数和费曼传播子

引理8.10.1.

$$[\theta(t), \overbrace{\frac{(i\zeta)^{2s}}{2^{2s-1}} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots}^{2s}] = -\frac{(i\zeta)^{2s}}{2^{2s-1}} \sum_{n=0}^{2s-1} C_{2s}^n \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j \cdots}^n (i\zeta)^{2s-n} [\partial_\pi^{2s-n} \theta(t)] \overbrace{\partial_i \partial_j \cdots}^n$$

$$\begin{aligned} \text{证明: } & [\theta(t), \overbrace{\frac{(i\zeta)^{2s}}{2^{2s-1}} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots}^{2s}] \\ &= \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots}^{2s} [\theta(t), \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s}] \\ &= -\frac{(i\zeta)^{2s}}{2^{2s-1}} \sum_{n=0}^{2s-1} C_{2s}^n \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j \cdots}^n (i\zeta)^{2s-n} [\partial_\pi^{2s-n} \theta(t)] \overbrace{\partial_i \partial_j \cdots}^n \\ &= -\frac{i^{2s}}{2^{2s-1}} \sum_{n=0}^{2s-1} \zeta^n C_{2s}^n \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j \cdots}^n [\partial_t^{2s-n} \theta(t)] \overbrace{\partial_i \partial_j \cdots}^n \end{aligned}$$

□

推论8.10.1.

$$\left\{ \begin{aligned} \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}(s; x) &:= \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots}^{2s} \Delta(x) \\ \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(+)}(s; x) &:= \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots}^{2s} \Delta^{(+)}(x) \\ \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(-)}(s; x) &:= \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots}^{2s} \Delta^{(-)}(x) \\ \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(l)}(s; x) &:= \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots}^{2s} \Delta^{(l)}(x) \end{aligned} \right.$$

推论8.10.2.

$$\left\{ \begin{aligned} \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(c)}(s; x) &:= \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots}^{2s} \Delta^{(c)}(x) - \frac{i^{2s}}{2^{2s-1}} \sum_{n=0}^{2s-1} \zeta^n C_{2s}^n \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j \cdots}^n [\partial_t^{2s-1-n} \delta(t)] \overbrace{\partial_i \partial_j \cdots}^n \Delta(x) \\ \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(F)}(s; x) &= i \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(c)}(s; x) \\ &:= \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots}^{2s} \Delta_F(x) - \frac{i^{2s+1}}{2^{2s-1}} \sum_{n=0}^{2s-1} \zeta^n C_{2s}^n \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j \cdots}^n [\partial_t^{2s-1-n} \delta(t)] \overbrace{\partial_i \partial_j \cdots}^n \Delta(x) \end{aligned} \right.$$

推论8.10.3.

$$\left\{ \begin{aligned} \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(ret)}(s; x) &:= \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots}^{2s} \Delta^{(ret)}(x) - \frac{i^{2s}}{2^{2s-1}} \sum_{n=0}^{2s-1} \zeta^n C_{2s}^n \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j \cdots}^n [\partial_t^{2s-1-n} \delta(t)] \overbrace{\partial_i \partial_j \cdots}^n \Delta(x) \\ \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(adv)}(s; x) &:= \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots \partial_a \partial_b \cdots}^{2s} \Delta^{(adv)}(x) - \frac{i^{2s}}{2^{2s-1}} \sum_{n=0}^{2s-1} \zeta^n C_{2s}^n \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j \cdots}^n [\partial_t^{2s-1-n} \delta(t)] \overbrace{\partial_i \partial_j \cdots}^n \Delta(x) \end{aligned} \right.$$

推论8.10.4.

$$\left\{ \begin{aligned}
& \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(c)}(s; x) := \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a}^{2s} \overbrace{(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b}^{2s} \cdots \partial_a \partial_b \cdots \Delta^{(c)}(x) \\
& - \frac{i^{2s+1}}{2^{2s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{[(2s-2-n)/2]} \zeta^n C_{2s}^n C_{2s-1-n}^{2l+1} \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j}^n \cdots \partial_i \partial_j \cdots \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\
& \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(F)}(s; x) = i \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(c)}(s; x) := \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a}^{2s} \overbrace{(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b}^{2s} \cdots \partial_a \partial_b \cdots \Delta^F(x) \\
& - \frac{i^{2s+1}}{2^{2s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{[(2s-2-n)/2]} \zeta^n C_{2s}^n C_{2s-1-n}^{2l+1} \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j}^n \cdots \partial_i \partial_j \cdots \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\
& \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(F)}(s; p) = -i \frac{(-\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a}^{2s} \overbrace{(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b}^{2s} \cdots p_a p_b \cdots + \dots
\end{aligned} \right.$$

推论8.10.5.

$$\left\{ \begin{aligned}
& \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(ret)}(s; x) := \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a}^{2s} \overbrace{(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b}^{2s} \cdots \partial_a \partial_b \cdots \Delta^{(ret)}(x) \\
& - \frac{i^{2s+1}}{2^{2s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{[(2s-2-n)/2]} \zeta^n C_{2s}^n C_{2s-1-n}^{2l+1} \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j}^n \cdots \partial_i \partial_j \cdots \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x) \\
& \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(adv)}(s; x) := \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a}^{2s} \overbrace{(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b}^{2s} \cdots \partial_a \partial_b \cdots \Delta^{(adv)}(x) \\
& - \frac{i^{2s+1}}{2^{2s-1}} \sum_{n=0}^{2s-2} \sum_{l=0}^{[(2s-2-n)/2]} \zeta^n C_{2s}^n C_{2s-1-n}^{2l+1} \overbrace{\sigma_{A_\zeta A'_\zeta}^i \sigma_{B_\zeta B'_\zeta}^j}^n \cdots \partial_i \partial_j \cdots \nabla^{2l} \partial_t^{2s-2-n-2l} \delta^4(x)
\end{aligned} \right.$$

引理8.10.2.  $\Delta_{\underbrace{A_\zeta B_\zeta \cdots E_\zeta F_\zeta \cdots Z_\zeta A'_\zeta B'_\zeta \cdots E'_\zeta F'_\zeta \cdots Z'_\zeta}_{2s}}(s; x)|_{t=0}$ 

$$= i \frac{(i\zeta)^{2s+1}}{2^{2s-1}} \sum_{k=0}^{[s-\frac{1}{2}]} \frac{(2s)!}{(2s-2k-1)!(2k)!} \overbrace{(\sigma \cdot \nabla)_{A_\zeta A'_\zeta}}^{2s-2k-1} \overbrace{(\sigma \cdot \nabla)_{B_\zeta B'_\zeta}}^{2k} \cdots \delta_{E_\zeta E'_\zeta} \delta_{F_\zeta F'_\zeta} \cdots \nabla^{2k} \delta_{Z_\zeta Z'_\zeta} \delta^3(\vec{r})$$

推论8.10.6.

$$\left\{ \begin{aligned}
& (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a \partial^a \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}(s; x) = 0 & (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a \partial^a \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(c)}(s; x) = -\zeta \delta(t) \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}(s; x)|_{t=0} \\
& (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a \partial^a \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(+)}(s; x) = 0 & (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a \partial^a \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(ret)}(s; x) = -\zeta \delta(t) \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}(s; x)|_{t=0} \\
& (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a \partial^a \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(-)}(s; x) = 0 & (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a \partial^a \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(adv)}(s; x) = -\zeta \delta(t) \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}(s; x)|_{t=0} \\
& (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a \partial^a \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(l)}(s; x) = 0 & (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a \partial^a \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}^{(F)}(s; x) = -i\zeta \delta(t) \Delta_{\underbrace{A_\zeta B_\zeta \cdots A'_\zeta B'_\zeta}_{2s}}(s; x)|_{t=0}
\end{aligned} \right.$$

推论8.10.7.

$$\left\{ \begin{aligned}
& (\sigma, -i\zeta)_a \partial^a \Delta_{A_\zeta A'_\zeta}(\frac{1}{2}; x) = 0 & (\sigma, -i\zeta)_a \partial^a \Delta_{A_\zeta A'_\zeta}^{(c)}(\frac{1}{2}; x) = i\zeta \delta_{A_\zeta A'_\zeta} \delta^4(x) \\
& (\sigma, -i\zeta)_a \partial^a \Delta_{A_\zeta A'_\zeta}(\frac{1}{2}; x) = 0 & (\sigma, -i\zeta)_a \partial^a \Delta_{A_\zeta A'_\zeta}^{(ret)}(\frac{1}{2}; x) = i\zeta \delta_{A_\zeta A'_\zeta} \delta^4(x) \\
& (\sigma, -i\zeta)_a \partial^a \Delta_{A_\zeta A'_\zeta}(\frac{1}{2}; x) = 0 & (\sigma, -i\zeta)_a \partial^a \Delta_{A_\zeta A'_\zeta}^{(adv)}(\frac{1}{2}; x) = i\zeta \delta_{A_\zeta A'_\zeta} \delta^4(x) \\
& (\sigma, -i\zeta)_a \partial^a \Delta_{A_\zeta A'_\zeta}(\frac{1}{2}; x) = 0 & (\sigma, -i\zeta)_a \partial^a \Delta_{A_\zeta A'_\zeta}^{(F)}(\frac{1}{2}; x) = -\zeta \delta_{A_\zeta A'_\zeta} \delta^4(x)
\end{aligned} \right.$$

# 第二十五章 有质量复粒子的场协变量子化方案

自我评述：本章和下一章描述的有质量复粒子，正反粒子不同，数学本质上是复函数，区别于马约拉纳粒子；马约拉纳粒子，正反粒子相同，数学本质上是实函数，后面章节会详细讨论。有质量粒子方案采用与无质量粒子方案相反的步骤，先证明一般自旋粒子情形，再分别研究自旋 $-\frac{1}{2}, 1, \frac{3}{2}, 2$ 的特殊情形。这样做的理由：一是协变量子化新方案经过之前的研究，总体上已经比较明确。二是先证明一般情形，后面的特殊情形就不需再做证明，省去很多麻烦与篇幅，内容更紧凑，也可以更专注于物理。为了证明一般情形，必须先研究Dirac方程自旋基的性质，所以本章前半部分主要是研究Dirac方程自旋基的内容，后半部分才是一般自旋粒子情形的证明。但Dirac方程完整的协变量子化方案还是放在后面章节去研究。在本章按统一的方式对所有有质量自旋复粒子建立了相应的量子场论。与无质量粒子一样，也无需知道哈密顿量，就可以按统一的新程式对各种有质量自旋粒子进行了量子化，给出了统一的量子化对易规则和能量动量算符形式，给出了部分量子彭加莱代数。与无质量粒子一样，角动量算符也只取得部分成功，没有彻底解决，仍需努力。

## 1 基础准备部分

### 1.1 Dirac自旋基的引入

#### 1.1.1 Dirac电子方程<sup>[4]</sup>平面波的四维傅里叶解法

定理1.1.1.  $(\gamma^a \partial_a + m)\psi(\vec{r}, t) = 0$

$$\Leftrightarrow \begin{cases} \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{2E_{\vec{p}}} [a(\vec{p}, E_{\vec{p}}) e^{i(\vec{p}\cdot\vec{r} - E_{\vec{p}}t)} + a(-\vec{p}, -E_{\vec{p}}) e^{-i(\vec{p}\cdot\vec{r} - E_{\vec{p}}t)}] d^3\vec{p} \\ (i\gamma^a p_a + m)a(\vec{p}, E_{\vec{p}}) = 0, (-i\gamma^a p_a + m)a(-\vec{p}, -E_{\vec{p}}) = 0 \end{cases}$$

证明:  $(\gamma^a \partial_a + m)\psi(\vec{r}, t) = 0$

$$\begin{aligned} &\Rightarrow \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} (i\gamma^a p_a + m)\psi(\vec{p}, E) e^{ip\cdot x} d^3\vec{p} dE = 0 \\ &\Leftrightarrow (i\gamma^a p_a + m)\psi(\vec{p}, E) = 0 \Leftrightarrow (i\gamma^a p_a - m)(i\gamma^a p_a + m)\psi(\vec{p}, E) = 0, (i\gamma^a p_a + m)\psi(\vec{p}, E) = 0 \\ &\Leftrightarrow (E^2 - \vec{p}^2 - m^2)\psi(\vec{p}, E) = 0, (i\gamma^a p_a + m)\psi(\vec{p}, E) = 0 \\ &\Leftrightarrow \psi(\vec{p}, E) = a(\vec{p}, E)\delta(E^2 - \vec{p}^2 - m^2) + \psi_0(\vec{p}, E)\delta_{E^2, \vec{p}^2 + m^2}, (i\gamma^a p_a + m)\psi(\vec{p}, E) = 0 \\ &\Rightarrow \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} [a(\vec{p}, E)\delta(E^2 - \vec{p}^2 - m^2) + \psi_0(\vec{p}, E)\delta_{E^2, \vec{p}^2 + m^2}] e^{ip\cdot x} d^3\vec{p} dE \\ &\Leftrightarrow \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} a(\vec{p}, E)\delta(E^2 - \vec{p}^2 - m^2) e^{ip\cdot x} d^3\vec{p} dE \\ &\Leftrightarrow \begin{cases} \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{2E_{\vec{p}}} [a(\vec{p}, E_{\vec{p}}) e^{i(\vec{p}\cdot\vec{r} - E_{\vec{p}}t)} + a(\vec{p}, -E_{\vec{p}}) e^{i(\vec{p}\cdot\vec{r} + E_{\vec{p}}t)}] d^3\vec{p} \\ (i\gamma^a p_a + m)a(\vec{p}, E_{\vec{p}}) = 0, (i\gamma \cdot \vec{p} + \gamma^4 E_{\vec{p}} + m)a(\vec{p}, -E_{\vec{p}}) = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{2E_{\vec{p}}} [a(\vec{p}, E_{\vec{p}}) e^{i(\vec{p}\cdot\vec{r} - E_{\vec{p}}t)} + a(-\vec{p}, -E_{\vec{p}}) e^{-i(\vec{p}\cdot\vec{r} - E_{\vec{p}}t)}] d^3\vec{p} \\ (i\gamma^a p_a + m)a(\vec{p}, E_{\vec{p}}) = 0, (-i\gamma^a p_a + m)a(-\vec{p}, -E_{\vec{p}}) = 0 \end{cases} \quad \square \end{aligned}$$

定理1.1.2.  $(i\gamma^a p_a + m)a(\vec{p}, E_{\vec{p}}) = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), (i\gamma^a p_a + m) = \begin{bmatrix} m & -\varsigma E + \sigma \cdot \vec{p} \\ -\varsigma E - \sigma \cdot \vec{p} & m \end{bmatrix}$

$$\Leftrightarrow \begin{cases} a(\vec{p}, E_{\vec{p}}) = \begin{bmatrix} m\varphi(\vec{p}) \\ (\varsigma E_{\vec{p}} + \sigma \cdot \vec{p})\varphi(\vec{p}) \end{bmatrix} = (-i\gamma^a p_a + m) \begin{bmatrix} \varphi(\vec{p}) \\ 0 \end{bmatrix} = (-i\gamma^a p_a + m) \begin{bmatrix} 0 \\ \frac{\varsigma E_{\vec{p}} + \sigma \cdot \vec{p}}{m} \varphi(\vec{p}) \end{bmatrix} \\ a(\vec{p}, E_{\vec{p}}) = \begin{bmatrix} (\varsigma E_{\vec{p}} - \sigma \cdot \vec{p})\eta(\vec{p}) \\ m\eta(\vec{p}) \end{bmatrix} = (-i\gamma^a p_a + m) \begin{bmatrix} 0 \\ \eta(\vec{p}) \end{bmatrix} = (-i\gamma^a p_a + m) \begin{bmatrix} \frac{\varsigma E_{\vec{p}} - \sigma \cdot \vec{p}}{m} \eta(\vec{p}) \\ 0 \end{bmatrix} \end{cases}$$

$$\text{推论1.1.1. } (i\gamma^a p_a - m)a(-\vec{p}, -E_{\vec{p}}) = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), (i\gamma^a p_a - m) = \begin{bmatrix} -m & -\varsigma E + \sigma \cdot \vec{p} \\ -\varsigma E - \sigma \cdot \vec{p} & -m \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} a(-\vec{p}, -E_{\vec{p}}) = \begin{bmatrix} -m\varphi(-\vec{p}) \\ (\varsigma E_{\vec{p}} + \sigma \cdot \vec{p})\varphi(-\vec{p}) \end{bmatrix} = (-i\gamma^a p_a - m) \begin{bmatrix} \varphi(-\vec{p}) \\ 0 \end{bmatrix} = (-i\gamma^a p_a - m) \begin{bmatrix} 0 \\ \frac{-\varsigma E_{\vec{p}} - \sigma \cdot \vec{p}}{m}\varphi(-\vec{p}) \end{bmatrix} \\ a(-\vec{p}, -E_{\vec{p}}) = \begin{bmatrix} (\varsigma E_{\vec{p}} - \sigma \cdot \vec{p})\eta(-\vec{p}) \\ -m\eta(-\vec{p}) \end{bmatrix} = (-i\gamma^a p_a - m) \begin{bmatrix} 0 \\ \eta(-\vec{p}) \end{bmatrix} = (-i\gamma^a p_a - m) \begin{bmatrix} \frac{-\varsigma E_{\vec{p}} + \sigma \cdot \vec{p}}{m}\eta(-\vec{p}) \\ 0 \end{bmatrix} \end{cases}$$

自我评述：从上可以看出Dirac方程的平面波解有多种等价的表达形式。自旋基的直观选取有好多种，本质上有无限种选法，但它们本质上也是表象等价的，差一个么正变换。无论选取哪一个基，物理是等价一致的。但选的好，计算就方便。但要注意对于无质量粒子，以上几种表达不一定是等价的。

### 1.1.2 非归一化Dirac自旋基(适合任意质量情形)

推论1.1.2.  $(\gamma^a \partial_a + m)\psi(\vec{r}, t) = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x)$

$$\Leftrightarrow \psi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{2E} [a(\vec{p}, E)e^{i(\vec{p}\cdot\vec{r}-Et)} + a(-\vec{p}, -E)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$a(\vec{p}, E) = (-i\gamma^a p_a + m) \begin{bmatrix} \varphi(\vec{p}) \\ 0 \end{bmatrix}, a(-\vec{p}, -E) = (-i\gamma^a p_a - m) \begin{bmatrix} \eta(\vec{p}) \\ 0 \end{bmatrix}$$

$\gamma^a$ 本章采用以上约定，另有说明除外。

$$\text{定义1.1.1. } X(\vec{p}, \frac{\kappa}{2}) := (-i\gamma^a p_a + m) \begin{bmatrix} \lambda(\hat{p}, \frac{\kappa}{2}) \\ 0 \end{bmatrix}, Y(\vec{p}, \frac{\kappa}{2}) := (-i\gamma^a p_a - m) \begin{bmatrix} \lambda(\hat{p}, \frac{\kappa}{2}) \\ 0 \end{bmatrix}$$

$$\text{推论1.1.3. } X(\vec{p}, \frac{\kappa}{2}) = \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \begin{bmatrix} m \\ \varsigma E + \kappa|\vec{p}| \end{bmatrix}, Y(\vec{p}, \frac{\kappa}{2}) = \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \begin{bmatrix} -m \\ \varsigma E + \kappa|\vec{p}| \end{bmatrix}$$

### 1.1.3 归一化无质量Dirac自旋基

$$\text{定义1.1.2. } X(\vec{p}, \frac{\kappa}{2}) = Y(\vec{p}, \frac{\kappa}{2}) := -i\gamma^a p_a \begin{bmatrix} \lambda(\hat{p}, \frac{\kappa}{2}) \\ 0 \end{bmatrix}$$

$$\text{推论1.1.4. } X(\vec{p}, \frac{\kappa}{2}) = Y(\vec{p}, \frac{\kappa}{2}) = 2\varsigma|\vec{p}|\lambda(\hat{p}, \frac{\kappa}{2}) \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}, X(\vec{p}, -\frac{\kappa}{2}) = Y(\vec{p}, -\frac{\kappa}{2}) = 0$$

$$\text{推论1.1.5. } \bar{X}(\vec{p}, \frac{\kappa}{2}) = \bar{Y}(\vec{p}, \frac{\kappa}{2}) = \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \bar{X}(\vec{p}, -\frac{\kappa}{2}) = \bar{Y}(\vec{p}, -\frac{\kappa}{2}) = 0$$

自我评述：直接用 $m \rightarrow 0$ 得到无质量情形，并不全面，这只是一组解，还有另一组解。事实上无质量情形需要重新分析。

### 1.1.4 Dirac电荷基的定义

$$\text{定义1.1.3. } \mu(\vec{p}, \frac{\kappa}{2}) := \begin{bmatrix} \sqrt{\frac{E-\kappa\varsigma|\vec{p}|}{2m}} \\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{2m}} \end{bmatrix}, \nu(\vec{p}, \frac{\kappa}{2}) := \begin{bmatrix} -\sqrt{\frac{E-\kappa\varsigma|\vec{p}|}{2m}} \\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{2m}} \end{bmatrix}$$

$$\text{推论1.1.6. } \mu(\vec{p}, \frac{\kappa}{2}) := \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\frac{m}{E+\kappa\varsigma|\vec{p}|}} \\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{m}} \end{bmatrix}, \nu(\vec{p}, \frac{\kappa}{2}) := \frac{1}{\sqrt{2}} \begin{bmatrix} -\sqrt{\frac{m}{E+\kappa\varsigma|\vec{p}|}} \\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{m}} \end{bmatrix}$$

$$\text{定义1.1.4. } \tilde{\mu}(\vec{p}, \frac{\kappa}{2}) := \begin{bmatrix} \sqrt{\frac{E-\kappa\varsigma|\vec{p}|}{2E}} \\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{2E}} \end{bmatrix}, \tilde{\nu}(\vec{p}, \frac{\kappa}{2}) := \begin{bmatrix} -\sqrt{\frac{E-\kappa\varsigma|\vec{p}|}{2E}} \\ \varsigma\sqrt{\frac{E+\kappa\varsigma|\vec{p}|}{2E}} \end{bmatrix}$$

$$\text{推论1.1.7. } \tilde{\mu}(\vec{p}, \frac{\kappa}{2}) = \sqrt{\frac{m}{E}}\mu(\vec{p}, \frac{\kappa}{2}), \tilde{\nu}(\vec{p}, \frac{\kappa}{2}) = \sqrt{\frac{m}{E}}\nu(\vec{p}, \frac{\kappa}{2})$$

自我评述：为何称为电荷基跟后面的具体分析有关，也许这样称呼并不合适。

### 1.1.5 归一化Dirac自旋基

$$\text{推论1.1.8. } u(\vec{p}, \frac{\kappa}{2}) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2m(E+\kappa|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + \kappa|\vec{p}| \end{bmatrix}, v(\vec{p}, \frac{\kappa}{2}) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2m(E+\kappa|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + \kappa|\vec{p}| \end{bmatrix}$$

$$\text{推论1.1.9. } u(\vec{p}, \frac{\kappa}{2}) = \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \mu(\vec{p}, \frac{\kappa}{2}), v(\vec{p}, \frac{\kappa}{2}) = \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \nu(\vec{p}, \frac{\kappa}{2})$$

$$\text{推论1.1.10. } u(\vec{p}, h) = -\varsigma\gamma_5 v(\vec{p}, h), v(\vec{p}, h) = -\varsigma\gamma_5 u(\vec{p}, h), h = -\frac{1}{2}, \frac{1}{2}$$

$$\text{定理1.1.3. } (i\gamma^a p_a + m)u(\vec{p}, h) = 0, (i\gamma^a p_a - m)v(\vec{p}, h) = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x, \varsigma I \otimes \sigma_z)$$

$$\text{定义1.1.5. } \tilde{u}(\vec{p}, \frac{\kappa}{2}) := \sqrt{\frac{m}{E}} u(\vec{p}, \frac{\kappa}{2}), \tilde{v}(\vec{p}, \frac{\kappa}{2}) := \sqrt{\frac{m}{E}} v(\vec{p}, \frac{\kappa}{2})$$

$$\text{推论1.1.11. } \tilde{u}(\vec{p}, \frac{\kappa}{2}) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2E(E+\kappa|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + \kappa|\vec{p}| \end{bmatrix}, \tilde{v}(\vec{p}, \frac{\kappa}{2}) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2E(E+\kappa|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + \kappa|\vec{p}| \end{bmatrix}$$

$$\text{推论1.1.12. } \tilde{u}(\vec{p}, \frac{\kappa}{2}; m=0) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2E(E+\kappa|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + \kappa|\vec{p}| \end{bmatrix}, \tilde{v}(\vec{p}, \frac{\kappa}{2}; m=0) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2E(E+\kappa|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + \kappa|\vec{p}| \end{bmatrix}$$

自我评述：为何定义两种归一化自旋基，与两种不同的归一化方式相对应。又为何如此选择自旋基主要理由有两点，一是分解为两个基的直积，可以简化很多计算；二是其中一个基选为螺旋度，可以充分利用之前的螺旋度分析成果，也可以大大简化计算。

### 1.1.6 新电荷算符的定义

$$\text{定义1.1.6. } \hat{Q}(\vec{p}) := \frac{i\gamma^a p_a}{m}, \hat{q}(\vec{p}, \kappa) := \frac{-\varsigma E \sigma_x + i\kappa|\vec{p}|\sigma_y}{m}$$

$$\text{引理1.1.1. } i\gamma^a p_a = \begin{bmatrix} 0 & -\varsigma E + \sigma \cdot \vec{p} \\ -\varsigma E - \sigma \cdot \vec{p} & 0 \end{bmatrix} = -\varsigma E I \otimes \sigma_x + i\sigma \cdot \vec{p} \otimes \sigma_y, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x)$$

$$\text{定理1.1.4. } \hat{Q}(\vec{p})u(\vec{p}, \frac{\kappa}{2}) = -u(\vec{p}, \frac{\kappa}{2}), \hat{Q}(\vec{p})v(\vec{p}, \frac{\kappa}{2}) = v(\vec{p}, \frac{\kappa}{2}), \hat{q}(\vec{p}, \kappa)\mu(\vec{p}, \frac{\kappa}{2}) = -\mu(\vec{p}, \frac{\kappa}{2}), \hat{q}(\vec{p}, \kappa)\nu(\vec{p}, \frac{\kappa}{2}) = \nu(\vec{p}, \frac{\kappa}{2})$$

$$\begin{aligned} \text{证明: } \hat{Q}(\vec{p})u(\vec{p}, \frac{\kappa}{2}) &= \frac{i\gamma^a p_a}{m} u(\vec{p}, \frac{\kappa}{2}) \\ &= (-\varsigma \frac{E}{m} I \otimes \sigma_x + i \frac{1}{m} \sigma \cdot \vec{p} \otimes \sigma_y) \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \mu(\vec{p}, \frac{\kappa}{2}) = (I \otimes \frac{-\varsigma E \sigma_x + i\kappa|\vec{p}|\sigma_y}{m}) (\lambda(\hat{p}, \frac{\kappa}{2})) \otimes \mu(\vec{p}, \frac{\kappa}{2}) \\ &= -\lambda(\hat{p}, \frac{\kappa}{2}) \otimes \mu(\vec{p}, \frac{\kappa}{2}) = -u(\vec{p}, \frac{\kappa}{2}) \end{aligned} \quad \square$$

$$\begin{aligned} \text{证明: } \hat{Q}(\vec{p})v(\vec{p}, \frac{\kappa}{2}) &= \frac{i\gamma^a p_a}{m} v(\vec{p}, \frac{\kappa}{2}) \\ &= (-\varsigma \frac{E}{m} I \otimes \sigma_x + i \frac{1}{m} \sigma \cdot \vec{p} \otimes \sigma_y) \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \nu(\vec{p}, \frac{\kappa}{2}) = (I \otimes \frac{-\varsigma E \sigma_x + i\kappa|\vec{p}|\sigma_y}{m}) (\lambda(\hat{p}, \frac{\kappa}{2})) \otimes \nu(\vec{p}, \frac{\kappa}{2}) \\ &= \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \nu(\vec{p}, \frac{\kappa}{2}) = v(\vec{p}, \frac{\kappa}{2}) \end{aligned} \quad \square$$

### 1.1.7 Dirac自旋基是自旋、螺旋度和电荷三个算符的共同本征态

性质1.1.1.

$$\begin{cases} \sigma^2(\frac{1}{2}) \otimes I u(\vec{p}, \frac{\kappa}{2}) = \frac{1}{2}(\frac{1}{2} + 1)u(\vec{p}, \frac{\kappa}{2}) & \sigma^2(\frac{1}{2}) \otimes I v(\vec{p}, \frac{\kappa}{2}) = \frac{1}{2}(\frac{1}{2} + 1)v(\vec{p}, \frac{\kappa}{2}) \\ \sigma(\frac{1}{2}) \cdot \hat{p} \otimes I u(\vec{p}, \frac{\kappa}{2}) = \frac{\kappa}{2}u(\vec{p}, \frac{\kappa}{2}) & \sigma(\frac{1}{2}) \cdot \hat{p} \otimes I v(\vec{p}, \frac{\kappa}{2}) = \frac{\kappa}{2}v(\vec{p}, \frac{\kappa}{2}) \\ \hat{Q}(\vec{p})u(\vec{p}, \frac{\kappa}{2}) = -u(\vec{p}, \frac{\kappa}{2}) & \hat{Q}(\vec{p})v(\vec{p}, \frac{\kappa}{2}) = v(\vec{p}, \frac{\kappa}{2}) \\ \text{描述电子: } (s, h; Q) = (\frac{1}{2}; \frac{\kappa}{2}, -1) & \text{描述正电子: } (s, h; Q) = (\frac{1}{2}; \frac{\kappa}{2}, 1) \end{cases}$$

## 1.2 四维矢量自旋基的引入

### 1.2.1 四维矢量自旋基

$$\text{推论1.2.1. } \lambda_m(\hat{p}, -1) = \lambda_m^*(\hat{p}, 1), \lambda_m(\hat{p}, 0) = -\lambda_m^*(\hat{p}, 0), \lambda_m(\hat{p}, 1) = \lambda_m^*(\hat{p}, -1)$$

$$\text{定义1.2.1. } \varepsilon_a(\vec{p}, \kappa) := [i\lambda_m(\vec{p}, \kappa), 0]_a, \varepsilon_a(\vec{p}, 0) := \frac{1}{m}[iE\lambda_m(\vec{p}, 0), i|\vec{p}|]_a, \bar{\varepsilon}_a(\vec{p}, h) := \varepsilon_a^+(\vec{p}, h)\eta_a^a$$

$$\text{推论1.2.2. } \begin{cases} \lambda_m(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, 1) = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ -1 \\ 0 \end{bmatrix} \\ \lambda_m(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, 0) = \begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix} \\ \lambda_m(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, -1) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ -1 \\ 0 \end{bmatrix} \end{cases}$$

$$\text{推论1.2.3. } \varepsilon_a\left(\begin{bmatrix} 0 \\ |\vec{p}| \end{bmatrix}, 1\right) := \frac{1}{\sqrt{2}}[-1, -i, 0, 0]_a, \varepsilon_a\left(\begin{bmatrix} 0 \\ |\vec{p}| \end{bmatrix}, 0\right) := \frac{1}{m}[0, 0, E, i|\vec{p}|]_a, \varepsilon_a\left(\begin{bmatrix} 0 \\ |\vec{p}| \end{bmatrix}, -1\right) := \frac{1}{\sqrt{2}}[1, -i, 0, 0]_a$$

$$\text{推论1.2.4. } \eta_{aa'}\varepsilon^{+a'}(\vec{p}, \kappa) = -\varepsilon_a(\vec{p}, -\kappa), \eta_{aa'}\varepsilon^{+a'}(\vec{p}, 0) = \varepsilon_a(\vec{p}, 0), \eta_{aa'}\varepsilon^{+a'}(\vec{p}, h) = (-1)^h\varepsilon_a(\vec{p}, -h)$$

$$\text{定理1.2.1. } \varepsilon^+(\vec{p}, h)\varepsilon(\vec{p}, h') = \left(\frac{E^2+p^2}{m^2}\right)^{1-|h|}\delta_{hh'}, \sum_{h=1}^{-1}\varepsilon_a(\vec{p}, h)\varepsilon_a^+(\vec{p}, h) = \eta_{aa'} + \frac{p_a p_{a'}}{m^2}, \sum_{h=1}^{-1}h\varepsilon(\vec{p}, h)\varepsilon^+(\vec{p}, h) = R \cdot \hat{p}$$

$$\text{定理1.2.2. } \bar{\varepsilon}(\vec{p}, h)\varepsilon(\vec{p}, h') = \delta_{hh'}, \sum_{h=1}^{-1}\varepsilon_a(\vec{p}, h)\bar{\varepsilon}_b(\vec{p}, h) = \delta_{ab} + \frac{p_a p_b}{m^2}, \sum_{h=1}^{-1}h\varepsilon(\vec{p}, h)\bar{\varepsilon}(\vec{p}, h) = R \cdot \hat{p}$$

$$\text{推论1.2.5. } (R \cdot \hat{p})\varepsilon(\vec{p}, h) = h\varepsilon(\vec{p}, h), (R \cdot \hat{p})\frac{P^{[a]}}{m} = 0; R^2\varepsilon(\vec{p}, h) = 1(1+1)\varepsilon(\vec{p}, h)$$

$$\text{推论1.2.6. } (L \cdot \hat{p})\varepsilon(\vec{p}, \kappa) = 0, (L \cdot \hat{p})\varepsilon(\vec{p}, 0) = -\frac{P^{[a]}}{m}, (L \cdot \hat{p})\frac{P^{[a]}}{m} = -\varepsilon(\vec{p}, 0)$$

$$\text{推论1.2.7. } \begin{cases} (\sigma_+ \cdot \hat{p})\varepsilon(\vec{p}, \kappa) = \kappa\varepsilon(\vec{p}, \kappa), (\sigma_+ \cdot \hat{p})\varepsilon(\vec{p}, 0) = -\frac{P^{[a]}}{m}, (\sigma_+ \cdot \hat{p})\frac{P^{[a]}}{m} = -\varepsilon(\vec{p}, 0) \\ (\sigma_- \cdot \hat{p})\varepsilon(\vec{p}, \kappa) = \kappa\varepsilon(\vec{p}, \kappa), (\sigma_- \cdot \hat{p})\varepsilon(\vec{p}, 0) = \frac{P^{[a]}}{m}, (\sigma_- \cdot \hat{p})\frac{P^{[a]}}{m} = \varepsilon(\vec{p}, 0) \end{cases}$$

自我评述：为何如此选择自旋基，与后面的具体分析相关，事实上我是在后面的具体分析中提炼出这个结果，然后放到这里进行必要的先行研究，从而使后面的章节可以更专注于物理本身。

### 1.2.2 复矢量自旋基和四维矢量自旋基之间的关系I

$$\text{推论1.2.8. } \begin{cases} [R \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, \kappa) = -i\kappa\varepsilon(\vec{p}, h), [R \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, 0) = 0, [R \cdot \lambda_m(\vec{p}, 0)]\frac{P^{[a]}}{m} = 0 \\ [L \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, \kappa) = 0, [L \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, 0) = \frac{iP^{[a]}}{m}, [L \cdot \lambda_m(\vec{p}, 0)]\frac{P^{[a]}}{m} = i\varepsilon(\vec{p}, 0) \end{cases}$$

$$\text{推论1.2.9. } \begin{cases} [R \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 1) = [\vec{0}, 0], [R \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, -1) = -[\lambda_m(\vec{p}, 0), 0] \\ [R \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 0) = -\frac{E}{m}[\lambda_m(\vec{p}, 1), 0], [R \cdot \lambda_m(\vec{p}, 1)]\frac{P^{[a]}}{m} = -\frac{|\vec{p}|}{m}[\lambda_m(\vec{p}, 1), 0] \\ [L \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 1) = [\vec{0}, 0], [L \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, -1) = [\vec{0}, 1] \\ [L \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 0) = -\frac{|\vec{p}|}{m}[\lambda_m(\vec{p}, 1), 0], [L \cdot \lambda_m(\vec{p}, 1)]\frac{P^{[a]}}{m} = -\frac{E}{m}[\lambda_m(\vec{p}, 1), 0] \end{cases}$$

$$\text{推论1.2.10. } \begin{cases} [R \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, -1) = [\vec{0}, 0], [R \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 1) = [\lambda_m(\vec{p}, 0), 0] \\ [R \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 0) = \frac{E}{m}[\lambda_m(\vec{p}, -1), 0], [R \cdot \lambda_m(\vec{p}, -1)]\frac{P^{[a]}}{m} = \frac{|\vec{p}|}{m}[\lambda_m(\vec{p}, -1), 0] \\ [L \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, -1) = [\vec{0}, 0], [L \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 1) = [\vec{0}, 1] \\ [L \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 0) = -\frac{|\vec{p}|}{m}[\lambda_m(\vec{p}, -1), 0], [L \cdot \lambda_m(\vec{p}, -1)]\frac{P^{[a]}}{m} = -\frac{E}{m}[\lambda_m(\vec{p}, -1), 0] \end{cases}$$

### 1.2.3 复矢量自旋基和四维矢量自旋基之间的关系II

$$\text{推论1.2.11. } \begin{cases} [\sigma_+ \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, \kappa) = -i\kappa\varepsilon(\vec{p}, h), [\sigma_+ \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, 0) = \frac{iP^{[a]}}{m}, [\sigma_+ \cdot \lambda_m(\vec{p}, 0)]\frac{P^{[a]}}{m} = i\varepsilon(\vec{p}, 0) \\ [\sigma_- \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, \kappa) = -i\kappa\varepsilon(\vec{p}, h), [\sigma_- \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, 0) = -\frac{iP^{[a]}}{m}, [\sigma_- \cdot \lambda_m(\vec{p}, 0)]\frac{P^{[a]}}{m} = -i\varepsilon(\vec{p}, 0) \end{cases}$$

$$\text{推论1.2.12. } \begin{cases} [\sigma_+ \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 1) = [\vec{0}, 0], [\sigma_+ \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, -1) = -[\lambda_m(\vec{p}, 0), -1] \\ [\sigma_+ \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 0) = -\frac{E+|\vec{p}|}{m}[\lambda_m(\vec{p}, 1), 0], [\sigma_+ \cdot \lambda_m(\vec{p}, 1)]\frac{P^{[a]}}{m} = -\frac{E+|\vec{p}|}{m}[\lambda_m(\vec{p}, 1), 0] \\ [\sigma_- \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 1) = [\vec{0}, 0], [\sigma_- \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, -1) = -[\lambda_m(\vec{p}, 0), 1] \\ [\sigma_- \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 0) = -\frac{E-|\vec{p}|}{m}[\lambda_m(\vec{p}, 1), 0], [\sigma_- \cdot \lambda_m(\vec{p}, 1)]\frac{P^{[a]}}{m} = \frac{E-|\vec{p}|}{m}[\lambda_m(\vec{p}, 1), 0] \end{cases}$$

$$\text{推论1.2.13. } \begin{cases} [\sigma_+ \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, -1) = [\vec{0}, 0], [\sigma_+ \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 1) = [\lambda_m(\vec{p}, 0), 1] \\ [\sigma_+ \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 0) = \frac{E-|\vec{p}|}{m}[\lambda_m(\vec{p}, -1), 0], [\sigma_+ \cdot \lambda_m(\vec{p}, -1)]\frac{P^{[a]}}{m} = -\frac{E-|\vec{p}|}{m}[\lambda_m(\vec{p}, -1), 0] \\ [\sigma_- \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, -1) = [\vec{0}, 0], [\sigma_- \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 1) = [\lambda_m(\vec{p}, 0), -1] \\ [\sigma_- \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 0) = \frac{E+|\vec{p}|}{m}[\lambda_m(\vec{p}, -1), 0], [\sigma_- \cdot \lambda_m(\vec{p}, -1)]\frac{P^{[a]}}{m} = \frac{E+|\vec{p}|}{m}[\lambda_m(\vec{p}, -1), 0] \end{cases}$$



## 1.2.4 复矢量自旋基和四维矢量自旋基之间的关系III

$$\text{推论1.2.14.} \begin{cases} [\sigma_+ \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, \kappa) = -i\kappa\varepsilon(\vec{p}, h), [\sigma_+ \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, 0) = \frac{ip_{[a]}}{m}, [\sigma_+ \cdot \lambda_m(\vec{p}, 0)]\frac{p_{[a]}}{m} = i\varepsilon(\vec{p}, 0) \\ [\sigma_- \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, \kappa) = -i\kappa\varepsilon(\vec{p}, h), [\sigma_- \cdot \lambda_m(\vec{p}, 0)]\varepsilon(\vec{p}, 0) = -\frac{ip_{[a]}}{m}, [\sigma_- \cdot \lambda_m(\vec{p}, 0)]\frac{p_{[a]}}{m} = -i\varepsilon(\vec{p}, 0) \end{cases}$$

$$\text{推论1.2.15.} \begin{cases} [\sigma_+ \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 1) = [\vec{0}, 0], [\sigma_+ \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, -1) = i\frac{E+|\vec{p}|}{m}[\varepsilon(\vec{p}, 0) - \frac{p_{[a]}}{m}] \\ [\sigma_+ \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 0) = i\frac{E+|\vec{p}|}{m}\varepsilon(\vec{p}, 1), [\sigma_+ \cdot \lambda_m(\vec{p}, 1)]\frac{p_{[a]}}{m} = i\frac{E+|\vec{p}|}{m}\varepsilon(\vec{p}, 1) \\ [\sigma_- \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 1) = [\vec{0}, 0], [\sigma_- \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, -1) = i\frac{E-|\vec{p}|}{m}[\varepsilon(\vec{p}, 0) + \frac{p_{[a]}}{m}] \\ [\sigma_- \cdot \lambda_m(\vec{p}, 1)]\varepsilon(\vec{p}, 0) = i\frac{E-|\vec{p}|}{m}\varepsilon(\vec{p}, 1), [\sigma_- \cdot \lambda_m(\vec{p}, 1)]\frac{p_{[a]}}{m} = -i\frac{E-|\vec{p}|}{m}\varepsilon(\vec{p}, 1) \end{cases}$$

$$\text{推论1.2.16.} \begin{cases} [\sigma_+ \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, -1) = [\vec{0}, 0], [\sigma_+ \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 1) = -i\frac{E-|\vec{p}|}{m}[\varepsilon(\vec{p}, 0) + \frac{p_{[a]}}{m}] \\ [\sigma_+ \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 0) = -i\frac{E-|\vec{p}|}{m}\varepsilon(\vec{p}, -1), [\sigma_+ \cdot \lambda_m(\vec{p}, -1)]\frac{p_{[a]}}{m} = i\frac{E-|\vec{p}|}{m}\varepsilon(\vec{p}, -1) \\ [\sigma_- \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, -1) = [\vec{0}, 0], [\sigma_- \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 1) = -i\frac{E+|\vec{p}|}{m}[\varepsilon(\vec{p}, 0) - \frac{p_{[a]}}{m}] \\ [\sigma_- \cdot \lambda_m(\vec{p}, -1)]\varepsilon(\vec{p}, 0) = -i\frac{E+|\vec{p}|}{m}\varepsilon(\vec{p}, -1), [\sigma_- \cdot \lambda_m(\vec{p}, -1)]\frac{p_{[a]}}{m} = -i\frac{E+|\vec{p}|}{m}\varepsilon(\vec{p}, -1) \end{cases}$$

## 1.2.5 复矢量自旋基和四维矢量自旋基之间的关系IV

$$\text{推论1.2.17.} \begin{cases} [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, \kappa)]\lambda_m^\alpha(\vec{p}, 0) = -i\kappa\varepsilon^a(\vec{p}, h), [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 0)]\lambda_m^\alpha(\vec{p}, 0) = \frac{ip^a}{m}, [\sigma_{+\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p}, 0) = i\varepsilon^a(\vec{p}, 0) \\ [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p}, \kappa)]\lambda_m^\alpha(\vec{p}, 0) = -i\kappa\varepsilon^a(\vec{p}, h), [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p}, 0)]\lambda_m^\alpha(\vec{p}, 0) = -\frac{ip^a}{m}, [\sigma_{-\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p}, 0) = -i\varepsilon^a(\vec{p}, 0) \end{cases}$$

$$\text{推论1.2.18.} \begin{cases} [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 1)]\lambda_m^\alpha(\vec{p}, 1) = [\vec{0}, 0], [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, -1)]\lambda_m^\alpha(\vec{p}, 1) = i\frac{E+|\vec{p}|}{m}[\varepsilon^a(\vec{p}, 0) - \frac{p^a}{m}] \\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 0)]\lambda_m^\alpha(\vec{p}, 1) = i\frac{E+|\vec{p}|}{m}\varepsilon^a(\vec{p}, 1), [\sigma_{+\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p}, 1) = i\frac{E+|\vec{p}|}{m}\varepsilon^a(\vec{p}, 1) \\ [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p}, 1)]\lambda_m^\alpha(\vec{p}, 1) = [\vec{0}, 0], [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p}, -1)]\lambda_m^\alpha(\vec{p}, 1) = i\frac{E-|\vec{p}|}{m}[\varepsilon^a(\vec{p}, 0) + \frac{p^a}{m}] \\ [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p}, 0)]\lambda_m^\alpha(\vec{p}, 1) = i\frac{E-|\vec{p}|}{m}\varepsilon^a(\vec{p}, 1), [\sigma_{-\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p}, 1) = -i\frac{E-|\vec{p}|}{m}\varepsilon^a(\vec{p}, 1) \end{cases}$$

$$\text{推论1.2.19.} \begin{cases} [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, -1)]\lambda_m^\alpha(\vec{p}, -1) = [\vec{0}, 0], [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 1)]\lambda_m^\alpha(\vec{p}, -1) = -i\frac{E-|\vec{p}|}{m}[\varepsilon^a(\vec{p}, 0) + \frac{p^a}{m}] \\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 0)]\lambda_m^\alpha(\vec{p}, -1) = -i\frac{E-|\vec{p}|}{m}\varepsilon^a(\vec{p}, -1), [\sigma_{+\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p}, -1) = i\frac{E-|\vec{p}|}{m}\varepsilon^a(\vec{p}, -1) \\ [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p}, -1)]\lambda_m^\alpha(\vec{p}, -1) = [\vec{0}, 0], [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p}, 1)]\lambda_m^\alpha(\vec{p}, -1) = -i\frac{E+|\vec{p}|}{m}[\varepsilon^a(\vec{p}, 0) - \frac{p^a}{m}] \\ [\sigma_{-\alpha}^{ab}\varepsilon_b(\vec{p}, 0)]\lambda_m^\alpha(\vec{p}, -1) = -i\frac{E+|\vec{p}|}{m}\varepsilon^a(\vec{p}, -1), [\sigma_{-\alpha}^{ab}\frac{p_b}{m}]\lambda_m^\alpha(\vec{p}, -1) = -i\frac{E+|\vec{p}|}{m}\varepsilon^a(\vec{p}, -1) \end{cases}$$

## 1.2.6 复矢量自旋基和四维矢量自旋基之间的关系V

推论1.2.20.

$$\begin{cases} [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 1)]\lambda_m^\alpha(\vec{p}, 0) = -i\varepsilon^a(\vec{p}, 1), [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 0)]\lambda_m^\alpha(\vec{p}, 0) = \frac{ip^a}{m}, [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, -1)]\lambda_m^\alpha(\vec{p}, 0) = i\varepsilon^a(\vec{p}, -1) \\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 1)]\lambda_m^\alpha(\vec{p}, 1) = [\vec{0}, 0], [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 0)]\lambda_m^\alpha(\vec{p}, 1) = i\frac{E+|\vec{p}|}{m}\varepsilon^a(\vec{p}, 1) \\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, -1)]\lambda_m^\alpha(\vec{p}, 1) = i\frac{E+|\vec{p}|}{m}[\varepsilon^a(\vec{p}, 0) - \frac{p^a}{m}] \\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 1)]\lambda_m^\alpha(\vec{p}, -1) = -i\frac{E-|\vec{p}|}{m}[\varepsilon^a(\vec{p}, 0) + \frac{p^a}{m}], [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, 0)]\lambda_m^\alpha(\vec{p}, -1) = -i\frac{E-|\vec{p}|}{m}\varepsilon^a(\vec{p}, -1) \\ [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, -1)]\lambda_m^\alpha(\vec{p}, -1) = [\vec{0}, 0] \end{cases}$$

推论1.2.21.

$$\begin{aligned} & \sum_{h, h'=1}^{-1} [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, h)]\lambda_m^\alpha(\vec{p}, h')\{[\sigma_{+\alpha'}^{a'b'}\varepsilon_{b'}(\vec{p}, h)]\lambda_m^{\alpha'}(\vec{p}, h')\}^+ \\ &= \sum_{h=1}^{-1} [\sigma_{+\alpha}^{ab}\varepsilon_b(\vec{p}, h)]\delta^{\alpha\alpha'}\{[\sigma_{+\alpha'}^{a'b'}\varepsilon_{b'}(\vec{p}, h)]\}^+ \\ &= -\delta^{\alpha\alpha'}\sigma_{+\alpha}^{ab}\sigma_{+\alpha'}^{a'b'}\sum_{h=1}^{-1}\varepsilon_b(\vec{p}, h)\varepsilon_{b'}^+(\vec{p}, h) \end{aligned}$$

$$\begin{aligned}
&= -(-\delta^{aa'}\delta^{bb'} + \delta^{ab'}\delta^{ba'} + \varepsilon^{aba'b'}) \sum_{h=1}^{-1} \varepsilon_b(\vec{p}, h) \varepsilon_{b'}^+(\vec{p}, h) \\
&= 3\delta^{aa'} - \sum_{h=1}^{-1} \varepsilon^{a'}(\vec{p}, h) \varepsilon^{+a}(\vec{p}, h) = 3\delta^{aa'} - \sum_{h=1}^{-1} \varepsilon^a(\vec{p}, h) \varepsilon^{+a'}(\vec{p}, h)
\end{aligned}$$

推论1.2.22.

$$\begin{aligned}
&\sum_{h, h'=1}^{-1} [\sigma_{-\alpha}^{ab} \varepsilon_b(\vec{p}, h)] \lambda_m^\alpha(\vec{p}, h') \{[\sigma_{-\alpha'}^{a'b'} \varepsilon_{b'}(\vec{p}, h)] \lambda_m^{\alpha'}(\vec{p}, h')\}^+ \\
&= \sum_{h=1}^{-1} [\sigma_{-\alpha}^{ab} \varepsilon_b(\vec{p}, h)] \delta^{\alpha\alpha'} \{[\sigma_{-\alpha'}^{a'b'} \varepsilon_{b'}(\vec{p}, h)]\}^+ \\
&= -\delta^{\alpha\alpha'} \sigma_{-\alpha}^{ab} \sigma_{-\alpha'}^{a'b'} \sum_{h=1}^{-1} \varepsilon_b(\vec{p}, h) \varepsilon_{b'}^+(\vec{p}, h) \\
&= -(-\delta^{aa'}\delta^{bb'} + \delta^{ab'}\delta^{ba'} - \varepsilon^{aba'b'}) \sum_{h=1}^{-1} \varepsilon_b(\vec{p}, h) \varepsilon_{b'}^+(\vec{p}, h) \\
&= 3\delta^{aa'} - \sum_{h=1}^{-1} \varepsilon^{a'}(\vec{p}, h) \varepsilon^{+a}(\vec{p}, h) = 3\delta^{aa'} - \sum_{h=1}^{-1} \varepsilon^a(\vec{p}, h) \varepsilon^{+a'}(\vec{p}, h)
\end{aligned}$$

### 1.3 Dirac自旋基的数学分析

#### 1.3.1 二维自旋基之间的等价关系

$$\text{性质1.3.1.} \quad \begin{cases} \lambda^*(\hat{p}, -\frac{\kappa}{2}) = -i\kappa\sigma_y \lambda(\hat{p}, \frac{\kappa}{2}), \lambda^+(\hat{p}, -\frac{\kappa}{2}) = i\kappa\lambda^T(\hat{p}, \frac{\kappa}{2})\sigma_y \\ \lambda(\hat{p}, \frac{\kappa}{2}) = i\kappa\sigma_y \lambda^*(\hat{p}, -\frac{\kappa}{2}), \lambda^T(\hat{p}, \frac{\kappa}{2}) = -i\kappa\lambda^+(\hat{p}, -\frac{\kappa}{2})\sigma_y \end{cases}$$

$$\text{性质1.3.2.} \quad \begin{cases} \mu^*(\vec{p}, -\frac{\kappa}{2}) = \varsigma\sigma_x \mu(\vec{p}, \frac{\kappa}{2}), \mu^+(\vec{p}, -\frac{\kappa}{2}) = \varsigma\mu^T(\vec{p}, \frac{\kappa}{2})\sigma_x \\ \nu^*(\vec{p}, -\frac{\kappa}{2}) = -\varsigma\sigma_x \nu(\vec{p}, \frac{\kappa}{2}), \nu^+(\vec{p}, -\frac{\kappa}{2}) = -\varsigma\nu^T(\vec{p}, \frac{\kappa}{2})\sigma_x \end{cases}$$

$$\text{性质1.3.3.} \quad \begin{cases} \mu(\vec{p}, \frac{\kappa}{2}) = \varsigma\sigma_x \mu^*(\vec{p}, -\frac{\kappa}{2}), \mu^T(\vec{p}, \frac{\kappa}{2}) = \varsigma\mu^+(\vec{p}, -\frac{\kappa}{2})\sigma_x \\ \nu(\vec{p}, \frac{\kappa}{2}) = -\varsigma\sigma_x \nu^*(\vec{p}, -\frac{\kappa}{2}), \nu^T(\vec{p}, \frac{\kappa}{2}) = -\varsigma\nu^+(\vec{p}, -\frac{\kappa}{2})\sigma_x \end{cases}$$

#### 1.3.2 Dirac自旋基之间的等价关系

$$\text{性质1.3.4.} \quad \begin{cases} u(\vec{p}, \frac{\kappa}{2}) = i\kappa\varsigma\sigma_y \otimes \sigma_x u^*(\vec{p}, -\frac{\kappa}{2}) = \kappa\gamma_2\gamma_5 u^*(\vec{p}, -\frac{\kappa}{2}) \\ v(\vec{p}, \frac{\kappa}{2}) = -i\kappa\varsigma\sigma_y \otimes \sigma_x v^*(\vec{p}, -\frac{\kappa}{2}) = -\kappa\gamma_2\gamma_5 v^*(\vec{p}, -\frac{\kappa}{2}) \end{cases}$$

$$\text{性质1.3.5.} \quad \begin{cases} u^*(\vec{p}, -\frac{\kappa}{2}) = -i\kappa\varsigma\sigma_y \otimes \sigma_x u(\vec{p}, \frac{\kappa}{2}) = -\kappa\gamma_2\gamma_5 u(\vec{p}, \frac{\kappa}{2}) \\ v^*(\vec{p}, -\frac{\kappa}{2}) = i\kappa\varsigma\sigma_y \otimes \sigma_x v(\vec{p}, \frac{\kappa}{2}) = \kappa\gamma_2\gamma_5 v(\vec{p}, \frac{\kappa}{2}) \end{cases}$$

$$\text{性质1.3.6.} \quad \begin{cases} u^+(\vec{p}, -\frac{\kappa}{2}) = i\kappa\varsigma u^T(\vec{p}, \frac{\kappa}{2})\sigma_y \otimes \sigma_x = \kappa u^T(\vec{p}, \frac{\kappa}{2})\gamma_2\gamma_5 \\ v^+(\vec{p}, -\frac{\kappa}{2}) = -i\kappa\varsigma v^T(\vec{p}, \frac{\kappa}{2})\sigma_y \otimes \sigma_x = -i\kappa v^T(\vec{p}, \frac{\kappa}{2})\gamma_2\gamma_5 \end{cases}$$

$$\text{性质1.3.7.} \quad \begin{cases} u^T(\vec{p}, \frac{\kappa}{2}) = -i\kappa\varsigma u^+(\vec{p}, -\frac{\kappa}{2})\sigma_y \otimes \sigma_x = -\kappa u^+(\vec{p}, -\frac{\kappa}{2})\gamma_2\gamma_5 \\ v^T(\vec{p}, \frac{\kappa}{2}) = i\kappa\varsigma v^+(\vec{p}, -\frac{\kappa}{2})\sigma_y \otimes \sigma_x = \kappa v^+(\vec{p}, -\frac{\kappa}{2})\gamma_2\gamma_5 \end{cases}$$

#### 1.3.3 Dirac自旋基的完备性分析

$$\text{推论1.3.1.} \quad \begin{cases} \mu(\vec{p}, \frac{\kappa}{2}) \mu^+(\vec{p}, -\frac{\kappa}{2}) = \frac{1}{2m} \begin{bmatrix} m & \varsigma E - \kappa |\vec{p}| \\ \varsigma E + \kappa |\vec{p}| & m \end{bmatrix} = \frac{1}{2} (I + \varsigma \frac{E}{m} \sigma_x - i\kappa \frac{|\vec{p}|}{m} \sigma_y) \\ \mu(\vec{p}, \frac{\kappa}{2}) \mu^+(\vec{p}, \frac{\kappa}{2}) = \frac{\varsigma}{2m} \begin{bmatrix} \varsigma E - \kappa |\vec{p}| & m \\ \varsigma E + \kappa |\vec{p}| & m \end{bmatrix} = \frac{\varsigma}{2} (I + \varsigma \frac{E}{m} \sigma_x - i\kappa \frac{|\vec{p}|}{m} \sigma_y) \sigma_x \end{cases}$$

$$\text{推论1.3.2.} \quad \begin{cases} u(\vec{p}, \frac{\kappa}{2}) u^+(\vec{p}, -\frac{\kappa}{2}) = \frac{1}{4} [\kappa(\sigma \cdot \hat{p} + I) i\sigma_y] \otimes (I + \varsigma \frac{E}{m} \sigma_x - i\kappa \frac{|\vec{p}|}{m} \sigma_y) \\ u(\vec{p}, \frac{\kappa}{2}) u^+(\vec{p}, \frac{\kappa}{2}) = \frac{1}{4} [(\kappa\sigma \cdot \hat{p} + I) \otimes (I + \varsigma \frac{E}{m} \sigma_x - i\kappa \frac{|\vec{p}|}{m} \sigma_y)] (\varsigma I \otimes \sigma_x) \end{cases}$$

$$\text{推论1.3.3. } \sum_{h=1/2}^{-1/2} [u(\vec{p}, h)\bar{u}(\vec{p}, h) - v(\vec{p}, h)\bar{v}(\vec{p}, h)] = I_4, \sum_{h=1/2}^{-1/2} [u(\vec{p}, h)\bar{u}(\vec{p}, h) + v(\vec{p}, h)\bar{v}(\vec{p}, h)] = \frac{-i\gamma^a p_a}{m}$$

$$\text{推论1.3.4. } \sum_{h=1/2}^{-1/2} [u(\vec{p}, h)u^+(\vec{p}, h) + v(-\vec{p}, h)v^+(-\vec{p}, h)] = \frac{E}{m}$$

### 1.3.4 Dirac方程准投影算子<sup>[4]</sup>

$$\text{定义1.3.1. } \Lambda_+(\frac{1}{2}) := \sum_{h=1/2}^{-1/2} u(\vec{p}, h)u^+(\vec{p}, h) = \frac{(m-i\gamma^a p_a)\gamma_4}{2m}, \Lambda_-(\frac{1}{2}) := \sum_{h=1/2}^{-1/2} v(\vec{p}, h)v^+(\vec{p}, h) = \frac{(-m-i\gamma^a p_a)\gamma_4}{2m}$$

### 1.3.5 二维自旋基的正交性质

$$\text{定义1.3.2. } \hat{p}_a := (\hat{p}, i)$$

性质1.3.8.

$$\begin{cases} \lambda^+(\hat{p}, \frac{\kappa}{2})(\sigma, i\kappa)_a \lambda(\hat{p}, \frac{\kappa}{2}) = \kappa \hat{p}_a \\ \mu^+(\vec{p}, \frac{\kappa}{2})(\sigma, I)_a \mu(\vec{p}, \frac{\kappa}{2}) = \frac{1}{m}(\varsigma m, 0, -\kappa\varsigma|\vec{p}|, E)_a \\ \nu^+(\vec{p}, \frac{\kappa}{2})(\sigma, I)_a \nu(\vec{p}, \frac{\kappa}{2}) = \frac{1}{m}(-\varsigma m, 0, -\kappa\varsigma|\vec{p}|, E)_a \end{cases} \quad \begin{cases} \lambda^+(\hat{p}, -\frac{\kappa}{2})(\sigma, i\kappa)_a \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2}[\lambda_m(\hat{p}, \kappa), 0]_a \\ \mu^+(\vec{p}, -\frac{\kappa}{2})(\sigma, I)_a \mu(\vec{p}, \frac{\kappa}{2}) = \frac{1}{m}(\varsigma E, -i\kappa|\vec{p}|, 0, m)_a \\ \nu^+(\vec{p}, -\frac{\kappa}{2})(\sigma, I)_a \nu(\vec{p}, \frac{\kappa}{2}) = \frac{1}{m}(-\varsigma E, i\kappa|\vec{p}|, 0, m)_a \end{cases}$$

### 1.3.6 Dirac自旋基的正交性质

性质1.3.9.

$$\begin{cases} u^+(\vec{p}, \frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_x]u(\vec{p}, \frac{\kappa}{2}) = -v^+(\vec{p}, \frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_x]v(\vec{p}, \frac{\kappa}{2}) = \kappa\varsigma\hat{p}_a \\ u^+(\vec{p}, \frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_y]u(\vec{p}, \frac{\kappa}{2}) = v^+(\vec{p}, \frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_y]v(\vec{p}, \frac{\kappa}{2}) = 0 \\ u^+(\vec{p}, \frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_z]u(\vec{p}, \frac{\kappa}{2}) = v^+(\vec{p}, \frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_z]v(\vec{p}, \frac{\kappa}{2}) = -\frac{\varsigma p_a}{m} \\ u^+(\vec{p}, \frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes I]u(\vec{p}, \frac{\kappa}{2}) = v^+(\vec{p}, \frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes I]v(\vec{p}, \frac{\kappa}{2}) = \frac{\kappa E \hat{p}_a}{m} \end{cases}$$

性质1.3.10.

$$\begin{cases} u^+(\vec{p}, -\frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_x]u(\vec{p}, \frac{\kappa}{2}) = -v^+(\vec{p}, -\frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_x]v(\vec{p}, \frac{\kappa}{2}) = -\kappa\varsigma\sqrt{2}\frac{E}{m}\varepsilon_a(\vec{p}, \kappa) \\ u^+(\vec{p}, -\frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_y]u(\vec{p}, \frac{\kappa}{2}) = -v^+(\vec{p}, -\frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_y]v(\vec{p}, \frac{\kappa}{2}) = i\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_a(\vec{p}, \kappa) \\ u^+(\vec{p}, -\frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_z]u(\vec{p}, \frac{\kappa}{2}) = v^+(\vec{p}, -\frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes \sigma_z]v(\vec{p}, \frac{\kappa}{2}) = 0 \\ u^+(\vec{p}, -\frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes I]u(\vec{p}, \frac{\kappa}{2}) = v^+(\vec{p}, -\frac{\kappa}{2})[(\sigma, i\kappa)_a \otimes I]v(\vec{p}, \frac{\kappa}{2}) = -\kappa\sqrt{2}\varepsilon_a(\vec{p}, \kappa) \end{cases}$$

推论1.3.5.

$$\begin{cases} \bar{u}(\vec{p}, h)u(\vec{p}, h') = \delta_{hh'}, \bar{v}(\vec{p}, h)v(\vec{p}, h') = -\delta_{hh'}, \bar{u}(\vec{p}, h)v(\vec{p}, h') = 0, \bar{v}(\vec{p}, h)u(\vec{p}, h') = 0 \\ u^+(\vec{p}, h)u(\vec{p}, h') = \frac{E}{m}\delta_{hh'}, v^+(\vec{p}, h)v(\vec{p}, h') = \frac{E}{m}\delta_{hh'}, u^+(\vec{p}, h)v(-\vec{p}, h') = 0, v^+(\vec{p}, h)u(-\vec{p}, h') = 0 \\ \Lambda_+(\vec{p}, \frac{1}{2}) := \sum_{h=1/2}^{-1/2} u(\vec{p}, h)\bar{u}(\vec{p}, h) = \frac{m-i\gamma^a p_a}{2m}, \Lambda_-(\vec{p}, \frac{1}{2}) := \sum_{h=1/2}^{-1/2} v(\vec{p}, h)\bar{v}(\vec{p}, h) = \frac{-m-i\gamma^a p_a}{2m} \end{cases}$$

### 1.3.7 Dirac自旋基性质推论I

性质1.3.11.

$$\begin{cases} u^+(\vec{p}, \frac{\kappa}{2})\gamma_a u(\vec{p}, \frac{\kappa}{2}) = -v^+(\vec{p}, \frac{\kappa}{2})\gamma_a v(\vec{p}, \frac{\kappa}{2}) = (\vec{0}, I) \\ u^+(\vec{p}, \frac{\kappa}{2})\gamma_i \gamma_j u(\vec{p}, \frac{\kappa}{2}) = v^+(\vec{p}, \frac{\kappa}{2})\gamma_i \gamma_j v(\vec{p}, \frac{\kappa}{2}) = \frac{E}{m}(\delta_{ij} + i\kappa\varepsilon_{ijk}\hat{p}^k) \\ u^+(\vec{p}, \frac{\kappa}{2})\gamma_4 \gamma_a u(\vec{p}, \frac{\kappa}{2}) = v^+(\vec{p}, \frac{\kappa}{2})\gamma_4 \gamma_a v(\vec{p}, \frac{\kappa}{2}) = -i\frac{p_a}{m} \\ u^+(\vec{p}, \frac{\kappa}{2})\gamma_a \gamma_4 u(\vec{p}, \frac{\kappa}{2}) = v^+(\vec{p}, \frac{\kappa}{2})\gamma_a \gamma_4 v(\vec{p}, \frac{\kappa}{2}) = i\frac{p_a^*}{m} \\ u^+(\vec{p}, \frac{\kappa}{2})\gamma_4 \gamma_5 u(\vec{p}, \frac{\kappa}{2}) = v^+(\vec{p}, \frac{\kappa}{2})\gamma_4 \gamma_5 v(\vec{p}, \frac{\kappa}{2}) = 0 \end{cases}$$

性质1.3.12.

$$\begin{cases} u^+(\vec{p}, \frac{\kappa}{2})\gamma_a u(\vec{p}, \frac{\kappa}{2}) = -v^+(\vec{p}, \frac{\kappa}{2})\gamma_a v(\vec{p}, \frac{\kappa}{2}) = (\vec{0}, I) \\ u^+(\vec{p}, -\frac{\kappa}{2})\gamma_a u(\vec{p}, \frac{\kappa}{2}) = -v^+(\vec{p}, -\frac{\kappa}{2})\gamma_a v(\vec{p}, \frac{\kappa}{2}) = i\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_a(\vec{p}, \kappa) \end{cases}$$

### 1.3.8 Dirac自旋基性质推论II

性质1.3.13.

$$\begin{cases} \bar{u}(\vec{p}, \frac{\kappa}{2})u(\vec{p}, \frac{\kappa}{2}) = -\bar{v}(\vec{p}, \frac{\kappa}{2})v(\vec{p}, \frac{\kappa}{2}) = I \\ \bar{u}(\vec{p}, \frac{\kappa}{2})\gamma_a u(\vec{p}, \frac{\kappa}{2}) = \bar{v}(\vec{p}, \frac{\kappa}{2})\gamma_a v(\vec{p}, \frac{\kappa}{2}) = -i\frac{p_a}{m} \\ \bar{u}(\vec{p}, \frac{\kappa}{2})\gamma_i\gamma_j u(\vec{p}, \frac{\kappa}{2}) = -\bar{v}(\vec{p}, \frac{\kappa}{2})\gamma_i\gamma_j v(\vec{p}, \frac{\kappa}{2}) = \delta_{ij} + i\kappa\varepsilon_{ijk}\hat{p}^k \\ \bar{u}(\vec{p}, \frac{\kappa}{2})\gamma_4\gamma_a u(\vec{p}, \frac{\kappa}{2}) = -\bar{v}(\vec{p}, \frac{\kappa}{2})\gamma_4\gamma_a v(\vec{p}, \frac{\kappa}{2}) = (\vec{0}, I) \\ \bar{u}(\vec{p}, \frac{\kappa}{2})\gamma_a\gamma_4 u(\vec{p}, \frac{\kappa}{2}) = -\bar{v}(\vec{p}, \frac{\kappa}{2})\gamma_a\gamma_4 v(\vec{p}, \frac{\kappa}{2}) = (\vec{0}, I) \end{cases}$$

性质1.3.14.

$$\begin{cases} \bar{u}(\vec{p}, \frac{\kappa}{2})u(\vec{p}, \frac{\kappa}{2}) = -\bar{v}(\vec{p}, \frac{\kappa}{2})v(\vec{p}, \frac{\kappa}{2}) = I \\ \bar{u}(\vec{p}, \frac{\kappa}{2})\gamma_a u(\vec{p}, \frac{\kappa}{2}) = \bar{v}(\vec{p}, \frac{\kappa}{2})\gamma_a v(\vec{p}, \frac{\kappa}{2}) = -i\frac{p_a}{m} \\ \bar{u}(\vec{p}, \frac{\kappa}{2})\gamma_a\gamma_b u(\vec{p}, \frac{\kappa}{2}) = -\bar{v}(\vec{p}, \frac{\kappa}{2})\gamma_a\gamma_b v(\vec{p}, \frac{\kappa}{2}) = \delta_{ab} + i\kappa\varepsilon_{abc}\hat{p}^c \\ \bar{u}(\vec{p}, \frac{\kappa}{2})S_{ab}(e, \varsigma)u(\vec{p}, \frac{\kappa}{2}) = -\bar{v}(\vec{p}, \frac{\kappa}{2})S_{ab}(e, \varsigma)v(\vec{p}, \frac{\kappa}{2}) = \frac{\kappa}{2}\varepsilon_{abc4}\hat{p}^c \end{cases}$$

## 1.4 Dirac自旋基与四维矢量基之间关系的分析

### 1.4.1 Dirac自旋基 $u(\vec{p}, \frac{\kappa}{2})$ 与四维矢量基 $\varepsilon_a(\vec{p}, h)$ 之间的等价变换

定义1.4.1.  $\mathbb{X}_a = [im\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)\partial^b]C$ ,  $\mathbb{X}_a(p) = i[m\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)p^b]C$

性质1.4.1.

$$\begin{cases} u^+(\vec{p}, -\frac{\kappa}{2})\mathbb{X}_a(p)u^*(\vec{p}, -\frac{\kappa}{2}) = 2\sqrt{2}\frac{E^2}{m}\varepsilon_a^+(\vec{p}, -\kappa) \\ u^+(\vec{p}, -\frac{\kappa}{2})im\gamma_a C u^*(\vec{p}, -\frac{\kappa}{2}) = \sqrt{2}m\varepsilon_a^+(\vec{p}, -\kappa) \\ u^+(\vec{p}, -\frac{\kappa}{2})[-2iS_{ab}(e, \varsigma)p^b]C u^*(\vec{p}, -\frac{\kappa}{2}) = \sqrt{2}\frac{E^2+\hat{p}^2}{m}\varepsilon_a^+(\vec{p}, -\kappa) \end{cases}$$

证明:  $u^+(\vec{p}, -\frac{\kappa}{2})im\gamma_a C u^*(\vec{p}, -\frac{\kappa}{2})$   
 $= -i\kappa\varsigma u^+(\vec{p}, -\frac{\kappa}{2})im\gamma_a C\sigma_y \otimes \sigma_x u(\vec{p}, \frac{\kappa}{2})$   
 $= \kappa u^+(\vec{p}, -\frac{\kappa}{2})m\gamma_a(I \otimes \sigma_y)u(\vec{p}, \frac{\kappa}{2})$   
 $= -i\sqrt{2}[m\lambda_m(\vec{p}, \kappa), 0]_a$   
 $= -\sqrt{2}m\varepsilon_a(\vec{p}, \kappa) = \sqrt{2}m\varepsilon_a^+(\vec{p}, \kappa)$  □

证明:  $u^+(\vec{p}, -\frac{\kappa}{2})[-2iS_{ab}(e, \varsigma)p^b]C u^*(\vec{p}, -\frac{\kappa}{2})$   
 $= -i\kappa\varsigma u^+(\vec{p}, -\frac{\kappa}{2})[-2iS_{ab}(e, \varsigma)p^b]C\sigma_y \otimes \sigma_x u(\vec{p}, \frac{\kappa}{2})$   
 $= \kappa u^+(\vec{p}, -\frac{\kappa}{2})i\gamma_a\gamma_b p^b(I \otimes \sigma_y)u(\vec{p}, \frac{\kappa}{2})$   
 $= [\kappa\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_{ijk}p^j\lambda_m^k(\vec{p}, \kappa) - i\sqrt{2}\frac{E^2}{m}\lambda_m(\vec{p}, \kappa), 0]_a$   
 $= [i\kappa\sqrt{2}\frac{\hat{p}^2}{m}\varepsilon_{ijk}\lambda_m^j(\vec{p}, 0)\lambda_m^k(\vec{p}, \kappa) - i\sqrt{2}\frac{E^2}{m}\lambda_m(\vec{p}, \kappa), 0]_a$   
 $= -i\sqrt{2}\frac{\hat{p}^2}{m}[\lambda_m(\vec{p}, \kappa), 0]_a - i\sqrt{2}\frac{E^2}{m}[\lambda_m(\vec{p}, \kappa), 0]_a$   
 $= -i\sqrt{2}\frac{E^2+\hat{p}^2}{m}[\lambda_m(\vec{p}, \kappa), 0]_a = -\sqrt{2}\frac{E^2+\hat{p}^2}{m}\varepsilon_a(\vec{p}, \kappa) = \sqrt{2}\frac{E^2+\hat{p}^2}{m}\varepsilon_a^+(\vec{p}, \kappa)$  □

证明:  $u^+(\vec{p}, -\frac{\kappa}{2})\mathbb{X}_a(p)u^*(\vec{p}, -\frac{\kappa}{2})$   
 $= u^+(\vec{p}, -\frac{\kappa}{2})i[m\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)p^b]C u^*(\vec{p}, -\frac{\kappa}{2})$   
 $= -i\kappa\varsigma u^+(\vec{p}, -\frac{\kappa}{2})i[m\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)p^b]C\sigma_y \otimes \sigma_x u(\vec{p}, \frac{\kappa}{2})$   
 $= \kappa u^+(\vec{p}, -\frac{\kappa}{2})[m\gamma_a(\varsigma) + i\gamma_a\gamma_b p^b](I \otimes \sigma_y)u(\vec{p}, \frac{\kappa}{2})$   
 $= -i\sqrt{2}[m\lambda_m(\hat{p}, \kappa), 0]_a + \kappa\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_{ijk}p^j\lambda_m^k(\hat{p}, \kappa) - i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p}, \kappa)$   
 $= -i\sqrt{2}[m\lambda_m(\hat{p}, \kappa), 0]_a + i\kappa\sqrt{2}\frac{\hat{p}^2}{m}\varepsilon_{ijk}\lambda_m^j(\hat{p}, 0)\lambda_m^k(\hat{p}, \kappa) - i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p}, \kappa)$   
 $= -i\sqrt{2}[m\lambda_m(\hat{p}, \kappa), 0]_a - i\sqrt{2}\frac{\hat{p}^2}{m}[\lambda_m(\hat{p}, \kappa), 0]_a - i\sqrt{2}\frac{E^2}{m}[\lambda_m(\hat{p}, \kappa), 0]_a$   
 $= -i2\sqrt{2}\frac{E^2}{m}[\lambda_m(\hat{p}, \kappa), 0]_a = -2\sqrt{2}\frac{E^2}{m}\varepsilon_a(\vec{p}, \kappa) = 2\sqrt{2}\frac{E^2}{m}\varepsilon_a^+(\vec{p}, \kappa)$  □

推论1.4.1.

$$\begin{cases} \varepsilon_a^+(\vec{p}, \kappa) = \frac{i}{\sqrt{2}} u^+(\vec{p}, \frac{\kappa}{2}) \gamma_a C u^*(\vec{p}, \frac{\kappa}{2}) = \frac{m}{2\sqrt{2}E^2} u^+(\vec{p}, \frac{\kappa}{2}) \mathbb{X}_a(p) u^*(\vec{p}, \frac{\kappa}{2}) \\ \varepsilon_a(\vec{p}, \kappa) = -\frac{i}{\sqrt{2}} u^T(\vec{p}, \frac{\kappa}{2}) \bar{C} \gamma_a u(\vec{p}, \frac{\kappa}{2}) = \frac{m}{2\sqrt{2}E^2} u^T(\vec{p}, \frac{\kappa}{2}) \mathbb{X}_a^+(p) u(\vec{p}, \frac{\kappa}{2}) \end{cases}$$

推论1.4.2.

$$\begin{cases} \varepsilon^{+a}(\vec{p}, \kappa) = \frac{i}{\sqrt{2}} U^{+\lambda_\kappa \mu_\kappa}(\vec{p}, \kappa) (\gamma^a C)_{\lambda_\kappa \mu_\kappa} = \frac{m}{2\sqrt{2}E^2} U^{+\lambda_\kappa \mu_\kappa}(\vec{p}, \kappa) \mathbb{X}_{\lambda_\kappa \mu_\kappa}^a(p) \\ \varepsilon_a(\vec{p}, \kappa) = -\frac{i}{\sqrt{2}} (\bar{C} \gamma_a)^{\lambda_\kappa \mu_\kappa} U_{\lambda_\kappa \mu_\kappa}(\vec{p}, \kappa) = \frac{m}{2\sqrt{2}E^2} \mathbb{X}_a^{+\lambda_\kappa \mu_\kappa}(p) U_{\lambda_\kappa \mu_\kappa}(\vec{p}, \kappa) \end{cases}$$

性质1.4.2.  $u^+(\vec{p}, \frac{\kappa}{2}) \mathbb{X}_a(p) u^*(\vec{p}, -\frac{\kappa}{2}) = 2iE[\lambda_m(\hat{p}, 0), 0]$

证明:  $u^+(\vec{p}, \frac{\kappa}{2}) \mathbb{X}_a(p) u^*(\vec{p}, -\frac{\kappa}{2})$

$$\begin{aligned} &= u^+(\vec{p}, \frac{\kappa}{2}) i[m\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)p^b] C u^*(\vec{p}, -\frac{\kappa}{2}) \\ &= -i\kappa u^+(\vec{p}, \frac{\kappa}{2}) i[m\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)p^b] C \sigma_y \otimes \sigma_x u(\vec{p}, \frac{\kappa}{2}) \\ &= \kappa u^+(\vec{p}, \frac{\kappa}{2}) [m\gamma_a(\varsigma) + i\gamma_a \gamma_b p^b] (I \otimes \sigma_y) u(\vec{p}, \frac{\kappa}{2}) \\ &= (E\hat{p}, -i|\vec{p}|) + (E\hat{p}, i|\vec{p}|) \\ &= (2E\hat{p}, 0) = 2iE[\lambda_m(\hat{p}, 0), 0] \end{aligned}$$

□

推论1.4.3.  $u^+(\vec{p}, \frac{\kappa}{2}) im\gamma_a C u^*(\vec{p}, -\frac{\kappa}{2}) = m\varepsilon_a^+(\vec{p}, 0)$ ,  $u^+(\vec{p}, \frac{\kappa}{2}) [-2iS_{ab}(e, \varsigma)p^b C] u^*(\vec{p}, -\frac{\kappa}{2}) = m\varepsilon_a(\vec{p}, 0)$

推论1.4.4.  $u^+(\vec{p}, \frac{\kappa}{2}) \mathbb{X}_a(p) u^*(\vec{p}, -\frac{\kappa}{2}) = [2E\hat{p}, 0]$ ,  $u^+(\vec{p}, \frac{\kappa}{2}) \mathbb{X}_a(-p) u^*(\vec{p}, -\frac{\kappa}{2}) = [0, -2i|\vec{p}|]$

推论1.4.5.

$$\begin{cases} \varepsilon_a^+(\vec{p}, 0) = iu^+(\vec{p}, \frac{\kappa}{2}) \gamma_a C u^*(\vec{p}, -\frac{\kappa}{2}), [i\lambda_m(\hat{p}, 0), 0]_a = \frac{1}{2E} u^+(\vec{p}, \frac{\kappa}{2}) \mathbb{X}_a(p) u^*(\vec{p}, -\frac{\kappa}{2}) \\ \varepsilon_a(\vec{p}, 0) = -iu^T(\vec{p}, -\frac{\kappa}{2}) \bar{C} \gamma_a u(\vec{p}, \frac{\kappa}{2}), [i\lambda_m(\hat{p}, 0), 0]_a = \frac{1}{2E} u^T(\vec{p}, -\frac{\kappa}{2}) \mathbb{X}_a^+(p) u(\vec{p}, \frac{\kappa}{2}) \end{cases}$$

推论1.4.6.

$$\begin{cases} \varepsilon_a^+(\vec{p}, 0) = \frac{i}{\sqrt{2}} U^{+\lambda_\kappa \mu_\kappa}(\vec{p}, 0) (\gamma^a C)_{\lambda_\kappa \mu_\kappa}, [i\lambda_m(\hat{p}, 0), 0]_a = \frac{1}{2\sqrt{2}E} U^{+\lambda_\kappa \mu_\kappa}(\vec{p}, 0) \mathbb{X}_{\lambda_\kappa \mu_\kappa}^a(p) \\ \varepsilon_a(\vec{p}, 0) = -\frac{i}{\sqrt{2}} (\bar{C} \gamma_a)^{\lambda_\kappa \mu_\kappa} U_{\lambda_\kappa \mu_\kappa}(\vec{p}, 0), [i\lambda_m(\hat{p}, 0), 0]_a = \frac{1}{2\sqrt{2}E} \mathbb{X}_a^{+\lambda_\kappa \mu_\kappa}(p) U_{\lambda_\kappa \mu_\kappa}(\vec{p}, 0) \end{cases}$$

推论1.4.7.

$$\begin{cases} \varepsilon^{+a}(\vec{p}, h) = \frac{i}{\sqrt{2}} U^{+\lambda_\kappa \mu_\kappa}(\vec{p}, h) (\gamma^a C)_{\lambda_\kappa \mu_\kappa}, [-i\lambda_m^+(\hat{p}, h), 0]_a = (\frac{m}{E})^{|h|} \frac{1}{2\sqrt{2}E} U^{+\lambda_\kappa \mu_\kappa}(\vec{p}, h) \mathbb{X}_{\lambda_\kappa \mu_\kappa}^a(p) \\ \varepsilon_a(\vec{p}, h) = -\frac{i}{\sqrt{2}} (\bar{C} \gamma_a)^{\lambda_\kappa \mu_\kappa} U_{\lambda_\kappa \mu_\kappa}(\vec{p}, h), [i\lambda_m(\hat{p}, h), 0]_a = (\frac{m}{E})^{|h|} \frac{1}{2\sqrt{2}E} \mathbb{X}_a^{+\lambda_\kappa \mu_\kappa}(p) U_{\lambda_\kappa \mu_\kappa}(\vec{p}, h) \end{cases}$$

推论1.4.8.

$$\begin{cases} \lambda_m^+(\hat{p}, h) = (\frac{m}{E})^{|h|} \frac{i}{2\sqrt{2}E} U^{+\lambda_\kappa \mu_\kappa}(\vec{p}, h) \mathbb{X}_{\lambda_\kappa \mu_\kappa}(p) & \begin{cases} 0 = (\frac{m}{E})^{|h|} \frac{i}{2\sqrt{2}E} U^{+\lambda_\kappa \mu_\kappa}(\vec{p}, h) \mathbb{X}_{\lambda_\kappa \mu_\kappa}^\pi(p) \\ 0 = -(\frac{m}{E})^{|h|} \frac{i}{2\sqrt{2}E} \mathbb{X}_\pi^{+\lambda_\kappa \mu_\kappa}(p) U_{\lambda_\kappa \mu_\kappa}(\vec{p}, h) \end{cases} \\ \lambda_m(\hat{p}, h) = -(\frac{m}{E})^{|h|} \frac{i}{2\sqrt{2}E} \mathbb{X}^{+\lambda_\kappa \mu_\kappa}(p) U_{\lambda_\kappa \mu_\kappa}(\vec{p}, h) & \end{cases}$$

推论1.4.9.

$$\begin{cases} 0 = U^{+\lambda_\kappa \mu_\kappa}(\vec{p}, 0) \mathbb{X}_{\lambda_\kappa \mu_\kappa}(-p) & \begin{cases} |\vec{p}| = \frac{i}{2\sqrt{2}} U^{+\lambda_\kappa \mu_\kappa}(\vec{p}, 0) \mathbb{X}_{\lambda_\kappa \mu_\kappa}^\pi(-p) \\ |\vec{p}| = -\frac{i}{2\sqrt{2}} \mathbb{X}_\pi^{+\lambda_\kappa \mu_\kappa}(-p) U_{\lambda_\kappa \mu_\kappa}(\vec{p}, 0) \end{cases} \\ 0 = \mathbb{X}^{+\lambda_\kappa \mu_\kappa}(-p) U_{\lambda_\kappa \mu_\kappa}(\vec{p}, 0) & \end{cases}$$

### 1.4.2 Dirac自旋基 $v(\vec{p}, \frac{\kappa}{2})$ 与四维矢量基 $\varepsilon_a(\vec{p}, h)$ 之间的等价变换

性质1.4.3.

$$\begin{cases} v^+(\hat{p}, -\frac{\kappa}{2}) \mathbb{X}_a(-p) v^*(\vec{p}, -\frac{\kappa}{2}) = -2\sqrt{2} \frac{E^2}{m} \varepsilon_a^+(\vec{p}, -\kappa) \\ v^+(\hat{p}, -\frac{\kappa}{2}) im\gamma_a(\varsigma) C v^*(\vec{p}, -\frac{\kappa}{2}) = -\sqrt{2} m \varepsilon_a^+(\vec{p}, -\kappa) \\ v^+(\hat{p}, -\frac{\kappa}{2}) 2iS_{ab}(e, \varsigma) p^b C v^*(\vec{p}, -\frac{\kappa}{2}) = -\sqrt{2} \frac{E^2 + \vec{p}^2}{m} \varepsilon_a^+(\vec{p}, -\kappa) \end{cases}$$

证明:  $v^+(\hat{p}, -\frac{\kappa}{2}) im\gamma_a(\varsigma) C v^*(\vec{p}, -\frac{\kappa}{2})$

$$\begin{aligned} &= i\kappa v^+(\hat{p}, -\frac{\kappa}{2}) im\gamma_a(\varsigma) C \sigma_y \otimes \sigma_x v(\vec{p}, \frac{\kappa}{2}) \\ &= -\kappa v^+(\hat{p}, -\frac{\kappa}{2}) m\gamma_a(\varsigma) (I \otimes \sigma_y) v(\vec{p}, \frac{\kappa}{2}) \\ &= i\sqrt{2} [m\lambda_m(\hat{p}, \kappa), 0]_a = \sqrt{2} m \varepsilon_a(\vec{p}, \kappa) = -\sqrt{2} m \varepsilon_a^+(\vec{p}, -\kappa) \end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } v^+(\hat{p}, -\frac{\kappa}{2})2iS_{ab}(e, \varsigma)p^b C v^*(\vec{p}, -\frac{\kappa}{2}) \\
&= i\kappa\varsigma v^+(\hat{p}, -\frac{\kappa}{2})2iS_{ab}(e, \varsigma)p^b C \sigma_y \otimes \sigma_x v(\vec{p}, \frac{\kappa}{2}) \\
&= -\kappa v^+(\hat{p}, -\frac{\kappa}{2})[-i\gamma_a \gamma_b p^b](I \otimes \sigma_y)v(\vec{p}, \frac{\kappa}{2}) \\
&= -\kappa\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_{ijk}p^j\lambda_m^k(\hat{p}, \kappa) + i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p}, \kappa) \\
&= -i\kappa\sqrt{2}\frac{\vec{p}^2}{m}\varepsilon_{ijk}\lambda_m^j(\hat{p}, 0)\lambda_m^k(\hat{p}, \kappa) + i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p}, \kappa) \\
&= +i\sqrt{2}\frac{\vec{p}^2}{m}\lambda_m(\hat{p}, \kappa) + i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p}, \kappa) \\
&= i\sqrt{2}\frac{E^2+\vec{p}^2}{m}[\lambda_m(\hat{p}, \kappa), 0]_a = \sqrt{2}\frac{E^2+\vec{p}^2}{m}\varepsilon_a(\vec{p}, \kappa) = -\sqrt{2}\frac{E^2+\vec{p}^2}{m}\varepsilon_a^+(\vec{p}, -\kappa)
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } v^+(\hat{p}, -\frac{\kappa}{2})\mathbb{X}_a(-p)v^*(\vec{p}, -\frac{\kappa}{2}) \\
&= v^+(\hat{p}, -\frac{\kappa}{2})i[m\gamma_a(\varsigma) + 2S_{ab}(e, \varsigma)p^b]C v^*(\vec{p}, -\frac{\kappa}{2}) \\
&= i\kappa\varsigma v^+(\hat{p}, -\frac{\kappa}{2})i[m\gamma_a(\varsigma) + 2S_{ab}(e, \varsigma)p^b]C \sigma_y \otimes \sigma_x v(\vec{p}, \frac{\kappa}{2}) \\
&= -\kappa v^+(\hat{p}, -\frac{\kappa}{2})[m\gamma_a(\varsigma) - i\gamma_a \gamma_b p^b](I \otimes \sigma_y)v(\vec{p}, \frac{\kappa}{2}) \\
&= i\sqrt{2}[m\lambda_m(\hat{p}, \kappa), 0]_a - \kappa\sqrt{2}\frac{|\vec{p}|}{m}\varepsilon_{ijk}p^j\lambda_m^k(\hat{p}, \kappa) + i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p}, \kappa) \\
&= i\sqrt{2}[m\lambda_m(\hat{p}, \kappa), 0]_a - i\kappa\sqrt{2}\frac{\vec{p}^2}{m}\varepsilon_{ijk}\lambda_m^j(\hat{p}, 0)\lambda_m^k(\hat{p}, \kappa) + i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p}, \kappa) \\
&= i\sqrt{2}[m\lambda_m(\hat{p}, \kappa), 0]_a + i\sqrt{2}\frac{\vec{p}^2}{m}\lambda_m(\hat{p}, \kappa) + i\sqrt{2}\frac{E^2}{m}\lambda_m(\hat{p}, \kappa) \\
&= i2\sqrt{2}\frac{E^2}{m}[\lambda_m(\hat{p}, \kappa), 0]_a = 2\sqrt{2}\frac{E^2}{m}\varepsilon_a(\vec{p}, \kappa) = -2\sqrt{2}\frac{E^2}{m}\varepsilon_a^+(\vec{p}, -\kappa)
\end{aligned}$$

□

推论1.4.10.

$$\begin{cases} -\varepsilon_a^+(\vec{p}, \kappa) = \frac{i}{\sqrt{2}}v^+(\hat{p}, \frac{\kappa}{2})\gamma_a C v^*(\vec{p}, \frac{\kappa}{2}) = \frac{m}{2\sqrt{2}E^2}v^+(\hat{p}, \frac{\kappa}{2})\mathbb{X}_a(p)v^*(\vec{p}, \frac{\kappa}{2}) \\ -\varepsilon_a(\vec{p}, \kappa) = -\frac{i}{\sqrt{2}}v^T(\vec{p}, \frac{\kappa}{2})\bar{C}\gamma_a v(\vec{p}, \frac{\kappa}{2}) = \frac{m}{2\sqrt{2}E^2}v^T(\vec{p}, \frac{\kappa}{2})\mathbb{X}_a^+(p)v(\vec{p}, \frac{\kappa}{2}) \end{cases}$$

推论1.4.11.

$$\begin{cases} -\varepsilon^{+\alpha}(\vec{p}, \kappa) = \frac{i}{\sqrt{2}}V^{+\lambda_s\mu_s}(\vec{p}, \kappa)(\gamma^\alpha C)_{\lambda_s\mu_s} = \frac{m}{2\sqrt{2}E^2}V^{+\lambda_s\mu_s}(\vec{p}, \kappa)\mathbb{X}_{\lambda_s\mu_s}^\alpha(-p) \\ -\varepsilon_a(\vec{p}, \kappa) = -\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_s\mu_s}V_{\lambda_s\mu_s}(\vec{p}, \kappa) = \frac{m}{2\sqrt{2}E^2}\mathbb{X}_a^{+\lambda_s\mu_s}(-p)V_{\lambda_s\mu_s}(\vec{p}, \kappa) \end{cases}$$

$$\text{性质1.4.4. } v^+(\hat{p}, \frac{\kappa}{2})\mathbb{X}_a(-p)v^*(\vec{p}, -\frac{\kappa}{2}) = -2iE[\lambda_m(\hat{p}, 0), 0]_a$$

$$\begin{aligned}
& \text{证明: } v^+(\hat{p}, \frac{\kappa}{2})\mathbb{X}_a(-p)v^*(\vec{p}, -\frac{\kappa}{2}) \\
&= v^+(\hat{p}, \frac{\kappa}{2})i[m\gamma_a(\varsigma) + 2S_{ab}(e, \varsigma)p^b]C v^*(\vec{p}, -\frac{\kappa}{2}) \\
&= i\kappa\varsigma v^+(\hat{p}, \frac{\kappa}{2})i[m\gamma_a(\varsigma) + 2S_{ab}(e, \varsigma)p^b]C \sigma_y \otimes \sigma_x v(\vec{p}, \frac{\kappa}{2}) \\
&= -\kappa v^+(\hat{p}, \frac{\kappa}{2})[m\gamma_a(\varsigma) - i\gamma_a \gamma_b p^b](I \otimes \sigma_y)v(\vec{p}, \frac{\kappa}{2}) \\
&= -[E\hat{p}, -i|\vec{p}|] - [E\hat{p}, i|\vec{p}|] \\
&= -2iE[\lambda_m(\hat{p}, 0), 0]_a
\end{aligned}$$

□

$$\text{推论1.4.12. } v^+(\hat{p}, \frac{\kappa}{2})im\gamma_a C v^*(\vec{p}, -\frac{\kappa}{2}) = -m\varepsilon_a^+(\vec{p}, 0), v^+(\hat{p}, \frac{\kappa}{2})[2iS_{ab}(e, \varsigma)p^b C]v^*(\vec{p}, -\frac{\kappa}{2}) = -m\varepsilon_a(\vec{p}, 0)$$

$$\text{推论1.4.13. } v^+(\hat{p}, \frac{\kappa}{2})\mathbb{X}_a(-p)v^*(\vec{p}, -\frac{\kappa}{2}) = -[2E\hat{p}, 0], v^+(\hat{p}, \frac{\kappa}{2})\mathbb{X}_a(p)v^*(\vec{p}, -\frac{\kappa}{2}) = -[0, -2i|\vec{p}|]$$

推论1.4.14.

$$\begin{cases} -\varepsilon_a^+(\vec{p}, 0) = iv^+(\hat{p}, \frac{\kappa}{2})\gamma_a C v^*(\vec{p}, -\frac{\kappa}{2}), -[i\lambda_m(\hat{p}, 0), 0]_a = \frac{1}{2E}v^+(\hat{p}, \frac{\kappa}{2})\mathbb{X}_a(-p)v^*(\vec{p}, -\frac{\kappa}{2}) \\ -\varepsilon_a(\vec{p}, 0) = -iv^T(\vec{p}, -\frac{\kappa}{2})\bar{C}\gamma_a v(\vec{p}, \frac{\kappa}{2}), -[i\lambda_m(\hat{p}, 0), 0]_a = \frac{1}{2E}v^T(\vec{p}, -\frac{\kappa}{2})\mathbb{X}_a^+(-p)v(\vec{p}, \frac{\kappa}{2}) \end{cases}$$

推论1.4.15.

$$\begin{cases} -\varepsilon_a^+(\vec{p}, 0) = \frac{i}{\sqrt{2}}V^{+\lambda_s\mu_s}(\vec{p}, 0)(\gamma^\alpha C)_{\lambda_s\mu_s}, -[i\lambda_m(\hat{p}, 0), 0]_a = \frac{1}{2\sqrt{2}E}V^{+\lambda_s\mu_s}(\vec{p}, 0)\mathbb{X}_{\lambda_s\mu_s}^\alpha(-p) \\ -\varepsilon_a(\vec{p}, 0) = -\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_s\mu_s}V_{\lambda_s\mu_s}(\vec{p}, 0), -[i\lambda_m(\hat{p}, 0), 0]_a = \frac{1}{2\sqrt{2}E}\mathbb{X}_a^{+\lambda_s\mu_s}(-p)V_{\lambda_s\mu_s}(\vec{p}, 0) \end{cases}$$

推论1.4.16.

$$\begin{cases} -\varepsilon^{+\alpha}(\vec{p}, h) = \frac{i}{\sqrt{2}}V^{+\lambda_s\mu_s}(\vec{p}, h)(\gamma^\alpha C)_{\lambda_s\mu_s}, -[i\lambda_m^+(\hat{p}, h), 0]_a = (\frac{m}{E})^{|h|}\frac{1}{2\sqrt{2}E}V^{+\lambda_s\mu_s}(\vec{p}, h)\mathbb{X}_{\lambda_s\mu_s}^\alpha(-p) \\ -\varepsilon_a(\vec{p}, h) = -\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_s\mu_s}V_{\lambda_s\mu_s}(\vec{p}, h), -[i\lambda_m(\hat{p}, h), 0]_a = (\frac{m}{E})^{|h|}\frac{1}{2\sqrt{2}E}\mathbb{X}_a^{+\lambda_s\mu_s}(-p)V_{\lambda_s\mu_s}(\vec{p}, h) \end{cases}$$

推论1.4.17.

$$\begin{cases} -\lambda_m^+(\hat{p}, h) = \left(\frac{m}{E}\right)^{|h|} \frac{i}{2\sqrt{2E}} V^{+\lambda_c \mu_c}(\vec{p}, h) \mathbb{X}_{\lambda_c \mu_c}(-p) \\ -\lambda_m(\hat{p}, h) = -\left(\frac{m}{E}\right)^{|h|} \frac{i}{2\sqrt{2E}} \mathbb{X}^{+\lambda_c \mu_c}(-p) V_{\lambda_c \mu_c}(\vec{p}, h) \end{cases} \quad \begin{cases} 0 = \left(\frac{m}{E}\right)^{|h|} \frac{i}{2\sqrt{2E}} V^{+\lambda_c \mu_c}(\vec{p}, h) \mathbb{X}_{\lambda_c \mu_c}^\pi(-p) \\ 0 = -\left(\frac{m}{E}\right)^{|h|} \frac{i}{2\sqrt{2E}} \mathbb{X}^{+\lambda_c \mu_c}(-p) V_{\lambda_c \mu_c}^\pi(\vec{p}, h) \end{cases}$$

推论1.4.18.

$$\begin{cases} 0 = V^{+\lambda_c \mu_c}(\vec{p}, 0) \mathbb{X}_{\lambda_c \mu_c}(p) \\ 0 = \mathbb{X}^{+\lambda_c \mu_c}(p) V_{\lambda_c \mu_c}(\vec{p}, 0) \end{cases} \quad \begin{cases} -|\vec{p}| = \frac{i}{2\sqrt{2}} V^{+\lambda_c \mu_c}(\vec{p}, 0) \mathbb{X}_{\lambda_c \mu_c}^\pi(p) \\ -|\vec{p}| = -\frac{i}{2\sqrt{2}} \mathbb{X}^{+\lambda_c \mu_c}(p) V_{\lambda_c \mu_c}(\vec{p}, 0) \end{cases}$$

### 1.4.3 Dirac自旋基与四维矢量基之间的美妙关系

性质1.4.5.  $[\sigma \cdot \lambda_m(\hat{p}, \frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = 0$ ,  $[\sigma \cdot \lambda_m(\hat{p}, -\frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2} \lambda(\hat{p}, -\frac{\kappa}{2})$ ,  $[\sigma \cdot \lambda_m(\hat{p}, 0)] \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa \lambda(\hat{p}, \frac{\kappa}{2})$

证明:  $\lambda^+(\hat{p}, \frac{\kappa}{2})(\sigma, i\kappa)_a \lambda(\hat{p}, \frac{\kappa}{2}) = \kappa \hat{p}_a$ ,  $\lambda^+(\hat{p}, -\frac{\kappa}{2})(\sigma, i\kappa)_a \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2} [\lambda_m(\hat{p}, \kappa), 0]_a$   
 $\Rightarrow \lambda^+(\hat{p}, \frac{\kappa}{2}) \sigma_k \lambda(\hat{p}, \frac{\kappa}{2}) = \kappa \hat{p}_k$ ,  $\lambda^+(\hat{p}, -\frac{\kappa}{2}) \sigma_k \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2} \lambda_{mk}(\hat{p}, \frac{\kappa}{2})$   
 $\Rightarrow \lambda^+(\hat{p}, \frac{\kappa}{2}) [\sigma \cdot \lambda_m(\hat{p}, \frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = 0$ ,  $\lambda^+(\hat{p}, -\frac{\kappa}{2}) [\sigma \cdot \lambda_m(\hat{p}, \frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = 0$   
 $\Rightarrow \lambda(\hat{p}, \frac{\kappa}{2}) \lambda^+(\hat{p}, \frac{\kappa}{2}) [\sigma \cdot \lambda_m(\hat{p}, \frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = 0$ ,  $\lambda(\hat{p}, -\frac{\kappa}{2}) \lambda^+(\hat{p}, -\frac{\kappa}{2}) [\sigma \cdot \lambda_m(\hat{p}, \frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = 0$   
 $\Rightarrow [\lambda(\hat{p}, \frac{\kappa}{2}) \lambda^+(\hat{p}, \frac{\kappa}{2}) + \lambda(\hat{p}, -\frac{\kappa}{2}) \lambda^+(\hat{p}, -\frac{\kappa}{2})] [\sigma \cdot \lambda_m(\hat{p}, \frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = 0$   
 $\Rightarrow [\sigma \cdot \lambda_m(\hat{p}, \frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = 0$  □

证明:  $\lambda^+(\hat{p}, \frac{\kappa}{2})(\sigma, i\kappa)_a \lambda(\hat{p}, \frac{\kappa}{2}) = \kappa \hat{p}_a$ ,  $\lambda^+(\hat{p}, -\frac{\kappa}{2})(\sigma, i\kappa)_a \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2} [\lambda_m(\hat{p}, \kappa), 0]_a$   
 $\Rightarrow \lambda^+(\hat{p}, \frac{\kappa}{2}) \sigma_k \lambda(\hat{p}, \frac{\kappa}{2}) = \kappa \hat{p}_k$ ,  $\lambda^+(\hat{p}, -\frac{\kappa}{2}) \sigma_k \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2} \lambda_{mk}(\hat{p}, \frac{\kappa}{2})$   
 $\Rightarrow \lambda^+(\hat{p}, \frac{\kappa}{2}) [\sigma \cdot \lambda_m(\hat{p}, -\frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = 0$ ,  $\lambda^+(\hat{p}, -\frac{\kappa}{2}) [\sigma \cdot \lambda_m(\hat{p}, -\frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2}$   
 $\Rightarrow \lambda(\hat{p}, \frac{\kappa}{2}) \lambda^+(\hat{p}, -\frac{\kappa}{2}) [\sigma \cdot \lambda_m(\hat{p}, -\frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = 0$ ,  $\lambda(\hat{p}, -\frac{\kappa}{2}) \lambda^+(\hat{p}, -\frac{\kappa}{2}) [\sigma \cdot \lambda_m(\hat{p}, -\frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2} \lambda(\hat{p}, -\frac{\kappa}{2})$   
 $\Rightarrow [\lambda(\hat{p}, \frac{\kappa}{2}) \lambda^+(\hat{p}, \frac{\kappa}{2}) + \lambda(\hat{p}, -\frac{\kappa}{2}) \lambda^+(\hat{p}, -\frac{\kappa}{2})] [\sigma \cdot \lambda_m(\hat{p}, -\frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2} \lambda(\hat{p}, -\frac{\kappa}{2})$   
 $\Rightarrow [\sigma \cdot \lambda_m(\hat{p}, -\frac{\kappa}{2})] \lambda(\hat{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2} \lambda(\hat{p}, -\frac{\kappa}{2})$  □

性质1.4.6.

$$\begin{cases} [\gamma \cdot \lambda_m(\hat{p}, \kappa)] u(\vec{p}, \frac{\kappa}{2}) = 0, [\gamma \cdot \lambda_m(\hat{p}, -\kappa)] u(\vec{p}, \frac{\kappa}{2}) = -\kappa\sqrt{2} \gamma_5 u(\vec{p}, -\frac{\kappa}{2}) \\ [\gamma \cdot \lambda_m(\hat{p}, 0)] u(\vec{p}, \frac{\kappa}{2}) = -i\kappa(I \otimes \sigma_y) u(\vec{p}, \frac{\kappa}{2}) \\ [\gamma \cdot \lambda_m(\hat{p}, \kappa)] v(\vec{p}, \frac{\kappa}{2}) = 0, [\gamma \cdot \lambda_m(\hat{p}, -\kappa)] v(\vec{p}, \frac{\kappa}{2}) = \kappa\sqrt{2} \gamma_5 v(\vec{p}, -\frac{\kappa}{2}) \\ [\gamma \cdot \lambda_m(\hat{p}, 0)] v(\vec{p}, \frac{\kappa}{2}) = -i\kappa(I \otimes \sigma_y) v(\vec{p}, \frac{\kappa}{2}) \end{cases}$$

性质1.4.7.

$$\begin{cases} [\gamma^a \varepsilon_a(\vec{p}, \kappa)] u(\vec{p}, \frac{\kappa}{2}) = 0, [\gamma^a \varepsilon_a(\vec{p}, -\kappa)] u(\vec{p}, \frac{\kappa}{2}) = -i\kappa\sqrt{2} \gamma_5 u(\vec{p}, -\frac{\kappa}{2}), [\gamma^a \varepsilon_a(\vec{p}, 0)] u(\vec{p}, \frac{\kappa}{2}) = -i\kappa \gamma_5 u(\vec{p}, \frac{\kappa}{2}) \\ [\gamma^a \varepsilon_a(\vec{p}, \kappa)] v(\vec{p}, \frac{\kappa}{2}) = 0, [\gamma^a \varepsilon_a(\vec{p}, -\kappa)] v(\vec{p}, \frac{\kappa}{2}) = i\kappa\sqrt{2} \gamma_5 v(\vec{p}, -\frac{\kappa}{2}), [\gamma^a \varepsilon_a(\vec{p}, 0)] v(\vec{p}, \frac{\kappa}{2}) = i\kappa \gamma_5 v(\vec{p}, \frac{\kappa}{2}) \end{cases}$$

## 1.5 二阶Bargmann-Wigner自旋基与四维矢量基之间的关系

### 1.5.1 二阶Bargmann-Wigner自旋基分解为四维矢量基

定理1.5.1.  $U_{\lambda_c \mu_c}(\vec{p}, h) = \frac{1}{2\sqrt{2m}} \mathbb{X}_{\lambda_c \mu_c}^a(p) \varepsilon_a(\vec{p}, h)$ ,  $V_{\lambda_c \mu_c}(\vec{p}, h) = \frac{1}{2\sqrt{2m}} \mathbb{X}_{\lambda_c \mu_c}^a(-p) \tilde{\varepsilon}_a(\vec{p}, h)$

证明:  $\frac{1}{2\sqrt{2m}} \mathbb{X}_{\lambda_c \mu_c}^a(p) \varepsilon_a(\vec{p}, \kappa)$   
 $= \frac{1}{2\sqrt{2m}} \mathbb{X}^a(p) \varepsilon_a(\vec{p}, \kappa) = \frac{i\zeta}{2\sqrt{2m}} \mathbb{X}(p) \cdot \lambda_m(\vec{p}, \kappa)$   
 $= \frac{-\zeta}{2\sqrt{2m}} \{m\gamma - \frac{i}{2} [\gamma^a p_a, \gamma]\} C \cdot \lambda_m(\vec{p}, \kappa)$   
 $= \frac{-\zeta}{2\sqrt{2m}} \{m\gamma_j - \frac{i}{2} [\gamma_i p^i + \gamma_4 iE, \gamma_j]\} C \lambda_m^j(\vec{p}, \kappa)$   
 $= \frac{-\zeta}{2\sqrt{2m}} [(m + E\gamma_4)\gamma_j + \varepsilon_{ijk} p^i \sigma^k \otimes I] C \lambda_m^j(\vec{p}, \kappa)$   
 $= \frac{i\zeta}{2\sqrt{2m}} [i(m + E\gamma_4)\sigma_j \sigma_y \lambda_m^j(\vec{p}, \kappa) \otimes \sigma_x - i\kappa |\vec{p}| \sigma_j \sigma_y \lambda_m^j(\vec{p}, \kappa) \otimes \sigma_z]$   
 $= -\frac{1}{\sqrt{2}} \sigma_j \sigma_y \lambda_m^j(\vec{p}, \kappa) \otimes \frac{\zeta}{2m} [(m\sigma_x + \zeta E) - \kappa |\vec{p}| \sigma_z]$   
 $= \lambda(\hat{p}, \frac{\zeta}{2}) \lambda^T(\hat{p}, \frac{\zeta}{2}) \otimes \mu(\vec{p}, \frac{\kappa}{2}) \mu^T(\vec{p}, \frac{\kappa}{2})$

$$= u(\vec{p}, \frac{\kappa}{2})u^T(\vec{p}, \frac{\kappa}{2})$$

$$= U_{\lambda_s\mu_s}(\vec{p}, \kappa)$$

□

证明:  $\frac{1}{2\sqrt{2m}}\mathbb{X}_{\lambda_s\mu_s}^a(p)\varepsilon_a(\vec{p}, 0)$

$$= \frac{1}{2\sqrt{2m}}\mathbb{X}^a(p)\varepsilon_a(\vec{p}, 0) = \frac{i\varsigma}{2\sqrt{2m}}\mathbb{X}(p) \cdot \frac{E}{m}\lambda_m(\vec{p}, 0) + \frac{i\varsigma}{2\sqrt{2m}}\mathbb{X}^\pi(p)\frac{|\vec{p}|}{m}$$

$$= \frac{-\varsigma}{2\sqrt{2m}}\{m\gamma_j - \frac{i}{2}[\gamma^a p_a, \gamma_j]\}C \cdot \frac{E}{m}\lambda_m(\vec{p}, 0) + \frac{-\varsigma}{2\sqrt{2m}}(m\gamma_4 - i\gamma^j p_j \gamma_4)C\frac{|\vec{p}|}{m}$$

$$= \frac{-\varsigma}{2\sqrt{2m}}\{m\gamma_j - \frac{i}{2}[\gamma_i p^i + \gamma_4 iE, \gamma_j]\}C\frac{E}{m}\lambda_m^j(\vec{p}, 0) + \frac{1}{2\sqrt{2m}}(m - i\gamma_i p^i)\gamma_2\frac{|\vec{p}|}{m}$$

$$= \frac{-\varsigma}{2\sqrt{2m}}[(m + E\gamma_4)\gamma_j + \varepsilon_{ijk}p^i\sigma^k \otimes I]C\frac{E}{m}\lambda_m^j(\vec{p}, 0) + \frac{1}{2\sqrt{2m}}(m\sigma_y \otimes \sigma_y - i\sigma_i\sigma_y p^i \otimes I)\frac{|\vec{p}|}{m}$$

$$= \frac{-\varsigma}{2\sqrt{2m}}(m + E\gamma_4)\sigma_j\sigma_y\frac{E}{m}\lambda_m^j(\vec{p}, 0) \otimes \sigma_x + \frac{1}{2\sqrt{2m}}(m\sigma_y \otimes \sigma_y - i\sigma_i\sigma_y p^i \otimes I)\frac{|\vec{p}|}{m}$$

$$= -\frac{1}{\sqrt{2}}\sigma_j\sigma_y\lambda_m^j(\vec{p}, 0) \otimes \frac{\varsigma}{2}(\frac{E}{m}\sigma_x + \varsigma\frac{E^2 - \vec{p}^2}{m^2}) + \frac{1}{2\sqrt{2}}\frac{|\vec{p}|}{m}\sigma_y \otimes \sigma_y$$

$$= -\frac{1}{2\sqrt{2}}[\sigma_j\sigma_y\lambda_m^j(\vec{p}, 0) \otimes (\varsigma\frac{E}{m}\sigma_x + I) - \frac{|\vec{p}|}{m}\sigma_y \otimes \sigma_y]$$

$$= U_{\lambda_s\mu_s}(\vec{p}, 0)$$

□

证明:  $U(\vec{p}, 0) = \frac{1}{\sqrt{2}}[u(\vec{p}, \frac{\kappa}{2})u^T(\vec{p}, -\frac{\kappa}{2}) + u(\vec{p}, -\frac{\kappa}{2})u^T(\vec{p}, \frac{\kappa}{2})]$

$$= \frac{1}{\sqrt{2}}[\lambda(\hat{p}, \frac{\kappa}{2})\lambda^T(\hat{p}, -\frac{\kappa}{2}) \otimes \mu(\vec{p}, \frac{\kappa}{2})\mu^T(\vec{p}, -\frac{\kappa}{2}) + \lambda(\hat{p}, -\frac{\kappa}{2})\lambda^T(\hat{p}, \frac{\kappa}{2}) \otimes \mu(\vec{p}, -\frac{\kappa}{2})\mu^T(\vec{p}, \frac{\kappa}{2})]$$

$$= \frac{1}{\sqrt{2}}[\frac{i}{2}(\sigma \cdot \hat{p} + \kappa I)\sigma_y \otimes \frac{1}{2}(I + \varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y) + \frac{i}{2}(\sigma \cdot \hat{p} - \kappa I)\sigma_y \otimes \frac{1}{2}(I + \varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y)]$$

$$= \frac{i}{4\sqrt{2}}[(\sigma \cdot \hat{p} + \kappa I)\sigma_y \otimes (I + \varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y) + (\sigma \cdot \hat{p} - \kappa I)\sigma_y \otimes (I + \varsigma\frac{E}{m}\sigma_x + i\kappa\frac{|\vec{p}|}{m}\sigma_y)]$$

$$= \frac{i}{2\sqrt{2}}[(\sigma\sigma_y \cdot \hat{p}) \otimes (I + \varsigma\frac{E}{m}\sigma_x) + \sigma_y \otimes (-i\frac{|\vec{p}|}{m}\sigma_y)]$$

$$= -\frac{1}{2\sqrt{2}}\{[\sigma\sigma_y \cdot \lambda(\hat{p}, 0)] \otimes (I + \varsigma\frac{E}{m}\sigma_x) - \frac{|\vec{p}|}{m}\sigma_y \otimes \sigma_y\}$$

□

### 1.5.2 Dirac自旋基与四维矢量基等价变换关系的小结与理顺

推论1.5.1.

$$\begin{cases} U_{\lambda_s\mu_s}(\vec{p}, h) = \frac{1}{2\sqrt{2m}}\mathbb{X}_{\lambda_s\mu_s}^a(p)\varepsilon_a(\vec{p}, h), V_{\lambda_s\mu_s}(\vec{p}, h) = -\frac{1}{2\sqrt{2m}}\mathbb{X}_{\lambda_s\mu_s}^a(-p)\varepsilon_a(\vec{p}, h) \\ \varepsilon_a(\vec{p}, h) = -\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_s\mu_s}U_{\lambda_s\mu_s}(\vec{p}, h) = \frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_s\mu_s}V_{\lambda_s\mu_s}(\vec{p}, h) \end{cases}$$

推论1.5.2.

$$\begin{cases} U_{\lambda_s\mu_s}(\vec{p}, h) = -\frac{i}{4m}\mathbb{X}_{\lambda_s\mu_s}^a(p)(\bar{C}\gamma_a)^{\lambda'_s\mu'_s}U_{\lambda'_s\mu'_s}(\hat{p}, h), V_{\lambda_s\mu_s}(\vec{p}, h) = -\frac{i}{4m}\mathbb{X}_{\lambda_s\mu_s}^a(-p)(\bar{C}\gamma_a)^{\lambda'_s\mu'_s}V_{\lambda'_s\mu'_s}(\hat{p}, h) \\ \varepsilon_a(\vec{p}, h) = -\frac{i}{4m}(\bar{C}\gamma_a)^{\lambda_s\mu_s}\mathbb{X}_{\lambda_s\mu_s}^b(p)\varepsilon_b(\vec{p}, h) = -\frac{i}{4m}(\bar{C}\gamma_a)^{\lambda_s\mu_s}\mathbb{X}_{\lambda_s\mu_s}^b(-p)\varepsilon_b(\vec{p}, h) \end{cases}$$

推论1.5.3.

$$\begin{cases} [i\lambda_m(\hat{p}, h), 0]_a = (\frac{m}{E})|h|\frac{1}{2\sqrt{2E}}\mathbb{X}_a^{+\lambda_s\mu_s}(p)U_{\lambda_s\mu_s}(\vec{p}, h) = -(\frac{m}{E})|h|\frac{1}{2\sqrt{2E}}\mathbb{X}_a^{+\lambda_s\mu_s}(-p)V_{\lambda_s\mu_s}(\vec{p}, h) \\ [i\lambda_m(\hat{p}, h), 0]_a = (\frac{m}{E})|h|\frac{1}{8mE}\mathbb{X}_a^{+\lambda_s\mu_s}(p)\mathbb{X}_{\lambda_s\mu_s}^b(p)\varepsilon_b(\vec{p}, h) = (\frac{m}{E})|h|\frac{1}{8mE}\mathbb{X}_a^{+\lambda_s\mu_s}(-p)\mathbb{X}_{\lambda_s\mu_s}^b(-p)\varepsilon_b(\vec{p}, h) \end{cases}$$

推论1.5.4.

$$\begin{cases} \mathbb{X}_\pi^{+\lambda_s\mu_s}(p)U_{\lambda_s\mu_s}(\vec{p}, h) = \mathbb{X}_\pi^{+\lambda_s\mu_s}(-p)V_{\lambda_s\mu_s}(\vec{p}, h) = 0 \\ \lambda_m(\hat{p}, h) = -(\frac{m}{E})|h|\frac{i}{2\sqrt{2E}}\mathbb{X}^{+\lambda_s\mu_s}(p)U_{\lambda_s\mu_s}(\vec{p}, h) = (\frac{m}{E})|h|\frac{i}{2\sqrt{2E}}\mathbb{X}^{+\lambda_s\mu_s}(-p)V_{\lambda_s\mu_s}(\vec{p}, h) \end{cases}$$

推论1.5.5.

$$\begin{cases} \mathbb{X}^{+\lambda_s\mu_s}(-p)U_{\lambda_s\mu_s}(\vec{p}, 0) = \mathbb{X}^{+\lambda_s\mu_s}(p)V_{\lambda_s\mu_s}(\vec{p}, 0) = 0 \\ -\frac{i}{2\sqrt{2}}\mathbb{X}_\pi^{+\lambda_s\mu_s}(-p)U_{\lambda_s\mu_s}(\vec{p}, 0) = \frac{i}{2\sqrt{2}}\mathbb{X}_\pi^{+\lambda_s\mu_s}(p)V_{\lambda_s\mu_s}(\vec{p}, 0) = |\vec{p}| \end{cases}$$

定理1.5.2.  $(\bar{C}\gamma_a)^{\lambda_s\mu_s}\mathbb{X}_{\lambda_s\mu_s}^b(p) = (\bar{C}\gamma_a)^{\lambda_s\mu_s}\mathbb{X}_{\lambda_s\mu_s}^b(-p) = 4im\delta_a^b$

定理1.5.3.  $\mathbb{X}_a^{+\lambda_s\mu_s}(p)\mathbb{X}_{b\lambda_s\mu_s}(p) = \mathbb{X}_a^{+\lambda_s\mu_s}(-p)\mathbb{X}_{b\lambda_s\mu_s}(-p) = 8E^2\delta_{ab} - 4p_a p_b^+$

证明:  $\mathbb{X}_a^{+\lambda_s\mu_s}(p)\mathbb{X}_{b\lambda_s\mu_s}(p)$

$$= tr[\mathbb{X}_a(p)\mathbb{X}_b(p)]$$

$$= tr\{\bar{C}[m\gamma_a - 2S_{ac}(e, \varsigma)p^{+c}][m\gamma_b - 2S_{bd}(e, \varsigma)p^d]C\}$$

$$= tr\{[m\gamma_a - 2S_{ac}(e, \varsigma)p^{+c}][m\gamma_b - 2S_{bd}(e, \varsigma)p^d]\}$$

$$= m^2 tr(\gamma_a \gamma_b) + 4tr[S_{ac}(e, \varsigma)S_{bd}(e, \varsigma)p^{+c}p^d]$$



$$\begin{aligned}
&= 4m^2\delta_{ab} + 4(\delta_{ab}\delta_{dc} - \delta_{ad}\delta_{bc})p^+c p^d \\
&= 4m^2\delta_{ab} + 4(\delta_{ab}p_c^+ p^c - p_a p_b^+) \\
&= 8E^2\delta_{ab} - 4p_a p_b^+
\end{aligned}$$

□

$$\text{推论1.5.6. } \mathbb{X}_a^{+\lambda_c\mu_c}(p)\mathbb{X}_{\lambda_c\mu_c}^b(p) = \mathbb{X}_a^{+\lambda_c\mu_c}(-p)\mathbb{X}_{\lambda_c\mu_c}^b(-p) = 8E^2\delta_a^b - 4p_a p^{+b}$$

$$\text{猜想1.5.1. } -\frac{i}{4m}\mathbb{X}_{\lambda_c\mu_c}^a(p)(\bar{C}\gamma_a)^{\lambda'_c\mu'_c} = -\frac{i}{4m}\mathbb{X}_{\lambda_c\mu_c}^a(-p)(\bar{C}\gamma_a)^{\lambda'_c\mu'_c} = \frac{1}{(2!)^2}\delta_{\{\lambda_c\mu_c\}}^{\{\lambda'_c\mu'_c\}}$$

### 1.5.3 有质量自旋-1粒子准投影算子分解为Dirac准投影算子

定理1.5.4.

$$\begin{cases}
\Lambda_+(\vec{p}, 1) := \sum_{h=1}^{-1} U_{\lambda_c\mu_c}(\vec{p}, h)U_{\lambda'_c\mu'_c}^+(\vec{p}, h) = \frac{1}{(2!)^2}\Lambda_{+\{\lambda_c(\lambda'_c(\vec{p}, \frac{1}{2})\Lambda_{+\mu_c\mu'_c})\}}(\vec{p}, \frac{1}{2}) \\
\Lambda_-(\vec{p}, 1) := \sum_{h=1}^{-1} V_{\lambda_c\mu_c}(\vec{p}, h)V_{\lambda'_c\mu'_c}^+(\vec{p}, h) = \frac{1}{(2!)^2}\Lambda_{-\{\lambda_c(\lambda'_c(\vec{p}, \frac{1}{2})\Lambda_{-\mu_c\mu'_c})\}}(\vec{p}, \frac{1}{2})
\end{cases}$$

$$\text{证明: } \Lambda_+(\vec{p}, 1) := \sum_{h=1}^{-1} U_{\lambda_c\mu_c}(\vec{p}, h)U_{\lambda'_c\mu'_c}^+(\vec{p}, h)$$

=

$$\begin{aligned}
&u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2}) \\
&+ \frac{1}{2}[u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\mu_c}(\vec{p}, -\frac{1}{2}) + u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\lambda_c}(\vec{p}, -\frac{1}{2})][u_{\lambda'_c}^+(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2}) + u_{\mu'_c}^+(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})] \\
&+ u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})
\end{aligned}$$

=

$$\begin{aligned}
&\frac{1}{4}\{[u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\mu_c}(\vec{p}, \frac{1}{2}) + u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\lambda_c}(\vec{p}, \frac{1}{2})][u_{\lambda'_c}^+(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + u_{\mu'_c}^+(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2})] \\
&+ [u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\mu_c}(\vec{p}, -\frac{1}{2}) + u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\lambda_c}(\vec{p}, -\frac{1}{2})][u_{\lambda'_c}^+(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2}) + u_{\mu'_c}^+(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})] \\
&+ [u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\mu_c}(\vec{p}, -\frac{1}{2}) + u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\lambda_c}(\vec{p}, -\frac{1}{2})][u_{\lambda'_c}^+(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2}) + u_{\mu'_c}^+(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})] \\
&+ [u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\mu_c}(\vec{p}, -\frac{1}{2}) + u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\lambda_c}(\vec{p}, -\frac{1}{2})][u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2}) + u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})]\}
\end{aligned}$$

=

$$\begin{aligned}
&\frac{1}{4}\{u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2})[u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})] \\
&+ u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2})[u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2}) + u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})] \\
&+ u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2})[u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})] \\
&+ u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2})[u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})] \\
&+ [u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})]u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2}) \\
&+ [u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2}) + u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})]u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2}) \\
&+ [u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})]u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2}) \\
&+ [u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})]u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})
\end{aligned}$$

=

$$\begin{aligned}
&\frac{1}{4}\{[u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})][u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})] \\
&+ [u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})][u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2}) + u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})] \\
&+ [u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2}) + u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})][u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})] \\
&+ [u_{\mu_c}(\vec{p}, \frac{1}{2})u_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + u_{\mu_c}(\vec{p}, -\frac{1}{2})u_{\mu'_c}^+(\vec{p}, -\frac{1}{2})][u_{\lambda_c}(\vec{p}, \frac{1}{2})u_{\lambda'_c}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_c}(\vec{p}, -\frac{1}{2})u_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})] \\
&= \frac{1}{4}[\Lambda_{+\lambda_c\lambda'_c}(\vec{p}, \frac{1}{2})\Lambda_{+\mu_c\mu'_c}(\vec{p}, \frac{1}{2}) + \Lambda_{+\lambda_c\lambda'_c}(\vec{p}, \frac{1}{2})\Lambda_{+\mu_c\lambda'_c}(\vec{p}, \frac{1}{2}) + \Lambda_{+\mu_c\lambda'_c}(\vec{p}, \frac{1}{2})\Lambda_{+\lambda_c\mu'_c}(\vec{p}, \frac{1}{2}) + \Lambda_{+\mu_c\mu'_c}(\vec{p}, \frac{1}{2})\Lambda_{+\lambda_c\lambda'_c}(\vec{p}, \frac{1}{2})] \\
&= \frac{1}{(2!)^2}\Lambda_{+\{\lambda_c(\lambda'_c(\vec{p}, \frac{1}{2})\Lambda_{+\mu_c\mu'_c})\}}(\vec{p}, \frac{1}{2})
\end{aligned}$$

□

### 1.5.4 解析证明一个重要定理

$$\text{定理1.5.5. } \mathbb{X}_{\lambda_c\mu_c}^a(p)(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})\mathbb{X}_{\lambda'_c\mu'_c}^{+a'}(p) = \frac{1}{2}[(m - i\gamma^a p_a)\gamma^4]_{\{\lambda_c(\lambda'_c[(m - i\gamma^b p_b)\gamma^4]_{\mu_c\mu'_c})\}}$$

$$\text{证明: } \mathbb{X}_{\lambda_c\mu_c}^a(p)(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})\mathbb{X}_{\lambda'_c\mu'_c}^{+a'}(p)$$

$$= \mathbb{X}_{\lambda_c\mu_c}^a(p) \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h)\varepsilon_{a'}^+(\vec{p}, h)\mathbb{X}_{\lambda'_c\mu'_c}^{+a'}(p)$$

$$\begin{aligned}
&= 8m^2 \sum_{h=1}^{-1} U_{\lambda_c \mu_c}(\vec{p}, h) U_{\lambda'_c \mu'_c}^+(\vec{p}, h) \\
&= 8m^2 \frac{1}{(2!)^2} \Lambda_{+\{\lambda_c \mu_c\}}(\vec{p}, \frac{1}{2}) \Lambda_{+\{\mu'_c \lambda'_c\}}(\vec{p}, \frac{1}{2}) \\
&= \frac{1}{2} [(m - i\gamma^a p_a) \gamma^4]_{\{\lambda_c \mu_c\}} [(m - i\gamma^b p_b) \gamma^4]_{\{\mu'_c \lambda'_c\}}
\end{aligned}$$

□

## 1.6 Dirac方程的协变反对易规则

### 1.6.1 Dirac方程及其分离形式 [4, 5]

定义1.6.1.  $\gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x)$ ,  $-\gamma^a \gamma^4 = i\varsigma(\sigma \otimes \sigma_z, i\varsigma)$ ,  $\gamma^4 \gamma^a = i\varsigma(\sigma \otimes \sigma_z, -i\varsigma)$ ,  $\gamma^4 \prec \gamma_{\lambda_c \lambda'_c}^4$ ,  $\gamma_4 \prec \gamma_{\lambda'_c \lambda_c}^4$

推论1.6.1.  $(\gamma^a \partial_a + m) \psi_{\lambda_c}(x) = 0 \Leftrightarrow [(\sigma \otimes \sigma_z, -i\varsigma)^a \partial_a - imI \otimes \sigma_x] \psi(x) = 0$

$$\text{推论1.6.2. } \begin{cases} (\gamma^a \partial_a + m)_{\kappa_c} \lambda_c \psi_{\lambda_c}(x) = 0 \\ \psi_{\lambda_c}(x) = [\lambda_{A_c}(x), \eta^{A'_c}(x)]^T \end{cases} \Leftrightarrow \begin{cases} (\sigma, -i\varsigma)_a^{A'_c A_c} \partial^a \lambda_{A_c}(x) = im \eta^{A'_c}(x) \\ (\sigma, i\varsigma)_{A'_c A_c}^a \partial_a \eta^{A'_c}(x) = -im \lambda_{A_c}(x) \end{cases}$$

### 1.6.2 Dirac方程的协变反对易规则

推论1.6.3.

$$\{\psi_{\lambda_c}(x), \bar{\psi}^{\mu_c}(x')\} = i(m - \gamma^a \partial_a)_{\lambda_c \mu_c} \Delta(x - x') \Leftrightarrow \{\psi_{\lambda_c}(x), \psi_{\lambda'_c}^+(x')\} = i[(m - \gamma^a \partial_a) \gamma^4]_{\lambda_c \lambda'_c} \Delta(x - x')$$

推论1.6.4.

$$\begin{cases} \{\psi_{\lambda_c}(x), \psi_{\lambda'_c}^+(x')\} = i[(m - \gamma^a \partial_a) \gamma^4]_{\lambda_c \lambda'_c} \Delta(x - x') \\ \{\psi_{\lambda_c}(x), \psi_{\lambda'_c}(x')\} = 0, \{\psi_{\lambda_c}^+(x), \psi_{\lambda'_c}^+(x')\} = 0 \\ \psi_{\lambda_c}(x) = [\lambda_{A_c}(x), \eta^{A'_c}(x)]^T \\ \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x) \\ S_{ab}(e, \varsigma) = -\frac{i}{4} [\gamma_a, \gamma_b] = S_{ab}(\varsigma) \oplus S_{ab}(-\varsigma) \\ S_{ab}(\varsigma) = \frac{i}{2} \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma} = -\frac{i}{4} (\sigma, i\varsigma)_{[a} (\sigma, -i\varsigma)_{b]} \end{cases} \Leftrightarrow \begin{cases} \{\lambda_{A_c}(x), \lambda_{A'_c}^+(x')\} = -\varsigma(\sigma, i\varsigma)_{A'_c A_c}^a \partial_a \Delta(x - x') \\ \{\eta^{A'_c}(x), \eta_{A_c}^+(x')\} = \varsigma(\sigma, -i\varsigma)_{A'_c A_c}^a \partial^a \Delta(x - x') \\ \{\lambda_{A_c}(x), \eta_{A_c}^{B_c}(x')\} = i\varsigma m \delta_{A_c}^{B_c} \Delta(x - x') \\ \{\eta^{A'_c}(x), \lambda_{B'_c}^+(x')\} = i\varsigma m \delta_{A'_c}^{B'_c} \Delta(x - x') \\ \{\lambda_{A_c}(x), \lambda_{B_c}(x')\} = 0, \{\eta^{A'_c}(x), \eta_{B'_c}^+(x')\} = 0 \\ \{\lambda_{A'_c}^+(x), \lambda_{B'_c}^+(x')\} = 0, \{\eta_{A_c}^+(x), \eta_{B_c}^+(x')\} = 0 \\ \{\lambda_{A_c}(x), \eta^{A'_c}(x')\} = 0, \{\lambda_{A'_c}^+(x), \eta_{A_c}^+(x')\} = 0 \end{cases}$$

以上内容是基础部分，其推理和结论适用于所有章节，特别是适用于以下章节和下一章。

## 2 重要的组合学恒等式

### 2.1 基本恒等式

推论2.1.1.  $C_n^m = C_{n-1}^{m-1} + C_{n-1}^m$

推论2.1.2.

$$\begin{cases} \sum_{k=1}^n k = \frac{1}{2!} n(n+1), \sum_{k=1}^n k^3 = [\frac{1}{2!} n(n+1)]^2 \\ \sum_{k=1}^n k^2 = \frac{1}{3!} n(n+1)(2n+1), \sum_{k=1}^n k^4 = \frac{1}{5!} 2n(2n+1)(2n+2)(3n^2+3n-1) \end{cases}$$

性质2.1.1.

$$\begin{cases} \sum_{i=0}^n i^p = \frac{1}{p+1} \sum_{k=0}^p (-1)^k C_{p+1}^k B_k n^{p+1-k}, B_k = \delta_{k0} - \frac{1}{k+1} \sum_{j=0}^{k-1} C_{k+1}^j B_j, \frac{z}{e^z-1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \\ B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{2k+1} = 0 (k \geq 1) \end{cases}$$

### 2.2 一组之前章节的组合学恒等式

性质2.2.1.  $\sum_{l=r}^n (-1)^{n-l} C_{n+1}^l C_l^r = C_{n+1}^r$ ,  $\sum_{l=r}^n (-1)^{n-l} C_{n+1}^l C_{l+1}^{r+1} = C_{n+2}^{r+1} - \delta_{nr}$

性质2.2.2.  $\sum_{l=0}^{n-r} (-1)^{n-r-l} C_{n+1}^{l+r} C_{l+r}^r = C_{n+1}^r$ ,  $\sum_{l=0}^{n-r} (-1)^{n-r-l} C_{n+1}^{l+r} C_{l+r+1}^{r+1} = C_{n+2}^{r+1} - \delta_{nr}$

性质2.2.3.  $\sum_{l=0}^n (-1)^{n-l} C_{n+1+r}^{l+r} C_{l+r}^r = C_{n+1+r}^r$ ,  $\sum_{l=0}^n (-1)^{n-l} C_{n+1+r}^{l+r} C_{l+r+1}^{r+1} = C_{n+1+r+1}^{r+1} - \delta_{n+r,r}$

### 2.3 一个组合学恒等式及其证明

性质2.3.1. 
$$\sum_{h'=n'}^{-n'} C_{n+h}^{n'+h'} C_{n-h}^{n'-h'} = C_{2n}^{2n'}$$

证明:  $(1+x)^{n+h}(1+x)^{n-h}|2n' \text{次项} = \sum_{h'=n'}^{-n'} C_{n+h}^{n'+h'} x^{n'+h'} C_{n-h}^{n'-h'} x^{n'-h'} = (1+x)^{2n}|2n' \text{次项} = C_{2n}^{2n'} x^{2n'}$

$\Leftrightarrow \sum_{h'=n'}^{-n'} C_{n+h}^{n'+h'} C_{n-h}^{n'-h'} = C_{2n}^{2n'}$  □

### 2.4 一个重要组合学恒等式及其证明

引理2.4.1. 
$$\sum_{a+b=k} C_a^c C_b^d = C_{k+1}^{c+d+1} \Leftrightarrow \sum_{j=0}^{k-c-d} C_{c+j}^c C_{k-c-j}^d = C_{k+1}^{c+d+1}$$

证明:  $\sum_{j=0}^0 C_{c+j}^c C_{d-j}^d = C_c^c C_d^d = C_{c+d+1}^{c+d+1}$  □

证明:  $\sum_{j=0}^1 C_{c+j}^c C_{d+1-j}^d = C_c^c C_{d+1}^d + C_{c+1}^c C_d^d = C_{c+d+2}^{c+d+1}$  □

证明:  $\sum_{j=0}^{k-c-d} C_{c+j}^c C_{k-c-j}^d, c \geq 1$

$= \sum_{j=0}^{k-c-d} (C_{c+j-1}^{c-1} + C_{c+j-1}^c) C_{k-c-j}^d$

$= \sum_{j=0}^{k-1-(c-1)-d} C_{c-1+j}^{c-1} C_{k-1-(c-1)-j}^d + \sum_{j=-1}^{k-1-c-d} C_{c+j}^c C_{k-1-c-j}^d$

$= \sum_{j=0}^{k-1-(c-1)-d} C_{c-1+j}^{c-1} C_{k-1-(c-1)-j}^d + \sum_{j=0}^{k-1-c-d} C_{c+j}^c C_{k-1-c-j}^d$

$= C_k^{c+d} + C_k^{c+d+1} = C_{k+1}^{c+d+1}$  □

证明:  $\sum_{j=0}^{k-0-d} C_{0+j}^0 C_{k-0-j}^d = \sum_{j=0}^{k-d} C_j^0 C_{k-j}^d = \sum_{j=0}^{k-d} C_{k-j}^d = \sum_{j=d}^k C_j^d = C_{k+1}^{d+1} = C_{k+1}^{0+d+1}, c=0$  □

以上引理肯定是正确的，我大约一年前在书上看见过这个公式，是采用排列组合的方法证明的，但太抽象不够直观，难于看懂。现在我终于找到了以上直观且严格的方法证明了它(2023.10.02)，本质上是数学归纳法。以下是更简洁的证法，但不够严谨、有点含混，但能更能简单明白地看清证明的关键思路。

证明:  $\sum_{a+b=k} C_a^c C_b^d = \sum_{a+b=k} (C_a^c C_{b-1}^{d-1} + C_a^c C_{b-1}^d) = \sum_{a+b-1=k-1} (C_a^c C_{b-1}^{d-1} + C_a^c C_{b-1}^d) = C_k^{c+d} + C_k^{c+d} = C_{k+1}^{c+d+1}$  □

### 2.5 以上重要组合学恒等式的推论

推论2.5.1. 
$$\sum_{h'=n'}^{-n'} C_{n+h'+h}^{n'+h'+h} C_{n-h'-h}^{n'-h'-h} = \sum_{h'=n'}^{-n'} C_{n+h'+h}^{n-n'+h} C_{n-h'-h}^{n-n'-h} = C_{2n+1}^{2n'}; n' - n \leq h \leq n - n', n \geq n'$$

推论2.5.2. 
$$\sum_{h'=s'}^{-s'} C_{s+h'+h}^{s'+h'+h} C_{s-h'-h}^{s'-h'-h} = \sum_{h'=s'}^{-s'} C_{s+h'+h}^{s-s'+h} C_{s-h'-h}^{s-s'-h} = C_{2s+1}^{2s'}; s' - s \leq h \leq s - s', s \geq s'$$

推论2.5.3. 
$$\sum_{h'=s'}^{-s'} C_{s+h'}^{s'+h'} C_{s-h'}^{s'-h'} = C_{2s+1}^{2s'}$$

推论2.5.4.

$\sum_{h'=s}^{-s} C_{s+h'}^{s+h'} C_{s-h'}^{s-h'} = C_{2s+1}^{2s} [\Leftrightarrow] \sum_{h'=s}^{-s} C_{s+h'}^0 C_{s-h'}^0 = C_{2s+1}^1 [\Leftrightarrow] \sum_{k=0}^{2s} C_k^0 C_{2s-k}^0 = C_{2s+1}^1$

$\sum_{h'=s-1}^{1-s} C_{s+h'}^{s-1+h'} C_{s-h'}^{s-1-h'} = C_{2s+1}^{2s-2} [\Leftrightarrow] \sum_{h'=s-1}^{1-s} C_{s+h'}^1 C_{s-h'}^1 = C_{2s+1}^3 [\Leftrightarrow] \sum_{k=1}^{2s-1} C_k^1 C_{2s-k}^1 = C_{2s+1}^3$

$\sum_{h'=s-2}^{2-s} C_{s+h'}^{s-2+h'} C_{s-h'}^{s-2-h'} = C_{2s+1}^{2s-4} [\Leftrightarrow] \sum_{h'=s-2}^{2-s} C_{s+h'}^2 C_{s-h'}^2 = C_{2s+1}^5 [\Leftrightarrow] \sum_{k=2}^{2s-2} C_k^2 C_{2s-k}^2 = C_{2s+1}^5$

$$\cdots \\ \sum_{h'=s-l}^{l-s} C_{s+h'}^{s-l+h'} C_{s-h'}^{s-l-h'} = C_{2s+1}^{2s-2l} [\Leftrightarrow] \sum_{h'=s-l}^{l-s} C_{s+h'}^l C_{s-h'}^l = C_{2s+1}^{2l+1} [\Leftrightarrow] \sum_{k=l}^{2s-l} C_k^l C_{2s-k}^l = C_{2s+1}^{2l+1}$$

$$\text{推论2.5.5. } \sum_{a+b=n} C_a^c C_b^d = C_{n+1}^{c+d+1} \Rightarrow \sum_{k=l}^{n-l} C_k^l C_{n-k}^l = C_{n+1}^{2l+1}$$

推论2.5.6.

$$\begin{cases} C_{n+h}^2 + C_{n+h-1}^1 C_{n-h+1}^1 + C_{n-h+2}^2 = \sum_{h'=1}^{-1} C_{(n+h')+(h-1)}^{1+h'} C_{(n-h')-(h-1)}^{1-h'} = C_{2n+1}^2 \\ C_{n+h}^{2n'} C_{n-h}^0 + C_{n+h-1}^{2n'-1} C_{n-h+1}^1 + C_{n+h-2}^{2n'-2} C_{n-h+2}^2 + \cdots + C_{n+h-2n'}^0 C_{n-h+2n'}^{2n'} = \sum_{h'=n'}^{-n'} C_{(n+h')+(h-n')}^{n'+h'} C_{(n-h')-(h-n')}^{n'-h'} = C_{2n+1}^{2n'} \end{cases}$$

推论2.5.7.

$$\begin{cases} \sum_{h'=1}^{-1} C_{(n+h')+(h)}^{1+h'} C_{(n-h')-h}^{1-h'} = C_{2n+1}^2, \sum_{h'=2}^{-2} C_{(n+h')+(h)}^{2+h'} C_{(n-h')-h}^{2-h'} = C_{2n+1}^4, \sum_{h'=n'}^{-n'} C_{(n+h')+(h)}^{n'+h'} C_{(n-h')-h}^{n'-h'} = C_{2n+1}^{2n'} \\ \sum_{h'=1}^{-1} C_{(n-h')+(h)}^{1-h'} C_{(n+h')-h}^{1+h'} = C_{2n+1}^2, \sum_{h'=2}^{-2} C_{(n-h')+(h)}^{2-h'} C_{(n+h')-h}^{2+h'} = C_{2n+1}^4, \sum_{h'=n'}^{-n'} C_{(n-h')+(h)}^{n'-h'} C_{(n+h')-h}^{n'+h'} = C_{2n+1}^{2n'} \end{cases}$$

### 3 Bargmann-Wigner方程<sup>[17]</sup>的自旋基和平面波解

#### 3.1 Dirac方程自旋基的广义二项式定理及其推论

定理3.1.1.

$$\begin{aligned} & \sum_{h=s}^{-s} C_{2s}^{s-h} \underbrace{u_{\{\lambda_\zeta, \frac{1}{2}\}} u_{\mu_\zeta, \frac{1}{2}} \cdots u_{\sigma_\zeta, \frac{1}{2}} u_{\tau_\zeta, \frac{1}{2}}}_{s+h} \underbrace{u_{\lambda'_\zeta, \frac{1}{2}} u_{\mu'_\zeta, \frac{1}{2}} \cdots u_{\sigma'_\zeta, \frac{1}{2}} u_{\tau'_\zeta, \frac{1}{2}}}_{s-h} \\ &= \sum_{h=1/2}^{-1/2} u_{\{\lambda_\zeta, h\}} u_{\{\lambda'_\zeta, h\}} \sum_{h=1/2}^{-1/2} u_{\mu_\zeta, h} u_{\mu'_\zeta, h} \cdots \sum_{h=1/2}^{-1/2} u_{\sigma_\zeta, h} u_{\sigma'_\zeta, h} \sum_{h=1/2}^{-1/2} u_{\tau_\zeta, h} u_{\tau'_\zeta, h} \end{aligned}$$

推论3.1.1.

$$\begin{aligned} & \sum_{h=s}^{-s} C_{2s}^{s-h} \underbrace{u_{\{1_\zeta, \frac{1}{2}\}} u_{1'_\zeta, \frac{1}{2}} \cdots u_{1_\zeta, \frac{1}{2}} u_{1'_\zeta, \frac{1}{2}}}_{s+h} \underbrace{u_{1_\zeta, \frac{1}{2}} u_{1'_\zeta, \frac{1}{2}} \cdots u_{1_\zeta, \frac{1}{2}} u_{1'_\zeta, \frac{1}{2}}}_{s-h} \\ &= \sum_{h=1/2}^{-1/2} u_{\{1_\zeta, h\}} u_{\{1'_\zeta, h\}} \sum_{h=1/2}^{-1/2} u_{1_\zeta, h} u_{1'_\zeta, h} \cdots \sum_{h=1/2}^{-1/2} u_{1_\zeta, h} u_{1'_\zeta, h} \sum_{h=1/2}^{-1/2} u_{1_\zeta, h} u_{1'_\zeta, h} \\ &\Leftrightarrow \sum_{h=s}^{-s} C_{2s}^{s-h} [u_{1_\zeta, \frac{1}{2}} u_{1'_\zeta, \frac{1}{2}}]^{s+h} [u_{1_\zeta, \frac{1}{2}} u_{1'_\zeta, \frac{1}{2}}]^{s-h} = [u_{1_\zeta, \frac{1}{2}} u_{1'_\zeta, \frac{1}{2}} + u_{1_\zeta, \frac{1}{2}} u_{1'_\zeta, \frac{1}{2}}]^{2s} \end{aligned}$$

推论3.1.2.

$$\begin{aligned} & \sum_{h=s}^{-s} C_{2s}^{s-h} \underbrace{u_{\{2_\zeta, \frac{1}{2}\}} u_{2'_\zeta, \frac{1}{2}} \cdots u_{2_\zeta, \frac{1}{2}} u_{2'_\zeta, \frac{1}{2}}}_{s+h} \underbrace{u_{2_\zeta, \frac{1}{2}} u_{2'_\zeta, \frac{1}{2}} \cdots u_{2_\zeta, \frac{1}{2}} u_{2'_\zeta, \frac{1}{2}}}_{s-h} \\ &= \sum_{h=1/2}^{-1/2} u_{\{2_\zeta, h\}} u_{\{2'_\zeta, h\}} \sum_{h=1/2}^{-1/2} u_{2_\zeta, h} u_{2'_\zeta, h} \cdots \sum_{h=1/2}^{-1/2} u_{2_\zeta, h} u_{2'_\zeta, h} \sum_{h=1/2}^{-1/2} u_{2_\zeta, h} u_{2'_\zeta, h} \\ &\Leftrightarrow \sum_{h=s}^{-s} C_{2s}^{s-h} [u_{2_\zeta, \frac{1}{2}} u_{2'_\zeta, \frac{1}{2}}]^{s+h} [u_{2_\zeta, \frac{1}{2}} u_{2'_\zeta, \frac{1}{2}}]^{s-h} = [u_{2_\zeta, \frac{1}{2}} u_{2'_\zeta, \frac{1}{2}} + u_{2_\zeta, \frac{1}{2}} u_{2'_\zeta, \frac{1}{2}}]^{2s} \end{aligned}$$

以上两个推论正好就是二项式展开定理。

引理3.1.1.

$$\begin{aligned} & [u_{\lambda_\zeta, \frac{1}{2}} u_{\lambda'_\zeta, \frac{1}{2}} + u_{\lambda_\zeta, \frac{1}{2}} u_{\lambda'_\zeta, \frac{1}{2}}] [u_{\mu_\zeta, \frac{1}{2}} u_{\mu'_\zeta, \frac{1}{2}} + u_{\mu_\zeta, \frac{1}{2}} u_{\mu'_\zeta, \frac{1}{2}}] \\ & \neq \\ & [u_{\mu_\zeta, \frac{1}{2}} u_{\mu'_\zeta, \frac{1}{2}} + u_{\mu_\zeta, \frac{1}{2}} u_{\mu'_\zeta, \frac{1}{2}}] [u_{\lambda_\zeta, \frac{1}{2}} u_{\lambda'_\zeta, \frac{1}{2}} + u_{\lambda_\zeta, \frac{1}{2}} u_{\lambda'_\zeta, \frac{1}{2}}] \end{aligned}$$

## 引理3.1.2.

$$\begin{aligned}
& [v_{\lambda_c}(\vec{p}, \frac{1}{2})v_{\lambda'_c}^+(\vec{p}, \frac{1}{2}) + v_{\lambda_c}(\vec{p}, -\frac{1}{2})v_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})][v_{\mu_c}(\vec{p}, \frac{1}{2})v_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + v_{\mu_c}(\vec{p}, -\frac{1}{2})v_{\mu'_c}^+(\vec{p}, -\frac{1}{2})] \\
& \neq \\
& [v_{\mu_c}(\vec{p}, \frac{1}{2})v_{\lambda'_c}^+(\vec{p}, \frac{1}{2}) + v_{\mu_c}(\vec{p}, -\frac{1}{2})v_{\lambda'_c}^+(\vec{p}, -\frac{1}{2})][v_{\lambda_c}(\vec{p}, \frac{1}{2})v_{\mu'_c}^+(\vec{p}, \frac{1}{2}) + v_{\lambda_c}(\vec{p}, -\frac{1}{2})v_{\mu'_c}^+(\vec{p}, -\frac{1}{2})]
\end{aligned}$$

## 推论3.1.3.

$$\begin{aligned}
& \Lambda_{+\lambda_c\lambda'_c}(\vec{p}, \frac{1}{2})\Lambda_{+\mu_c\mu'_c}(\vec{p}, \frac{1}{2}) \neq \Lambda_{+\mu_c\lambda'_c}(\vec{p}, \frac{1}{2})\Lambda_{+\lambda_c\mu'_c}(\vec{p}, \frac{1}{2}) \\
& \Lambda_{-\lambda_c\lambda'_c}(\vec{p}, \frac{1}{2})\Lambda_{-\mu_c\mu'_c}(\vec{p}, \frac{1}{2}) \neq \Lambda_{-\mu_c\lambda'_c}(\vec{p}, \frac{1}{2})\Lambda_{-\lambda_c\mu'_c}(\vec{p}, \frac{1}{2})
\end{aligned}$$

## 推论3.1.4.

$$\begin{cases}
\frac{1}{[(2s)!]^2} \underbrace{\Lambda_{+\{\lambda_c\lambda'_c\}}(\vec{p}, \frac{1}{2})\Lambda_{+\{\mu_c\mu'_c\}}(\vec{p}, \frac{1}{2}) \cdot \cdot \cdot \Lambda_{+\tau_c\tau'_c}(\vec{p}, \frac{1}{2})}_{2s} \neq \underbrace{\Lambda_{+\lambda_c\lambda'_c}(\vec{p}, \frac{1}{2})\Lambda_{+\mu_c\mu'_c}(\vec{p}, \frac{1}{2}) \cdot \cdot \cdot \Lambda_{+\tau_c\tau'_c}(\vec{p}, \frac{1}{2})}_{2s} \\
\frac{1}{[(2s)!]^2} \underbrace{\Lambda_{-\{\lambda_c\lambda'_c\}}(\vec{p}, \frac{1}{2})\Lambda_{-\{\mu_c\mu'_c\}}(\vec{p}, \frac{1}{2}) \cdot \cdot \cdot \Lambda_{-\tau_c\tau'_c}(\vec{p}, \frac{1}{2})}_{2s} \neq \underbrace{\Lambda_{-\lambda_c\lambda'_c}(\vec{p}, \frac{1}{2})\Lambda_{-\mu_c\mu'_c}(\vec{p}, \frac{1}{2}) \cdot \cdot \cdot \Lambda_{-\tau_c\tau'_c}(\vec{p}, \frac{1}{2})}_{2s}
\end{cases}$$

## 3.2 Bargmann-Wigner方程平面波解的合理推测(本章节后面将会给出严格的证明)

$$\text{定理3.2.1. } (\gamma^a \partial_a + m)_{\kappa_c} \psi_{\lambda_c\mu_c \cdot \cdot \cdot \tau_c}^{\lambda_c}(\vec{r}, t) = 0, \psi_{\lambda_c\mu_c \cdot \cdot \cdot \tau_c}(\vec{r}, t) = \frac{1}{(2s)!} \psi_{\{\lambda_c\mu_c \cdot \cdot \tau_c\}}(\vec{r}, t)$$

$$\psi_{\lambda_c\mu_c \cdot \cdot \tau_c}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=s}^{-s} E^{s-\frac{1}{2}} \sqrt{\frac{m^{2s}}{E}} [a(\vec{p}, h) \underbrace{U_{\lambda_c\mu_c \cdot \cdot \tau_c}}_{2s}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) \underbrace{V_{\lambda_c\mu_c \cdot \cdot \tau_c}}_{2s}(\vec{p}, h) e^{-ip \cdot x}] d^3\vec{p}$$

$$\underbrace{U_{\lambda_c\mu_c \cdot \cdot \tau_c}}_{2s}(\vec{p}, h) = \frac{1}{(2s)!} \sqrt{C_{2s}^{s-h}} \underbrace{u_{\{\lambda_c(\vec{p}, \frac{1}{2})\mu_c(\vec{p}, \frac{1}{2})\}}_{s+h} \cdot \cdot \cdot u_{\sigma_c(\vec{p}, -\frac{1}{2})\tau_c(\vec{p}, -\frac{1}{2})}}_{s-h}$$

$$\underbrace{V_{\lambda_c\mu_c \cdot \cdot \tau_c}}_{2s}(\vec{p}, h) = \frac{1}{(2s)!} \sqrt{C_{2s}^{s-h}} \underbrace{v_{\{\lambda_c(\vec{p}, \frac{1}{2})\mu_c(\vec{p}, \frac{1}{2})\}}_{s+h} \cdot \cdot \cdot v_{\sigma_c(\vec{p}, -\frac{1}{2})\tau_c(\vec{p}, -\frac{1}{2})}}_{s-h}$$

## 推论3.2.1.

$$\begin{cases}
a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m^{2s}}{E}} U^{+\{\lambda_c\mu_c \cdot \cdot \tau_c\}}(\vec{p}, h) \psi_{\lambda_c\mu_c \cdot \cdot \tau_c}(\vec{r}, t) e^{-ip \cdot x} d^3\vec{r} \\
b^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m^{2s}}{E}} V^{+\{\lambda_c\mu_c \cdot \cdot \tau_c\}}(\vec{p}, h) \psi_{\lambda_c\mu_c \cdot \cdot \tau_c}(\vec{r}, t) e^{ip \cdot x} d^3\vec{r}
\end{cases}$$

$$\text{推论3.2.2. } (\gamma^a \partial_a + m)_{\kappa_c} \psi_{\lambda_c\mu_c \cdot \cdot \tau_c}^{\lambda_c}(\vec{r}, t) = 0, \psi_{\lambda_c\mu_c \cdot \cdot \tau_c}(\vec{r}, t) = \frac{1}{(2s)!} \psi_{\{\lambda_c\mu_c \cdot \cdot \tau_c\}}(\vec{r}, t)$$

$$\psi_{\lambda_c\mu_c \cdot \cdot \tau_c}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=s}^{-s} E^{s-\frac{1}{2}} [a(\vec{p}, h) \underbrace{\tilde{U}_{\lambda_c\mu_c \cdot \cdot \tau_c}}_{2s}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) \underbrace{\tilde{V}_{\lambda_c\mu_c \cdot \cdot \tau_c}}_{2s}(\vec{p}, h) e^{-ip \cdot x}] d^3\vec{p}$$

$$\begin{cases}
a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \tilde{U}^{+\{\lambda_c\mu_c \cdot \cdot \tau_c\}}(\vec{p}, h) \psi_{\lambda_c\mu_c \cdot \cdot \tau_c}(\vec{r}, t) e^{-ip \cdot x} d^3\vec{r} \\
b^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \tilde{V}^{+\{\lambda_c\mu_c \cdot \cdot \tau_c\}}(\vec{p}, h) \psi_{\lambda_c\mu_c \cdot \cdot \tau_c}(\vec{r}, t) e^{ip \cdot x} d^3\vec{r}
\end{cases}$$

自我评述：以上的表达式与无质量粒子的平面波解形式十分相似，所以物理学本质上是有统一表述的，但要注意无质量粒子的平面波解不能通过以上有质量粒子的平面波解通过  $m \rightarrow 0$  得到，这说明无质量粒子与有质量粒子有本质的区别。

## 3.3 实表象下Bargmann-Wigner方程的平面波解

$$\text{定理3.3.1. } (\gamma_s^a \partial_a + m)_{\kappa_c} \psi_{\lambda_c\mu_c \cdot \cdot \tau_c}^{\lambda_c}(\vec{r}, t) = 0, \psi_{\lambda_c\mu_c \cdot \cdot \tau_c}(\vec{r}, t) = \frac{1}{(2s)!} \psi_{\{\lambda_c\mu_c \cdot \cdot \tau_c\}}(\vec{r}, t)$$

$$\psi_{\lambda_c\mu_c \cdot \cdot \tau_c}(\vec{r}, t)$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=s}^{-s} \sqrt{\frac{m^{2s}}{E}} [a(\vec{p}, h) \underbrace{U_{\lambda_c\mu_c \cdot \cdot \tau_c}}_{2s}(\vec{p}, h) e^{ip \cdot x} - (-1)^{s+h} \underbrace{U_{\lambda_c\mu_c \cdot \cdot \tau_c}^+}_{2s}(\vec{p}, -h) e^{-ip \cdot x}] d^3\vec{p}$$

$$U_{s \underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \underbrace{u_{s\{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{s\mu_\zeta(\vec{p}, \frac{1}{2})\cdots}}}_{s+h} \underbrace{u_{s\sigma_\zeta(\vec{p}, -\frac{1}{2})u_{s\tau_\zeta(\vec{p}, -\frac{1}{2})}}}_{s-h}$$

$$V_{s \underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = -(-1)^{s-h} \zeta^{2s} U_{s \underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}^+(\vec{p}, -h)$$

推论3.3.1.

$$\begin{cases} a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m^{2s}}{E}} U_{s \underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s}}^+(\vec{p}, h) \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r} \\ b^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m^{2s}}{E}} U_{s \underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}, h) \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{r}, t) e^{ip \cdot x} d^3 \vec{r} \end{cases}$$

### 3.4 Bargmann-Wigner方程的自旋基

定义3.4.1.

$$\begin{cases} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) := \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \underbrace{u_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots}}}_{s+h} \underbrace{u_{\sigma_\zeta(\vec{p}, -\frac{1}{2})u_{\tau_\zeta(\vec{p}, -\frac{1}{2})}}}_{s-h} \\ V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) := \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \underbrace{v_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})v_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots}}}_{s+h} \underbrace{v_{\sigma_\zeta(\vec{p}, -\frac{1}{2})v_{\tau_\zeta(\vec{p}, -\frac{1}{2})}}}_{s-h} \end{cases}$$

定义3.4.2.

$$\begin{cases} \tilde{U}_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) := \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \underbrace{\tilde{u}_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})\tilde{u}_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots}}}_{s+h} \underbrace{\tilde{u}_{\sigma_\zeta(\vec{p}, -\frac{1}{2})\tilde{u}_{\tau_\zeta(\vec{p}, -\frac{1}{2})}}}_{s-h} \\ \tilde{V}_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) := \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \underbrace{\tilde{v}_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})\tilde{v}_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots}}}_{s+h} \underbrace{\tilde{v}_{\sigma_\zeta(\vec{p}, -\frac{1}{2})\tilde{v}_{\tau_\zeta(\vec{p}, -\frac{1}{2})}}}_{s-h} \end{cases}$$

推论3.4.1.

$$\begin{cases} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = \sqrt{C_{2s}^{h-s}} [u_{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots} u_{\sigma_\zeta(\vec{p}, -\frac{1}{2})u_{\tau_\zeta(\vec{p}, -\frac{1}{2})} + \cdots] \\ V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = \sqrt{C_{2s}^{h-s}} [v_{\lambda_\zeta(\vec{p}, \frac{1}{2})v_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots} v_{\sigma_\zeta(\vec{p}, -\frac{1}{2})v_{\tau_\zeta(\vec{p}, -\frac{1}{2})} + \cdots] \end{cases}$$

推论3.4.2.  $u(\vec{p}, h) = -\varsigma \gamma_5 v(\vec{p}, h), v(\vec{p}, h) = -\varsigma \gamma_5 u(\vec{p}, h), h = -\frac{1}{2}, \frac{1}{2}$

$$\text{推论3.4.3. } \begin{cases} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = (-\varsigma)^{2s} \overbrace{\gamma_5 \otimes \gamma_5 \cdots}^{2s} V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) \\ V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = (-\varsigma)^{2s} \overbrace{\gamma_5 \otimes \gamma_5 \cdots}^{2s} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) \end{cases}$$

### 3.5 Bargmann-Wigner方程的两种自旋基之间的关系

$$\text{推论3.5.1. } \begin{cases} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}^+(\vec{p}, h) = (-1)^{s+h} \zeta^{2s} \overbrace{\sigma_y \otimes \sigma_y \cdots}^{4s} V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, -h) \\ V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}^+(\vec{p}, h) = (-1)^{s-h} \zeta^{2s} \overbrace{\sigma_y \otimes \sigma_y \cdots}^{4s} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, -h) \end{cases}$$

$$\begin{aligned} \text{证明: } U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}^+(\vec{p}, h) &= \frac{1}{(2s)!} \sqrt{C_{2s}^{s-h}} \underbrace{u_{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots}}_{s+h} \underbrace{u_{\sigma_\zeta(\vec{p}, -\frac{1}{2})u_{\tau_\zeta(\vec{p}, -\frac{1}{2})}}_{s-h} \\ &= (-1)^{s+h} \zeta^{2s} \overbrace{\sigma_y \otimes \sigma_y \cdots}^{4s} \frac{1}{(2s)!} \sqrt{C_{2s}^{s-h}} \underbrace{v_{\{\lambda_\zeta(\vec{p}, -\frac{1}{2})v_{\mu_\zeta(\vec{p}, -\frac{1}{2})\cdots}}}_{s+h} \underbrace{v_{\sigma_\zeta(\vec{p}, \frac{1}{2})v_{\tau_\zeta(\vec{p}, \frac{1}{2})}}}_{s-h} \\ &= (-1)^{s+h} \zeta^{2s} \overbrace{\sigma_y \otimes \sigma_y \cdots}^{4s} V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, -h) \end{aligned}$$

□

$$\text{推论3.5.2.} \left\{ \begin{array}{l} U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, -h) = (-1)^{s-h} \zeta^{2s} \overbrace{\sigma_y \otimes \sigma_y \dots}^{4s} V_{\lambda_s \mu_s \dots \sigma_s \tau_s}^+(\vec{p}, h) \\ V_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, -h) = (-1)^{s+h} \zeta^{2s} \overbrace{\sigma_y \otimes \sigma_y \dots}^{4s} U_{\lambda_s \mu_s \dots \sigma_s \tau_s}^+(\vec{p}, h) \end{array} \right.$$

$$\text{推论3.5.3.} \left\{ \begin{array}{l} U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, -h) = (-1)^{s-h} \zeta^{2s} \overbrace{(C\gamma_4) \otimes (C\gamma_4) \dots}^{2s} V_{\lambda_s \mu_s \dots \sigma_s \tau_s}^+(\vec{p}, h) \\ V_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, -h) = (-1)^{s+h} \zeta^{2s} \overbrace{(C\gamma_4) \otimes (C\gamma_4) \dots}^{2s} U_{\lambda_s \mu_s \dots \sigma_s \tau_s}^+(\vec{p}, h) \end{array} \right.$$

### 3.6 Bargmann-Wigner方程自旋基的正交性质(可以直接看出来)

推论3.6.1.

$$\left\{ \begin{array}{l} \overbrace{U_{\lambda_s \mu_s \dots \sigma_s \tau_s}}^{2s}(\vec{p}, h) \underbrace{U_{\lambda_s \mu_s \dots \sigma_s \tau_s}}_{2s}(\vec{p}, h') = \delta_{hh'}, \overbrace{U_{\lambda_s \mu_s \dots \sigma_s \tau_s}}^{2s}(\vec{p}, h) \underbrace{V_{\lambda_s \mu_s \dots \sigma_s \tau_s}}_{2s}(\vec{p}, h') = 0 \\ \overbrace{V_{\lambda_s \mu_s \dots \sigma_s \tau_s}}^{2s}(\vec{p}, h) \underbrace{V_{\lambda_s \mu_s \dots \sigma_s \tau_s}}_{2s}(\vec{p}, h') = \delta_{hh'}, \overbrace{V_{\lambda_s \mu_s \dots \sigma_s \tau_s}}^{2s}(\vec{p}, h) \underbrace{U_{\lambda_s \mu_s \dots \sigma_s \tau_s}}_{2s}(\vec{p}, h') = 0 \end{array} \right.$$

推论3.6.2.

$$\left\{ \begin{array}{l} U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h) \underbrace{U_{\lambda_s \mu_s \dots \sigma_s \tau_s}}_{2s}(\vec{p}, h') = \left(\frac{E}{m}\right)^{2s} \delta_{hh'}, U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h) \underbrace{V_{\lambda_s \mu_s \dots \sigma_s \tau_s}}_{2s}(-\vec{p}, h') = 0 \\ V_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h) \underbrace{V_{\lambda_s \mu_s \dots \sigma_s \tau_s}}_{2s}(\vec{p}, h') = \left(\frac{E}{m}\right)^{2s} \delta_{hh'}, V_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h) \underbrace{U_{\lambda_s \mu_s \dots \sigma_s \tau_s}}_{2s}(-\vec{p}, h') = 0 \end{array} \right.$$

## 4 Bargmann-Wigner方程自旋基的递推关系

### 4.1 Bargmann-Wigner方程自旋基递推关系(枚举试探法)

定理4.1.1.  $U_{\lambda_s \mu_s \dots \rho_s \sigma_s \tau_s}(\vec{p}, s + \frac{1}{2} - l)$

$$= \frac{1}{\sqrt{C_{2s+1}^l}} \left[ \sqrt{C_{2s}^{l-1}} U_{\lambda_s \mu_s \dots \rho_s \sigma_s}(\vec{p}, s - l + 1) u_{\tau_s}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2s}^l} U_{\lambda_s \mu_s \dots \rho_s \sigma_s}(\vec{p}, s - l) u_{\tau_s}(\vec{p}, \frac{1}{2}) \right]$$

证明:

$$\begin{aligned} & U_{\lambda_s \mu_s \dots \rho_s \sigma_s \tau_s}(\vec{p}, s + \frac{1}{2}) \\ &= \frac{1}{\sqrt{(2s+1)!(2s+1)!(0)!}} \underbrace{u_{\lambda_s}(\vec{p}, \frac{1}{2}) u_{\mu_s}(\vec{p}, \frac{1}{2}) \dots u_{\rho_s}(\vec{p}, -\frac{1}{2}) u_{\sigma_s}(\vec{p}, -\frac{1}{2})}_{2s+1} u_{\tau_s}(\vec{p}, -\frac{1}{2}) \\ &= \underbrace{u_{\lambda_s}(\vec{p}, \frac{1}{2}) u_{\mu_s}(\vec{p}, \frac{1}{2}) \dots u_{\rho_s}(\vec{p}, \frac{1}{2})}_{2s+1} u_{\sigma_s}(\vec{p}, \frac{1}{2}) u_{\tau_s}(\vec{p}, \frac{1}{2}) \\ &= C_{2s}^\phi u_{\lambda_s}(\vec{p}, \frac{1}{2}) u_{\mu_s}(\vec{p}, \frac{1}{2}) \dots u_{\rho_s}(\vec{p}, \frac{1}{2}) u_{\sigma_s}(\vec{p}, \frac{1}{2}) u_{\tau_s}(\vec{p}, \frac{1}{2}) \\ &= \frac{1}{\sqrt{C_{2s+1}^0}} C_{2s}^0 U_{\lambda_s \mu_s \dots \rho_s \sigma_s}(\vec{p}, s) u_{\tau_s}(\vec{p}, \frac{1}{2}) \end{aligned}$$

证明:

$$\begin{aligned} & U_{\lambda_s \mu_s \dots \rho_s \sigma_s \tau_s}(\vec{p}, s - \frac{1}{2}) \\ &= \frac{1}{\sqrt{(2s+1)!(2s)!(1)!}} \underbrace{u_{\lambda_s}(\vec{p}, \frac{1}{2}) u_{\mu_s}(\vec{p}, \frac{1}{2}) \dots u_{\rho_s}(\vec{p}, -\frac{1}{2})}_{2s} u_{\sigma_s}(\vec{p}, -\frac{1}{2}) u_{\tau_s}(\vec{p}, -\frac{1}{2}) \\ &= \frac{1}{\sqrt{(2s+1)!(2s)!(1)!}} \underbrace{u_{\lambda_s}(\vec{p}, \frac{1}{2}) u_{\mu_s}(\vec{p}, \frac{1}{2}) \dots u_{\rho_s}(\vec{p}, \frac{1}{2})}_{2s+1} u_{\sigma_s}(\vec{p}, \frac{1}{2}) u_{\tau_s}(\vec{p}, -\frac{1}{2}) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{C_{2s+1}^1}} \\
&\{ [C_{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, \frac{1}{2})u_{\sigma_\zeta}(\vec{p}, \frac{1}{2})}^{2s} u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + C_{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, \frac{1}{2})u_{\sigma_\zeta}(\vec{p}, \frac{1}{2})}^{2s} u_{\tau_\zeta}(\vec{p}, \frac{1}{2})] \} \\
&= \frac{1}{\sqrt{C_{2s+1}^1}} [\sqrt{C_{2s}^0} U_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta}(\vec{p}, s) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2s}^1} U_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta}(\vec{p}, s-1) u_{\tau_\zeta}(\vec{p}, \frac{1}{2})] \quad \square
\end{aligned}$$

证明:

$$\begin{aligned}
&U_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta \tau_\zeta}(\vec{p}, s - \frac{3}{2}) \\
&= \frac{1}{\sqrt{(2s+1)!(2s-1)!(2)!}} \underbrace{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, -\frac{1}{2})u_{\sigma_\zeta}(\vec{p}, -\frac{1}{2})}_{2s-1} u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \frac{1}{\sqrt{(2s+1)!(2s-1)!(2)!}} \underbrace{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, \frac{1}{2})u_{\sigma_\zeta}(\vec{p}, -\frac{1}{2})}_{2s+1} u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \frac{1}{\sqrt{C_{2s+1}^2}} C_{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, \frac{1}{2})u_{\sigma_\zeta}(\vec{p}, \frac{1}{2})u_{\tau_\zeta}(\vec{p}, \frac{1}{2})}^{(\vec{p}, -\frac{1}{2}), (\vec{p}, -\frac{1}{2})} \\
&= \frac{1}{\sqrt{C_{2s+1}^2}} \\
&[C_{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, \frac{1}{2})u_{\sigma_\zeta}(\vec{p}, \frac{1}{2})}^{2s} u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + C_{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, \frac{1}{2})u_{\sigma_\zeta}(\vec{p}, \frac{1}{2})}^{2s} u_{\tau_\zeta}(\vec{p}, \frac{1}{2})] \\
&= \frac{1}{\sqrt{C_{2s+1}^2}} [\sqrt{C_{2s}^1} U_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta}(\vec{p}, s-1) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2s}^2} U_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta}(\vec{p}, s-2) u_{\tau_\zeta}(\vec{p}, \frac{1}{2})] \quad \square
\end{aligned}$$

证明:

$$\begin{aligned}
&U_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta \tau_\zeta}(\vec{p}, s + \frac{1}{2} - l) \\
&= \frac{1}{\sqrt{(2s+1)!(2s+1-l)!(l)!}} \underbrace{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, -\frac{1}{2})u_{\sigma_\zeta}(\vec{p}, -\frac{1}{2})}_{2s+1-l} u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \frac{1}{\sqrt{(2s+1)!(2s-l+1)!(l)!}} \underbrace{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, \frac{1}{2})u_{\sigma_\zeta}(\vec{p}, -\frac{1}{2})}_{2s+1} u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \frac{1}{\sqrt{C_{2s+1}^l}} C_{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, \frac{1}{2})u_{\sigma_\zeta}(\vec{p}, \frac{1}{2})u_{\tau_\zeta}(\vec{p}, \frac{1}{2})}^{(\vec{p}, -\frac{1}{2}), \dots, (\vec{p}, -\frac{1}{2})} \\
&= \frac{1}{\sqrt{C_{2s+1}^l}} \\
&[C_{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, \frac{1}{2})u_{\sigma_\zeta}(\vec{p}, \frac{1}{2})}^{l-1} u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + C_{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\rho_\zeta}(\vec{p}, \frac{1}{2})u_{\sigma_\zeta}(\vec{p}, \frac{1}{2})}^l u_{\tau_\zeta}(\vec{p}, \frac{1}{2})] \\
&= \frac{1}{\sqrt{C_{2s+1}^l}} [\sqrt{C_{2s}^{l-1}} U_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta}(\vec{p}, s-l+1) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2s}^l} U_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta}(\vec{p}, s-l) u_{\tau_\zeta}(\vec{p}, \frac{1}{2})] \quad \square
\end{aligned}$$

定理4.1.2.  $V_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta \tau_\zeta}(\vec{p}, s + \frac{1}{2} - l)$ 

$$\begin{aligned}
&= \frac{1}{\sqrt{C_{2s+1}^l}} [\sqrt{C_{2s}^{l-1}} V_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta}(\vec{p}, s-l+1) v_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2s}^l} V_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta}(\vec{p}, s-l) v_{\tau_\zeta}(\vec{p}, \frac{1}{2})]
\end{aligned}$$

证明:

$$\begin{aligned}
&V_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta \tau_\zeta}(\vec{p}, s + \frac{1}{2} - l) \\
&= \frac{1}{\sqrt{(2s+1)!(2s+1-l)!(l)!}} \underbrace{v_{\lambda_\zeta}(\vec{p}, \frac{1}{2})v_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots v_{\rho_\zeta}(\vec{p}, -\frac{1}{2})v_{\sigma_\zeta}(\vec{p}, -\frac{1}{2})}_{2s+1-l} v_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \frac{1}{\sqrt{(2s+1)!(2s-l+1)!(l)!}} \underbrace{v_{\lambda_\zeta}(\vec{p}, \frac{1}{2})v_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots v_{\rho_\zeta}(\vec{p}, \frac{1}{2})v_{\sigma_\zeta}(\vec{p}, -\frac{1}{2})}_{2s+1} v_{\tau_\zeta}(\vec{p}, -\frac{1}{2})
\end{aligned}$$



$$\begin{aligned}
&= \frac{1}{\sqrt{C_{2s+1}^l}} C_{\overbrace{(\vec{p}, -\frac{1}{2}), \dots, (\vec{p}, -\frac{1}{2})}^l} \\
&\quad \underbrace{v_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) v_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots v_{\rho_\zeta}(\vec{p}, \frac{1}{2}) v_{\sigma_\zeta}(\vec{p}, \frac{1}{2}) v_{\tau_\zeta}(\vec{p}, \frac{1}{2})}_{2s+1} \\
&= \frac{1}{\sqrt{C_{2s+1}^l}} \left[ C_{\overbrace{(\vec{p}, -\frac{1}{2}), \dots, (\vec{p}, -\frac{1}{2})}^{l-1}} \right. \\
&\quad \underbrace{v_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) v_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots v_{\rho_\zeta}(\vec{p}, \frac{1}{2}) v_{\sigma_\zeta}(\vec{p}, \frac{1}{2})}_{2s} v_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + C_{\overbrace{(\vec{p}, -\frac{1}{2}), \dots, (\vec{p}, -\frac{1}{2})}^l} \\
&\quad \left. \underbrace{v_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) v_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots v_{\rho_\zeta}(\vec{p}, \frac{1}{2}) v_{\sigma_\zeta}(\vec{p}, \frac{1}{2})}_{2s} v_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \right] \\
&= \frac{1}{\sqrt{C_{2s+1}^l}} \left[ \sqrt{C_{2s}^{l-1}} \underbrace{V_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta}(\vec{p}, s-l+1)}_{2s} v_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2s}^l} \underbrace{V_{\lambda_\zeta \mu_\zeta \cdots \rho_\zeta \sigma_\zeta}(\vec{p}, s-l)}_{2s} v_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \right] \quad \square
\end{aligned}$$

## 4.2 Bargmann-Wigner方程U-自旋基的分解(组合学方法)

$$\text{定理4.2.1. } U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s} \lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}(\vec{p}, h) = \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+h}^{s'+h'} C_{s-h}^{s'-h'}}}{\sqrt{C_{2s}^{2s'}}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2(s-s')}}(\vec{p}, h-h') U_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}_{2s'}}(\vec{p}, h')$$

$$\begin{aligned}
\text{证明: } U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s} \lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}(\vec{p}, h) &= \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \\
&\sum_{h'=s'}^{-s'} C_{s+h}^{s'+h'} C_{s-h}^{s'-h'} \underbrace{u_{\{\lambda_\zeta(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})\}}(\vec{p}, -\frac{1}{2})}_{(s-s')+(h-h')} \\
&\quad \underbrace{u_{\{\lambda'_\zeta(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\sigma'_\zeta}(\vec{p}, -\frac{1}{2}) u_{\tau'_\zeta}(\vec{p}, -\frac{1}{2})\}}(\vec{p}, -\frac{1}{2})}_{(s-s')-(h-h')} \\
&\quad \underbrace{u_{\{\lambda'_\zeta(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\sigma'_\zeta}(\vec{p}, -\frac{1}{2}) u_{\tau'_\zeta}(\vec{p}, -\frac{1}{2})\}}(\vec{p}, -\frac{1}{2})}_{(s'+h')} \\
&\quad \underbrace{u_{\{\lambda'_\zeta(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\sigma'_\zeta}(\vec{p}, -\frac{1}{2}) u_{\tau'_\zeta}(\vec{p}, -\frac{1}{2})\}}(\vec{p}, -\frac{1}{2})}_{(s'-h')} \\
&= \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \\
&\sum_{h'=s'}^{-s'} C_{s+h}^{s'+h'} C_{s-h}^{s'-h'} \sqrt{[2(s-s')]! [(s-s')+(h-h')]! [(s-s')-(h-h')]!} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2(s-s')}}(\vec{p}, h-h') \\
&\quad \sqrt{(2s')!(s'+h')!(s'-h')!} U_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}_{2s'}}(\vec{p}, h') \\
&= \frac{\sqrt{[2(s-s')]!} \sqrt{(2s')!}}{\sqrt{(2s)!}} \\
&\sum_{h'=s'}^{-s'} C_{s+h}^{s'+h'} C_{s-h}^{s'-h'} \frac{\sqrt{[(s-s')+(h-h')]! [(s-s')-(h-h')]!} \sqrt{(s'+h')!(s'-h')!}}{\sqrt{(s+h)!(s-h)!}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2(s-s')}}(\vec{p}, h-h') U_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}_{2s'}}(\vec{p}, h') \\
&= \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+h}^{s'+h'} C_{s-h}^{s'-h'}}}{\sqrt{C_{2s}^{2s'}}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2(s-s')}}(\vec{p}, h-h') U_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}_{2s'}}(\vec{p}, h') \quad \square
\end{aligned}$$

$$\text{推论4.2.1. } U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2n} \lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}(\vec{p}, h) = \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+h}^{n'+h'} C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2(n-n')}}(\vec{p}, h-h') U_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}_{2n'}}(\vec{p}, h')$$

$$\begin{aligned}
\text{推论4.2.2. } U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta \sigma_\zeta \tau_\zeta}_{2n}}(\vec{p}, h) &= \frac{\sqrt{C_{2n}^{2n}}}{\sqrt{C_{2n}^2}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n-2}}(\vec{p}, h-1) U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1) \\
&+ \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n-2}}(\vec{p}, h) U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta}_{2n-2}}(\vec{p}, h+1) U_{\sigma_\zeta \tau_\zeta}(\vec{p}, -1)
\end{aligned}$$

$$\begin{aligned}
\text{推论4.2.3. } U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) &= \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+h'} C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h-h') U_{\tau_\zeta}(\vec{p}, h') \\
&= \frac{\sqrt{s+h}}{\sqrt{2s}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h-\frac{1}{2}) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h+\frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})
\end{aligned}$$

## 4.3 Bargmann-Wigner方程V-自旋基的分解(组合学方法)

$$\text{定理4.3.1. } V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s} \lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}(\vec{p}, h) = \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+h}^{s'+h'} C_{s-h}^{s'-h'}}}{\sqrt{C_{2s}^{2s'}}} V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2(s-s')}}(\vec{p}, h-h') V_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}_{2s'}}(\vec{p}, h')$$

$$\begin{aligned}
\text{证明: } V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \lambda'_\zeta \mu'_\zeta \dots \sigma'_\zeta \tau'_\zeta}(\vec{p}, h) &= \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \\
\sum_{h'=s'}^{-s'} C_{s+h}^{s'+h'} C_{s-h}^{s'-h'} &\underbrace{v_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})v_{\mu_\zeta(\vec{p}, \frac{1}{2})\dots v_{\sigma_\zeta(\vec{p}, -\frac{1}{2})v_{\tau_\zeta}\}(\vec{p}, -\frac{1}{2})}}}_{(s-s')+(h-h')} \underbrace{v_{\{\lambda'_\zeta(\vec{p}, \frac{1}{2})v_{\mu'_\zeta(\vec{p}, \frac{1}{2})\dots v_{\sigma'_\zeta(\vec{p}, -\frac{1}{2})v_{\tau'_\zeta}\}(\vec{p}, -\frac{1}{2})}}}_{(s-s')-(h-h')} \underbrace{v_{\{\lambda'_\zeta(\vec{p}, \frac{1}{2})v_{\mu'_\zeta(\vec{p}, \frac{1}{2})\dots v_{\sigma'_\zeta(\vec{p}, -\frac{1}{2})v_{\tau'_\zeta}\}(\vec{p}, -\frac{1}{2})}}}_{(s'+h')} \underbrace{v_{\{\lambda'_\zeta(\vec{p}, \frac{1}{2})v_{\mu'_\zeta(\vec{p}, \frac{1}{2})\dots v_{\sigma'_\zeta(\vec{p}, -\frac{1}{2})v_{\tau'_\zeta}\}(\vec{p}, -\frac{1}{2})}}}_{(s'-h')} \\
&= \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \\
\sum_{h'=s'}^{-s'} C_{s+h}^{s'+h'} C_{s-h}^{s'-h'} &\sqrt{[2(s-s')]![(s-s')+(h-h')]![(s-s')-(h-h')]!} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h-h')}_{2(s-s')} \\
\sqrt{(2s')!(s'+h')!(s'-h')!} &\underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots \sigma'_\zeta \tau'_\zeta}(\vec{p}, h')}_{2s'} \\
&= \frac{\sqrt{[2(s-s')]!}\sqrt{(2s)!}}{\sqrt{(2s)!}} \\
\sum_{h'=s'}^{-s'} C_{s+h}^{s'+h'} C_{s-h}^{s'-h'} &\frac{\sqrt{[(s-s')+(h-h')]![(s-s')-(h-h')]!}\sqrt{(s'+h')!(s'-h')!}}{\sqrt{(s+h)!(s-h)!}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h-h')}_{2(s-s')} \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots \sigma'_\zeta \tau'_\zeta}(\vec{p}, h')}_{2s'} \\
&= \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+h}^{s'+h'} C_{s-h}^{s'-h'}}}{\sqrt{C_{2s}^{2s'}}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h-h')}_{2(s-s')} \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots \sigma'_\zeta \tau'_\zeta}(\vec{p}, h')}_{2s'}
\end{aligned}$$

□

$$\text{推论4.3.1. } V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \lambda'_\zeta \mu'_\zeta \dots \sigma'_\zeta \tau'_\zeta}(\vec{p}, h) = \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+h}^{n'+h'} C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h-h')}_{2(n-n')} \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots \sigma'_\zeta \tau'_\zeta}(\vec{p}, h')}_{2n'}$$

$$\begin{aligned}
\text{推论4.3.2. } V_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta \sigma_\zeta \tau_\zeta}(\vec{p}, h) &= \frac{\sqrt{C_{n+h}^{2n}}}{\sqrt{C_{2n}^{2n}}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta}(\vec{p}, h-1)}_{2n-2} V_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1) \\
&+ \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta}(\vec{p}, h)}_{2n-2} V_{\sigma_\zeta \tau_\zeta}(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta}(\vec{p}, h+1)}_{2n-2} V_{\sigma_\zeta \tau_\zeta}(\vec{p}, -1)
\end{aligned}$$

$$\begin{aligned}
\text{推论4.3.3. } V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h) &= \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+h'} C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}(\vec{p}, h-h')}_{2s-1} V_{\tau_\zeta}(\vec{p}, h') \\
&= \frac{\sqrt{s+h}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}(\vec{p}, h-\frac{1}{2})}_{2s-1} v_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}(\vec{p}, h+\frac{1}{2})}_{2s-1} v_{\tau_\zeta}(\vec{p}, -\frac{1}{2})
\end{aligned}$$

#### 4.4 Bargmann-Wigner方程自旋基的合成

$$\text{推论4.4.1. } \frac{\sqrt{C_{s+h'}^{s'+h'} C_{s-h'}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-h')}_{2s} = \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta \dots \tau_\zeta}(\vec{p}, h)}_{2s} \underbrace{\bar{U}^{\rho_\zeta \sigma_\zeta \dots \tau_\zeta}(\vec{p}, h')}_{2s'}, -s-s' \leq h \leq s+s'$$

$$\text{推论4.4.2. } \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-h')}_{2s} = \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta \dots \tau_\zeta}(\vec{p}, h)}_{2s} \underbrace{\bar{V}^{\rho_\zeta \sigma_\zeta \dots \tau_\zeta}(\vec{p}, h')}_{2s'}, -s-s' \leq h \leq s+s'$$

#### 4.5 Bargmann-Wigner方程自旋基合成的推论

$$\text{推论4.5.1. } \begin{cases} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, s+\frac{1}{2}-l)}_{2s+1} u^{+\tau_\zeta}(\vec{p}, -\frac{1}{2}) = \frac{\sqrt{l}}{\sqrt{2s+1}} \frac{E}{m} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, s-l+1)}_{2s} \\ \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, s+\frac{1}{2}-l)}_{2s+1} u^{+\tau_\zeta}(\vec{p}, \frac{1}{2}) = \frac{\sqrt{2s+1-l}}{\sqrt{2s+1}} \frac{E}{m} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, s-l)}_{2s} \end{cases}$$

$$\text{推论4.5.2. } \begin{cases} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, s+\frac{1}{2}-l)}_{2s+1} v^{+\tau_\zeta}(\vec{p}, -\frac{1}{2}) = \frac{\sqrt{l}}{\sqrt{2s+1}} \frac{E}{m} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, s-l+1)}_{2s} \\ \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, s+\frac{1}{2}-l)}_{2s+1} v^{+\tau_\zeta}(\vec{p}, \frac{1}{2}) = \frac{\sqrt{2s+1-l}}{\sqrt{2s+1}} \frac{E}{m} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, s-l)}_{2s} \end{cases}$$

## 5 Bargmann-Wigner方程的准投影算子

### 5.1 Bargmann-Wigner方程准投影算子的定义和性质

定义5.1.1.

$$\begin{cases} \Lambda_{+\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s} \underbrace{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}, s) := \sum_{h=s}^{-s} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}_{2s}}^+(\vec{p}, h) \\ \Lambda_{-\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s} \underbrace{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}, s) := \sum_{h=s}^{-s} V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}, h) V_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}_{2s}}^+(\vec{p}, h) \end{cases}$$

推论5.1.1.

$$\Lambda_{\pm \underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s} \underbrace{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}, s) = \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{\pm \{\lambda_\zeta(\lambda'_\zeta(\vec{p}, \frac{1}{2})\}_{2s} \Lambda_{\pm \mu_\zeta \mu'_\zeta(\vec{p}, \frac{1}{2})\}_{2s}} \cdot \Lambda_{\pm \tau_\zeta \tau'_\zeta(\vec{p}, \frac{1}{2})\}_{2s}}}_{2s}}$$

从对称指标广义二项式定理便可以直接得到以上推论。

推论5.1.2.

$$\begin{cases} \sum_{h=s}^{-s} (-1)^{s-h} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}, h) V_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}, -h) \\ = \frac{\zeta^{2s}}{[(2s)!]^2} \underbrace{(\Lambda_+ \bar{C} \gamma_4)_{\{\lambda_\zeta(\lambda'_\zeta(\vec{p}, \frac{1}{2})\}_{2s}} (\Lambda_+ \bar{C} \gamma_4)_{\{\mu_\zeta \mu'_\zeta(\vec{p}, \frac{1}{2})\}_{2s}} \cdot (\Lambda_+ \bar{C} \gamma_4)_{\{\tau_\zeta \tau'_\zeta(\vec{p}, \frac{1}{2})\}_{2s}}}_{2s} \\ \sum_{h=s}^{-s} (-1)^{s+h} V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}, -h) \\ = \frac{\zeta^{2s}}{[(2s)!]^2} \underbrace{(\Lambda_- \bar{C} \gamma_4)_{\{\lambda_\zeta(\lambda'_\zeta(\vec{p}, \frac{1}{2})\}_{2s}} (\Lambda_- \bar{C} \gamma_4)_{\{\mu_\zeta \mu'_\zeta(\vec{p}, \frac{1}{2})\}_{2s}} \cdot (\Lambda_- \bar{C} \gamma_4)_{\{\tau_\zeta \tau'_\zeta(\vec{p}, \frac{1}{2})\}_{2s}}}_{2s} \end{cases}$$

推论5.1.3.

$$\begin{cases} \Lambda_{\pm \underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta \cdots \lambda'_\zeta \cdots \mu'_\zeta \cdots \tau'_\zeta}_{2n_1 \ 2n_2 \ 2n_k} \cdots}_{2n_1 \ 2n_2 \ 2n_k}}(\vec{p}, s) = \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{\pm \{\lambda_\zeta \cdots (\lambda'_\zeta \cdots (\vec{p}, n_1)\}_{2n_1} \Lambda_{\pm \mu_\zeta \cdots \mu'_\zeta \cdots (\vec{p}, n_2)\}_{2n_2}} \cdots}_{2n_1 \ 2n_2 \ 2n_k}} \cdot \underbrace{\Lambda_{\pm \tau_\zeta \cdots \tau'_\zeta \cdots (\vec{p}, n_k)\}_{2n_k}}}_{2n_k} \\ s = n_1 + n_2 + \cdots + n_k \end{cases}$$

推论5.1.4.

$$\begin{aligned} & \Lambda_{\pm \underbrace{\lambda_\zeta \mu_\zeta \cdots \lambda'_\zeta \mu'_\zeta \cdots}_{2n}}(\vec{p}, n) \\ &= \frac{1}{[(2n)!]^2} \underbrace{\Lambda_{\pm \{\lambda_\zeta(\lambda'_\zeta(\vec{p}, \frac{1}{2})\}_{2n} \Lambda_{\pm \mu_\zeta \mu'_\zeta(\vec{p}, \frac{1}{2})\}_{2n}} \cdots}}_{2n} = \frac{1}{(2\sqrt{2}m)^{2n}} \frac{1}{[(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta(\pm p)\}_{2n}}^a}_{n} \cdot \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta(\pm p)\}_{2n}}^{+a'}}_{n} \cdot \underbrace{\Lambda_{maa'}(\vec{p}, 1)}_{n} \cdots \end{aligned}$$

推论5.1.5.

$$\begin{aligned} & \Lambda_{\pm \underbrace{\lambda_\zeta \mu_\zeta \cdots \lambda'_\zeta \mu'_\zeta \cdots}_{2n}}(\vec{p}, n) = \frac{1}{(2m)^{2n}} \frac{1}{[(2n)!]^2} \underbrace{[(\pm m - i\gamma^a p_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(\pm m - i\gamma^b p_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \cdots})\}_{2n}}}_{2n} \\ &= \frac{1}{(2\sqrt{2}m)^{2n}} \frac{1}{[(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta(\pm p)\}_{2n}}^a}_{n} \cdot \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta(\pm p)\}_{2n}}^{+a'}}_{n} \cdot \underbrace{(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})}_{n} \cdots \end{aligned}$$

推论5.1.6.

$$\begin{cases} \sum_{h=s}^{-s} (-1)^{s-h} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}}(\vec{p}, h) V_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots}_{2s}}(\vec{p}, -h) = \frac{1}{(2m)^{2s}} \frac{\zeta^{2s}}{[(2s)!]^2} \underbrace{[(m - i\gamma^a p_a) C]_{\{\lambda_\zeta(\lambda'_\zeta[(m - i\gamma^b p_b) C]_{\mu_\zeta \mu'_\zeta \cdots})\}_{2s}}}_{2s} \\ \sum_{h=s}^{-s} (-1)^{s+h} V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots}_{2s}}(\vec{p}, -h) = \frac{1}{(2m)^{2s}} \frac{\zeta^{2s}}{[(2s)!]^2} \underbrace{[(-m - i\gamma^a p_a) C]_{\{\lambda_\zeta(\lambda'_\zeta[(-m - i\gamma^b p_b) C]_{\mu_\zeta \mu'_\zeta \cdots})\}_{2s}}}_{2s} \end{cases}$$

### 5.2 Bargmann-Wigner方程真正的两类投影算子

$$\text{定义5.2.1. } \tilde{\Lambda}_{\pm \underbrace{\lambda_\zeta \mu_\zeta \cdots \lambda'_\zeta \mu'_\zeta \cdots}_{2s}}(\vec{p}, s) := \left(\frac{m}{E}\right)^{2s} \Lambda_{\pm \underbrace{\lambda_\zeta \mu_\zeta \cdots \lambda'_\zeta \mu'_\zeta \cdots}_{2s}}(\vec{p}, s)$$

$$\text{定义5.2.2. } \bar{\Lambda}_{\pm \lambda_{\zeta} \mu_{\zeta} \dots}^{\overbrace{\eta_{\zeta} \xi_{\zeta} \dots}^{2s}}(\vec{p}, s) := \Lambda_{\pm \lambda_{\zeta} \mu_{\zeta} \dots \lambda'_{\zeta} \mu'_{\zeta} \dots}(\vec{p}, s) \overbrace{\gamma_{\lambda'_{\zeta}}^4 \eta_{\zeta} \gamma_{\mu'_{\zeta}}^4 \xi_{\zeta} \dots}^{2s}.$$

### 5.3 Bargmann-Wigner方程准投影算子的递推关系(严格证明)

$$\begin{aligned} \text{定理5.3.1. } & \sum_{h=s}^{-s} U_{\lambda_{\zeta} \mu_{\zeta} \dots}(\vec{p}, h) \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots}(\vec{p}, h) \\ &= \frac{2s+1}{2s+2s'+1} \sum_{h''=s+s'}^{-s-s'} [U_{\lambda_{\zeta} \mu_{\zeta} \dots \rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h'') \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots \rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h'')] \sum_{h'=s'}^{-s'} [U_{\rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h') \bar{U}_{\rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h')] \end{aligned}$$

$$\begin{aligned} \text{证明: } & \frac{C^{s'+h'}_{s+s'+h} C^{s'-h'}_{s+s'-h}}{C^{2s'}_{2(s+s')}} U_{\lambda_{\zeta} \mu_{\zeta} \dots}(\vec{p}, h-h') \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots}(\vec{p}, h-h') \\ &= U_{\lambda_{\zeta} \mu_{\zeta} \dots \rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h) \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots \rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h) U_{\rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h') \bar{U}_{\rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h'); -s-s' \leq h \leq s+s', -s' \leq h' \leq s' \\ &\Rightarrow \sum_{h=s+s'}^{-s-s'} \sum_{h'=s'}^{-s'} \frac{C^{s'+h'}_{s+s'+h} C^{s'-h'}_{s+s'-h}}{C^{2s'}_{2(s+s')}} U_{\lambda_{\zeta} \mu_{\zeta} \dots}(\vec{p}, h-h') \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots}(\vec{p}, h-h') \\ &= \sum_{h=s+s'}^{-s-s'} [U_{\lambda_{\zeta} \mu_{\zeta} \dots \rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h) \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots \rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h)] \sum_{h'=s'}^{-s'} [U_{\rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h') \bar{U}_{\rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h')] \\ &\Leftrightarrow \sum_{h'=s'}^{-s'} \sum_{h''=s+s'-h'}^{-s-s'-h'} \frac{C^{s'+h'+h''}_{s+s'+h'+h''} C^{s'-h'-h''}_{s+s'-h'-h''}}{C^{2s'}_{2(s+s')}} U_{\lambda_{\zeta} \mu_{\zeta} \dots}(\vec{p}, h'') \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots}(\vec{p}, h'') \\ &= \sum_{h=s+s'}^{-s-s'} [U_{\lambda_{\zeta} \mu_{\zeta} \dots \rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h) \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots \rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h)] \sum_{h'=s'}^{-s'} [U_{\rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h') \bar{U}_{\rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h')] \\ &\Leftrightarrow \sum_{h'=s'}^{-s'} \sum_{h''=s}^{-s} \frac{C^{s'+h'+h''}_{s+s'+h'+h''} C^{s'-h'-h''}_{s+s'-h'-h''}}{C^{2s'}_{2(s+s')}} U_{\lambda_{\zeta} \mu_{\zeta} \dots}(\vec{p}, h'') \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots}(\vec{p}, h'') \\ &= \sum_{h=s+s'}^{-s-s'} [U_{\lambda_{\zeta} \mu_{\zeta} \dots \rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h) \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots \rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h)] \sum_{h'=s'}^{-s'} [U_{\rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h') \bar{U}_{\rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h')] \\ &\Leftrightarrow \sum_{h'=s}^{-s} \sum_{h''=s'}^{-s'} \frac{C^{s'+h'+h''}_{s+s'+h'+h''} C^{s'-h'-h''}_{s+s'-h'-h''}}{C^{2s'}_{2(s+s')}} U_{\lambda_{\zeta} \mu_{\zeta} \dots}(\vec{p}, h'') \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots}(\vec{p}, h'') \\ &= \sum_{h=s+s'}^{-s-s'} [U_{\lambda_{\zeta} \mu_{\zeta} \dots \rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h) \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots \rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h)] \sum_{h'=s'}^{-s'} [U_{\rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h') \bar{U}_{\rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h')] \\ &\Leftrightarrow \sum_{h''=s}^{-s} \frac{C^{2s'}_{2(s+s'+1)}}{C^{2s'}_{2(s+s')}} U_{\lambda_{\zeta} \mu_{\zeta} \dots}(\vec{p}, h'') \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots}(\vec{p}, h'') \\ &= \sum_{h=s+s'}^{-s-s'} [U_{\lambda_{\zeta} \mu_{\zeta} \dots \rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h) \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots \rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h)] \sum_{h'=s'}^{-s'} [U_{\rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h') \bar{U}_{\rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h')] \\ &\Leftrightarrow \sum_{h''=s}^{-s} U_{\lambda_{\zeta} \mu_{\zeta} \dots}(\vec{p}, h'') \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots}(\vec{p}, h'') \\ &= \frac{2s+1}{2s+2s'+1} \sum_{h=s+s'}^{-s-s'} [U_{\lambda_{\zeta} \mu_{\zeta} \dots \rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h) \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots \rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h)] \sum_{h'=s'}^{-s'} [U_{\rho'_{\zeta} \sigma'_{\zeta} \dots}(\vec{p}, h') \bar{U}_{\rho_{\zeta} \sigma_{\zeta} \dots}(\vec{p}, h')] \\ &\Leftrightarrow \sum_{h=s}^{-s} U_{\lambda_{\zeta} \mu_{\zeta} \dots}(\vec{p}, h) \bar{U}_{\lambda'_{\zeta} \mu'_{\zeta} \dots}(\vec{p}, h) \end{aligned}$$

$$= \frac{2s+1}{2s+2s'+1} \sum_{h''=s+s'}^{-s-s'} [U_{\lambda_s \mu_s \dots \rho_s \sigma_s \dots}(\vec{p}, h'') \bar{U}_{\lambda'_s \mu'_s \dots \rho'_s \sigma'_s \dots}(\vec{p}, h'')] \sum_{h'=s'}^{-s'} [U_{\rho'_s \sigma'_s \dots}(\vec{p}, h') \bar{U}_{\rho_s \sigma_s \dots}(\vec{p}, h')] \quad \square$$

$$\begin{aligned} \text{定理5.3.2.} & \sum_{h=s}^{-s} U_{\lambda_s \mu_s \dots}(\vec{p}, h) U_{\lambda'_s \mu'_s \dots}^+(\vec{p}, h) \\ &= \frac{2s+1}{2s+2s'+1} \left(\frac{m}{E}\right)^{4s'} \sum_{h''=s+s'}^{-s-s'} [U_{\lambda_s \mu_s \dots \rho_s \sigma_s \dots}(\vec{p}, h'') U_{\lambda'_s \mu'_s \dots \rho'_s \sigma'_s \dots}^+(\vec{p}, h'')] \sum_{h'=s'}^{-s'} [U_{\rho'_s \sigma'_s \dots}(\vec{p}, h') U_{\rho_s \sigma_s \dots}^+(\vec{p}, h')] \end{aligned}$$

自我评述：一个猜想经过多年后终于又得到了严格的证明，诀窍在于对于一个特殊组合学公式的利用。

## 5.4 Bargmann-Wigner方程准投影算子之间的关系

$$\text{定理5.4.1.} \quad \begin{cases} \Lambda_{\pm \lambda_s \mu_s \dots \lambda'_s \mu'_s \dots}(\vec{p}, s) = \frac{2s+1}{2s+2} \left(\frac{m}{E}\right)^2 \Lambda_{\pm \lambda_s \mu_s \dots \tau_s \lambda'_s \mu'_s \dots \tau'_s}(\vec{p}, s + \frac{1}{2}) \Lambda_{\pm \tau_s \tau'_s}(\vec{p}, \frac{1}{2}) \\ \Lambda_{\pm \lambda_s \mu_s \dots \tau_s \lambda'_s \mu'_s \dots \tau'_s}(\vec{p}, s + \frac{1}{2}) = \frac{1}{[(2s+1)!]^2} \Lambda_{\pm \{\lambda_s \mu_s \dots \lambda'_s \mu'_s \dots\}}(\vec{p}, s) \Lambda_{\pm \tau_s \tau'_s}(\vec{p}, \frac{1}{2}) \end{cases}$$

$$\text{定理5.4.2.} \quad \begin{cases} \Lambda_{\pm \lambda_s \mu_s \dots \lambda'_s \mu'_s \dots}(\vec{p}, s) = \frac{2s+1}{2(s+l)+1} \left(\frac{m}{E}\right)^{4l} \Lambda_{\pm \lambda_s \mu_s \dots \rho_s \tau_s \lambda'_s \mu'_s \dots \rho'_s \tau'_s}(\vec{p}, s+l) \Lambda_{\pm \rho_s \tau_s \rho'_s \tau'_s}(\vec{p}, l) \\ \Lambda_{\pm \lambda_s \mu_s \dots \tau_s \lambda'_s \mu'_s \dots \tau'_s}(\vec{p}, s+l) = \frac{1}{[2(s+l)!]^2} \Lambda_{\pm \{\lambda_s \mu_s \dots \lambda'_s \mu'_s \dots\}}(\vec{p}, s) \Lambda_{\pm \rho_s \tau_s \rho'_s \tau'_s}(\vec{p}, l) \end{cases}$$

## 6 Bargmann-Wigner方程的对易规则

### 6.1 Bargmann-Wigner方程的协变对易规则

定理6.1.1.  $[a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}')$ ,  $[b(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}')$ ,  $[rest]_{-2s+1} = 0$

$$\Rightarrow \begin{cases} [\psi_{\lambda_s \mu_s \dots}(x), \psi_{\lambda'_s \mu'_s \dots}^+(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}^{2s} \Delta(x - x') \\ [\psi_{\lambda_s \mu_s \dots}^{(+)}(x), \psi_{\lambda'_s \mu'_s \dots}^{(++)}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}^{2s} \Delta^{(+)}(x - x') \\ [\psi_{\lambda_s \mu_s \dots}^{(-)}(x), \psi_{\lambda'_s \mu'_s \dots}^{(-+)}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}^{2s} \Delta^{(-)}(x - x') \\ [rest]_{-2s+1} = 0 \end{cases}$$

$$\text{证明: } [\psi_{\lambda_s \mu_s \dots}(x), \psi_{\lambda'_s \mu'_s \dots}^+(x')]_{-2s+1} = \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' \sum_{h, h'=s}^{-s} (EE')^{s-\frac{1}{2}} \sqrt{\frac{m^2}{EE'}}^{2s}$$

$$\begin{aligned} & [[a(\vec{p}, h) U_{\lambda_s \mu_s \dots}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) V_{\lambda_s \mu_s \dots}(\vec{p}, h) e^{-ip \cdot x}, [a^+(\vec{p}', h') U_{\lambda'_s \mu'_s \dots}^+(\vec{p}', h') e^{-ip' \cdot x'} + b(\vec{p}', h') V_{\lambda'_s \mu'_s \dots}^+(\vec{p}', h') e^{ip' \cdot x'}]] \\ &= \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' \sum_{h, h'=s}^{-s} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} \\ & [U_{\lambda_s \mu_s \dots}(\vec{p}, h) U_{\lambda'_s \mu'_s \dots}^+(\vec{p}', h') [a(\vec{p}, h), a^+(\vec{p}', h')] e^{i(p \cdot x - p' \cdot x')} + V_{\lambda_s \mu_s \dots}(\vec{p}, h) V_{\lambda'_s \mu'_s \dots}^+(\vec{p}', h') [b^+(\vec{p}, h), b(\vec{p}', h')] e^{-i(p \cdot x - p' \cdot x')} \\ &= \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' \sum_{h, h'=s}^{-s} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} \\ & [U_{\lambda_s \mu_s \dots}(\vec{p}, h) U_{\lambda'_s \mu'_s \dots}^+(\vec{p}', h') \delta_{hh'} \delta^3(\vec{p} - \vec{p}') e^{i(p \cdot x - p' \cdot x')} + (-1)^{2s+1} V_{\lambda_s \mu_s \dots}(\vec{p}, h) V_{\lambda'_s \mu'_s \dots}^+(\vec{p}', h') \delta_{hh'} \delta^3(\vec{p} - \vec{p}') e^{-i(p \cdot x - p' \cdot x')}] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} \left[ \sum_{h=s}^{-s} \underbrace{U_{\lambda_s \mu_s \dots}}_{2s}(\vec{p}, h) \underbrace{U_{\lambda'_s \mu'_s \dots}^+}_{2s}(\vec{p}, h) e^{ip \cdot (x-x')} + (-1)^{2s+1} \sum_{h=s}^{-s} \underbrace{V_{\lambda_s \mu_s \dots}}_{2s}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots}^+}_{2s}(\vec{p}, h) e^{-ip \cdot (x-x')} \right] \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} \left[ \Lambda_{+\lambda_s \mu_s \dots \lambda'_s \mu'_s \dots}(\vec{p}, s) e^{ip \cdot (x-x')} + (-1)^{2s+1} \Lambda_{-\lambda_s \mu_s \dots \lambda'_s \mu'_s \dots}(\vec{p}, s) e^{-ip \cdot (x-x')} \right] \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} \left[ \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{+\{\lambda_s(\lambda'_s(\vec{p}, \frac{1}{2})\Lambda_{+\mu_s \mu'_s}(\vec{p}, \frac{1}{2}))\}}}_{2s} \dots e^{ip \cdot (x-x')} \right. \\
&\quad \left. + (-1)^{2s+1} \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{-\{\lambda_s(\lambda'_s(\vec{p}, \frac{1}{2})\Lambda_{-\mu_s \mu'_s}(\vec{p}, \frac{1}{2}))\}}}_{2s} \dots e^{-ip \cdot (x-x')} \right] \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} e^{ip \cdot (x-x')} \right. \\
&\quad \left. + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \underbrace{[(-m + \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(-m + \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} e^{-ip \cdot (x-x')} \right\} \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} \frac{-i}{(2\pi)^3} \int d^3\vec{p} \frac{1}{2E} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} \Delta(x-x') \\
&= \frac{i(2m)^{2s}}{2^{2s-1}} \Lambda_{+\lambda_s \mu_s \dots \lambda'_s \mu'_s \dots}(-i\partial, s) \Delta(x-x')
\end{aligned}$$

□

证明:  $[\psi_{\lambda_s \mu_s \dots}^{(+)}(x), \psi_{\lambda'_s \mu'_s \dots}^{(+)+}(x')]_{-2s+1}$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' \sum_{h, h'=s}^{-s} (EE')^{s-\frac{1}{2}} \sqrt{\frac{m^2}{EE'}} [a(\vec{p}, h) \underbrace{U_{\lambda_s \mu_s \dots}}_{2s}(\vec{p}, h) e^{ip \cdot x}, a^+(\vec{p}', h') \underbrace{U_{\lambda'_s \mu'_s \dots}^+}_{2s}(\vec{p}', h') e^{-ip' \cdot x'}] \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' \sum_{h, h'=s}^{-s} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} [U_{\lambda_s \mu_s \dots}(\vec{p}, h) \underbrace{U_{\lambda'_s \mu'_s \dots}^+}_{2s}(\vec{p}', h') [a(\vec{p}, h), a^+(\vec{p}', h')] e^{i(p \cdot x - p' \cdot x')} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' \sum_{h, h'=s}^{-s} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} U_{\lambda_s \mu_s \dots}(\vec{p}, h) \underbrace{U_{\lambda'_s \mu'_s \dots}^+}_{2s}(\vec{p}', h') \delta_{hh'} \delta^3(\vec{p} - \vec{p}') e^{i(p \cdot x - p' \cdot x')} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} \sum_{h=s}^{-s} \underbrace{U_{\lambda_s \mu_s \dots}}_{2s}(\vec{p}, h) \underbrace{U_{\lambda'_s \mu'_s \dots}^+}_{2s}(\vec{p}, h) e^{ip \cdot (x-x')} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} \Lambda_{+\lambda_s \mu_s \dots \lambda'_s \mu'_s \dots}(\vec{p}, s) e^{ip \cdot (x-x')} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{+\{\lambda_s(\lambda'_s(\vec{p}, \frac{1}{2})\Lambda_{+\mu_s \mu'_s}(\vec{p}, \frac{1}{2}))\}}}_{2s} \dots e^{ip \cdot (x-x')} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} e^{ip \cdot (x-x')} \right. \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} \frac{-i}{(2\pi)^3} \int d^3\vec{p} \frac{1}{2E} e^{ip \cdot (x-x')} \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} \Delta^{(+)}(x-x') \\
&= \frac{i(2m)^{2s}}{2^{2s-1}} \Lambda_{+\lambda_s \mu_s \dots \lambda'_s \mu'_s \dots}(-i\partial, s) \Delta^{(+)}(x-x')
\end{aligned}$$

□

证明:  $[\psi_{\lambda_s \mu_s \dots}^{(-)}(x), \psi_{\lambda'_s \mu'_s \dots}^{(-)+}(x')]_{-2s+1}$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' \sum_{h, h'=s}^{-s} (EE')^{s-\frac{1}{2}} \sqrt{\frac{m^2}{EE'}} [b^+(\vec{p}, h) \underbrace{V_{\lambda_s \mu_s \dots}}_{2s}(\vec{p}, h) e^{-ip \cdot x}, b(\vec{p}', h') \underbrace{V_{\lambda'_s \mu'_s \dots}^+}_{2s}(\vec{p}', h') e^{ip' \cdot x'}]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' \sum_{h, h'=s}^{-s} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} \underbrace{V_{\lambda_s \mu_s \dots}}_{2s}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots}^+}_{2s}(\vec{p}', h') [b^+(\vec{p}, h), b(\vec{p}', h')] e^{-i(p \cdot x - p' \cdot x')} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} d^3\vec{p}' \sum_{h, h'=s}^{-s} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} (-1)^{2s+1} \underbrace{V_{\lambda_s \mu_s \dots}}_{2s}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots}^+}_{2s}(\vec{p}', h') \delta_{hh'} \delta^3(\vec{p} - \vec{p}') e^{-i(p \cdot x - p' \cdot x')} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} (-1)^{2s+1} \sum_{h=s}^{-s} \underbrace{V_{\lambda_s \mu_s \dots}}_{2s}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots}^+}_{2s}(\vec{p}, h) e^{-ip \cdot (x - x')} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} (-1)^{2s+1} \Lambda_{-\lambda_s \mu_s \dots \lambda'_s \mu'_s \dots}(\vec{p}, s) e^{-ip \cdot (x - x')} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} (-1)^{2s+1} \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{-\{\lambda_s(\lambda'_s(\vec{p}, \frac{1}{2})\Lambda_{-\mu_s \mu'_s}(\vec{p}, \frac{1}{2})\dots}}_{2s}} \dots e^{-ip \cdot (x - x')} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{m^{2s}}{E} (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})}}^{2s}} e^{-ip \cdot (x - x')} \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})}}^{2s}} \frac{i}{(2\pi)^3} \int d^3\vec{p} \frac{1}{2E} e^{-ip \cdot (x - x')} \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})}}^{2s}} \Delta^{(-)}(x - x') \\
&= \frac{i(2m)^{2s}}{2^{2s-1}} \Lambda_{+\lambda_s \mu_s \dots \lambda'_s \mu'_s \dots}(-i\partial, s) \Delta^{(-)}(x - x') \quad \square
\end{aligned}$$

## 6.2 Bargmann-Wigner方程协变对易规则的反向推理

定理6.2.1.

$$\begin{cases} [\psi_{\lambda_s \mu_s \dots}(x), \psi_{\lambda'_s \mu'_s \dots}^+(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})}}^{2s}} \Delta(x - x') \\ [rest]_{-2s+1} = 0 \end{cases}$$

$$\Rightarrow [a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [b(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [rest]_{-2s+1} = 0$$

下面给出几个主要对易括号的详细证明过程。

证明:  $[a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1}$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) U^{\lambda'_s \mu'_s \dots}(\vec{p}', h') [\psi_{\lambda_s \mu_s \dots}(x), \psi_{\lambda'_s \mu'_s \dots}^+(x')]_{-2s+1} e^{-i(p \cdot x - p' \cdot x')} d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) U^{\lambda'_s \mu'_s \dots}(\vec{p}', h') \\
&\quad \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})}}^{2s}} \Delta(x - x') e^{-i(p \cdot x - p' \cdot x')} d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{r} d^3\vec{r}' \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) U^{\lambda'_s \mu'_s \dots}(\vec{p}', h') \\
&\quad \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})}}^{2s}} \left\{ \frac{-i}{(2\pi)^3} \int \frac{1}{2E_0} [e^{ip_0 \cdot (x - x')} - e^{-ip_0 \cdot (x - x')}] d^3\vec{p}_0 \right\} e^{-i(p \cdot x - p' \cdot x')} \\
&= \left[ \frac{1}{(2\pi)^3} \right]^2 \int \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) U^{\lambda'_s \mu'_s \dots}(\vec{p}', h') \\
&\quad \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - i\gamma^b p_{0b}) \gamma^4]_{\mu_s \mu'_s \dots})}}^{2s}} e^{ip_0 \cdot (x - x')} e^{-i(p \cdot x - p' \cdot x')} d^3\vec{r} d^3\vec{r}' \right. \\
&\quad \left. + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_s(\lambda'_s[(-m - i\gamma^b p_{0b}) \gamma^4]_{\mu_s \mu'_s \dots})}}^{2s}} e^{-ip_0 \cdot (x - x')} e^{-i(p \cdot x - p' \cdot x')} \right\} d^3\vec{r} d^3\vec{r}' d^3\vec{p}_0 \\
&= \left[ \frac{1}{(2\pi)^3} \right]^2 \int d^3\vec{r} d^3\vec{r}' d^3\vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s}
\end{aligned}$$

$$\begin{aligned}
& U^{+\overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}}(\vec{p}, h) U^{\overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}}_{2s}(\vec{p}_0, h_0) \underbrace{U^+_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}}_{2s}(\vec{p}_0, h_0) e^{i(p_0-p)\cdot x} e^{-i(p_0-p')\cdot x'} \right. \\
& + (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}}_{2s}(\vec{p}_0, h_0) \underbrace{V^+_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}}_{2s}(\vec{p}_0, h_0) e^{-i(p_0+p)\cdot x} e^{i(p_0+p')\cdot x'} \left. \right\} \\
& = \int d^3\vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left( \frac{m^2}{EE'} \right)^{2s} \\
& U^{+\overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}}(\vec{p}, h) U^{\overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}}_{2s}(\vec{p}_0, h_0) \underbrace{U^+_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}}_{2s}(\vec{p}_0, h_0) \delta^3(\vec{p}_0 - \vec{p}) \delta^3(\vec{p}_0 - \vec{p}') \right. \\
& + (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}}_{2s}(\vec{p}_0, h_0) \underbrace{V^+_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}}_{2s}(\vec{p}_0, h_0) e^{2iE_0(t-t')} \delta^3(\vec{p}_0 + \vec{p}) \delta^3(\vec{p}_0 + \vec{p}') \left. \right\} \\
& = \delta^3(\vec{p} - \vec{p}') \left( \frac{m}{E} \right)^{4s} U^{+\overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}}(\vec{p}, h) U^{\overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}}(\vec{p}, h') \\
& \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}}_{2s}(\vec{p}, h_0) \underbrace{U^+_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}}_{2s}(\vec{p}, h_0) + (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}}_{2s}(-\vec{p}, h_0) \underbrace{V^+_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}}_{2s}(-\vec{p}, h_0) e^{2iE(t-t')} \right\} \\
& = \delta^3(\vec{p} - \vec{p}') \left( \sum_{h_0=s}^{-s} \delta_{hh_0} \delta_{h'h_0} + 0 \right) \\
& = \delta_{hh'} \delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

证明:  $[b^+(\vec{p}, h), b(\vec{p}', h')]_{-2s+1}$ 

$$\begin{aligned}
& = \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left( \frac{m}{EE'} \right)^{2s} V^{+\overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}}(\vec{p}, h) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}}(\vec{p}', h') [\psi_{\overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}}(x), \psi^+_{\overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}}(x')]_{-2s+1} e^{i(p\cdot x - p'\cdot x')} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left( \frac{m}{EE'} \right)^{2s} V^{+\overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}}(\vec{p}, h) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}}(\vec{p}', h') \\
& \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \cdots})\}}}_{2s} \Delta(x - x') e^{i(p\cdot x - p'\cdot x')} d^3\vec{r} d^3\vec{r}' \\
& = \frac{1}{(2\pi)^3} \int d^3\vec{r} d^3\vec{r}' \sqrt{EE'} \left( \frac{m}{EE'} \right)^{2s} V^{+\overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}}(\vec{p}, h) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}}(\vec{p}', h') \\
& \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \cdots})\}}}_{2s} \left\{ \frac{-i}{(2\pi)^3} \int \frac{1}{2E_0} [e^{ip_0\cdot(x-x')} - e^{-ip_0\cdot(x-x')}] d^3\vec{p}_0 \right\} e^{i(p\cdot x - p'\cdot x')} \\
& = \left[ \frac{1}{(2\pi)^3} \right]^2 \int \frac{\sqrt{EE'}}{E_0} \left( \frac{m^2}{EE'} \right)^{2s} V^{+\overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}}(\vec{p}, h) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}}(\vec{p}', h') \\
& \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - i\gamma^b p_{0b}) \gamma^4]_{\mu_\zeta \mu'_\zeta \cdots})\}}}_{2s} e^{ip_0\cdot(x-x')} e^{i(p\cdot x - p'\cdot x')} d^3\vec{r} d^3\vec{r}' \right. \\
& + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(-m - i\gamma^b p_{0b}) \gamma^4]_{\mu_\zeta \mu'_\zeta \cdots})\}}}_{2s} e^{-ip_0\cdot(x-x')} e^{i(p\cdot x - p'\cdot x')} \left. \right\} d^3\vec{r} d^3\vec{r}' d^3\vec{p}_0 \\
& = \left[ \frac{1}{(2\pi)^3} \right]^2 \int d^3\vec{r} d^3\vec{r}' d^3\vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left( \frac{m^2}{EE'} \right)^{2s} \\
& V^{+\overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}}(\vec{p}, h) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}}_{2s}(\vec{p}_0, h_0) \underbrace{U^+_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}}_{2s}(\vec{p}_0, h_0) e^{i(p_0+p)\cdot x} e^{-i(p_0+p')\cdot x'} \right. \\
& + (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}}_{2s}(\vec{p}_0, h_0) \underbrace{V^+_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}}_{2s}(\vec{p}_0, h_0) e^{-i(p_0-p)\cdot x} e^{i(p_0-p')\cdot x'} \left. \right\} \\
& = \int d^3\vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left( \frac{m^2}{EE'} \right)^{2s} \\
& V^{+\overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}}(\vec{p}, h) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}}_{2s}(\vec{p}_0, h_0) \underbrace{U^+_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}}_{2s}(\vec{p}_0, h_0) e^{-2iE_0(t-t')} \delta^3(\vec{p}_0 + \vec{p}) \delta^3(\vec{p}_0 + \vec{p}') \right.
\end{aligned}$$



$$\begin{aligned}
& + (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_s \mu_s \dots \tau_s}}_{2s}(\vec{p}_0, h_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}^+(\vec{p}_0, h_0) \delta^3(\vec{p}_0 - \vec{p}) \delta^3(\vec{p}_0 - \vec{p}') \\
& = \delta^3(\vec{p} - \vec{p}') \left(\frac{m}{E}\right)^{4s} V^{+\lambda_s \mu_s \dots}(\vec{p}, h) V^{\lambda'_s \mu'_s \dots}(\vec{p}, h') \\
& \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_s \mu_s \dots \tau_s}}_{2s}(-\vec{p}, h_0) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}^+(-\vec{p}, h_0) e^{-2iE(t-t')} + (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_s \mu_s \dots \tau_s}}_{2s}(\vec{p}, h_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}^+(\vec{p}, h_0) \right\} \\
& = (-1)^{2s+1} \delta^3(\vec{p} - \vec{p}') \left(0 + \sum_{h_0=s}^{-s} \delta_{hh_0} \delta_{h'h_0}\right) \\
& = (-1)^{2s+1} \delta_{hh'} \delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

证明:  $[a(\vec{p}, h), b(\vec{p}', h')]_{-2s+1}$ 

$$\begin{aligned}
& = \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', h') [\psi_{\lambda_s \mu_s \dots}(x), \psi_{\lambda'_s \mu'_s \dots}^+(x')]_{-2s+1} e^{-i(p \cdot x + p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', h') \\
& \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} \Delta(x - x') e^{-i(p \cdot x + p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
& = \frac{1}{(2\pi)^3} \int d^3 \vec{r} d^3 \vec{r}' \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', h') \\
& \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} \left\{ \frac{-i}{(2\pi)^3} \int \frac{1}{2E_0} [e^{ip_0 \cdot (x-x')} - e^{-ip_0 \cdot (x-x')}] d^3 \vec{p}_0 \right\} e^{-i(p \cdot x + p' \cdot x')} \\
& = \left[\frac{1}{(2\pi)^3}\right]^2 \int \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', h') \\
& \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - i\gamma^b p_{0b}) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} e^{ip_0 \cdot (x-x')} e^{-i(p \cdot x + p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \right. \\
& + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_s(\lambda'_s [(-m - i\gamma^b p_{0b}) \gamma^4]_{\mu_s \mu'_s \dots})\}}}_{2s} e^{-ip_0 \cdot (x-x')} e^{-i(p \cdot x + p' \cdot x')} \left. \right\} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}_0 \\
& = \left[\frac{1}{(2\pi)^3}\right]^2 \int d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
& U^{+\lambda_s \mu_s \dots}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_s \mu_s \dots \tau_s}}_{2s}(\vec{p}_0, h_0) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}^+(\vec{p}_0, h_0) e^{i(p_0 - p) \cdot x} e^{-i(p_0 + p') \cdot x'} \right. \\
& + (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_s \mu_s \dots \tau_s}}_{2s}(\vec{p}_0, h_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}^+(\vec{p}_0, h_0) e^{-i(p_0 + p) \cdot x} e^{i(p_0 - p') \cdot x'} \left. \right\} \\
& = \int d^3 \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
& U^{+\lambda_s \mu_s \dots}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_s \mu_s \dots \tau_s}}_{2s}(\vec{p}_0, h_0) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}^+(\vec{p}_0, h_0) e^{-iE_0 t'} \delta^3(\vec{p}_0 - \vec{p}) \delta^3(\vec{p}_0 + \vec{p}') \right. \\
& + (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_s \mu_s \dots \tau_s}}_{2s}(\vec{p}_0, h_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}^+(\vec{p}_0, h_0) e^{iE_0 t'} \delta^3(\vec{p}_0 + \vec{p}) \delta^3(\vec{p}_0 - \vec{p}') \left. \right\} \\
& = \delta^3(\vec{p} + \vec{p}') \left(\frac{m}{E}\right)^{4s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', h') \\
& \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_s \mu_s \dots \tau_s}}_{2s}(\vec{p}, h_0) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}^+(\vec{p}, h_0) + (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_s \mu_s \dots \tau_s}}_{2s}(\vec{p}, h_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}^+(\vec{p}, h_0) e^{2iE(t-t')} \right\} \\
& = 0 + 0 = 0
\end{aligned}$$

□

自我评述：以上的证法类似于Penrose方程情形，不再基于等时对易规则，直接基于协变对易规则进行计算，看似更难了，其实也是更简单了，因为等时对易规则不容易算出，即使算出了也难于使用，而协变对易规

则本身已知且很有规律，且可以分解为自旋基的乘积，从而变得简单了。整个证明过程基本上只依赖于自旋基的性质，没有高难度的复杂计算。以上给出了最难的三个对易括号的证明，其他的几个对易括号可容易证出，不再详细列出。事实上，等时对易规则是协变对易规则的特例，所以等时对易规则的证法也可以采用以上的证法(取 $t = t'$ 即可)。

### 6.3 Bargmann-Wigner方程协变对易规则的小结

结合以上两节的证明，便得到以下重要的定理。

定理6.3.1.

$$\left\{ \begin{aligned} [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}(x')]_{-2s+1} &= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})}}^{2s} \Delta(x - x') \\ [rest]_{-2s+1} &= 0 \end{aligned} \right.$$

$$\Leftrightarrow [a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [b(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [rest]_{-2s+1} = 0$$

定理6.3.2.  $[\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}(x')]_{-2s+1} = 2im^{2s} \Lambda_{+\lambda_\zeta \mu_\zeta \dots \tau_\zeta \lambda'_\zeta \mu'_\zeta \dots \tau_\zeta}(-i\partial, s)$

### 6.4 Bargmann-Wigner方程的对易函数、因果函数和费曼传播子

引理6.4.1.  $\overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})}}^{2s}$

$$= \sum_{n=0}^{2s} C_{2s}^n \overbrace{[-(\gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [-(\gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots} [m \gamma^4]_{\eta_\zeta \eta'_\zeta} [m \gamma^4]_{\xi_\zeta \xi'_\zeta \dots})}}^n \overbrace{[m \gamma^4]_{\eta_\zeta \eta'_\zeta} [m \gamma^4]_{\xi_\zeta \xi'_\zeta \dots}}^{2s-n}$$

$$= \sum_{n=0}^{2s} (-1)^n m^{2s-n} C_{2s}^n \overbrace{(\gamma^a \gamma^4)_{\{\lambda_\zeta (\lambda'_\zeta (\gamma^b \gamma^4)_{\mu_\zeta \mu'_\zeta \dots} (\gamma^4)_{\eta_\zeta \eta'_\zeta} (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots})}}^n \overbrace{\partial_a \partial_b \dots}^n$$

引理6.4.2.  $\overbrace{[\theta(t), [(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})}}^{2s}]_{-2s+1}$

$$= \sum_{n=0}^{2s} (-1)^n m^{2s-n} C_{2s}^n \overbrace{[\theta(t), (\gamma^a \gamma^4)_{\{\lambda_\zeta (\lambda'_\zeta (\gamma^b \gamma^4)_{\mu_\zeta \mu'_\zeta \dots} (\gamma^4)_{\eta_\zeta \eta'_\zeta} (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots})}}^n \overbrace{\partial_a \partial_b \dots}^n$$

$$= \sum_{n=0}^{2s} (-1)^n m^{2s-n} C_{2s}^n \sum_{l=0}^{n-1} C_n^l \overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta (\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots} \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots (\gamma^4)_{\eta_\zeta \eta'_\zeta} (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots})}}^l \overbrace{[\partial_\pi^{n-l} \theta(t)] \partial_i \partial_j \dots}^{2s-n}$$

$$= \sum_{n=0}^{2s} \sum_{l=0}^{n-1} (-1)^n m^{2s-n} C_{2s}^n C_n^l \overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta (\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots} \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots (\gamma^4)_{\eta_\zeta \eta'_\zeta} (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots})}}^l \overbrace{[\partial_\pi^{n-l} \theta(t)] \partial_i \partial_j \dots}^{2s-n}$$

$$= \sum_{n=0}^{2s} \sum_{l=0}^{n-1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta (\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots} \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots (\gamma^4)_{\eta_\zeta \eta'_\zeta} (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots})}}^l \overbrace{[\partial_t^{n-1-l} \delta(t)] \partial_i \partial_j \dots}^{2s-n}$$

推论6.4.1.

$$\left\{ \begin{aligned} \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(s; x) &:= \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})}}^{2s} \Delta(x) \\ \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(+)}(s; x) &:= \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})}}^{2s} \Delta^{(+)}(x) \\ \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(-)}(s; x) &:= \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})}}^{2s} \Delta^{(-)}(x) \\ \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(l)}(s; x) &:= \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})}}^{2s} \Delta^{(l)}(x) \end{aligned} \right.$$

## 推论6.4.2.

$$\begin{aligned}
& \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(c)}(s; x) := \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta^{(c)}(x) \\
& + \frac{2^{1-2s}}{[(2s)!]^2} \sum_{n=0}^{2s} \sum_{l=0}^{n-1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta(\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots}}\}}}^l \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{n-l} \overbrace{(\gamma^4)_{\eta_\zeta \eta'_\zeta (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots}}}^{2s-n}} \overbrace{[\partial_t^{n-1-l} \delta(t)] \partial_i \partial_j \dots}^l \Delta(x) \\
& \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(F)}(s; x) := \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta_F(x) \\
& + i \frac{2^{1-2s}}{[(2s)!]^2} \sum_{n=0}^{2s} \sum_{l=0}^{n-1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta(\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots}}\}}}^l \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{n-l} \overbrace{(\gamma^4)_{\eta_\zeta \eta'_\zeta (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots}}}^{2s-n}} \overbrace{[\partial_t^{n-1-l} \delta(t)] \partial_i \partial_j \dots}^l \Delta(x) \\
& = i \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(c)}(s; x)
\end{aligned}$$

## 推论6.4.3.

$$\begin{aligned}
& \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(ret)}(s; x) := \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta^{(ret)}(x) \\
& + \frac{2^{1-2s}}{[(2s)!]^2} \sum_{n=0}^{2s} \sum_{l=0}^{n-1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta(\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots}}\}}}^l \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{n-l} \overbrace{(\gamma^4)_{\eta_\zeta \eta'_\zeta (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots}}}^{2s-n}} \overbrace{[\partial_t^{n-1-l} \delta(t)] \partial_i \partial_j \dots}^l \Delta(x) \\
& \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(adv)}(s; x) := \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta^{(adv)}(x) \\
& + \frac{2^{1-2s}}{[(2s)!]^2} \sum_{n=0}^{2s} \sum_{l=0}^{n-1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta(\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots}}\}}}^l \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{n-l} \overbrace{(\gamma^4)_{\eta_\zeta \eta'_\zeta (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots}}}^{2s-n}} \overbrace{[\partial_t^{n-1-l} \delta(t)] \partial_i \partial_j \dots}^l \Delta(x)
\end{aligned}$$

$$\text{引理6.4.3. } \Delta(x) \partial_t^n \delta(t) = \sum_{l=0}^{[(n-1)/2]} C_n^{2l+1} (\nabla^2 - m^2)^l \partial_t^{n-2l-1} \delta^4(x)$$

$$\text{推论6.4.4. } \Delta(x) \partial_t^{n-1-l} \delta(t) = \sum_{r=0}^{[(n-l-2)/2]} C_{n-1-l}^{2r+1} (\nabla^2 - m^2)^r \partial_t^{n-l-2-2r} \delta^4(x)$$

## 推论6.4.5.

$$\begin{aligned}
& \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(c)}(s; x) := \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta^{(c)}(x) + \frac{2^{1-2s}}{[(2s)!]^2} \sum_{n=0}^{2s} \sum_{l=0}^{n-2} \sum_{r=0}^{[(n-l-2)/2]} \\
& C_{n-l-1}^{2r+1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta(\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots}}\}}}^l \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{n-l} \overbrace{(\gamma^4)_{\eta_\zeta \eta'_\zeta (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots}}}^{2s-n}} \overbrace{\partial_i \partial_j \dots}^l \cdot (\nabla^2 - m^2)^r \partial_t^{n-l-2-2r} \delta^4(x) \\
& \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(F)}(s; x) := \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta_F(x) + i \frac{2^{1-2s}}{[(2s)!]^2} \sum_{n=0}^{2s} \sum_{l=0}^{n-2} \sum_{r=0}^{[(n-l-2)/2]} \\
& C_{n-l-1}^{2r+1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta(\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots}}\}}}^l \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{n-l} \overbrace{(\gamma^4)_{\eta_\zeta \eta'_\zeta (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots}}}^{2s-n}} \overbrace{\partial_i \partial_j \dots}^l \cdot (\nabla^2 - m^2)^r \partial_t^{n-l-2-2r} \delta^4(x) \\
& = i \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(c)}(s; x) \\
& \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(F)}(s; p) = -i \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - i\gamma^b p_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \frac{1}{p^2 + m^2 - i\varepsilon} + \dots
\end{aligned}$$

## 推论6.4.6.

$$\left\{ \begin{aligned} \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(ret)}(s; x) &:= \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta^{(ret)}(x) + \frac{2^{1-2s}}{[(2s)!]^2} \sum_{n=0}^{2s} \sum_{l=0}^{n-2} \sum_{r=0}^{[(n-l-2)/2]} \\ C_{n-l-1}^{2r+1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} &\overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta(\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots} \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots (\gamma^4)_{\eta_\zeta \eta'_\zeta} (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots})\}}}^l \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots (\gamma^4)_{\eta_\zeta \eta'_\zeta} (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots}}^{n-l} \overbrace{\partial_i \partial_j \dots}^l \cdot (\nabla^2 - m^2)^r \partial_t^{n-l-2-2r} \delta^4(x) \\ \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(adv)}(s; x) &:= \frac{2^{1-2s}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta^{(adv)}(x) + \frac{2^{1-2s}}{[(2s)!]^2} \sum_{n=0}^{2s} \sum_{l=0}^{n-2} \sum_{r=0}^{[(n-l-2)/2]} \\ C_{n-l-1}^{2r+1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} &\overbrace{(\gamma^i \gamma^4)_{\{\lambda_\zeta(\lambda'_\zeta (\gamma^j \gamma^4)_{\mu_\zeta \mu'_\zeta \dots} \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots (\gamma^4)_{\eta_\zeta \eta'_\zeta} (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots})\}}}^l \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots (\gamma^4)_{\eta_\zeta \eta'_\zeta} (\gamma^4)_{\xi_\zeta \xi'_\zeta \dots}}^{n-l} \overbrace{\partial_i \partial_j \dots}^l \cdot (\nabla^2 - m^2)^r \partial_t^{n-l-2-2r} \delta^4(x) \end{aligned} \right.$$

引理6.4.4.  $\Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(s; x)|_{t=0}$

$$= \frac{-i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \sum_{l=0}^{[s-\frac{1}{2}]} [C_{2s}^{2l+1} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_\zeta(\lambda'_\zeta (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_\zeta \mu'_\zeta \dots} \delta_{\tau_\zeta \tau'_\zeta})\}}}^{2s-2l-1}] (m^2 - \nabla^2)^l \delta^3(\vec{r})$$

推论6.4.7.

$$\left\{ \begin{aligned} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(c)}(s; x) &= 0 & \left\{ \begin{aligned} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(c)}(s; x) &= -i\gamma^4 \delta(t) \Delta_{\kappa_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(s; x)|_{t=0} \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(+)}(s; x) &= 0 & \left\{ \begin{aligned} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(ret)}(s; x) &= -i\gamma^4 \delta(t) \Delta_{\kappa_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(s; x)|_{t=0} \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(-)}(s; x) &= 0 & \left\{ \begin{aligned} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(adv)}(s; x) &= -i\gamma^4 \delta(t) \Delta_{\kappa_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(s; x)|_{t=0} \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(l)}(s; x) &= 0 & \left\{ \begin{aligned} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}^{(F)}(s; x) &= \gamma^4 \delta(t) \Delta_{\kappa_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(s; x)|_{t=0} \end{aligned} \right. \end{aligned} \right. \end{aligned} \right.$$

推论6.4.8.

$$\left\{ \begin{aligned} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \lambda'_\zeta}(\frac{1}{2}; x) &= 0 & \left\{ \begin{aligned} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \lambda'_\zeta}^{(c)}(\frac{1}{2}; x) &= -\gamma_{\kappa_\zeta \lambda'_\zeta}^4 \delta^4(x) \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \lambda'_\zeta}^{(+)}(\frac{1}{2}; x) &= 0 & \left\{ \begin{aligned} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \lambda'_\zeta}^{(ret)}(\frac{1}{2}; x) &= -\gamma_{\kappa_\zeta \lambda'_\zeta}^4 \delta^4(x) \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \lambda'_\zeta}^{(-)}(\frac{1}{2}; x) &= 0 & \left\{ \begin{aligned} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \lambda'_\zeta}^{(adv)}(\frac{1}{2}; x) &= -\gamma_{\kappa_\zeta \lambda'_\zeta}^4 \delta^4(x) \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \lambda'_\zeta}^{(l)}(\frac{1}{2}; x) &= 0 & \left\{ \begin{aligned} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Delta_{\lambda_\zeta \lambda'_\zeta}^{(F)}(\frac{1}{2}; x) &= -i\gamma_{\kappa_\zeta \lambda'_\zeta}^4 \delta^4(x) \end{aligned} \right. \end{aligned} \right. \end{aligned} \right.$$

## 6.5 可分离表象下Bargmann-Wigner方程的协变量子化规则的重要推论

定义6.5.1.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \dots} = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \psi_{\lambda_\zeta \mu_\zeta \dots} = \Gamma_{\lambda_\zeta \mu_\zeta \dots}^{K_\zeta} \psi_{K_\zeta}(s)$

推论6.5.1.

$$\begin{aligned} [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} &= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \\ \Rightarrow [\psi_{A_\zeta B_\zeta C_\zeta \dots}(x), \psi_{A'_\zeta B'_\zeta C'_\zeta \dots}^+(x')]_{-2s+1} &= i \frac{(i\varsigma)^{2s}}{2^{2s-1}} (\sigma, i\varsigma)_{A_\zeta A'_\zeta}^a (\sigma, i\varsigma)_{B_\zeta B'_\zeta}^b \overbrace{\partial_a \partial_b \dots}^{2s} \Delta(x - x') \end{aligned}$$

证明:

$$\begin{aligned} [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} &= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \\ \Leftrightarrow [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1}, \gamma^a &= (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x) \\ &= i \frac{(i\varsigma)^{2s}}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[-imI \otimes \sigma(x) + (\sigma \otimes \sigma_z, i\varsigma)^a \partial_a]_{\{\lambda_\zeta(\lambda'_\zeta [-imI \otimes \sigma(x) + (\sigma \otimes \sigma_z, i\varsigma)^b \partial_b]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \end{aligned}$$

$$\begin{aligned} \Rightarrow [\psi_{\underbrace{A_\zeta B_\zeta C_\zeta \cdots}_{2s}}(x), \psi_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \cdots}_{2s}}^+(x')]_{-2s+1} &= i \frac{(i\zeta)^{2s}}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{(\sigma, i\zeta)_{\{A_\zeta(A'_\zeta(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots\}}^a}^{2s}} \overbrace{\partial_a \partial_b \cdots}^{2s} \Delta(x-x') \\ \Leftrightarrow [\psi_{\underbrace{A_\zeta B_\zeta C_\zeta \cdots}_{2s}}(x), \psi_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \cdots}_{2s}}^+(x')]_{-2s+1} &= i \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a}^{2s} \overbrace{(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b}^{2s} \partial_a \partial_b \cdots \Delta(x-x') \end{aligned} \quad \square$$

## 6.6 Bargmann-Wigner方程对易规则两种描述的等价性证明

$$\text{引理6.6.1. } 2\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(p)(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})\mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(p) = [(m - i\gamma^a p_a)\gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(m - i\gamma^b p_b)\gamma^4]_{\mu_\zeta \mu'_\zeta})\}}$$

$$\text{引理6.6.2. } 2\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x)(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2})\mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x')\Delta(x-x') = [(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b)\gamma^4]_{\mu_\zeta \mu'_\zeta})\}}\Delta(x-x')$$

定理6.6.1.

$$\begin{aligned} [\psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2n}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots}_{2n}}^+(x')] &= \frac{i}{2^{2n-1}} \frac{1}{[(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b)\gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots\}}}_{2n} \Delta(x-x') \\ \Leftrightarrow [\psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2n}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots}_{2n}}^+(x')] &= \frac{i}{2^{2n-1}} \frac{1}{[(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a}_{n} \cdots \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{+a'}}_{n} \cdots \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_{n} \cdots \Delta(x-x') \end{aligned}$$

证明:

$$\begin{aligned} &[\psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2n}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots}_{2n}}^+(x')] \\ &= \frac{i}{2^{2n-1} [(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b)\gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots\}}}_{2n} \Delta(x-x') \\ &= \frac{i}{2^{4n-1} [(2n)!]^2} \underbrace{\{[(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b)\gamma^4]_{\mu_\zeta \mu'_\zeta})\}} \cdots\}}_{n} \Delta(x-x') \\ &= \frac{i}{2^{3n-1} [(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a}_{n} \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^b}_{n} \cdots \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{+a'}}_{n} \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{+b'}}_{n} \cdots \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_{n} \underbrace{[\eta_{bb'} - \frac{\partial_b \partial_{b'}}{m^2}]}_{n} \cdots \Delta(x-x') \\ &= \frac{i}{2^{3n-1} (n!)^2 [(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a}_{n} \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^b}_{n} \cdots \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{+a'}}_{n} \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{+b'}}_{n} \cdots \underbrace{[\eta_{\{a(a' - \frac{\partial_a \partial_{a'}}{m^2})\}}]}_{n} \underbrace{[\eta_{\{b(b' - \frac{\partial_b \partial_{b'}}{m^2})\}}]}_{n} \cdots \Delta(x-x') \end{aligned} \quad \square$$

定理6.6.2.

$$\begin{aligned} &\{\psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2n+1}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}_{2n+1}}^+(x')\} \\ &= \frac{i}{2^{2n}} \frac{1}{[(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b)\gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots [(m - \gamma^c \partial_c)\gamma^4]_{\tau_\zeta \tau'_\zeta})\}}}_{2n+1} \Delta(x-x') \\ \Leftrightarrow &\{\psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2n+1}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}_{2n+1}}^+(x')\} \\ &= \frac{i}{2^{3n} [(2n+1)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a}_{n} \cdots \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{+a'}}_{n} \cdots \underbrace{[(m - \gamma^c \partial_c)\gamma^4]_{\tau_\zeta \tau'_\zeta}}_{n} \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_{n} \cdots \Delta(x-x') \end{aligned}$$

证明:

$$\begin{aligned} &\{\psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}_{2n+1}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}_{2n+1}}^+(x')\} \\ &= \frac{i}{2^{2n} [(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b)\gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots [(m - \gamma^c \partial_c)\gamma^4]_{\tau_\zeta \tau'_\zeta})\}}}_{2n+1} \Delta(x-x') \\ &= \frac{i}{2^{4n} [(2n+1)!]^2} \underbrace{\{[(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b)\gamma^4]_{\mu_\zeta \mu'_\zeta})\}} \cdots [(m - \gamma^c \partial_c)\gamma^4]_{\tau_\zeta \tau'_\zeta}\}}_{n} \Delta(x-x') \\ &= \frac{i}{2^{3n} [(2n+1)!]^2} \end{aligned}$$

$$\begin{aligned} & \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a(x) \mathbb{X}_{\{\eta_\zeta \xi_\zeta\}}^b(x)}_n \cdot \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{+a'}(x') \mathbb{X}_{\{\eta'_\zeta \xi'_\zeta\}}^{+b'}(x')}_n \cdot \underbrace{[(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_n [\eta_{\{a(a' - \frac{\partial_a \partial_{a'}^+}{m^2})\}} [\eta_{\{b(b' - \frac{\partial_b \partial_{b'}^+}{m^2})\}}] \cdot \cdot] \Delta(x - x') \\ &= \frac{i}{2^{3n} [(2n+1)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{+a'}(x')}_n \cdot \underbrace{[(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_n [\eta_{\{aa' - \frac{\partial_a \partial_{a'}^+}{m^2}\}}] \cdot \cdot \Delta(x - x') \end{aligned} \quad \square$$

## 6.7 有质量玻色子对易规则小结

定理6.7.1.  $n \geq 0$

$$\begin{aligned} & [a(\vec{p}, h; n), a^+(\vec{p}', h'; n)] = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [b(\vec{p}, h; n), b^+(\vec{p}', h'; n)] = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [rest] = 0 \\ & \Leftrightarrow \underbrace{[\psi_{\lambda_\zeta \mu_\zeta \dots}(x)]}_{2n} \cdot \underbrace{[\psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]}_{2n} = \frac{i}{2^{2n-1} [(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta \mu_\zeta\}}}_{2n} \underbrace{[(m - \gamma^b \partial_b) \gamma^4]_{\{\mu_\zeta \mu'_\zeta\}}}_{2n} \Delta(x - x'), [rest] = 0 \\ & \Leftrightarrow \underbrace{[\psi_{\lambda_\zeta \mu_\zeta \dots}(x)]}_{2n} \cdot \underbrace{[\psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]}_{2n} = \frac{i}{2^{3n-1} [(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{+a'}(x')}_n \cdot \underbrace{[\eta_{\{aa' - \frac{\partial_a \partial_{a'}^+}{m^2}\}}]_n}_{2n} \Delta(x - x'), [rest] = 0 \end{aligned}$$

## 6.8 有质量费米子反对易规则小结

定理6.8.1.  $n \geq 0$

$$\begin{aligned} & \{a(\vec{p}, h; n + \frac{1}{2}), a^+(\vec{p}', h'; n + \frac{1}{2})\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), \{b(\vec{p}, h; n + \frac{1}{2}), b^+(\vec{p}', h'; n + \frac{1}{2})\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), \{rest\} = 0 \\ & \Leftrightarrow \underbrace{\{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(x')\}}_{2n+1} \\ &= \frac{i}{2^{2n} [(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta \mu_\zeta\}}}_{2n+1} \underbrace{[(m - \gamma^b \partial_b) \gamma^4]_{\{\mu_\zeta \mu'_\zeta\}}}_{2n+1} \cdot \underbrace{[(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_{2n+1} \Delta(x - x'), \{rest\} = 0 \\ & \Leftrightarrow \underbrace{\{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(x')\}}_{2n+1} \\ &= \frac{i}{2^{3n} [(2n+1)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{+a'}(x')}_n \cdot \underbrace{[(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_n [\eta_{\{aa' - \frac{\partial_a \partial_{a'}^+}{m^2}\}}] \cdot \cdot \Delta(x - x'), \{rest\} = 0 \end{aligned}$$

## 7 Bargmann-Wigner方程<sup>[17]</sup>各种量子算符的提取

### 7.1 Bargmann-Wigner方程的等时对易规则

定理7.1.1.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{r}, t) = 0, \psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{r}, t) = \frac{1}{(2s)!} \psi_{\{\lambda_\zeta \mu_\zeta \dots \tau_\zeta\}}(\vec{r}, t)$

$$\begin{aligned} \psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{r}, t) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=s}^{-s} E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} [a(\vec{p}, h) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}_{2s}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}_{2s}(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p} \\ \begin{cases} a(\vec{p}, h) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m}{E}}^{2s} U^{+\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}, h) \psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r} \\ b^+(\vec{p}, s) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m}{E}}^{2s} V^{+\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}, h) \psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{r}, t) e^{ip \cdot x} d^3 \vec{r} \end{cases} \end{aligned}$$

定理7.1.2.  $[\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(\vec{r}', t)]_{-2s+1}$

$$= \frac{1}{2^{2s-1} [(2s)!]^2} \sum_{l=0}^{[s-\frac{1}{2}]} [C_{2s}^{2l+1} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_\zeta \mu_\zeta\}} (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\mu_\zeta \mu'_\zeta\}} \delta_{\tau_\zeta \tau'_\zeta}}^{2s-2l-1}] (m^2 - \nabla^2)^l \delta^3(\vec{r} - \vec{r}')$$

证明:  $[\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(\vec{r}', t)]_{-2s+1}$

$$= \frac{i}{2^{2s-1} [(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta \mu_\zeta\}} [(m - \gamma^b \partial_b) \gamma^4]_{\{\mu_\zeta \mu'_\zeta\}} \cdot [(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}^{2s} \Delta(x - x')|_{t=t'}$$

$$\begin{aligned}
&= \frac{1}{2^{2s-1}} \frac{1}{[(2s)!]^2} \sum_{l=0}^{[s-\frac{1}{2}]} [C_{2s}^{2l+1} \overbrace{(m\gamma^4 - \vec{\gamma}^4 \cdot \nabla)_{\{\lambda_\zeta(\lambda'_\zeta(m\gamma^4 - \vec{\gamma}^4 \cdot \nabla)_{\mu_\zeta \mu'_\zeta} \cdots \delta_{\tau_\zeta} \tau'_\zeta)\}}^{2s-2l-1}}] (m^2 - \nabla^2)^l \delta^3(\vec{r} - \vec{r}') \\
&= \frac{1}{2^{2s-1}} \frac{1}{[(2s)!]^2} \sum_{l=0}^{[s-\frac{1}{2}]} [C_{2s}^{2l+1} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_\zeta(\lambda'_\zeta(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_\zeta \mu'_\zeta} \cdots \delta_{\tau_\zeta} \tau'_\zeta)\}}^{2s-2l-1}}] (m^2 - \nabla^2)^l \delta^3(\vec{r} - \vec{r}') \quad \square
\end{aligned}$$

## 7.2 Bargmann-Wigner方程能量算符的提取

$$\begin{aligned}
\text{引理7.2.1.} & \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E]_{\{\lambda_\zeta(\lambda'_\zeta[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E]_{\mu_\zeta \mu'_\zeta} \cdots \})}^{2s}} \\
&= \sum_{l=0}^{2s} C_{2s}^l E^l \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\{\lambda_\zeta(\lambda'_\zeta(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\mu_\zeta \mu'_\zeta} \cdots \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \cdots \})}^{2s-l}} \overbrace{\cdots}^l
\end{aligned}$$

$$\begin{aligned}
\text{引理7.2.2.} & \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E]_{\{\lambda_\zeta(\lambda'_\zeta[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E]_{\mu_\zeta \mu'_\zeta} \cdots \})}^{2s}} \\
&= \sum_{l=0}^{2s} (-1)^l C_{2s}^l E^l \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\{\lambda_\zeta(\lambda'_\zeta(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\mu_\zeta \mu'_\zeta} \cdots \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \cdots \})}^{2s-l}} \overbrace{\cdots}^l
\end{aligned}$$

引理7.2.3.

$$\begin{aligned}
& \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E]_{\{\lambda_\zeta(\lambda'_\zeta[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E]_{\mu_\zeta \mu'_\zeta} \cdots \})}^{2s}} \\
&+ \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E]_{\{\lambda_\zeta(\lambda'_\zeta[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E]_{\mu_\zeta \mu'_\zeta} \cdots \})}^{2s}} \\
&= 2 \sum_{l=0}^{[s]} C_{2s}^{2l} E^{2l} \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\{\lambda_\zeta(\lambda'_\zeta(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\mu_\zeta \mu'_\zeta} \cdots \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \cdots \})}^{2s-2l}} \overbrace{\cdots}^{2l}
\end{aligned}$$

引理7.2.4.

$$\begin{aligned}
& \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E]_{\{\lambda_\zeta(\lambda'_\zeta[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E]_{\mu_\zeta \mu'_\zeta} \cdots \})}^{2s}} \\
&- \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E]_{\{\lambda_\zeta(\lambda'_\zeta[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E]_{\mu_\zeta \mu'_\zeta} \cdots \})}^{2s}} \\
&= 2 \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} E^{2l+1} \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\{\lambda_\zeta(\lambda'_\zeta(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\mu_\zeta \mu'_\zeta} \cdots \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \cdots \})}^{2s-2l-1}} \overbrace{\cdots}^{2l+1}
\end{aligned}$$

定理7.2.1.

$$H(s) = \int \sum_{h=s}^{-s} E [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} = \int \psi^+ \overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(m^2 - \nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdots}}_{2s}(\vec{r}, t) d^3 \vec{r}$$

$$\text{证明: } \int \sum_{h=s}^{-s} E [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int \frac{m^{2s}}{E^{4s-2}} \psi^+ \overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}(\vec{r}, t) \psi \overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}(\vec{r}', t) \\
&\sum_{h=s}^{-s} [U \overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}(\vec{p}, h) U^+ \overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}(\vec{p}, h) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} + (-1)^{2s} V \overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}(\vec{p}, h) V^+ \overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}(\vec{p}, h) e^{-i\vec{p} \cdot (\vec{r} - \vec{r}')} ] d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int \frac{m^{2s}}{E^{4s-2}} \psi^+ \overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}(\vec{r}, t) \psi \overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}(\vec{r}', t) \\
&\sum_{h=s}^{-s} [U \overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}(\vec{p}, h) U^+ \overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}(\vec{p}, h) + (-1)^{2s} V \overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}(-\vec{p}, h) V^+ \overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}(-\vec{p}, h)] e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \frac{m^{2s}}{E^{4s-2}} \psi^+ \overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}(\vec{r}, t) \psi \overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}(\vec{r}', t) \frac{1}{(2m)^{2s} [(2s)!]^2} \\
&\{ \overbrace{[(m - i\gamma^a p_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(m - i\gamma^b p_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots \})}^{2s}} + \overbrace{[(m - i\gamma^a p_a^+) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta[(m - i\gamma^b p_b^+) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots \})}^{2s}} \} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \\
&= \frac{1}{2^{2s} [(2s)!]^2} \frac{1}{(2\pi)^3} \int d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \psi^+ \overbrace{\lambda_\zeta \mu_\zeta \cdots}^{2s}(\vec{r}, t) \psi \overbrace{\lambda'_\zeta \mu'_\zeta \cdots}^{2s}(\vec{r}', t) \frac{1}{E^{4s-2}}
\end{aligned}$$

$$\begin{aligned}
& \overbrace{\{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E]_{\{\lambda_\zeta(\lambda'_\zeta [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E]_{\mu_\zeta \mu'_\zeta} \dots})\}}\}^{2s} \\
& + \overbrace{\{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E]_{\{\lambda_\zeta(\lambda'_\zeta [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E]_{\mu_\zeta \mu'_\zeta} \dots})\}}\}^{2s} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \\
& = \frac{1}{2^{2s-1} [(2s)!]^2} \frac{1}{(2\pi)^3} \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \overbrace{\psi^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{r}', t)}^{2s} \frac{1}{E^{4s-2}} \\
& \sum_{l=0}^{[s]} C_{2s}^{2l} E^{2l} \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\{\lambda_\zeta(\lambda'_\zeta (m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\mu_\zeta \mu'_\zeta} \dots \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots})\}}^{2s-2l} \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{2l} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \\
& = \frac{1}{2^{2s-1} [(2s)!]^2} \frac{1}{(2\pi)^3} \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \overbrace{\psi^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{r}', t)}^{2s} \\
& \sum_{l=0}^{[s]} C_{2s}^{2l} \frac{1}{(E^2)^{2s-1-l}} \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\{\lambda_\zeta(\lambda'_\zeta (m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\mu_\zeta \mu'_\zeta} \dots \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots})\}}^{2s-2l} \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{2l} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \\
& = \frac{1}{2^{2s-1} [(2s)!]^2} \frac{1}{(2\pi)^3} \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \overbrace{\psi^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{r}', t)}^{2s} \\
& \sum_{l=0}^{[s]} C_{2s}^{2l} \frac{1}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_\zeta(\lambda'_\zeta (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_\zeta \mu'_\zeta} \dots \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots})\}}^{2s-2l} \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{2l} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \\
& = \frac{1}{2^{2s-1} [(2s)!]^2} \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \overbrace{\psi^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{r}', t)}^{2s} \\
& \sum_{l=0}^{[s]} C_{2s}^{2l} \frac{1}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_\zeta(\lambda'_\zeta (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_\zeta \mu'_\zeta} \dots \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots})\}}^{2s-2l} \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{2l} \delta^3(\vec{r} - \vec{r}') d^3 \vec{r} d^3 \vec{r}' \\
& = \frac{1}{2^{2s-1} [(2s)!]^2} \\
& \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \sum_{l=0}^{[s]} C_{2s}^{2l} \frac{\overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_\zeta(\lambda'_\zeta (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_\zeta \mu'_\zeta} \dots \delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots})\}}^{2s-2l}}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{2l} \overbrace{\psi^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{r}, t)}^{2s} d^3 \vec{r}' \\
& = \frac{1}{2^{2s-1}} \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \sum_{l=0}^{[s]} C_{2s}^{2l} \frac{\overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\lambda_\zeta \lambda'_\zeta (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_\zeta \mu'_\zeta} \dots}}^{2s-2l}}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{2l} \overbrace{\psi^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{r}, t)}^{2s} d^3 \vec{r}' \\
& = \frac{1}{2^{2s-1}} \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \sum_{l=0}^{[s]} C_{2s}^{2l} \frac{(i\partial_t)^{2s-2l}}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{\delta_{\lambda_\zeta \lambda'_\zeta} \delta_{\mu_\zeta \mu'_\zeta} \dots}^{2s-2l} \overbrace{\delta_{\rho_\zeta \rho'_\zeta} \delta_{\tau_\zeta \tau'_\zeta} \dots}^{2l} \overbrace{\psi^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{r}, t)}^{2s} d^3 \vec{r}' \\
& = \frac{1}{2^{2s-1}} \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \sum_{l=0}^{[s]} C_{2s}^{2l} \frac{(i\partial_t)^{2s-2l}}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{\delta_{\lambda_\zeta \lambda'_\zeta} \delta_{\mu_\zeta \mu'_\zeta} \dots}^{2s} \overbrace{\psi^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{r}, t)}^{2s} d^3 \vec{r}' \\
& = \frac{1}{2^{2s-1}} \sum_{l=0}^{[s]} C_{2s}^{2l} \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \frac{(i\partial_t)^{2s-2l}}{(m^2 - \nabla^2)^{2s-1-l}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) d^3 \vec{r}' \\
& = \frac{1}{2^{2s-1}} \sum_{l=0}^{[s]} C_{2s}^{2l} \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(m^2 - \nabla^2)^{2s-1-l}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) d^3 \vec{r}' \\
& = \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(m^2 - \nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) d^3 \vec{r}' \quad \square
\end{aligned}$$

定理7.2.2.

$$H(n) = \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \frac{1}{(m^2 - \nabla^2)^{n-1}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2n}(\vec{r}, t) d^3 \vec{r}, \quad H(n + \frac{1}{2}) = \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \frac{i\partial_t}{(m^2 - \nabla^2)^n} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2n+1}(\vec{r}, t) d^3 \vec{r}$$

### 7.3 Bargmann-Wigner方程动量算符的提取

定理7.3.1.

$$P(s) = \int \sum_{h=s}^{-s} \vec{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} = \int \psi^{+\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t) \frac{-i\nabla(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) d^3 \vec{r}$$



$$\begin{aligned}
& \text{证明: } \int \sum_{h=s}^{-s} \bar{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int \frac{m^{2s}}{E^{4s-1}} \bar{\psi} \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) \\
& \sum_{h=s}^{-s} [U^{\lambda_s \mu_s \cdots}(\vec{p}, h) U^{+\lambda'_s \mu'_s \cdots}(\vec{p}, h) e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} + (-1)^{2s} V^{\lambda_s \mu_s \cdots}(\vec{p}, h) V^{+\lambda'_s \mu'_s \cdots}(\vec{p}, h) e^{-i\vec{p} \cdot (\vec{r} - \vec{r}')} ] d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int \frac{m^{2s}}{E^{4s-1}} \bar{\psi} \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) \\
& \sum_{h=s}^{-s} [U^{\lambda_s \mu_s \cdots}(\vec{p}, h) U^{+\lambda'_s \mu'_s \cdots}(\vec{p}, h) - (-1)^{2s} V^{\lambda_s \mu_s \cdots}(-\vec{p}, h) V^{+\lambda'_s \mu'_s \cdots}(-\vec{p}, h)] e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \\
&= \frac{1}{(2\pi)^3} \int d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \frac{m^{2s}}{E^{4s-1}} \bar{\psi} \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) \frac{1}{(2m)^{2s} [(2s)!]^2} \\
& \left\{ [(m - i\gamma^a p_a) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - i\gamma^b p_b) \gamma^4]_{\mu_s \mu'_s \cdots})\}} - [(m - i\gamma^a p_a^+) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - i\gamma^b p_b^+) \gamma^4]_{\mu_s \mu'_s \cdots})\}} \right\} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \\
&= \frac{1}{2^{2s} [(2s)!]^2} \frac{1}{(2\pi)^3} \int d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p} \bar{\psi} \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) \frac{1}{E^{4s-1}} \vec{p} \\
& \left\{ [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E]_{\{\lambda_s(\lambda'_s [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E]_{\mu_s \mu'_s \cdots})\}} \right. \\
& \left. - [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E]_{\{\lambda_s(\lambda'_s [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E]_{\mu_s \mu'_s \cdots})\}} \right\} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \\
&= \frac{1}{2^{2s-1} [(2s)!]^2} \frac{1}{(2\pi)^3} \int \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) \frac{1}{E^{4s-1}} \vec{p} \\
& \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} E^{2l+1} \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\{\lambda_s(\lambda'_s (m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\mu_s \mu'_s \cdots} \delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s \cdots})\}}}^{2s-2l-1} \overbrace{e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}}^{2l+1} \\
&= \frac{1}{2^{2s-1} [(2s)!]^2} \frac{1}{(2\pi)^3} \int \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) \\
& \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \frac{\vec{p}}{(E^2)^{2s-1-l}} \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\{\lambda_s(\lambda'_s (m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})_{\mu_s \mu'_s \cdots} \delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s \cdots})\}}}^{2s-2l-1} \overbrace{e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}}^{2l+1} \\
&= \frac{1}{2^{2s-1} [(2s)!]^2} \frac{1}{(2\pi)^3} \int \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) \\
& \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \frac{-i\nabla}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_s(\lambda'_s (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_s \mu'_s \cdots} \delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s \cdots})\}}}^{2s-2l-1} \overbrace{e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}}^{2l+1} \\
&= \frac{1}{2^{2s-1} [(2s)!]^2} \int \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) \\
& \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \frac{-i\nabla}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_s(\lambda'_s (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_s \mu'_s \cdots} \delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s \cdots})\}}}^{2s-2l-1} \overbrace{\delta^3(\vec{r} - \vec{r}') d^3 \vec{r} d^3 \vec{r}'}^{2l+1} \\
&= \frac{1}{2^{2s-1} [(2s)!]^2} \\
& \int \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \frac{-i\nabla (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_s(\lambda'_s (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_s \mu'_s \cdots} \delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s \cdots})\}}}^{2s-2l-1} \overbrace{\psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) d^3 \vec{r}'}^{2l+1} \\
&= \frac{1}{2^{2s-1}} \int \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \frac{-i\nabla (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\lambda_s \lambda'_s (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_s \mu'_s \cdots}}^{2s-2l-1}}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{\delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s \cdots}}^{2l+1} \overbrace{\psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) d^3 \vec{r}'}^{2s} \\
&= \frac{1}{2^{2s-1}} \int \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \frac{-i\nabla (i\partial_t)^{2s-2l-1}}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{\delta_{\lambda_s \lambda'_s} \delta_{\mu_s \mu'_s \cdots}}^{2s-2l-1} \overbrace{\delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s \cdots}}^{2l+1} \overbrace{\psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) d^3 \vec{r}'}^{2s} \\
&= \frac{1}{2^{2s-1}} \int \psi^{+\lambda_s \mu_s \cdots}(\vec{r}, t) \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \frac{-i\nabla (i\partial_t)^{2s-2l-1}}{(m^2 - \nabla^2)^{2s-1-l}} \overbrace{\delta_{\lambda_s \lambda'_s} \delta_{\mu_s \mu'_s \cdots}}^{2s} \overbrace{\psi^{\lambda'_s \mu'_s \cdots}(\vec{r}', t) d^3 \vec{r}'}^{2s}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^{2s-1}} \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{-i\nabla(i\partial_t)^{2s-2l-1}}{(m^2-\nabla^2)^{2s-1-l}} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2s}(\vec{r}, t) d^3\vec{r} \\
&= \frac{1}{2^{2s-1}} \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{-i\nabla(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2s}(\vec{r}, t) d^3\vec{r} \\
&= \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{-i\nabla(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2s}(\vec{r}, t) d^3\vec{r}
\end{aligned}$$

□

定理7.3.2.

$$P(n) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2n}}(\vec{r}, t) \frac{(-i\nabla)(i\partial_t)}{(m^2-\nabla^2)^n} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2n}(\vec{r}, t) d^3\vec{r}, P(n + \frac{1}{2}) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2n+1}}(\vec{r}, t) \frac{-i\nabla}{(m^2-\nabla^2)^n} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2n+1}(\vec{r}, t) d^3\vec{r}$$

## 7.4 Bargmann-Wigner方程能量动量算符的小结

$$\text{定理7.4.1. } P_u(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2s}(\vec{r}, t) d^3\vec{r}$$

## 7.5 Bargmann-Wigner方程的各种物理算符

定理7.5.1.

$$P_u(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2s}(\vec{r}, t) d^3\vec{r} = \int \sum_h p_u [a^+(\vec{p}, h)a(\vec{p}, h) + (-1)^{2s}b(\vec{p}, h)b^+(\vec{p}, h)] d^3\vec{p}$$

$$\text{证明: } P_u(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2s}(\vec{r}, t) d^3\vec{r}$$

$$= \int \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_{h'} E'^{s-\frac{1}{2}} \sqrt{\frac{m}{E'}}^{2s} [a^+(\vec{p}', h')U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h')e^{-i(\vec{p}'\cdot\vec{r}-E't)} + b(\vec{p}', h')V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h')e^{i(\vec{p}'\cdot\vec{r}-E't)}] d^3\vec{p}'$$

$$\frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_h E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} \frac{p_u E^{2s-1}}{(E^2)^{2s-1}} [a(\vec{p}, h)U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)e^{ip\cdot x} + (-1)^{2s}b^+(\vec{p}, h)V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)e^{-ip\cdot x}] d^3\vec{p} d^3\vec{r}$$

$$= \frac{1}{(2\pi)^3} \int \sum_{h, h'} \left(\frac{E'}{E}\right)^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'E}}^{2s} [a^+(\vec{p}', h')U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h')e^{-i(\vec{p}'\cdot\vec{r}-E't)} + b(\vec{p}', h')V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h')e^{i(\vec{p}'\cdot\vec{r}-E't)}]$$

$$p_u [a(\vec{p}, h)U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)e^{ip\cdot x} + (-1)^{2s}b^+(\vec{p}, h)V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)e^{-ip\cdot x}] d^3\vec{p}' d^3\vec{p} d^3\vec{r}$$

$$= \int d^3\vec{p}' d^3\vec{p} \sum_{h, h'} \left(\frac{m}{E}\right)^{2s} p_u$$

$$\{\delta^3(\vec{p}'-\vec{p})[a^+(\vec{p}', h')a(\vec{p}, h)U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h')U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) + (-1)^{2s}b(\vec{p}', h')b^+(\vec{p}, h)V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h')V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)]$$

$$+ \delta^3(\vec{p}'+\vec{p})[(-1)^{2s}e^{2iEt}a^+(-\vec{p}', h')b^+(\vec{p}, h)U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}', h')V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)$$

$$+ e^{-2iEt}b(\vec{p}', h')a(\vec{p}, h)V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}', h')U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)]\}$$

$$= \int \sum_h p_u [a^+(\vec{p}, h)a(\vec{p}, h) + (-1)^{2s}b(\vec{p}, h)b^+(\vec{p}, h)] d^3\vec{p}$$

□

定理7.5.2.

$$Q(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2s}(\vec{r}, t) d^3\vec{r} = \int \sum_h [a^+(\vec{p}, h)a(\vec{p}, h) + (-1)^{2s-1}b(\vec{p}, h)b^+(\vec{p}, h)] d^3\vec{p}$$

$$\text{证明: } Q(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(m^2-\nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2s}(\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned}
&= \int \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_{h'} E'^{s-\frac{1}{2}} \sqrt{\frac{m}{E'}}^{2s} [a^+(\vec{p}', h') U^{+\lambda_s \mu_s \dots}(\vec{p}', h') e^{-i(\vec{p}' \cdot \vec{r} - E' t)} + b(\vec{p}', h') V^{+\lambda_s \mu_s \dots}(\vec{p}', h') e^{i(\vec{p}' \cdot \vec{r} - E' t)}] d^3 \vec{p}' \\
&\frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} \frac{E^{2s-1}}{(E^2)^{2s-1}} [a(\vec{p}, h) U_{\lambda_s \mu_s \dots}(\vec{p}, h) e^{i\vec{p} \cdot \vec{x}} + (-1)^{2s-1} b^+(\vec{p}, h) V_{\lambda_s \mu_s \dots}(\vec{p}, h) e^{-i\vec{p} \cdot \vec{x}}] d^3 \vec{p} d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int \sum_{h, h'} \left(\frac{E'}{E}\right)^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'E}}^{2s} [a^+(\vec{p}', h') U^{+\lambda_s \mu_s \dots}(\vec{p}', h') e^{-i(\vec{p}' \cdot \vec{r} - E' t)} + b(\vec{p}', h') V^{+\lambda_s \mu_s \dots}(\vec{p}', h') e^{i(\vec{p}' \cdot \vec{r} - E' t)}] \\
&[a(\vec{p}, h) U_{\lambda_s \mu_s \dots}(\vec{p}, h) e^{i\vec{p} \cdot \vec{x}} + (-1)^{2s-1} b^+(\vec{p}, h) V_{\lambda_s \mu_s \dots}(\vec{p}, h) e^{-i\vec{p} \cdot \vec{x}}] d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r} \\
&= \int d^3 \vec{p}' d^3 \vec{p} \sum_{h, h'} \left(\frac{m}{E}\right)^{2s} \\
&\{\delta^3(\vec{p}' - \vec{p}) [a^+(\vec{p}', h') a(\vec{p}, h) U^{+\lambda_s \mu_s \dots}(\vec{p}', h') U_{\lambda_s \mu_s \dots}(\vec{p}, h) + (-1)^{2s-1} b(\vec{p}', h') b^+(\vec{p}, h) V^{+\lambda_s \mu_s \dots}(\vec{p}', h') V_{\lambda_s \mu_s \dots}(\vec{p}, h)] \\
&+ \delta^3(\vec{p}' + \vec{p}) [(-1)^{2s-1} e^{2iEt} a^+(-\vec{p}', h') b^+(\vec{p}, h) U^{+\lambda_s \mu_s \dots}(-\vec{p}', h') V_{\lambda_s \mu_s \dots}(\vec{p}, h) \\
&+ e^{-2iEt} b(\vec{p}', h') a(\vec{p}, h) V^{+\lambda_s \mu_s \dots}(-\vec{p}', h') U_{\lambda_s \mu_s \dots}(\vec{p}, h)]\} \\
&= \int \sum_h [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s-1} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} \quad \square
\end{aligned}$$

定理7.5.3.

$$N(s) = \int \psi^{+\lambda_s \mu_s \dots}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi_{\lambda_s \mu_s \dots}(\vec{r}, t) d^3 \vec{r} = \int \sum_h [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p}$$

$$\begin{aligned}
\text{证明: } N(s) &= \int \psi^{+\lambda_s \mu_s \dots}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi_{\lambda_s \mu_s \dots}(\vec{r}, t) d^3 \vec{r} \\
&= \int \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_{h'} E'^{s-\frac{1}{2}} \sqrt{\frac{m}{E'}}^{2s} [a^+(\vec{p}', h') U^{+\lambda_s \mu_s \dots}(\vec{p}', h') e^{-i(\vec{p}' \cdot \vec{r} - E' t)} + b(\vec{p}', h') V^{+\lambda_s \mu_s \dots}(\vec{p}', h') e^{i(\vec{p}' \cdot \vec{r} - E' t)}] d^3 \vec{p}' \\
&\frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} \frac{E^{2s}}{E^{4s-1}} [a(\vec{p}, h) U_{\lambda_s \mu_s \dots}(\vec{p}, h) e^{i\vec{p} \cdot \vec{x}} + (-1)^{2s} b^+(\vec{p}, h) V_{\lambda_s \mu_s \dots}(\vec{p}, h) e^{-i\vec{p} \cdot \vec{x}}] d^3 \vec{p} d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int \sum_{h, h'} \left(\frac{E'}{E}\right)^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'E}}^{2s} [a^+(\vec{p}', h') U^{+\lambda_s \mu_s \dots}(\vec{p}', h') e^{-i(\vec{p}' \cdot \vec{r} - E' t)} + b(\vec{p}', h') V^{+\lambda_s \mu_s \dots}(\vec{p}', h') e^{i(\vec{p}' \cdot \vec{r} - E' t)}] \\
&[a(\vec{p}, h) U_{\lambda_s \mu_s \dots}(\vec{p}, h) e^{i\vec{p} \cdot \vec{x}} + (-1)^{2s} b^+(\vec{p}, h) V_{\lambda_s \mu_s \dots}(\vec{p}, h) e^{-i\vec{p} \cdot \vec{x}}] d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r} \\
&= \int d^3 \vec{p}' d^3 \vec{p} \sum_{h, h'} \left(\frac{m}{E}\right)^{2s} \\
&\{\delta^3(\vec{p}' - \vec{p}) [a^+(\vec{p}', h') a(\vec{p}, h) U^{+\lambda_s \mu_s \dots}(\vec{p}', h') U_{\lambda_s \mu_s \dots}(\vec{p}, h) + (-1)^{2s} b(\vec{p}', h') b^+(\vec{p}, h) V^{+\lambda_s \mu_s \dots}(\vec{p}', h') V_{\lambda_s \mu_s \dots}(\vec{p}, h)] \\
&+ \delta^3(\vec{p}' + \vec{p}) [(-1)^{2s} e^{2iEt} a^+(-\vec{p}', h') b^+(\vec{p}, h) U^{+\lambda_s \mu_s \dots}(-\vec{p}', h') V_{\lambda_s \mu_s \dots}(\vec{p}, h) \\
&+ e^{-2iEt} b(\vec{p}', h') a(\vec{p}, h) V^{+\lambda_s \mu_s \dots}(-\vec{p}', h') U_{\lambda_s \mu_s \dots}(\vec{p}, h)]\} \\
&= \int \sum_h [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} \quad \square
\end{aligned}$$

定理7.5.4.

$$\vec{S}(s) = \int \psi^{+\lambda_s \mu_s \dots}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\lambda_s \mu_s \dots}(\vec{r}, t) d^3 \vec{r} = \int \sum_h \hat{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p}$$

$$\text{证明: } \vec{S}(s) = \int \psi^{+\lambda_s \mu_s \dots}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\lambda_s \mu_s \dots}(\vec{r}, t) d^3 \vec{r}$$

$$\begin{aligned}
&= \int \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_{h'} E'^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'}}^{2s} [a^+(\vec{p}', h') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') e^{-i(\vec{p}' \cdot \vec{r} - E' t)} + b(\vec{p}', h') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') e^{i(\vec{p}' \cdot \vec{r} - E' t)}] d^3 \vec{p}' \\
&\frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h E^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E}}^{2s} \frac{\hat{p} E^{2s-1}}{(E^2)^{2s-1}} [a(\vec{p}, h) U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) e^{ip \cdot x} + (-1)^{2s} b^+(\vec{p}, h) V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p} d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int \sum_{h, h'} \left(\frac{E'}{E}\right)^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'E}}^{2s} \hat{p} [a^+(\vec{p}', h') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') e^{-i(\vec{p}' \cdot \vec{r} - E' t)} + b(\vec{p}', h') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') e^{i(\vec{p}' \cdot \vec{r} - E' t)}] \\
&[a(\vec{p}, h) U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) e^{ip \cdot x} + (-1)^{2s} b^+(\vec{p}, h) V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r} \\
&= \int d^3 \vec{p}' d^3 \vec{p} \sum_{h, h'} \left(\frac{m}{E}\right)^{2s} \hat{p} \\
&\{\delta^3(\vec{p} - \vec{p}') [a^+(\vec{p}', h') a(\vec{p}, h) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) + (-1)^{2s} b(\vec{p}', h') b^+(\vec{p}, h) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)] \\
&+ \delta^3(\vec{p} + \vec{p}') [(-1)^{2s} e^{2iEt} a^+(-\vec{p}', h') b^+(\vec{p}, h) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}', h') V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) \\
&+ e^{-2iEt} b(\vec{p}', h') a(\vec{p}, h) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}', h') U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)]\} \\
&= \int \sum_h \hat{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} \quad \square
\end{aligned}$$

定理7.5.5.

$$\vec{M}(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) d^3 \vec{r} = \int \sum_h \hat{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s-1} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p}$$

$$\begin{aligned}
\text{证明: } \vec{M}(s) &= \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) d^3 \vec{r} \\
&= \int \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_{h'} E'^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'}}^{2s} [a^+(\vec{p}', h') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') e^{-i(\vec{p}' \cdot \vec{r} - E' t)} + b(\vec{p}', h') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') e^{i(\vec{p}' \cdot \vec{r} - E' t)}] d^3 \vec{p}' \\
&\frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_h E^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E}}^{2s} \frac{\hat{p} E^{2s}}{E^{4s-1}} [a(\vec{p}, h) U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) e^{ip \cdot x} + (-1)^{2s-1} b^+(\vec{p}, h) V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p} d^3 \vec{r} \\
&= \frac{1}{(2\pi)^3} \int \sum_{h, h'} \left(\frac{E'}{E}\right)^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'E}}^{2s} \hat{p} [a^+(\vec{p}', h') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') e^{-i(\vec{p}' \cdot \vec{r} - E' t)} + b(\vec{p}', h') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') e^{i(\vec{p}' \cdot \vec{r} - E' t)}] \\
&[a(\vec{p}, h) U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) e^{ip \cdot x} + (-1)^{2s-1} b^+(\vec{p}, h) V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p}' d^3 \vec{p} d^3 \vec{r} \\
&= \int d^3 \vec{p}' d^3 \vec{p} \sum_{h, h'} \left(\frac{m}{E}\right)^{2s} \hat{p} \\
&\{\delta^3(\vec{p} - \vec{p}') [a^+(\vec{p}', h') a(\vec{p}, h) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) + (-1)^{2s-1} b(\vec{p}', h') b^+(\vec{p}, h) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', h') V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)] \\
&+ \delta^3(\vec{p} + \vec{p}') [(-1)^{2s-1} e^{2iEt} a^+(-\vec{p}', h') b^+(\vec{p}, h) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}', h') V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h) \\
&+ e^{-2iEt} b(\vec{p}', h') a(\vec{p}, h) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}', h') U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, h)]\} \\
&= \int \sum_h \hat{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s-1} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} \quad \square
\end{aligned}$$

## 7.6 Bargmann-Wigner方程的量子方程

$$\text{定理7.6.1. } \begin{cases} (\gamma^a \partial_a + m)_{\kappa_s} \psi_{\overbrace{\lambda_s \mu_s \dots \tau_s}^{2s}}(\vec{r}, t) = 0 \\ P_u(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{-i\partial_u (i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) d^3 \vec{r} \end{cases} \Rightarrow -i\partial_u \psi_{\overbrace{\lambda_s \mu_s \dots}^{2n}}(\vec{r}, t) = [\psi_{\overbrace{\lambda_s \mu_s \dots}^{2n}}(\vec{r}, t), P_u]$$

证明:  $[\psi_{\lambda_s \mu_s} \dots (\vec{r}, t), P_u(s)]$

$$\begin{aligned}
&= [\psi_{\lambda_s \mu_s} \dots (\vec{r}, t), \int \psi_{\lambda'_s \mu'_s}^+ \dots (\vec{r}', t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\lambda'_s \mu'_s} \dots (\vec{r}', t) d^3 \vec{r}'] \\
&= \int [\psi_{\lambda_s \mu_s} \dots (\vec{r}, t), \psi_{\lambda'_s \mu'_s}^+ \dots (\vec{r}', t)]_{-2s+1} \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\lambda'_s \mu'_s} \dots (\vec{r}', t) d^3 \vec{r}' \\
&= \int \frac{1}{2^{2s-1}} \frac{1}{[(2s)!]^2} \sum_{l=0}^{[s-\frac{1}{2}]} [C_{2s}^{2l+1} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_s(\lambda'_s(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_s \mu'_s} \dots \delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s} \dots \}}\}}^{2s-2l-1} \overbrace{\delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s} \dots}^{2l+1}]] \\
&\quad (m^2 - \nabla^2)^l \delta^3(\vec{r} - \vec{r}') \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-l}} \psi_{\lambda'_s \mu'_s} \dots (\vec{r}', t) d^3 \vec{r}' \\
&= \frac{1}{2^{2s-1}} \frac{1}{[(2s)!]^2} \sum_{l=0}^{[s-\frac{1}{2}]} [C_{2s}^{2l+1} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_s(\lambda'_s(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_s \mu'_s} \dots \delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s} \dots \}}\}}^{2s-2l-1} \overbrace{\delta_{\rho_s \rho'_s} \delta_{\tau_s \tau'_s} \dots}^{2l+1}]] \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-l-1}} \psi_{\lambda'_s \mu'_s} \dots (\vec{r}', t) \\
&= \frac{1}{2^{2s-1}} \frac{1}{[(2s)!]^2} \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} (i\partial_t)^{2s-2l-1} \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-l-1}} \psi_{\lambda_s \mu_s} \dots (\vec{r}, t) \\
&= \frac{1}{2^{2s-1}} \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \frac{-i\partial_u(i\partial_t)^{4s-2}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\lambda_s \mu_s} \dots (\vec{r}, t) \\
&= -i\partial_u \psi_{\lambda_s \mu_s} \dots (\vec{r}, t)
\end{aligned}$$

□

## 7.7 Bargmann-Wigner方程玻色子能量动量算符

定理7.7.1.

$$P_u(n) = \int \psi^+ \overbrace{\lambda_s \mu_s \dots}^{2n} (\vec{r}, t) \frac{-i\partial_u(i\partial_t)}{(m^2 - \nabla^2)^n} \psi_{\lambda_s \mu_s} \dots (\vec{r}, t) d^3 \vec{r}, P_u(n + \frac{1}{2}) = \int \psi^+ \overbrace{\lambda_s \mu_s \dots}^{2n+1} (\vec{r}, t) \frac{-i\partial_u}{(m^2 - \nabla^2)^n} \psi_{\lambda_s \mu_s} \dots (\vec{r}, t) d^3 \vec{r}$$

$$\text{定理7.7.2.} \left\{ \begin{aligned}
P_u(n) &= \int \psi^+ \overbrace{\lambda_s \mu_s \dots}^{2n} (\vec{r}, t) \frac{[-i\nabla, i\gamma^4(\vec{\gamma} \cdot \nabla + m)] \gamma^4 (\vec{\gamma} \cdot \nabla + m)}{(m^2 - \nabla^2)^n} \lambda_s \eta_s \psi_{\eta_s \mu_s} \dots (\vec{r}, t) d^3 \vec{r} \\
&= \int \psi^+ \overbrace{\lambda_s \mu_s \dots}^{2n} (\vec{r}, t) \frac{[-i\nabla \gamma^4 (\vec{\gamma} \cdot \nabla + m), i(m^2 - \nabla^2)]}{(m^2 - \nabla^2)^n} \lambda_s \eta_s \psi_{\eta_s \mu_s} \dots (\vec{r}, t) d^3 \vec{r} \\
P_u(n + \frac{1}{2}) &= \int \psi^+ \overbrace{\lambda_s \mu_s \dots}^{2n+1} (\vec{r}, t) \frac{[-i\nabla, i\gamma^4(\vec{\gamma} \cdot \nabla + m)]}{(m^2 - \nabla^2)^n} \lambda_s \eta_s \psi_{\eta_s \mu_s} \dots (\vec{r}, t) d^3 \vec{r}
\end{aligned} \right.$$

## 7.8 Bargmann-Wigner方程的玻色子量子方程

定理7.8.1.  $(\gamma^a \partial_a + m)_{\kappa_s} \lambda_s \psi_{\lambda_s \mu_s} \dots (\vec{r}, t) = 0 \Rightarrow -i\partial_u \psi_{\lambda_s \mu_s} \dots (\vec{r}, t) = [\psi_{\lambda_s \mu_s} \dots (\vec{r}, t), P_u(n)]$

证明:  $[\psi_{\lambda_s \mu_s} \dots (\vec{r}, t), H]$

$$\begin{aligned}
&= [\psi_{\lambda_s \mu_s} \dots (\vec{r}, t), \int \psi_{\lambda'_s \mu'_s}^+ \dots (\vec{r}', t) \frac{1}{(m^2 - \nabla'^2)^{n-1}} \psi_{\lambda'_s \mu'_s} \dots (\vec{r}', t) d^3 \vec{r}'] \\
&= \int [\psi_{\lambda_s \mu_s} \dots (\vec{r}, t), \psi_{\lambda'_s \mu'_s}^+ \dots (\vec{r}', t) \frac{1}{(m^2 - \nabla'^2)^{n-1}} \psi_{\lambda'_s \mu'_s} \dots (\vec{r}', t)] d^3 \vec{r}' \\
&= \int [\psi_{\lambda_s \mu_s} \dots (\vec{r}, t), \psi_{\lambda'_s \mu'_s}^+ \dots (\vec{r}', t)] \frac{1}{(m^2 - \nabla'^2)^{n-1}} \psi_{\lambda'_s \mu'_s} \dots (\vec{r}', t) d^3 \vec{r}'
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{1}{2^{2n-1}} \frac{1}{[(2n)!]^2} \sum_{l=0}^{[n-\frac{1}{2}]} [C_{2n}^{2l+1} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)}^{2n-2l-1} \{ \lambda_{\zeta} (\lambda'_{\zeta} (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_{\zeta} \mu'_{\zeta}} \cdots \delta_{\rho_{\zeta} \rho'_{\zeta}} \delta_{\tau_{\zeta} \tau'_{\zeta}} \cdots \} \}^{2l+1}] \\
&(m^2 - \nabla^2)^l \delta^3(\vec{r} - \vec{r}') \frac{1}{(m^2 - \nabla'^2)^{n-1}} \overbrace{\psi^{\lambda'_{\zeta} \mu'_{\zeta}} \cdots}^{2n}(\vec{r}', t) d^3 \vec{r}' \\
&= \frac{1}{2^{2n-1}} \frac{1}{[(2n)!]^2} \sum_{l=0}^{[n-\frac{1}{2}]} [C_{2n}^{2l+1} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)}^{2n-2l-1} \{ \lambda_{\zeta} (\lambda'_{\zeta} (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_{\zeta} \mu'_{\zeta}} \cdots \delta_{\rho_{\zeta} \rho'_{\zeta}} \delta_{\tau_{\zeta} \tau'_{\zeta}} \cdots \} \}^{2l+1}] \frac{1}{(m^2 - \nabla^2)^{n-1}} \overbrace{\psi^{\lambda'_{\zeta} \mu'_{\zeta}} \cdots}^{2n}(\vec{r}, t) \\
&= i \partial_t \underbrace{\psi_{\lambda_{\zeta} \mu_{\zeta}} \cdots}_{2n}(\vec{r}, t) \quad \square
\end{aligned}$$

证明:  $[\psi_{\lambda_{\zeta} \mu_{\zeta}} \cdots(\vec{r}, t), P]$

$$\begin{aligned}
&= [\underbrace{\psi_{\lambda_{\zeta} \mu_{\zeta}} \cdots}_{2n}(\vec{r}, t), \int \underbrace{\psi^{\lambda'_{\zeta} \mu'_{\zeta}} \cdots}_{2n}(\vec{r}', t) \frac{-i \nabla' \gamma^4 (\vec{\gamma} \cdot \nabla' + m) \lambda'_{\zeta}}{(m^2 - \nabla'^2)^n} \overbrace{\psi^{\eta'_{\zeta} \mu'_{\zeta}} \cdots}^{2n}(\vec{r}', t) d^3 \vec{r}'] \\
&= \int [\underbrace{\psi_{\lambda_{\zeta} \mu_{\zeta}} \cdots}_{2n}(\vec{r}, t), \underbrace{\psi^{\lambda'_{\zeta} \mu'_{\zeta}} \cdots}_{2n}(\vec{r}', t) \frac{-i \nabla' \gamma^4 (\vec{\gamma} \cdot \nabla' + m) \lambda'_{\zeta}}{(m^2 - \nabla'^2)^n} \overbrace{\psi^{\eta'_{\zeta} \mu'_{\zeta}} \cdots}^{2n}(\vec{r}', t)] d^3 \vec{r}' \\
&= \int [\underbrace{\psi_{\lambda_{\zeta} \mu_{\zeta}} \cdots}_{2n}(\vec{r}, t), \underbrace{\psi^{\lambda'_{\zeta} \mu'_{\zeta}} \cdots}_{2n}(\vec{r}', t)] \frac{-i \nabla' \gamma^4 (\vec{\gamma} \cdot \nabla' + m) \lambda'_{\zeta}}{(m^2 - \nabla'^2)^n} \overbrace{\psi^{\eta'_{\zeta} \mu'_{\zeta}} \cdots}^{2n}(\vec{r}', t) d^3 \vec{r}' \\
&= \int \frac{1}{2^{2n-1}} \frac{1}{[(2n)!]^2} \sum_{l=0}^{[n-\frac{1}{2}]} [C_{2n}^{2l+1} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)}^{2n-2l-1} \{ \lambda_{\zeta} (\lambda'_{\zeta} (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_{\zeta} \mu'_{\zeta}} \cdots \delta_{\rho_{\zeta} \rho'_{\zeta}} \delta_{\tau_{\zeta} \tau'_{\zeta}} \cdots \} \}^{2l+1}] \\
&(m^2 - \nabla^2)^l \delta^3(\vec{r} - \vec{r}') \frac{-i \nabla' \gamma^4 (\vec{\gamma} \cdot \nabla' + m) \lambda'_{\zeta}}{(m^2 - \nabla'^2)^n} \overbrace{\psi^{\eta'_{\zeta} \mu'_{\zeta}} \cdots}^{2n}(\vec{r}', t) d^3 \vec{r}' \\
&= \frac{1}{2^{2n-1}} \frac{1}{[(2n)!]^2} \\
&\sum_{l=0}^{[n-\frac{1}{2}]} [C_{2n}^{2l+1} \overbrace{(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)}^{2n-2l-1} \{ \lambda_{\zeta} (\lambda'_{\zeta} (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\mu_{\zeta} \mu'_{\zeta}} \cdots \delta_{\rho_{\zeta} \rho'_{\zeta}} \delta_{\tau_{\zeta} \tau'_{\zeta}} \cdots \} \}^{2l+1}] \frac{-i \nabla \gamma^4 (\vec{\gamma} \cdot \nabla + m) \lambda'_{\zeta}}{(m^2 - \nabla^2)^{n-l}} \overbrace{\psi^{\eta'_{\zeta} \mu'_{\zeta}} \cdots}^{2n}(\vec{r}, t) \\
&= -i \nabla \underbrace{\psi_{\lambda_{\zeta} \mu_{\zeta}} \cdots}_{2n}(\vec{r}, t) \quad \square
\end{aligned}$$

# 第二十六章 有质量复粒子的势协变量子化方案

自我评述：本章换成势的角度描述有质量复粒子，复粒子正反粒子不同，数学本质上是复函数，区别于马约拉纳粒子；马约拉纳粒子，正反粒子相同，数学本质上是实函数，后面章节会详细讨论。有质量粒子方案采用与无质量粒子方案相反的步骤，先证明一般自旋粒子情形，再分别研究自旋 $-\frac{1}{2}, 1, \frac{3}{2}, 2$ 的特殊情形。这样做的理由：一是协变量子化新方案经过之前的研究，总体上已经比较明确。二是先证明一般情形，后面的特殊情形就不需再做证明，省去很多麻烦与篇幅，内容更紧凑，也可以更专注于物理。在本章按统一的方式对所有有质量自旋复粒子建立了相应势的量子场论。与无质量粒子一样，也无需知道哈密顿量，就可以按统一的新程式对各种有质量自旋粒子进行了量子化，给出了统一的量子化势对易规则和能量动量算符形式，给出了部分量子彭加莱代数。与无质量粒子一样，角动量算符也只取得部分成功，没有彻底解决，仍需努力。

## 1 Klein-Gordon方程的对易规则

### 1.1 有质量自旋-n的Bargmann-Wigner方程等价于Klein-Gordon方程<sup>[17, 19, 22]</sup>

定义1.1.1.  $\mathbb{X}_a = [im\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)\partial^b]C$ ,  $\mathbb{X}_a(p) = i[m\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)p^b]C$

$$\text{定理1.1.1.} \quad \left\{ \begin{array}{l} [\gamma^a(\varsigma)\partial_a + m]\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}(x) = 0 \\ \psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}(x) \text{全对称} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (-\partial^c\partial_c + m^2)\underbrace{A_{ab\dots}}_n(x) = 0 \\ \delta^{ab}\underbrace{A_{ab\dots}}_n(x) = 0, \partial^a\underbrace{A_{ab\dots}}_n(x) = 0, \underbrace{A_{ab\dots}}_n(x) \text{全对称} \\ \psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}(x) = \frac{1}{2^n} \underbrace{\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a \mathbb{X}_{\eta_\varsigma\xi_\varsigma}^b \dots}_n \cdot \underbrace{A_{ab\dots}}_n(x) \end{array} \right.$$

$$\psi_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=-n}^{-n} E^{n-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2n} [a(\vec{p}, h)\underbrace{U_{\lambda_\varsigma\mu_\varsigma \dots}}_n(\vec{p}, h)e^{ip \cdot x} + b^+(\vec{p}, h)\underbrace{V_{\lambda_\varsigma\mu_\varsigma \dots}}_n(\vec{p}, h)e^{-ip \cdot x}] d^3\vec{p}$$

$$\underbrace{A_{ab\dots}}_n(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=-n}^{-n} \frac{1}{\sqrt{2^n E}} [a(\vec{p}, h)\underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h)e^{ip \cdot x} + b^+(\vec{p}, h)\underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h)e^{-ip \cdot x}] d^3\vec{p}$$

自我评述：把各自的平面波解代入以上两个等价的方程，并利用傅立叶分量相等性便可容易得到以下两个推论。这好比上面的方程是宏观结构，而下面的方程是微观结构，是数学原子。

推论1.1.1.

$$\left\{ \begin{array}{l} (i\gamma^a p_a + m)\underbrace{U_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}}_n(\vec{p}, h) = 0 \\ \underbrace{U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}}_n(\vec{p}, h) \text{全对称}, \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \\ = \frac{1}{(i\sqrt{2})^n} \underbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma} (\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma} \dots}_n \cdot \underbrace{U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}}_n(\vec{p}, h) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (p^c p_c + m^2)\underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) = 0, \delta^{ab}\underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) = 0 \\ p^a \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) = 0, \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \text{全对称} \\ \underbrace{U_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}}_n(\vec{p}, h) \\ = \frac{1}{(2\sqrt{2}m)^n} \underbrace{\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(p) \mathbb{X}_{\eta_\varsigma\xi_\varsigma}^b(p) \dots}_n \cdot \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \end{array} \right.$$

推论1.1.2.

$$\left\{ \begin{array}{l} (-i\gamma^a p_a + m)\underbrace{V_{[\lambda_\varsigma]\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}}_n(\vec{p}, h) = 0 \\ \underbrace{V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}}_n(\vec{p}, h) \text{全对称}, \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) \\ = \frac{1}{(i\sqrt{2})^n} \underbrace{(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma} (\bar{C}\gamma_b)^{\eta_\varsigma\xi_\varsigma} \dots}_n \cdot \underbrace{V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}}_n(\vec{p}, h) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (p^c p_c + m^2)\underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) = 0, \delta^{ab}\underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) = 0 \\ p^a \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) = 0, \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) \text{全对称} \\ \underbrace{V_{\lambda_\varsigma\mu_\varsigma\eta_\varsigma\xi_\varsigma \dots}}_n(\vec{p}, h) \\ = \frac{1}{(2\sqrt{2}m)^n} \underbrace{\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(-p) \mathbb{X}_{\eta_\varsigma\xi_\varsigma}^b(-p) \dots}_n \cdot \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) \end{array} \right.$$

## 1.2 Bargmann-Wigner方程玻色自旋基分解为自旋-1基

证明:

$$\begin{aligned}
& U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}(\vec{p}, n) \\
&= \frac{1}{\sqrt{(2n)!(2n)!(0)!}} \underbrace{u_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots u_{\sigma_\zeta(\vec{p}, -\frac{1}{2})u_{\tau_\zeta(\vec{p}, -\frac{1}{2})\}}}_{2n}}}_{0} \\
&= \underbrace{u_{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots u_{\sigma_\zeta(\vec{p}, \frac{1}{2})u_{\tau_\zeta(\vec{p}, \frac{1}{2})}}}_{2n}} \\
&= \underbrace{U_{\lambda_\zeta \mu_\zeta}(\vec{p}, 1) \cdots U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1)}_n = \frac{1}{\sqrt{(n!n!0)!}} \left(\frac{1}{2\sqrt{2m}}\right)^n \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(p)}_n \cdot \underbrace{\mathbb{X}_{\sigma_\zeta \tau_\zeta}^d(p)}_n \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_{d\}}(\vec{p}, 1)}_n} \\
&= \frac{1}{n! \sqrt{C_{2n}^0}} \left(\frac{1}{2\sqrt{2m}}\right)^n \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(p)}_n \cdot \underbrace{\mathbb{X}_{\sigma_\zeta \tau_\zeta}^d(p)}_n \sqrt{2^0} C_n^0 C_{n-0}^0 \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_{d\}}(\vec{p}, 1)}_n}
\end{aligned}$$

□

证明:

$$\begin{aligned}
& U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}(\vec{p}, n-1) \\
&= \frac{1}{\sqrt{(2n)!(2n-1)!(1)!}} \underbrace{u_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots u_{\sigma_\zeta(\vec{p}, -\frac{1}{2})u_{\tau_\zeta(\vec{p}, -\frac{1}{2})\}}}_{2n-1}}}_{1} \\
&= \frac{1}{\sqrt{(2n)!(2n-1)!(1)!}} \underbrace{u_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots u_{\sigma_\zeta(\vec{p}, \frac{1}{2})u_{\tau_\zeta(\vec{p}, -\frac{1}{2})\}}}_{2n-1}}}_{1} \\
&= \frac{1}{\sqrt{C_{2n}^1}} \left[ \underbrace{u_{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots u_{\sigma_\zeta(\vec{p}, \frac{1}{2})u_{\tau_\zeta(\vec{p}, -\frac{1}{2})}}}_{2n}} + \underbrace{u_{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots u_{\sigma_\zeta(\vec{p}, -\frac{1}{2})u_{\tau_\zeta(\vec{p}, \frac{1}{2})}}}_{2n}} \right] \\
&+ \cdots \\
&+ \left[ \underbrace{u_{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, -\frac{1}{2})\cdots u_{\sigma_\zeta(\vec{p}, \frac{1}{2})u_{\tau_\zeta(\vec{p}, \frac{1}{2})}}}_{2n}} + \underbrace{u_{\lambda_\zeta(\vec{p}, -\frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots u_{\sigma_\zeta(\vec{p}, \frac{1}{2})u_{\tau_\zeta(\vec{p}, \frac{1}{2})}}}_{2n}} \right] \\
&= \frac{1}{\sqrt{C_n^1}} \left[ \underbrace{U_{\lambda_\zeta \mu_\zeta}(\vec{p}, 1) U_{\eta_\zeta \xi_\zeta}(\vec{p}, 1) \cdots U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 0)}_{2n} + \underbrace{U_{\lambda_\zeta \mu_\zeta}(\vec{p}, 1) U_{\eta_\zeta \xi_\zeta}(\vec{p}, 0) \cdots U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1)}_{2n} \right] \\
&+ \cdots + \left[ \underbrace{U_{\lambda_\zeta \mu_\zeta}(\vec{p}, 0) U_{\eta_\zeta \xi_\zeta}(\vec{p}, 1) \cdots U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1)}_{2n} \right] \\
&= \frac{1}{\sqrt{n!(n-1)!1!}} \left(\frac{1}{2\sqrt{2m}}\right)^n \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(p)}_n \underbrace{\mathbb{X}_{\eta_\zeta \xi_\zeta}^b(p)}_n \cdot \underbrace{\mathbb{X}_{\sigma_\zeta \tau_\zeta}^d(p)}_n \underbrace{\varepsilon_{\{a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1) \cdots \varepsilon_c(\vec{p}, 1)\varepsilon_d\}}(\vec{p}, 0)}_n} \\
&= \frac{1}{n! \sqrt{C_{2n}^1}} \left(\frac{1}{2\sqrt{2m}}\right)^n \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(p)}_n \underbrace{\mathbb{X}_{\eta_\zeta \xi_\zeta}^b(p)}_n \cdot \underbrace{\mathbb{X}_{\sigma_\zeta \tau_\zeta}^d(p)}_n \sqrt{2^1} C_n^1 C_{n-1}^0 \underbrace{\varepsilon_{\{a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1) \cdots \varepsilon_c(\vec{p}, 1)\varepsilon_d\}}(\vec{p}, 0)}_n}
\end{aligned}$$

□

证明:

$$\begin{aligned}
& U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}(\vec{p}, n-2) \\
&= \frac{1}{\sqrt{(2n)!(2n-2)!(2)!}} \underbrace{u_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots u_{\sigma_\zeta(\vec{p}, -\frac{1}{2})u_{\tau_\zeta(\vec{p}, -\frac{1}{2})\}}}_{2n-2}}}_{2} \\
&= \frac{1}{\sqrt{(2n)!(2n-2)!(2)!}} \underbrace{u_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})u_{\mu_\zeta(\vec{p}, \frac{1}{2})\cdots u_{\sigma_\zeta(\vec{p}, -\frac{1}{2})u_{\tau_\zeta(\vec{p}, -\frac{1}{2})\}}}_{2n-2}}}_{2} \\
&= \frac{1}{\sqrt{C_{2n}^2}} C_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}^{(\vec{p}, -\frac{1}{2}), (\vec{p}, -\frac{1}{2})} \\
&= \frac{1}{\sqrt{C_{2n}^2}} \left[ \underbrace{C_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}^{(\vec{p}, -\frac{1}{2}), (\vec{p}, -\frac{1}{2})_{1,3,5,\dots}}}_{2n} + \underbrace{C_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}^{(\vec{p}, -\frac{1}{2}), (\vec{p}, -\frac{1}{2})_{rest}}}_{2n} \right] \\
&= \frac{1}{\sqrt{C_{2n}^2}} \left[ \sqrt{2^0} C_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}^{(\vec{p}, -1)} \underbrace{U_{\lambda_\zeta \mu_\zeta}(\vec{p}, 1) \cdots U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1)}_n + \sqrt{2^2} C_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}^{(\vec{p}, 0), (\vec{p}, 0)} \underbrace{U_{\lambda_\zeta \mu_\zeta}(\vec{p}, 1) \cdots U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1)}_n \right] \\
&= \frac{1}{n! \sqrt{C_{2n}^2}} \left(\frac{1}{2\sqrt{2m}}\right)^n \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(p)}_n \underbrace{\mathbb{X}_{\eta_\zeta \xi_\zeta}^b(p)}_n \cdot \underbrace{\mathbb{X}_{\sigma_\zeta \tau_\zeta}^d(p)}_n \\
&= \left[ \sqrt{2^0} C_n^0 C_{n-0}^0 \underbrace{\varepsilon_{\{a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1) \cdots \varepsilon_c(\vec{p}, 1)\varepsilon_d\}}(\vec{p}, -1)}_n + \sqrt{2^2} C_n^2 C_{n-2}^0 \underbrace{\varepsilon_{\{a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1) \cdots \varepsilon_c(\vec{p}, 0)\varepsilon_d\}}(\vec{p}, 0)}_n \right]
\end{aligned}$$

□



证明:

$$\begin{aligned}
& \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2n}(\vec{p}, n-3) \\
&= \frac{1}{\sqrt{(2n)!(2n-3)!(3)!}} \underbrace{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})}_{2n-3} \\
&= \frac{1}{\sqrt{(2n)!(2n-3)!(3)!}} \underbrace{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) u_{\xi_\zeta}(\vec{p}, -\frac{1}{2}) u_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})}_{2n} \\
&= \frac{1}{\sqrt{C_{2n}^3}} C_{\lambda_\zeta(\vec{p}, \frac{1}{2}), \mu_\zeta(\vec{p}, \frac{1}{2}), \sigma_\zeta(\vec{p}, \frac{1}{2}), \tau_\zeta(\vec{p}, \frac{1}{2})}^{(\vec{p}, -\frac{1}{2}), (\vec{p}, -\frac{1}{2}), (\vec{p}, -\frac{1}{2})} \\
&= \frac{1}{\sqrt{C_{2n}^3}} \left[ C_{\lambda_\zeta(\vec{p}, \frac{1}{2}), \mu_\zeta(\vec{p}, \frac{1}{2}), \sigma_\zeta(\vec{p}, \frac{1}{2}), \tau_\zeta(\vec{p}, \frac{1}{2})}^{(\vec{p}, -\frac{1}{2}), (\vec{p}, -\frac{1}{2}), (\vec{p}, -\frac{1}{2}), 1, 3, 5, \dots, (\vec{p}, -\frac{1}{2})} + C_{\lambda_\zeta(\vec{p}, \frac{1}{2}), \mu_\zeta(\vec{p}, \frac{1}{2}), \sigma_\zeta(\vec{p}, \frac{1}{2}), \tau_\zeta(\vec{p}, \frac{1}{2})}^{(\vec{p}, -\frac{1}{2}), (\vec{p}, -\frac{1}{2}), (\vec{p}, -\frac{1}{2})|_{rest}} \right] \\
&= \frac{1}{\sqrt{C_{2n}^3}} \left[ \sqrt{2} P_{\lambda_\zeta \mu_\zeta}^{(\vec{p}, -1), (\vec{p}, 0)}(\vec{p}, 1) \cdots U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1) + \sqrt{2^3} C_{\lambda_\zeta \mu_\zeta}^{(\vec{p}, 0), (\vec{p}, 0), (\vec{p}, 0)}(\vec{p}, 1) \cdots U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1) \right] \\
&= \frac{1}{n! \sqrt{C_{2n}^3}} \left( \frac{1}{2\sqrt{2m}} \right)^n \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(p) \mathbb{X}_{\eta_\zeta \xi_\zeta}^b(p) \cdots \mathbb{X}_{\sigma_\zeta \tau_\zeta}^d(p)}_n \\
& \left[ \sqrt{2^1} C_n^1 C_{n-1}^1 \underbrace{\varepsilon_{\{a}(\vec{p}, 1) \varepsilon_{b}(\vec{p}, 1) \cdots \varepsilon_{c}(\vec{p}, 0) \varepsilon_{d\}}(\vec{p}, -1)}_n + \sqrt{2^3} C_n^3 C_{n-3}^0 \underbrace{\varepsilon_{\{a}(\vec{p}, 1) \varepsilon_{b}(\vec{p}, 0) \cdots \varepsilon_{c}(\vec{p}, 0) \varepsilon_{d\}}(\vec{p}, 0)}_n \right] \quad \square
\end{aligned}$$

一般情形:

定理1.2.1.

$$\begin{cases}
\underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2n}(\vec{p}, n-2k) = \frac{1}{n! \sqrt{C_{2n}^{2k}}} \left( \frac{1}{2\sqrt{2m}} \right)^n \sum_{l=0}^{k|(n-k)} \sqrt{2^{2l}} C_n^{2l} C_{n-2l}^{k-l} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^{a_1}(p) \mathbb{X}_{\eta_\zeta \xi_\zeta}^{a_2}(p) \cdots \mathbb{X}_{\sigma_\zeta \tau_\zeta}^{a_n}(p)}_n \\
\varepsilon_{\{a_1}(\vec{p}, -1) \cdots \varepsilon_{a_{k-l}}(\vec{p}, -1) | \varepsilon_{a_{k-l+1}}(\vec{p}, 0) \cdots \varepsilon_{a_k}(\vec{p}, 0) | \varepsilon_{a_{k+1}}(\vec{p}, 1) \cdots \varepsilon_{a_n\}}(\vec{p}, 1) \\
\underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2n}(\vec{p}, n-2k-1) = \frac{1}{n! \sqrt{C_{2n}^{2k+1}}} \left( \frac{1}{2\sqrt{2m}} \right)^n \sum_{l=0}^{k|(n-1-k)} \sqrt{2^{2l+1}} C_n^{2l+1} C_{n-2l-1}^{k-l} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^{a_1}(p) \mathbb{X}_{\eta_\zeta \xi_\zeta}^{a_2}(p) \cdots \mathbb{X}_{\sigma_\zeta \tau_\zeta}^{a_n}(p)}_n \\
\varepsilon_{\{a_1}(\vec{p}, -1) \cdots \varepsilon_{a_{k-l}}(\vec{p}, -1) | \varepsilon_{a_{k-l+1}}(\vec{p}, 0) \cdots \varepsilon_{a_k}(\vec{p}, 0) | \varepsilon_{a_{k+1}}(\vec{p}, 1) \cdots \varepsilon_{a_n\}}(\vec{p}, 1)
\end{cases}$$

证明:

$$\begin{aligned}
& \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2n}(\vec{p}, n-2k) \\
&= \frac{1}{\sqrt{(2n)!(2n-2k)!(2k)!}} \underbrace{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \cdots u_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})}_{2n-2k} \\
&= \frac{1}{\sqrt{C_{2n}^{2k}}} C_{\lambda_\zeta(\vec{p}, \frac{1}{2}), \mu_\zeta(\vec{p}, \frac{1}{2}), \sigma_\zeta(\vec{p}, \frac{1}{2}), \tau_\zeta(\vec{p}, \frac{1}{2})}^{(\vec{p}, -\frac{1}{2}), \dots, (\vec{p}, -\frac{1}{2})} \\
&= \frac{1}{\sqrt{C_{2n}^{2k}}} \sum_{l=0}^{k|(n-k)} C_{\lambda_\zeta(\vec{p}, \frac{1}{2}), \mu_\zeta(\vec{p}, \frac{1}{2}), \sigma_\zeta(\vec{p}, \frac{1}{2}), \tau_\zeta(\vec{p}, \frac{1}{2})}^{(\vec{p}, -\frac{1}{2}), \dots, (\vec{p}, -\frac{1}{2}) |_{1,3,\dots}^{2k-2l}, \dots, (\vec{p}, -\frac{1}{2}), \dots, (\vec{p}, -\frac{1}{2}) |_{1,3,\dots}^{2l}} \\
&= \frac{1}{\sqrt{C_{2n}^{2k}}} \sum_{l=0}^{k|(n-k)} \sqrt{2^{2l}} C_{\lambda_\zeta \mu_\zeta}^{(\vec{p}, -1), \dots, (\vec{p}, -1) |_{1,3,\dots}^{k-l}, \dots, (\vec{p}, 0), \dots, (\vec{p}, 0) |_{1,3,\dots}^{2l}}(\vec{p}, 1) \cdots U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1) \\
&= \frac{1}{n! \sqrt{C_{2n}^{2k}}} \left( \frac{1}{2\sqrt{2m}} \right)^n \sum_{l=0}^{k|(n-k)} \sqrt{2^{2l}} C_n^{2l} C_{n-2l}^{k-l} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^{a_1}(p) \mathbb{X}_{\eta_\zeta \xi_\zeta}^{a_2}(p) \cdots \mathbb{X}_{\sigma_\zeta \tau_\zeta}^{a_n}(p)}_n \\
& \varepsilon_{\{a_1}(\vec{p}, -1) \cdots \varepsilon_{a_{k-l}}(\vec{p}, -1) | \varepsilon_{a_{k-l+1}}(\vec{p}, 0) \cdots \varepsilon_{a_k}(\vec{p}, 0) | \varepsilon_{a_{k+1}}(\vec{p}, 1) \cdots \varepsilon_{a_n\}}(\vec{p}, 1) \quad \square
\end{aligned}$$

证明:

$$\begin{aligned}
& U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, n-2k-1) \\
&= \frac{1}{\sqrt{(2n)!(2n-2k-1)!(2k+1)!}} \underbrace{u_{\lambda_s}(\vec{p}, \frac{1}{2}) u_{\mu_s}(\vec{p}, \frac{1}{2}) \cdots u_{\sigma_s}(\vec{p}, -\frac{1}{2})}_{2n-2k-1} u_{\tau_s}(\vec{p}, -\frac{1}{2}) \\
&= \frac{1}{\sqrt{C_{2n}^{2k+1}}} C_{\lambda_s(\vec{p}, \frac{1}{2}) \mu_s(\vec{p}, \frac{1}{2}) \dots \sigma_s(\vec{p}, -\frac{1}{2}) \tau_s(\vec{p}, -\frac{1}{2})} \\
&= \frac{1}{\sqrt{C_{2n}^{2k+1}}} \sum_{l=0}^{k|(n-1-k)} \underbrace{C_{\lambda_s(\vec{p}, \frac{1}{2}) \mu_s(\vec{p}, \frac{1}{2}) \dots \sigma_s(\vec{p}, -\frac{1}{2}) \tau_s(\vec{p}, -\frac{1}{2})}}_{2n} \underbrace{C_{\lambda_s(\vec{p}, -\frac{1}{2}) \mu_s(\vec{p}, -\frac{1}{2}) \dots \sigma_s(\vec{p}, -\frac{1}{2}) \tau_s(\vec{p}, -\frac{1}{2})}}_{2l+1} \underbrace{C_{\lambda_s(\vec{p}, -\frac{1}{2}) \mu_s(\vec{p}, -\frac{1}{2}) \dots \sigma_s(\vec{p}, -\frac{1}{2}) \tau_s(\vec{p}, -\frac{1}{2})}}_{2l+1} \\
&= \frac{1}{\sqrt{C_{2n}^{2k+1}}} \sum_{l=0}^{k|(n-1-k)} \sqrt{2^{2l+1}} C_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, 1) \cdot U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, 1) \\
&= \frac{1}{n! \sqrt{C_{2n}^{2k+1}}} \left( \frac{1}{2\sqrt{2m}} \right)^n \sum_{l=0}^{k|(n-1-k)} \sqrt{2^{2l+1}} C_n^{2l+1} C_{n-2l-1}^{k-l} \underbrace{\mathbb{X}_{\lambda_s \mu_s}^{a_1}(p) \mathbb{X}_{\eta_s \xi_s}^{a_2}(p) \cdots \mathbb{X}_{\sigma_s \tau_s}^{a_n}(p)}_n \\
& \varepsilon_{\{a_1(\vec{p}, -1) \cdots \varepsilon_{a_{k-l}}(\vec{p}, -1) | \varepsilon_{a_{k-l+1}}(\vec{p}, 0) \cdots \varepsilon_{a_k}(\vec{p}, 0) | \varepsilon_{a_{k+1}}(\vec{p}, 1) \cdots \varepsilon_{a_n}(\vec{p}, 1)\}}
\end{aligned}$$

□

### 1.3 Klein-Gordon方程自旋基的分解

$$\text{定理1.3.1. } \varepsilon_{a \dots bc \dots d}(\vec{p}, h) = \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+h}^{n'+h'} C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} \varepsilon_{a \dots b}(\vec{p}, h-h') \varepsilon_{c \dots d}(\vec{p}, h')$$

$$\begin{aligned}
& \text{证明: } U_{\lambda_s \mu_s \dots \sigma_s \tau_s \lambda'_s \mu'_s \dots \sigma'_s \tau'_s}(\vec{p}, h) = \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+h}^{n'+h'} C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h-h') U_{\lambda'_s \mu'_s \dots \sigma'_s \tau'_s}(\vec{p}, h') \\
& \Rightarrow \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_s \mu_s} \cdots (\bar{C}\gamma_b)^{\sigma_s \tau_s} (\bar{C}\gamma_c)^{\lambda'_s \mu'_s} \cdots (\bar{C}\gamma_d)^{\sigma'_s \tau'_s}}^n U_{\lambda_s \mu_s \dots \sigma_s \tau_s \lambda'_s \mu'_s \dots \sigma'_s \tau'_s}(\vec{p}, h) \\
&= \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_s \mu_s} \cdots (\bar{C}\gamma_b)^{\sigma_s \tau_s} (\bar{C}\gamma_c)^{\lambda'_s \mu'_s} \cdots (\bar{C}\gamma_d)^{\sigma'_s \tau'_s}}^n \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+h}^{n'+h'} C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h-h') U_{\lambda'_s \mu'_s \dots \sigma'_s \tau'_s}(\vec{p}, h') \\
& \Leftrightarrow \varepsilon_{a \dots bc \dots d}(\vec{p}, h) = \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+h}^{n'+h'} C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} \varepsilon_{a \dots b}(\vec{p}, h-h') \varepsilon_{c \dots d}(\vec{p}, h')
\end{aligned}$$

□

#### 推论1.3.1.

$$\varepsilon_{a \dots bc}(\vec{p}, h) = \frac{\sqrt{C_{n+h}^{n+h}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{a \dots b}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^{n+h} C_{n-h}^{n-h}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{a \dots b}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^{n-h}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{a \dots b}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1)$$

#### 定理1.3.2.

$$\varepsilon_{a_1 \dots a_n}(\vec{p}, h) = \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_2=1}^{-1} \cdots \sum_{h_n=1}^{-1} \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_1)! \cdots (1+h_n)!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_1)! \cdots (1-h_n)!}} \varepsilon_{a_1}(\vec{p}, h_1) \cdots \varepsilon_{a_n}(\vec{p}, h_n); h_1 := h - \sum_{i=2}^n h_i$$

$$\begin{aligned}
& \text{证明: } \varepsilon_{a_1 a_2 \dots a_n}(\vec{p}, h) = \sum_{h_n=1}^{-1} \frac{\sqrt{C_{n+h}^{1+h_n} C_{n-h}^{1-h_n}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{a_1 a_2 \dots a_{n-1}}(\vec{p}, h-h_n) \varepsilon_{a_n}(\vec{p}, h_n) \\
&= \sum_{h_n=1}^{-1} \frac{\sqrt{C_{n+h}^{1+h_n} C_{n-h}^{1-h_n}}}{\sqrt{C_{2n}^{2n}}} \sum_{h_{n-1}=1}^{-1} \frac{\sqrt{C_{(n-1)+h-h_n}^{1+h_{n-1}} C_{(n-1)-(h-h_n)}^{1-h_{n-1}}}}{\sqrt{C_{2(n-1)}^{2(n-1)}}} \varepsilon_{a_1 a_2 \dots a_{n-2}}(\vec{p}, h-h_n-h_{n-1}) \varepsilon_{a_{n-1}}(\vec{p}, h_{n-1}) \varepsilon_{a_n}(\vec{p}, h_n) \\
&= \frac{\sqrt{2!2!(2n-4)!}}{\sqrt{(2n)!}} \sum_{h_n=1}^{-1} \sum_{h_{n-1}=1}^{-1} \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_n)!(1+h_{n-1})![(n+h)-(1+h_n)-(1+h_{n-1})!]}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_n)!(1-h_{n-1})![(n-h)-(1-h_n)-(1-h_{n-1})!]}} \\
& \varepsilon_{a_1 a_2 \dots a_{n-2}}(\vec{p}, h-h_n-h_{n-1}) \varepsilon_{a_{n-1}}(\vec{p}, h_{n-1}) \varepsilon_{a_n}(\vec{p}, h_n) \\
&= \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_n=1}^{-1} \sum_{h_{n-1}=1}^{-1} \cdots \sum_{h_2=1}^{-1} \varepsilon_{a_1}(\vec{p}, h-h_n-h_{n-1} \cdots -h_2) \varepsilon_{a_2}(\vec{p}, h_2) \cdots \varepsilon_{a_{n-1}}(\vec{p}, h_{n-1}) \varepsilon_{a_n}(\vec{p}, h_n) \\
& \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_n)!(1+h_{n-1})! \cdots (1+h_2)![(n+h)-(1+h_n)-(1+h_{n-1}) \cdots -(1+h_2)]}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_n)!(1-h_{n-1})! \cdots (1-h_2)![(n-h)-(1-h_n)-(1-h_{n-1}) \cdots -(1-h_2)]}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_2=1}^{-1} \cdots \sum_{h_{n-1}=1}^{-1} \sum_{h_n=1}^{-1} \varepsilon_{a_1}(\vec{p}, h_1) \varepsilon_{a_2}(\vec{p}, h_2) \cdots \varepsilon_{a_{n-1}}(\vec{p}, h_{n-1}) \varepsilon_{a_n}(\vec{p}, h_n) \\
&\quad \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_1)!(1+h_2)\cdots(1+h_{n-1})!(1+h_n)!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_1)!(1-h_2)\cdots(1-h_{n-1})!(1-h_n)!}}; h_1 := h - \sum_{i=2}^n h_i \\
&= \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_2=1}^{-1} \cdots \sum_{h_n=1}^{-1} \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_1)\cdots(1+h_n)!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_1)\cdots(1-h_n)!}} \varepsilon_{a_1}(\vec{p}, h_1) \cdots \varepsilon_{a_n}(\vec{p}, h_n); h_1 := h - \sum_{i=2}^n h_i \quad \square
\end{aligned}$$

推论1.3.2.  $\varepsilon_{a_1 \cdots a_n}(\vec{p}, h); h_1 := h - \sum_{i=2}^n h_i$

$$\begin{aligned}
&= \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_2=1}^{-1} \cdots \sum_{h_n=1}^{-1} \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_1)\cdots(1+h_n)!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_1)\cdots(1-h_n)!}} \varepsilon_{a_1}(\vec{p}, h_1) \cdots \varepsilon_{a_n}(\vec{p}, h_n) [\delta(h_1 - 1) + \delta(h_1) + \delta(h_1 + 1)] \\
&= \frac{\sqrt{(2!)^n}}{\sqrt{(2n)!}} \sum_{h_1=1}^{-1} \sum_{h_2=1}^{-1} \cdots \sum_{h_n=1}^{-1} \frac{\sqrt{(n+h)!}}{\sqrt{(1+h_1)\cdots(1+h_n)!}} \frac{\sqrt{(n-h)!}}{\sqrt{(1-h_1)\cdots(1-h_n)!}} \varepsilon_{a_1}(\vec{p}, h_1) \varepsilon_{a_2}(\vec{p}, h_2) \cdots \varepsilon_{a_n}(\vec{p}, h_n) \delta(h - \sum_{i=1}^n h_i)
\end{aligned}$$

## 1.4 数学归纳法严格证明Klein-Gordon方程自旋基的完整分解

定理1.4.1.

$$\begin{cases} \varepsilon_{\underbrace{a \cdots b \cdots c \cdots}_n}(\vec{p}, n - 2k) = \frac{1}{\sqrt{C_{2n}^{2k}}} \sum_{i=0}^{\min(k, n-k)} \frac{2^i}{(n-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-k-i} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{2i} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{k-i} \\ \varepsilon_{\underbrace{a \cdots b \cdots c \cdots}_n}(\vec{p}, n - 2k - 1) = \frac{1}{\sqrt{C_{2n}^{2k+1}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i \sqrt{2}}{(n-k-i-1)!(2i+1)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-k-i-1} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{2i+1} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{k-i} \end{cases}$$

证明: 采用数学归纳法证明此定理。

第一步:  $n' = 1$ 时成立:

$$\begin{cases} \varepsilon_{\underbrace{a \cdots b \cdots c \cdots}_1}(\vec{p}, 1 - 2k) = \frac{1}{\sqrt{C_2^{2k}}} \sum_{i=0}^{\min(k, 1-k)} \frac{2^i}{(1-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{1-k-i} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{2i} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{k-i} \\ \varepsilon_{\underbrace{a \cdots b \cdots c \cdots}_1}(\vec{p}, 1 - 2k - 1) = \frac{1}{\sqrt{C_2^{2k+1}}} \sum_{i=0}^{\min(k, 1-1-k)} \frac{2^i \sqrt{2}}{(1-k-i-1)!(2i+1)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{1-k-i-1} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{2i+1} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{k-i} \end{cases}$$

第二步: 假设 $n' = n - 1$ 时成立:

$$\begin{cases} \varepsilon_{\underbrace{a \cdots b \cdots c \cdots}_{n-1}}(\vec{p}, (n-1) - 2k) = \frac{1}{\sqrt{C_{2n-2}^{2k}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i}{(n-1-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-1-k-i} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{2i} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{k-i} \\ \varepsilon_{\underbrace{a \cdots b \cdots c \cdots}_{n-1}}(\vec{p}, (n-1) - 2k - 1) = \frac{1}{\sqrt{C_{2n-2}^{2k+1}}} \sum_{i=0}^{\min(k, n-1-1-k)} \frac{2^i \sqrt{2}}{(n-1-k-i-1)!(2i+1)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-1-k-i-1} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{2i+1} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{k-i} \end{cases}$$

第三步:  $n' = n$ 时, 1:

$$\begin{aligned}
&n! \varepsilon_{\underbrace{a \cdots b \cdots c \cdots d}_{n}}(\vec{p}, h) \\
&= \frac{\sqrt{C_{2n}^{n+h}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{a \cdots b \cdots c}_{n-1}}(\vec{p}, h-1) \varepsilon_d(\vec{p}, 1) + \frac{\sqrt{C_{2n}^{n+h}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{a \cdots b \cdots c}_{n-1}}(\vec{p}, h) \varepsilon_d(\vec{p}, 0) + \frac{\sqrt{C_{2n}^{n-h}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{a \cdots b \cdots c}_{n-1}}(\vec{p}, h+1) \varepsilon_d(\vec{p}, -1) \\
&\Rightarrow n! \varepsilon_{\underbrace{a \cdots b \cdots c \cdots d}_{n}}(\vec{p}, n - 2k) = \frac{\sqrt{C_{2n}^{2n-2k}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{a \cdots b \cdots c}_{n-1}}(\vec{p}, (n-1) - 2k) \varepsilon_d(\vec{p}, 1) \\
&+ \frac{\sqrt{C_{2n}^{2n-2k}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{a \cdots b \cdots c}_{n-1}}(\vec{p}, (n-1) - 2(k-1) - 1) \varepsilon_d(\vec{p}, 0) + \frac{\sqrt{C_{2n}^{2n-2k}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{a \cdots b \cdots c}_{n-1}}(\vec{p}, (n-1) - 2(k-1)) \varepsilon_d(\vec{p}, -1) \\
&\Leftrightarrow n! \varepsilon_{\underbrace{a \cdots b \cdots c \cdots d}_{n}}(\vec{p}, n - 2k) \\
&= \frac{\sqrt{C_{2n}^{2n-2k}}}{\sqrt{C_{2n}^{2n}}} \frac{1}{\sqrt{C_{2n-2}^{2k}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i}{(n-1-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-1-k-i} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{2i} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{k-i} \varepsilon_d(\vec{p}, 1) \\
&+ \frac{\sqrt{C_{2n}^{2n-2k}}}{\sqrt{C_{2n}^{2n}}} \frac{1}{\sqrt{C_{2n-2}^{2k-1}}} \sum_{i=0}^{\min(k-1, n-1-1-k+1)} \frac{2^i \sqrt{2}}{(n-1-k+1-i-1)!(2i+1)!(k-1-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-1-k+1-i-1} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{2i+1} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{k-1-i} \varepsilon_d(\vec{p}, 0) \\
&+ \frac{\sqrt{C_{2n}^{2n-2k}}}{\sqrt{C_{2n}^{2n}}} \frac{1}{\sqrt{C_{2n-2}^{2k-2}}} \sum_{i=0}^{\min(k-1, n-1-k+1)} \frac{2^i}{(n-1-k+1-i)!(2i)!(k-1-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-1-k+1-i} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{2i} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{k-1-i} \varepsilon_d(\vec{p}, -1)
\end{aligned}$$

$$\begin{aligned}
n\varepsilon_{\underbrace{a \cdot b \cdot c \cdot d}_{n}}(\vec{p}, n-2k) &= \frac{\sqrt{C_{2n-2k}^2}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i}{(n-1-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-1-k-i} \varepsilon_{d\}}(\vec{p}, 1) \\
&+ \frac{\sqrt{C_{2n-2k}^1 C_{2k}^1}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k-1}}} \sum_{i=0}^{\min(k-1, n-1-k)} \frac{2^i \sqrt{2}}{(n-1-k-i)!(2i+1)!(k-1-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-1-k-i} \varepsilon_{d\}}(\vec{p}, 0) \\
&+ \frac{\sqrt{C_{2k}^2}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k-2}}} \sum_{i=0}^{\min(k-1, n-k)} \frac{2^i}{(n-k-i)!(2i)!(k-1-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-k-i} \varepsilon_{d\}}(\vec{p}, -1) \\
\Leftrightarrow n\varepsilon_{\underbrace{a \cdot b \cdot c}_{n}}(\vec{p}, n-2k) &= \frac{\sqrt{C_{2n-2k}^2}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i}{(n-1-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c\}}}_{n-k-i}(\vec{p}, -1) \\
&+ \frac{\sqrt{C_{2n-2k}^1 C_{2k}^1}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k-1}}} \sum_{i=0}^{\min(k-1, n-1-k)} \frac{2^i \sqrt{2}}{(n-1-k-i)!(2i+1)!(k-1-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c\}}}_{n-1-k-i}(\vec{p}, -1) \\
&+ \frac{\sqrt{C_{2k}^2}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k-2}}} \sum_{i=0}^{\min(k-1, n-k)} \frac{2^i}{(n-k-i)!(2i)!(k-1-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c\}}}_{n-k-i}(\vec{p}, -1) \\
\Leftrightarrow n\varepsilon_{\underbrace{a \cdot b \cdot c}_{n}}(\vec{p}, n-2k) &= \frac{\sqrt{C_{2n-2k}^2}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i}{(n-1-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c\}}}_{n-k-i}(\vec{p}, -1) \\
&+ \frac{\sqrt{C_{2n-2k}^1 C_{2k}^1}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k-1}}} \sum_{i=1}^{\min(k, n-k)} \frac{2^{i-1} \sqrt{2}}{(n-k-i)!(2i-1)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c\}}}_{n-k-i}(\vec{p}, -1) \\
&+ \frac{\sqrt{C_{2k}^2}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k-2}}} \sum_{i=0}^{\min(k-1, n-k)} \frac{2^i}{(n-k-i)!(2i)!(k-1-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c\}}}_{n-k-i}(\vec{p}, -1) \\
&= \frac{n}{\sqrt{C_{2n}^2}} \sum_{i=0}^{\min(k, n-k)} \frac{2^i}{(n-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c\}}}_{n-k-i}(\vec{p}, -1) \\
\Leftrightarrow \varepsilon_{\underbrace{a \cdot b \cdot c}_{n}}(\vec{p}, n-2k) &= \frac{1}{\sqrt{C_{2n}^2}} \sum_{i=0}^{\min(k, n-k)} \frac{2^i}{(n-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c\}}}_{n-k-i}(\vec{p}, -1)
\end{aligned}$$

同理  $n' = n$  时, 2:

$$\begin{aligned}
n!\varepsilon_{\underbrace{a \cdot b \cdot c \cdot d}_{n}}(\vec{p}, h) &= \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \cdot b \cdot c}_{n-1}}(\vec{p}, h-1) \varepsilon_{d\}}(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \cdot b \cdot c}_{n-1}}(\vec{p}, h) \varepsilon_{d\}}(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \cdot b \cdot c}_{n-1}}(\vec{p}, h+1) \varepsilon_{d\}}(\vec{p}, -1) \\
\Rightarrow n!\varepsilon_{\underbrace{a \cdot b \cdot c \cdot d}_{n}}(\vec{p}, n-2k-1) &= \frac{\sqrt{C_{n+n-2k-1}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \cdot b \cdot c}_{n-1}}(\vec{p}, (n-1)-2k-1) \varepsilon_{d\}}(\vec{p}, 1) + \frac{\sqrt{C_{n+n-2k-1}^1 C_{n-n+2k+1}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \cdot b \cdot c}_{n-1}}(\vec{p}, (n-1)-2k) \varepsilon_{d\}}(\vec{p}, 0) + \\
&\frac{\sqrt{C_{n-n+2k+1}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \cdot b \cdot c}_{n-1}}(\vec{p}, (n-1)-2(k-1)-1) \varepsilon_{d\}}(\vec{p}, -1) \\
&= \frac{\sqrt{C_{n+n-2k-1}^2}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k+1}}} \sum_{i=0}^{\min(k, n-1-1-k)} \frac{2^i \sqrt{2}}{(n-1-k-i-1)!(2i+1)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-1-k-i-1} \varepsilon_{d\}}(\vec{p}, 1) \\
&+ \frac{\sqrt{C_{n+n-2k-1}^1 C_{n-n+2k+1}^1}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i}{(n-1-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-1-k-i} \varepsilon_{d\}}(\vec{p}, 0) \\
&+ \frac{\sqrt{C_{n-n+2k+1}^2}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k-1}}} \sum_{i=0}^{\min(k-1, n-1-1-k+1)} \frac{2^i \sqrt{2}}{(n-1-k+1-i-1)!(2i+1)!(k-1-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c(\vec{p}, -1)\}}}_{n-1-k+1-i-1} \varepsilon_{d\}}(\vec{p}, -1) \\
\Leftrightarrow n\varepsilon_{\underbrace{a \cdot b \cdot c}_{n}}(\vec{p}, n-2k-1) &= \frac{\sqrt{C_{n-2k-1}^2}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k+1}}} \sum_{i=0}^{\min(k, n-2-k)} \frac{2^i \sqrt{2}}{(n-2-k-i)!(2i+1)!(k-i)!} \underbrace{\varepsilon_{\{a(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_c\}}}_{n-1-k-i}(\vec{p}, -1)
\end{aligned}$$

$$\begin{aligned}
& + \frac{\sqrt{C_{2n-2k-1}^1 C_{2k+1}^1}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i}{(n-1-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_{\{a}(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_{c\}}(\vec{p}, -1)}_{\substack{n-1-k-i \\ 2i+1 \\ k-i}} \\
& + \frac{\sqrt{C_{2k+1}^2}}{\sqrt{C_{2n}^2}} \frac{1}{\sqrt{C_{2n-2}^{2k-1}}} \sum_{i=0}^{\min(k-1, n-1-k)} \frac{2^i \sqrt{2}}{(n-1-k-i)!(2i+1)!(k-1-i)!} \underbrace{\varepsilon_{\{a}(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_{c\}}(\vec{p}, -1)}_{\substack{n-1-k-i \\ 2i+1 \\ k-i}} \\
& = \frac{n}{\sqrt{C_{2n}^{2k+1}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i \sqrt{2}}{(n-k-i-1)!(2i+1)!(k-i)!} \underbrace{\varepsilon_{\{a}(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_{c\}}(\vec{p}, -1)}_{\substack{n-k-i-1 \\ 2i+1 \\ k-i}} \\
& \Leftrightarrow \underbrace{\varepsilon_{a \cdots b \cdots c}(\vec{p}, n-2k-1)}_n = \frac{1}{\sqrt{C_{2n}^{2k+1}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i \sqrt{2}}{(n-k-i-1)!(2i+1)!(k-i)!} \underbrace{\varepsilon_{\{a}(\vec{p}, 1) \cdots \varepsilon_b(\vec{p}, 0) \cdots \varepsilon_{c\}}(\vec{p}, -1)}_{\substack{n-k-i-1 \\ 2i+1 \\ k-i}}
\end{aligned}$$

此步证明了  $n' = n$  时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。  $\square$

以前只是靠直觉、猜测、试探和验证得到了这个定理，时隔两年多(2019-2022)我终于用数学归纳法严格证明了以上定理，看来长久坚持与持续不断的深入思考十分重要，有时候比兴趣更重要。这为完全证明 Behrends-Fronsdal 投影算子公式奠定了基础。

推论1.4.1.

$$\begin{cases} \underbrace{\varepsilon_{a \cdots b \cdots c}(\vec{p}, n-2k)}_n \underbrace{\eta_{a'}^a \cdots \eta_{b'}^b \cdots \eta_{c'}^c \cdots}_n = (-1)^{n-2k} \underbrace{\varepsilon_{a' \cdots b' \cdots c'}(\vec{p}, 2k-n)}_n \\ \underbrace{\varepsilon_{a \cdots b \cdots c}(\vec{p}, n-2k-1)}_n \underbrace{\eta_{a'}^a \cdots \eta_{b'}^b \cdots \eta_{c'}^c \cdots}_n = (-1)^{n-2k-1} \underbrace{\varepsilon_{a' \cdots b' \cdots c'}(\vec{p}, 2k+1-n)}_n \end{cases}$$

$$\text{推论1.4.2. } \underbrace{\bar{\varepsilon}_{abc}(\vec{p}, h)}_n := \underbrace{\varepsilon_{abc}(\vec{p}, h)}_n \underbrace{\eta_{a'}^a \eta_{b'}^b \eta_{c'}^c \cdots}_n = (-1)^h \underbrace{\varepsilon_{a' b' c'}(\vec{p}, -h)}_n$$

## 1.5 Klein-Gordon方程准投影算子的递推关系

推论1.5.1.

$$\begin{cases} \underbrace{\varepsilon_{a \cdots bc}(\vec{p}, h)}_n \bar{\varepsilon}^c(\vec{p}, 1) = \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \underbrace{\varepsilon_{a \cdots b}(\vec{p}, h-1)}_{n-1} \\ \underbrace{\varepsilon_{a \cdots bc}(\vec{p}, h)}_n \bar{\varepsilon}^c(\vec{p}, 0) = \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \underbrace{\varepsilon_{a \cdots b}(\vec{p}, h)}_{n-1} \\ \underbrace{\varepsilon_{a \cdots bc}(\vec{p}, h)}_n \bar{\varepsilon}^c(\vec{p}, -1) = \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \underbrace{\varepsilon_{a \cdots b}(\vec{p}, h+1)}_{n-1} \end{cases}$$

$$[\Rightarrow] \sum_{h=(n-1)}^{-(n-1)} \underbrace{\varepsilon_{a \cdots b}(\vec{p}, h)}_{n-1} \underbrace{\bar{\varepsilon}_{a' \cdots b'}(\vec{p}, h)}_{n-1} = \frac{2(n-1)+1}{2n+1} \left[ \sum_{h=n}^{-n} \underbrace{\varepsilon_{a \cdots bc}(\vec{p}, h)}_n \underbrace{\bar{\varepsilon}_{a' \cdots b' c'}(\vec{p}, h)}_n \right] \left[ \sum_{h'=1}^{-1} \varepsilon^{c'}(\vec{p}, h') \bar{\varepsilon}^c(\vec{p}, h') \right]$$

$$\text{推论1.5.2. } \sum_{h=(n-n')}^{-(n-n')} \underbrace{\varepsilon_{a \cdots b}(\vec{p}, h)}_{n-n'} \underbrace{\bar{\varepsilon}_{a' \cdots b'}(\vec{p}, h)}_{n-n'}$$

$$= \frac{2(n-n')+1}{2n+1} \left[ \sum_{h=n}^{-n} \underbrace{\varepsilon_{a \cdots bc \cdots d}(\vec{p}, h)}_n \underbrace{\bar{\varepsilon}_{a' \cdots b' c' \cdots d'}(\vec{p}, h)}_n \right] \left[ \sum_{h'=1}^{-1} \varepsilon^{c'}(\vec{p}, h') \bar{\varepsilon}^c(\vec{p}, h') \right] \cdot \left[ \sum_{h'=1}^{-1} \varepsilon^{d'}(\vec{p}, h') \bar{\varepsilon}^d(\vec{p}, h') \right]$$

推论1.5.3.

$$\begin{cases} \sum_{h=(n-n')}^{-(n-n')} \underbrace{\varepsilon_{a \cdots b}(\vec{p}, h)}_{n-n'} \underbrace{\bar{\varepsilon}_{a' \cdots b'}(\vec{p}, h)}_{n-n'} = \frac{2(n-n')+1}{2n+1} \left[ \sum_{h=n}^{-n} \underbrace{\varepsilon_{a \cdots bc \cdots d}(\vec{p}, h)}_n \underbrace{\bar{\varepsilon}_{a' \cdots b' c' \cdots d'}(\vec{p}, h)}_n \right] \overbrace{\delta^{cc'} \cdots \delta^{dd'}}^{n'} \\ \sum_{h=(n-n')}^{-(n-n')} \underbrace{\varepsilon_{a \cdots b}(\vec{p}, h)}_{n-n'} \underbrace{\varepsilon_{a' \cdots b'}^+(\vec{p}, h)}_{n-n'} = \frac{2(n-n')+1}{2n+1} \left[ \sum_{h=n}^{-n} \underbrace{\varepsilon_{a \cdots bc \cdots d}(\vec{p}, h)}_n \underbrace{\varepsilon_{a' \cdots b' c' \cdots d'}^+(\vec{p}, h)}_n \right] \overbrace{\eta^{cc'} \cdots \eta^{dd'}}^{n'} \end{cases}$$

定理1.5.1.

$$\sum_{h=(n-n')}^{-(n-n')} \varepsilon_{a \cdot \cdot b}(\vec{p}, h) \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, h) = \frac{2n+1-2n'}{2n+1} \left[ \sum_{h=n}^{-n} \varepsilon_{a \cdot \cdot bc \cdot \cdot d}(\vec{p}, h) \bar{\varepsilon}_{a' \cdot \cdot b' c' \cdot \cdot d'}(\vec{p}, h) \right] \left[ \sum_{h'=n'}^{-n'} \varepsilon^{c' \cdot \cdot d'}(\vec{p}, h') \bar{\varepsilon}^{c \cdot \cdot d}(\vec{p}, h') \right]$$

证明:

$$\begin{aligned} \varepsilon_{a \cdot \cdot bc \cdot \cdot d}(\vec{p}, h) \bar{\varepsilon}^{c \cdot \cdot d}(\vec{p}, h') &= \frac{\sqrt{C_{n+h}^{n'+h'} C_{n-h}^{n'-h'}}}{\sqrt{C_{2n}^{2n'}}} \varepsilon_{a \cdot \cdot b}(\vec{p}, h-h') \\ \Rightarrow \varepsilon_{a \cdot \cdot bc \cdot \cdot d}(\vec{p}, h) \bar{\varepsilon}_{a' \cdot \cdot b' c' \cdot \cdot d'}(\vec{p}, h) \varepsilon^{c' \cdot \cdot d'}(\vec{p}, h') \bar{\varepsilon}^{c \cdot \cdot d}(\vec{p}, h') &= \frac{C_{n+h}^{n'+h'} C_{n-h}^{n'-h'}}{C_{2n}^{2n'}} \varepsilon_{a \cdot \cdot b}(\vec{p}, h-h') \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, h-h') \\ \Rightarrow \sum_{h=n}^{-n} \sum_{h'=n'}^{-n'} \varepsilon_{a \cdot \cdot bc \cdot \cdot d}(\vec{p}, h) \bar{\varepsilon}_{a' \cdot \cdot b' c' \cdot \cdot d'}(\vec{p}, h) \varepsilon^{c' \cdot \cdot d'}(\vec{p}, h') \bar{\varepsilon}^{c \cdot \cdot d}(\vec{p}, h') &= \sum_{h=n}^{-n} \sum_{h'=n'}^{-n'} \frac{C_{n+h}^{n'+h'} C_{n-h}^{n'-h'}}{C_{2n}^{2n'}} \varepsilon_{a \cdot \cdot b}(\vec{p}, h-h') \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, h-h') \\ &= \sum_{l=0}^{2n'} \frac{C_{n+(n-l)}^{n'+(n'-l)} C_{n-(n-l)}^{n'-(n'-l)}}{C_{2n}^{2n'}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n-n') \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, n-n') \\ &+ \sum_{l=0}^{2n'} \frac{C_{n+(n-1-l)}^{n'+(n'-1-l)} C_{n-(n-1-l)}^{n'-(n'-1-l)}}{C_{2n}^{2n'}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n-n'-1) \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, n-n'-1) \\ &+ \sum_{l=0}^{2n'} \frac{C_{n+(n-2-l)}^{n'+(n'-2-l)} C_{n-(n-2-l)}^{n'-(n'-2-l)}}{C_{2n}^{2n'}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n-n'-2) \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, n-n'-2) \\ &+ \dots \\ &+ \sum_{l=0}^{2n'} \frac{C_{n+[n-2(n-n')-l]}^{n'+[n'-2(n-n')-l]} C_{n-[n-2(n-n')-l]}^{n'-[n'-2(n-n')-l]}}{C_{2n}^{2n'}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n-n'-2(n-n')) \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, n-n'-2(n-n')) \\ &= \sum_{h=n}^{2n'-n} \sum_{l=0}^{2n'} \frac{C_{n+(h-l)}^{n'+(n'-l)} C_{n-(h-l)}^{n'-(n'-l)}}{C_{2n}^{2n'}} \varepsilon_{a \cdot \cdot b}(\vec{p}, h-n') \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, h-n') \\ &= \sum_{h=n}^{2n'-n} \sum_{h'=n'}^{-n'} \frac{C_{n+(h-n'+h')}^{n'+h'} C_{n-(h-n'+h')}^{n'-h'}}{C_{2n}^{2n'}} \varepsilon_{a \cdot \cdot b}(\vec{p}, h-n') \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, h-n') \\ &= \sum_{h=n-n'}^{n'-n} \sum_{h'=n'}^{-n'} \frac{C_{(n+h')+(h-n')}^{n'+h'} C_{(n-h')-h}^{n'-h'}}{C_{2n}^{2n'}} \varepsilon_{a \cdot \cdot b}(\vec{p}, h) \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, h) \\ \Rightarrow \sum_{h=(n-n')}^{-(n-n')} \varepsilon_{a \cdot \cdot b}(\vec{p}, h) \bar{\varepsilon}_{a' \cdot \cdot b'}(\vec{p}, h) &= \frac{2n+1-2n'}{2n+1} \left[ \sum_{h=n}^{-n} \varepsilon_{a \cdot \cdot bc \cdot \cdot d}(\vec{p}, h) \bar{\varepsilon}_{a' \cdot \cdot b' c' \cdot \cdot d'}(\vec{p}, h) \right] \left[ \sum_{h'=n'}^{-n'} \varepsilon^{c' \cdot \cdot d'}(\vec{p}, h') \bar{\varepsilon}^{c \cdot \cdot d}(\vec{p}, h') \right] \quad \square \end{aligned}$$

## 1.6 推导到自旋-n粒子Klein-Gordon方程的平面波解

$$\text{定理1.6.1. } (-\partial^c \partial_c + m^2) A_{ab \cdot \cdot}(x) = 0, A_{ab \cdot \cdot}(x) = \left( \frac{1}{2im} \right)^n \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \dots \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}^n(x)$$

$$A_{ab \cdot \cdot}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} [a(\vec{p}, h) \varepsilon_{ab \cdot \cdot}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) \bar{\varepsilon}_{ab \cdot \cdot}(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p}$$

$$\varepsilon_{ab \cdot \cdot}(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \dots U_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}^{2n}(\vec{p}, h)$$

$$\bar{\varepsilon}_{ab \cdot \cdot}(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \dots V_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}^{2n}(\vec{p}, h)$$

$$\text{证明: } [A_{ab \cdot \cdot}(x), A_{a'b' \cdot \cdot}^+(x')]$$

$$= \frac{1}{(2\pi)^{3/2}} \int \sum_{h, h'=n}^{-n} d^3 \vec{p} d^3 \vec{p}' \frac{1}{\sqrt{2^n E}} \frac{1}{\sqrt{2^n E'}}$$

$$\begin{aligned}
& [a(\vec{p}, h) \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) e^{ip\cdot x} + b^+(\vec{p}, h) \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) e^{-ip\cdot x}, a^+(\vec{p}', h') \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}', h') e^{-ip'\cdot x'} + b(\vec{p}', h') \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}', h') e^{ip'\cdot x'}] \\
&= \frac{1}{(2\pi)^{3/2}} \int \sum_{h, h'=n}^{-n} d^3\vec{p} d^3\vec{p}' \frac{1}{\sqrt{2^n E}} \frac{1}{\sqrt{2^n E'}} \\
& \{ \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \underbrace{\varepsilon_{ab\dots}}_n^+(\vec{p}', h') [a(\vec{p}, h), a^+(\vec{p}', h')] e^{ip\cdot x} e^{-ip'\cdot x'} + \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) \underbrace{\tilde{\varepsilon}_{ab\dots}}_n^+(\vec{p}', h') [b^+(\vec{p}, h), b(\vec{p}', h')] e^{-ip\cdot x} e^{ip'\cdot x'} \} \\
&= \frac{1}{(2\pi)^{3/2}} \int \sum_{h, h'=n}^{-n} d^3\vec{p} d^3\vec{p}' \frac{1}{\sqrt{2^n E}} \frac{1}{\sqrt{2^n E'}} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \underbrace{\varepsilon_{ab\dots}}_n^+(\vec{p}', h') \delta_{hh'} \delta(\vec{p} - \vec{p}') (e^{ip\cdot x} e^{-ip'\cdot x'} - e^{-ip\cdot x} e^{ip'\cdot x'}) \\
&= \frac{i}{2^{n-1}} \int \left[ \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \underbrace{\varepsilon_{ab\dots}}_n^+(\vec{p}, h) \right] \left\{ \frac{1}{(2\pi)^{3/2}} \frac{-i}{2E} [e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}] d^3\vec{p} \right\} \\
&= \frac{i}{2^{n-1}} \left[ \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab\dots}}_n(-i\partial, h) \underbrace{\varepsilon_{ab\dots}}_n^+(-i\partial, h) \right] \int \left\{ \frac{1}{(2\pi)^{3/2}} \frac{-i}{2E} [e^{ip\cdot(x-x')} - e^{-ip\cdot(x-x')}] d^3\vec{p} \right\} \\
&= \frac{i}{2^{n-1}} \left[ \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab\dots}}_n(-i\partial, h) \underbrace{\varepsilon_{ab\dots}}_n^+(-i\partial, h) \right] \Delta(x - x') \quad \square
\end{aligned}$$

### 1.7 自旋-n粒子Klein-Gordon方程自旋基的正确猜想(最初的想法仍保留)

定理1.7.1.

$$\begin{cases}
U_{\underbrace{\lambda_s \mu_s \eta_s \xi_s}_{2n}}(\vec{p}, h) = \frac{1}{(2\sqrt{2m})^n} \underbrace{\mathbb{X}_{\lambda_s \mu_s}^a(p) \mathbb{X}_{\eta_s \xi_s}^b(p)}_n \cdot \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \\
[\Rightarrow] \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_s \mu_s} (\bar{C}\gamma_b)^{\eta_s \xi_s} \dots}^n U_{\underbrace{\lambda_s \mu_s \eta_s \xi_s}_{2n}}(\vec{p}, h) \\
V_{\underbrace{\lambda_s \mu_s \eta_s \xi_s}_{2n}}(\vec{p}, h) = \frac{1}{(2\sqrt{2m})^n} \underbrace{\mathbb{X}_{\lambda_s \mu_s}^a(-p) \mathbb{X}_{\eta_s \xi_s}^b(-p)}_n \cdot \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) \\
[\Rightarrow] \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_s \mu_s} (\bar{C}\gamma_b)^{\eta_s \xi_s} \dots}^n V_{\underbrace{\lambda_s \mu_s \eta_s \xi_s}_{2n}}(\vec{p}, h)
\end{cases}$$

推论1.7.1.

$$\begin{cases}
\sum_{h=n}^{-n} U_{\underbrace{\lambda_s \mu_s}_{2n}}(\vec{p}, h) U_{\underbrace{\lambda'_s \mu'_s}_{2n}}^+(\vec{p}, h) = \frac{1}{(2\sqrt{2m})^{2n}} \underbrace{\mathbb{X}_{\lambda_s \mu_s}^a(p) \mathbb{X}_{\eta_s \xi_s}^b(p)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_s \mu'_s}^{+a'}(p) \mathbb{X}_{\eta'_s \xi'_s}^{+b'}(p)}_n \cdot \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \underbrace{\varepsilon_{a'b'\dots}^+}_{2n}(\vec{p}, h) \\
\sum_{h=n}^{-n} V_{\underbrace{\lambda_s \mu_s}_{2n}}(\vec{p}, h) V_{\underbrace{\lambda'_s \mu'_s}_{2n}}^+(\vec{p}, h) = \frac{1}{(2\sqrt{2m})^{2n}} \underbrace{\mathbb{X}_{\lambda_s \mu_s}^a(p) \mathbb{X}_{\eta_s \xi_s}^b(p)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_s \mu'_s}^{+a'}(p) \mathbb{X}_{\eta'_s \xi'_s}^{+b'}(p)}_n \cdot \sum_{h=n}^{-n} \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) \underbrace{\tilde{\varepsilon}_{a'b'\dots}^+}_{2n}(\vec{p}, h)
\end{cases}$$

推论1.7.2.

$$\begin{cases}
\sum_{h=n}^{-n} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \underbrace{\varepsilon_{a'b'\dots}^+}_{2n}(\vec{p}, h) = \frac{1}{2^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_s \mu_s} (\bar{C}\gamma_b)^{\eta_s \xi_s} \dots}^n \overbrace{(\gamma_{a'} C)^{\lambda'_s \mu'_s} (\gamma_{b'} C)^{\eta'_s \xi'_s} \dots}^n \sum_{h=n}^{-n} U_{\underbrace{\lambda_s \mu_s}_{2n}}(\vec{p}, h) U_{\underbrace{\lambda'_s \mu'_s}_{2n}}^+(\vec{p}, h) \\
\sum_{h=n}^{-n} \underbrace{\tilde{\varepsilon}_{ab\dots}}_n(\vec{p}, h) \underbrace{\tilde{\varepsilon}_{a'b'\dots}^+}_{2n}(\vec{p}, h) = \frac{1}{2^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_s \mu_s} (\bar{C}\gamma_b)^{\eta_s \xi_s} \dots}^n \overbrace{(\gamma_{a'} C)^{\lambda'_s \mu'_s} (\gamma_{b'} C)^{\eta'_s \xi'_s} \dots}^n \sum_{h=n}^{-n} V_{\underbrace{\lambda_s \mu_s}_{2n}}(\vec{p}, h) V_{\underbrace{\lambda'_s \mu'_s}_{2n}}^+(\vec{p}, h)
\end{cases}$$

定理1.7.2.  $\varepsilon_{\underbrace{ab\dots}}_n(\vec{p}, h) = (-1)^n \tilde{\varepsilon}_{\underbrace{ab\dots}}_n(\vec{p}, h)$

$$\begin{aligned}
\text{证明: } \varepsilon_{\underbrace{ab\dots}}_n(\vec{p}, h) &= \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_s \mu_s} (\bar{C}\gamma_b)^{\eta_s \xi_s} \dots}^n U_{\underbrace{\lambda_s \mu_s \eta_s \xi_s}_{2n}}(\vec{p}, h) \\
&= (-s)^{2n} \frac{1}{(i\sqrt{2})^n} \overbrace{(\gamma_5 \bar{C} \gamma_a \gamma_5)^{\lambda_s \mu_s} (\gamma_5 \bar{C} \gamma_b \gamma_5)^{\eta_s \xi_s} \dots}^n V_{\underbrace{\lambda_s \mu_s \eta_s \xi_s}_{2n}}(\vec{p}, h) \\
&= (-1)^n \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_s \mu_s} (\bar{C}\gamma_b)^{\eta_s \xi_s} \dots}^n V_{\underbrace{\lambda_s \mu_s \eta_s \xi_s}_{2n}}(\vec{p}, h) \\
&= (-1)^n \tilde{\varepsilon}_{\underbrace{ab\dots}}_n(\vec{p}, h) \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } \varepsilon^{+ab\dots}(\vec{p}, h') \varepsilon_{ab\dots}(\vec{p}, h) \\
&= \frac{1}{(-i\sqrt{2})^n} \overbrace{(\gamma^a C)_{\lambda'_\zeta \mu'_\zeta} (\gamma^b C)_{\eta'_\zeta \xi'_\zeta} \dots}^n \cdot U^{+\lambda'_\zeta \mu'_\zeta \eta'_\zeta \xi'_\zeta \dots}(\vec{p}, h') \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots}^n U_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}(\vec{p}, h) \\
&= \frac{1}{2^n} \overbrace{(\gamma^a C)_{\lambda'_\zeta \mu'_\zeta} (\gamma^b C)_{\eta'_\zeta \xi'_\zeta} \dots}^n \overbrace{(\bar{C} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots}^n \cdot U^{+\lambda'_\zeta \mu'_\zeta \eta'_\zeta \xi'_\zeta \dots}(\vec{p}, h') \underbrace{U_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}_{2n}(\vec{p}, h)
\end{aligned}$$

□

$$\text{定理1.7.3. } \varepsilon_{ab\dots}^+(\vec{p}, h) = (-1)^h \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}^n \varepsilon_{a'b'\dots}(\vec{p}, -h)$$

$$\begin{aligned}
& \text{证明: } \varepsilon_{ab\dots}^+(\vec{p}, h) = \frac{1}{(-i\sqrt{2})^n} \overbrace{(\bar{C} \gamma_a^*)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b^*)^{\eta_\zeta \xi_\zeta} \dots}^n U_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}^+(\vec{p}, h) \\
&= \frac{1}{(-i\sqrt{2})^n} \overbrace{(\bar{C} \gamma_a^*)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b^*)^{\eta_\zeta \xi_\zeta} \dots}^n (-1)^{n+h} \zeta^{2n} \overbrace{\sigma_y \otimes \sigma_y \dots}^{4n} V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, -h) \\
&= \frac{(-1)^{n+h}}{(-i\sqrt{2})^n} \overbrace{(\gamma_2 \bar{C} \gamma_a^* \gamma_2)^{\lambda_\zeta \mu_\zeta} (\gamma_2 \bar{C} \gamma_b^* \gamma_2)^{\eta_\zeta \xi_\zeta} \dots}^n V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, -h) \\
&= \frac{(-1)^{n+h}}{(i\sqrt{2})^n} \overbrace{(\bar{C} \gamma_a^* \eta_a^{a'})^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b^* \eta_b^{b'})^{\eta_\zeta \xi_\zeta} \dots}^n V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, -h) \\
&= \frac{(-1)^{n+h}}{(i\sqrt{2})^n} \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}^n \overbrace{(\bar{C} \gamma_a^*)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b^*)^{\eta_\zeta \xi_\zeta} \dots}^n V_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}(\vec{p}, -h) \\
&= (-1)^h \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}^n \varepsilon_{a'b'\dots}(\vec{p}, -h)
\end{aligned}$$

□

$$\text{猜想1.7.1. } \varepsilon_{a\dots b\dots c\dots}(\vec{p}, n-2k) = (-1)^n \tilde{\varepsilon}_{a\dots b\dots c\dots}(\vec{p}, n-2k)$$

$$\begin{aligned}
&= \frac{1}{\sqrt{C_{2n}^{2k}}} \sum_{i=0}^{\min(k, n-k)} \frac{2^i}{(n-k-i)!(2i)!(k-i)!} \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} \\
&= \frac{1}{\sqrt{C_{2n}^{2k}}} \frac{1}{n!} \sum_{i=0}^{\min(k, n-k)} 2^i C_n^{2i} C_{n-2i}^{k-i} \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} \\
&= \frac{1}{\sqrt{C_{2n}^{2k}}} \frac{1}{n!} [\sqrt{2^0} C_n^0 C_{n-0}^k \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} + \sqrt{2^2} C_n^2 C_{n-2}^{k-1} \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} \\
&+ \sqrt{2^4} C_n^4 C_{n-4}^{k-2} \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} + \sqrt{2^6} C_n^6 C_{n-6}^{k-3} \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} + \dots]
\end{aligned}$$

$$\text{猜想1.7.2. } \varepsilon_{a\dots b\dots c\dots}(\vec{p}, n-2k-1) = (-1)^n \tilde{\varepsilon}_{a\dots b\dots c\dots}(\vec{p}, n-2k-1)$$

$$\begin{aligned}
&= \frac{1}{\sqrt{C_{2n}^{2k+1}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i \sqrt{2}}{(n-k-i-1)!(2i+1)!(k-i)!} \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} \\
&= \frac{1}{\sqrt{C_{2n}^{2k+1}}} \frac{1}{n!} \sum_{i=0}^{\min(k, n-1-k)} \sqrt{2^{2i+1}} C_n^{2i+1} C_{n-2i-1}^{k-i} \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} \\
&= \frac{1}{\sqrt{C_{2n}^{2k+1}}} \frac{1}{n!} [\sqrt{2^1} C_n^1 C_{n-1}^k \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} + \sqrt{2^3} C_n^3 C_{n-3}^{k-1} \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} \\
&+ \sqrt{2^5} C_n^5 C_{n-5}^{k-2} \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} + \sqrt{2^7} C_n^7 C_{n-7}^{k-3} \varepsilon_{\{a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c\}(\vec{p}, -1)} + \dots]
\end{aligned}$$



推论1.7.3.  $\delta^{ab}\varepsilon_{ab\dots}(\vec{p}, h) = 0, p^a\varepsilon_{ab\dots}(\vec{p}, h) = 0, \varepsilon_{ab\dots}$  全对称

## 1.8 自旋-n粒子Klein-Gordon方程的平面波解

推论1.8.1.  $A_{ab\dots}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} \varepsilon_{ab\dots}(\vec{p}, h) [a(\vec{p}, h)e^{ip\cdot x} + (-1)^n b^+(\vec{p}, h)e^{-ip\cdot x}] d^3\vec{p}$

## 2 K-G方程自旋基和准投影算子的几个例子

### 2.1 自旋-1粒子Klein-Gordon方程的准投影算子

推论2.1.1.  $\varepsilon_{a'}^+(\vec{p}, h)\eta_a^{a'} = (-1)^h \varepsilon_a(\vec{p}, -h)$

定理2.1.1. 
$$\begin{cases} \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h)\varepsilon_{a'}^+(\vec{p}, h) = \eta_{aa'} + \frac{p_a p_{a'}}{m^2} \\ \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h)\varepsilon_{a'}^+(\vec{p}, h)\eta_b^{a'} = \sum_{h=1}^{-1} (-1)^h \varepsilon_a(\vec{p}, h)\varepsilon_b(\vec{p}, -h) = \delta_{ab} + \frac{p_a p_b}{m^2} \end{cases}$$

推论2.1.2.  $[-\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, -1) + \varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, 0) - \varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, 1)] = \delta_{ab} + \frac{p_a p_b}{m^2}$

推论2.1.3.  $p^a \varepsilon_a(\vec{p}, h) = 0$

### 2.2 自旋-1粒子C-K自旋基和1-自旋基的关系

引理2.2.1.

$$\begin{cases} [\sigma_+^a \varepsilon_a(\vec{p}, \kappa)]\lambda(\hat{p}, \kappa) = 0, [\sigma_+^a \varepsilon_a(\vec{p}, -\kappa)]\lambda(\hat{p}, \kappa) = -i\kappa\sqrt{2}\gamma_5 u(\vec{p}, -\frac{\kappa}{2}), [\sigma_+^a \varepsilon_a(\vec{p}, 0)]\lambda(\hat{p}, \kappa) = -i\kappa\gamma_5 \lambda(\hat{p}, \kappa) \\ [\gamma^a \varepsilon_a(\vec{p}, \kappa)]v(\vec{p}, \frac{\kappa}{2}) = 0, [\gamma^a \varepsilon_a(\vec{p}, -\kappa)]v(\vec{p}, \frac{\kappa}{2}) = i\kappa\sqrt{2}\gamma_5 v(\vec{p}, -\frac{\kappa}{2}), [\gamma^a \varepsilon_a(\vec{p}, 0)]v(\vec{p}, \frac{\kappa}{2}) = i\kappa\gamma_5 v(\vec{p}, \frac{\kappa}{2}) \end{cases}$$

### 2.3 自旋-2粒子Klein-Gordon方程的准投影算子

性质2.3.1.

$$\begin{cases} |2, 2\rangle = |1\rangle \otimes |1\rangle; \\ |2, 1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle); \\ |2, 0\rangle = \frac{1}{\sqrt{6}}(|-1\rangle \otimes |1\rangle + 2|0\rangle \otimes |0\rangle + |1\rangle \otimes |-1\rangle); \\ |2, -1\rangle = \frac{1}{\sqrt{2}}(|-1\rangle \otimes |0\rangle + |0\rangle \otimes |-1\rangle); \\ |2, -2\rangle = |-1\rangle \otimes |-1\rangle; \end{cases}$$

性质2.3.2.

$$\begin{cases} \varepsilon_{ab}(\vec{p}, 2) = \varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1) \\ \varepsilon_{ab}(\vec{p}, 1) = \frac{1}{\sqrt{2}}[\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, 0) + \varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, 1)] \\ \varepsilon_{ab}(\vec{p}, 0) = \frac{1}{\sqrt{6}}[\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, -1) + \varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, 1) + 2\varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, 0)] \\ \varepsilon_{ab}(\vec{p}, -1) = \frac{1}{\sqrt{2}}[\varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, 0) + \varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, -1)] \\ \varepsilon_{ab}(\vec{p}, -2) = \varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, -1) \\ \delta^{ab}\varepsilon_{ab}(\vec{p}, h) = 0, p^a \varepsilon_{ab}(\vec{p}, h) = 0, \varepsilon_{ab}(\vec{p}, h) = \varepsilon_{ba}(\vec{p}, h) \end{cases}$$

推论2.3.1.  $\varepsilon_{a'b'}^+(\vec{p}, h)\eta_a^{a'}\eta_b^{b'} = (-1)^h \varepsilon_{ab}(\vec{p}, -h)$

定理2.3.1.  $\sum_{h=2}^{-2} \varepsilon_{ab}(\vec{p}, h)\varepsilon_{a'b'}^+(\vec{p}, h) = \frac{1}{4}\{[\eta_{\{a(a'} + \frac{p_{\{a}p_{a'}}}{m^2}][\eta_{b\}b'}] + \frac{p_{b\}p_{b'}}{m^2}\} - \frac{1}{3}[\delta_{\{ab\} + \frac{p_{\{a}p_{b\}}}{m^2}][\delta_{(a'b')} + \frac{p_{(a'}p_{b')}}{m^2}]}$

证明:  $\sum_{h=2}^{-2} \varepsilon_{ab}(\vec{p}, h)\varepsilon_{a'b'}^+(\vec{p}, h)$

$$= \frac{1}{12}\{12\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1)\varepsilon_{a'}^+(\vec{p}, 1)\varepsilon_{b'}^+(\vec{p}, 1) + 6[\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, 0) + \varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, 1)][\varepsilon_{a'}^+(\vec{p}, 1)\varepsilon_{b'}^+(\vec{p}, 0) + \varepsilon_{a'}^+(\vec{p}, 0)\varepsilon_{b'}^+(\vec{p}, 1)]\}$$

$$\begin{aligned}
& + 2[\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, -1) + \varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, 1) + 2\varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, 0)] \\
& [\varepsilon_{a'}^+(\vec{p}, 1)\varepsilon_{b'}^+(\vec{p}, -1) + \varepsilon_{a'}^+(\vec{p}, -1)\varepsilon_{b'}^+(\vec{p}, 1) + 2\varepsilon_{a'}^+(\vec{p}, 0)\varepsilon_{b'}^+(\vec{p}, 0)] \\
& + 6[\varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, 0) + \varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, -1)][\varepsilon_a^+(\vec{p}, -1)\varepsilon_b^+(\vec{p}, 0) + \varepsilon_a^+(\vec{p}, 0)\varepsilon_b^+(\vec{p}, -1)] \\
& + 12\varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, -1)\varepsilon_{a'}^+(\vec{p}, -1)\varepsilon_{b'}^+(\vec{p}, -1)\} \\
& = \frac{1}{12}\{ \\
& 3[\varepsilon_a(\vec{p}, 1)\varepsilon_{a'}^+(\vec{p}, 1) + \varepsilon_a(\vec{p}, 0)\varepsilon_{a'}^+(\vec{p}, 0) + \varepsilon_a(\vec{p}, -1)\varepsilon_{a'}^+(\vec{p}, -1)][\varepsilon_b(\vec{p}, 1)\varepsilon_{b'}^+(\vec{p}, 1) + \varepsilon_b(\vec{p}, 0)\varepsilon_{b'}^+(\vec{p}, 0) + \varepsilon_b(\vec{p}, -1)\varepsilon_{b'}^+(\vec{p}, -1)] \\
& + \\
& 3[\varepsilon_a(\vec{p}, 1)\varepsilon_{b'}^+(\vec{p}, 1) + \varepsilon_a(\vec{p}, 0)\varepsilon_{b'}^+(\vec{p}, 0) + \varepsilon_a(\vec{p}, -1)\varepsilon_{b'}^+(\vec{p}, -1)][\varepsilon_b(\vec{p}, 1)\varepsilon_{a'}^+(\vec{p}, 1) + \varepsilon_b(\vec{p}, 0)\varepsilon_{a'}^+(\vec{p}, 0) + \varepsilon_b(\vec{p}, -1)\varepsilon_{a'}^+(\vec{p}, -1)] \\
& + \\
& 3[\varepsilon_b(\vec{p}, 1)\varepsilon_{a'}^+(\vec{p}, 1) + \varepsilon_b(\vec{p}, 0)\varepsilon_{a'}^+(\vec{p}, 0) + \varepsilon_b(\vec{p}, -1)\varepsilon_{a'}^+(\vec{p}, -1)][\varepsilon_a(\vec{p}, 1)\varepsilon_{b'}^+(\vec{p}, 1) + \varepsilon_a(\vec{p}, 0)\varepsilon_{b'}^+(\vec{p}, 0) + \varepsilon_a(\vec{p}, -1)\varepsilon_{b'}^+(\vec{p}, -1)] \\
& + \\
& 3[\varepsilon_b(\vec{p}, 1)\varepsilon_{b'}^+(\vec{p}, 1) + \varepsilon_b(\vec{p}, 0)\varepsilon_{b'}^+(\vec{p}, 0) + \varepsilon_b(\vec{p}, -1)\varepsilon_{b'}^+(\vec{p}, -1)][\varepsilon_a(\vec{p}, 1)\varepsilon_{a'}^+(\vec{p}, 1) + \varepsilon_a(\vec{p}, 0)\varepsilon_{a'}^+(\vec{p}, 0) + \varepsilon_a(\vec{p}, -1)\varepsilon_{a'}^+(\vec{p}, -1)] \\
& - \\
& 4[-\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, -1) + \varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, 0) - \varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, 1)] \\
& [-\varepsilon_{a'}^+(\vec{p}, 1)\varepsilon_{b'}^+(\vec{p}, -1) + \varepsilon_{a'}^+(\vec{p}, 0)\varepsilon_{b'}^+(\vec{p}, 0) - \varepsilon_{a'}^+(\vec{p}, -1)\varepsilon_{b'}^+(\vec{p}, 1)]\} \\
& = \frac{1}{4}\{[\eta_{\{a(a'+\frac{p_{\{a}p_{\{a'}^+\}}}{m^2})\}}][\eta_{\{b\}b'}\} + \frac{p_{\{b\}p_{\{b'}^+\}}}{m^2}] - \frac{4}{3}[\delta_{ab} + \frac{p_{\{a}p_{\{b\}}}{m^2}][\delta_{a'b'} + \frac{p_{\{a'}^+p_{\{b'}^+\}}}{m^2}]\} \\
& = \frac{1}{4}\{[\eta_{\{a(a'+\frac{p_{\{a}p_{\{a'}^+\}}}{m^2})\}}][\eta_{\{b\}b'}\} + \frac{p_{\{b\}p_{\{b'}^+\}}}{m^2}] - \frac{1}{3}[\delta_{\{ab\}} + \frac{p_{\{a}p_{\{b\}}}{m^2}][\delta_{\{a'b'}\}} + \frac{p_{\{a'}^+p_{\{b'}^+\}}}{m^2}]\} \quad \square
\end{aligned}$$

**推论2.3.2.**

$$\sum_{h=2}^{-2} \varepsilon_{a_1 a_2}(\vec{p}, h)\varepsilon_{b_1 b_2}^+(\vec{p}, h)\eta_{b_1}^{b_1'}\eta_{b_2}^{b_2'} = \frac{1}{4}\{[\delta_{\{a_1(b_1 + \frac{p_{\{a_1}p_{\{b_1\}}}{m^2})\}}][\delta_{\{a_2\}b_2\}} + \frac{p_{\{a_2\}p_{\{b_2\}}}{m^2}] - \frac{1}{3}[\delta_{\{a_1 a_2\}} + \frac{p_{\{a_1}p_{\{a_2\}}}{m^2}][\delta_{\{b_1 b_2\}} + \frac{p_{\{b_1}p_{\{b_2\}}}{m^2}]\}$$

**2.4 另一种证法****性质2.4.1.**

$$\left\{ \begin{aligned}
\varepsilon_{a_1 a_2}(\vec{p}, 2) &= \varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{a_2}(\vec{p}, 1) \\
\varepsilon_{a_1 a_2}(\vec{p}, 1) &= \frac{1}{\sqrt{2}}[\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{a_2}(\vec{p}, 0) + \varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{a_2}(\vec{p}, 1)] \\
\varepsilon_{a_1 a_2}(\vec{p}, 0) &= \frac{1}{\sqrt{6}}[\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{a_2}(\vec{p}, -1) + \varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{a_2}(\vec{p}, 1) + 2\varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{a_2}(\vec{p}, 0)] \\
\varepsilon_{a_1 a_2}(\vec{p}, -1) &= \frac{1}{\sqrt{2}}[\varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{a_2}(\vec{p}, 0) + \varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{a_2}(\vec{p}, -1)] \\
\varepsilon_{a_1 a_2}(\vec{p}, -2) &= \varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{a_2}(\vec{p}, -1) \\
\delta^{a_1 a_2} \varepsilon_{a_1 a_2}(\vec{p}, h) &= 0, p^{a_1} \varepsilon_{a_1 a_2}(\vec{p}, h) = 0, \varepsilon_{a_1 a_2}(\vec{p}, h) = \varepsilon_{a_2 a_1}(\vec{p}, h)
\end{aligned} \right.$$

**性质2.4.2.**

$$\left\{ \begin{aligned}
\varepsilon_{b_1 b_2}(\vec{p}, 2) &= \varepsilon_{b_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, 1) \\
\varepsilon_{b_1 b_2}(\vec{p}, 1) &= \frac{1}{\sqrt{2}}[\varepsilon_{b_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, 0) + \varepsilon_{b_1}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, 1)] \\
\varepsilon_{b_1 b_2}(\vec{p}, 0) &= \frac{1}{\sqrt{6}}[\varepsilon_{b_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{b_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1) + 2\varepsilon_{b_1}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, 0)] \\
\varepsilon_{b_1 b_2}(\vec{p}, -1) &= \frac{1}{\sqrt{2}}[\varepsilon_{b_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 0) + \varepsilon_{b_1}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, -1)] \\
\varepsilon_{b_1 b_2}(\vec{p}, -2) &= \varepsilon_{b_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, -1) \\
\delta^{b_1 b_2} \varepsilon_{b_1 b_2}(\vec{p}, h) &= 0, p^{b_1} \varepsilon_{b_1 b_2}(\vec{p}, h) = 0, \varepsilon_{b_1 b_2}(\vec{p}, h) = \varepsilon_{b_2 b_1}(\vec{p}, h)
\end{aligned} \right.$$

$$\text{推论2.4.1. } \varepsilon_a\left(\begin{bmatrix} 0 \\ |\vec{p}| \end{bmatrix}, 1\right) := \frac{1}{\sqrt{2}}[-1, -i, 0, 0]_a, \varepsilon_a\left(\begin{bmatrix} 0 \\ |\vec{p}| \end{bmatrix}, 0\right) := \frac{1}{m}[0, 0, E, i|\vec{p}|]_a, \varepsilon_a\left(\begin{bmatrix} 0 \\ |\vec{p}| \end{bmatrix}, -1\right) := \frac{1}{\sqrt{2}}[1, -i, 0, 0]_a$$

**证明:**

$$\begin{aligned}
& 2[\varepsilon_{a_1 a_2}(\vec{p}, 1)\varepsilon_{b_1 b_2}(\vec{p}, 1) + \varepsilon_{a_1 a_2}(\vec{p}, -1)\varepsilon_{b_1 b_2}(\vec{p}, -1)] \\
& = [\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{a_2}(\vec{p}, 0) + \varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{a_2}(\vec{p}, 1)][\varepsilon_{b_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 0) + \varepsilon_{b_1}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, -1)] \\
& + [\varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{a_2}(\vec{p}, 0) + \varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{a_2}(\vec{p}, -1)][\varepsilon_{b_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, 0) + \varepsilon_{b_1}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, 1)] \\
& = [\varepsilon_{a_2}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{a_2}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1)]\varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{b_1}(\vec{p}, 0)
\end{aligned}$$



$$\begin{aligned}
& [-\varepsilon_{b_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{b_1}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, 0) - \varepsilon_{b_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1)] \\
& = \frac{1}{12} \{ \\
& 3[-\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{b_1}(\vec{p}, -1) + \varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{b_1}(\vec{p}, 0) - \varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{b_1}(\vec{p}, 1)] \\
& [-\varepsilon_{a_2}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{a_2}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, 0) - \varepsilon_{a_2}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1)] \\
& + \\
& 3[-\varepsilon_{a_2}(\vec{p}, 1)\varepsilon_{b_1}(\vec{p}, -1) + \varepsilon_{a_2}(\vec{p}, 0)\varepsilon_{b_1}(\vec{p}, 0) - \varepsilon_{a_2}(\vec{p}, -1)\varepsilon_{b_1}(\vec{p}, 1)] \\
& [-\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, 0) - \varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1)] \\
& + \\
& 3[-\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, 0) - \varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1)] \\
& [-\varepsilon_{a_2}(\vec{p}, 1)\varepsilon_{b_1}(\vec{p}, -1) + \varepsilon_{a_2}(\vec{p}, 0)\varepsilon_{b_1}(\vec{p}, 0) - \varepsilon_{a_2}(\vec{p}, -1)\varepsilon_{b_1}(\vec{p}, 1)] \\
& + \\
& 3[-\varepsilon_{a_2}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{a_2}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, 0) - \varepsilon_{a_2}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1)] \\
& [-\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{b_1}(\vec{p}, -1) + \varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{b_1}(\vec{p}, 0) - \varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{b_1}(\vec{p}, 1)] \\
& - \\
& 4[-\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{a_2}(\vec{p}, -1) + \varepsilon_{a_1}(\vec{p}, 0)\varepsilon_{a_2}(\vec{p}, 0) - \varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{a_2}(\vec{p}, 1)] \\
& [-\varepsilon_{b_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{b_1}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, 0) - \varepsilon_{b_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1)] \\
& = \frac{1}{4} \{ [\delta_{\{a_1(b_1 + \frac{P_{\{a_1}P_{\{b_1}}\}}{m^2})\}}] [\delta_{\{a_2\}b_2}] + \frac{P_{\{a_2\}P_{\{b_2}\}}}{m^2} \} - \frac{1}{3} [\delta_{\{a_1a_2\}} + \frac{P_{\{a_1}P_{\{a_2}\}}}{m^2}] [\delta_{(b_1b_2)} + \frac{P_{(b_1)P_{(b_2)}}}{m^2}] \}
\end{aligned}$$

□

$$\begin{aligned}
& \text{推论2.4.2. } 2[\varepsilon_{a_1a_2}(\vec{p}, 1)\varepsilon_{b_1b_2}(\vec{p}, 1) + \varepsilon_{a_1a_2}(\vec{p}, -1)\varepsilon_{b_1b_2}(\vec{p}, -1)] \\
& = -P_{\{a_1(b_1[\varepsilon_{a_2}(\vec{p}, 0)\varepsilon_{b_2}(\vec{p}, 0)] + \varepsilon_{\{a_1}(\vec{p}, 0)\varepsilon_{a_2}\}(\vec{p}, 0)\varepsilon_{(b_1}(\vec{p}, 0)\varepsilon_{b_2)}(\vec{p}, 0)}
\end{aligned}$$

推论2.4.3.

$$\begin{aligned}
& [\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{b_1}(\vec{p}, -1) + \varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{b_1}(\vec{p}, 1)][\varepsilon_{a_2}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{a_2}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1)] \\
& + \\
& [\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1)][\varepsilon_{a_2}(\vec{p}, 1)\varepsilon_{b_1}(\vec{p}, -1) + \varepsilon_{a_2}(\vec{p}, -1)\varepsilon_{b_1}(\vec{p}, 1)] \\
& = 2[\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{a_2}(\vec{p}, 1)][\varepsilon_{b_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, -1)] + 2[\varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{a_2}(\vec{p}, -1)][\varepsilon_{b_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, 1)] \\
& + [\varepsilon_{a_1}(\vec{p}, 1)\varepsilon_{a_2}(\vec{p}, -1) + \varepsilon_{a_1}(\vec{p}, -1)\varepsilon_{a_2}(\vec{p}, 1)][\varepsilon_{b_1}(\vec{p}, 1)\varepsilon_{b_2}(\vec{p}, -1) + \varepsilon_{b_1}(\vec{p}, -1)\varepsilon_{b_2}(\vec{p}, 1)]
\end{aligned}$$

$$\begin{aligned}
& \text{推论2.4.4. } 2\varepsilon_{\{a_1}(\vec{p}, 1)\varepsilon_{b_1}\}(\vec{p}, -1)\varepsilon_{(a_2}(\vec{p}, 1)\varepsilon_{b_2)}(\vec{p}, -1) + 2\varepsilon_{\{a_1}(\vec{p}, 1)\varepsilon_{b_2}\}(\vec{p}, -1)\varepsilon_{(a_2}(\vec{p}, 1)\varepsilon_{b_1)}(\vec{p}, -1) \\
& = \varepsilon_{\{a_1}(\vec{p}, 1)\varepsilon_{a_2}\}(\vec{p}, 1)\varepsilon_{\{b_1}(\vec{p}, -1)\varepsilon_{b_2}\}(\vec{p}, -1) + \varepsilon_{\{a_1}(\vec{p}, -1)\varepsilon_{a_2}\}(\vec{p}, -1)\varepsilon_{\{b_1}(\vec{p}, 1)\varepsilon_{b_2}\}(\vec{p}, 1) \\
& + 2\varepsilon_{\{a_1}(\vec{p}, 1)\varepsilon_{a_2}\}(\vec{p}, -1)\varepsilon_{\{b_1}(\vec{p}, 1)\varepsilon_{b_2}\}(\vec{p}, -1)
\end{aligned}$$

推论2.4.5.  $Q_{\{a_1(b_1}Q_{a_2)b_2}$ 

$$= \varepsilon_{\{a_1}(\vec{p}, 1)\varepsilon_{a_2}\}(\vec{p}, 1)\varepsilon_{(b_1}(\vec{p}, -1)\varepsilon_{b_2)}(\vec{p}, -1) + \varepsilon_{\{a_1}(\vec{p}, -1)\varepsilon_{a_2}\}(\vec{p}, -1)\varepsilon_{(b_1}(\vec{p}, 1)\varepsilon_{b_2)}(\vec{p}, 1) + 2Q_{a_1a_2}Q_{b_1b_2}$$

推论2.4.6.

$$\begin{aligned}
& P_{\{a_1(b_1}P_{a_2)b_2} = [Q_{\{a_1(b_1} - \varepsilon_{\{a_1}(\vec{p}, 0)\varepsilon_{(b_1}(\vec{p}, 0)]][Q_{a_2\}b_2) - \varepsilon_{a_2}\}(\vec{p}, 0)\varepsilon_{b_2)}(\vec{p}, 0)] \\
& = Q_{\{a_1(b_1}Q_{a_2)b_2} - 2Q_{\{a_1(b_1}[\varepsilon_{a_2}\}(\vec{p}, 0)\varepsilon_{b_2)}(\vec{p}, 0)] + \varepsilon_{\{a_1}(\vec{p}, 0)\varepsilon_{a_2}\}(\vec{p}, 0)\varepsilon_{(b_1}(\vec{p}, 0)\varepsilon_{b_2)}(\vec{p}, 0)
\end{aligned}$$

## 2.5 自旋-3粒子Klein-Gordon方程的CG系数和自旋基

推论2.5.1.

$$\left\{ \begin{array}{l} \langle 2, 2; 1, 1|2, 1; 3, 3 \rangle = 1 \\ \langle 2, 2; 1, 0|2, 1; 3, 2 \rangle = \frac{1}{\sqrt{3}}, \langle 2, 1; 1, 1|2, 1; 3, 2 \rangle = \frac{\sqrt{2}}{\sqrt{3}} \\ \langle 2, 2; 1, -1|2, 1; 3, 1 \rangle = \frac{1}{\sqrt{15}}, \langle 2, 1; 1, 0|2, 1; 3, 1 \rangle = \frac{\sqrt{8}}{\sqrt{15}}, \langle 2, 0; 1, 1|2, 1; 3, 1 \rangle = \frac{\sqrt{6}}{\sqrt{15}} \\ \langle 2, 1; 1, -1|2, 1; 3, 0 \rangle = \frac{1}{\sqrt{3}}, \langle 2, 0; 1, 0|2, 1; 3, 0 \rangle = \frac{\sqrt{3}}{\sqrt{15}}, \langle 2, -1; 1, 1|2, 1; 3, 0 \rangle = \frac{1}{\sqrt{5}} \\ \langle 2, -2; 1, 1|2, 1; 3, -1 \rangle = \frac{1}{\sqrt{15}}, \langle 2, -1; 1, 0|2, 1; 3, -1 \rangle = \frac{\sqrt{8}}{\sqrt{15}}, \langle 2, 0; 1, -1|2, 1; 3, -1 \rangle = \frac{\sqrt{6}}{\sqrt{15}} \\ \langle 2, -2; 1, 0|2, 1; 3, -2 \rangle = \frac{1}{\sqrt{3}}, \langle 2, -1; 1, -1|2, 1; 3, -2 \rangle = \frac{\sqrt{2}}{\sqrt{3}} \\ \langle 2, -2; 1, -1|2, 1; 3, -3 \rangle = 1 \end{array} \right.$$

推论2.5.2.

$$\left\{ \begin{array}{l} \varepsilon_{abc}(\vec{p}, 3) = \varepsilon_{ab}(\vec{p}, 2)\varepsilon_c(\vec{p}, 1) = \frac{1}{3!}\varepsilon_{\{a}(\vec{p}, 1)\varepsilon_b(\vec{p}, 1)\varepsilon_{c\}}(\vec{p}, 1) \\ \varepsilon_{abc}(\vec{p}, 2) = \frac{1}{\sqrt{3}}\varepsilon_{ab}(\vec{p}, 2)\varepsilon_c(\vec{p}, 0) + \frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p}, 1)\varepsilon_c(\vec{p}, 1) = \frac{\sqrt{3}}{3!}\varepsilon_{\{a}(\vec{p}, 1)\varepsilon_b(\vec{p}, 1)\varepsilon_{c\}}(\vec{p}, 0) \\ \varepsilon_{abc}(\vec{p}, 1) = \frac{1}{\sqrt{15}}\varepsilon_{ab}(\vec{p}, 2)\varepsilon_c(\vec{p}, -1) + \frac{\sqrt{8}}{\sqrt{15}}\varepsilon_{ab}(\vec{p}, 1)\varepsilon_c(\vec{p}, 0) + \frac{\sqrt{6}}{\sqrt{15}}\varepsilon_{ab}(\vec{p}, 0)\varepsilon_c(\vec{p}, 1) \\ = \frac{6}{3!\sqrt{15}}\varepsilon_{\{a}(\vec{p}, 1)\varepsilon_b(\vec{p}, 0)\varepsilon_{c\}}(\vec{p}, 0) + \frac{3}{3!\sqrt{15}}\varepsilon_{\{a}(\vec{p}, 1)\varepsilon_b(\vec{p}, 1)\varepsilon_{c\}}(\vec{p}, -1) \\ \varepsilon_{abc}(\vec{p}, 0) = \frac{1}{\sqrt{5}}\varepsilon_{ab}(\vec{p}, 1)\varepsilon_c(\vec{p}, -1) + \frac{\sqrt{3}}{\sqrt{5}}\varepsilon_{ab}(\vec{p}, 0)\varepsilon_c(\vec{p}, 0) + \frac{1}{\sqrt{5}}\varepsilon_{ab}(\vec{p}, -1)\varepsilon_c(\vec{p}, 1) \\ = \frac{6}{3!\sqrt{10}}\varepsilon_{\{a}(\vec{p}, 1)\varepsilon_b(\vec{p}, 0)\varepsilon_{c\}}(\vec{p}, -1) + \frac{2}{3!\sqrt{10}}\varepsilon_{\{a}(\vec{p}, 0)\varepsilon_b(\vec{p}, 0)\varepsilon_{c\}}(\vec{p}, 0) \\ \varepsilon_{abc}(\vec{p}, -1) = \frac{1}{\sqrt{15}}\varepsilon_{ab}(\vec{p}, -2)\varepsilon_c(\vec{p}, 1) + \frac{\sqrt{8}}{\sqrt{15}}\varepsilon_{ab}(\vec{p}, -1)\varepsilon_c(\vec{p}, 0) + \frac{\sqrt{6}}{\sqrt{15}}\varepsilon_{ab}(\vec{p}, 0)\varepsilon_c(\vec{p}, -1) \\ = \frac{6}{3!\sqrt{15}}\varepsilon_{\{a}(\vec{p}, -1)\varepsilon_b(\vec{p}, 0)\varepsilon_{c\}}(\vec{p}, 0) + \frac{3}{3!\sqrt{15}}\varepsilon_{\{a}(\vec{p}, -1)\varepsilon_b(\vec{p}, -1)\varepsilon_{c\}}(\vec{p}, 1) \\ \varepsilon_{abc}(\vec{p}, -2) = \frac{1}{\sqrt{3}}\varepsilon_{ab}(\vec{p}, -2)\varepsilon_c(\vec{p}, 0) + \frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p}, -1)\varepsilon_c(\vec{p}, -1) = \frac{\sqrt{3}}{3!}\varepsilon_{\{a}(\vec{p}, -1)\varepsilon_b(\vec{p}, -1)\varepsilon_{c\}}(\vec{p}, 0) \\ \varepsilon_{abc}(\vec{p}, -3) = \varepsilon_{ab}(\vec{p}, -2)\varepsilon_c(\vec{p}, -1) = \frac{1}{3!}\varepsilon_{\{a}(\vec{p}, -1)\varepsilon_b(\vec{p}, -1)\varepsilon_{c\}}(\vec{p}, -1) \end{array} \right.$$

## 2.6 自旋-n粒子Klein-Gordon方程的准投影算子

定义2.6.1.  $\bar{\varepsilon}_a(\vec{p}, h) := \varepsilon_a^+(\vec{p}, h)\eta_a^{\prime}, \bar{\varepsilon}_{ab}(\vec{p}, h) := \varepsilon_{a'b'}^+(\vec{p}, h)\eta_a^{\prime}\eta_b^{\prime}, P_{ab} := \delta_{ab} + \frac{p_a p_b}{m^2}$

推论2.6.1.

$$\left\{ \begin{array}{l} \sum_{h=1}^{-1} \varepsilon_{a_1}(\vec{p}, h)\bar{\varepsilon}_{b_1}(\vec{p}, h) = P_{a_1 b_1}, \sum_{h=1}^{-1} -|h|\varepsilon_{a_1}(\vec{p}, h)\bar{\varepsilon}_{b_1}(\vec{p}, h) := Q_{a_1 b_1} = \varepsilon_{\{a_1}(\vec{p}, 1)\varepsilon_{b_1\}}(\vec{p}, -1) \\ \sum_{h=2}^{-2} \varepsilon_{a_1 a_2}(\vec{p}, h)\bar{\varepsilon}_{b_1 b_2}(\vec{p}, h) = \frac{1}{(2!)^2} [P_{\{a_1(b_1 P_{a_2} b_2)} - \frac{1}{3}P_{\{a_1 a_2\}}P_{(b_1 b_2)}] \end{array} \right.$$

猜想2.6.1.

$$\left\{ \begin{array}{l} \sum_{h=n}^{-n} \varepsilon_{a_1 a_2 \dots a_n}(\vec{p}, h)\bar{\varepsilon}_{b_1 b_2 \dots b_n}(\vec{p}, h) = \frac{1}{(n!)^2} \sum_{r=0}^{[n/2]} A_{n,r} [P_{\{a_1 a_2} P_{(b_1 b_2} \dots P_{a_{2r-1} a_{2r}} P_{b_{2r-1} b_{2r}}] [P_{a_{2r+1} b_{2r+1}} \dots P_{a_n} b_n)] \\ A_{n,r} = (-\frac{1}{2})^r \frac{n!(2n-2r-1)!!}{r!(n-2r)!(2n-1)!!} = (-\frac{1}{2})^r \frac{1}{r!} \frac{n(n-1) \dots (n-2r+1)}{(2n-1)(2n-3) \dots (2n-2r+1)} = (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} = (-1)^r C_{2n}^{-n} C_{2n-2r}^r \\ A_{n,0} = 1, A_{n,1} = -\frac{n(n-1)}{2(2n-1)}, A_{n,2} = \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)}, \dots \end{array} \right.$$

以上Behrends和Fronsdal构造出来的公式<sup>[52, 53] [19]</sup>, 并没有严格地被证明, 本质上还是一个猜想, 它是后面很多重要结论的前提条件. 需要严格证明它, 但目前我还做不到.

定义2.6.2.  $P_{a_1 a_2 \dots a_n b_1 b_2 \dots b_n}(n) := \sum_{h=n}^{-n} \varepsilon_{a_1 a_2 \dots a_n}(\vec{p}, h)\bar{\varepsilon}_{b_1 b_2 \dots b_n}(\vec{p}, h)$

猜想2.6.2.

$$\begin{aligned} P_{a_1 a_2 \dots a_n b_1 b_2 \dots b_n}(n) &= \frac{1}{(n!)^2} \sum_{r=0}^{[n/2]} (-\frac{1}{2})^r \frac{1}{r!} \frac{n(n-1) \dots (n-2r+1)}{(2n-1)(2n-3) \dots (2n-2r+1)} [P_{\{a_1 a_2} P_{(b_1 b_2} \dots P_{a_{2r-1} a_{2r}} P_{b_{2r-1} b_{2r}}] [P_{a_{2r+1} b_{2r+1}} \dots P_{a_n} b_n)] \\ &= \frac{1}{(2n)!} \sum_{r=0}^{[n/2]} (-1)^r C_{2n}^r C_{2n-2r}^n [P_{\{a_1 a_2} P_{(b_1 b_2} \dots P_{a_{2r-1} a_{2r}} P_{b_{2r-1} b_{2r}}] [P_{a_{2r+1} b_{2r+1}} \dots P_{a_n} b_n)] \end{aligned}$$

猜想2.6.3.  $P_{a_1 a_2 \dots a_n b_1 b_2 \dots b_n}(n) = P_{\{a_1 a_2 \dots a_{n-1}, (b_1 b_2 \dots b_{n-1})(n-1)P_{a_n\}b_n\}}$   
 $+ P_{\{a_1 a_2 \dots a_{n-1}, a_n\}(b_1 b_2 \dots b_{n-2})(n-1)P_{b_{n-1}b_n\}} + P_{\{a_1 a_2 \dots a_{n-2}(b_n, b_1 b_2 \dots b_{n-1})(n-1)P_{a_{n-1}a_n\}}$

猜想2.6.4.

$$\begin{cases} \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab \dots}}_n(\vec{p}, h) \underbrace{\varepsilon_{a'b' \dots}^+}_n(\vec{p}, h) = \frac{n+1}{2n+3} \sum_{h=n+1}^{-(n+1)} \sum_{h'=1}^{-1} \underbrace{\varepsilon_{ab \dots c}}_{n+1}(\vec{p}, h) \sigma_+^c \lambda_m(\vec{p}, h') [\underbrace{\varepsilon_{a'b' \dots c'}}_{n+1}(\vec{p}, h) \sigma_+^c \lambda_m(\vec{p}, h')]^+ \\ \sum_{h=n+1/2}^{-(n+1/2)} \underbrace{\tilde{\varepsilon}_{ab \dots [\tau_\zeta]}}_n(\vec{p}, h) \underbrace{\tilde{\varepsilon}_{a'b' \dots [\tau'_\zeta]}^+}_n(\vec{p}, h) = \frac{n+1}{2n+3} \sum_{h=n+1}^{-(n+1)} \sum_{h'=1/2}^{-1/2} \underbrace{\varepsilon_{ab \dots c}}_{n+1}(\vec{p}, h) \gamma^c u(\vec{p}, h') [\underbrace{\varepsilon_{a'b' \dots c'}}_{n+1}(\vec{p}, h) \gamma^c u(\vec{p}, h')]^+ \end{cases}$$

### 3 Rarita-Schwinger方程的反对易规则

3.1 有质量自旋 $-n + \frac{1}{2}$ 的Bargmann-Wigner方程等价于Rarita-Schwinger方程 [17, 19, 20]

定理3.1.1.

$$\begin{cases} (\gamma^a \partial_a + m) \psi_{\underbrace{[\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta]}_{2n+1}}(x) = 0 \\ \psi_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n+1}}(x) \text{全对称} \end{cases} \Leftrightarrow \begin{cases} (\gamma^c \partial_c + m) A_{\underbrace{ab \dots [\tau_\zeta]}_n}(x) = 0 \\ \delta^{ab} A_{\underbrace{ab \dots [\tau_\zeta]}_n}(x) = 0, \gamma^a A_{\underbrace{ab \dots [\tau_\zeta]}_n}(x) = 0, A_{\underbrace{ab \dots [\tau_\zeta]}_n}(x) \text{全对称} \\ \psi_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n+1}}(x) = \frac{1}{2^n} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \mathbb{X}_{\eta_\zeta \xi_\zeta}^b}_{n} \cdot A_{\underbrace{ab \dots \tau_\zeta}_n}(x) \end{cases}$$

$$\begin{aligned} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2n}(\vec{r}, t) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n+1/2}^{-(n+1/2)} E^n \sqrt{\frac{m}{E}}^{2n+1} [a(\vec{p}, h) U_{\underbrace{\lambda_\zeta \mu_\zeta \dots}}_{2n}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) V_{\underbrace{\lambda_\zeta \mu_\zeta \dots}}_{2n}(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p} \\ A_{ab \dots \tau_\zeta}(\vec{r}, t) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{\sqrt{m}}{\sqrt{2^n E}} [\varepsilon_{ab \dots \tau_\zeta}(\vec{p}, h) a(\vec{p}, h) e^{ip \cdot x} + \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}, h) b^+(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p} \end{aligned}$$

自我评述: 与玻色子情形一样处理, 把各自的平面波解代入以上两个等价的方程, 并利用傅立叶分量相等性也可容易得到以下两个推论。

推论3.1.1.

$$\begin{cases} (i\gamma^a p_a + m) U_{\underbrace{[\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta]}_{2n+1}}(\vec{p}, h) = 0 \\ U_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n+1}}(\vec{p}, h) \text{全对称}, \varepsilon_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) \\ = \frac{1}{(i\sqrt{2})^n} \underbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \dots}_{n} \cdot U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2n+1}}(\vec{p}, h) \end{cases} \Leftrightarrow \begin{cases} (i\gamma^c p_c + m) \varepsilon_{\underbrace{ab \dots [\tau_\zeta]}_n}(\vec{p}, h) = 0 \\ \varepsilon_{\underbrace{ab \dots [\tau_\zeta]}_n}(\vec{p}, h) \text{全对称}, \delta^{ab} \varepsilon_{\underbrace{ab \dots [\tau_\zeta]}_n}(\vec{p}, h) = 0, \\ \gamma^a \varepsilon_{\underbrace{ab \dots [\tau_\zeta]}_n}(\vec{p}, h) = 0, p^a \varepsilon_{\underbrace{ab \dots [\tau_\zeta]}_n}(\vec{p}, h) = 0 \\ U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2n+1}}(\vec{p}, h) \\ = \frac{1}{(2\sqrt{2}m)^n} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(p) \mathbb{X}_{\eta_\zeta \xi_\zeta}^b(p)}_n \cdot \varepsilon_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) \end{cases}$$

推论3.1.2.

$$\begin{cases} (-i\gamma^a p_a + m) V_{\underbrace{[\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta]}_{2n+1}}(\vec{p}, h) = 0 \\ V_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n+1}}(\vec{p}, h) \text{全对称}, \tilde{\varepsilon}_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) \\ = \frac{1}{(i\sqrt{2})^n} \underbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \dots}_{n} \cdot V_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2n+1}}(\vec{p}, h) \end{cases} \Leftrightarrow \begin{cases} (-i\gamma^c p_c + m) \tilde{\varepsilon}_{\underbrace{ab \dots [\tau_\zeta]}_n}(\vec{p}, h) = 0 \\ \tilde{\varepsilon}_{\underbrace{ab \dots [\tau_\zeta]}_n}(\vec{p}, h) \text{全对称}, \delta^{ab} \tilde{\varepsilon}_{\underbrace{ab \dots [\tau_\zeta]}_n}(\vec{p}, h) = 0 \\ \gamma^a \tilde{\varepsilon}_{\underbrace{ab \dots [\tau_\zeta]}_n}(\vec{p}, h) = 0, p^a \tilde{\varepsilon}_{\underbrace{ab \dots [\tau_\zeta]}_n}(\vec{p}, h) = 0 \\ V_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2n+1}}(\vec{p}, h) \\ = \frac{1}{(2\sqrt{2}m)^n} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(-p) \mathbb{X}_{\eta_\zeta \xi_\zeta}^b(-p)}_n \cdot \tilde{\varepsilon}_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) \end{cases}$$

推论3.1.3.  $\Lambda_-(\vec{p}, \frac{1}{2})\gamma_4\varepsilon_{ab\dots[\tau_\zeta]}(\vec{p}, h) = 0, \Lambda_+(\vec{p}, \frac{1}{2})\gamma_4\tilde{\varepsilon}_{ab\dots[\tau_\zeta]}(\vec{p}, h) = 0$

### 3.2 自旋 $s = n + \frac{1}{2}$ 粒子 Rarita-Schwinger 方程的自旋基

定理3.2.1.

$$\left\{ \begin{aligned} U_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) &= \frac{1}{(2\sqrt{2m})^n} \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a(p)\mathbb{X}_{\eta_\zeta\xi_\zeta}^b(p)}_n \cdot \varepsilon_{ab\dots\tau_\zeta}(\vec{p}, h) \\ [\Rightarrow]\varepsilon_{ab\dots\tau_\zeta}(\vec{p}, h) &= \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta}}^n \dots U_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) \\ V_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) &= \frac{1}{(2\sqrt{2m})^n} \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a(p)\mathbb{X}_{\eta_\zeta\xi_\zeta}^b(p)}_n \cdot \tilde{\varepsilon}_{ab\dots\tau_\zeta}(\vec{p}, h) \\ [\Rightarrow]\tilde{\varepsilon}_{ab\dots\tau_\zeta}(\vec{p}, h) &= \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta}}^n \dots V_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) \end{aligned} \right.$$

定理3.2.2.

$$\left\{ \begin{aligned} U_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) &= \frac{1}{(2\sqrt{2m})^n} \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a(p)\mathbb{X}_{\eta_\zeta\xi_\zeta}^b(p)}_n \cdot \varepsilon_{ab\dots\tau_\zeta}(\vec{p}, h) \\ [\Rightarrow]\varepsilon_{ab\dots\tau_\zeta}(\vec{p}, h) &= \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta}}^n \dots U_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) \\ V_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) &= \frac{1}{(2\sqrt{2m})^n} \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a(p)\mathbb{X}_{\eta_\zeta\xi_\zeta}^b(p)}_n \cdot \tilde{\varepsilon}_{ab\dots\tau_\zeta}(\vec{p}, h) \\ [\Rightarrow]\tilde{\varepsilon}_{ab\dots\tau_\zeta}(\vec{p}, h) &= \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta}}^n \dots V_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) \end{aligned} \right.$$

推论3.2.1.

$$\left\{ \begin{aligned} &\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} U_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) U_{\lambda'_\zeta\mu'_\zeta\dots\tau'_\zeta}^+(\vec{p}, h) \\ &= \frac{1}{(2\sqrt{2m})^{2n}} \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a(p)\mathbb{X}_{\eta_\zeta\xi_\zeta}^b(p)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_\zeta\mu'_\zeta}^{+a'}(p)\mathbb{X}_{\eta'_\zeta\xi'_\zeta}^{+b'}(p)}_n \cdot \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \varepsilon_{ab\dots\tau_\zeta}(\vec{p}, h) \varepsilon_{a'b'\dots\tau'_\zeta}^+(\vec{p}, h) \\ &\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} V_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) V_{\lambda'_\zeta\mu'_\zeta\dots\tau'_\zeta}^+(\vec{p}, h) \\ &= \frac{1}{(2\sqrt{2m})^{2n}} \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a(p)\mathbb{X}_{\eta_\zeta\xi_\zeta}^b(p)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_\zeta\mu'_\zeta}^{+a'}(p)\mathbb{X}_{\eta'_\zeta\xi'_\zeta}^{+b'}(p)}_n \cdot \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \tilde{\varepsilon}_{ab\dots\tau_\zeta}(\vec{p}, h) \tilde{\varepsilon}_{a'b'\dots\tau'_\zeta}^+(\vec{p}, h) \end{aligned} \right.$$

推论3.2.2.

$$\left\{ \begin{aligned} &\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \varepsilon_{ab\dots\tau_\zeta}(\vec{p}, h) \varepsilon_{a'b'\dots\tau'_\zeta}^+(\vec{p}, h) \\ &= \frac{1}{2^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta}}^n \dots \overbrace{(\gamma_{a'}C)^{\lambda'_\zeta\mu'_\zeta}(\gamma_{b'}C)^{\eta'_\zeta\xi'_\zeta}}^n \cdot \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} U_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) U_{\lambda'_\zeta\mu'_\zeta\dots\tau'_\zeta}^+(\vec{p}, h) \\ &\sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \tilde{\varepsilon}_{ab\dots\tau_\zeta}(\vec{p}, h) \tilde{\varepsilon}_{a'b'\dots\tau'_\zeta}^+(\vec{p}, h) \\ &= \frac{1}{2^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta}}^n \dots \overbrace{(\gamma_{a'}C)^{\lambda'_\zeta\mu'_\zeta}(\gamma_{b'}C)^{\eta'_\zeta\xi'_\zeta}}^n \cdot \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} V_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p}, h) V_{\lambda'_\zeta\mu'_\zeta\dots\tau'_\zeta}^+(\vec{p}, h) \end{aligned} \right.$$

**定理3.2.3.**  $\varepsilon_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) = -\varsigma(-1)^n \gamma_5 \tau_\zeta^{\sigma_\zeta} \tilde{\varepsilon}_{\underbrace{ab \dots \sigma_\zeta}_n}(\vec{p}, h)$

$$\begin{aligned} \text{证明: } \varepsilon_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) &= \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \dots}^n U_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n}}(\vec{p}, h) \\ &= (-\varsigma)^{2n+1} \frac{1}{(i\sqrt{2})^n} \overbrace{(\gamma_5 \bar{C}\gamma_a \gamma_5)^{\lambda_\zeta \mu_\zeta} (\gamma_5 \bar{C}\gamma_b \gamma_5)^{\eta_\zeta \xi_\zeta} \dots \gamma_5 \tau_\zeta^{\sigma_\zeta}}^n V_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \sigma_\zeta}_{2n}}(\vec{p}, h) \\ &= -\varsigma(-1)^n \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \dots \gamma_5 \tau_\zeta^{\sigma_\zeta}}^n V_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \sigma_\zeta}_{2n}}(\vec{p}, h) \\ &= -\varsigma(-1)^n \gamma_5 \tau_\zeta^{\sigma_\zeta} \tilde{\varepsilon}_{\underbrace{ab \dots \sigma_\zeta}_n}(\vec{p}, h) \end{aligned}$$

□

**定理3.2.4.**  $\varepsilon_{\underbrace{a'b' \dots \tau'_\zeta}_n}(\vec{p}, h) = (-1)^{h-\frac{1}{2}} (\gamma_2 \gamma_5)_{\tau'_\zeta} \overbrace{\eta_{a'}^a \eta_{b'}^b \dots}^n \varepsilon_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, -h)$

$$\begin{aligned} \text{证明: } \varepsilon_{\underbrace{a'b' \dots \tau'_\zeta}_n}(\vec{p}, h) &= \frac{1}{(-i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_{a'}^*)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_{b'}^*)^{\eta_\zeta \xi_\zeta} \dots}^n U_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau'_\zeta}_{2n}}^+(\vec{p}, h) \\ &= \frac{1}{(-i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_{a'}^*)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_{b'}^*)^{\eta_\zeta \xi_\zeta} \dots}^n (-1)^{n+\frac{1}{2}+h} \zeta^{2n+1} \overbrace{\sigma_y \otimes \sigma_y \dots}^{4n+2} V_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau'_\zeta}_{2n}}(\vec{p}, -h) \\ &= \varsigma \frac{(-1)^{n+\frac{1}{2}+h}}{(-i\sqrt{2})^n} \gamma_2 \tau'_\zeta \overbrace{(\gamma_2 \bar{C}\gamma_{a'}^* \gamma_2)^{\lambda_\zeta \mu_\zeta} (\gamma_2 \bar{C}\gamma_{b'}^* \gamma_2)^{\eta_\zeta \xi_\zeta} \dots}^n V_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n}}(\vec{p}, -h) \\ &= \varsigma \frac{(-1)^{n+\frac{1}{2}+h}}{(i\sqrt{2})^n} \gamma_2 \tau'_\zeta \overbrace{(\bar{C}\gamma_a \eta_{a'}^a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b \eta_{b'}^b)^{\eta_\zeta \xi_\zeta} \dots}^n V_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n}}(\vec{p}, -h) \\ &= \varsigma \frac{(-1)^{n+\frac{1}{2}+h}}{(i\sqrt{2})^n} \gamma_2 \tau'_\zeta \overbrace{\eta_{a'}^a \eta_{b'}^b \dots}^n \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \dots}^n V_{\underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n}}(\vec{p}, -h) \\ &= \varsigma(-1)^{n+\frac{1}{2}+h} \gamma_2 \tau'_\zeta \overbrace{\eta_{a'}^a \eta_{b'}^b \dots}^n \tilde{\varepsilon}_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, -h) \\ &= (-1)^{h-\frac{1}{2}} (\gamma_2 \gamma_5)_{\tau'_\zeta} \overbrace{\eta_{a'}^a \eta_{b'}^b \dots}^n \varepsilon_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, -h) \end{aligned}$$

□

### 3.3 自旋 $s = n + \frac{1}{2}$ 粒子 Rarita-Schwinger 方程的平面波解

**推论3.3.1.**

$$A_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{\sqrt{m}}{\sqrt{2^n E}} [\varepsilon_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) a(\vec{p}, h) e^{ip \cdot x} + \tilde{\varepsilon}_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) b^+(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p}$$

**证明:**  $\{A_{\underbrace{ab \dots \tau_\zeta}_n}(x), A_{\underbrace{a'b' \dots \tau'_\zeta}_n}^+(x')\}$

$$\begin{aligned} &= \frac{1}{(2\pi)^{3/2}} \int \sum_{h, h'=n}^{-n} d^3 \vec{p} d^3 \vec{p}' \frac{\sqrt{m}}{\sqrt{2^n E}} \frac{\sqrt{m}}{\sqrt{2^n E'}} \\ &\{a(\vec{p}, h) \varepsilon_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) \tilde{\varepsilon}_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) e^{-ip \cdot x}, a^+(\vec{p}', h') \varepsilon_{\underbrace{ab \dots \tau'_\zeta}_n}^+(\vec{p}', h') e^{-ip' \cdot x'} + b(\vec{p}', h') \tilde{\varepsilon}_{\underbrace{ab \dots \tau'_\zeta}_n}^+(\vec{p}', h') e^{ip' \cdot x'}\} \\ &= \frac{1}{(2\pi)^{3/2}} \int \sum_{h, h'=n}^{-n} d^3 \vec{p} d^3 \vec{p}' \frac{\sqrt{m}}{\sqrt{2^n E}} \frac{\sqrt{m}}{\sqrt{2^n E'}} e^{-ip \cdot x} e^{ip' \cdot x'} \\ &\{\varepsilon_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) \varepsilon_{\underbrace{ab \dots \tau'_\zeta}_n}^+(\vec{p}', h') \{a(\vec{p}, h), a^+(\vec{p}', h')\} e^{ip \cdot x} e^{-ip' \cdot x'} + \tilde{\varepsilon}_{\underbrace{ab \dots \tau_\zeta}_n}(\vec{p}, h) \tilde{\varepsilon}_{\underbrace{ab \dots \tau'_\zeta}_n}^+(\vec{p}', h') \{b^+(\vec{p}, h), b(\vec{p}', h')\}\} \\ &= \frac{1}{(2\pi)^{3/2}} \int \sum_{h, h'=n}^{-n} d^3 \vec{p} d^3 \vec{p}' \frac{\sqrt{m}}{\sqrt{2^n E}} \frac{\sqrt{m}}{\sqrt{2^n E'}} \delta_{hh'} \delta(\vec{p} - \vec{p}') \end{aligned}$$



$$\begin{aligned}
& \left[ \underbrace{\varepsilon_{ab \dots \tau_\zeta}}_n(\vec{p}, h) \underbrace{\varepsilon_{ab \dots \tau'_\zeta}^+}_{n'}(\vec{p}', h') e^{ip \cdot x} e^{-ip' \cdot x'} + \underbrace{\tilde{\varepsilon}_{ab \dots \tau_\zeta}}_n(\vec{p}, h) \underbrace{\tilde{\varepsilon}_{ab \dots \tau'_\zeta}^+}_{n'}(\vec{p}', h') e^{-ip \cdot x} e^{ip' \cdot x'} \right] \\
&= \frac{im}{2^{n-1}} \int_{h=n}^{-n} \left\{ \frac{1}{(2\pi)^{3/2}} \frac{-i}{2E} \left[ \underbrace{\varepsilon_{ab \dots \tau_\zeta}}_n(\vec{p}, h) \underbrace{\varepsilon_{ab \dots \tau'_\zeta}^+}_{n'}(\vec{p}, h) e^{ip \cdot (x-x')} + \underbrace{\tilde{\varepsilon}_{ab \dots \tau_\zeta}}_n(\vec{p}, h) \underbrace{\tilde{\varepsilon}_{ab \dots \tau'_\zeta}^+}_{n'}(\vec{p}, h) e^{-ip \cdot (x-x')} \right] d^3 \vec{p} \right\} \\
&= \frac{im}{2^{n-1}} \left[ \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab \dots \tau_\zeta}}_n(-i\partial, h) \underbrace{\varepsilon_{ab \dots \tau'_\zeta}^+}_{n'}(-i\partial, h) \right] \int \left\{ \frac{1}{(2\pi)^{3/2}} \frac{-i}{2E} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3 \vec{p} \right\} \\
&= \frac{im}{2^{n-1}} \left[ \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab \dots \tau_\zeta}}_n(-i\partial, h) \underbrace{\varepsilon_{ab \dots \tau'_\zeta}^+}_{n'}(-i\partial, h) \right] \Delta(x-x') \quad \square
\end{aligned}$$

### 3.4 自旋-\$\frac{n}{2}\$粒子Rarita-Schwinger方程的自旋基

定理3.4.1.

$$\begin{cases}
U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) \\
= \frac{1}{\sqrt{C_{2n+1}^l}} \left[ \sqrt{C_{2n}^{l-1}} U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l + 1) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2n}^l} U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \right] \\
V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) \\
= \frac{1}{\sqrt{C_{2n+1}^l}} \left[ \sqrt{C_{2n}^{l-1}} V_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l + 1) v_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2n}^l} V_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l) v_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \right]
\end{cases}$$

推论3.4.1.

$$\begin{cases}
\frac{m}{E} u^{+\tau_\zeta}(\vec{p}, -\frac{1}{2}) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) = \frac{\sqrt{C_{2n}^{l-1}}}{\sqrt{C_{2n+1}^l}} U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l + 1) = \sqrt{\frac{l}{2n+1}} U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l + 1) \\
\frac{m}{E} u^{+\tau_\zeta}(\vec{p}, \frac{1}{2}) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) = \frac{\sqrt{C_{2n}^l}}{\sqrt{C_{2n+1}^l}} U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l) = \sqrt{\frac{2n+1-l}{2n+1}} U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l) \\
\frac{m}{E} v^{+\tau_\zeta}(\vec{p}, -\frac{1}{2}) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) = \frac{\sqrt{C_{2n}^{l-1}}}{\sqrt{C_{2n+1}^l}} V_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l + 1) = \sqrt{\frac{l}{2n+1}} V_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l + 1) \\
\frac{m}{E} v^{+\tau_\zeta}(\vec{p}, \frac{1}{2}) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) = \frac{\sqrt{C_{2n}^l}}{\sqrt{C_{2n+1}^l}} V_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l) = \sqrt{\frac{2n+1-l}{2n+1}} V_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta}(\vec{p}, n - l)
\end{cases}$$

$$\text{定理3.4.2. } \Lambda_{\pm \lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta}(\vec{p}, n) = \frac{2n+1}{2n+2} \left(\frac{m}{E}\right)^2 \Lambda_{\pm \lambda_\zeta \mu_\zeta \dots \tau_\zeta \lambda'_\zeta \mu'_\zeta}(\vec{p}, n + \frac{1}{2}) \Lambda_{\pm}^{\tau'_\zeta \tau_\zeta}(\vec{p}, \frac{1}{2})$$

推论3.4.2.

$$\begin{cases}
\varepsilon_{ab \dots \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) = \frac{1}{\sqrt{C_{2n+1}^l}} \left[ \sqrt{C_{2n}^{l-1}} \varepsilon_{ab \dots}(\vec{p}, n - l + 1) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2n}^l} \varepsilon_{ab \dots}(\vec{p}, n - l) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \right] \\
\tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) = \frac{1}{\sqrt{C_{2n+1}^l}} \left[ \sqrt{C_{2n}^{l-1}} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n - l + 1) v_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2n}^l} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n - l) v_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \right]
\end{cases}$$

推论3.4.3.

$$\begin{cases}
\frac{m}{E} u^{+\tau_\zeta}(\vec{p}, -\frac{1}{2}) \varepsilon_{ab \dots \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) = \frac{\sqrt{C_{2n}^{l-1}}}{\sqrt{C_{2n+1}^l}} \varepsilon_{ab \dots}(\vec{p}, n - l + 1) = \sqrt{\frac{l}{2n+1}} \varepsilon_{ab \dots}(\vec{p}, n - l + 1) \\
\frac{m}{E} u^{+\tau_\zeta}(\vec{p}, \frac{1}{2}) \varepsilon_{ab \dots \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) = \frac{\sqrt{C_{2n}^l}}{\sqrt{C_{2n+1}^l}} \varepsilon_{ab \dots}(\vec{p}, n - l) = \sqrt{\frac{2n+1-l}{2n+1}} \varepsilon_{ab \dots}(\vec{p}, n - l) \\
\frac{m}{E} v^{+\tau_\zeta}(\vec{p}, -\frac{1}{2}) \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) = \frac{\sqrt{C_{2n}^{l-1}}}{\sqrt{C_{2n+1}^l}} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n - l + 1) = \sqrt{\frac{l}{2n+1}} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n - l + 1) \\
\frac{m}{E} v^{+\tau_\zeta}(\vec{p}, \frac{1}{2}) \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) = \frac{\sqrt{C_{2n}^l}}{\sqrt{C_{2n+1}^l}} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n - l) = \sqrt{\frac{2n+1-l}{2n+1}} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n - l)
\end{cases}$$

$$\text{定理3.4.3. } \sum_{h=n}^{-n} \varepsilon_{ab \dots}(\vec{p}, h) \varepsilon_{a'b' \dots}^+(\vec{p}, h) = \frac{2n+1}{2n+2} \left(\frac{m}{E}\right)^2 \sum_{h=n+1/2}^{-(n+1/2)} \varepsilon_{ab \dots \tau_\zeta}(\vec{p}, h) \varepsilon_{a'b' \dots \tau'_\zeta}^+(\vec{p}, h) \Lambda_+^{\tau'_\zeta \tau_\zeta}(\vec{p}, \frac{1}{2})$$

$$\text{证明: } \left(\frac{m}{E}\right)^2 \sum_{l=0}^{2n+1} \left[ u^{\tau'_\zeta}(\vec{p}, -\frac{1}{2}) u^{+\tau_\zeta}(\vec{p}, -\frac{1}{2}) + u^{\tau'_\zeta}(\vec{p}, \frac{1}{2}) u^{+\tau_\zeta}(\vec{p}, \frac{1}{2}) \right] \varepsilon_{ab \dots \tau_\zeta}(\vec{p}, n + \frac{1}{2} - l) \varepsilon_{a'b' \dots \tau'_\zeta}^+(\vec{p}, n + \frac{1}{2} - l)$$

$$\begin{aligned}
&= \left(\frac{m}{E}\right)^2 \sum_{h=n+1/2}^{-(n+1/2)} \underbrace{\varepsilon_{ab\dots\tau_\zeta}}_n(\vec{p}, h) \varepsilon_{a'b'\dots\tau'_\zeta}^+(\vec{p}, h) \Lambda_+^{\tau'_\zeta\tau_\zeta} \\
&= \sum_{l=0}^{2n+1} \left[ \frac{l}{2n+1} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, n-l+1) \varepsilon_{a'b'\dots}^+(\vec{p}, n-l+1) + \frac{2n+1-l}{2n+1} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, n-l) \varepsilon_{a'b'\dots}^+(\vec{p}, n-l) \right] \\
&= \frac{2n+2}{2n+1} \sum_{l=1}^{2n+1} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, n-l+1) \varepsilon_{a'b'\dots}^+(\vec{p}, n-l+1) \\
&= \frac{2n+2}{2n+1} \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \varepsilon_{a'b'\dots}^+(\vec{p}, h)
\end{aligned}$$

□

$$\begin{aligned}
\text{证明: } &\sum_{l=0}^{2n+1} \underbrace{\varepsilon_{ab\dots\tau_\zeta}}_n(\vec{p}, n+\frac{1}{2}-l) \varepsilon_{a'b'\dots\tau'_\zeta}^+(\vec{p}, n+\frac{1}{2}-l) \\
&= \sum_{l=0}^{2n+1} \frac{1}{C_{2n+1}^l} \left[ \sqrt{C_{2n}^{l-1}} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, n-l+1) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2n}^l} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, n-l) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \right] \\
&\left[ \sqrt{C_{2n}^{l-1}} \varepsilon_{a'b'\dots}(\vec{p}, n-l+1) u_{\tau'_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2n}^l} \varepsilon_{a'b'\dots}(\vec{p}, n-l) u_{\tau'_\zeta}(\vec{p}, \frac{1}{2}) \right]^+ \\
?? = &\sum_{l=0}^{2n+1} \frac{1}{C_{2n+1}^l} \left[ C_{2n}^{l-1} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, n-l+1) \varepsilon_{a'b'\dots}^+(\vec{p}, n-l+1) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) u_{\tau'_\zeta}^+(\vec{p}, -\frac{1}{2}) \right. \\
&\left. + \sum_{l=0}^{2n+1} \frac{1}{C_{2n+1}^l} C_{2n}^l \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, n-l) \varepsilon_{a'b'\dots}^+(\vec{p}, n-l) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) u_{\tau'_\zeta}^+(\vec{p}, \frac{1}{2}) \right]
\end{aligned}$$

□

## 4 R-S方程自旋基和准投影算子的几个例子

### 4.1 自旋-1粒子R-S自旋基和Dirac自旋基的关系

引理4.1.1.

$$\begin{cases} [\gamma^a \varepsilon_a(\vec{p}, \kappa)] u(\vec{p}, \frac{\kappa}{2}) = 0, [\gamma^a \varepsilon_a(\vec{p}, -\kappa)] u(\vec{p}, \frac{\kappa}{2}) = -i\kappa \sqrt{2} \gamma_5 u(\vec{p}, -\frac{\kappa}{2}), [\gamma^a \varepsilon_a(\vec{p}, 0)] u(\vec{p}, \frac{\kappa}{2}) = -i\kappa \gamma_5 u(\vec{p}, \frac{\kappa}{2}) \\ [\gamma^a \varepsilon_a(\vec{p}, \kappa)] v(\vec{p}, \frac{\kappa}{2}) = 0, [\gamma^a \varepsilon_a(\vec{p}, -\kappa)] v(\vec{p}, \frac{\kappa}{2}) = i\kappa \sqrt{2} \gamma_5 v(\vec{p}, -\frac{\kappa}{2}), [\gamma^a \varepsilon_a(\vec{p}, 0)] v(\vec{p}, \frac{\kappa}{2}) = i\kappa \gamma_5 v(\vec{p}, \frac{\kappa}{2}) \end{cases}$$

性质4.1.1.

$$\begin{cases} u(\vec{p}, \frac{1}{2}) = -\frac{i\zeta}{\sqrt{2}} [\gamma^a \varepsilon_a(\vec{p}, 1)] v(\vec{p}, -\frac{1}{2}) = i\zeta [\gamma^a \varepsilon_a(\vec{p}, 0)] v(\vec{p}, \frac{1}{2}) \\ u(\vec{p}, -\frac{1}{2}) = \frac{i\zeta}{\sqrt{2}} [\gamma^a \varepsilon_a(\vec{p}, -1)] v(\vec{p}, \frac{1}{2}) = -i\zeta [\gamma^a \varepsilon_a(\vec{p}, 0)] v(\vec{p}, -\frac{1}{2}) \\ v(\vec{p}, \frac{1}{2}) = \frac{i\zeta}{\sqrt{2}} [\gamma^a \varepsilon_a(\vec{p}, 1)] u(\vec{p}, -\frac{1}{2}) = -i\zeta [\gamma^a \varepsilon_a(\vec{p}, 0)] u(\vec{p}, \frac{1}{2}) \\ v(\vec{p}, -\frac{1}{2}) = -\frac{i\zeta}{\sqrt{2}} [\gamma^a \varepsilon_a(\vec{p}, -1)] u(\vec{p}, \frac{1}{2}) = i\zeta [\gamma^a \varepsilon_a(\vec{p}, 0)] u(\vec{p}, -\frac{1}{2}) \end{cases}$$

定理4.1.1.

$$\begin{cases} \sum_{h=1/2}^{-1/2} u_{\tau_\zeta}(\vec{p}, h) u_{\tau'_\zeta}^+(\vec{p}, h) = \frac{1}{3} \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \varepsilon_{a'}^+(\vec{p}, h) \gamma^a \Lambda_-(\vec{p}, \frac{1}{2}) \gamma^{a'} \\ \sum_{h=1/2}^{-1/2} v_{\tau_\zeta}(\vec{p}, h) v_{\tau'_\zeta}^+(\vec{p}, h) = \frac{1}{3} \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \varepsilon_{a'}^+(\vec{p}, h) \gamma^a \Lambda_+(\vec{p}, \frac{1}{2}) \gamma^{a'} \end{cases}$$

$$\begin{aligned}
\text{证明: } &\sum_{h=1/2}^{-1/2} u_{\tau_\zeta}(\vec{p}, h) u_{\tau'_\zeta}^+(\vec{p}, h) \\
&= \frac{1}{3} \sum_{h=2}^{-2} \{ [\varepsilon_a(\vec{p}, h) \gamma^a v(\vec{p}, \frac{1}{2})] [\varepsilon_{a'}(\vec{p}, h) \gamma^{a'} v(\vec{p}, \frac{1}{2})]^+ + [\varepsilon_a(\vec{p}, h) \gamma^a v(\vec{p}, -\frac{1}{2})] [\varepsilon_{a'}(\vec{p}, h) \gamma^{a'} v(\vec{p}, -\frac{1}{2})]^+ \} \\
&= \frac{1}{3} \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \varepsilon_{a'}^+(\vec{p}, h) \gamma^a \Lambda_-(\vec{p}, \frac{1}{2}) \gamma^{a'}
\end{aligned}$$

□

$$\begin{aligned}
\text{证明: } &\sum_{h=1/2}^{-1/2} v_{\tau_\zeta}(\vec{p}, h) v_{\tau'_\zeta}^+(\vec{p}, h) \\
&= \frac{1}{3} \sum_{h=2}^{-2} \{ [\varepsilon_a(\vec{p}, h) \gamma^a u(\vec{p}, \frac{1}{2})] [\varepsilon_{a'}(\vec{p}, h) \gamma^{a'} u(\vec{p}, \frac{1}{2})]^+ + [\varepsilon_a(\vec{p}, h) \gamma^a u(\vec{p}, -\frac{1}{2})] [\varepsilon_{a'}(\vec{p}, h) \gamma^{a'} u(\vec{p}, -\frac{1}{2})]^+ \}
\end{aligned}$$

$$= \frac{1}{3} \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \varepsilon_{a'}^+(\vec{p}, h) \gamma^a \Lambda_+(\vec{p}, \frac{1}{2}) \gamma^{a'}$$

□

以上定理直接验证也是成立的。

## 4.2 自旋-2粒子R-S自旋基和Dirac自旋基的关系

引理4.2.1.

$$\begin{cases} \varepsilon_{ab}(\vec{p}, 2) \gamma^b u(\vec{p}, \frac{1}{2}) = 0 \\ \varepsilon_{ab}(\vec{p}, 1) \gamma^b u(\vec{p}, \frac{1}{2}) = -\frac{i}{\sqrt{2}} \varepsilon_a(\vec{p}, 1) \gamma_5 u(\vec{p}, \frac{1}{2}) \\ \varepsilon_{ab}(\vec{p}, 0) \gamma^b u(\vec{p}, \frac{1}{2}) = -\frac{i}{\sqrt{3}} [\varepsilon_a(\vec{p}, 1) \gamma_5 u(\vec{p}, -\frac{1}{2}) + \sqrt{2} \varepsilon_a(\vec{p}, 0) \gamma_5 u(\vec{p}, \frac{1}{2})] \\ \varepsilon_{ab}(\vec{p}, -1) \gamma^b u(\vec{p}, \frac{1}{2}) = -\frac{i}{\sqrt{2}} [\varepsilon_a(\vec{p}, -1) \gamma_5 u(\vec{p}, \frac{1}{2}) + \sqrt{2} \varepsilon_a(\vec{p}, 0) \gamma_5 u(\vec{p}, -\frac{1}{2})] \\ \varepsilon_{ab}(\vec{p}, -2) \gamma^b u(\vec{p}, \frac{1}{2}) = -i\sqrt{2} \varepsilon_a(\vec{p}, -1) \gamma_5 u(\vec{p}, -\frac{1}{2}) \end{cases}$$

引理4.2.2.

$$\begin{cases} \varepsilon_{ab}(\vec{p}, 2) \gamma^b u(\vec{p}, -\frac{1}{2}) = i\sqrt{2} \varepsilon_a(\vec{p}, 1) \gamma_5 u(\vec{p}, \frac{1}{2}) \\ \varepsilon_{ab}(\vec{p}, 1) \gamma^b u(\vec{p}, -\frac{1}{2}) = \frac{i}{\sqrt{2}} [\varepsilon_a(\vec{p}, 1) \gamma_5 u(\vec{p}, -\frac{1}{2}) + \sqrt{2} \varepsilon_a(\vec{p}, 0) \gamma_5 u(\vec{p}, \frac{1}{2})] \\ \varepsilon_{ab}(\vec{p}, 0) \gamma^b u(\vec{p}, -\frac{1}{2}) = \frac{i}{\sqrt{3}} [\varepsilon_a(\vec{p}, -1) \gamma_5 u(\vec{p}, \frac{1}{2}) + \sqrt{2} \varepsilon_a(\vec{p}, 0) \gamma_5 u(\vec{p}, -\frac{1}{2})] \\ \varepsilon_{ab}(\vec{p}, -1) \gamma^b u(\vec{p}, -\frac{1}{2}) = \frac{i}{\sqrt{2}} \varepsilon_a(\vec{p}, -1) \gamma_5 u(\vec{p}, -\frac{1}{2}) \\ \varepsilon_{ab}(\vec{p}, -2) \gamma^b u(\vec{p}, -\frac{1}{2}) = 0 \end{cases}$$

引理4.2.3.

$$\begin{cases} \varepsilon_{ab}(\vec{p}, 2) \gamma^b v(\vec{p}, \frac{1}{2}) = 0 \\ \varepsilon_{ab}(\vec{p}, 1) \gamma^b v(\vec{p}, \frac{1}{2}) = \frac{i}{\sqrt{2}} \varepsilon_a(\vec{p}, 1) \gamma_5 v(\vec{p}, \frac{1}{2}) \\ \varepsilon_{ab}(\vec{p}, 0) \gamma^b v(\vec{p}, \frac{1}{2}) = \frac{i}{\sqrt{3}} [\varepsilon_a(\vec{p}, 1) \gamma_5 v(\vec{p}, -\frac{1}{2}) + \sqrt{2} \varepsilon_a(\vec{p}, 0) \gamma_5 v(\vec{p}, \frac{1}{2})] \\ \varepsilon_{ab}(\vec{p}, -1) \gamma^b v(\vec{p}, \frac{1}{2}) = \frac{i}{\sqrt{2}} [\varepsilon_a(\vec{p}, -1) \gamma_5 v(\vec{p}, \frac{1}{2}) + \sqrt{2} \varepsilon_a(\vec{p}, 0) \gamma_5 v(\vec{p}, -\frac{1}{2})] \\ \varepsilon_{ab}(\vec{p}, -2) \gamma^b v(\vec{p}, \frac{1}{2}) = i\sqrt{2} \varepsilon_a(\vec{p}, -1) \gamma_5 v(\vec{p}, -\frac{1}{2}) \end{cases}$$

引理4.2.4.

$$\begin{cases} \varepsilon_{ab}(\vec{p}, 2) \gamma^b v(\vec{p}, -\frac{1}{2}) = -i\sqrt{2} \varepsilon_a(\vec{p}, 1) \gamma_5 v(\vec{p}, \frac{1}{2}) \\ \varepsilon_{ab}(\vec{p}, 1) \gamma^b v(\vec{p}, -\frac{1}{2}) = -\frac{i}{\sqrt{2}} [\varepsilon_a(\vec{p}, 1) \gamma_5 v(\vec{p}, -\frac{1}{2}) + \sqrt{2} \varepsilon_a(\vec{p}, 0) \gamma_5 v(\vec{p}, \frac{1}{2})] \\ \varepsilon_{ab}(\vec{p}, 0) \gamma^b v(\vec{p}, -\frac{1}{2}) = -\frac{i}{\sqrt{3}} [\varepsilon_a(\vec{p}, -1) \gamma_5 v(\vec{p}, \frac{1}{2}) + \sqrt{2} \varepsilon_a(\vec{p}, 0) \gamma_5 v(\vec{p}, -\frac{1}{2})] \\ \varepsilon_{ab}(\vec{p}, -1) \gamma^b v(\vec{p}, -\frac{1}{2}) = -\frac{i}{\sqrt{2}} \varepsilon_a(\vec{p}, -1) \gamma_5 v(\vec{p}, -\frac{1}{2}) \\ \varepsilon_{ab}(\vec{p}, -2) \gamma^b v(\vec{p}, -\frac{1}{2}) = 0 \end{cases}$$

## 4.3 自旋- $\frac{3}{2}$ 粒子Rarita-Schwinger方程的准投影算子

性质4.3.1.

$$\begin{cases} \varepsilon_{a\tau_\zeta}(\vec{p}, \frac{3}{2}) = \varepsilon_a(\vec{p}, 1) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \\ \varepsilon_{a\tau_\zeta}(\vec{p}, \frac{1}{2}) = \frac{1}{\sqrt{3}} [\varepsilon_a(\vec{p}, 1) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{2} \varepsilon_a(\vec{p}, 0) u_{\tau_\zeta}(\vec{p}, \frac{1}{2})] \\ \varepsilon_{a\tau_\zeta}(\vec{p}, -\frac{1}{2}) = \frac{1}{\sqrt{3}} [\varepsilon_a(\vec{p}, -1) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \sqrt{2} \varepsilon_a(\vec{p}, 0) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})] \\ \varepsilon_{a\tau_\zeta}(\vec{p}, -\frac{3}{2}) = \varepsilon_a(\vec{p}, -1) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\ \gamma^a \varepsilon_{a[\tau_\zeta]}(\vec{p}, h) = 0 \end{cases}$$

性质4.3.2.

$$\begin{cases} \tilde{\varepsilon}_{a\tau_\zeta}(\vec{p}, \frac{3}{2}) = -\varepsilon_a(\vec{p}, 1)v_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \\ \tilde{\varepsilon}_{a\tau_\zeta}(\vec{p}, \frac{1}{2}) = -\frac{1}{\sqrt{3}}[\varepsilon_a(\vec{p}, 1)v_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p}, 0)v_{\tau_\zeta}(\vec{p}, \frac{1}{2})] \\ \tilde{\varepsilon}_{a\tau_\zeta}(\vec{p}, -\frac{1}{2}) = -\frac{1}{\sqrt{3}}[\varepsilon_a(\vec{p}, -1)v_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \sqrt{2}\varepsilon_a(\vec{p}, 0)v_{\tau_\zeta}(\vec{p}, -\frac{1}{2})] \\ \tilde{\varepsilon}_{a\tau_\zeta}(\vec{p}, -\frac{3}{2}) = -\varepsilon_a(\vec{p}, -1)v_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\ \gamma^a \tilde{\varepsilon}_{a[\tau_\zeta]}(\vec{p}, h) = 0 \end{cases}$$

推论4.3.1.

$$\begin{cases} \varepsilon_{a[\tau_\zeta]}(\vec{p}, \frac{3}{2}) = -\frac{i\zeta}{\sqrt{2}}\varepsilon_{ab}(\vec{p}, 2)\gamma^b v(\vec{p}, -\frac{1}{2}) = i\zeta\sqrt{2}\varepsilon_a(\vec{p}, 1)\gamma_5 v(\vec{p}, \frac{1}{2}) \\ \varepsilon_{a[\tau_\zeta]}(\vec{p}, \frac{1}{2}) = -i\zeta\frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p}, 1)\gamma^b v(\vec{p}, -\frac{1}{2}) = i\zeta\varepsilon_{ab}(\vec{p}, 0)\gamma^b v(\vec{p}, \frac{1}{2}) \\ \varepsilon_{a[\tau_\zeta]}(\vec{p}, -\frac{1}{2}) = i\zeta\frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p}, -1)\gamma^b v(\vec{p}, \frac{1}{2}) = -i\zeta\varepsilon_{ab}(\vec{p}, 0)\gamma^b v(\vec{p}, -\frac{1}{2}) \\ \varepsilon_{a[\tau_\zeta]}(\vec{p}, -\frac{3}{2}) = \frac{i\zeta}{\sqrt{2}}\varepsilon_{ab}(\vec{p}, -2)\gamma^b v(\vec{p}, \frac{1}{2}) = -i\zeta\sqrt{2}\varepsilon_a(\vec{p}, -1)\gamma_5 v(\vec{p}, -\frac{1}{2}) \\ \gamma^a \varepsilon_{a[\tau_\zeta]}(\vec{p}, h) = 0 \end{cases}$$

推论4.3.2.

$$\begin{cases} \tilde{\varepsilon}_{a[\tau_\zeta]}(\vec{p}, \frac{3}{2}) = -\frac{i\zeta}{\sqrt{2}}\varepsilon_{ab}(\vec{p}, 2)\gamma^b u(\vec{p}, -\frac{1}{2}) = i\zeta\sqrt{2}\varepsilon_a(\vec{p}, 1)\gamma_5 u(\vec{p}, \frac{1}{2}) \\ \tilde{\varepsilon}_{a[\tau_\zeta]}(\vec{p}, \frac{1}{2}) = -i\zeta\frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p}, 1)\gamma^b u(\vec{p}, -\frac{1}{2}) = i\zeta\varepsilon_{ab}(\vec{p}, 0)\gamma^b u(\vec{p}, \frac{1}{2}) \\ \tilde{\varepsilon}_{a[\tau_\zeta]}(\vec{p}, -\frac{1}{2}) = i\zeta\frac{\sqrt{2}}{\sqrt{3}}\varepsilon_{ab}(\vec{p}, -1)\gamma^b u(\vec{p}, \frac{1}{2}) = -i\zeta\varepsilon_{ab}(\vec{p}, 0)\gamma^b u(\vec{p}, -\frac{1}{2}) \\ \tilde{\varepsilon}_{a[\tau_\zeta]}(\vec{p}, -\frac{3}{2}) = \frac{i\zeta}{\sqrt{2}}\varepsilon_{ab}(\vec{p}, -2)\gamma^b u(\vec{p}, \frac{1}{2}) = -i\zeta\sqrt{2}\varepsilon_a(\vec{p}, -1)\gamma_5 u(\vec{p}, -\frac{1}{2}) \\ \gamma^a \tilde{\varepsilon}_{a[\tau_\zeta]}(\vec{p}, h) = 0 \end{cases}$$

定理4.3.1.

$$\begin{cases} \sum_{h=3/2}^{-3/2} \varepsilon_{a[\tau_\zeta]}(\vec{p}, h)\varepsilon_{a'[\tau_\zeta]}^+(\vec{p}, h) = \frac{2}{5} \sum_{h=2}^{-2} \varepsilon_{ab}(\vec{p}, h)\varepsilon_{a'b'}^+(\vec{p}, h)\gamma^b \Lambda_-(\vec{p}, \frac{1}{2})\gamma^{b'} \\ \sum_{h=3/2}^{-3/2} \tilde{\varepsilon}_{a[\tau_\zeta]}(\vec{p}, h)\tilde{\varepsilon}_{a'[\tau_\zeta]}^+(\vec{p}, h) = \frac{2}{5} \sum_{h=2}^{-2} \varepsilon_{ab}(\vec{p}, h)\varepsilon_{a'b'}^+(\vec{p}, h)\gamma^b \Lambda_+(\vec{p}, \frac{1}{2})\gamma^{b'} \end{cases}$$

$$\text{证明: } \sum_{h=3/2}^{-3/2} \varepsilon_{a[\tau_\zeta]}(\vec{p}, h)\varepsilon_{a'[\tau_\zeta]}^+(\vec{p}, h)$$

$$= \frac{2}{5} \sum_{h=2}^{-2} \{[\varepsilon_{ab}(\vec{p}, h)\gamma^b v(\vec{p}, \frac{1}{2})][\varepsilon_{a'b'}(\vec{p}, h)\gamma^{b'} v(\vec{p}, \frac{1}{2})]^+ + [\varepsilon_{ab}(\vec{p}, h)\gamma^b v(\vec{p}, -\frac{1}{2})][\varepsilon_{a'b'}(\vec{p}, h)\gamma^{b'} v(\vec{p}, -\frac{1}{2})]^+\}$$

$$= \frac{2}{5} \sum_{h=2}^{-2} \varepsilon_{ab}(\vec{p}, h)\varepsilon_{a'b'}^+(\vec{p}, h)\gamma^b \Lambda_-(\vec{p}, \frac{1}{2})\gamma^{b'} \quad \square$$

$$\text{证明: } \sum_{h=3/2}^{-3/2} \tilde{\varepsilon}_{a[\tau_\zeta]}(\vec{p}, h)\tilde{\varepsilon}_{a'[\tau_\zeta]}^+(\vec{p}, h)$$

$$= \frac{2}{5} \sum_{h=2}^{-2} \{[\varepsilon_{ab}(\vec{p}, h)\gamma^b u(\vec{p}, \frac{1}{2})][\varepsilon_{a'b'}(\vec{p}, h)\gamma^{b'} u(\vec{p}, \frac{1}{2})]^+ + [\varepsilon_{ab}(\vec{p}, h)\gamma^b u(\vec{p}, -\frac{1}{2})][\varepsilon_{a'b'}(\vec{p}, h)\gamma^{b'} u(\vec{p}, -\frac{1}{2})]^+\}$$

$$= \frac{2}{5} \sum_{h=2}^{-2} \varepsilon_{ab}(\vec{p}, h)\varepsilon_{a'b'}^+(\vec{p}, h)\gamma^b \Lambda_+(\vec{p}, \frac{1}{2})\gamma^{b'} \quad \square$$

#### 4.4 自旋- $n + \frac{1}{2}$ 粒子Rarita-Schwinger方程的准投影算子猜想证明

引理4.4.1.

$$\begin{cases} [\gamma^a \varepsilon_a(\vec{p}, \kappa)]u(\vec{p}, \frac{\kappa}{2}) = 0, [\gamma^a \varepsilon_a(\vec{p}, -\kappa)]u(\vec{p}, \frac{\kappa}{2}) = i\sqrt{2}\kappa\zeta v(\vec{p}, -\frac{\kappa}{2}), [\gamma^a \varepsilon_a(\vec{p}, 0)]u(\vec{p}, \frac{\kappa}{2}) = i\kappa\zeta v(\vec{p}, \frac{\kappa}{2}) \\ [\gamma^a \varepsilon_a(\vec{p}, \kappa)]v(\vec{p}, \frac{\kappa}{2}) = 0, [\gamma^a \varepsilon_a(\vec{p}, -\kappa)]v(\vec{p}, \frac{\kappa}{2}) = -i\sqrt{2}\kappa\zeta u(\vec{p}, -\frac{\kappa}{2}), [\gamma^a \varepsilon_a(\vec{p}, 0)]v(\vec{p}, \frac{\kappa}{2}) = -i\kappa\zeta u(\vec{p}, \frac{\kappa}{2}) \end{cases}$$

引理4.4.2.  $\varepsilon_{\underbrace{a \dots b \dots c}_{n+1}}(\vec{p}, n+1)\gamma^c u(\vec{p}, \frac{1}{2})$

$$= \frac{1}{\sqrt{C_{2(n+1)}^0}} \frac{1}{(n+1)!} \sqrt{2^0} C_{n+1}^0 C_{n+1-0}^0 \underbrace{\varepsilon_{\{a(\vec{p}, 1) \dots}_{n+1}}}_{n+1} \cdot \underbrace{\varepsilon_{\{b(\vec{p}, 0) \dots\}_0}}_0 \cdot \underbrace{\varepsilon_{\{c(\vec{p}, -1) \dots\}_0}}_0 \gamma^c u(\vec{p}, \frac{1}{2})$$

$$= \underbrace{\varepsilon_a(\vec{p}, 1) \cdot \varepsilon_b(\vec{p}, 1)}_n \cdot \varepsilon_c(\vec{p}, 1) \gamma^c u(\vec{p}, \frac{1}{2}) = 0$$

$$\begin{aligned} \text{引理4.4.3. } & \varepsilon_{a \dots b \dots c}(\vec{p}, n+1) \gamma^c u(\vec{p}, -\frac{1}{2}) \\ &= \frac{1}{\sqrt{C_{2(n+1)}^0}} \frac{1}{(n+1)!} \sqrt{2^0} C_{n+1}^0 C_{n+1-0}^0 \underbrace{\varepsilon_{\{a}(\vec{p}, 1)}_{n+1}}_0 \cdot \underbrace{\varepsilon_b(\vec{p}, 0)}_0 \cdot \underbrace{\varepsilon_{c\}}_0(\vec{p}, -1) \gamma^c u(\vec{p}, -\frac{1}{2}) \\ &= \varepsilon_a(\vec{p}, 1) \cdot \varepsilon_b(\vec{p}, 1) \cdot \varepsilon_c(\vec{p}, 1) \gamma^c u(\vec{p}, \frac{1}{2}) \\ &= -i\zeta \sqrt{2} (-1)^n \underbrace{\tilde{\varepsilon}_a(\vec{p}, 1) \cdot \tilde{\varepsilon}_b(\vec{p}, 1)}_n \cdot v(\vec{p}, \frac{1}{2}) \\ &= -i\zeta \sqrt{2} (-1)^n \tilde{\varepsilon}_{a \dots b \dots [\tau_c]}(\vec{p}, n + \frac{1}{2}) \end{aligned}$$

$$\begin{aligned} \text{引理4.4.4. } & \varepsilon_{a \dots b \dots c \dots}(\vec{p}, n) \gamma^c u(\vec{p}, \frac{1}{2}) \\ &= \frac{1}{\sqrt{C_{2(n+1)}^1}} \frac{1}{(n+1)!} \sqrt{2^1} C_{n+1}^1 C_n^0 \underbrace{\varepsilon_{\{a}(\vec{p}, 1)}_n}_1 \cdot \underbrace{\varepsilon_b(\vec{p}, 0)}_1 \cdot \underbrace{\varepsilon_{c\}}_0(\vec{p}, -1) \gamma^c u(\vec{p}, \frac{1}{2}) \\ &= \frac{1}{\sqrt{n+1}} \\ & \underbrace{[\varepsilon_a(\vec{p}, 1) \cdot \varepsilon_b(\vec{p}, 1) \cdot \varepsilon_c(\vec{p}, 0) + \dots + \varepsilon_a(\vec{p}, 1) \cdot \varepsilon_b(\vec{p}, 0) \cdot \varepsilon_c(\vec{p}, 1) + \dots + \varepsilon_a(\vec{p}, 0) \cdot \varepsilon_b(\vec{p}, 1) \cdot \varepsilon_c(\vec{p}, 1)]}_{n+1} \gamma^c u(\vec{p}, \frac{1}{2}) \\ &= \frac{1}{\sqrt{n+1}} \varepsilon_a(\vec{p}, 1) \cdot \varepsilon_b(\vec{p}, 1) \cdot \varepsilon_c(\vec{p}, 0) \gamma^c u(\vec{p}, \frac{1}{2}) \\ &= \frac{i\zeta}{\sqrt{n+1}} (-1)^n \underbrace{\tilde{\varepsilon}_a(\vec{p}, 1) \cdot \tilde{\varepsilon}_b(\vec{p}, 1)}_n \cdot v(\vec{p}, \frac{1}{2}) \\ &= \frac{i\zeta}{\sqrt{n+1}} (-1)^n \tilde{\varepsilon}_{a \dots b \dots [\tau_c]}(\vec{p}, n + \frac{1}{2}) \end{aligned}$$

$$\begin{aligned} \text{引理4.4.5. } & \varepsilon_{a \dots b \dots c \dots}(\vec{p}, n) \gamma^c u(\vec{p}, -\frac{1}{2}) \\ &= \frac{1}{\sqrt{C_{2(n+1)}^1}} \frac{1}{(n+1)!} \sqrt{2^1} C_{n+1}^1 C_n^0 \underbrace{\varepsilon_{\{a}(\vec{p}, 1)}_n}_1 \cdot \underbrace{\varepsilon_b(\vec{p}, 0)}_1 \cdot \underbrace{\varepsilon_{c\}}_0(\vec{p}, -1) \gamma^c u(\vec{p}, -\frac{1}{2}) \\ &= \frac{1}{\sqrt{n+1}} \\ & \underbrace{[\varepsilon_a(\vec{p}, 1) \cdot \varepsilon_b(\vec{p}, 1) \cdot \varepsilon_c(\vec{p}, 0) + \dots + \varepsilon_a(\vec{p}, 1) \cdot \varepsilon_b(\vec{p}, 0) \cdot \varepsilon_c(\vec{p}, 1) + \dots + \varepsilon_a(\vec{p}, 0) \cdot \varepsilon_b(\vec{p}, 1) \cdot \varepsilon_c(\vec{p}, 1)]}_{n+1} \gamma^c u(\vec{p}, -\frac{1}{2}) \\ &= \frac{1}{\sqrt{n+1}} \\ & \{ \underbrace{\varepsilon_a(\vec{p}, 1) \cdot \varepsilon_b(\vec{p}, 1) \cdot \varepsilon_c(\vec{p}, 0)}_n \gamma^c u(\vec{p}, -\frac{1}{2}) + \underbrace{[\varepsilon_a(\vec{p}, 0) \cdot \varepsilon_b(\vec{p}, 1) \cdot \dots + \dots + \varepsilon_a(\vec{p}, 1) \cdot \varepsilon_b(\vec{p}, 0) \cdot \dots]}_n \varepsilon_c(\vec{p}, 1) \gamma^c u(\vec{p}, -\frac{1}{2}) \} \\ &= -\frac{i\zeta}{\sqrt{n+1}} (-1)^n \{ \underbrace{\tilde{\varepsilon}_a(\vec{p}, 1) \cdot \tilde{\varepsilon}_b(\vec{p}, 1)}_n \cdot v(\vec{p}, -\frac{1}{2}) + \sqrt{2} \underbrace{[\tilde{\varepsilon}_a(\vec{p}, 0) \cdot \tilde{\varepsilon}_b(\vec{p}, 1) \cdot \dots + \dots + \tilde{\varepsilon}_a(\vec{p}, 1) \cdot \tilde{\varepsilon}_b(\vec{p}, 0) \cdot \dots]}_n v(\vec{p}, \frac{1}{2}) \} \\ &= -\frac{i\zeta}{\sqrt{n+1}} (-1)^n \frac{1}{n!} [\sqrt{2^0} C_n^0 C_{n-0}^0 \underbrace{\tilde{\varepsilon}_{\{a}(\vec{p}, 1)}_n}_n \cdot v(\vec{p}, -\frac{1}{2}) + \sqrt{2^1} C_n^1 C_{n-1}^0 \underbrace{\tilde{\varepsilon}_{\{a}(\vec{p}, 0) \cdot \tilde{\varepsilon}_b(\vec{p}, 1) \cdot \dots}_n}_n v(\vec{p}, \frac{1}{2})] \\ &= -\frac{i\zeta \sqrt{2n+1}}{\sqrt{n+1}} (-1)^n \frac{1}{\sqrt{C_{2n+1}^1}} [\sqrt{C_{2n}^0} \tilde{\varepsilon}_{a \dots b \dots} v(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2n}^1} \tilde{\varepsilon}_{a \dots b \dots} v(\vec{p}, \frac{1}{2})] \\ &= -\frac{i\zeta \sqrt{2n+1}}{\sqrt{n+1}} (-1)^n \tilde{\varepsilon}_{a \dots b \dots [\tau_c]}(\vec{p}, n - \frac{1}{2}) \end{aligned}$$

$$\begin{aligned} \text{引理4.4.6. } & \varepsilon_{a \dots b \dots c}(\vec{p}, n-1) \gamma^c u(\vec{p}, \frac{1}{2}) \\ &= \frac{1}{\sqrt{C_{2(n+1)}^2}} \frac{1}{(n+1)!} \\ & [\sqrt{2^0} C_{n+1}^0 C_{n+1-0}^1 \underbrace{\varepsilon_{\{a}(\vec{p}, 1)}_n}_0 \cdot \underbrace{\varepsilon_b(\vec{p}, 0)}_0 \cdot \underbrace{\varepsilon_{c\}}_1(\vec{p}, -1) + \sqrt{2^2} C_{n+1}^2 C_{n-1}^0 \underbrace{\varepsilon_{\{a}(\vec{p}, 1)}_{n-1}}_{n-1} \cdot \underbrace{\varepsilon_b(\vec{p}, 0)}_2 \cdot \underbrace{\varepsilon_{c\}}_0(\vec{p}, -1)] \gamma^c u(\vec{p}, \frac{1}{2}) \\ &= \frac{1}{\sqrt{C_{2(n+1)}^2}} (-1)^n \underbrace{[\tilde{\varepsilon}_a(\vec{p}, 1) \cdot \tilde{\varepsilon}_b(\vec{p}, 1) \cdot \varepsilon_c(\vec{p}, -1)]}_n \gamma^c u(\vec{p}, \frac{1}{2}) + \sqrt{2} \sqrt{2^1} \frac{1}{(n-1)!} \underbrace{\tilde{\varepsilon}_{\{a}(\vec{p}, 0) \cdot \tilde{\varepsilon}_b(\vec{p}, 1) \cdot \dots}_n}_n \varepsilon_c(\vec{p}, 0) \gamma^c u(\vec{p}, \frac{1}{2}) \\ &= \frac{i\sqrt{2(2n+1)}\zeta}{\sqrt{C_{2(n+1)}^2}} (-1)^n \frac{1}{\sqrt{C_{2n+1}^2}} \end{aligned}$$

$$\begin{aligned}
& [\sqrt{2^0} C_n^0 C_{n-0}^0 \frac{1}{n!} \underbrace{\tilde{\varepsilon}_{\{a}(\vec{p}, 1) \cdot \tilde{\varepsilon}_b(\vec{p}, 1) \cdot \cdot \}_{n}}_{n} v(\vec{p}, -\frac{1}{2}) + \sqrt{2^1} C_n^1 C_{n-1}^0 \frac{1}{n!} \underbrace{\tilde{\varepsilon}_{\{a}(\vec{p}, 0) \cdot \tilde{\varepsilon}_b(\vec{p}, 1) \cdot \cdot \}_{n}}_{n} v(\vec{p}, \frac{1}{2})] \\
&= \frac{i\sqrt{2}}{\sqrt{n+1}} (-1)^n \frac{1}{\sqrt{C_{2n+1}^2}} [\sqrt{C_{2n}^0} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b \cdot \cdot}_{n+1}}_{n+1} v(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2n}^1} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b \cdot \cdot}_{n+1}}_{n+1} v(\vec{p}, \frac{1}{2})] \\
&= \frac{i\sqrt{2}}{\sqrt{n+1}} (-1)^n \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b \cdot \cdot [\tau_c]}_{n+1}}_{n+1}(\vec{p}, n - \frac{1}{2})
\end{aligned}$$

定理4.4.1.

$$\begin{cases} \gamma^c u(\vec{p}, \frac{1}{2}) \underbrace{\tilde{\varepsilon}_{a \cdot \cdot bc}_{n+1}}_{n+1}(\vec{p}, h) = -i\zeta \frac{\sqrt{n+1-h}}{\sqrt{n+1}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b[\tau_c]}_{n+1}}_{n+1}(\vec{p}, h + \frac{1}{2}) \\ \gamma^c u(\vec{p}, -\frac{1}{2}) \underbrace{\tilde{\varepsilon}_{a \cdot \cdot bc}_{n+1}}_{n+1}(\vec{p}, h) = i\zeta \frac{\sqrt{n+1+h}}{\sqrt{n+1}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b[\tau_c]}_{n+1}}_{n+1}(\vec{p}, h - \frac{1}{2}) \end{cases} \quad \begin{cases} \gamma^c v(\vec{p}, \frac{1}{2}) \underbrace{\varepsilon_{a \cdot \cdot bc}_{n+1}}_{n+1}(\vec{p}, h) = -i\zeta \frac{\sqrt{n+1-h}}{\sqrt{n+1}} \underbrace{\varepsilon_{a \cdot \cdot b[\tau_c]}_{n+1}}_{n+1}(\vec{p}, h + \frac{1}{2}) \\ \gamma^c v(\vec{p}, -\frac{1}{2}) \underbrace{\varepsilon_{a \cdot \cdot bc}_{n+1}}_{n+1}(\vec{p}, h) = i\zeta \frac{\sqrt{n+1+h}}{\sqrt{n+1}} \underbrace{\varepsilon_{a \cdot \cdot b[\tau_c]}_{n+1}}_{n+1}(\vec{p}, h - \frac{1}{2}) \end{cases}$$

证明:  $\gamma^c u(\vec{p}, \frac{1}{2}) \underbrace{\tilde{\varepsilon}_{a \cdot \cdot bc}_{n+1}}_{n+1}(\vec{p}, h)$

$$\begin{aligned}
&= \frac{\sqrt{C_{n+1+h}^2 C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h-1) \gamma^c u(\vec{p}, \frac{1}{2}) \tilde{\varepsilon}_c(\vec{p}, 1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h) \gamma^c u(\vec{p}, \frac{1}{2}) \tilde{\varepsilon}_c(\vec{p}, 0) \\
&+ \frac{\sqrt{C_{n+1-h}^2 C_{n+1+h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h+1) \gamma^c u(\vec{p}, \frac{1}{2}) \tilde{\varepsilon}_c(\vec{p}, -1) \\
&= -\frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h) i\zeta v(\vec{p}, \frac{1}{2}) - \frac{\sqrt{C_{n+1-h}^2 C_{n+1+h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h+1) i\sqrt{2}\zeta v(\vec{p}, -\frac{1}{2}) \\
&= -i\zeta \frac{\sqrt{n+1-h}}{\sqrt{n+1}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b[\tau_c]}_{n+1}}_{n+1}(\vec{p}, h + \frac{1}{2}) \quad \square
\end{aligned}$$

证明:  $\gamma^c u(\vec{p}, -\frac{1}{2}) \underbrace{\tilde{\varepsilon}_{a \cdot \cdot bc}_{n+1}}_{n+1}(\vec{p}, h)$

$$\begin{aligned}
&= \frac{\sqrt{C_{n+1+h}^2 C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h-1) \gamma^c u(\vec{p}, -\frac{1}{2}) \tilde{\varepsilon}_c(\vec{p}, 1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h) \gamma^c u(\vec{p}, -\frac{1}{2}) \tilde{\varepsilon}_c(\vec{p}, 0) \\
&+ \frac{\sqrt{C_{n+1-h}^2 C_{n+1+h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h+1) \gamma^c u(\vec{p}, -\frac{1}{2}) \tilde{\varepsilon}_c(\vec{p}, -1) \\
&= \frac{\sqrt{C_{n+1+h}^2 C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h-1) i\sqrt{2}\zeta v(\vec{p}, \frac{1}{2}) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h) i\zeta v(\vec{p}, -\frac{1}{2}) \\
&= i\zeta \frac{\sqrt{n+1+h}}{\sqrt{n+1}} \underbrace{\tilde{\varepsilon}_{a \cdot \cdot b[\tau_c]}_{n+1}}_{n+1}(\vec{p}, h - \frac{1}{2}) \quad \square
\end{aligned}$$

证明:  $\gamma^c v(\vec{p}, \frac{1}{2}) \underbrace{\varepsilon_{a \cdot \cdot bc}_{n+1}}_{n+1}(\vec{p}, h)$

$$\begin{aligned}
&= \frac{\sqrt{C_{n+1+h}^2 C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\varepsilon_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h-1) \gamma^c v(\vec{p}, \frac{1}{2}) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \underbrace{\varepsilon_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h) \gamma^c v(\vec{p}, \frac{1}{2}) \varepsilon_c(\vec{p}, 0) \\
&+ \frac{\sqrt{C_{n+1-h}^2 C_{n+1+h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\varepsilon_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h+1) \gamma^c v(\vec{p}, \frac{1}{2}) \varepsilon_c(\vec{p}, -1) \\
&= -\frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \underbrace{\varepsilon_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h) i\zeta u(\vec{p}, \frac{1}{2}) - \frac{\sqrt{C_{n+1-h}^2 C_{n+1+h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\varepsilon_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h+1) i\sqrt{2}\zeta u(\vec{p}, -\frac{1}{2}) \\
&= -i\zeta \frac{\sqrt{n+1-h}}{\sqrt{n+1}} \underbrace{\varepsilon_{a \cdot \cdot b[\tau_c]}_{n+1}}_{n+1}(\vec{p}, h + \frac{1}{2}) \quad \square
\end{aligned}$$

证明:  $\gamma^c v(\vec{p}, -\frac{1}{2}) \underbrace{\varepsilon_{a \cdot \cdot bc}_{n+1}}_{n+1}(\vec{p}, h)$

$$\begin{aligned}
&= \frac{\sqrt{C_{n+1+h}^2 C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\varepsilon_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h-1) \gamma^c v(\vec{p}, -\frac{1}{2}) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \underbrace{\varepsilon_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h) \gamma^c v(\vec{p}, -\frac{1}{2}) \varepsilon_c(\vec{p}, 0) \\
&+ \frac{\sqrt{C_{n+1-h}^2 C_{n+1+h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\varepsilon_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h+1) \gamma^c v(\vec{p}, -\frac{1}{2}) \varepsilon_c(\vec{p}, -1) \\
&= \frac{\sqrt{C_{n+1+h}^2 C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \underbrace{\varepsilon_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h-1) i\sqrt{2}\zeta u(\vec{p}, \frac{1}{2}) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \underbrace{\varepsilon_{a \cdot \cdot b}_{n+1}}_{n+1}(\vec{p}, h) i\zeta u(\vec{p}, -\frac{1}{2}) \\
&= i\zeta \frac{\sqrt{n+1+h}}{\sqrt{n+1}} \underbrace{\varepsilon_{a \cdot \cdot b[\tau_c]}_{n+1}}_{n+1}(\vec{p}, h - \frac{1}{2}) \quad \square
\end{aligned}$$

推论4.4.1.

$$\begin{cases} \underbrace{\tilde{\varepsilon}_{a \dots b \dots c}}_{n+1}(\vec{p}, n+1-l)\gamma^c u(\vec{p}, \frac{1}{2}) = -i\zeta \frac{\sqrt{l}}{\sqrt{n+1}} \underbrace{\tilde{\varepsilon}_{a \dots b[\tau_\zeta]}}_n(\vec{p}, n-l+\frac{3}{2}) \\ \underbrace{\tilde{\varepsilon}_{a \dots b \dots c}}_{n+1}(\vec{p}, n+1-l)\gamma^c u(\vec{p}, -\frac{1}{2}) = i\zeta \frac{\sqrt{2n+2-l}}{\sqrt{n+1}} \underbrace{\tilde{\varepsilon}_{a \dots b[\tau_\zeta]}}_n(\vec{p}, n-l+\frac{1}{2}) \\ \underbrace{\varepsilon_{a \dots b \dots c}}_{n+1}(\vec{p}, n+1-l)\gamma^c v(\vec{p}, \frac{1}{2}) = -i\zeta \frac{\sqrt{l}}{\sqrt{n+1}} \varepsilon_{a \dots b[\tau_\zeta]}(\vec{p}, n-l+\frac{3}{2}) \\ \underbrace{\varepsilon_{a \dots b \dots c}}_{n+1}(\vec{p}, n+1-l)\gamma^c v(\vec{p}, -\frac{1}{2}) = i\zeta \frac{\sqrt{2n+2-l}}{\sqrt{n+1}} \varepsilon_{a \dots b[\tau_\zeta]}(\vec{p}, n-l+\frac{1}{2}) \end{cases}$$

推论4.4.2.

$$\begin{cases} \sum_{h=n+1/2}^{-(n+1/2)} \underbrace{\varepsilon_{ab \dots [\tau_\zeta]}}_n(\vec{p}, h) \varepsilon_{a'b' \dots [\tau'_\zeta]}^+(\vec{p}, h) = \frac{n+1}{2n+3} \sum_{h=n+1}^{-(n+1)} \sum_{h'=1/2}^{-1/2} [\underbrace{\varepsilon_{ab \dots c}}_{n+1}(\vec{p}, h) \gamma^c v(\vec{p}, h')] [\underbrace{\varepsilon_{a'b' \dots c'}}_{n+1}(\vec{p}, h) \gamma^{c'} v(\vec{p}, h')] + \\ \sum_{h=n+1/2}^{-(n+1/2)} \underbrace{\tilde{\varepsilon}_{ab \dots [\tau_\zeta]}}_n(\vec{p}, h) \tilde{\varepsilon}_{a'b' \dots [\tau'_\zeta]}^+(\vec{p}, h) = \frac{n+1}{2n+3} \sum_{h=n+1}^{-(n+1)} \sum_{h'=1/2}^{-1/2} [\underbrace{\tilde{\varepsilon}_{ab \dots c}}_{n+1}(\vec{p}, h) \gamma^c u(\vec{p}, h')] [\underbrace{\tilde{\varepsilon}_{a'b' \dots c'}}_{n+1}(\vec{p}, h) \gamma^{c'} u(\vec{p}, h')] + \end{cases}$$

推论4.4.3.

$$\begin{cases} \sum_{h=n+1/2}^{-(n+1/2)} \underbrace{\varepsilon_{ab \dots [\tau_\zeta]}}_n(\vec{p}, h) \varepsilon_{a'b' \dots [\tau'_\zeta]}^+(\vec{p}, h) = \frac{1}{2} \frac{2n+2}{2n+3} \sum_{h=n+1}^{-(n+1)} \underbrace{\varepsilon_{ab \dots c}}_{n+1}(\vec{p}, h) \varepsilon_{a'b' \dots c'}^+(\vec{p}, h) \gamma^c \Lambda_-(\vec{p}, \frac{1}{2}) \gamma^{c'} \\ \sum_{h=n+1/2}^{-(n+1/2)} \underbrace{\tilde{\varepsilon}_{ab \dots [\tau_\zeta]}}_n(\vec{p}, h) \tilde{\varepsilon}_{a'b' \dots [\tau'_\zeta]}^+(\vec{p}, h) = \frac{1}{2} \frac{2n+2}{2n+3} \sum_{h=n+1}^{-(n+1)} \underbrace{\tilde{\varepsilon}_{ab \dots c}}_{n+1}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \dots c'}^+(\vec{p}, h) \gamma^c \Lambda_+(\vec{p}, \frac{1}{2}) \gamma^{c'} \end{cases}$$

## 4.5 自旋 $-n + \frac{1}{2}$ 粒子Rarita-Schwinger方程自旋基的合成

推论4.5.1.

$$\begin{cases} \underbrace{\varepsilon_{ab \dots \tau_\zeta}}_n(\vec{p}, n+\frac{1}{2}-l) = \frac{1}{\sqrt{C_{2n+1}^l}} [\sqrt{C_{2n}^{l-1}} \underbrace{\varepsilon_{ab \dots}}_n(\vec{p}, n-l+1) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2n}^l} \underbrace{\varepsilon_{ab \dots}}_n(\vec{p}, n-l) u_{\tau_\zeta}(\vec{p}, \frac{1}{2})] \\ \underbrace{\tilde{\varepsilon}_{ab \dots \tau_\zeta}}_n(\vec{p}, n+\frac{1}{2}-l) = \frac{1}{\sqrt{C_{2n+1}^l}} [\sqrt{C_{2n}^{l-1}} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n-l+1) v_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \sqrt{C_{2n}^l} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n-l) v_{\tau_\zeta}(\vec{p}, \frac{1}{2})] \end{cases}$$

推论4.5.2.

$$\begin{cases} \frac{m}{E} u^{+\tau_\zeta}(\vec{p}, -\frac{1}{2}) \underbrace{\varepsilon_{ab \dots \tau_\zeta}}_n(\vec{p}, n+\frac{1}{2}-l) = \frac{\sqrt{C_{2n}^{l-1}}}{\sqrt{C_{2n+1}^l}} \underbrace{\varepsilon_{ab \dots}}_n(\vec{p}, n-l+1) = \sqrt{\frac{l}{2n+1}} \underbrace{\varepsilon_{ab \dots}}_n(\vec{p}, n-l+1) \\ \frac{m}{E} u^{+\tau_\zeta}(\vec{p}, \frac{1}{2}) \underbrace{\varepsilon_{ab \dots \tau_\zeta}}_n(\vec{p}, n+\frac{1}{2}-l) = \frac{\sqrt{C_{2n}^l}}{\sqrt{C_{2n+1}^l}} \underbrace{\varepsilon_{ab \dots}}_n(\vec{p}, n-l) = \sqrt{\frac{2n+1-l}{2n+1}} \underbrace{\varepsilon_{ab \dots}}_n(\vec{p}, n-l) \\ \frac{m}{E} v^{+\tau_\zeta}(\vec{p}, -\frac{1}{2}) \underbrace{\tilde{\varepsilon}_{ab \dots \tau_\zeta}}_n(\vec{p}, n+\frac{1}{2}-l) = \frac{\sqrt{C_{2n}^{l-1}}}{\sqrt{C_{2n+1}^l}} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n-l+1) = \sqrt{\frac{l}{2n+1}} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n-l+1) \\ \frac{m}{E} v^{+\tau_\zeta}(\vec{p}, \frac{1}{2}) \underbrace{\tilde{\varepsilon}_{ab \dots \tau_\zeta}}_n(\vec{p}, n+\frac{1}{2}-l) = \frac{\sqrt{C_{2n}^l}}{\sqrt{C_{2n+1}^l}} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n-l) = \sqrt{\frac{2n+1-l}{2n+1}} \tilde{\varepsilon}_{ab \dots}(\vec{p}, n-l) \end{cases}$$

$$\text{推论4.5.3. } \varepsilon_a(\vec{p}, \kappa) = -\frac{i}{\sqrt{2}} u^T(\vec{p}, \frac{\kappa}{2}) \bar{C} \gamma_a u(\vec{p}, \frac{\kappa}{2}), \varepsilon_a(\vec{p}, 0) = -i u^T(\vec{p}, \frac{\kappa}{2}) \bar{C} \gamma_a u(\vec{p}, \frac{\kappa}{2})$$

推论4.5.4.  $\varepsilon_{a \dots bc}(\vec{p}, h)$

$$= \frac{\sqrt{C_{2n+1+h}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \dots b}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \dots b}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \dots b}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1)$$

定理4.5.1.  $\varepsilon_{a \dots bc}(\vec{p}, n+1-l)$

$$= -\frac{i}{\sqrt{2}} u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_c \frac{\sqrt{2n+2-l}}{\sqrt{2n+2}} \varepsilon_{a \dots b[\tau_\zeta]}(\vec{p}, n+\frac{1}{2}-l) - \frac{i}{\sqrt{2}} u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_c \frac{\sqrt{l}}{\sqrt{2n+2}} \varepsilon_{a \dots b[\tau_\zeta]}(\vec{p}, n+\frac{3}{2}-l)$$

证明:

$$\varepsilon_{a \dots bc}(\vec{p}, n+1-l)$$

$$= \frac{\sqrt{C_{2n+2-l}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \dots b}(\vec{p}, n-l) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{2n+2-l}^1 C_l^1}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \dots b}(\vec{p}, n+1-l) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_l^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \dots b}(\vec{p}, n+2-l) \varepsilon_c(\vec{p}, -1)$$

$$\begin{aligned}
&= \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n-l) \left[ -\frac{i}{\sqrt{2}} u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_c u(\vec{p}, \frac{1}{2}) \right] \\
&+ \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+1-l) \left\{ \left[ -\frac{i}{2} u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_c u(\vec{p}, \frac{1}{2}) \right] + \left[ -\frac{i}{2} u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_c u(\vec{p}, -\frac{1}{2}) \right] \right\} \\
&+ \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+2-l) \left[ -\frac{i}{\sqrt{2}} u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_c u(\vec{p}, -\frac{1}{2}) \right] \\
&= -\frac{i}{\sqrt{2}} u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_c \left\{ \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n-l) u(\vec{p}, \frac{1}{2}) + \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+1-l) u(\vec{p}, -\frac{1}{2}) \right\} \\
&- \frac{i}{\sqrt{2}} u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_c \left\{ \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+1-l) u(\vec{p}, \frac{1}{2}) + \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+2-l) u(\vec{p}, -\frac{1}{2}) \right\} \\
&= -\frac{i}{\sqrt{2}} u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_c \left\{ \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \frac{\sqrt{C_{2n+1}^2}}{\sqrt{C_{2n+1}^2}} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n-l) u(\vec{p}, \frac{1}{2}) + \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \frac{\sqrt{C_{2n+1}^2}}{\sqrt{C_{2n+1}^2}} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+1-l) u(\vec{p}, -\frac{1}{2}) \right\} \\
&- \frac{i}{\sqrt{2}} u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_c \left\{ \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \frac{\sqrt{C_{2n+1}^2}}{\sqrt{C_{2n+1}^2}} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+1-l) u(\vec{p}, \frac{1}{2}) + \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \frac{\sqrt{C_{2n+1}^2}}{\sqrt{C_{2n+1}^2}} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+2-l) u(\vec{p}, -\frac{1}{2}) \right\} \\
&= -\frac{i}{\sqrt{2}} u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_c \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \frac{\sqrt{C_{2n+1}^2}}{\sqrt{C_{2n+1}^2}} \left[ -\frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n+1}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n-l) u(\vec{p}, \frac{1}{2}) + \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n+1}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+1-l) u(\vec{p}, -\frac{1}{2}) \right] \\
&- \frac{i}{\sqrt{2}} u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_c \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \frac{\sqrt{C_{2n+1}^2}}{\sqrt{C_{2n+1}^2}} \left\{ \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n+1}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+1-l) u(\vec{p}, \frac{1}{2}) + \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n+1}^2}} \varepsilon_{a \cdot \cdot b}(\vec{p}, n+2-l) u(\vec{p}, -\frac{1}{2}) \right\} \\
&= -\frac{i}{\sqrt{2}} u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_c \frac{\sqrt{2n+2-l}}{\sqrt{2n+2}} \varepsilon_{a \cdot \cdot b[\tau_c]}(\vec{p}, n+\frac{1}{2}-l) - \frac{i}{\sqrt{2}} u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_c \frac{\sqrt{l}}{\sqrt{2n+2}} \varepsilon_{a \cdot \cdot b[\tau_c]}(\vec{p}, n+\frac{3}{2}-l) \\
&= -\frac{i}{\sqrt{2}} u^T(\vec{p}, \frac{1}{2}) \gamma_4 \gamma_2 \gamma_c \frac{\sqrt{2n+2-l}}{\sqrt{2n+2}} \varepsilon_{a \cdot \cdot b[\tau_c]}(\vec{p}, n+\frac{1}{2}-l) - \frac{i}{\sqrt{2}} u^T(\vec{p}, -\frac{1}{2}) \gamma_4 \gamma_2 \gamma_c \frac{\sqrt{l}}{\sqrt{2n+2}} \varepsilon_{a \cdot \cdot b[\tau_c]}(\vec{p}, n+\frac{3}{2}-l) \quad \square
\end{aligned}$$

$$\text{推论4.5.5. } \varepsilon_{a \cdot \cdot bc}(\vec{p}, h) = -\frac{i}{2} \left[ \frac{\sqrt{n+1+h}}{\sqrt{n+1}} u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_c \varepsilon_{a \cdot \cdot b[\tau_c]}(\vec{p}, h-\frac{1}{2}) + \frac{\sqrt{n+1-h}}{\sqrt{n+1}} u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_c \varepsilon_{a \cdot \cdot b[\tau_c]}(\vec{p}, h+\frac{1}{2}) \right]$$

$$\text{推论4.5.6. } \varepsilon_{a \cdot \cdot bc}(\vec{p}, h) = -\frac{i}{2} \left[ \frac{\sqrt{n+1+h}}{\sqrt{n+1}} \varepsilon_{a \cdot \cdot b \tau_c}(\vec{p}, h-\frac{1}{2}) u_{\sigma_c}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{n+1-h}}{\sqrt{n+1}} \varepsilon_{a \cdot \cdot b \tau_c}(\vec{p}, h+\frac{1}{2}) u_{\sigma_c}(\hat{p}, -\frac{1}{2}) \right] (\bar{C} \gamma_c)^{\tau_c \sigma_c}$$

## 5 各种势准投影算子的缩减模式

### 5.1 各种势准投影算子的正式定义

定义5.1.1.

$$\left\{ \begin{aligned}
\Lambda_{m \underbrace{ab \cdot \cdot}_{n} \underbrace{a'b' \cdot \cdot}_{n}}(\vec{p}, n) &:= \sum_{h=n}^{-n} \varepsilon_{ab \cdot \cdot}(\vec{p}, h) \varepsilon_{a'b' \cdot \cdot}^+(\vec{p}, h) = \sum_{h=n}^{-n} \tilde{\varepsilon}_{ab \cdot \cdot}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \cdot \cdot}^+(\vec{p}, h) \\
\Lambda_{+m \underbrace{ab \cdot \cdot}_{n} \tau_c \underbrace{a'b' \cdot \cdot}_{n} \tau_c'}(\vec{p}, n+\frac{1}{2}) &:= \sum_{h=n+1/2}^{-(n+1/2)} \varepsilon_{ab \cdot \cdot \tau_c}(\vec{p}, h) \varepsilon_{a'b' \cdot \cdot \tau_c'}^+(\vec{p}, h) \\
\Lambda_{-m \underbrace{ab \cdot \cdot}_{n} \tau_c \underbrace{a'b' \cdot \cdot}_{n} \tau_c'}(\vec{p}, n+\frac{1}{2}) &:= \sum_{h=n+1/2}^{-(n+1/2)} \tilde{\varepsilon}_{ab \cdot \cdot \tau_c}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \cdot \cdot \tau_c'}^+(\vec{p}, h)
\end{aligned} \right.$$

### 5.2 各种势准投影算子之间的关系-最简缩减模式

定理5.2.1.

$$\left\{ \begin{aligned}
\sum_{h=n}^{-n} \varepsilon_{ab \cdot \cdot}(\vec{p}, h) \varepsilon_{a'b' \cdot \cdot}^+(\vec{p}, h) &= \frac{2n+1}{2n+2} \left( \frac{m}{E} \right)^2 \sum_{h=n+1/2}^{-(n+1/2)} \varepsilon_{ab \cdot \cdot \tau_c}(\vec{p}, h) \varepsilon_{a'b' \cdot \cdot \tau_c'}^+(\vec{p}, h) \Lambda_{+}^{\tau_c' \tau_c}(\vec{p}, \frac{1}{2}) \\
\sum_{h=n+1/2}^{-(n+1/2)} \varepsilon_{ab \cdot \cdot [\tau_c]}(\vec{p}, h) \varepsilon_{a'b' \cdot \cdot [\tau_c]}^+(\vec{p}, h) &= \frac{2n+2}{2n+3} \sum_{h=n+1}^{-(n+1)} \varepsilon_{ab \cdot \cdot c}(\vec{p}, h) \varepsilon_{a'b' \cdot \cdot c'}^+(\vec{p}, h) \gamma^c \Lambda_{-}(\vec{p}, \frac{1}{2}) \gamma^{c'}
\end{aligned} \right.$$

定理5.2.2.

$$\left\{ \begin{aligned}
\sum_{h=n}^{-n} \tilde{\varepsilon}_{ab \cdot \cdot}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \cdot \cdot}^+(\vec{p}, h) &= \frac{2n+1}{2n+2} \left( \frac{m}{E} \right)^2 \sum_{h=n+1/2}^{-(n+1/2)} \tilde{\varepsilon}_{ab \cdot \cdot \tau_c}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \cdot \cdot \tau_c'}^+(\vec{p}, h) \Lambda_{-}^{\tau_c' \tau_c}(\vec{p}, \frac{1}{2}) \\
\sum_{h=n+1/2}^{-(n+1/2)} \tilde{\varepsilon}_{ab \cdot \cdot [\tau_c]}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \cdot \cdot [\tau_c]}^+(\vec{p}, h) &= \frac{2n+2}{2n+3} \sum_{h=n+1}^{-(n+1)} \tilde{\varepsilon}_{ab \cdot \cdot c}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \cdot \cdot c'}^+(\vec{p}, h) \gamma^c \Lambda_{+}(\vec{p}, \frac{1}{2}) \gamma^{c'}
\end{aligned} \right.$$



## 推论5.2.1.

$$\begin{cases} \sum_{h=n}^{-n} \varepsilon_{ab \dots}(\vec{p}, h) \varepsilon_{a'b' \dots}^+(\vec{p}, h) = \frac{2n+1}{2n+3} \sum_{h=n+1}^{-(n+1)} \varepsilon_{ab \dots c}(\vec{p}, h) \varepsilon_{a'b' \dots c'}^+(\vec{p}, h) \eta^{cc'} \\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \varepsilon_{ab \dots \tau_\zeta}(\vec{p}, h) \varepsilon_{a'b' \dots \tau'_\zeta}^+(\vec{p}, h) = \frac{2n+2}{2n+4} \sum_{h=n+\frac{3}{2}}^{-(n+\frac{3}{2})} \varepsilon_{ab \dots c \tau_\zeta}(\vec{p}, h) \varepsilon_{a'b' \dots c' \tau'_\zeta}^+(\vec{p}, h) \eta^{cc'} \end{cases}$$

$$\begin{aligned} \text{证明: } & \sum_{h=n}^{-n} \varepsilon_{ab \dots}(\vec{p}, h) \varepsilon_{a'b' \dots}^+(\vec{p}, h) \\ &= \frac{2n+1}{2n+2} \left(\frac{m}{E}\right)^2 \sum_{h=n+1/2}^{-(n+1/2)} \varepsilon_{ab \dots \tau_\zeta}(\vec{p}, h) \varepsilon_{a'b' \dots \tau'_\zeta}^+(\vec{p}, h) \Lambda_+^{\tau'_\zeta \tau_\zeta}(\vec{p}, \frac{1}{2}) \\ &= \frac{2n+1}{4n+6} \left(\frac{m}{E}\right)^2 \sum_{h=n+1}^{-(n+1)} \varepsilon_{ab \dots c}(\vec{p}, h) \varepsilon_{a'b' \dots c'}^+(\vec{p}, h) [\gamma^c \Lambda_-(\vec{p}, \frac{1}{2}) \gamma^{c'}]_{\tau_\zeta \tau'_\zeta} \Lambda_+^{\tau'_\zeta \tau_\zeta}(\vec{p}, \frac{1}{2}) \\ &= \frac{2n+1}{4n+6} \left(\frac{m}{E}\right)^2 \sum_{h=n+1}^{-(n+1)} \varepsilon_{ab \dots c}(\vec{p}, h) \varepsilon_{a'b' \dots c'}^+(\vec{p}, h) \text{tr}[\gamma^c \Lambda_-(\vec{p}, \frac{1}{2}) \gamma^{c'} \Lambda_+(\vec{p}, \frac{1}{2})] \\ &= \frac{2n+1}{2n+3} \sum_{h=n+1}^{-(n+1)} \varepsilon_{ab \dots c}(\vec{p}, h) \varepsilon_{a'b' \dots c'}^+(\vec{p}, h) \eta^{cc'} \end{aligned} \quad \square$$

$$\begin{aligned} \text{证明: } & \sum_{h=n+1/2}^{-(n+1/2)} \varepsilon_{ab \dots \tau_\zeta}(\vec{p}, h) \varepsilon_{a'b' \dots \tau'_\zeta}^+(\vec{p}, h) \\ &= \frac{2n+2}{2n+3} \frac{1}{2} \sum_{h=n+1}^{-(n+1)} \varepsilon_{ab \dots c}(\vec{p}, h) \varepsilon_{a'b' \dots c'}^+(\vec{p}, h) \gamma^c \Lambda_-(\vec{p}, \frac{1}{2}) \gamma^{c'} \\ &= \frac{2n+2}{2n+3} \frac{1}{2} \frac{2n+3}{2} \frac{2n+3}{2n+5} \sum_{h=n+2}^{-(n+2)} \varepsilon_{ab \dots cd}(\vec{p}, h) \varepsilon_{a'b' \dots c'd'}^+(\vec{p}, h) \gamma^c \Lambda_-(\vec{p}, \frac{1}{2}) \gamma^{c'} \eta^{dd'} \\ &= \frac{2n+2}{2n+5} \frac{2n+5}{2n+4} \frac{1}{2} \frac{2n+4}{2n+5} \sum_{h=n+2}^{-(n+2)} \varepsilon_{ab \dots cd}(\vec{p}, h) \varepsilon_{a'b' \dots c'd'}^+(\vec{p}, h) \gamma^c \Lambda_-(\vec{p}, \frac{1}{2}) \gamma^{c'} \eta^{dd'} \\ &= \frac{2n+2}{2n+4} \frac{1}{2} \frac{2n+4}{2n+5} \sum_{h=n+2}^{-(n+2)} \varepsilon_{ab \dots cd}(\vec{p}, h) \varepsilon_{a'b' \dots c'd'}^+(\vec{p}, h) \gamma^d \Lambda_-(\vec{p}, \frac{1}{2}) \gamma^{d'} \eta^{cc'} \\ &= \frac{2n+2}{2n+4} \sum_{h=n+3/2}^{-(n+3/2)} \varepsilon_{ab \dots c \tau_\zeta}(\vec{p}, h) \varepsilon_{a'b' \dots c' \tau'_\zeta}^+(\vec{p}, h) \eta^{cc'} \end{aligned} \quad \square$$

## 推论5.2.2.

$$\begin{cases} \sum_{h=n}^{-n} \tilde{\varepsilon}_{ab \dots}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \dots}^+(\vec{p}, h) = \frac{2n+1}{2n+3} \sum_{h=n+1}^{-(n+1)} \tilde{\varepsilon}_{ab \dots c}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \dots c'}^+(\vec{p}, h) \eta^{cc'} \\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \dots \tau'_\zeta}^+(\vec{p}, h) = \frac{2n+2}{2n+4} \sum_{h=n+\frac{3}{2}}^{-(n+\frac{3}{2})} \tilde{\varepsilon}_{ab \dots c \tau_\zeta}(\vec{p}, h) \tilde{\varepsilon}_{a'b' \dots c' \tau'_\zeta}^+(\vec{p}, h) \eta^{cc'} \end{cases}$$

## 推论5.2.3.

$$\begin{cases} \sum_{h=n}^{-n} \varepsilon_{a \dots}(\vec{p}, h) \varepsilon_{a' \dots}^+(\vec{p}, h) = \frac{2n+1}{2(n+m)+1} \sum_{h=n+m}^{-(n+m)} \varepsilon_{a \dots c \dots d}(\vec{p}, h) \varepsilon_{a' \dots c' \dots d'}^+(\vec{p}, h) \eta^{cc'} \dots \eta^{dd'} \\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \varepsilon_{a \dots \tau_\zeta}(\vec{p}, h) \varepsilon_{a' \dots \tau'_\zeta}^+(\vec{p}, h) = \frac{2n+2}{2(n+m)+2} \sum_{h=n+m+\frac{1}{2}}^{-(n+m+\frac{1}{2})} \varepsilon_{a \dots c \dots d \tau_\zeta}(\vec{p}, h) \varepsilon_{a' \dots c' \dots d' \tau'_\zeta}^+(\vec{p}, h) \eta^{cc'} \dots \eta^{dd'} \end{cases}$$

## 推论5.2.4.

$$\begin{cases} \sum_{h=n}^{-n} \tilde{\varepsilon}_{a \dots a'}(\vec{p}, h) \tilde{\varepsilon}_{a' \dots a}^+(\vec{p}, h) = \frac{2n+1}{2(n+m)+1} \sum_{h=n+m}^{-(n+m)} \tilde{\varepsilon}_{a \dots c \dots d}(\vec{p}, h) \tilde{\varepsilon}_{a' \dots c' \dots d'}^+(\vec{p}, h) \underbrace{\eta^{cc'} \dots \eta^{dd'}}_m \\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \tilde{\varepsilon}_{a \dots \tau_\zeta}(\vec{p}, h) \tilde{\varepsilon}_{a' \dots \tau'_\zeta}^+(\vec{p}, h) = \frac{2n+2}{2(n+m)+2} \sum_{h=n+m+\frac{1}{2}}^{-(n+m+\frac{1}{2})} \tilde{\varepsilon}_{a \dots c \dots d \tau_\zeta}(\vec{p}, h) \tilde{\varepsilon}_{a' \dots c' \dots d' \tau'_\zeta}^+(\vec{p}, h) \underbrace{\eta^{cc'} \dots \eta^{dd'}}_m \end{cases}$$

### 5.3 各种势准投影算子最简缩减模式的重新整理

定理5.3.1.

$$\begin{cases} \Lambda_{m \underbrace{ab \dots a'}_n \underbrace{b' \dots a'}_n}(\vec{p}, n) = \frac{2n+1}{2n+2} \left(\frac{m}{E}\right)^2 \Lambda_{\pm m \underbrace{ab \dots \tau_\zeta} \underbrace{a' b' \dots \tau'_\zeta}}(\vec{p}, n + \frac{1}{2}) \Lambda_{\pm}^{\tau'_\zeta \tau_\zeta}(\vec{p}, \frac{1}{2}) \\ \Lambda_{\pm m \underbrace{ab \dots [\tau_\zeta] a' b' \dots [\tau'_\zeta]}}(\vec{p}, n + \frac{1}{2}) = \frac{2n+2}{2n+3} \frac{1}{2} \Lambda_{m \underbrace{ab \dots c} \underbrace{a' b' \dots c'}}(\vec{p}, n + 1) \gamma^c \Lambda_{\mp}(\vec{p}, \frac{1}{2}) \gamma^{c'} \end{cases}$$

推论5.3.1.

$$\begin{cases} \Lambda_{m \underbrace{ab \dots a'}_n \underbrace{b' \dots a'}_n}(\vec{p}, n) = \frac{2n+1}{2n+3} \Lambda_{m \underbrace{ab \dots c} \underbrace{a' b' \dots c'}}(\vec{p}, n + 1) \eta^{cc'} \\ \Lambda_{\pm m \underbrace{ab \dots \tau_\zeta} \underbrace{a' b' \dots \tau'_\zeta}}(\vec{p}, n + \frac{1}{2}) = \frac{2n+2}{2n+4} \Lambda_{\pm m \underbrace{ab \dots c} \underbrace{a' b' \dots c'}}(\vec{p}, n + \frac{3}{2}) \eta^{cc'} \end{cases}$$

推论5.3.2.

$$\begin{cases} \Lambda_{m \underbrace{ab \dots a'}_n \underbrace{b' \dots a'}_n}(\vec{p}, n) = \frac{2n+1}{2(n+l)+1} \Lambda_{m \underbrace{ab \dots c \dots d} \underbrace{a' b' \dots c' \dots d'}}(\vec{p}, n + l) \underbrace{\eta^{cc'} \dots \eta^{dd'}}_l \\ \Lambda_{\pm m \underbrace{ab \dots \tau_\zeta} \underbrace{a' b' \dots \tau'_\zeta}}(\vec{p}, n + \frac{1}{2}) = \frac{2(n+\frac{1}{2})+1}{2(n+l+\frac{1}{2})+1} \Lambda_{\pm m \underbrace{ab \dots c \dots d} \underbrace{a' b' \dots c' \dots d'}}(\vec{p}, n + l + \frac{1}{2}) \underbrace{\eta^{cc'} \dots \eta^{dd'}}_l \end{cases}$$

推论5.3.3.

$$\begin{cases} \Lambda_{m \underbrace{ab \dots a'}_n \underbrace{b' \dots a'}_n}(\vec{p}, n) = \frac{2n+1}{2(n+l+\frac{1}{2})+1} \left(\frac{m}{E}\right)^2 \Lambda_{\pm m \underbrace{ab \dots c \dots d} \underbrace{a' b' \dots c' \dots d'}}(\vec{p}, n + l + \frac{1}{2}) \underbrace{\eta^{cc'} \dots \eta^{dd'}}_l \Lambda_{\pm}^{\tau'_\zeta \tau_\zeta}(\vec{p}, \frac{1}{2}) \\ \Lambda_{\pm m \underbrace{ab \dots [\tau_\zeta] a' b' \dots [\tau'_\zeta]}}(\vec{p}, n + \frac{1}{2}) = \frac{2(n+\frac{1}{2})+1}{2(n+l+\frac{1}{2})+1} \frac{1}{2} \Lambda_{m \underbrace{ab \dots c \dots d} \underbrace{a' b' \dots c' \dots d'}}(\vec{p}, n + l + \frac{1}{2}) \underbrace{\eta^{cc'} \dots \eta^{dd'}}_l \gamma^e \Lambda_{\mp}(\vec{p}, \frac{1}{2}) \gamma^{e'} \end{cases}$$

### 5.4 各种势准投影算子之间的关系-物理缩减模式

推论5.4.1.

$$\begin{cases} \sum_{h=n}^{-n} \varepsilon_{ab \dots}(\vec{p}, h) \varepsilon_{a' b' \dots}^+(\vec{p}, h) = \frac{2n+1}{2n+3} \sum_{h=n+1}^{-(n+1)} \varepsilon_{ab \dots c}(\vec{p}, h) \varepsilon_{a' b' \dots c'}^+(\vec{p}, h) (\eta^{cc'} + \frac{p^c p^{c'}}{m^2}) \\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \varepsilon_{ab \dots \tau_\zeta}(\vec{p}, h) \varepsilon_{a' b' \dots \tau'_\zeta}^+(\vec{p}, h) = \frac{2n+2}{2n+4} \sum_{h=n+\frac{3}{2}}^{-(n+\frac{3}{2})} \varepsilon_{ab \dots c \tau_\zeta}(\vec{p}, h) \varepsilon_{a' b' \dots c' \tau'_\zeta}^+(\vec{p}, h) (\eta^{cc'} + \frac{p^c p^{c'}}{m^2}) \end{cases}$$

推论5.4.2.

$$\begin{cases} \sum_{h=n}^{-n} \tilde{\varepsilon}_{ab \dots}(\vec{p}, h) \tilde{\varepsilon}_{a' b' \dots}^+(\vec{p}, h) = \frac{2n+1}{2n+3} \sum_{h=n+1}^{-(n+1)} \tilde{\varepsilon}_{ab \dots c}(\vec{p}, h) \tilde{\varepsilon}_{a' b' \dots c'}^+(\vec{p}, h) (\eta^{cc'} + \frac{p^c p^{c'}}{m^2}) \\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}, h) \tilde{\varepsilon}_{a' b' \dots \tau'_\zeta}^+(\vec{p}, h) = \frac{2n+2}{2n+4} \sum_{h=n+\frac{3}{2}}^{-(n+\frac{3}{2})} \tilde{\varepsilon}_{ab \dots c \tau_\zeta}(\vec{p}, h) \tilde{\varepsilon}_{a' b' \dots c' \tau'_\zeta}^+(\vec{p}, h) (\eta^{cc'} + \frac{p^c p^{c'}}{m^2}) \end{cases}$$

推论5.4.3.

$$\begin{cases} \sum_{h=n}^{-n} \varepsilon_{a \dots}(\vec{p}, h) \varepsilon_{a' \dots}^+(\vec{p}, h) = \frac{2n+1}{2(n+m)+1} \sum_{h=n+m}^{-(n+m)} \varepsilon_{a \dots c \dots d}(\vec{p}, h) \varepsilon_{a' \dots c' \dots d'}^+(\vec{p}, h) \underbrace{(\eta^{cc'} + \frac{p^c p^{c'}}{m^2}) \dots (\eta^{dd'} + \frac{p^d p^{d'}}{m^2})}_m \\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \varepsilon_{a \dots \tau_\zeta}(\vec{p}, h) \varepsilon_{a' \dots \tau'_\zeta}^+(\vec{p}, h) \\ = \frac{2n+2}{2(n+m)+2} \sum_{h=n+m+\frac{1}{2}}^{-(n+m+\frac{1}{2})} \varepsilon_{a \dots c \dots d \tau_\zeta}(\vec{p}, h) \varepsilon_{a' \dots c' \dots d' \tau'_\zeta}^+(\vec{p}, h) \underbrace{(\eta^{cc'} + \frac{p^c p^{c'}}{m^2}) \dots (\eta^{dd'} + \frac{p^d p^{d'}}{m^2})}_m \end{cases}$$

推论5.4.4.

$$\left\{ \begin{aligned} \sum_{h=n}^{-n} \tilde{\varepsilon}_{a \dots}(\vec{p}, h) \tilde{\varepsilon}_{a' \dots}^+(\vec{p}, h) &= \frac{2n+1}{2(n+m)+1} \sum_{h=n+m}^{-(n+m)} \tilde{\varepsilon}_{a \dots c \dots d}(\vec{p}, h) \tilde{\varepsilon}_{a' \dots c' \dots d'}^+(\vec{p}, h) \underbrace{(\eta^{cc'} + \frac{p^c p^{c'}}{m^2}) \cdot (\eta^{dd'} + \frac{p^d p^{d'}}{m^2})}_m \\ \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \tilde{\varepsilon}_{a \dots} \tau_\zeta(\vec{p}, h) \tilde{\varepsilon}_{a' \dots}^+ \tau'_\zeta(\vec{p}, h) \\ &= \frac{2n+2}{2(n+m)+2} \sum_{h=n+m+\frac{1}{2}}^{-(n+m+\frac{1}{2})} \tilde{\varepsilon}_{a \dots c \dots d} \tau_\zeta(\vec{p}, h) \tilde{\varepsilon}_{a' \dots c' \dots d'}^+ \tau'_\zeta(\vec{p}, h) \underbrace{(\eta^{cc'} + \frac{p^c p^{c'}}{m^2}) \cdot (\eta^{dd'} + \frac{p^d p^{d'}}{m^2})}_m \end{aligned} \right.$$

## 5.5 各种势准投影算子物理缩减模式的重新整理

定理5.5.1.

$$\left\{ \begin{aligned} \Lambda_{m \underbrace{ab \dots}_{n} \underbrace{a'b' \dots}_{n}}(\vec{p}, n) &= \frac{2n+1}{2n+3} \Lambda_{m \underbrace{ab \dots}_{n+1} \underbrace{a'b' \dots}_{n+1}}(\vec{p}, n+1) \Lambda_m^{cc'}(\vec{p}, 1) \\ \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_\zeta \underbrace{a'b' \dots}_{n} \tau'_\zeta}(\vec{p}, n + \frac{1}{2}) &= \frac{2n+2}{2n+4} \Lambda_{\pm m \underbrace{ab \dots}_{n+1} \tau_\zeta \underbrace{a'b' \dots}_{n+1} \tau'_\zeta}(\vec{p}, n + \frac{3}{2}) \Lambda_m^{cc'}(\vec{p}, 1) \end{aligned} \right.$$

推论5.5.1.

$$\left\{ \begin{aligned} \Lambda_{m \underbrace{ab \dots}_{n} \underbrace{a'b' \dots}_{n}}(\vec{p}, n) &= \frac{2n+1}{2(n+l)+1} \Lambda_{m \underbrace{ab \dots c \dots d}_{n+l} \underbrace{a'b' \dots c' \dots d'}_{n+l}}(\vec{p}, n+l) \underbrace{\Lambda_m^{cc'}(\vec{p}, 1) \cdot \Lambda_m^{dd'}(\vec{p}, 1)}_l \\ \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_\zeta \underbrace{a'b' \dots}_{n} \tau'_\zeta}(\vec{p}, n + \frac{1}{2}) &= \frac{2(n+\frac{1}{2})+1}{2(n+l+\frac{1}{2})+1} \Lambda_{\pm m \underbrace{ab \dots c \dots d}_{n+l} \tau_\zeta \underbrace{a'b' \dots c' \dots d'}_{n+l} \tau'_\zeta}(\vec{p}, n+l + \frac{1}{2}) \underbrace{\Lambda_m^{cc'}(\vec{p}, 1) \cdot \Lambda_m^{dd'}(\vec{p}, 1)}_l \end{aligned} \right.$$

推论5.5.2.

$$\left\{ \begin{aligned} \Lambda_{m \underbrace{ab \dots}_{n} \underbrace{a'b' \dots}_{n}}(\vec{p}, n) &= \frac{2n+1}{2(n+2)+1} \Lambda_{m \underbrace{ab \dots cd}_{n+2} \underbrace{a'b' \dots c'd'}_{n+2}}(\vec{p}, n+2) \Lambda_m^{cdc'd'}(\vec{p}, 2) \\ \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_\zeta \underbrace{a'b' \dots}_{n} \tau'_\zeta}(\vec{p}, n + \frac{1}{2}) &= \frac{2(n+\frac{1}{2})+1}{2(n+2+\frac{1}{2})+1} \Lambda_{\pm m \underbrace{ab \dots cd}_{n+2} \tau_\zeta \underbrace{a'b' \dots c'd'}_{n+2} \tau'_\zeta}(\vec{p}, n+2 + \frac{1}{2}) \Lambda_m^{cdc'd'}(\vec{p}, 2) \end{aligned} \right.$$

推论5.5.3.

$$\left\{ \begin{aligned} \Lambda_{m \underbrace{ab \dots}_{n} \underbrace{a'b' \dots}_{n}}(\vec{p}, n) &= \frac{2n+1}{2(n+l)+1} \Lambda_{m \underbrace{ab \dots c \dots d}_{n+l} \underbrace{a'b' \dots c' \dots d'}_{n+l}}(\vec{p}, n+l) \underbrace{\Lambda_m^{c \dots d c' \dots d'}}_l(\vec{p}, l) \\ \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_\zeta \underbrace{a'b' \dots}_{n} \tau'_\zeta}(\vec{p}, n + \frac{1}{2}) &= \frac{2(n+\frac{1}{2})+1}{2(n+l+\frac{1}{2})+1} \Lambda_{\pm m \underbrace{ab \dots c \dots d}_{n+l} \tau_\zeta \underbrace{a'b' \dots c' \dots d'}_{n+l} \tau'_\zeta}(\vec{p}, n+l + \frac{1}{2}) \underbrace{\Lambda_m^{c \dots d c' \dots d'}}_l(\vec{p}, l) \end{aligned} \right.$$

推论5.5.4.

$$\left\{ \begin{aligned} \Lambda_{m \underbrace{ab \dots}_{n} \underbrace{a'b' \dots}_{n}}(\vec{p}, n) &= \frac{2n+1}{2(n+l+\frac{1}{2})+1} \left(\frac{m}{E}\right)^2 \Lambda_{\pm m \underbrace{ab \dots c \dots d}_{n+l} \tau_\zeta \underbrace{a'b' \dots c' \dots d'}_{n+l} \tau'_\zeta}(\vec{p}, n+l + \frac{1}{2}) \underbrace{\Lambda_m^{c \dots d c' \dots d'}}_l(\vec{p}, l) \Lambda_{\pm}^{\tau'_\zeta \tau_\zeta}(\vec{p}, \frac{1}{2}) \\ \Lambda_{\pm m \underbrace{ab \dots}_{n} [\tau_\zeta] \underbrace{a'b' \dots}_{n} [\tau'_\zeta]}(\vec{p}, n + \frac{1}{2}) &= \frac{2(n+\frac{1}{2})+1}{2(n+1+l)+1} \frac{1}{2} \Lambda_{m \underbrace{ab \dots c \dots de}_{n+1+l} \underbrace{a'b' \dots c' \dots d' e'}_{n+1+l}}(\vec{p}, n+1+l) \underbrace{\Lambda_m^{c \dots d c' \dots d'}}_l(\vec{p}, l) \gamma^e \Lambda_{\mp}(\vec{p}, \frac{1}{2}) \gamma^{e'} \end{aligned} \right.$$

## 5.6 各种势准投影算子的通用性质

性质5.6.1.

$$\left\{ \begin{aligned} \Lambda_{m \underbrace{ab \dots}_{n} \underbrace{a'b' \dots}_{n}}(\vec{p}, n) &= \frac{1}{n!} \Lambda_m \{ \underbrace{ab \dots}_{n} \} \underbrace{a'b' \dots}_{n}(\vec{p}, n) = \frac{1}{n!} \Lambda_m \underbrace{ab \dots}_{n} \cdot \underbrace{(a'b' \dots)}_n(\vec{p}, n) = \frac{1}{(n!)^2} \Lambda_m \{ \underbrace{ab \dots}_{n} \} \{ \underbrace{a'b' \dots}_{n} \}(\vec{p}, n) \\ \delta^{ab} \Lambda_{m \underbrace{ab \dots}_{n} \underbrace{a'b' \dots}_{n}}(\vec{p}, n) &= \delta^{a'b'} \Lambda_{m \underbrace{ab \dots}_{n} \underbrace{a'b' \dots}_{n}}(\vec{p}, n) = 0, p^a \Lambda_{m \underbrace{ab \dots}_{n} \underbrace{a'b' \dots}_{n}}(\vec{p}, n) = p^{+a'} \Lambda_{m \underbrace{ab \dots}_{n} \underbrace{a'b' \dots}_{n}}(\vec{p}, n) = 0 \end{aligned} \right.$$

性质5.6.2.

$$\begin{cases}
\Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_{\zeta} \underbrace{a'b' \dots}_{n} \tau'_{\zeta}}(\vec{p}, n + \frac{1}{2}) = \frac{1}{n!} \Lambda_{\pm m \{ab \dots\}_{n} \tau_{\zeta} \{a'b' \dots\}_{n} \tau'_{\zeta}}(\vec{p}, n + \frac{1}{2}) \\
= \frac{1}{n!} \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_{\zeta} \underbrace{(a'b' \dots)_{n}}_{n}} \tau'_{\zeta}(\vec{p}, n + \frac{1}{2}) = \frac{1}{(n!)^2} \Lambda_{\pm m \{ab \dots\}_{n} \tau_{\zeta} \{a'b' \dots\}_{n} \tau'_{\zeta}}(\vec{p}, n + \frac{1}{2}) \\
\delta^{ab} \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_{\zeta} \underbrace{a'b' \dots}_{n} \tau'_{\zeta}}(\vec{p}, n + \frac{1}{2}) = \delta^{a'b'} \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_{\zeta} \underbrace{a'b' \dots}_{n} \tau'_{\zeta}}(\vec{p}, n + \frac{1}{2}) = 0 \\
p^a \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_{\zeta} \underbrace{a'b' \dots}_{n} \tau'_{\zeta}}(\vec{p}, n + \frac{1}{2}) = p^{+a'} \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_{\zeta} \underbrace{a'b' \dots}_{n} \tau'_{\zeta}}(\vec{p}, n + \frac{1}{2}) = 0 \\
\gamma^a \Lambda_{\pm m \underbrace{ab \dots}_{n} [\tau_{\zeta}] \underbrace{a'b' \dots}_{n} \tau'_{\zeta}}(\vec{p}, n + \frac{1}{2}) = 0, \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_{\zeta} \underbrace{a'b' \dots}_{n} [\tau'_{\zeta}]}(\vec{p}, n + \frac{1}{2}) \gamma^{a'} = 0 \\
(\pm i \gamma^c p_c + m) \Lambda_{\pm m \underbrace{ab \dots}_{n} [\tau_{\zeta}] \underbrace{a'b' \dots}_{n} \tau'_{\zeta}}(\vec{p}, n + \frac{1}{2}) = 0, \Lambda_{\pm m \underbrace{ab \dots}_{n} \tau_{\zeta} \underbrace{a'b' \dots}_{n} [\tau'_{\zeta}]}(\vec{p}, n + \frac{1}{2}) (\pm i \gamma^{c'} p_{c'} + m) = 0
\end{cases}$$

## 6 势对易规则的直接解法(等价变换法)

### 6.1 引理

引理6.1.1.  $\mathbb{X}_{\{\lambda_{\zeta} \mu_{\zeta}\}}^a(p) \mathbb{X}_{\{\eta_{\zeta} \xi_{\zeta}\}}^b(p) [\delta_{ab} + \frac{p_a p_b}{m^2}] = 0$

### 6.2 势 $A_{abc}$ 对易规则的变换求法

定理6.2.1.

$$\begin{cases}
[\psi_{\lambda_{\zeta} \mu_{\zeta} \dots}^{\dots}(x), \psi_{\lambda'_{\zeta} \mu'_{\zeta} \dots}^{\dots}(x')] = \frac{i}{2^{3n-1} [(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_{\zeta} \mu_{\zeta}\}}^a(x) \dots}_{n} \underbrace{\mathbb{X}_{\{\lambda'_{\zeta} \mu'_{\zeta}\}}^{+a'}(x') \dots}_{n} \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \dots \Delta(x - x') \\
A_{\underbrace{ab \dots}_n}(x) = \frac{1}{(i2m)^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_{\zeta} \mu_{\zeta}} (\bar{C}\gamma_b)^{\eta_{\zeta} \xi_{\zeta}} \dots}_{n} \underbrace{\psi_{\lambda_{\zeta} \mu_{\zeta} \eta_{\zeta} \xi_{\zeta} \dots}}_{2n}(x), \psi_{\lambda_{\zeta} \mu_{\zeta} \dots}(x) = \frac{1}{(2n)!} \underbrace{\psi_{\{\lambda_{\zeta} \mu_{\zeta} \dots\}}}_{2n}(x) \\
A_{\underbrace{a'b' \dots}_n}^+(x) = \frac{1}{(-i2m)^n} \overbrace{(\gamma_{a'} C)^{\lambda'_{\zeta} \mu'_{\zeta}} (\gamma_{b'} C)^{\eta'_{\zeta} \xi'_{\zeta}} \dots}_{n} \underbrace{\psi_{\lambda'_{\zeta} \mu'_{\zeta} \eta'_{\zeta} \xi'_{\zeta} \dots}}_{2n}^+(x), \psi_{\lambda'_{\zeta} \mu'_{\zeta} \dots}^+(x) = \frac{1}{(2n)!} \underbrace{\psi_{\{\lambda'_{\zeta} \mu'_{\zeta} \dots\}}}_{2n}^+(x)
\end{cases}$$

$$\begin{aligned}
&\Rightarrow [A_{\underbrace{ab \dots}_n}(x), A_{\underbrace{a'b' \dots}_n}^+(x')] = i \frac{1}{2^{5n-1} m^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_{\zeta} \mu_{\zeta}} (\bar{C}\gamma_b)^{\eta_{\zeta} \xi_{\zeta}} \dots}_{n} \overbrace{(\gamma_{a'} C)^{\lambda'_{\zeta} \mu'_{\zeta}} (\gamma_{b'} C)^{\eta'_{\zeta} \xi'_{\zeta}} \dots}_{n} \\
&\frac{1}{[(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_{\zeta} \mu_{\zeta}\}}^c(x) \mathbb{X}_{\{\eta_{\zeta} \xi_{\zeta}\}}^d(x) \dots}_{n} \underbrace{\mathbb{X}_{\{\lambda'_{\zeta} \mu'_{\zeta}\}}^{+c'}(x') \mathbb{X}_{\{\eta'_{\zeta} \xi'_{\zeta}\}}^{+d'}(x') \dots}_{n} \underbrace{[\eta_{cc'} - \frac{\partial_c \partial_{c'}}{m^2}][\eta_{dd'} - \frac{\partial_d \partial_{d'}}{m^2}]}_n \dots \Delta(x - x') \\
&= i \frac{1}{2^{4n-1} m^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_{\zeta} \mu_{\zeta}} (\bar{C}\gamma_b)^{\eta_{\zeta} \xi_{\zeta}} \dots}_{n} \overbrace{(\gamma_{a'} C)^{\lambda'_{\zeta} \mu'_{\zeta}} (\gamma_{b'} C)^{\eta'_{\zeta} \xi'_{\zeta}} \dots}_{n} \\
&\frac{1}{[(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_{\zeta} \mu_{\zeta}\}} \underbrace{[(m - \gamma^b \partial_b) \gamma^4]_{\{\eta_{\zeta} \xi_{\zeta}\}}}_{2n}}_{2n} \Delta(x - x')
\end{aligned}$$

证明:  $[A_{\underbrace{ab \dots}_n}(x), A_{\underbrace{a'b' \dots}_n}^+(x')]$

$$\begin{aligned}
&= \frac{1}{(2m)^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_{\zeta} \mu_{\zeta}} (\bar{C}\gamma_b)^{\eta_{\zeta} \xi_{\zeta}} \dots}_{n} \overbrace{(\gamma_{a'} C)^{\lambda'_{\zeta} \mu'_{\zeta}} (\gamma_{b'} C)^{\eta'_{\zeta} \xi'_{\zeta}} \dots}_{n} \cdot [\psi_{\lambda_{\zeta} \mu_{\zeta} \eta_{\zeta} \xi_{\zeta} \dots}(x), \psi_{\lambda'_{\zeta} \mu'_{\zeta} \eta'_{\zeta} \xi'_{\zeta} \dots}^+(x')] \\
&= \frac{1}{(2m)^{2n}} \frac{i}{2^{3n-1} [(2n)!]^2} \overbrace{(\bar{C}\gamma_a)^{\lambda_{\zeta} \mu_{\zeta}} (\bar{C}\gamma_b)^{\eta_{\zeta} \xi_{\zeta}} \dots}_{n} \overbrace{(\gamma_{a'} C)^{\lambda'_{\zeta} \mu'_{\zeta}} (\gamma_{b'} C)^{\eta'_{\zeta} \xi'_{\zeta}} \dots}_{n} \\
&\underbrace{\mathbb{X}_{\{\lambda_{\zeta} \mu_{\zeta}\}}^c(x) \mathbb{X}_{\{\eta_{\zeta} \xi_{\zeta}\}}^d(x) \dots}_{n} \underbrace{\mathbb{X}_{\{\lambda'_{\zeta} \mu'_{\zeta}\}}^{+c'}(x') \mathbb{X}_{\{\eta'_{\zeta} \xi'_{\zeta}\}}^{+d'}(x') \dots}_{n} \underbrace{[\eta_{cc'} - \frac{\partial_c \partial_{c'}}{m^2}][\eta_{dd'} - \frac{\partial_d \partial_{d'}}{m^2}]}_n \dots \Delta(x - x') \\
&= i \frac{1}{2^{5n-1} m^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_{\zeta} \mu_{\zeta}} (\bar{C}\gamma_b)^{\eta_{\zeta} \xi_{\zeta}} \dots}_{n} \overbrace{(\gamma_{a'} C)^{\lambda'_{\zeta} \mu'_{\zeta}} (\gamma_{b'} C)^{\eta'_{\zeta} \xi'_{\zeta}} \dots}_{n} \\
&\frac{1}{[(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_{\zeta} \mu_{\zeta}\}}^c(x) \mathbb{X}_{\{\eta_{\zeta} \xi_{\zeta}\}}^d(x) \dots}_{n} \underbrace{\mathbb{X}_{\{\lambda'_{\zeta} \mu'_{\zeta}\}}^{+c'}(x') \mathbb{X}_{\{\eta'_{\zeta} \xi'_{\zeta}\}}^{+d'}(x') \dots}_{n} \underbrace{[\eta_{cc'} - \frac{\partial_c \partial_{c'}}{m^2}][\eta_{dd'} - \frac{\partial_d \partial_{d'}}{m^2}]}_n \dots \Delta(x - x')
\end{aligned}$$

$$= i \frac{1}{2^{4n-1} m^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \cdots (\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \cdots}^n \cdot \overbrace{\Delta(x-x')}^n$$

$$\frac{1}{[(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\lambda_\zeta \mu_\zeta} [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots}_{2n} \Delta(x-x')$$

□

### 6.3 势 $A_{abc \cdots \tau_\zeta}$ 反对易规则的变换求法

定理6.3.1.

$$\left\{ \begin{aligned} & \left\{ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}(x)}_{2n+1}, \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}^+(x')}_{2n+1} \right\} \\ &= \frac{i}{2^{3n} [(2n+1)!]^2} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{a'}(x')}_n \cdot \underbrace{[(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_n \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdot \Delta(x-x') \\ & A_{ab \cdots \tau_\zeta}(x) = \frac{1}{(i2m)^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \cdots}_{2n+1} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \cdots \tau_\zeta}(x)}_{2n+1}, \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}(x)}_{2n+1} = \frac{1}{(2n+1)!} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdots \tau_\zeta}(x)}_{2n+1} \\ & A_{a'b' \cdots \tau'_\zeta}^+(x) = \frac{1}{(-i2m)^n} \overbrace{(\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \cdots}_{2n+1} \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \eta'_\zeta \xi'_\zeta \cdots \tau'_\zeta}^+(x)}_{2n+1}, \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}^+(x)}_{2n+1} = \frac{1}{(2n+1)!} \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \cdots \tau'_\zeta}^+(x)}_{2n+1} \end{aligned} \right.$$

$$\Rightarrow \left\{ \underbrace{A_{ab \cdots \tau_\zeta}(x)}_n, \underbrace{A_{a'b' \cdots \tau'_\zeta}^+(x')}_n \right\}$$

$$= i \frac{1}{2^{5n} m^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \cdots (\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \cdots}^n \cdot \overbrace{\Delta(x-x')}^n$$

$$\frac{1}{[(2n+1)!]^2} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{a'}(x')}_n \cdot \underbrace{[(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_n \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdot \Delta(x-x')$$

$$= i \frac{1}{2^{4n} m^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \cdots (\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \cdots}^n \cdot \overbrace{\Delta(x-x')}^n$$

$$\frac{1}{[(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\lambda_\zeta \mu_\zeta} [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots [(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_{2n+1} \Delta(x-x')$$

证明:  $\left\{ \underbrace{A_{ab \cdots \tau_\zeta}(x)}_n, \underbrace{A_{a'b' \cdots \tau'_\zeta}^+(x')}_n \right\}$ 

$$= \frac{1}{(2m)^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \cdots (\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \cdots}_{2n+1} \cdot \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \cdots \tau_\zeta}(x)}_{2n+1}, \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \eta'_\zeta \xi'_\zeta \cdots \tau'_\zeta}^+(x')}_{2n+1}$$

$$= i \frac{1}{2^{5n} m^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \cdots (\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \cdots}^n \cdot \overbrace{\Delta(x-x')}^n$$

$$\frac{1}{[(2n+1)!]^2} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{a'}(x')}_n \cdot \underbrace{[(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_n \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdot \Delta(x-x')$$

$$= i \frac{1}{2^{4n} m^{2n}} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \cdots (\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \cdots}^n \cdot \overbrace{\Delta(x-x')}^n$$

$$\frac{1}{[(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\lambda_\zeta \mu_\zeta} [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots [(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_{2n+1} \Delta(x-x')$$

□

### 6.4 Bargmann-Wigner全对称方程的等时量子化规则

定理6.4.1.

$$\left[ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdots}(x)}_{2s}, \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \cdots}^+(x')}_{2s} \right]_{-2s+1} = i \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma \otimes \sigma_z, i\zeta)^a_{\lambda_\zeta \mu_\zeta} (\sigma \otimes \sigma_z, i\zeta)^b_{\mu'_\zeta \mu'_\zeta} \cdots}}_{2s} \underbrace{\partial_a \partial_b \cdots \Delta(x-x')}_{2s}$$

$$\Rightarrow \left[ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdots \xi_\zeta \eta_\zeta \cdots \tau_\zeta}(\vec{r}, t)}_{2s}, \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \cdots \xi'_\zeta \eta'_\zeta \cdots \tau'_\zeta}^+(\vec{r}', t)}_{2s} \right]_{-2s+1}$$

$$= - \frac{(i\zeta)^{2s+1}}{2^{2s-1}} \sum_{k=0}^{[s-\frac{1}{2}]} \frac{(2s)!}{(2s-2k-1)!(2k)!} \underbrace{[(\sigma \cdot \nabla) \otimes \sigma_z]_{\lambda_\zeta \lambda'_\zeta} [(\sigma \cdot \nabla) \otimes \sigma_z]_{\mu_\zeta \mu'_\zeta} \cdots}_{2s-2k-1} \underbrace{\delta_{\xi_\zeta \xi'_\zeta} \delta_{\eta_\zeta \eta'_\zeta} \cdots}_{2k} \nabla^2 \delta_{\tau_\zeta \tau'_\zeta} \delta^3(\vec{r} - \vec{r}')$$

## 7 有质量粒子协变对易规则的总结与梳理

### 7.1 有质量玻色子协变对易规则梳理

$$\text{定义7.1.1.} \quad \begin{cases} \hat{P}_{a_1 \cdots a_n a'_1 \cdots a'_n}(n) = \frac{1}{(n!)^2} \sum_{P(a)}^{P(b)} \sum_{r=0}^{[n/2]} k_r \hat{P}_{a_1 a_2} \hat{P}_{a'_1 a'_2} \cdots \hat{P}_{a_{2r-1} a_{2r}} \hat{P}_{a'_{2r-1} a'_{2r}} \prod_{i=2r+1}^n \hat{P}_{a_i a'_i} \\ k_r = \left(-\frac{1}{2}\right)^r \frac{n!(2n-2r-1)!!}{r!(n-2r)!(2n-1)!!} \end{cases}$$

$$\text{定义7.1.2.} \quad \begin{cases} \hat{P}_{a_1 \cdots a_n b_1 \cdots b_n}(n) = \frac{1}{(n!)^2} \sum_{P(a)}^{P(b)} \sum_{r=0}^{[n/2]} k_r \hat{P}_{a_1 a_2} \hat{P}_{b_1 b_2} \cdots \hat{P}_{a_{2r-1} a_{2r}} \hat{P}_{b_{2r-1} b_{2r}} \prod_{i=2r+1}^n \hat{P}_{a_i b_i} \\ \hat{P}_{a_1 \cdots a_n b_1 \cdots b_n}(n) := \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \cdots \eta_{b_n}^{a'_n} \hat{P}_{a_1 \cdots a_n a'_1 \cdots a'_n}(n) \end{cases}$$

$$\text{定理7.1.1.} \quad [A_{a_1 a_2 \cdots a_n}(x), \bar{A}_{b_1 b_2 \cdots b_n}(x')] = i \hat{P}_{a_1 a_2 \cdots a_n b_1 b_2 \cdots b_n}(n) \Delta(x-x'), \quad \bar{A}_{b_1 b_2 \cdots b_n} := \eta_{b_1}^{b'_1} \eta_{b_2}^{b'_2} \cdots \eta_{b_n}^{b'_n} A_{b'_1 b'_2 \cdots b'_n}^+$$

$$(\Downarrow)$$

$$\text{定理7.1.2.} \quad [A_{a_1 a_2 \cdots a_n}(x), A_{a'_1 a'_2 \cdots a'_n}^+(x')] = i \hat{P}_{a_1 a_2 \cdots a_n a'_1 a'_2 \cdots a'_n}(n) \Delta(x-x')$$

$$(\Downarrow)$$

$$\text{定理7.1.3.} \quad [A_{\underbrace{ab \cdots}_n}(x), A_{\underbrace{a'b' \cdots}_n}^+(x')] = \frac{1}{m^{2n}} \frac{i}{2^{5n-1} [(2n)!]^2} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \cdots}^n \overbrace{(\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \cdots}^n$$

$$\underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^c(x) \mathbb{X}_{\eta_\zeta \xi_\zeta}^d(x) \cdots}_{n} \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{c'}(x') \mathbb{X}_{\eta'_\zeta \xi'_\zeta}^{d'}(x') \cdots}_{n} \underbrace{[\eta_{cc'} - \frac{\partial_c \partial_{c'}}{m^2}][\eta_{dd'} - \frac{\partial_d \partial_{d'}}{m^2}]}_n \cdots \Delta(x-x')$$

$$(\Downarrow)$$

$$\text{定理7.1.4.} \quad [A_{\underbrace{ab \cdots}_n}(x), A_{\underbrace{a'b' \cdots}_n}^+(x')] = \frac{1}{m^{2n}} \frac{i}{2^{4n-1} [(2n)!]^2} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \xi_\zeta} \cdots}^n \overbrace{(\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \cdots}^n$$

$$\underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta \mu_\zeta\}} \{[(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots\}}_{2n} \Delta(x-x')$$

$$(\Downarrow)$$

$$\text{定理7.1.5.} \quad [\psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2n}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots}_{2n}}^+(x')] = \frac{i}{2^{3n-1} [(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a(x) \cdots}_{n} \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{a'}(x') \cdots}_{n} \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdots \Delta(x-x')$$

$$(\Downarrow)$$

$$\text{定理7.1.6.} \quad [\psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2n}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots}_{2n}}^+(x')] = \frac{i}{2^{2n-1} [(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta \mu_\zeta\}} \{[(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots\}}_{2n} \Delta(x-x')$$

$$(\Downarrow)$$

$$\text{定理7.1.7.} \quad [\psi_{\underbrace{A_\zeta B_\zeta \cdots}_{2n}}(x), \psi_{\underbrace{A'_\zeta B'_\zeta \cdots}_{2n}}^+(x')] = i \frac{(i\zeta)^{2n}}{2^{2n-1} [(2n)!]^2} \overbrace{(\sigma, i\zeta)_{\{A_\zeta (A'_\zeta (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \cdots\}}^a}^{2n}} \overbrace{\partial_a \partial_b \cdots}_{2n} \Delta(x-x')$$

$$(\Downarrow)$$

$$\text{定理7.1.8.} \quad [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i \frac{(-1)^{2n}}{2^{2n-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(n) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x-x')$$

势 $A$ 和场 $\psi$ 的对易关系都有两种等价表述方式，且互为前提和互为因果。既可以从势对易关系推出一切，也可以从场对易关系推出一切。这说明对于有质量粒子，势和场两种描述方案是完全等价的。并且可以从有质量粒子的对易规则推得与无质量粒子完全类似的对易规则，反之则不行。

## 7.2 有质量玻色子对易规则与准投影算子之间的关系

$$\begin{aligned} \text{推论7.2.1. } & \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab\dots}}_n(\vec{p}, h) \underbrace{\varepsilon_{a'b'\dots}^+}_{n'}(\vec{p}, h) \\ &= \frac{1}{2^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta} \dots (\bar{C}\gamma_{a'})^{+\lambda'_\zeta\mu'_\zeta} (\bar{C}\gamma_{b'})^{+\eta'_\zeta\xi'_\zeta} \dots}}^n \sum_{h=n}^{-n} \underbrace{U_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta} \dots}_{2n}(\vec{p}, h) \underbrace{U_{\lambda'_\zeta\mu'_\zeta\eta'_\zeta\xi'_\zeta} \dots}_{2n}(\vec{p}, h) \end{aligned}$$

$$\text{定理7.2.1. } [\psi_{\lambda_\zeta\mu_\zeta\dots}(x), \psi_{\lambda'_\zeta\mu'_\zeta\dots}^+(x')]_{-2s+1} = 2im^{2s} \sum_{h=s}^{-s} \underbrace{U_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta} \dots}_{2n}(-i\partial, h) \underbrace{U_{\lambda'_\zeta\mu'_\zeta\eta'_\zeta\xi'_\zeta} \dots}_{2n}(-i\partial, h)$$

$$\text{定理7.2.2. } [A_{\underbrace{ab\dots}_n}(x), A_{\underbrace{a'b'\dots}_n}^+(x')] = \frac{i}{2^{2n-1}} \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab\dots}}_n(-i\partial, h) \underbrace{\varepsilon_{a'b'\dots}^+}_{n'}(-i\partial, h)$$

$$\begin{aligned} \text{证明: } & [A_{\underbrace{ab\dots}_n}(x), A_{\underbrace{a'b'\dots}_n}^+(x')] = \frac{1}{m^{2n}} \frac{i}{2^{4n-1} [(2n)!]^2} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta} \dots (\gamma_{a'} C)^{\lambda'_\zeta\mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta\xi'_\zeta} \dots}}^n \\ & \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta\mu'_\zeta} \dots\}}}_{2n} \Delta(x - x') \\ &= \frac{i}{2^{2n-1}} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta} \dots (\gamma_{a'} C)^{\lambda'_\zeta\mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta\xi'_\zeta} \dots}}^n \sum_{h=n}^{-n} \underbrace{U_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta} \dots}_{2n}(-i\partial, h) \underbrace{U_{\lambda'_\zeta\mu'_\zeta\eta'_\zeta\xi'_\zeta} \dots}_{2n}(-i\partial, h) \Delta(x - x') \\ &= \frac{i}{2^{2n-1}} \sum_{h=n}^{-n} \underbrace{\varepsilon_{ab\dots}}_n(-i\partial, h) \underbrace{\varepsilon_{a'b'\dots}^+}_{n'}(-i\partial, h) \end{aligned} \quad \square$$

## 7.3 有质量费米子协变反对易规则梳理

$$\text{定义7.3.1. } \hat{P}_{a_1\dots a_n\tau_\zeta a'_1\dots a'_n\tau'_\zeta}(n + \frac{1}{2}) = \frac{n+1}{2n+3} \hat{P}_{aa_1\dots a_n a'_1\dots a'_n}(n + 1) [\gamma^a (-m - \gamma^c \partial_c) \gamma^4 \gamma^{a'}]_{\tau_\zeta\tau'_\zeta}, \gamma^{a'} = \gamma^a \eta_a^{a'}$$

$$\text{定义7.3.2. } \begin{cases} \hat{P}_{a_1\dots a_n\tau_\zeta b_1\dots b_n\tau'_\zeta}(n + \frac{1}{2}) = \frac{n+1}{2n+3} \hat{P}_{aa_1\dots a_n bb_1\dots b_n}(n + 1) [\gamma^a (m + \gamma^c \partial_c) \gamma^b \gamma^4]_{\tau_\zeta\tau'_\zeta} \\ \hat{P}_{a_1\dots a_n\tau_\zeta b_1\dots b_n\tau'_\zeta}(n + \frac{1}{2}) := \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \dots \eta_{b_n}^{a'_n} \hat{P}_{a_1\dots a_n\tau_\zeta a'_1\dots a'_n\tau'_\zeta}(n + \frac{1}{2}) \end{cases}$$

$$\text{推论7.3.1. } \hat{P}_{a_1\dots a_n\tau_\zeta b_1\dots b_n\tau'_\zeta}(n + \frac{1}{2}) = \frac{n+1}{2n+3} \hat{P}_{aa_1\dots a_n bb_1\dots b_n}(n + 1) [(m - \gamma^c \partial_c) \gamma^a \gamma^b \gamma^4]_{\tau_\zeta\tau'_\zeta}$$

$$\text{定理7.3.1. } \{A_{a_1 a_2 \dots a_n \tau_\zeta}(x), \bar{A}_{b_1 b_2 \dots b_n \tau'_\zeta}(x')\} = i \hat{P}_{a_1 \dots a_n \tau_\zeta b_1 \dots b_n \tau'_\zeta}(n + \frac{1}{2}) \Delta(x - x')$$

[⇔]

$$\text{定理7.3.2. } \{A_{a_1 a_2 \dots a_n \tau_\zeta}(x), A_{a'_1 a'_2 \dots a'_n \tau'_\zeta}^+(x')\} = i \hat{P}_{a_1 \dots a_n \tau_\zeta a'_1 \dots a'_n \tau'_\zeta}(n + \frac{1}{2}) \Delta(x - x')$$

[⇔]

$$\text{定理7.3.3. } \{A_{\underbrace{ab\dots}_{n}} \tau_\zeta(x), A_{\underbrace{a'b'\dots}_{n'}} \tau'_\zeta(x')\} = i \frac{1}{m^{2n}} \frac{1}{2^{5n} [(2n+1)!]^2} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta} \dots (\gamma_{a'} C)^{\lambda'_\zeta\mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta\xi'_\zeta} \dots}}^n$$

$$\underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a(x) \dots \mathbb{X}_{\lambda'_\zeta\mu'_\zeta}^{a'}(x')}_{n} \cdot \underbrace{[(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta\tau'_\zeta}}_{n} \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \dots}_{n} \Delta(x - x')$$

[⇔]

$$\text{定理7.3.4. } \{A_{\underbrace{ab\dots}_{n}} \tau_\zeta(x), A_{\underbrace{a'b'\dots}_{n'}} \tau'_\zeta(x')\} = \frac{1}{m^{2n}} \frac{i}{2^{4n} [(2n+1)!]^2} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta\mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta\xi_\zeta} \dots (\gamma_{a'} C)^{\lambda'_\zeta\mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta\xi'_\zeta} \dots}}^n$$

$$\underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta\mu'_\zeta} \dots [(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta\tau'_\zeta}\}}}_{2n+1} \Delta(x - x')$$

[⇔]

$$\begin{aligned} \text{定理7.3.5. } & \{ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2n+1}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}}_{2n+1}(x') \} \\ &= \frac{i}{2^{3n} [(2n+1)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a}_n(x) \cdot \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{a'}}_n(x') \cdot \underbrace{[(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_n \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \Delta(x - x') \end{aligned}$$

[⇕]

$$\text{定理7.3.6. } \{ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2n+1}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}}_{2n+1}(x') \} = \frac{i}{2^{2n} [(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2n+1} \Delta(x - x')$$

[⇓]

$$\text{定理7.3.7. } \{ \underbrace{\psi_{A_\zeta B_\zeta \dots}}_{2n+1}(x), \underbrace{\psi_{A'_\zeta B'_\zeta \dots}}_{2n+1}(x') \} = i \frac{(i\zeta)^{2n+1}}{2^{2n} [(2n+1)!]^2} \overbrace{(\sigma, i\zeta)_{\{A_\zeta (A'_\zeta (\sigma, i\zeta)_{B_\zeta B'_\zeta \dots})\}}^{2n+1}} \overbrace{\partial_a \partial_b \dots}_{2n+1} \Delta(x - x')$$

[⇕]

$$\text{定理7.3.8. } \{ \psi_{k_\zeta}(x), \psi_{k'_\zeta}(x') \} = i \frac{(-1)^{2n+1}}{2^{n-1/2}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} \overbrace{(n + \frac{1}{2}) \partial_a \partial_b \partial_c \dots}_{2n+1} \Delta(x - x')$$

势A和场ψ的反对易关系都有两种等价表述方式，且互为前提和互为因果。既可以从势反对易关系推出一切，也可以从场反对易关系推出一切。这说明对于有质量粒子，势和场两种描述方案是完全等价的。并且可以从有质量粒子的对易规则推得与无质量粒子完全类似的反对易规则，反之则不行。

#### 7.4 有质量费米子对易规则与准投影算子之间的关系

$$\begin{aligned} \text{推论7.4.1. } & \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \underbrace{\varepsilon_{ab \dots \tau_\zeta}}_n(\vec{p}, h) \underbrace{\varepsilon_{a'b' \dots \tau'_\zeta}}_n(\vec{p}, h) \\ &= \frac{1}{2^n} \underbrace{(\bar{C} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots (\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \dots}_n \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}_{2n+1}(\vec{p}, h) \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+}_{2n+1}(\vec{p}, h) \end{aligned}$$

$$\text{定理7.4.1. } [\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}}_{2s}(x')]_{-2s+1} = 2im^{2s} \sum_{h=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(-i\partial, h) \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(-i\partial, h)$$

$$\text{定理7.4.2. } \{ \underbrace{A_{ab \dots \tau_\zeta}}_n(x), \underbrace{A_{a'b' \dots \tau'_\zeta}^+}_n(x') \} = \frac{im}{2^{n-1}} \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \underbrace{\varepsilon_{ab \dots \tau_\zeta}}_n(-i\partial, h) \underbrace{\varepsilon_{a'b' \dots \tau'_\zeta}^+}_n(-i\partial, h)$$

$$\begin{aligned} \text{证明: } & \{ \underbrace{A_{ab \dots \tau_\zeta}}_n(x), \underbrace{A_{a'b' \dots \tau'_\zeta}^+}_n(x') \} = \frac{1}{m^{2n}} \frac{i}{2^{4n} [(2n+1)!]^2} \overbrace{(\bar{C} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots (\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \dots}_n \\ & \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2n+1} \underbrace{[(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_n \Delta(x - x') \\ &= \frac{im}{2^{2n-1}} \overbrace{(\bar{C} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots (\gamma_{a'} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} C)^{\eta'_\zeta \xi'_\zeta} \dots}_n \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}_{2n+1}(-i\partial, h) \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+}_{2n+1}(-i\partial, h) \Delta(x - x') \\ &= \frac{im}{2^{n-1}} \sum_{h=n+\frac{1}{2}}^{-(n+\frac{1}{2})} \underbrace{\varepsilon_{ab \dots \tau_\zeta}}_n(-i\partial, h) \underbrace{\varepsilon_{a'b' \dots \tau'_\zeta}^+}_n(-i\partial, h) \end{aligned}$$

□



# 第二十七章 有质量实粒子的协变量子化方案

自我评述：有质量实粒子，即马约拉纳粒子，正反粒子相同，数学本质上完全可以由实函数描述，也可以用复函数描述，但必须满足马约拉纳条件。对于Bargmann-Wigner 方程或Dirac方程描述的粒子，一般来说既可以描述带荷的复粒子，也可以描述不带荷的马约拉纳粒子。两种情形的主对易规则形式一致，但其余对易或反对易括号，对带荷的复粒子一般为零；对不带荷的马约拉纳粒子，其余对易或反对易括号由主对易规则和马约拉纳条件自然得到，一般不为零。在以后章节一般只讨论复粒子情形，也只给出主对易规则，不再专门讨论马约拉纳粒子情形，若要得到马约拉纳粒子情形的量子场论，只需在复粒子情形加上马约拉纳条件即可自然得到。在本章按统一的方式对所有自旋的马约拉纳粒子建立了相应的量子场论。与复粒子一样，也无需知道哈密顿量，就可以按统一的新程式对各种有质量自旋粒子进行了量子化，给出了统一的场、势量子化对易规则和能量动量算符形式，给出了部分量子彭加莱代数。与复粒子一样，角动量算符也只取得部分成功，没有彻底解决，仍需努力，角动量算符问题是新量子化程式的一个亟待解决的难题。

## 1 Majorana方程

### 1.1 实表象和Dirac分离表象下的Majorana方程 [4]

定义1.1.1.

$$\begin{cases} (\gamma_s^a \partial_a + m)\psi_s = 0, \gamma_s^a = (\sigma_{-\kappa} \sigma_{\kappa y}, \varsigma \sigma_{\kappa x}), \psi_s^* = \psi_s \\ (\gamma^a \partial_a + m)\psi = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \psi^* = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi \\ \psi_s = S_s(\kappa, \theta)\psi, S_s(\kappa, \theta) := e^{i\theta} S_{em}(\kappa) \\ S_s^T(\kappa, \theta) S_s(\kappa, \theta) = e^{2i\theta} S_{em}^T(\kappa) S_{em}(\kappa) = -e^{2i\theta} \sigma_y \otimes \sigma_y \end{cases}, S_{em}(\kappa) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & -i & -i & 0 \\ 0 & -\kappa & \kappa & 0 \end{bmatrix}$$

### 1.2 实表象和Dirac分离表象下的Majorana条件

推论1.2.1.  $\psi_s = \psi_s^* \Leftrightarrow \psi^* = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi, -\sigma_y \otimes \sigma_y = \bar{C} \gamma_4$

$\theta$ 是调节相位参数，一般取0或 $\pi/2$ 。

## 2 Majorana B-W方程

### 2.1 实表象和Dirac分离表象下的Majorana B-W方程 [17]

定义2.1.1.

$$\begin{cases} (\gamma_s^a \partial_a + m)_{\kappa_s} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2s}(\vec{r}, t) = 0, \gamma_s^a = (\sigma_{-\kappa} \sigma_{\kappa y}, \varsigma \sigma_{\kappa x}), \psi_s^* = \psi_s \\ (\gamma^a \partial_a + m)_{\kappa_s} \underbrace{\psi_{\lambda_s \mu_s \dots}}_{2s}(\vec{r}, t) = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \psi^* = (-1)^{2s} e^{4si\theta} \underbrace{\sigma_y \otimes \dots \otimes \sigma_y}_{4s} \psi \end{cases}$$

### 2.2 实表象和Dirac分离表象下的Majorana条件

推论2.2.1.

$$\psi_s = \psi_s^* \Leftrightarrow \psi^* = (-1)^{2s} e^{4si\theta} \underbrace{\sigma_y \otimes \dots \otimes \sigma_y}_{4s} \psi = e^{4si\theta} \underbrace{(\bar{C} \gamma_4) \otimes (\bar{C} \gamma_4) \dots}_{2s} \psi$$

$\theta$ 是调节相位参数，一般取0或 $\pi/2$ 。

### 3 分离表象下Majorana B-W方程的平面波解

#### 3.1 引理

$$\text{引理3.1.1. } \sum_{h=s}^{-s} b^+(\vec{p}, h) V_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{p}, h) = (-1)^{2s} e^{-4si\theta} \sum_{h=s}^{-s} a^+(\vec{p}, h) \overbrace{\sigma_y \otimes \sigma_y \dots}^{4s} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}^+(\vec{p}, h)$$

$$\Leftrightarrow b^+(\vec{p}, h) = \zeta^{2s} e^{-4si\theta} (-1)^{s+h} a^+(\vec{p}, -h) \Leftrightarrow b(\vec{p}, h) = \zeta^{2s} e^{4si\theta} (-1)^{s+h} a(\vec{p}, -h)$$

$$\text{证明: } \sum_{h=s}^{-s} b^+(\vec{p}, h) V_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{p}, h) = (-1)^{2s} e^{-4si\theta} \sum_{h=s}^{-s} a^+(\vec{p}, h) \overbrace{\sigma_y \otimes \sigma_y \dots}^{4s} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}^+(\vec{p}, h)$$

$$= \zeta^{2s} e^{-4si\theta} \sum_{h=s}^{-s} (-1)^{s+h} a^+(\vec{p}, -h) V_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{p}, h)$$

$$\Leftrightarrow b^+(\vec{p}, h) = \zeta^{2s} e^{-4si\theta} (-1)^{s+h} a^+(\vec{p}, -h) \Leftrightarrow b(\vec{p}, h) = \zeta^{2s} e^{4si\theta} (-1)^{s+h} a(\vec{p}, -h) \quad \square$$

#### 引理3.1.2.

$$\left\{ \begin{array}{l} [a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ [a(\vec{p}, h), a(\vec{p}', h')]_{-2s+1} = 0 \\ [a^+(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = 0 \\ b^+(\vec{p}, h) = \zeta^{2s} e^{-4si\theta} (-1)^{s+h} a^+(\vec{p}, -h) \\ b(\vec{p}, h) = \zeta^{2s} e^{4si\theta} (-1)^{s+h} a(\vec{p}, -h) \end{array} \right. \Rightarrow \left\{ \begin{array}{l} [b(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ [b(\vec{p}, h), b(\vec{p}', h')]_{-2s+1} = 0 \\ [b^+(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = 0 \\ [a(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \zeta^{2s} e^{-4si\theta} (-1)^{s-h} \delta_{-h, h'} \delta^3(\vec{p} - \vec{p}') \\ [a^+(\vec{p}, h), b(\vec{p}', h')]_{-2s+1} = -\zeta^{2s} e^{4si\theta} (-1)^{s-h} \delta_{-h, h'} \delta^3(\vec{p} - \vec{p}') \\ [a(\vec{p}, h), b(\vec{p}', h')]_{-2s+1} = 0 \\ [a^+(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = 0 \end{array} \right.$$

#### 3.2 分离表象下Majorana B-W方程<sup>[17]</sup>的平面波解(证明还需补上)

定理3.2.1.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) = 0, \gamma^a = (\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x)$

$$\left\{ \begin{array}{l} \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) = \frac{1}{(2s)!} \psi_{\underbrace{\{\lambda_\zeta \mu_\zeta \dots \tau_\zeta\}_{2s}}_{2s}}(\vec{r}, t), \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}^+(\vec{r}, t) = (-1)^{2s} e^{4si\theta} \overbrace{\sigma_y \otimes \dots \otimes \sigma_y}^{4s} \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) \\ \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \sum_{h=s}^{-s} \sqrt{\frac{m^{2s}}{E}} [a(\vec{p}, h) U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) V_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p} \\ b^+(\vec{p}, h) = \zeta^{2s} e^{-4si\theta} (-1)^{s+h} a^+(\vec{p}, -h) \\ \left\{ \begin{array}{l} a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m^{2s}}{E}} U^{+\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{p}, h) \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r} \\ b^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m^{2s}}{E}} V^{+\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{p}, h) \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) e^{ip \cdot x} d^3 \vec{r} \end{array} \right. \end{array} \right.$$

推论3.2.1.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) = 0, \gamma^a = (\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x)$

$$\psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) = \frac{1}{(2s)!} \psi_{\underbrace{\{\lambda_\zeta \mu_\zeta \dots \tau_\zeta\}_{2s}}_{2s}}(\vec{r}, t), \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}^+(\vec{r}, t) = (-1)^{2s} e^{4si\theta} \overbrace{\sigma_y \otimes \dots \otimes \sigma_y}^{4s} \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t)$$

$$\psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \sum_{h=s}^{-s} \sqrt{\frac{m^{2s}}{E}} [a(\vec{p}, h) U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{p}, h) e^{ip \cdot x} + (-1)^{2s} e^{-4si\theta} a^+(\vec{p}, h) \overbrace{\sigma_y \otimes \sigma_y \dots}^{4s} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}^+(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p}$$

$$a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m^{2s}}{E}} U^{+\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{p}, h) \psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r}$$

推论3.2.2.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) = 0, \gamma^a = (\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x)$

$$\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) = \frac{1}{(2s)!} \underbrace{\psi_{\{\lambda_\zeta \mu_\zeta \dots\}}}_{2s}(\vec{r}, t), \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}^+}_{2s}(\vec{r}, t) = (-1)^{2s} e^{4si\theta} \overbrace{\sigma_y \otimes \dots \otimes \sigma_y}^{4s} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t)$$

$$\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int \sum_{h=s}^{-s} E^{s-\frac{1}{2}} [a(\vec{p}, h) \underbrace{\tilde{U}_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, h) e^{ip \cdot x} + (-1)^{2s} e^{-4si\theta} a^+(\vec{p}, h) \overbrace{\sigma_y \otimes \sigma_y \dots}^{4s} \underbrace{\tilde{U}_{\lambda_\zeta \mu_\zeta \dots}^+}_{2s}(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p}$$

$$a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \overbrace{\tilde{U}^+_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}^{2s}(\vec{p}, h) \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}_{2s}(\vec{r}, t) e^{-ip \cdot x} d^3 \vec{r}$$

### 3.3 分离表象下Majorana B-W方程的协变对易规则

定理3.3.1.

$$\begin{cases} [a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [b(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ [a(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \zeta^{2s} e^{-4si\theta} (-1)^{s-h} \delta_{-h, h'} \delta^3(\vec{p} - \vec{p}') \\ [a^+(\vec{p}, h), b(\vec{p}', h')]_{-2s+1} = -\zeta^{2s} e^{4si\theta} (-1)^{s+h} \delta_{-h, h'} \delta^3(\vec{p} - \vec{p}') \\ [rest]_{-2s+1} = 0 \end{cases}$$

$$\Rightarrow \Rightarrow \begin{cases} [\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2s} \Delta(x - x') \\ [\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{e^{-4si\theta}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2s} \Delta(x - x') \\ [\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}^+}_{2s}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{e^{4si\theta}}{[(2s)!]^2} \overbrace{[C^+(m - \gamma^a \partial_a^+)]_{\{\lambda_\zeta (\lambda'_\zeta [C^+(m - \gamma^b \partial_b^+)]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2s} \Delta(x - x') \end{cases}$$

证明:  $[\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(x')]_{-2s+1} = \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' \sum_{h, h'=s}^{-s} (EE')^{s-\frac{1}{2}} \sqrt{\frac{m^2}{EE'}}^{2s}$

$$[[a(\vec{p}, h) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, h) e^{-ip \cdot x}, [a^+(\vec{p}', h') \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}', h') e^{-ip' \cdot x'} + b(\vec{p}', h') \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}', h') e^{ip' \cdot x'}]$$

$$= \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' \sum_{h, h'=s}^{-s} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} \{ [U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}', h')]_{-2s+1} [a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} e^{i(p \cdot x - p' \cdot x')} + V_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}', h') [b^+(\vec{p}, h), b(\vec{p}', h')]_{-2s+1} e^{-i(p \cdot x - p' \cdot x')} \}$$

$$= \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' \sum_{h, h'=s}^{-s} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} [U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}', h')]_{-2s+1} \delta_{hh'} \delta^3(\vec{p} - \vec{p}') e^{i(p \cdot x - p' \cdot x')} + (-1)^{2s+1} V_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}', h') \delta_{hh'} \delta^3(\vec{p} - \vec{p}') e^{-i(p \cdot x - p' \cdot x')}$$

$$= \frac{1}{(2\pi)^3} \int d^3 \vec{p} \frac{m^{2s}}{E} [\sum_{h=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, h) \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}, h) e^{ip \cdot (x-x')} + (-1)^{2s+1} \sum_{h=s}^{-s} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, h) \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}, h) e^{-ip \cdot (x-x')}]$$

$$= \frac{1}{(2\pi)^3} \int d^3 \vec{p} \frac{m^{2s}}{E} [\Lambda_+ \underbrace{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}_{2s}(\vec{p}, s) e^{ip \cdot (x-x')} + (-1)^{2s+1} \Lambda_- \underbrace{\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}_{2s}(\vec{p}, s) e^{-ip \cdot (x-x')}]$$

$$= \frac{1}{(2\pi)^3} \int d^3 \vec{p} \frac{m^{2s}}{E} [\frac{1}{[(2s)!]^2} \underbrace{\Lambda_+ \{\lambda_\zeta (\lambda'_\zeta (\vec{p}, \frac{1}{2}) \Lambda_+ \mu_\zeta \mu'_\zeta (\vec{p}, \frac{1}{2}) \dots\}}_{2s} e^{ip \cdot (x-x')} + (-1)^{2s+1} \frac{1}{[(2s)!]^2} \underbrace{\Lambda_- \{\lambda_\zeta (\lambda'_\zeta (\vec{p}, \frac{1}{2}) \Lambda_- \mu_\zeta \mu'_\zeta (\vec{p}, \frac{1}{2}) \dots\}}_{2s} e^{-ip \cdot (x-x')}]$$

$$= \frac{1}{(2\pi)^3} \int d^3 \vec{p} \frac{m^{2s}}{E} \{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2s} e^{ip \cdot (x-x')} \}$$

$$\begin{aligned}
& + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m + \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(-m + \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \cdot)\}}^{2s}} e^{-ip \cdot (x-x')} \} \\
& = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \cdot)\}}^{2s}} \frac{-i}{(2\pi)^3} \int d^3 \vec{p} \frac{1}{2E} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] \\
& = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \cdot)\}}^{2s}} \Delta(x-x') \\
& = \frac{i(2m)^{2s}}{2^{2s-1}} \Lambda_{+\lambda_\zeta \mu_\zeta \cdot \cdot \cdot \lambda'_\zeta \mu'_\zeta \cdot \cdot \cdot} (-i\partial, s) \Delta(x-x')
\end{aligned}$$

□

$$\begin{aligned}
\text{证明: } & [\psi_{\lambda_\zeta \mu_\zeta \cdot \cdot \cdot}(x), \psi_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \cdot}(x')]_{-2s+1} = \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' \sum_{h, h'=s}^{-s} (EE')^{s-\frac{1}{2}} \sqrt{\frac{m^2}{EE'}}^{2s} \\
& [a(\vec{p}, h) U_{\lambda_\zeta \mu_\zeta \cdot \cdot \cdot}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) V_{\lambda_\zeta \mu_\zeta \cdot \cdot \cdot}(\vec{p}, h) e^{-ip \cdot x} \\
& , a(\vec{p}', h') U_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \cdot}(\vec{p}', h') e^{ip' \cdot x'} + b^+(\vec{p}', h') V_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \cdot}(\vec{p}', h') e^{-ip' \cdot x'}]_{-2s+1} \\
& = \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' \sum_{h, h'=s}^{-s} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} \\
& [U_{\lambda_\zeta \mu_\zeta \cdot \cdot \cdot}(\vec{p}, h) V_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \cdot}(\vec{p}', h') [a(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} e^{i(p \cdot x - p' \cdot x')} \\
& + V_{\lambda_\zeta \mu_\zeta \cdot \cdot \cdot}(\vec{p}, h) U_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \cdot}(\vec{p}', h') [b^+(\vec{p}, h), a(\vec{p}', h')]_{-2s+1} e^{-i(p \cdot x - p' \cdot x')} \\
& = \frac{1}{(2\pi)^3} \int d^3 \vec{p} d^3 \vec{p}' \sum_{h, h'=s}^{-s} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} \zeta^{2s} e^{-4si\theta} (-1)^{s-h} \delta_{-h, h'} \\
& [U_{\lambda_\zeta \mu_\zeta \cdot \cdot \cdot}(\vec{p}, h) V_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \cdot}(\vec{p}', h') \delta^3(\vec{p} - \vec{p}') e^{i(p \cdot x - p' \cdot x')} - V_{\lambda_\zeta \mu_\zeta \cdot \cdot \cdot}(\vec{p}, h) U_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \cdot}(\vec{p}', h') \delta^3(\vec{p} - \vec{p}') e^{-i(p \cdot x - p' \cdot x')}] \\
& = \frac{1}{(2\pi)^3} \int d^3 \vec{p} \sum_{h=s}^{-s} \frac{m^{2s}}{E} \zeta^{2s} e^{-4si\theta} \\
& [(-1)^{s-h} U_{\lambda_\zeta \mu_\zeta \cdot \cdot \cdot}(\vec{p}, h) V_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \cdot}(\vec{p}, -h) e^{ip \cdot (x-x')} + (-1)^{2s+1} (-1)^{s+h} V_{\lambda_\zeta \mu_\zeta \cdot \cdot \cdot}(\vec{p}, h) U_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \cdot}(\vec{p}, -h) e^{-ip \cdot (x-x')}] \\
& = \frac{1}{(2\pi)^3} \int d^3 \vec{p} \sum_{h=s}^{-s} \frac{m^{2s}}{E} e^{-4si\theta} \\
& \left[ \frac{1}{[(2s)!]^2} (\Lambda_+ \bar{C} \gamma_4)_{\{\lambda_\zeta(\lambda'_\zeta (\vec{p}, \frac{1}{2})) (\Lambda_+ \bar{C} \gamma_4)_{\mu_\zeta \mu'_\zeta} (\vec{p}, \frac{1}{2}) \cdot \cdot (\Lambda_+ \bar{C} \gamma_4)_{\tau_\zeta \tau'_\zeta} (\vec{p}, \frac{1}{2})\}} e^{ip \cdot (x-x')} \right. \\
& \left. + (-1)^{2s+1} \frac{1}{[(2s)!]^2} (\Lambda_- \bar{C} \gamma_4)_{\{\lambda_\zeta(\lambda'_\zeta (\vec{p}, \frac{1}{2})) (\Lambda_- \bar{C} \gamma_4)_{\mu_\zeta \mu'_\zeta} (\vec{p}, \frac{1}{2}) \cdot \cdot (\Lambda_- \bar{C} \gamma_4)_{\tau_\zeta \tau'_\zeta} (\vec{p}, \frac{1}{2})\}} e^{-ip \cdot (x-x')} \right] \\
& = e^{-4si\theta} \frac{1}{(2\pi)^3} \int d^3 \vec{p} \frac{m^{2s}}{E} \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \cdot)\}}^{2s}} e^{ip \cdot (x-x')} \right. \\
& \left. + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m + \gamma^a \partial_a) C]_{\{\lambda_\zeta(\lambda'_\zeta [(-m + \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \cdot)\}}^{2s}} e^{-ip \cdot (x-x')} \right\} \\
& = e^{-4si\theta} \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \cdot)\}}^{2s}} \frac{-i}{(2\pi)^3} \int d^3 \vec{p} \frac{1}{2E} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] \\
& = e^{-4si\theta} \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \cdot)\}}^{2s}} \Delta(x-x')
\end{aligned}$$

□

推论3.3.1.  $[a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}')$ ,  $[a(\vec{p}, h), a(\vec{p}', h')]_{-2s+1} = 0$ ,  $[a^+(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = 0$

$$\Rightarrow \begin{cases} [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \\ [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{e^{-4si\theta}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \\ [\psi_{\lambda_\zeta \mu_\zeta \dots}^+(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{e^{4si\theta}}{[(2s)!]^2} \overbrace{[C^+(m - \gamma^a \partial_a^+)]_{\{\lambda_\zeta(\lambda'_\zeta [C^+(m - \gamma^b \partial_b^+)]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \end{cases}$$

### 3.4 分离表象下Majorana B-W方程协变对易规则的反向推理

定理3.4.1.

$$\Rightarrow \begin{cases} [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \\ [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{e^{-4si\theta}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \\ [\psi_{\lambda_\zeta \mu_\zeta \dots}^+(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{e^{4si\theta}}{[(2s)!]^2} \overbrace{[C^+(m - \gamma^a \partial_a^+)]_{\{\lambda_\zeta(\lambda'_\zeta [C^+(m - \gamma^b \partial_b^+)]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \\ [a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [b(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ [a(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \zeta^{2s} e^{-4si\theta} (-1)^{s-h} \delta_{-h, h'} \delta^3(\vec{p} - \vec{p}') \\ [a^+(\vec{p}, h), b(\vec{p}', h')]_{-2s+1} = -\zeta^{2s} e^{4si\theta} (-1)^{s+h} \delta_{-h, h'} \delta^3(\vec{p} - \vec{p}') \\ [rest]_{-2s+1} = 0 \end{cases}$$

下面给出几个主要对易括号的详细证明过程。

$$\begin{aligned} & \text{证明: } [a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} \\ &= \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) U^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{p}', h') [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}(x')]_{-2s+1} e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) U^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{p}', h') \\ & \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int d^3 \vec{r} d^3 \vec{r}' \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) U^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{p}', h') \\ & \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \left\{ \frac{-i}{(2\pi)^3} \int \frac{1}{2E_0} [e^{ip_0 \cdot (x-x')} - e^{-ip_0 \cdot (x-x')}] d^3 \vec{p}_0 \right\} e^{-i(p \cdot x - p' \cdot x')} \\ &= \left[ \frac{1}{(2\pi)^3} \right]^2 \int \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} U^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) U^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{p}', h') \\ & \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - i\gamma^b p_{0b}) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} e^{ip_0 \cdot (x-x')} e^{-i(p \cdot x - p' \cdot x')} \right. \\ & \left. + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(-m - i\gamma^b p_{0b}) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} e^{-ip_0 \cdot (x-x')} e^{-i(p \cdot x - p' \cdot x')} \right\} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}_0 \\ &= \left[ \frac{1}{(2\pi)^3} \right]^2 \int d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\ & U^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) U^{\lambda'_\zeta \mu'_\zeta \dots}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}_0, h_0)}_{2s} \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(\vec{p}_0, h_0)}_{2s} e^{i(p_0 - p) \cdot x} e^{-i(p_0 - p') \cdot x'} \right\} \end{aligned}$$

$$\begin{aligned}
& + (-1)^{2s+1} \sum_{h_0=s}^{-s} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) V_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p}_0, h_0) e^{-i(p_0+p) \cdot x} e^{i(p_0+p') \cdot x'} \} \\
& = \int d^3 \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left( \frac{m^2}{EE'} \right)^{2s} \\
& U^+ \overbrace{\lambda_s \mu_s \dots \tau_s}^{2s}(\vec{p}, h) U^+ \overbrace{\lambda'_s \mu'_s \dots \tau'_s}^{2s}(\vec{p}', h') \{ \sum_{h_0=s}^{-s} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) U_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p}_0, h_0) \delta^3(\vec{p}_0 - \vec{p}) \delta^3(\vec{p}_0 - \vec{p}') \\
& + (-1)^{2s+1} \sum_{h_0=s}^{-s} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) V_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p}_0, h_0) e^{2iE_0(t-t')} \delta^3(\vec{p}_0 + \vec{p}) \delta^3(\vec{p}_0 + \vec{p}') \} \\
& = \delta^3(\vec{p} - \vec{p}') \left( \frac{m}{E} \right)^{4s} U^+ \overbrace{\lambda_s \mu_s \dots \tau_s}^{2s}(\vec{p}, h) U^+ \overbrace{\lambda'_s \mu'_s \dots \tau'_s}^{2s}(\vec{p}', h') \\
& \{ \sum_{h_0=s}^{-s} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, h_0) U_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p}, h_0) + (-1)^{2s+1} \sum_{h_0=s}^{-s} V_{\lambda_s \mu_s \dots \tau_s}(-\vec{p}, h_0) V_{\lambda'_s \mu'_s \dots \tau'_s}^+(-\vec{p}, h_0) e^{2iE(t-t')} \} \\
& = \delta^3(\vec{p} - \vec{p}') \left( \sum_{h_0=s}^{-s} \delta_{hh_0} \delta_{h'h_0} + 0 \right) \\
& = \delta_{hh'} \delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

证明:  $[b^+(\vec{p}, h), b(\vec{p}', h')]_{-2s+1}$

$$\begin{aligned}
& = \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left( \frac{m}{EE'} \right)^{2s} V^+ \overbrace{\lambda_s \mu_s \dots \tau_s}^{2s}(\vec{p}, h) V^+ \overbrace{\lambda'_s \mu'_s \dots \tau'_s}^{2s}(\vec{p}', h') [\psi_{\lambda_s \mu_s \dots \tau_s}(x), \psi_{\lambda'_s \mu'_s \dots \tau'_s}^+(x')]_{-2s+1} e^{i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
& = \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left( \frac{m}{EE'} \right)^{2s} V^+ \overbrace{\lambda_s \mu_s \dots \tau_s}^{2s}(\vec{p}, h) V^+ \overbrace{\lambda'_s \mu'_s \dots \tau'_s}^{2s}(\vec{p}', h') \\
& \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots \tau_s})}}^{2s} \Delta(x - x') e^{i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
& = \frac{1}{(2\pi)^3} \int d^3 \vec{r} d^3 \vec{r}' \sqrt{EE'} \left( \frac{m}{EE'} \right)^{2s} V^+ \overbrace{\lambda_s \mu_s \dots \tau_s}^{2s}(\vec{p}, h) V^+ \overbrace{\lambda'_s \mu'_s \dots \tau'_s}^{2s}(\vec{p}', h') \\
& \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^4]_{\mu_s \mu'_s \dots \tau_s})}}^{2s} \left\{ \frac{-i}{(2\pi)^3} \int \frac{1}{2E_0} [e^{ip_0 \cdot (x-x')} - e^{-ip_0 \cdot (x-x')}] d^3 \vec{p}_0 \right\} e^{i(p \cdot x - p' \cdot x')} \\
& = \left[ \frac{1}{(2\pi)^3} \right]^2 \int \frac{\sqrt{EE'}}{E_0} \left( \frac{m^2}{EE'} \right)^{2s} V^+ \overbrace{\lambda_s \mu_s \dots \tau_s}^{2s}(\vec{p}, h) V^+ \overbrace{\lambda'_s \mu'_s \dots \tau'_s}^{2s}(\vec{p}', h') \\
& \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_s(\lambda'_s [(m - i\gamma^b p_{0b}) \gamma^4]_{\mu_s \mu'_s \dots \tau_s})}}^{2s} e^{ip_0 \cdot (x-x')} e^{i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \right. \\
& + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_s(\lambda'_s [(-m - i\gamma^b p_{0b}) \gamma^4]_{\mu_s \mu'_s \dots \tau_s})}}^{2s} e^{-ip_0 \cdot (x-x')} e^{i(p \cdot x - p' \cdot x')} \left. \right\} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}_0 \\
& = \left[ \frac{1}{(2\pi)^3} \right]^2 \int d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left( \frac{m^2}{EE'} \right)^{2s} \\
& V^+ \overbrace{\lambda_s \mu_s \dots \tau_s}^{2s}(\vec{p}, h) V^+ \overbrace{\lambda'_s \mu'_s \dots \tau'_s}^{2s}(\vec{p}', h') \{ \sum_{h_0=s}^{-s} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) U_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p}_0, h_0) e^{i(p_0+p) \cdot x} e^{-i(p_0+p') \cdot x'} \\
& + (-1)^{2s+1} \sum_{h_0=s}^{-s} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) V_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p}_0, h_0) e^{-i(p_0-p) \cdot x} e^{i(p_0-p') \cdot x'} \} \\
& = \int d^3 \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left( \frac{m^2}{EE'} \right)^{2s} \\
& V^+ \overbrace{\lambda_s \mu_s \dots \tau_s}^{2s}(\vec{p}, h) V^+ \overbrace{\lambda'_s \mu'_s \dots \tau'_s}^{2s}(\vec{p}', h') \{ \sum_{h_0=s}^{-s} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) U_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p}_0, h_0) e^{-2iE_0(t-t')} \delta^3(\vec{p}_0 + \vec{p}) \delta^3(\vec{p}_0 + \vec{p}') \\
& + (-1)^{2s+1} \sum_{h_0=s}^{-s} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) V_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p}_0, h_0) \delta^3(\vec{p}_0 - \vec{p}) \delta^3(\vec{p}_0 - \vec{p}') \} \\
& = \delta^3(\vec{p} - \vec{p}') \left( \frac{m}{E} \right)^{4s} V^+ \overbrace{\lambda_s \mu_s \dots \tau_s}^{2s}(\vec{p}, h) V^+ \overbrace{\lambda'_s \mu'_s \dots \tau'_s}^{2s}(\vec{p}', h')
\end{aligned}$$

$$\begin{aligned}
& \left\{ \sum_{h_0=s}^{-s} U_{\lambda_s \mu_s \dots \tau_s}(-\vec{p}, h_0) U_{\lambda'_s \mu'_s \dots \tau'_s}^+(-\vec{p}, h_0) e^{-2iE(t-t')} + (-1)^{2s+1} \sum_{h_0=s}^{-s} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, h_0) V_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p}, h_0) \right\} \\
&= (-1)^{2s+1} \delta^3(\vec{p} - \vec{p}') (0 + \sum_{h_0=s}^{-s} \delta_{hh_0} \delta_{h'h_0}) \\
&= (-1)^{2s+1} \delta_{hh'} \delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

证明:  $[a(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1}$ 

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) V^{+\lambda'_s \mu'_s \dots}(\vec{p}', h') [\psi_{\lambda_s \mu_s \dots}(x), \psi_{\lambda'_s \mu'_s \dots}^+(x')]_{-2s+1} e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) V^{+\lambda'_s \mu'_s \dots}(\vec{p}', h') \\
&\frac{i}{2^{2s-1}} \frac{e^{-4si\theta}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) C]_{\mu_s \mu'_s \dots})\}}}_{2s} \Delta(x - x') e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int d^3 \vec{r} d^3 \vec{r}' \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) V^{+\lambda'_s \mu'_s \dots}(\vec{p}', h') \\
&\frac{i}{2^{2s-1}} \frac{e^{-4si\theta}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) C]_{\mu_s \mu'_s \dots})\}}}_{2s} \left\{ \frac{-i}{(2\pi)^3} \int \frac{1}{2E_0} [e^{ip_0 \cdot (x-x')} - e^{-ip_0 \cdot (x-x')}] d^3 \vec{p}_0 \right\} e^{-i(p \cdot x - p' \cdot x')} \\
&= \left[ \frac{1}{(2\pi)^3} \right]^2 \int \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) V^{+\lambda'_s \mu'_s \dots}(\vec{p}', h') \\
&\left\{ \frac{1}{(2m)^{2s}} \frac{e^{-4si\theta}}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_{0a}) C]_{\{\lambda_s(\lambda'_s [(m - i\gamma^b p_{0b}) C]_{\mu_s \mu'_s \dots})\}}}_{2s} e^{ip_0 \cdot (x-x')} e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \right. \\
&+ (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{e^{-4si\theta}}{[(2s)!]^2} \overbrace{[(-m - i\gamma^a p_{0a}) C]_{\{\lambda_s(\lambda'_s [(-m - i\gamma^b p_{0b}) C]_{\mu_s \mu'_s \dots})\}}}_{2s} e^{-ip_0 \cdot (x-x')} e^{-i(p \cdot x - p' \cdot x')} \left. \right\} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}_0 \\
&= \zeta^{2s} e^{-4si\theta} \left[ \frac{1}{(2\pi)^3} \right]^2 \int d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
&U^{+\lambda_s \mu_s \dots}(\vec{p}, h) V^{+\lambda'_s \mu'_s \dots}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} (-1)^{s-h_0} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) V_{\lambda'_s \mu'_s \dots \tau'_s}(\vec{p}_0, -h_0) e^{i(p_0 - p) \cdot x} e^{-i(p_0 - p') \cdot x'} \right. \\
&+ (-1)^{2s+1} \sum_{h_0=s}^{-s} (-1)^{s+h_0} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) U_{\lambda'_s \mu'_s \dots \tau'_s}(\vec{p}_0, -h_0) e^{-i(p_0 + p) \cdot x} e^{i(p_0 + p') \cdot x'} \left. \right\} \\
&= \zeta^{2s} e^{-4si\theta} \int d^3 \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
&U^{+\lambda_s \mu_s \dots}(\vec{p}, h) V^{+\lambda'_s \mu'_s \dots}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} (-1)^{s-h_0} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) V_{\lambda'_s \mu'_s \dots \tau'_s}(\vec{p}_0, -h_0) \delta^3(\vec{p}_0 - \vec{p}) \delta^3(\vec{p}_0 - \vec{p}') \right. \\
&+ (-1)^{2s+1} \sum_{h_0=s}^{-s} (-1)^{s+h_0} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, h_0) U_{\lambda'_s \mu'_s \dots \tau'_s}(\vec{p}_0, -h_0) \delta^3(\vec{p}_0 + \vec{p}) \delta^3(\vec{p}_0 + \vec{p}') \left. \right\} \\
&= \zeta^{2s} e^{-4si\theta} \left(\frac{m}{E}\right)^{4s} \\
&U^{+\lambda_s \mu_s \dots}(\vec{p}, h) V^{+\lambda'_s \mu'_s \dots}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} (-1)^{s-h_0} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, h_0) V_{\lambda'_s \mu'_s \dots \tau'_s}(\vec{p}, -h_0) \delta^3(\vec{p} - \vec{p}') \right. \\
&+ (-1)^{2s+1} \sum_{h_0=s}^{-s} (-1)^{s+h_0} V_{\lambda_s \mu_s \dots \tau_s}(-\vec{p}, h_0) U_{\lambda'_s \mu'_s \dots \tau'_s}(-\vec{p}, -h_0) \delta^3(\vec{p} + \vec{p}') \left. \right\} \\
&= \zeta^{2s} e^{-4si\theta} (-1)^{s-h} \delta_{-h, h'} \delta^3(\vec{p} - \vec{p}')
\end{aligned}$$

□

证明:  $[a(\vec{p}, h), b(\vec{p}', h')]_{-2s+1}$ 

$$\begin{aligned}
&= \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) V^{+\lambda'_s \mu'_s \dots}(\vec{p}', h') [\psi_{\lambda_s \mu_s \dots}(x), \psi_{\lambda'_s \mu'_s \dots}^+(x')]_{-2s+1} e^{-i(p \cdot x + p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}, h) V^{+\lambda'_s \mu'_s \dots}(\vec{p}', h')
\end{aligned}$$

$$\begin{aligned}
& \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \})}}^{2s} \Delta(x - x') e^{-i(p \cdot x + p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\
&= \frac{1}{(2\pi)^3} \int d^3 \vec{r} d^3 \vec{r}' \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\lambda_\zeta \mu_\zeta \cdot \cdot}(\vec{p}, h) V^{\lambda'_\zeta \mu'_\zeta \cdot \cdot}(\vec{p}', h') \\
& \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \})}}^{2s} \left\{ \frac{-i}{(2\pi)^3} \int \frac{1}{2E_0} [e^{ip_0 \cdot (x-x')} - e^{-ip_0 \cdot (x-x')}] d^3 \vec{p}_0 \right\} e^{-i(p \cdot x + p' \cdot x')} \\
&= \left[ \frac{1}{(2\pi)^3} \right]^2 \int \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} U^{+\lambda_\zeta \mu_\zeta \cdot \cdot}(\vec{p}, h) V^{\lambda'_\zeta \mu'_\zeta \cdot \cdot}(\vec{p}', h') \\
& \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - i\gamma^b p_{0b}) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \})}}^{2s} e^{ip_0 \cdot (x-x')} e^{-i(p \cdot x + p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \right. \\
&+ (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m - i\gamma^a p_{0a}) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(-m - i\gamma^b p_{0b}) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \})}}^{2s} e^{-ip_0 \cdot (x-x')} e^{-i(p \cdot x + p' \cdot x')} \left. \right\} d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}_0 \\
&= \left[ \frac{1}{(2\pi)^3} \right]^2 \int d^3 \vec{r} d^3 \vec{r}' d^3 \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
& U^{+\lambda_\zeta \mu_\zeta \cdot \cdot}(\vec{p}, h) V^{\lambda'_\zeta \mu'_\zeta \cdot \cdot}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdot \cdot \tau_\zeta}(\vec{p}_0, h_0)}_{2s} \underbrace{U_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \tau'_\zeta}^+(\vec{p}_0, h_0)}_{2s} e^{i(p_0-p) \cdot x} e^{-i(p_0+p') \cdot x'} \right. \\
&+ (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_\zeta \mu_\zeta \cdot \cdot \tau_\zeta}(\vec{p}_0, h_0)}_{2s} \underbrace{V_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \tau'_\zeta}^+(\vec{p}_0, h_0)}_{2s} e^{-i(p_0+p) \cdot x} e^{i(p_0-p') \cdot x'} \left. \right\} \\
&= \int d^3 \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
& U^{+\lambda_\zeta \mu_\zeta \cdot \cdot}(\vec{p}, h) V^{\lambda'_\zeta \mu'_\zeta \cdot \cdot}(\vec{p}', h') \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdot \cdot \tau_\zeta}(\vec{p}_0, h_0)}_{2s} \underbrace{U_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \tau'_\zeta}^+(\vec{p}_0, h_0)}_{2s} e^{-iE_0 t'} \delta^3(\vec{p}_0 - \vec{p}) \delta^3(\vec{p}_0 + \vec{p}') \right. \\
&+ (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_\zeta \mu_\zeta \cdot \cdot \tau_\zeta}(\vec{p}_0, h_0)}_{2s} \underbrace{V_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \tau'_\zeta}^+(\vec{p}_0, h_0)}_{2s} e^{iE_0 t'} \delta^3(\vec{p}_0 + \vec{p}) \delta^3(\vec{p}_0 - \vec{p}') \left. \right\} \\
&= \delta^3(\vec{p} + \vec{p}') \left(\frac{m}{E}\right)^{4s} U^{+\lambda_\zeta \mu_\zeta \cdot \cdot}(\vec{p}, h) V^{\lambda'_\zeta \mu'_\zeta \cdot \cdot}(\vec{p}', h') \\
& \left\{ \sum_{h_0=s}^{-s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdot \cdot \tau_\zeta}(\vec{p}, h_0)}_{2s} \underbrace{U_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \tau'_\zeta}^+(\vec{p}, h_0)}_{2s} + (-1)^{2s+1} \sum_{h_0=s}^{-s} \underbrace{V_{\lambda_\zeta \mu_\zeta \cdot \cdot \tau_\zeta}(\vec{p}, h_0)}_{2s} \underbrace{V_{\lambda'_\zeta \mu'_\zeta \cdot \cdot \tau'_\zeta}^+(\vec{p}, h_0)}_{2s} e^{2iE(t-t')} \right\} \\
&= 0 + 0 = 0
\end{aligned}$$

□

推论3.4.1.

$$\begin{cases}
\left[ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdot \cdot}(x)}_{2s}, \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \cdot \cdot}^+(x')}_{2s} \right]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \})}}^{2s} \Delta(x - x') \\
\left[ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdot \cdot}(x)}_{2s}, \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \cdot \cdot}^+(x')}_{2s} \right]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{e^{-4si\theta}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \})}}^{2s} \Delta(x - x') \\
\left[ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdot \cdot}^+(x)}_{2s}, \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \cdot \cdot}^+(x')}_{2s} \right]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{e^{4si\theta}}{[(2s)!]^2} \overbrace{[C^+(m - \gamma^a \partial_a^+)]_{\{\lambda_\zeta (\lambda'_\zeta [C^+(m - \gamma^b \partial_b^+)]_{\mu_\zeta \mu'_\zeta} \cdot \cdot \})}}^{2s} \Delta(x - x')
\end{cases}$$

$$\Rightarrow [a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [a(\vec{p}, h), a(\vec{p}', h')]_{-2s+1} = 0, [a^+(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = 0$$



### 3.5 分离表象下Majorana B-W方程协变对易规则的小结

定理3.5.1.

$$\left\{ \begin{aligned} [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} &= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \\ [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} &= \frac{i}{2^{2s-1}} \frac{e^{-4si\theta}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \\ [\psi_{\lambda_\zeta \mu_\zeta \dots}^+(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} &= \frac{i}{2^{2s-1}} \frac{e^{4si\theta}}{[(2s)!]^2} \overbrace{[C^+(m - \gamma^a \partial_a^+)]_{\{\lambda_\zeta (\lambda'_\zeta [C^+(m - \gamma^b \partial_b^+)]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x') \end{aligned} \right.$$

$$\Leftrightarrow [a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [a(\vec{p}, h), a(\vec{p}', h')]_{-2s+1} = 0, [a^+(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = 0$$

### 3.6 分离表象下Majorana B-W方程的协变量子化规则的重要推论

定义3.6.1.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \psi_{\lambda_\zeta \mu_\zeta \dots} = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \psi_{\lambda_\zeta \mu_\zeta \dots} = \Gamma_{\lambda_\zeta \mu_\zeta}^{K_\zeta} \cdot \psi_{K_\zeta}(s)$

推论3.6.1.

$$[\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x')$$

$$\Rightarrow [\psi_{A_\zeta B_\zeta C_\zeta \dots}(x), \psi_{A'_\zeta B'_\zeta C'_\zeta \dots}^+(x')]_{-2s+1} = i \frac{(i\varsigma)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\varsigma)_{A_\zeta A'_\zeta}^a (\sigma, i\varsigma)_{B_\zeta B'_\zeta}^b \dots \partial_a \partial_b \dots}^{2s} \Delta(x - x')$$

证明:

$$[\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x')$$

$$\Leftrightarrow [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1}, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x)$$

$$= i \frac{(i\varsigma)^{2s}}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[-imI \otimes \sigma(x) + (\sigma \otimes \sigma_z, i\varsigma)^a \partial_a]_{\{\lambda_\zeta (\lambda'_\zeta [-imI \otimes \sigma(x) + (\sigma \otimes \sigma_z, i\varsigma)^b \partial_b]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x')$$

$$\Rightarrow [\psi_{A_\zeta B_\zeta C_\zeta \dots}(x), \psi_{A'_\zeta B'_\zeta C'_\zeta \dots}^+(x')]_{-2s+1} = i \frac{(i\varsigma)^{2s}}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{(\sigma, i\varsigma)_{A_\zeta (A'_\zeta)}^a (\sigma, i\varsigma)_{B_\zeta (B'_\zeta)}^b \dots \partial_a \partial_b \dots}^{2s} \Delta(x - x')$$

$$\Leftrightarrow [\psi_{A_\zeta B_\zeta C_\zeta \dots}(x), \psi_{A'_\zeta B'_\zeta C'_\zeta \dots}^+(x')]_{-2s+1} = i \frac{(i\varsigma)^{2s}}{2^{2s-1}} \overbrace{(\sigma, i\varsigma)_{A_\zeta (A'_\zeta)}^a (\sigma, i\varsigma)_{B_\zeta (B'_\zeta)}^b \dots \partial_a \partial_b \dots}^{2s} \Delta(x - x') \quad \square$$

推论3.6.2.

$$[\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')] = \frac{i}{2^{2s-1}} \frac{e^{-4si\theta}}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x')$$

$$\Rightarrow [\psi_{A_\zeta B_\zeta C_\zeta \dots}(x), \psi_{A'_\zeta B'_\zeta C'_\zeta \dots}^+(x')]_{-2s+1} = \Delta(x - x')$$

推论3.6.3.

$$[\psi_{\lambda_\zeta \mu_\zeta \dots}^+(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')] = \frac{i}{2^{2s-1}} \frac{e^{4si\theta}}{[(2s)!]^2} \overbrace{[C^+(m - \gamma^a \partial_a^+)]_{\{\lambda_\zeta (\lambda'_\zeta [C^+(m - \gamma^b \partial_b^+)]_{\mu_\zeta \mu'_\zeta \dots})\}}}^{2s} \Delta(x - x')$$

$$\Rightarrow [\psi_{A_\zeta B_\zeta C_\zeta \dots}^+(x), \psi_{A'_\zeta B'_\zeta C'_\zeta \dots}^+(x')]_{-2s+1} = \Delta(x - x')$$

## 4 分离表象下Majorana B-W方程对易规则的两种等价表述

### 4.1 分离表象下Majorana B-W方程的等价协变对易规则

引理4.1.1.

$$\begin{cases} 2\mathbb{X}_{\lambda_s\mu_s}^a(p)(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})\mathbb{X}_{\lambda'_s\mu'_s}^{+a'}(p) = [(m - i\gamma^a p_a)\gamma^4]_{\{\lambda_s(\lambda'_s[(m - i\gamma^b p_b)\gamma^4]_{\mu_s\mu'_s})\}} \\ 2\mathbb{X}_{\lambda_s\mu_s}^a(p)(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})(\bar{C}\gamma_4\mathbb{X}^{+a'}\bar{C}\gamma_4)_{\lambda'_s\mu'_s}(p) = [(m - i\gamma^a p_a)C]_{\{\lambda_s(\lambda'_s[(m - i\gamma^b p_b)C]_{\mu_s\mu'_s})\}} \\ \left\{ \begin{aligned} 2\mathbb{X}_{\lambda_s\mu_s}^a(x)(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2})\mathbb{X}_{\lambda'_s\mu'_s}^{+a'}(x')\Delta(x - x') &= [(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b)\gamma^4]_{\mu_s\mu'_s})\}}\Delta(x - x') \\ 2\mathbb{X}_{\lambda_s\mu_s}^a(x)(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2})(\bar{C}\gamma_4\mathbb{X}^{+a'}\bar{C}\gamma_4)_{\lambda'_s\mu'_s}(x')\Delta(x - x') &= [(m - \gamma^a \partial_a)C]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b)C]_{\mu_s\mu'_s})\}}\Delta(x - x') \end{aligned} \right. \end{cases}$$

定理4.1.1.

$$\begin{cases} \left\{ \begin{aligned} [\psi_{\lambda_s\mu_s \dots}^{2n}(x), \psi_{\lambda'_s\mu'_s \dots}^{+2n}(x')] &= \frac{i}{2^{2n-1}} \frac{1}{[(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b)\gamma^4]_{\mu_s\mu'_s \dots})\}}}_{2n} \Delta(x - x') \\ \Leftrightarrow [\psi_{\lambda_s\mu_s \dots}^{2n}(x), \psi_{\lambda'_s\mu'_s \dots}^{+2n}(x')] &= \frac{i}{2^{3n-1}} \frac{1}{[(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_s\mu_s}(x) \dots \cdot \cdot \cdot}_{n}} \underbrace{\mathbb{X}_{\{\lambda'_s\mu'_s}^{+a'}(x') \dots \cdot \cdot \cdot}_{n}} \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \dots}_{n} \Delta(x - x') \end{aligned} \right. \\ \Leftrightarrow \left\{ \begin{aligned} [\psi_{\lambda_s\mu_s \dots}^{2n}(x), \psi_{\lambda'_s\mu'_s \dots}^{+2n}(x')] &= \frac{i}{2^{2n-1}} \frac{e^{-4ni\theta}}{[(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a)C]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b)C]_{\mu_s\mu'_s \dots})\}}}_{2n} \Delta(x - x') \\ \Leftrightarrow [\psi_{\lambda_s\mu_s \dots}^{2n}(x), \psi_{\lambda'_s\mu'_s \dots}^{+2n}(x')] &= \frac{i}{2^{3n-1}} \frac{e^{-4ni\theta}}{[(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_s\mu_s}(x) \dots \cdot \cdot \cdot}_{n}} \underbrace{(\bar{C}\gamma_4\mathbb{X}^{+a'}\bar{C}\gamma_4)_{\{\lambda'_s\mu'_s}(x') \dots \cdot \cdot \cdot}_{n}} \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \dots}_{n} \Delta(x - x') \end{aligned} \right. \end{cases}$$

定理4.1.2.

$$\begin{cases} \left\{ \begin{aligned} \{ \psi_{\lambda_s\mu_s \dots \tau_s}^{2n+1}(x), \psi_{\lambda'_s\mu'_s \dots \tau'_s}^{+2n+1}(x') \} \\ = \frac{i}{2^{2n}} \frac{1}{[(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b)\gamma^4]_{\mu_s\mu'_s \dots} \cdot \cdot \cdot [(m - \gamma^c \partial_c)\gamma^4]_{\tau_s\tau'_s})\}}}_{2n+1} \Delta(x - x') \\ \Leftrightarrow \{ \psi_{\lambda_s\mu_s \dots \tau_s}^{2n+1}(x), \psi_{\lambda'_s\mu'_s \dots \tau'_s}^{+2n+1}(x') \} \\ = \frac{i}{2^{3n}} \frac{1}{[(2n+1)!]^2} \underbrace{\mathbb{X}_{\{\lambda_s\mu_s}(x) \dots \cdot \cdot \cdot}_{n}} \underbrace{\mathbb{X}_{\{\lambda'_s\mu'_s}^{+a'}(x') \dots \cdot \cdot \cdot}_{n}} \underbrace{[(m - \gamma^c \partial_c)\gamma^4]_{\tau_s\tau'_s}} \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \dots}_{n} \Delta(x - x') \end{aligned} \right. \\ \Leftrightarrow \left\{ \begin{aligned} \{ \psi_{\lambda_s\mu_s \dots \tau_s}^{2n+1}(x), \psi_{\lambda'_s\mu'_s \dots \tau'_s}^{+2n+1}(x') \} \\ = \frac{i}{2^{2n}} \frac{e^{-(4n+2)\theta}}{[(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a)C]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b)C]_{\mu_s\mu'_s \dots} \cdot \cdot \cdot [(m - \gamma^c \partial_c)C]_{\tau_s\tau'_s})\}}}_{2n+1} \Delta(x - x') \\ \Leftrightarrow \{ \psi_{\lambda_s\mu_s \dots \tau_s}^{2n+1}(x), \psi_{\lambda'_s\mu'_s \dots \tau'_s}^{+2n+1}(x') \} \\ = \frac{ie^{-(4n+2)\theta}}{2^{3n}} \frac{1}{[(2n+1)!]^2} \underbrace{\mathbb{X}_{\{\lambda_s\mu_s}(x) \dots \cdot \cdot \cdot}_{n}} \underbrace{(\bar{C}\gamma_4\mathbb{X}^{+a'}\bar{C}\gamma_4)_{\{\lambda'_s\mu'_s}(x') \dots \cdot \cdot \cdot}_{n}} \underbrace{[(m - \gamma^c \partial_c)C]_{\tau_s\tau'_s}} \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \dots}_{n} \Delta(x - x') \end{aligned} \right. \end{cases}$$

### 4.2 分离表象下有质量Majorana玻色子对易规则小结

定理4.2.1.  $n \geq 0$

$$\begin{aligned} [a(\vec{p}, h; n), a^+(\vec{p}', h'; n)] &= \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [a(\vec{p}, h; n), a(\vec{p}', h'; n)] = 0, [a^+(\vec{p}, h; n), a^+(\vec{p}', h'; n)] = 0 \\ \Leftrightarrow [\psi_{\lambda_s\mu_s \dots}^{2n}(x), \psi_{\lambda'_s\mu'_s \dots}^{+2n}(x')] &= \frac{i}{2^{2n-1}} \frac{1}{[(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a)\gamma^4]_{\{\lambda_s(\lambda'_s[(m - \gamma^b \partial_b)\gamma^4]_{\mu_s\mu'_s \dots})\}}}_{2n} \Delta(x - x') \\ \Leftrightarrow [\psi_{\lambda_s\mu_s \dots}^{2n}(x), \psi_{\lambda'_s\mu'_s \dots}^{+2n}(x')] &= \frac{i}{2^{3n-1}} \frac{1}{[(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_s\mu_s}(x) \dots \cdot \cdot \cdot}_{n}} \underbrace{\mathbb{X}_{\{\lambda'_s\mu'_s}^{+a'}(x') \dots \cdot \cdot \cdot}_{n}} \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \dots}_{n} \Delta(x - x') \end{aligned}$$

### 4.3 分离表象下有质量Majorana费米子反对易规则小结

定理4.3.1.  $n \geq 0$

$$\begin{aligned} & \{a(\vec{p}, h; n + \frac{1}{2}), a^+(\vec{p}', h'; n + \frac{1}{2})\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), \{rest\} = 0 \\ & \Leftrightarrow \left\{ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}_{2n+1}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+}_{2n+1}(x') \right\} \\ & = \frac{i}{2^{2n}} \frac{1}{[(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\lambda_\zeta \mu_\zeta} \dots [(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta}}_{2n+1} \Delta(x - x') \\ & \Leftrightarrow \left\{ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}_{2n+1}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+}_{2n+1}(x') \right\} \\ & = \frac{i}{2^{3n} [(2n+1)!]^2} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a}_{n}(x) \dots \underbrace{\mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{a'}}_{n}(x') \dots [(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta} \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \Delta(x - x') \end{aligned}$$

### 4.4 分离表象下推导到自旋-n粒子Klein-Gordon方程的平面波解

定理4.4.1.  $(-\partial^c \partial_c + m^2) A_{\underbrace{ab \dots}_n}(x) = 0, A_{\underbrace{ab \dots}_n}(x) = (\frac{1}{2im})^n \overbrace{(\bar{C} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}^n(x)$

$$A_{\underbrace{ab \dots}_n}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} [a(\vec{p}, h) \varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) e^{i(\vec{p} \cdot \vec{r} - Et)} + b^+(\vec{p}, h) \tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^3 \vec{p}$$

$$\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots U_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}^n(\vec{p}, h)$$

$$\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots V_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}^n(\vec{p}, h)$$

## 5 分离表象下Majorana B-W方程各种量子算符

### 5.1 分离表象下Majorana B-W方程各种算符的提取

定理5.1.1.

$$\left\{ \begin{aligned} P_u(s) &= \int \psi^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{r}, t) \frac{-i\partial_u (i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) d^3 \vec{r} = \int \sum_h p_u [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} a(\vec{p}, h) a^+(\vec{p}, h)] d^3 \vec{p} \\ Q(s) &= \int \psi^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) d^3 \vec{r} = \int \sum_h [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s-1} a(\vec{p}, h) a^+(\vec{p}, h)] d^3 \vec{p} \\ N(s) &= \int \psi^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) d^3 \vec{r} = \int \sum_h [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} a(\vec{p}, h) a^+(\vec{p}, h)] d^3 \vec{p} \\ \vec{S}(s) &= \int \psi^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla} (i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) d^3 \vec{r} = \int \sum_h \hat{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} a(\vec{p}, h) a^+(\vec{p}, h)] d^3 \vec{p} \\ \vec{M}(s) &= \int \psi^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla} (i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{r}, t) d^3 \vec{r} = \int \sum_h \hat{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s-1} a(\vec{p}, h) a^+(\vec{p}, h)] d^3 \vec{p} \end{aligned} \right.$$

## 6 Klein-Gordon方程的对易规则

### 6.1 有质量自旋-n的Majorana B-W方程等价于Klein-Gordon方程 [17, 19, 22]

定义6.1.1.  $\mathbb{X}_a \equiv [im\gamma_a(\zeta) - 2S_{ab}(e, \zeta)\partial^b]C$

$$\text{定理6.1.1.} \quad \left\{ \begin{array}{l} [\gamma^a(\zeta)\partial_a + m]\psi_{[\lambda_\zeta]\mu_\zeta\eta_\zeta\xi_\zeta\cdots}(x) = 0 \\ \psi_{[\lambda_\zeta]\mu_\zeta\eta_\zeta\xi_\zeta\cdots}(x) \text{全对称} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (-\partial^c\partial_c + m^2)\underbrace{A_{ab\cdots}}_n(x) = 0 \\ \delta^{ab}\underbrace{A_{ab\cdots}}_n(x) = 0, \partial^a\underbrace{A_{ab\cdots}}_n(x) = 0, \underbrace{A_{ab\cdots}}_n(x) \text{全对称} \\ \psi_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\cdots}(x) = \frac{1}{2^n}\underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a\mathbb{X}_{\eta_\zeta\xi_\zeta}^b\cdots}_n\cdot\underbrace{A_{ab\cdots}}_n(x) \end{array} \right.$$

$$\psi_{\lambda_\zeta\mu_\zeta\cdots}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} E^{n-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2n} [a(\vec{p}, h)\underbrace{U_{\lambda_\zeta\mu_\zeta\cdots}}_{2n}(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)\underbrace{V_{\lambda_\zeta\mu_\zeta\cdots}}_{2n}(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\underbrace{A_{ab\cdots}}_n(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} [a(\vec{p}, h)\underbrace{\varepsilon_{ab\cdots}}_n(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)\underbrace{\tilde{\varepsilon}_{ab\cdots}}_n(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

## 6.2 自旋-n粒子Klein-Gordon方程的平面波解

$$\text{推论6.2.1.} \quad A_{ab\cdots}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} \varepsilon_{ab\cdots}(\vec{p}, h) [a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + (-1)^n b^+(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\text{推论6.2.2.} \quad A_{ab\cdots}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} \varepsilon_{ab\cdots}(\vec{p}, h) [a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + e^{-4ni\theta}(-1)^h a^+(\vec{p}, -h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\text{性质6.2.1.} \quad b^+(\vec{p}, h) = \zeta^{2s} e^{-4si\theta} (-1)^{s+h} a^+(\vec{p}, -h) \Rightarrow b(\vec{p}, h) = \zeta^{2s} e^{4si\theta} (-1)^{s+h} a(\vec{p}, -h)$$

$$\text{推论6.2.3.} \quad A_{ab\cdots}(\vec{r}, t) = e^{-4ni\theta} \underbrace{\eta_a^a \eta_b^b \cdots}_n \cdot A_{a'b'\cdots}^+(\vec{r}, t)$$

$$\begin{aligned} \text{证明: } & e^{-4ni\theta} \underbrace{\eta_a^a \eta_b^b \cdots}_n \cdot A_{a'b'\cdots}^+(\vec{r}, t) \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} \underbrace{\eta_a^a \eta_b^b \cdots}_n \cdot \varepsilon_{a'b'\cdots}^+(\vec{p}, h) [e^{-4ni\theta} a^+(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)} + (-1)^h a(\vec{p}, -h)e^{i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p} \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} (-1)^h \varepsilon_{ab\cdots}(\vec{p}, -h) [e^{-4ni\theta} a^+(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)} + (-1)^h a(\vec{p}, -h)e^{i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p} \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} \varepsilon_{ab\cdots}(\vec{p}, -h) [a(\vec{p}, -h)e^{i(\vec{p}\cdot\vec{r}-Et)} + (-1)^h e^{-4ni\theta} a^+(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p} \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} \varepsilon_{ab\cdots}(\vec{p}, h) [a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + (-1)^h e^{-4ni\theta} a^+(\vec{p}, -h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p} \\ &= A_{ab\cdots}(\vec{r}, t) \end{aligned}$$

□

从上面结论可知，为了保持势 $A_{ab\cdots}$ 的实性，参数 $\theta$ 取0或 $\pi/2$ 比较合适。

## 7 Rarita-Schwinger方程的对易规则

### 7.1 有质量自旋- $n + \frac{1}{2}$ 的Majorana B-W方程等价于Rarita-Schwinger方程 [17, 19, 20]

定理7.1.1.

$$\left\{ \begin{array}{l} (\gamma^a\partial_a + m)\psi_{[\lambda_\zeta]\mu_\zeta\eta_\zeta\xi_\zeta\cdots\tau_\zeta}(x) = 0 \\ \psi_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\cdots\tau_\zeta}(x) \text{全对称} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (\gamma^c\partial_c + m)\underbrace{A_{ab\cdots}[\tau_\zeta]}_n(x) = 0 \\ \delta^{ab}\underbrace{A_{ab\cdots}[\tau_\zeta]}_n(x) = 0, \gamma^a\underbrace{A_{ab\cdots}[\tau_\zeta]}_n(x) = 0, \underbrace{A_{ab\cdots}[\tau_\zeta]}_n(x) \text{全对称} \\ \psi_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\cdots\tau_\zeta}(x) = \frac{1}{2^n}\underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a\mathbb{X}_{\eta_\zeta\xi_\zeta}^b\cdots}_n\cdot\underbrace{A_{ab\cdots}[\tau_\zeta]}_n(x) \end{array} \right.$$

$$\psi_{\lambda_\zeta\mu_\zeta\cdots}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n+1/2}^{-(n+1/2)} E^{n-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2n+1} [a(\vec{p}, h)\underbrace{U_{\lambda_\zeta\mu_\zeta\cdots}}_{2n}(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)\underbrace{V_{\lambda_\zeta\mu_\zeta\cdots}}_{2n}(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$A_{ab\cdots\tau_\zeta}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{\sqrt{m}}{\sqrt{2^n E}} [\varepsilon_{ab\cdots\tau_\zeta}(\vec{p}, h)a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + \tilde{\varepsilon}_{ab\cdots\tau_\zeta}(\vec{p}, h)b^+(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

## 7.2 分离表象下自旋 $s = n + \frac{1}{2}$ 粒子 Rarita-Schwinger 方程的平面波解

$$\text{推论 7.2.1. } A_{ab \dots \tau_\zeta}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{\sqrt{m}}{\sqrt{2^n E}} [\varepsilon_{ab \dots \tau_\zeta}(\vec{p}, h) a(\vec{p}, h) e^{i(\vec{p} \cdot \vec{r} - Et)} + \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}, h) b^+(\vec{p}, h) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^3 \vec{p}$$

$$\text{推论 7.2.2. } A_{ab \dots \tau_\zeta}(\vec{r}, t)$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{\sqrt{m}}{\sqrt{2^n E}} \varepsilon_{ab \dots \sigma_\zeta}(\vec{p}, h) [\delta_{\tau_\zeta \sigma_\zeta} a(\vec{p}, h) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{h-\frac{1}{2}} e^{-(4n+2)i\theta} \gamma_{5\tau_\zeta \sigma_\zeta} a^+(\vec{p}, -h) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^3 \vec{p}$$

$$\text{定理 7.2.1. } A_{ab \dots \tau_\zeta}^+(\vec{r}, t) = -e^{(4n+2)i\theta} \overbrace{\eta_a^{\prime} \eta_b^{\prime} \dots}^n (\sigma_y \otimes \sigma_y)_{\tau_\zeta \tau_\zeta}^{\prime} A_{a'b' \dots \tau_\zeta}'(\vec{r}, t) = e^{(4n+2)i\theta} \overbrace{\eta_a^{\prime} \eta_b^{\prime} \dots}^n (\bar{C} \gamma_4)_{\tau_\zeta \tau_\zeta}^{\prime} A_{a'b' \dots \tau_\zeta}'(\vec{r}, t)$$

$$\text{证明: } -e^{-(4n+2)i\theta} \overbrace{\eta_a^{\prime} \eta_b^{\prime} \dots}^n (\sigma_y \otimes \sigma_y)_{\tau_\zeta \tau_\zeta}^{\prime} A_{a'b' \dots \tau_\zeta}'(\vec{r}, t)$$

$$= -\frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{\sqrt{m}}{\sqrt{2^n E}}$$

$$\overbrace{\eta_a^{\prime} \eta_b^{\prime} \dots}^n \gamma_{2\tau_\zeta \tau_\zeta}^{\prime} \varepsilon_{a'b' \dots \sigma_\zeta}^+(\vec{p}, h) [e^{-(4n+2)i\theta} \delta_{\tau_\zeta \sigma_\zeta}^{\prime} a^+(\vec{p}, h) e^{-i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{h-\frac{1}{2}} \gamma_{5\tau_\zeta \sigma_\zeta} a(\vec{p}, -h) e^{i(\vec{p} \cdot \vec{r} - Et)}] d^3 \vec{p}$$

$$= -\frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{\sqrt{m}}{\sqrt{2^n E}} \varepsilon_{ab \dots \tau_\zeta}(\vec{p}, -h) [(-1)^{h-\frac{1}{2}} e^{-(4n+2)i\theta} \gamma_{5\tau_\zeta \tau_\zeta}^{\prime} a^+(\vec{p}, h) e^{-i(\vec{p} \cdot \vec{r} - Et)} - \delta_{\tau_\zeta \tau_\zeta}^{\prime} a(\vec{p}, -h) e^{i(\vec{p} \cdot \vec{r} - Et)}] d^3 \vec{p}$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{\sqrt{m}}{\sqrt{2^n E}} \varepsilon_{ab \dots \sigma_\zeta}(\vec{p}, h) [(-1)^{h-\frac{1}{2}} e^{-(4n+2)i\theta} \gamma_{5\tau_\zeta \sigma_\zeta} a^+(\vec{p}, -h) e^{-i(\vec{p} \cdot \vec{r} - Et)} + \delta_{\tau_\zeta \sigma_\zeta} a(\vec{p}, h) e^{i(\vec{p} \cdot \vec{r} - Et)}] d^3 \vec{p}$$

$$= A_{ab \dots \tau_\zeta}(\vec{r}, t) \quad \square$$

从上面结论可知，为了保持势  $A_{ab \dots \tau_\zeta}$  的实性，参数  $\theta$  取 0 或  $\pi/2$  比较合适，但取 0 更简单。

## 7.3 分离表象下 Majorana B-W 方程的等时量子化规则

定理 7.3.1.

$$\begin{aligned} & [\psi_{\lambda_\zeta \mu_\zeta \dots}^-(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} = i \frac{(i\zeta)^{2s}}{2^{2s-1}} \overbrace{(\sigma \otimes \sigma_z, i\zeta)^a}_{2s} \{ \lambda_\zeta (\lambda'_\zeta (\sigma \otimes \sigma_z, i\zeta)^b_{\mu_\zeta \mu'_\zeta} \dots) \} \overbrace{\partial_a \partial_b \dots}_{2s} \Delta(x - x') \\ \Rightarrow & [\psi_{\lambda_\zeta \mu_\zeta \dots \xi_\zeta \eta_\zeta \dots \tau_\zeta}^-(\vec{r}, t), \psi_{\lambda'_\zeta \mu'_\zeta \dots \xi'_\zeta \eta'_\zeta \dots \tau'_\zeta}^+(\vec{r}', t)]_{-2s+1} \\ = & -\frac{(i\zeta)^{2s+1}}{2^{2s-1}} \sum_{k=0}^{[s-\frac{1}{2}]} \frac{(2s)!}{(2s-2k-1)!(2k)!} \overbrace{[(\sigma \cdot \nabla) \otimes \sigma_z]_{\lambda_\zeta \lambda'_\zeta} [(\sigma \cdot \nabla) \otimes \sigma_z]_{\mu_\zeta \mu'_\zeta} \dots}_{2s-2k-1} \overbrace{\delta_{\xi_\zeta \xi'_\zeta} \delta_{\eta_\zeta \eta'_\zeta} \dots}_{2k} \nabla^{2k} \delta_{\tau_\zeta \tau'_\zeta} \delta^3(\vec{r} - \vec{r}') \end{aligned}$$

## 8 分离表象下有质量 Majorana 粒子协变对易规则的总结与梳理

### 8.1 定义

定义 8.1.1.

$$\begin{aligned} & \Gamma_{\substack{b_1 b_2 \dots b'_1 b'_2 \dots \\ a_1 a_2 \dots a'_1 a'_2 \dots}}^n(p; n) \\ := & \overbrace{(\bar{C} \gamma_{a_1})^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_{a_2})^{\eta_\zeta \xi_\zeta} \dots (\gamma_{a'_1} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{a'_2} C)^{\eta'_\zeta \xi'_\zeta} \dots}_{n} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^{b_1}(p) \mathbb{X}_{\eta_\zeta \xi_\zeta}^{b_2}(p) \dots}_{n} \underbrace{\mathbb{X}_{(\lambda'_\zeta \mu'_\zeta)}^{+b'_1}(-p) \mathbb{X}_{\eta'_\zeta \xi'_\zeta}^{+b'_2}(-p) \dots}_{n} \\ & \Gamma_{\substack{b_1 b_2 \dots b'_1 b'_2 \dots \\ a_1 a_2 \dots a'_1 a'_2 \dots}}^n(x, x'; n) \\ := & \overbrace{(\bar{C} \gamma_{a_1})^{\lambda_\zeta \mu_\zeta} (\bar{C} \gamma_{a_2})^{\eta_\zeta \xi_\zeta} \dots (\gamma_{a'_1} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{a'_2} C)^{\eta'_\zeta \xi'_\zeta} \dots}_{n} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^{b_1}(x) \mathbb{X}_{\eta_\zeta \xi_\zeta}^{b_2}(x) \dots}_{n} \underbrace{\mathbb{X}_{(\lambda'_\zeta \mu'_\zeta)}^{+b'_1}(x') \mathbb{X}_{\eta'_\zeta \xi'_\zeta}^{+b'_2}(x') \dots}_{n} \end{aligned}$$

定义8.1.2.

$$\begin{aligned} & \Gamma_{\underbrace{a_1 a_2 \cdots a'_1 a'_2 \cdots}_n}(p; n) \\ & := \underbrace{(\bar{C}\gamma_{a_1})^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_{a_2})^{\eta_\zeta \xi_\zeta} \cdots (\gamma_{a'_1} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{a'_2} C)^{\eta'_\zeta \xi'_\zeta} \cdots}_{2n} [(m - i\gamma^a p_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - i\gamma^b p_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots})} \\ & \Gamma_{\underbrace{a_1 a_2 \cdots a'_1 a'_2 \cdots}_n}(x; n) \\ & := \underbrace{(\bar{C}\gamma_{a_1})^{\lambda_\zeta \mu_\zeta} (\bar{C}\gamma_{a_2})^{\eta_\zeta \xi_\zeta} \cdots (\gamma_{a'_1} C)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{a'_2} C)^{\eta'_\zeta \xi'_\zeta} \cdots}_{2n} [(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \cdots})} \end{aligned}$$

$$\text{推论8.1.1. } \Gamma_{\underbrace{a_1 a_2 \cdots a'_1 a'_2 \cdots}_n}(x; n) = \frac{1}{2^n} \Gamma_{\underbrace{b_1 b_2 \cdots b'_1 b'_2 \cdots}_n}(x, x'; n) \underbrace{[\eta_{b_1 b'_1} - \frac{\partial_{b_1} \partial_{b'_1}^+}{m^2}] [\eta_{b_2 b'_2} - \frac{\partial_{b_2} \partial_{b'_2}^+}{m^2}]}_n \cdots$$

## 8.2 分离表象下有质量Majorana玻色子协变对易规则梳理(取 $\theta = 0$ )

$$\text{定义8.2.1. } \hat{P}_{a_1 \cdots a_n \tau_\zeta b_1 \cdots b_n}(n) := \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \cdots \eta_{b_n}^{a'_n} \hat{P}_{a_1 \cdots a_n \tau_\zeta a'_1 \cdots a'_n}(n)$$

$$\text{定理8.2.1. } \begin{cases} [A_{a_1 a_2 \cdots a_n}(x), A_{a'_1 a'_2 \cdots a'_n}^+(x')] = i \hat{P}_{a_1 a_2 \cdots a_n a'_1 a'_2 \cdots a'_n}(n) \Delta(x - x') \\ [A_{a_1 a_2 \cdots a_n}(x), A_{b_1 b_2 \cdots b_n}(x')] = i \hat{P}_{a_1 a_2 \cdots a_n a'_1 a'_2 \cdots a'_n}(n) \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \cdots \eta_{b_n}^{a'_n} \Delta(x - x') \\ A_{a_1 a_2 \cdots a_n} = A_{a'_1 a'_2 \cdots a'_n}^+ \eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots \eta_{a_n}^{a'_n}, \underbrace{A_{a'_1 a'_2 \cdots}^+}_n = \underbrace{A_{a_1 a_2 \cdots}}_n \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots}_n \end{cases}$$

⟨⟩

$$\text{定理8.2.2. } \begin{cases} [A_{a_1 a_2 \cdots}(x), A_{a'_1 a'_2 \cdots}^+(x')] = \frac{im^{-2n}}{2^{4n-1} [(2n)!]^2} \Gamma_{a_1 a_2 \cdots a'_1 a'_2 \cdots}(x; n) \Delta(x - x') \\ [A_{a_1 a_2 \cdots}(x), A_{b_1 b_2 \cdots}(x')] = \frac{im^{-2n}}{2^{4n-1} [(2n)!]^2} \Gamma_{a_1 a_2 \cdots a'_1 a'_2 \cdots}(x; n) \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \cdots \Delta(x - x') \\ \underbrace{A_{a_1 a_2 \cdots}}_n = \underbrace{A_{a'_1 a'_2 \cdots}^+}_n \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots}_n, \underbrace{A_{a'_1 a'_2 \cdots}^+}_n = \underbrace{A_{a_1 a_2 \cdots}}_n \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots}_n \end{cases}$$

⟨⟩

$$\text{定理8.2.3. } \begin{cases} [A_{a_1 a_2 \cdots}(x), A_{a'_1 a'_2 \cdots}^+(x')] = \frac{im^{-2n}}{2^{5n-1} [(2n)!]^2} \Gamma_{\underbrace{b_1 b_2 \cdots b'_1 b'_2 \cdots}_n}(x, x'; n) \underbrace{[\eta_{b_1 b'_1} - \frac{\partial_{b_1} \partial_{b'_1}^+}{m^2}]}_n \cdots \Delta(x - x') \\ [A_{a_1 a_2 \cdots}(x), A_{b_1 b_2 \cdots}(x')] = \frac{im^{-2n}}{2^{5n-1} [(2n)!]^2} \Gamma_{\underbrace{c_1 c_2 \cdots c'_1 c'_2 \cdots}_n}(x, x'; n) \underbrace{[\eta_{c_1 c'_1} - \frac{\partial_{c_1} \partial_{c'_1}^+}{m^2}]}_n \cdots \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \cdots \Delta(x - x') \\ \underbrace{A_{a_1 a_2 \cdots}}_n = \underbrace{A_{a'_1 a'_2 \cdots}^+}_n \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots}_n, \underbrace{A_{a'_1 a'_2 \cdots}^+}_n = \underbrace{A_{a_1 a_2 \cdots}}_n \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots}_n \end{cases}$$

⟨⟩

$$\text{定理8.2.4. } \begin{cases} [\psi_{\lambda_\zeta \mu_\zeta \cdots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \cdots}^+(x')] = \frac{i}{2^{3n-1} [(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta \cdots\}}^a}_n(x) \cdots \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta \cdots\}}^{+a'}}_n(x') \cdots \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}^+}{m^2}]}_n \cdots \Delta(x - x') \\ [\psi_{\lambda_\zeta \mu_\zeta \cdots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \cdots}(x')] = \frac{i}{2^{3n-1} [(2n)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta \cdots\}}^a}_n(x) \cdots \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta \cdots\}}^{a'}}_n(x') \cdots \underbrace{[\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdots \Delta(x - x') \\ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdots}}_{2n} = \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \cdots}^+}_{2n} \underbrace{\gamma_{2\lambda_\zeta}^{\lambda'_\zeta} \gamma_{2\mu_\zeta}^{\mu'_\zeta} \cdots}_{2n}, \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \cdots}}_{2n} = \underbrace{\psi_{\lambda_\zeta \mu_\zeta \cdots}}_{2n} \underbrace{\gamma_{2\lambda'_\zeta}^{\lambda_\zeta} \gamma_{2\mu'_\zeta}^{\mu'_\zeta} \cdots}_{2n}, \mathbb{X}^a = \gamma_2 \mathbb{X}^{+a'} \gamma_2 \eta_{a'}^a \end{cases}$$

$$[\Downarrow]$$

$$\text{定理8.2.5.} \left\{ \begin{array}{l} [\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2n}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2n}(x')] = \frac{i}{2^{2n-1} [(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2n} \Delta(x - x') \\ [\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2n}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2n}(x')] = \frac{i}{2^{2n-1} [(2n)!]^2} \underbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2n} \Delta(x - x') \\ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2n} = \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2n} \underbrace{\gamma_2^{\lambda'_\zeta} \gamma_2^{\mu'_\zeta}}_{2n} \dots, \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2n} = \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2n} \underbrace{\gamma_2^{\lambda_\zeta} \gamma_2^{\mu_\zeta}}_{2n} \dots \end{array} \right.$$

$$[\Downarrow]$$

$$\text{定理8.2.6.} [\underbrace{\psi_{A_\zeta B_\zeta \dots}}_{2n}(x), \underbrace{\psi_{A'_\zeta B'_\zeta \dots}^+}_{2n}(x')] = i \frac{(i\zeta)^{2n}}{2^{2n-1} [(2n)!]^2} \underbrace{(\sigma, i\zeta)_{\{A_\zeta (A'_\zeta (\sigma, i\zeta)_{B_\zeta B'_\zeta \dots})\}}}_{2n} \underbrace{\partial_a \partial_b \dots}_{2n} \Delta(x - x')$$

$$[\Downarrow]$$

$$\text{定理8.2.7.} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i \frac{(-1)^{2n}}{2^{n-1}} \Gamma_{k_\zeta k'_\zeta}^{abc} \dots \underbrace{(n)}_{2n} \underbrace{\partial_a \partial_b \partial_c \dots}_{2s} \Delta(x - x')$$

势 $A$ 和场 $\psi$ 的对易关系都有两种等价表述方式，且互为前提和互为因果。既可以从势对易关系推出一切，也可以从场对易关系推出一切。这说明对于有质量粒子，势和场两种描述方案是完全等价的。并且可以从有质量粒子的对易规则推得与无质量粒子完全类似的对易规则，反之则不行。

### 8.3 分离表象下有质量Majorana费米子协变反对易规则梳理

$$\text{定理8.3.1.} \{A_{a_1 a_2 \dots a_n \tau_\zeta}(x), A_{a'_1 a'_2 \dots a'_n \tau'_\zeta}^+(x')\} = i \hat{P}_{a_1 \dots a_n \tau_\zeta a'_1 \dots a'_n \tau'_\zeta} (n + \frac{1}{2}) \Delta(x - x')$$

$$[\Downarrow]$$

定理8.3.2.

$$\left\{ \begin{array}{l} \{A_{a_1 a_2 \dots a_n \tau_\zeta}(x), A_{b_1 b_2 \dots b_n \sigma_\zeta}(x')\} = -i \hat{P}_{a_1 \dots a_n \tau_\zeta b_1 \dots b_n \sigma_\zeta} (n + \frac{1}{2}) \gamma_2^{\tau'_\zeta} \Delta(x - x') \\ A_{a_1 a_2 \dots a_n \tau_\zeta}(\vec{r}, t) = -A_{a'_1 a'_2 \dots a'_n \tau'_\zeta}^+(\vec{r}, t) \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \dots}_{n} \gamma_2^{\tau'_\zeta}, A_{a'_1 a'_2 \dots a'_n \tau'_\zeta}^+(\vec{r}, t) = -A_{a_1 a_2 \dots a_n \tau_\zeta}(\vec{r}, t) \underbrace{\eta_{a'_1}^{a_1} \eta_{a'_2}^{a_2} \dots}_{n} \gamma_2^{\tau_\zeta} \end{array} \right.$$

$$[\Downarrow]$$

定理8.3.3.

$$\left\{ \begin{array}{l} \{A_{a_1 a_2 \dots a_n \tau_\zeta}(x), A_{a'_1 a'_2 \dots a'_n \tau'_\zeta}^+(x')\} = \frac{im^{-2n}}{2^{5n} [(2n+1)!]^2} \\ \underbrace{(\bar{C} \gamma_{a_1})^{\lambda_\zeta \mu_\zeta} \dots (\gamma_{a'_1} C)^{\lambda'_\zeta \mu'_\zeta} \dots}_{n} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^{b_1} (x) \dots \mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{b'_1} (x') \dots}_{n} \cdot [(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta} \underbrace{[\eta_{b_1 b'_1} - \frac{\partial_{b_1} \partial_{b'_1}^+}{m^2}]}_{n} \Delta(x - x') \\ \{A_{a_1 a_2 \dots a_n \tau_\zeta}(x), A_{a'_1 a'_2 \dots a'_n \tau'_\zeta}^+(x')\} = \frac{i(im)^{-2n}}{2^{5n} [(2n+1)!]^2} \\ \underbrace{(\bar{C} \gamma_{a_1})^{\lambda_\zeta \mu_\zeta} \dots (\bar{C} \gamma_{a'_1})^{\lambda'_\zeta \mu'_\zeta} \dots}_{n} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^{b_1} (x) \dots \mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{b'_1} (x') \dots}_{n} \cdot [(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta} \underbrace{[\delta_{b_1 b'_1} - \frac{\partial_{b_1} \partial_{b'_1}}{m^2}]}_{n} \Delta(x - x') \\ A_{a_1 a_2 \dots a_n \tau_\zeta}(\vec{r}, t) = -A_{a'_1 a'_2 \dots a'_n \tau'_\zeta}^+(\vec{r}, t) \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \dots}_{n} \gamma_2^{\tau'_\zeta}, A_{a'_1 a'_2 \dots a'_n \tau'_\zeta}^+(\vec{r}, t) = -A_{a_1 a_2 \dots a_n \tau_\zeta}(\vec{r}, t) \underbrace{\eta_{a'_1}^{a_1} \eta_{a'_2}^{a_2} \dots}_{n} \gamma_2^{\tau_\zeta} \\ \bar{C} \gamma_a = -\gamma_2 \gamma_{a'} C \gamma_2 \eta_{a'}^a, \mathbb{X}^a = \gamma_2 \mathbb{X}^{+a'} \gamma_2 \eta_{a'}^a \end{array} \right.$$

$$[\Downarrow]$$

## 定理8.3.4.

$$\begin{cases}
\{A_{\underbrace{a_1 a_2 \dots \tau_\zeta}_n}(x), A_{\underbrace{a'_1 a'_2 \dots \tau'_\zeta}_n}(x')\} \\
= \frac{im^{-2n}}{2^{4n}[(2n+1)!]^2} \underbrace{(\bar{C}\gamma_{a_1})^{\lambda_\zeta \mu_\zeta} \dots (\gamma_{a'_1} C)^{\lambda'_\zeta \mu'_\zeta} \dots}_{2n+1} [(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta \dots [(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta})\}} \Delta(x - x') \\
\{A_{\underbrace{a_1 a_2 \dots \tau_\zeta}_n}(x), A_{\underbrace{a'_1 a'_2 \dots \tau'_\zeta}_n}(x')\} \\
= \frac{i(im)^{-2n}}{2^{4n}[(2n+1)!]^2} \underbrace{(\bar{C}\gamma_{a_1})^{\lambda_\zeta \mu_\zeta} \dots (\bar{C}\gamma_{a'_1})^{\lambda'_\zeta \mu'_\zeta} \dots}_{2n+1} [(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta(\lambda'_\zeta \dots [(m - \gamma^c \partial_c) C]_{\tau_\zeta \tau'_\zeta})\}} \Delta(x - x') \\
A_{\underbrace{a_1 a_2 \dots \tau_\zeta}_n}(\vec{r}, t) = -A_{\underbrace{a'_1 a'_2 \dots \tau'_\zeta}_n}(\vec{r}, t) \underbrace{\eta_{a'_1}^{a_1} \eta_{a'_2}^{a_2} \dots \gamma_{2\tau'_\zeta}}_{n}, A_{\underbrace{a'_1 a'_2 \dots \tau'_\zeta}_n}(\vec{r}, t) = -A_{\underbrace{a_1 a_2 \dots \tau_\zeta}_n}(\vec{r}, t) \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \dots \gamma_{2\tau_\zeta}}_{n}, \bar{C}\gamma_a = -\gamma_2 \gamma_{a'} C \gamma_2 \eta_a^{a'}
\end{cases}$$

(⇕)

## 定理8.3.5.

$$\begin{cases}
\{\psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots}_{2n+1}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \dots}_{2n+1}}(x')\} = \frac{i}{2^{3n}[(2n+1)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{a'}(x')}_n \cdot [(m - \gamma^c \partial_c) \gamma^4]_{\tau_\zeta \tau'_\zeta} [\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \cdot \Delta(x - x') \\
\{\psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots}_{2n+1}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \dots}_{2n+1}}(x')\} = \frac{i}{2^{3n}[(2n+1)!]^2} \underbrace{\mathbb{X}_{\{\lambda_\zeta \mu_\zeta\}}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\{\lambda'_\zeta \mu'_\zeta\}}^{a'}(x')}_n \cdot [(m - \gamma^c \partial_c) C]_{\tau_\zeta \tau'_\zeta} [\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \cdot \Delta(x - x') \\
\psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots}_{2n+1}} = -\psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \dots}_{2n+1}}^+ \cdot \underbrace{\gamma_{2\lambda'_\zeta} \gamma_{2\mu'_\zeta} \dots}_{2n+1}, \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \dots}_{2n+1}}^+ = -\psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots}_{2n+1}} \cdot \underbrace{\gamma_{2\lambda_\zeta} \gamma_{2\mu_\zeta} \dots}_{2n+1}, \mathbb{X}^a = \gamma_2 \mathbb{X}^{a'} \gamma_2 \eta_a^{a'}
\end{cases}$$

(⇕)

## 定理8.3.6.

$$\begin{cases}
\{\psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots}_{2n+1}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \dots}_{2n+1}}(x')\} = \frac{i}{2^{2n}[(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \dots})\}}_{2n+1} \Delta(x - x') \\
\{\psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots}_{2n+1}}(x), \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \dots}_{2n+1}}(x')\} = \frac{i}{2^{2n}[(2n+1)!]^2} \underbrace{[(m - \gamma^a \partial_a) C]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) C]_{\mu_\zeta \mu'_\zeta} \dots})\}}_{2n+1} \Delta(x - x') \\
\psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots}_{2n+1}} = -\psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \dots}_{2n+1}}^+ \cdot \underbrace{\gamma_{2\lambda'_\zeta} \gamma_{2\mu'_\zeta} \dots}_{2n+1}, \psi_{\underbrace{\lambda'_\zeta \mu'_\zeta \dots}_{2n+1}}^+ = -\psi_{\underbrace{\lambda_\zeta \mu_\zeta \dots}_{2n+1}} \cdot \underbrace{\gamma_{2\lambda_\zeta} \gamma_{2\mu_\zeta} \dots}_{2n+1}
\end{cases}$$

(⇕)

$$\text{定理8.3.7. } \{\psi_{\underbrace{A_\zeta B_\zeta \dots}_{2n+1}}(x), \psi_{\underbrace{A'_\zeta B'_\zeta \dots}_{2n+1}}(x')\} = i \frac{(is)^{2n+1}}{2^{2n}[(2n+1)!]^2} \underbrace{(\sigma, is)_{\{A_\zeta(A'_\zeta(\sigma, is)_{B_\zeta B'_\zeta}^b \dots)}^a}_{2n+1} \partial_a \partial_b \dots \Delta(x - x')$$

(⇕)

$$\text{定理8.3.8. } \{\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')\} = i \frac{(-1)^{2n+1}}{2^{n-1/2}} \Gamma_{\underbrace{k_\zeta k'_\zeta}_{2n+1}}^{abc} (n + \frac{1}{2}) \partial_a \partial_b \partial_c \dots \Delta(x - x')$$

势 $A$ 和场 $\psi$ 的反对易关系都有两种等价表述方式, 且互为前提和互为因果。既可以从势反对易关系推出一切, 也可以从场反对易关系推出一切。这说明对于有质量粒子, 势和场两种描述方案是完全等价的。并且可以从有质量粒子的对易规则推得与无质量粒子完全类似的反对易规则, 反之则不行。

## 8.4 实表象和Dirac分离表象的Majorana方程

$$\text{引理8.4.1. } S_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$S_y(\sigma_x, \sigma_y, \sigma_z) S_y^+ = (-\sigma_z, \sigma_y, \sigma_x), S_y^+(\sigma_x, \sigma_y, \sigma_z) S_y = (\sigma_z, \sigma_y, -\sigma_x)$$

$$I \otimes S_y[(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_z), -\varsigma I \otimes \sigma_x] I \otimes S_y^+ = [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_z]$$

$$I \otimes S_y^+[(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \varsigma I \otimes \sigma_x] I \otimes S_y = [(\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_z), -\varsigma I \otimes \sigma_x]$$



定义8.4.1.

$$\begin{cases} (\gamma_s^a \partial_a + m)\psi_s = 0, \gamma_s^a = (\sigma_{-\kappa} \sigma_{\kappa y}, \varsigma \sigma_{\kappa z}), \psi_s^* = \psi_s \\ (\gamma^a \partial_a + m)\psi = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x), \psi^* = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi \\ \psi_s = S_s(\kappa, \theta) \psi, S_s(\kappa, \theta) := e^{i\theta} S_{em}(\kappa) (I \otimes S_y^+) \\ S_s^T(\kappa, \theta) S_s(\kappa, \theta) = e^{2i\theta} S_{em}^T(\kappa) S_{em}(\kappa) = -e^{2i\theta} \sigma_y \otimes \sigma_y \end{cases}, S_{em}(\kappa) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & -i & -i & 0 \\ 0 & -\kappa & \kappa & 0 \end{bmatrix}, S_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

## 9 实表象下Bargmann-Wigner方程

### 9.1 实表象B-W自旋基

引理9.1.1.

$$\begin{cases} \overbrace{S_s \otimes S_s \cdots S_s}^{2s} \underbrace{U_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, h) = \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \underbrace{[S_s u_{\{\lambda_s(\vec{p}, \frac{1}{2})\}}]_{s+h}}_{s+h} \underbrace{[S_s u_{\mu_s(\vec{p}, \frac{1}{2})}] \cdots [S_s u_{\sigma_s(\vec{p}, -\frac{1}{2})}]_{s-h}}_{s-h} [S_s u_{\tau_s}(\vec{p}, -\frac{1}{2})] \\ \overbrace{S_s \otimes S_s \cdots S_s}^{2s} \underbrace{V_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, h) = \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \underbrace{[S_s v_{\{\lambda_s(\vec{p}, \frac{1}{2})\}}]_{s+h}}_{s+h} \underbrace{[S_s v_{\mu_s(\vec{p}, \frac{1}{2})}] \cdots [S_s v_{\sigma_s(\vec{p}, -\frac{1}{2})}]_{s-h}}_{s-h} [S_s v_{\tau_s}(\vec{p}, -\frac{1}{2})] \end{cases}$$

定义9.1.1.

$$\begin{cases} \underbrace{U_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, h) := \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \underbrace{u_{s\{\lambda_s(\vec{p}, \frac{1}{2})\}}}_{s+h} \underbrace{u_{s\mu_s(\vec{p}, \frac{1}{2})} \cdots u_{s\sigma_s(\vec{p}, -\frac{1}{2})}}_{s-h} u_{s\tau_s}(\vec{p}, -\frac{1}{2}) \\ \underbrace{V_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, h) := \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \underbrace{v_{s\{\lambda_s(\vec{p}, \frac{1}{2})\}}}_{s+h} \underbrace{v_{s\mu_s(\vec{p}, \frac{1}{2})} \cdots v_{s\sigma_s(\vec{p}, -\frac{1}{2})}}_{s-h} v_{s\tau_s}(\vec{p}, -\frac{1}{2}) \end{cases}$$

### 9.2 实表象下B-W自旋基之间关系

$$\text{推论9.2.1.} \begin{cases} \underbrace{U_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, h) = (-\varsigma)^{2s} \overbrace{\gamma_{s5} \otimes \gamma_{s5} \cdots \gamma_{s5}}^{2s} \underbrace{V_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, h) \\ \underbrace{V_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, h) = (-\varsigma)^{2s} \overbrace{\gamma_{s5} \otimes \gamma_{s5} \cdots \gamma_{s5}}^{2s} \underbrace{U_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, h) \end{cases}$$

$$\text{推论9.2.2.} \begin{cases} \underbrace{U_{\lambda_s \mu_s \cdots \sigma_s \tau_s}^+}_{2s}(\vec{p}, h) = (-1)^{s-h} \zeta^{2s} \underbrace{V_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, -h) \\ \underbrace{V_{\lambda_s \mu_s \cdots \sigma_s \tau_s}^+}_{2s}(\vec{p}, h) = (-1)^{s+h} \zeta^{2s} \underbrace{U_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, -h) \end{cases}$$

$$\text{推论9.2.3.} \begin{cases} \underbrace{U_{\lambda_s \mu_s \cdots \sigma_s \tau_s}^+}_{2s}(\vec{p}, h) = (-1)^{s+h} \overbrace{\gamma_{s5} \otimes \gamma_{s5} \cdots \gamma_{s5}}^{2s} \underbrace{U_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, -h) \\ \underbrace{V_{\lambda_s \mu_s \cdots \sigma_s \tau_s}^+}_{2s}(\vec{p}, h) = (-1)^{s-h} \overbrace{\gamma_{s5} \otimes \gamma_{s5} \cdots \gamma_{s5}}^{2s} \underbrace{V_{\lambda_s \mu_s \cdots \sigma_s \tau_s}}_{2s}(\vec{p}, -h) \end{cases}$$

### 9.3 实表象下B-W自旋基和矢量基之间的关系

推论9.3.1.

$$\begin{cases} \underbrace{U_{\lambda_s \mu_s \eta_s \xi_s \cdots}}_{2n}(\vec{p}, h) = \frac{1}{(2\sqrt{2m})^n} \underbrace{[S_s \mathbb{X}^a(p) S_s^T]_{\lambda_s \mu_s}}_n \underbrace{[S_s \mathbb{X}^b(p) S_s^T]_{\eta_s \xi_s \cdots}}_n \underbrace{\varepsilon_{ab} \cdots}_n(\vec{p}, h) \\ [\Rightarrow] \underbrace{\varepsilon_{ab} \cdots}_n(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(S_s^* \bar{C} \gamma_a S_s^+)^{\lambda_s \mu_s} (S_s^* \bar{C} \gamma_b S_s^+)^{\eta_s \xi_s \cdots}}^n \underbrace{U_{\lambda_s \mu_s \eta_s \xi_s \cdots}}_{2n}(\vec{p}, h) \\ \underbrace{V_{\lambda_s \mu_s \eta_s \xi_s \cdots}}_{2n}(\vec{p}, h) = \frac{1}{(2\sqrt{2m})^n} \underbrace{[S_s \mathbb{X}^a(-p) S_s^T]_{\lambda_s \mu_s}}_n \underbrace{[S_s \mathbb{X}^b(-p) S_s^T]_{\eta_s \xi_s \cdots}}_n \underbrace{\tilde{\varepsilon}_{ab} \cdots}_n(\vec{p}, h) \\ [\Rightarrow] \underbrace{\tilde{\varepsilon}_{ab} \cdots}_n(\vec{p}, h) = \frac{1}{(i\sqrt{2})^n} \overbrace{(S_s^* \bar{C} \gamma_a S_s^+)^{\lambda_s \mu_s} (S_s^* \bar{C} \gamma_b S_s^+)^{\eta_s \xi_s \cdots}}^n \underbrace{V_{\lambda_s \mu_s \eta_s \xi_s \cdots}}_{2n}(\vec{p}, h) \end{cases}$$

## 推论9.3.2.

$$\left\{ \begin{aligned} U_{s \underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n+1}}(\vec{p}, h) &= \frac{1}{(2\sqrt{2}m)^n} \underbrace{[S_s \mathbb{X}^a(p) S_s^T]_{\lambda_\zeta \mu_\zeta}}_n \underbrace{[S_s \mathbb{X}^b(p) S_s^T]_{\eta_\zeta \xi_\zeta}}_n \dots \underbrace{\varepsilon_{s ab \dots \tau_\zeta}}_n(\vec{p}, h) \\ [\Rightarrow] \varepsilon_{s \underbrace{ab \dots \tau_\zeta}}(\vec{p}, h) &= \frac{1}{(i\sqrt{2})^n} \underbrace{(S_s^* \bar{C} \gamma_a S_s^+)^{\lambda_\zeta \mu_\zeta} (S_s^* \bar{C} \gamma_b S_s^+)^{\eta_\zeta \xi_\zeta}}_n \dots U_{s \underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n+1}}(\vec{p}, h) \\ V_{s \underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n+1}}(\vec{p}, h) &= \frac{1}{(2\sqrt{2}m)^n} \underbrace{[S_s \mathbb{X}^a(-p) S_s^T]_{\lambda_\zeta \mu_\zeta}}_n \underbrace{[S_s \mathbb{X}^b(-p) S_s^T]_{\eta_\zeta \xi_\zeta}}_n \dots \underbrace{\tilde{\varepsilon}_{s ab \dots \tau_\zeta}}_n(\vec{p}, h) \\ [\Rightarrow] \tilde{\varepsilon}_{s \underbrace{ab \dots \tau_\zeta}}(\vec{p}, h) &= \frac{1}{(i\sqrt{2})^n} \underbrace{(S_s^* \bar{C} \gamma_a S_s^+)^{\lambda_\zeta \mu_\zeta} (S_s^* \bar{C} \gamma_b S_s^+)^{\eta_\zeta \xi_\zeta}}_n \dots V_{s \underbrace{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}_{2n+1}}(\vec{p}, h) \end{aligned} \right.$$

## 9.4 实表象下矢量基

## 定理9.4.1.

$$\left\{ \begin{aligned} \varepsilon_{\underbrace{ab \dots}_{n}}^+(\vec{p}, h) &= (-1)^h \underbrace{\eta_a^a \eta_b^{b'}}_n \dots \varepsilon_{\underbrace{a'b' \dots}_{n}}(\vec{p}, -h) \\ \varepsilon_{\underbrace{ab \dots}_{n}}(\vec{p}, h) &= (-1)^h \underbrace{\eta_a^a \eta_b^{b'}}_n \dots \varepsilon_{\underbrace{a'b' \dots}_{n}}^+(\vec{p}, -h) \\ \varepsilon_{\underbrace{ab \dots}_{n}}(\vec{p}, h) &= (-1)^n \tilde{\varepsilon}_{\underbrace{ab \dots}_{n}}(\vec{p}, h) \end{aligned} \right. \quad \left\{ \begin{aligned} \varepsilon_{\underbrace{sab \dots \tau'_\zeta}_{n}}^+(\vec{p}, h) &= (-1)^{h+\frac{1}{2}} e^{-2i\theta} \gamma_{s5\tau'_\zeta} \underbrace{\eta_a^a \eta_b^{b'}}_n \dots \varepsilon_{\underbrace{s a' b' \dots \tau'_\zeta}_{n}}(\vec{p}, -h) \\ \varepsilon_{\underbrace{sab \dots \tau'_\zeta}_{n}}(\vec{p}, h) &= (-1)^{h-\frac{1}{2}} e^{2i\theta} \gamma_{s5\tau'_\zeta} \underbrace{\eta_a^a \eta_b^{b'}}_n \dots \varepsilon_{\underbrace{s a' b' \dots \tau'_\zeta}_{n}}^+(\vec{p}, -h) \\ \varepsilon_{\underbrace{sab \dots \tau_\zeta}_{n}}(\vec{p}, h) &= -\varsigma (-1)^n \gamma_{s5\tau_\zeta} \sigma_\zeta \tilde{\varepsilon}_{\underbrace{sab \dots \sigma_\zeta}_{n}}(\vec{p}, h) \end{aligned} \right.$$

证明:  $\varepsilon_{\underbrace{sab \dots \tau_\zeta}_{n}}^+(\vec{p}, h)$ 

$$= (-1)^{h-\frac{1}{2}} (S_s^* \gamma_2 \gamma_5 S_s^+)_{\tau_\zeta} \underbrace{\eta_a^a \eta_b^{b'}}_n \dots \varepsilon_{\underbrace{s a' b' \dots \tau'_\zeta}_{n}}(\vec{p}, -h)$$

$$= -(-1)^{h-\frac{1}{2}} e^{-2i\theta} (S_s^* S_s^T S_s \gamma_5 S_s^+)_{\tau_\zeta} \underbrace{\eta_a^a \eta_b^{b'}}_n \dots \varepsilon_{\underbrace{s a' b' \dots \tau'_\zeta}_{n}}(\vec{p}, -h)$$

$$= (-1)^{h+\frac{1}{2}} e^{-2i\theta} \gamma_{s5\tau_\zeta} \underbrace{\eta_a^a \eta_b^{b'}}_n \dots \varepsilon_{\underbrace{s a' b' \dots \tau'_\zeta}_{n}}(\vec{p}, -h)$$

□

9.5 实表象下Majorana B-W方程<sup>[17]</sup>平面波解

## 定理9.5.1.

$$(\gamma_s^a \partial_a + m)_{\kappa_\zeta} \psi_{s \underbrace{\lambda_\zeta \mu_\zeta \dots}_{2s}}(\vec{r}, t) = 0, \psi_{s \underbrace{\lambda_\zeta \mu_\zeta \dots}_{2s}}(\vec{r}, t) = \frac{1}{(2s)!} \psi_{s \underbrace{\{\lambda_\zeta \mu_\zeta \dots\}_{2s}}(\vec{r}, t), \psi_{s \underbrace{\lambda_\zeta \mu_\zeta \dots}_{2s}}^+(\vec{r}, t) = \psi_{s \underbrace{\lambda_\zeta \mu_\zeta \dots}_{2s}}(\vec{r}, t)$$

$$\psi_{s \underbrace{\lambda_\zeta \mu_\zeta \dots}_{2s}}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=s}^{-s} \sqrt{\frac{m^{2s}}{E}} [a(\vec{p}, h) U_{s \underbrace{\lambda_\zeta \mu_\zeta \dots}_{2s}}(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + a^+(\vec{p}, h) U_{s \underbrace{\lambda_\zeta \mu_\zeta \dots}_{2s}}^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$U_{s \underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = \frac{1}{\sqrt{(2s)!(s+h)!(s-h)!}} \underbrace{u_{s \underbrace{\{\lambda_\zeta \mu_\zeta \dots\}_{s+h}}(\vec{p}, \frac{1}{2}) u_{s \underbrace{\mu_\zeta \dots}_{2s}}(\vec{p}, \frac{1}{2}) \dots u_{s \underbrace{\sigma_\zeta \tau_\zeta}_{s-h}}(\vec{p}, -\frac{1}{2})}_{s-h}}(\vec{p}, -\frac{1}{2})$$

$$a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m^{2s}}{E}} U_{s \underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}^+(\vec{p}, h) \psi_{s \underbrace{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}_{2s}}(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r}$$

## 9.6 实表象下Majorana B-W方程各种算符的提取

定理9.6.1.

$$\begin{cases}
P_u(s) = \int \psi_s^{\overbrace{\lambda_\zeta \mu_\zeta}^{2s}}(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_s^{\underbrace{\lambda_\zeta \mu_\zeta}_{2s}}(\vec{r}, t) d^3\vec{r} = \int \sum_h p_u [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} a(\vec{p}, h) a^+(\vec{p}, h)] d^3\vec{p} \\
Q(s) = \int \psi_s^{\overbrace{\lambda_\zeta \mu_\zeta}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_s^{\underbrace{\lambda_\zeta \mu_\zeta}_{2s}}(\vec{r}, t) d^3\vec{r} = \int \sum_h [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s-1} a(\vec{p}, h) a^+(\vec{p}, h)] d^3\vec{p} \\
N(s) = \int \psi_s^{\overbrace{\lambda_\zeta \mu_\zeta}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi_s^{\underbrace{\lambda_\zeta \mu_\zeta}_{2s}}(\vec{r}, t) d^3\vec{r} = \int \sum_h [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} a(\vec{p}, h) a^+(\vec{p}, h)] d^3\vec{p} \\
\vec{S}(s) = \int \psi_s^{\overbrace{\lambda_\zeta \mu_\zeta}^{2s}}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_s^{\underbrace{\lambda_\zeta \mu_\zeta}_{2s}}(\vec{r}, t) d^3\vec{r} = \int \sum_h \hat{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} a(\vec{p}, h) a^+(\vec{p}, h)] d^3\vec{p} \\
\vec{M}(s) = \int \psi_s^{\overbrace{\lambda_\zeta \mu_\zeta}^{2s}}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi_s^{\underbrace{\lambda_\zeta \mu_\zeta}_{2s}}(\vec{r}, t) d^3\vec{r} = \int \sum_h \hat{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s-1} a(\vec{p}, h) a^+(\vec{p}, h)] d^3\vec{p}
\end{cases}$$

## 9.7 实表象下Klein-Gordon方程和Rarita-Schwinger方程的平面波解

推论9.7.1.

$$\begin{cases}
A_{\underbrace{ab\dots}_{n}}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} [a(\vec{p}, h) \varepsilon_{\underbrace{ab\dots}_{n}}(\vec{p}, h) e^{ip \cdot x} + e^{-4ni\theta} a^+(\vec{p}, h) \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}_{n} \varepsilon_{\underbrace{a'b'\dots}_{n}}^+(\vec{p}, h) e^{-ip \cdot x}] d^3\vec{p} \\
A_{s \underbrace{ab\dots}_{n} \tau_\zeta}(\vec{r}, t) \\
= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{\sqrt{m}}{\sqrt{2^n E}} [a(\vec{p}, h) \varepsilon_{s \underbrace{ab\dots}_{n} \tau_\zeta}(\vec{p}, h) e^{ip \cdot x} + e^{-4ni\theta} a^+(\vec{p}, h) \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}_{n} \varepsilon_{\underbrace{sa'b'\dots}_{n} \tau_\zeta}^+(\vec{p}, h) e^{-ip \cdot x}] d^3\vec{p}
\end{cases}$$

推论9.7.2.

$$\begin{cases}
A_{\underbrace{ab\dots}_{n}}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} \varepsilon_{\underbrace{ab\dots}_{n}}(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p} \cdot \vec{r} - Et)} + e^{-4ni\theta} (-1)^h a^+(\vec{p}, -h) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^3\vec{p} \\
A_{s \underbrace{ab\dots}_{n} \tau_\zeta}(x) \\
= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{\sqrt{m}}{\sqrt{2^n E}} \varepsilon_{s \underbrace{ab\dots}_{n} \tau_\zeta}(\vec{p}, h) [a(\vec{p}, h) \delta_{\tau_\zeta}^{\sigma_\zeta} e^{ip \cdot x} + e^{-(4n+2)\theta} (-1)^{h-\frac{1}{2}} a^+(\vec{p}, -h) \gamma_{s5\tau_\zeta}^{\sigma_\zeta} e^{-ip \cdot x}] d^3\vec{p}
\end{cases}$$

推论9.7.3.

$$\begin{cases}
A_{\underbrace{ab\dots}_{n}}(\vec{r}, t) = e^{-4ni\theta} \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}_{n} A_{\underbrace{a'b'\dots}_{n}}^+(\vec{r}, t), A_{\underbrace{sab\dots}_{n}}^+(\vec{r}, t) = e^{4ni\theta} \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}_{n} A_{\underbrace{sa'b'\dots}_{n}}(\vec{r}, t) \\
A_{\underbrace{sab\dots}_{n} \tau_\zeta}(\vec{r}, t) = e^{-4ni\theta} \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}_{n} A_{\underbrace{sa'b'\dots}_{n} \tau_\zeta}^+(\vec{r}, t), A_{\underbrace{sab\dots}_{n} \tau_\zeta}^+(\vec{r}, t) = e^{4ni\theta} \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}_{n} A_{\underbrace{sa'b'\dots}_{n} \tau_\zeta}(\vec{r}, t)
\end{cases}$$

证明:  $A_{\underbrace{sab\dots}_{n} \tau_\zeta}^+(\vec{r}, t)$ 

$$\begin{aligned}
&= -e^{(4n+2)\theta} (S_s^*)_{\tau_\zeta}^{\sigma_\zeta} \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}_{n} (\sigma_y \otimes \sigma_y)_{\sigma_\zeta}^{\xi_\zeta} (S_s^+)_{\xi_\zeta}^{\eta_\zeta} A_{\underbrace{sa'b'\dots}_{n} \tau_\zeta}(\vec{r}, t) \\
&= e^{4ni\theta} (S_s^* S_s^T S_s S_s^+)_{\tau_\zeta}^{\sigma_\zeta} \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}_{n} A_{\underbrace{sa'b'\dots}_{n} \tau_\zeta}(\vec{r}, t) \\
&= e^{4ni\theta} \overbrace{\eta_a^{a'} \eta_b^{b'} \dots}_{n} A_{\underbrace{sa'b'\dots}_{n} \tau_\zeta}(\vec{r}, t)
\end{aligned}$$

□

## 9.8 实表象下Majorana B-W方程协变对易规则

推论9.8.1.

$$\left\{ \begin{aligned} & [\psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2s} \cdots (x), \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2s} \cdots (x')]_{-2s+1} = \frac{i}{2^{2s-1} [(2s)!]^2} \overbrace{[(m - \gamma_s^a \partial_a) \gamma_s^4] \{ \lambda_\zeta (\lambda'_\zeta [(m - \gamma_s^b \partial_b) \gamma_s^4]_{\mu_\zeta \mu'_\zeta} \cdots \})}^{2s} \Delta(x - x') \\ & [\psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2s} \cdots (x), \psi_s \underbrace{\lambda'_\zeta \mu'_\zeta}_{2s} \cdots (x')]_{-2s+1} = [\psi_s^+ \underbrace{\lambda_\zeta \mu_\zeta}_{2s} \cdots (x), \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2s} \cdots (x')]_{-2s+1} = [\psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2s} \cdots (x), \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2s} \cdots (x')]_{-2s+1} \\ & \psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2s} \cdots (x) = \psi_s^+ \underbrace{\lambda_\zeta \mu_\zeta}_{2s} \cdots (x) \end{aligned} \right.$$

$$\Leftrightarrow [a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [a(\vec{p}, h), a(\vec{p}', h')]_{-2s+1} = 0, [a^+(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = 0$$

## 10 实表象下有质量Majorana粒子协变对易规则的总结与梳理

### 10.1 实表象下有质量Majorana玻色子协变对易规则梳理(取 $\theta = 0$ )

$$\text{定理10.1.1.} \left\{ \begin{aligned} & [A_{a_1 a_2 \cdots a_n}(x), A_{a'_1 a'_2 \cdots a'_n}^+(x')] = i \hat{P}_{a_1 a_2 \cdots a_n a'_1 a'_2 \cdots a'_n}(n) \Delta(x - x') \\ & [A_{a_1 a_2 \cdots a_n}(x), A_{b_1 b_2 \cdots b_n}(x')] = i \hat{P}_{a_1 a_2 \cdots a_n a'_1 a'_2 \cdots a'_n}(n) \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \cdots \eta_{b_n}^{a'_n} \Delta(x - x') \\ & A_{a_1 a_2 \cdots a_n} = A_{a'_1 a'_2 \cdots a'_n}^+ \eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots \eta_{a_n}^{a'_n}, \underbrace{A_{a'_1 a'_2 \cdots}^+}_n = \underbrace{A_{a_1 a_2 \cdots}}_n \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots}_n \end{aligned} \right.$$

$$(\Leftrightarrow)$$

$$\text{定理10.1.2.} \left\{ \begin{aligned} & [A_{a_1 a_2 \cdots}(x), A_{a'_1 a'_2 \cdots}^+(x')] = \frac{im^{-2n}}{2^{4n-1} [(2n)!]^2} \Gamma_{a_1 a_2 \cdots a'_1 a'_2 \cdots}(x; n) \Delta(x - x') \\ & [A_{a_1 a_2 \cdots}(x), A_{b_1 b_2 \cdots}(x')] = \frac{im^{-2n}}{2^{4n-1} [(2n)!]^2} \Gamma_{a_1 a_2 \cdots a'_1 a'_2 \cdots}(x; n) \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \cdots \Delta(x - x') \\ & A_{a_1 a_2 \cdots} = A_{a'_1 a'_2 \cdots}^+ \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots}_n, A_{a'_1 a'_2 \cdots}^+ = A_{a_1 a_2 \cdots} \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots}_n \end{aligned} \right.$$

$$(\Leftrightarrow)$$

$$\text{定理10.1.3.} \left\{ \begin{aligned} & [A_{a_1 a_2 \cdots}(x), A_{a'_1 a'_2 \cdots}^+(x')] = \frac{im^{-2n}}{2^{5n-1} [(2n)!]^2} \Gamma_{a_1 a_2 \cdots a'_1 a'_2 \cdots}^{b_1 b_2 \cdots b'_1 b'_2 \cdots}(x, x'; n) \underbrace{[\eta_{b_1 b'_1} - \frac{\partial_{b_1} \partial_{b'_1}^+}{m^2}] \cdots}_n \Delta(x - x') \\ & [A_{a_1 a_2 \cdots}(x), A_{b_1 b_2 \cdots}(x')] = \frac{im^{-2n}}{2^{5n-1} [(2n)!]^2} \Gamma_{a_1 a_2 \cdots a'_1 a'_2 \cdots}^{c_1 c_2 \cdots c'_1 c'_2 \cdots}(x, x'; n) \underbrace{[\eta_{c_1 c'_1} - \frac{\partial_{c_1} \partial_{c'_1}^+}{m^2}] \cdots}_n \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \cdots \Delta(x - x') \\ & A_{a_1 a_2 \cdots} = A_{a'_1 a'_2 \cdots}^+ \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots}_n, A_{a'_1 a'_2 \cdots}^+ = A_{a_1 a_2 \cdots} \underbrace{\eta_{a_1}^{a'_1} \eta_{a_2}^{a'_2} \cdots}_n \end{aligned} \right.$$

$$(\Leftrightarrow)$$

$$\text{定理10.1.4.} \left\{ \begin{aligned} & [\psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots (x), \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots (x')] = \frac{i}{2^{3n-1} [(2n)!]^2} \underbrace{\mathbb{X}_s^a \{ \lambda_\zeta \mu_\zeta \}}_n(x) \cdots \underbrace{\mathbb{X}_s^{+a} \{ \lambda'_\zeta \mu'_\zeta \}}_n(x') \cdots \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}^+}{m^2}] \cdots}_n \Delta(x - x') \\ & [\psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots (x), \psi_s \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots (x')] = \frac{i}{2^{3n-1} [(2n)!]^2} \underbrace{\mathbb{X}_s^a \{ \lambda_\zeta \mu_\zeta \}}_n(x) \cdots \underbrace{\mathbb{X}_s^{+a'} \{ \lambda'_\zeta \mu'_\zeta \}}_n(x') \cdots \underbrace{[\eta_{aa'} - \frac{\partial_{[a} \partial_{a'}^+]}{m^2}] \cdots}_n \Delta(x - x') \\ & \psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots = \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots \underbrace{\delta_{\lambda_\zeta}^{\lambda'_\zeta} \delta_{\mu_\zeta}^{\mu'_\zeta} \cdots}_{2n}, \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots = \psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots \underbrace{\delta_{\lambda'_\zeta}^{\lambda_\zeta} \delta_{\mu'_\zeta}^{\mu_\zeta} \cdots}_{2n}, \mathbb{X}_s^a := S_s \mathbb{X}^a S_s^T \end{aligned} \right.$$

$$(\Leftrightarrow)$$

$$\text{定理10.1.5.} \left\{ \begin{array}{l} [\psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots(x), \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots(x')] = \frac{i}{2^{3n-1}[(2n)!]^2} \underbrace{\mathbb{X}_s^a \{\lambda_\zeta \mu_\zeta\}}_n \cdots \underbrace{\mathbb{X}_s^{a'} \{\lambda'_\zeta \mu'_\zeta\}}_n \cdots \underbrace{[\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdots \Delta(x-x') \\ [\psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots(x), \psi_s \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots(x')] = \frac{i}{2^{3n-1}[(2n)!]^2} \underbrace{\mathbb{X}_s^a \{\lambda_\zeta \mu_\zeta\}}_n \cdots \underbrace{\mathbb{X}_s^{a'} \{\lambda'_\zeta \mu'_\zeta\}}_n \cdots \underbrace{[\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdots \Delta(x-x') \\ \psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots = \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots \underbrace{\delta_{\lambda_\zeta}^{\lambda'_\zeta} \delta_{\mu_\zeta}^{\mu'_\zeta}}_{2n} \cdots, \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots = \psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots \underbrace{\delta_{\lambda'_\zeta}^{\lambda_\zeta} \delta_{\mu'_\zeta}^{\mu_\zeta}}_{2n} \cdots, \mathbb{X}_s^a := S_s \mathbb{X}^a S_s^T, \mathbb{X}_s^{a'} = \mathbb{X}_s^{+a'} \eta_a^{a'} \end{array} \right.$$

[⇕]

$$\text{定理10.1.6.} \left\{ \begin{array}{l} [\psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots(x), \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots(x')] = \frac{i}{2^{2n-1}[(2n)!]^2} \underbrace{[(m - \gamma_s^a \partial_a) \gamma_s^4]_{\{\lambda_\zeta \mu_\zeta\}}}_{2n} \underbrace{[(m - \gamma_s^b \partial_b) \gamma_s^4]_{\{\lambda'_\zeta \mu'_\zeta\}}}_{2n} \Delta(x-x') \\ [\psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots(x), \psi_s \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots(x')] = \frac{i}{2^{2n-1}[(2n)!]^2} \underbrace{[(m - \gamma_s^a \partial_a) \gamma_s^4]_{\{\lambda_\zeta \mu_\zeta\}}}_{2n} \underbrace{[(m - \gamma_s^b \partial_b) \gamma_s^4]_{\{\lambda'_\zeta \mu'_\zeta\}}}_{2n} \Delta(x-x') \\ \psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots = \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots \underbrace{\delta_{\lambda_\zeta}^{\lambda'_\zeta} \delta_{\mu_\zeta}^{\mu'_\zeta}}_{2n} \cdots, \psi_s^+ \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n} \cdots = \psi_s \underbrace{\lambda_\zeta \mu_\zeta}_{2n} \cdots \underbrace{\delta_{\lambda'_\zeta}^{\lambda_\zeta} \delta_{\mu'_\zeta}^{\mu_\zeta}}_{2n} \cdots \end{array} \right.$$

[⇓]

$$\text{定理10.1.7.} [\psi_{\underbrace{A_\zeta B_\zeta}_{2n}} \cdots(x), \psi_{\underbrace{A'_\zeta B'_\zeta}_{2n}}^+ \cdots(x')] = i \frac{(i\zeta)^{2n}}{2^{2n-1}[(2n)!]^2} \underbrace{(\sigma, i\zeta)_{\{A_\zeta(A'_\zeta)}^a}}_{2n} \underbrace{(\sigma, i\zeta)_{\{B_\zeta(B'_\zeta)}^b}}_{2n} \partial_a \partial_b \cdots \Delta(x-x')$$

[⇕]

$$\text{定理10.1.8.} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i \frac{(-1)^{2n}}{2^{2n-1}} \Gamma_{k_\zeta k'_\zeta}^{abc} \cdots \underbrace{(n)}_{2n} \underbrace{\partial_a \partial_b \partial_c}_{2s} \cdots \Delta(x-x')$$

$$\text{定义10.1.1.} \mathbb{X}^a \equiv [im\gamma^a(\zeta) - 2S^{ab}(e, \zeta)\partial_b]C, \mathbb{X}^a(p) \equiv i[m\gamma^a(\zeta) - 2S^{ab}(e, \zeta)p_b]C$$

$$\text{定义10.1.2.} \mathbb{X}_s^a \equiv [im\gamma_s^a(\zeta) - 2S_s^{ab}(e, \zeta)\partial_b]C_s, \mathbb{X}_s^a(p) \equiv i[m\gamma_s^a(\zeta) - 2S_s^{ab}(e, \zeta)p_b]C_s, C_s := -\gamma_s^2 \gamma_s^4 \gamma^2$$

势 $A$ 和场 $\psi$ 的对易关系都有两种等价表述方式, 且互为前提和互为因果。既可以从势对易关系推出一切, 也可以从场对易关系推出一切。这说明对于有质量粒子, 势和场两种描述方案是完全等价的。并且可以从有质量粒子的对易规则推得与无质量粒子完全类似的对易规则, 反之则不行。

## 10.2 实表象下有质量Majorana费米子协变反对易规则梳理

$$\text{定义10.2.1.} \hat{P}_{sa_1 \cdots a_n \tau_\zeta a'_1 \cdots a'_n \tau'_\zeta}(n + \frac{1}{2}) = \frac{n+1}{2n+3} \hat{P}_{aa_1 \cdots a_n a'_1 \cdots a'_n}(n+1) [\gamma_s^a (-m - \gamma_s^c \partial_c) \gamma_s^4 \gamma_s^{a'}]_{\tau_\zeta \tau'_\zeta}, \gamma_s^{a'} = \gamma_s^a \eta_a^{a'}$$

$$\text{定义10.2.2.} \left\{ \begin{array}{l} \hat{P}_{sa_1 \cdots a_n \tau_\zeta b_1 \cdots b_n \tau'_\zeta}(n + \frac{1}{2}) = \frac{n+1}{2n+3} \hat{P}_{aa_1 \cdots a_n bb_1 \cdots b_n}(n+1) [\gamma_s^a (m + \gamma_s^c \partial_c) \gamma_s^b \gamma_s^4]_{\tau_\zeta \tau'_\zeta} \\ \hat{P}_{sa_1 \cdots a_n \tau_\zeta b_1 \cdots b_n \tau'_\zeta}(n + \frac{1}{2}) := \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \cdots \eta_{b_n}^{a'_n} \hat{P}_{sa_1 \cdots a_n \tau_\zeta a'_1 \cdots a'_n \tau'_\zeta}(n + \frac{1}{2}) \end{array} \right.$$

$$\text{推论10.2.1.} \hat{P}_{sa_1 \cdots a_n \tau_\zeta b_1 \cdots b_n \tau'_\zeta}(n + \frac{1}{2}) = \frac{n+1}{2n+3} \hat{P}_{aa_1 \cdots a_n bb_1 \cdots b_n}(n+1) [(m - \gamma_s^c \partial_c) \gamma_s^a \gamma_s^b \gamma_s^4]_{\tau_\zeta \tau'_\zeta}$$

$$\text{定理10.2.1.} \{A_{sa_1 a_2 \cdots a_n \tau_\zeta}(x), A_{sa'_1 a'_2 \cdots a'_n \tau'_\zeta}^+(x')\} = i \hat{P}_{sa_1 \cdots a_n \tau_\zeta a'_1 \cdots a'_n \tau'_\zeta}(n + \frac{1}{2}) \Delta(x-x')$$

[⇕]

$$\text{定理10.2.2.} \left\{ \begin{array}{l} \{A_{sa_1 a_2 \cdots a_n \tau_\zeta}(x), A_{sb_1 b_2 \cdots b_n \tau'_\zeta}(x')\} = i \hat{P}_{sa_1 \cdots a_n \tau_\zeta b_1 \cdots b_n \tau'_\zeta}(n + \frac{1}{2}) \Delta(x-x') \\ A_{sab \cdots \tau_\zeta} = \underbrace{\eta_a^{a'} \eta_b^{b'}}_n \cdots A_{sa'b' \cdots \tau_\zeta}^+, A_{sab \cdots \tau_\zeta}^+ = \underbrace{\eta_a^{a'} \eta_b^{b'}}_n \cdots A_{sa'b' \cdots \tau_\zeta} \end{array} \right.$$

[⇕]

## 定理10.2.3.

$$\left\{ \begin{aligned}
& \{A_{s \underbrace{a_1 a_2 \dots \tau_\zeta}_n}(x), A_{s \underbrace{a'_1 a'_2 \dots \tau'_\zeta}_n}(x')\} = \frac{im^{-2n}}{2^{5n}[(2n+1)!]^2} \\
& \underbrace{(\gamma_s^4 \gamma_{sa_1})^{\lambda_\zeta \mu_\zeta}}_n \dots \underbrace{(\gamma_s^4 \gamma_{sa'_1})^{\lambda'_\zeta \mu'_\zeta}}_n \dots \underbrace{\mathbb{X}_s^{b_1}_{\{\lambda_\zeta \mu_\zeta\}}(x)}_n \dots \underbrace{\mathbb{X}_s^{b'_1}_{\{\lambda'_\zeta \mu'_\zeta\}}(x')}_n \dots \underbrace{[(m - \gamma_s^c \partial_c) \gamma_s^4]_{\tau_\zeta \tau'_\zeta}}_n \underbrace{[\eta_{b_1 b'_1} - \frac{\partial_{b_1} \partial_{b'_1}^+}{m^2}]_n} \dots \Delta(x - x') \\
& \{A_{s \underbrace{a_1 a_2 \dots \tau_\zeta}_n}(x), A_{s \underbrace{a'_1 a'_2 \dots \tau'_\zeta}_n}(x')\} = \frac{i(im)^{-2n}}{2^{5n}[(2n+1)!]^2} \\
& \underbrace{(\gamma_s^4 \gamma_{sa_1})^{\lambda_\zeta \mu_\zeta}}_n \dots \underbrace{(\gamma_s^4 \gamma_{sa'_1})^{\lambda'_\zeta \mu'_\zeta}}_n \dots \underbrace{\mathbb{X}_s^{b_1}_{\{\lambda_\zeta \mu_\zeta\}}(x)}_n \dots \underbrace{\mathbb{X}_s^{b'_1}_{\{\lambda'_\zeta \mu'_\zeta\}}(x')}_n \dots \underbrace{[(m - \gamma_s^c \partial_c) \gamma_s^4]_{\tau_\zeta \tau'_\zeta}}_n \underbrace{[\delta_{b_1 b'_1} - \frac{\partial_{b_1} \partial_{b'_1}^+}{m^2}]_n} \dots \Delta(x - x') \\
& A_{sab \dots \tau_\zeta} = \underbrace{\eta_a^{a'} \eta_b^{b'}}_n \dots A_{sa'b' \dots \tau_\zeta}^+, A_{sab \dots \tau_\zeta}^+ = \underbrace{\eta_a^{a'} \eta_b^{b'}}_n \dots A_{sa'b' \dots \tau_\zeta} \\
& S_s^* \bar{C} \gamma^a S_s^+ = \gamma_s^4 \gamma_s^a, S_s \gamma^{a'} C S_s^T = \gamma_s^{a'} \gamma_s^4, \mathbb{X}_s^a := S_s \mathbb{X}^a S_s^T, \mathbb{X}_s^a = \mathbb{X}_s^{+a'} \eta_{a'}^a
\end{aligned} \right.$$

[⇕]

## 定理10.2.4.

$$\left\{ \begin{aligned}
& \{A_{s \underbrace{a_1 a_2 \dots \tau_\zeta}_n}(x), A_{s \underbrace{a'_1 a'_2 \dots \tau'_\zeta}_n}(x')\} \\
& = \frac{im^{-2n}}{2^{4n}[(2n+1)!]^2} \underbrace{(\gamma_s^4 \gamma_{sa_1})^{\lambda_\zeta \mu_\zeta}}_n \dots \underbrace{(\gamma_{sa'_1}^4 \gamma_s^4)^{\lambda'_\zeta \mu'_\zeta}}_n \dots \underbrace{[(m - \gamma_s^a \partial_a) \gamma_s^4]_{\{\lambda_\zeta \mu_\zeta\}}}_{2n+1} \underbrace{[(m - \gamma_s^c \partial_c) \gamma_s^4]_{\tau_\zeta \tau'_\zeta}}_{2n+1} \Delta(x - x') \\
& \{A_{s \underbrace{a_1 a_2 \dots \tau_\zeta}_n}(x), A_{s \underbrace{a'_1 a'_2 \dots \tau'_\zeta}_n}(x')\} \\
& = \frac{i(im)^{-2n}}{2^{4n}[(2n+1)!]^2} \underbrace{(\gamma_s^4 \gamma_{sa_1})^{\lambda_\zeta \mu_\zeta}}_n \dots \underbrace{(\gamma_s^4 \gamma_{sa'_1})^{\lambda'_\zeta \mu'_\zeta}}_n \dots \underbrace{[(m - \gamma_s^a \partial_a) \gamma_s^4]_{\{\lambda_\zeta \mu_\zeta\}}}_{2n+1} \underbrace{[(m - \gamma_s^c \partial_c) \gamma_s^4]_{\tau_\zeta \tau'_\zeta}}_{2n+1} \Delta(x - x') \\
& A_{sab \dots \tau_\zeta} = \underbrace{\eta_a^{a'} \eta_b^{b'}}_n \dots A_{sa'b' \dots \tau_\zeta}^+, A_{sab \dots \tau_\zeta}^+ = \underbrace{\eta_a^{a'} \eta_b^{b'}}_n \dots A_{sa'b' \dots \tau_\zeta}, S_s^* \bar{C} \gamma^a S_s^+ = \gamma_s^4 \gamma_s^a, S_s \gamma^{a'} C S_s^T = \gamma_s^{a'} \gamma_s^4
\end{aligned} \right.$$

[⇕]

## 定理10.2.5.

$$\left\{ \begin{aligned}
& \{\psi_{s \underbrace{\lambda_\zeta \mu_\zeta}_{2n+1}}(x), \psi_{s \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n+1}}^+(x')\} \\
& = \frac{i}{2^{3n}[(2n+1)!]^2} \underbrace{\mathbb{X}_s^a_{\{\lambda_\zeta \mu_\zeta\}}(x)}_n \dots \underbrace{\mathbb{X}_s^{+a'}_{\{\lambda'_\zeta \mu'_\zeta\}}(x')}_n \dots \underbrace{[(m - \gamma_s^c \partial_c) \gamma_s^4]_{\tau_\zeta \tau'_\zeta}}_n \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}^+}{m^2}]_n} \dots \Delta(x - x') \\
& \{\psi_{s \underbrace{\lambda_\zeta \mu_\zeta}_{2n+1}}(x), \psi_{s \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n+1}}(x')\} \\
& = \frac{i}{2^{3n}[(2n+1)!]^2} \underbrace{\mathbb{X}_s^a_{\{\lambda_\zeta \mu_\zeta\}}(x)}_n \dots \underbrace{\mathbb{X}_s^{+a'}_{\{\lambda'_\zeta \mu'_\zeta\}}(x')}_n \dots \underbrace{[(m - \gamma_s^c \partial_c) \gamma_s^4]_{\tau_\zeta \tau'_\zeta}}_n \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}^+}{m^2}]_n} \dots \Delta(x - x') \\
& \psi_{s \underbrace{\lambda_\zeta \mu_\zeta}_{2n+1}} \dots = \psi_{s \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n+1}}^+ \dots \underbrace{\delta_{\lambda_\zeta}^{\lambda'_\zeta} \delta_{\mu_\zeta}^{\mu'_\zeta}}_{2n+1} \dots \psi_{s \underbrace{\lambda'_\zeta \mu'_\zeta}_{2n+1}}^+ \dots = \psi_{s \underbrace{\lambda_\zeta \mu_\zeta}_{2n+1}} \dots \underbrace{\delta_{\lambda'_\zeta}^{\lambda_\zeta} \delta_{\mu'_\zeta}^{\mu_\zeta}}_{2n+1} \dots, \mathbb{X}_s^a := S_s \mathbb{X}^a S_s^T
\end{aligned} \right.$$

[⇕]

## 定理10.2.6.

$$\left\{ \begin{aligned}
& \left\{ \underbrace{\psi_{s \lambda_\zeta \mu_\zeta \dots}(x)}_{2n+1}, \underbrace{\psi_{s \lambda'_\zeta \mu'_\zeta \dots}(x')}_{2n+1} \right\} \\
&= \frac{i}{2^{3n} [(2n+1)!]^2} \underbrace{\mathbb{X}_s^a \{ \lambda_\zeta \mu_\zeta \dots \}}_n \cdot \underbrace{\mathbb{X}_s^{a'} \{ \lambda'_\zeta \mu'_\zeta \dots \}}_n \cdot [(m - \gamma_s^c \partial_c) \gamma_s^4]_{\tau_\zeta \tau'_\zeta} \underbrace{[\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdot \Delta(x - x') \\
& \left\{ \underbrace{\psi_{s \lambda_\zeta \mu_\zeta \dots}(x)}_{2n+1}, \underbrace{\psi_{s \lambda'_\zeta \mu'_\zeta \dots}(x')}_{2n+1} \right\} \\
&= \frac{i}{2^{3n} [(2n+1)!]^2} \underbrace{\mathbb{X}_s^a \{ \lambda_\zeta \mu_\zeta \dots \}}_n \cdot \underbrace{\mathbb{X}_s^{a'} \{ \lambda'_\zeta \mu'_\zeta \dots \}}_n \cdot [(m - \gamma_s^c \partial_c) \gamma_s^4]_{\tau_\zeta \tau'_\zeta} \underbrace{[\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdot \Delta(x - x') \\
& \psi_{s \lambda_\zeta \mu_\zeta \dots} = \underbrace{\psi_{s \lambda'_\zeta \mu'_\zeta \dots}^+}_{2n+1} \cdot \underbrace{\delta_{\lambda_\zeta \lambda'_\zeta} \delta_{\mu_\zeta \mu'_\zeta}}_{2n+1} \cdot \underbrace{\psi_{s \lambda'_\zeta \mu'_\zeta \dots}^+}_{2n+1} = \underbrace{\psi_{s \lambda_\zeta \mu_\zeta \dots}}_{2n+1} \cdot \underbrace{\delta_{\lambda'_\zeta \lambda_\zeta} \delta_{\mu'_\zeta \mu_\zeta}}_{2n+1} \cdot \mathbb{X}_s^a := S_s \mathbb{X}_s^a S_s^T, \mathbb{X}_s^a = \mathbb{X}_s^{+a'} \eta_{a'}^a
\end{aligned} \right.$$

[⇕]

定理10.2.7.

$$\left\{ \begin{aligned}
& \left\{ \underbrace{\psi_{s \lambda_\zeta \mu_\zeta \dots}(x)}_{2n+1}, \underbrace{\psi_{s \lambda'_\zeta \mu'_\zeta \dots}(x')}_{2n+1} \right\} = \frac{i}{2^{2n} [(2n+1)!]^2} \underbrace{[(m - \gamma_s^a \partial_a) \gamma_s^4]_{\lambda_\zeta \lambda'_\zeta} \{ (m - \gamma_s^b \partial_b) \gamma_s^4 \}_{\mu_\zeta \mu'_\zeta \dots}}_{2n+1} \Delta(x - x') \\
& \left\{ \underbrace{\psi_{s \lambda_\zeta \mu_\zeta \dots}(x)}_{2n+1}, \underbrace{\psi_{s \lambda'_\zeta \mu'_\zeta \dots}(x')}_{2n+1} \right\} = \frac{i}{2^{2n} [(2n+1)!]^2} \underbrace{[(m - \gamma_s^a \partial_a) \gamma_s^4]_{\lambda_\zeta \lambda'_\zeta} \{ (m - \gamma_s^b \partial_b) \gamma_s^4 \}_{\mu_\zeta \mu'_\zeta \dots}}_{2n+1} \Delta(x - x') \\
& \psi_{s \lambda_\zeta \mu_\zeta \dots} = \underbrace{\psi_{s \lambda'_\zeta \mu'_\zeta \dots}^+}_{2n+1} \cdot \underbrace{\delta_{\lambda_\zeta \lambda'_\zeta} \delta_{\mu_\zeta \mu'_\zeta}}_{2n+1} \cdot \underbrace{\psi_{s \lambda'_\zeta \mu'_\zeta \dots}^+}_{2n+1} = \underbrace{\psi_{s \lambda_\zeta \mu_\zeta \dots}}_{2n+1} \cdot \underbrace{\delta_{\lambda'_\zeta \lambda_\zeta} \delta_{\mu'_\zeta \mu_\zeta}}_{2n+1}
\end{aligned} \right.$$

[⇓]

$$\text{定理10.2.8. } \left\{ \underbrace{\psi_{A_\zeta B_\zeta \dots}(x)}_{2n+1}, \underbrace{\psi_{A'_\zeta B'_\zeta \dots}(x')}_{2n+1} \right\} = i \frac{(i\zeta)^{2n+1}}{2^{2n} [(2n+1)!]^2} \underbrace{(\sigma, i\zeta)_{A_\zeta (A'_\zeta}^a}_{2n+1} \underbrace{(\sigma, i\zeta)_{B_\zeta (B'_\zeta}^b}_{2n+1} \partial_a \partial_b \dots \Delta(x - x')$$

[⇕]

$$\text{定理10.2.9. } \{ \psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x') \} = i \frac{(-1)^{2n+1}}{2^{n-1/2}} \Gamma_{k_\zeta k'_\zeta}^{abc \dots} \underbrace{(n + \frac{1}{2})}_{2n+1} \partial_a \partial_b \partial_c \dots \Delta(x - x')$$

势 $A$ 和场 $\psi$ 的反对易关系都有两种等价表述方式，且互为前提和互为因果。既可以从势反对易关系推出一切，也可以从场反对易关系推出一切。这说明对于有质量粒子，势和场两种描述方案是完全等价的。并且可以从有质量粒子的对易规则推得与无质量粒子完全类似的反对易规则，反之则不行。

# 第二十八章 各种对称和反对称方程的平面波解

自我评述：在本章中我对于各种全对称和全反对称方程平面波解提供了统一的解法，并为后续物理研究提供先导性的知识。

## 1 Bargmann-Wigner方程的平面波解

### 1.1 两个推论

推论1.1.1.

$$\begin{cases} U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h) = \frac{\sqrt{s+h}}{\sqrt{2s}} U_{\lambda_s \mu_s \dots \sigma_s}(\vec{p}, h - \frac{1}{2}) u_{\tau_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} U_{\lambda_s \mu_s \dots \sigma_s}(\vec{p}, h + \frac{1}{2}) u_{\tau_s}(\vec{p}, -\frac{1}{2}) \\ U_{\lambda_s \mu_s \dots \sigma_s \tau_s \eta_s}(\vec{p}, h) = \frac{\sqrt{s+1/2+h}}{\sqrt{2s+1}} U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h - \frac{1}{2}) u_{\eta_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1/2-h}}{\sqrt{2s+1}} U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h + \frac{1}{2}) u_{\eta_s}(\vec{p}, -\frac{1}{2}) \end{cases}$$

推论1.1.2.

$$\begin{cases} V_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h) = \frac{\sqrt{s+h}}{\sqrt{2s}} V_{\lambda_s \mu_s \dots \sigma_s}(\vec{p}, h - \frac{1}{2}) v_{\tau_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} V_{\lambda_s \mu_s \dots \sigma_s}(\vec{p}, h + \frac{1}{2}) v_{\tau_s}(\vec{p}, -\frac{1}{2}) \\ V_{\lambda_s \mu_s \dots \sigma_s \tau_s \eta_s}(\vec{p}, h) = \frac{\sqrt{s+1/2+h}}{\sqrt{2s+1}} V_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h - \frac{1}{2}) v_{\eta_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1/2-h}}{\sqrt{2s+1}} V_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h + \frac{1}{2}) v_{\eta_s}(\vec{p}, -\frac{1}{2}) \end{cases}$$

### 1.2 两个关于U-自旋基的引理

$$\begin{aligned} \text{引理1.2.1. } \sum_{h=s}^{-s} a_{\eta_s}(\vec{p}, h) U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h) &= \sum_{h=s}^{-s} a_{\tau_s}(\vec{p}, h) U_{\lambda_s \mu_s \dots \sigma_s \eta_s}(\vec{p}, h) \\ \Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}} a_{[\eta_s]}(\vec{p}, h) u_{\tau_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{2s}} a_{[\eta_s]}(\vec{p}, h-1) u_{\tau_s}(\vec{p}, -\frac{1}{2}) &= 0, -(s-1) \leq h \leq s \end{aligned}$$

$$\begin{aligned} \text{证明: } \sum_{h=s}^{-s} a_{\eta_s}(\vec{p}, h) U_{\lambda_s \mu_s \dots \sigma_s \tau_s}(\vec{p}, h) &= \sum_{h=s}^{-s} a_{\tau_s}(\vec{p}, h) U_{\lambda_s \mu_s \dots \sigma_s \eta_s}(\vec{p}, h) \\ \Leftrightarrow \sum_{h=s}^{-s} a_{\eta_s}(\vec{p}, h) \left[ \frac{\sqrt{s+h}}{\sqrt{2s}} U_{\lambda_s \mu_s \dots \sigma_s}(\vec{p}, h - \frac{1}{2}) u_{\tau_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} U_{\lambda_s \mu_s \dots \sigma_s}(\vec{p}, h + \frac{1}{2}) u_{\tau_s}(\vec{p}, -\frac{1}{2}) \right] \\ &= \sum_{h=s}^{-s} a_{\tau_s}(\vec{p}, h) \left[ \frac{\sqrt{s+h}}{\sqrt{2s}} U_{\lambda_s \mu_s \dots \sigma_s}(\vec{p}, h - \frac{1}{2}) u_{\eta_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} U_{\lambda_s \mu_s \dots \sigma_s}(\vec{p}, h + \frac{1}{2}) u_{\eta_s}(\vec{p}, -\frac{1}{2}) \right] \\ \Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}} a_{\eta_s}(\vec{p}, h) u_{\tau_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{2s}} a_{\eta_s}(\vec{p}, h-1) u_{\tau_s}(\vec{p}, -\frac{1}{2}) \\ &= \frac{\sqrt{s+h}}{\sqrt{2s}} a_{\tau_s}(\vec{p}, h) u_{\eta_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{2s}} a_{\tau_s}(\vec{p}, h-1) u_{\eta_s}(\vec{p}, -\frac{1}{2}), -(s-1) \leq h \leq s \end{aligned}$$

$$\Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}} a_{[\eta_s]}(\vec{p}, h) u_{\tau_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{2s}} a_{[\eta_s]}(\vec{p}, h-1) u_{\tau_s}(\vec{p}, -\frac{1}{2}) = 0, -(s-1) \leq h \leq s \quad \square$$

$$\text{引理1.2.2. } \frac{\sqrt{s+h}}{\sqrt{2s}} a_{[\eta_s]}(\vec{p}, h) u_{\tau_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{2s}} a_{[\eta_s]}(\vec{p}, h-1) u_{\tau_s}(\vec{p}, -\frac{1}{2}) = 0, -(s-1) \leq h \leq s$$

$$\Leftrightarrow \begin{cases} a_{\eta_s}(\vec{p}, h) = c_+(\vec{p}, h) u_{\eta_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}} c_+(\vec{p}, h-1) u_{\eta_s}(\vec{p}, -\frac{1}{2}), -(s-1) \leq h \leq s \\ a_{\eta_s}(\vec{p}, -s) = c_+(\vec{p}, -s) u_{\eta_s}(\vec{p}, \frac{1}{2}) + c_-(\vec{p}, -s) u_{\eta_s}(\vec{p}, -\frac{1}{2}), h = -s \end{cases}$$

$$\text{证明: } \frac{\sqrt{s+h}}{\sqrt{2s}} a_{[\eta_s]}(\vec{p}, h) u_{\tau_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{2s}} a_{[\eta_s]}(\vec{p}, h-1) u_{\tau_s}(\vec{p}, -\frac{1}{2}) = 0, -(s-1) \leq h \leq s$$

$$\begin{aligned} \Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}} \{ c_+(\vec{p}, h) u_{[\eta_s]}(\vec{p}, \frac{1}{2}) + c_-(\vec{p}, h) u_{[\eta_s]}(\vec{p}, -\frac{1}{2}) + d_+(\vec{p}, h) v_{[\eta_s]}(\vec{p}, \frac{1}{2}) + d_-(\vec{p}, h) v_{[\eta_s]}(\vec{p}, -\frac{1}{2}) \} u_{\tau_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{2s}} \\ \{ c_+(\vec{p}, h-1) u_{[\eta_s]}(\vec{p}, \frac{1}{2}) + c_-(\vec{p}, h-1) u_{[\eta_s]}(\vec{p}, -\frac{1}{2}) + d_+(\vec{p}, h-1) v_{[\eta_s]}(\vec{p}, \frac{1}{2}) + d_-(\vec{p}, h-1) v_{[\eta_s]}(\vec{p}, -\frac{1}{2}) \} u_{\tau_s}(\vec{p}, -\frac{1}{2}) = \\ 0, -(s-1) \leq h \leq s \\ \Leftrightarrow c_+(\vec{p}, h) u_{[\eta_s]}(\vec{p}, \frac{1}{2}) u_{\tau_s}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}} c_-(\vec{p}, h-1) u_{[\eta_s]}(\vec{p}, -\frac{1}{2}) u_{\tau_s}(\vec{p}, -\frac{1}{2}) \\ + [c_-(\vec{p}, h) - \frac{\sqrt{s+1-h}}{\sqrt{s+h}} c_+(\vec{p}, h-1)] u_{[\eta_s]}(\vec{p}, -\frac{1}{2}) u_{\tau_s}(\vec{p}, \frac{1}{2}) \end{aligned}$$



$$\begin{aligned}
& + d_+(\vec{p}, h)v_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right)u_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_-(\vec{p}, h-1)v_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)u_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) \\
& + d_-(\vec{p}, h)v_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)u_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p}, h-1)v_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right)u_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) = 0, -(s-1) \leq h \leq s \\
& \Leftrightarrow [c_-(\vec{p}, h) - \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p}, h-1)]u_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)u_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) \\
& + d_+(\vec{p}, h)v_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right)u_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_-(\vec{p}, h-1)v_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)u_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) \\
& + d_-(\vec{p}, h)v_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)u_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p}, h-1)v_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right)u_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) = 0, -(s-1) \leq h \leq s \\
& \Leftrightarrow \begin{cases} \frac{\sqrt{s+h}}{\sqrt{2s}}c_-(\vec{p}, h) = \frac{\sqrt{s+1-h}}{\sqrt{2s}}c_+(\vec{p}, h-1), -(s-1) \leq h \leq s \\ d_+(\vec{p}, h) = 0, d_-(\vec{p}, h) = 0, -s \leq h \leq s \end{cases} \\
& \Leftrightarrow \begin{cases} a_{\eta_\zeta}(\vec{p}, h) = c_+(\vec{p}, h)u_{\eta_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p}, h-1)u_{\eta_\zeta}\left(\vec{p}, -\frac{1}{2}\right), -(s-1) \leq h \leq s \\ a_{\eta_\zeta}(\vec{p}, -s) = c_+(\vec{p}, -s)u_{\eta_\zeta}\left(\vec{p}, \frac{1}{2}\right) + c_-(\vec{p}, -s)u_{\eta_\zeta}\left(\vec{p}, -\frac{1}{2}\right), h = -s \end{cases} \quad \square
\end{aligned}$$

### 1.3 两个关于V-自旋基的引理

$$\begin{aligned}
\text{引理1.3.1.} \quad & \sum_{h=s}^{-s} b_{\eta_\zeta}^+(\vec{p}, h)V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = \sum_{h=s}^{-s} b_{\tau_\zeta}^+(\vec{p}, h)V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \eta_\zeta}_{2s}}(\vec{p}, h) \\
& \Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}, h)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}, h-1)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) = 0, -(s-1) \leq h \leq s
\end{aligned}$$

$$\begin{aligned}
\text{证明:} \quad & \sum_{h=s}^{-s} b_{\eta_\zeta}^+(\vec{p}, h)V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = \sum_{h=s}^{-s} b_{\tau_\zeta}^+(\vec{p}, h)V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \eta_\zeta}_{2s}}(\vec{p}, h) \\
& \Leftrightarrow \sum_{h=s}^{-s} b_{\eta_\zeta}^+(\vec{p}, h)\left[\frac{\sqrt{s+h}}{\sqrt{2s}}V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta}_{2s-1}}\left(\vec{p}, h-\frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s-h}}{\sqrt{2s}}V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta}_{2s-1}}\left(\vec{p}, h+\frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right)\right] \\
& = \sum_{h=s}^{-s} b_{\tau_\zeta}^+(\vec{p}, h)\frac{\sqrt{s+h}}{\sqrt{2s}}V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta}_{2s-1}}\left(\vec{p}, h-\frac{1}{2}\right)v_{\eta_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s-h}}{\sqrt{2s}}V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta}_{2s-1}}\left(\vec{p}, h+\frac{1}{2}\right)v_{\eta_\zeta}\left(\vec{p}, -\frac{1}{2}\right) \\
& \Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}b_{\eta_\zeta}^+(\vec{p}, h)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{2s}}b_{\eta_\zeta}^+(\vec{p}, h-1)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) \\
& = \frac{\sqrt{s+h}}{\sqrt{2s}}b_{\tau_\zeta}^+(\vec{p}, h)v_{\eta_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{2s}}b_{\tau_\zeta}^+(\vec{p}, h-1)v_{\eta_\zeta}\left(\vec{p}, -\frac{1}{2}\right), -(s-1) \leq h \leq s \\
& \Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}, h)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}, h-1)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) = 0, -(s-1) \leq h \leq s \quad \square
\end{aligned}$$

$$\begin{aligned}
\text{引理1.3.2.} \quad & \frac{\sqrt{s+h}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}, h)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}, h-1)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) = 0, -(s-1) \leq h \leq s \\
& \Leftrightarrow \begin{cases} b_{\eta_\zeta}^+(\vec{p}, h) = d_+(\vec{p}, h)v_{\eta_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p}, h-1)v_{\eta_\zeta}\left(\vec{p}, -\frac{1}{2}\right), -(s-1) \leq h \leq s \\ b_{\eta_\zeta}^+(\vec{p}, -s) = d_+(\vec{p}, -s)v_{\eta_\zeta}\left(\vec{p}, \frac{1}{2}\right) + d_-(\vec{p}, -s)v_{\eta_\zeta}\left(\vec{p}, -\frac{1}{2}\right), h = -s \end{cases}
\end{aligned}$$

$$\text{证明:} \quad \frac{\sqrt{s+h}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}, h)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}, h-1)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) = 0, -(s-1) \leq h \leq s$$

$$\begin{aligned}
& \Leftrightarrow \frac{\sqrt{s+h}}{\sqrt{2s}}\{c_+(\vec{p}, h)u_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right) + c_-(\vec{p}, h)u_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right) + d_+(\vec{p}, h)v_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right) + d_-(\vec{p}, h)v_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)\}v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{2s}} \\
& \{c_+(\vec{p}, h-1)u_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right) + c_-(\vec{p}, h-1)u_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right) + d_+(\vec{p}, h-1)v_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right) + d_-(\vec{p}, h-1)v_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)\}v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) = \\
& 0, -(s-1) \leq h \leq s \\
& \Leftrightarrow d_+(\vec{p}, h)v_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_-(\vec{p}, h-1)v_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) \\
& + [d_-(\vec{p}, h) - \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p}, h-1)]v_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) \\
& + c_+(\vec{p}, h)u_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_-(\vec{p}, h-1)u_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) \\
& + c_-(\vec{p}, h)u_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p}, h-1)u_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) = 0, -(s-1) \leq h \leq s \\
& \Leftrightarrow [d_-(\vec{p}, h) - \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p}, h-1)]v_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) \\
& + c_+(\vec{p}, h)u_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_-(\vec{p}, h-1)u_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) \\
& + c_-(\vec{p}, h)u_{[\eta_\zeta]}\left(\vec{p}, -\frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, \frac{1}{2}\right) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p}, h-1)u_{[\eta_\zeta]}\left(\vec{p}, \frac{1}{2}\right)v_{\tau_\zeta}\left(\vec{p}, -\frac{1}{2}\right) = 0, -(s-1) \leq h \leq s \\
& \Leftrightarrow \begin{cases} \frac{\sqrt{s+h}}{\sqrt{2s}}d_-(\vec{p}, h) = \frac{\sqrt{s+1-h}}{\sqrt{2s}}d_+(\vec{p}, h-1), -(s-1) \leq h \leq s \\ c_+(\vec{p}, h) = 0, c_-(\vec{p}, h) = 0, -s \leq h \leq s \end{cases}
\end{aligned}$$

$$\Leftrightarrow \begin{cases} b_{\eta_\zeta}^+(\vec{p}, h) = d_+(\vec{p}, h)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p}, h-1)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}), -(s-1) \leq h \leq s \\ b_{\eta_\zeta}^+(\vec{p}, -s) = d_+(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + d_-(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}), h = -s \end{cases} \quad \square$$

### 1.3.1 两个重要定理

定理1.3.1.

$$\begin{cases} a_{\eta_\zeta}(\vec{p}, h) = c_+(\vec{p}, h)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p}, h-1)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}), -(s-1) \leq h \leq s \\ a_{\eta_\zeta}(\vec{p}, -s) = c_+(\vec{p}, -s)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + c_-(\vec{p}, -s)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}), h = -s \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum_{h=s}^{-s} a_{\eta_\zeta}(\vec{p}, h) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h) = \sum_{(h+1/2)=(s+1/2)}^{-(s+1/2)} a(\vec{p}, h + \frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, h + \frac{1}{2}) \\ a(\vec{p}, -s - \frac{1}{2}) := c_-(\vec{p}, -s), a(\vec{p}, h + \frac{1}{2}) := \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p}, h), -s \leq h \leq s \end{cases}$$

证明:

$$\begin{aligned} & \begin{cases} a_{\eta_\zeta}(\vec{p}, h) = c_+(\vec{p}, h)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p}, h-1)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}), -(s-1) \leq h \leq s \\ a_{\eta_\zeta}(\vec{p}, -s) = c_+(\vec{p}, -s)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + c_-(\vec{p}, -s)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}), h = -s \end{cases} \\ \Leftrightarrow & \sum_{h=s}^{-s} a_{\eta_\zeta}(\vec{p}, h) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h) = \sum_{h=s}^{-s+1} [c_+(\vec{p}, h)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p}, h-1)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2})] \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h) \\ & + \sum_{h=-s} [c_+(\vec{p}, -s)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + c_-(\vec{p}, -s)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2})] \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, -s) \\ = & \sum_{h=s}^{-s+1} c_+(\vec{p}, h)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h) + \sum_{h=s}^{-s+1} \frac{\sqrt{s+1-h}}{\sqrt{s+h}}c_+(\vec{p}, h-1)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h) \\ & + \sum_{h=-s} [c_+(\vec{p}, -s)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + c_-(\vec{p}, -s)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2})] \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, -s) \\ = & [\sum_{h=s-1}^{-s} c_+(\vec{p}, h)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h) + \sum_{h=s-1}^{-s} \frac{\sqrt{s-h}}{\sqrt{s+h+1}}c_+(\vec{p}, h)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h+1)] \\ & + [\sum_{h=s} c_+(\vec{p}, s)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, s) + \sum_{h=-s} c_-(\vec{p}, -s)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, -s)] \\ = & \sum_{h=s-1}^{-s} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p}, h) [\frac{\sqrt{s+h+1}}{\sqrt{2s+1}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s+1}}c_+(\vec{p}, h) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h+1)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2})] \\ & + [\sum_{h=s} c_+(\vec{p}, s)u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, s) + \sum_{h=-s} c_-(\vec{p}, -s)u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, -s)] \\ = & \sum_{h=s-1}^{-s} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p}, h) \\ & [ \frac{\sqrt{(s+\frac{1}{2})+(h+\frac{1}{2})}}{\sqrt{2s+1}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, (h+\frac{1}{2}) - \frac{1}{2})u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{(s+\frac{1}{2})-(h+\frac{1}{2})}}{\sqrt{2s+1}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, (h+\frac{1}{2}) + \frac{1}{2})u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) ] \\ & + [c_+(\vec{p}, s) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, s + \frac{1}{2}) + c_-(\vec{p}, -s) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, -s - \frac{1}{2})] \\ = & \sum_{(h+1/2)=(s-1/2)}^{-(s-1/2)} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p}, h) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, h + \frac{1}{2}) + c_+(\vec{p}, s) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, s + \frac{1}{2}) \\ & + c_-(\vec{p}, -s) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, -s - \frac{1}{2}) \\ = & \sum_{(h+1/2)=(s+1/2)}^{-(s-1/2)} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p}, h) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, h + \frac{1}{2}) + c_-(\vec{p}, -s) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, -s - \frac{1}{2}) \\ = & \sum_{(h+1/2)=(s+1/2)}^{-(s+1/2)} a(\vec{p}, h + \frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, h + \frac{1}{2}) \\ , a(\vec{p}, h + \frac{1}{2}) := & \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}c_+(\vec{p}, h), -s \leq h \leq s; a(\vec{p}, -s - \frac{1}{2}) := c_-(\vec{p}, -s) \end{aligned} \quad \square$$

定理1.3.2.

$$\begin{cases} b_{\eta_\zeta}^+(\vec{p}, h) = d_+(\vec{p}, h)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p}, h-1)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}), -(s-1) \leq h \leq s \\ b_{\eta_\zeta}^+(\vec{p}, -s) = d_+(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + d_-(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}), h = -s \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum_{h=s}^{-s} b_{\eta_\zeta}^+(\vec{p}, h)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h) = \sum_{(h+1/2)=(s+1/2)}^{-(s+1/2)} b^+(\vec{p}, h + \frac{1}{2})\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}, h + \frac{1}{2}) \\ b^+(\vec{p}, -s - \frac{1}{2}) := d_-(\vec{p}, -s), b^+(\vec{p}, h + \frac{1}{2}) := \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p}, h), -s \leq h \leq s \end{cases}$$

证明:

$$\begin{aligned} & \begin{cases} b_{\eta_\zeta}^+(\vec{p}, h) = d_+(\vec{p}, h)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p}, h-1)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}), -(s-1) \leq h \leq s \\ b_{\eta_\zeta}^+(\vec{p}, -s) = d_+(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + d_-(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}), h = -s \end{cases} \\ \Leftrightarrow & \sum_{h=s}^{-s} b_{\eta_\zeta}^+(\vec{p}, h)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h) = \sum_{h=s}^{-s+1} [d_+(\vec{p}, h)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p}, h-1)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})]\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h) \\ & + \sum_{h=-s} [d_+(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + d_-(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})]\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, -s) \\ = & \sum_{h=s}^{-s+1} d_+(\vec{p}, h)v_{\eta_\zeta}(\vec{p}, \frac{1}{2})\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h) + \sum_{h=s}^{-s+1} \frac{\sqrt{s+1-h}}{\sqrt{s+h}}d_+(\vec{p}, h-1)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h) \\ & + \sum_{h=-s} [d_+(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + d_-(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})]\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, -s) \\ = & [\sum_{h=s-1}^{-s} d_+(\vec{p}, h)v_{\eta_\zeta}(\vec{p}, \frac{1}{2})\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h) + \sum_{h=s-1}^{-s} \frac{\sqrt{s-h}}{\sqrt{s+h+1}}d_+(\vec{p}, h)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h+1)] \\ & + [\sum_{h=s} d_+(\vec{p}, s)v_{\eta_\zeta}(\vec{p}, \frac{1}{2})\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, s) + \sum_{h=-s} d_-(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, -s)] \\ = & \sum_{h=s-1}^{-s} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p}, h)[\frac{\sqrt{s+h+1}}{\sqrt{2s+1}}\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s+1}}d_+(\vec{p}, h)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h+1)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})] \\ & + [\sum_{h=s} d_+(\vec{p}, s)v_{\eta_\zeta}(\vec{p}, \frac{1}{2})\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, s) + \sum_{h=-s} d_-(\vec{p}, -s)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, -s)] \\ = & \sum_{h=s-1}^{-s} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p}, h) \\ & [\frac{\sqrt{(s+\frac{1}{2})+(h+\frac{1}{2})}}{\sqrt{2s+1}}\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, (h+\frac{1}{2})-\frac{1}{2})v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{(s+\frac{1}{2})-(h+\frac{1}{2})}}{\sqrt{2s+1}}\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, (h+\frac{1}{2})+\frac{1}{2})v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})] \\ & + [d_+(\vec{p}, s)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}, s+\frac{1}{2}) + d_-(\vec{p}, -s)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}, -s-\frac{1}{2})] \\ = & \sum_{(h+1/2)=(s-1/2)}^{-(s-1/2)} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p}, h)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}, h+\frac{1}{2}) + d_+(\vec{p}, s)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}, s+\frac{1}{2}) \\ & + d_-(\vec{p}, -s)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}, -s-\frac{1}{2}) \\ = & \sum_{(h+1/2)=(s+1/2)}^{-(s-1/2)} \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p}, h)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}, h+\frac{1}{2}) + d_-(\vec{p}, -s)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}, -s-\frac{1}{2}) \\ = & \sum_{(h+1/2)=(s+1/2)}^{-(s+1/2)} b^+(\vec{p}, h + \frac{1}{2})\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}, h + \frac{1}{2}) \\ , b^+(\vec{p}, h + \frac{1}{2}) := & \frac{\sqrt{2s+1}}{\sqrt{s+h+1}}d_+(\vec{p}, h), -s \leq h \leq s; b^+(\vec{p}, -s - \frac{1}{2}) := d_-(\vec{p}, -s) \quad \square \end{aligned}$$

#### 1.4 用数学归纳法严格求解Bargmann-Wigner方程的平面波解

定理1.4.1.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}(x) = 0, \psi_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}(x) = \frac{1}{(2s)!} \psi_{\{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\}}(x)$

$$\psi_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=s}^{-s} \frac{m^s}{\sqrt{E}} [a(\vec{p}, h)\underbrace{U_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h)e^{ip \cdot x} + b^+(\vec{p}, h)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h)e^{-ip \cdot x}] d^3\vec{p}$$

$$\underbrace{U_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}, h) = \frac{1}{(2s)!} \sqrt{C_{2s}^{s-h}} \underbrace{u_{\{\lambda_\zeta(\vec{p}, \frac{1}{2})\mu_\zeta(\vec{p}, \frac{1}{2})\dots u_{\sigma_\zeta(\vec{p}, -\frac{1}{2})\tau_\zeta(\vec{p}, -\frac{1}{2})\}}}_{s+h}(\vec{p}, \frac{1}{2})}_{s-h}$$



$$\left\{ \begin{aligned} \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{(h+1/2)=(s+1/2)}^{-(s+1/2)} \frac{m^s}{\sqrt{E}} \\ &[\sqrt{ma}(\vec{p}, h + \frac{1}{2}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, h + \frac{1}{2}) e^{ip \cdot x} + \sqrt{mb^+}(\vec{p}, h + \frac{1}{2}) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}, h + \frac{1}{2}) e^{-ip \cdot x}] d^3 \vec{p} \\ \sqrt{ma}(\vec{p}, -s - \frac{1}{2}) &:= c_-(\vec{p}, -s), \sqrt{ma}(\vec{p}, h + \frac{1}{2}) := \frac{\sqrt{2s+1}}{\sqrt{(s+\frac{1}{2})+(h+\frac{1}{2})}} c_+(\vec{p}, h), -s \leq h \leq s \\ \sqrt{mb^+}(\vec{p}, -s - \frac{1}{2}) &:= d_-(\vec{p}, -s), \sqrt{mb^+}(\vec{p}, h + \frac{1}{2}) := \frac{\sqrt{2s+1}}{\sqrt{(s+\frac{1}{2})+(h+\frac{1}{2})}} d_+(\vec{p}, h), -s \leq h \leq s \\ \psi_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta}(x) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=s+1/2}^{-(s+1/2)} \frac{m^{s+1/2}}{\sqrt{E}} [a(\vec{p}, h) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta}}_{2s+1}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta}}_{2s+1}(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p} \\ a(\vec{p}, -s - \frac{1}{2}) &:= \frac{c_-(\vec{p}, -s)}{\sqrt{m}}, a(\vec{p}, h) := \frac{\sqrt{2s+1}}{\sqrt{(s+\frac{1}{2})+h}} \frac{c_+(\vec{p}, h - \frac{1}{2})}{\sqrt{m}}, -s + \frac{1}{2} \leq h \leq s + \frac{1}{2} \\ b^+(\vec{p}, -s - \frac{1}{2}) &:= \frac{d_-(\vec{p}, -s)}{\sqrt{m}}, b^+(\vec{p}, h) := \frac{\sqrt{2s+1}}{\sqrt{(s+\frac{1}{2})+h}} \frac{d_+(\vec{p}, h - \frac{1}{2})}{\sqrt{m}}, -s + \frac{1}{2} \leq h \leq s + \frac{1}{2} \end{aligned} \right.$$

此步证明了  $n' = n$  时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

## 2 N+1维时空中Bargmann-Wigner方程的平面波解

### 2.1 N+1维时空中Bargmann-Wigner方程U-自旋基的性质

#### 2.1.1 关于对称性条件的U-自旋基引理

引理2.1.1.  $\sum_{n_1+\dots+n_l=2s} a_{\eta_\zeta}(\vec{p}; n_1, \dots, n_l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) = \sum_{n_1+\dots+n_l=2s} a_{\tau_\zeta}(\vec{p}; n_1, \dots, n_l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l)$

$$\Leftrightarrow \left\{ \begin{aligned} \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1 + 1, n_2, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2 + 1, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2, \dots, n_l + 1) u_{\tau_\zeta]}(\vec{p}; l) = 0 \end{aligned} \right.$$

证明:  $\sum_{n_1+\dots+n_l=2s} a_{\eta_\zeta}(\vec{p}; n_1, \dots, n_l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) = \sum_{n_1+\dots+n_l=2s} a_{\tau_\zeta}(\vec{p}; n_1, \dots, n_l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l)$

$$\Leftrightarrow \sum_{n_1+\dots+n_l=2s} a_{\eta_\zeta}(\vec{p}; n_1, \dots, n_l) \left[ \frac{\sqrt{n_1}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1 - 1, n_2, \dots, n_l) U_{\tau_\zeta}(\vec{p}; 1) \right.$$

$$+ \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1, n_2 - 1, \dots, n_l) U_{\tau_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_l}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1, n_2, \dots, n_l - 1) U_{\tau_\zeta}(\vec{p}; l) \left. \right]$$

$$= \sum_{n_1+\dots+n_l=2s} a_{\tau_\zeta}(\vec{p}; n_1, \dots, n_l) \left[ \frac{\sqrt{n_1}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1 - 1, n_2, \dots, n_l) U_{\eta_\zeta}(\vec{p}; 1) \right.$$

$$+ \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1, n_2 - 1, \dots, n_l) U_{\eta_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_l}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1, n_2, \dots, n_l - 1) U_{\eta_\zeta}(\vec{p}; l) \left. \right]$$

$$\Leftrightarrow \left\{ \begin{aligned} \frac{\sqrt{n_1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1) u_{\tau_\zeta]}(\vec{p}; l) = 0 \\ \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1 + 1, n_2 - 1, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2 - 1, \dots, n_l + 1) u_{\tau_\zeta]}(\vec{p}; l) = 0 \dots \dots \\ \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1 + 1, n_2, \dots, n_l - 1) u_{\tau_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2 + 1, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; l) = 0 \end{aligned} \right.$$

$$\Leftrightarrow \left\{ \begin{aligned} \frac{\sqrt{n_1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1) u_{\tau_\zeta]}(\vec{p}; l) = 0 \end{aligned} \right.$$

$$\Leftrightarrow \begin{cases} \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1+1, n_2, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2+1, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2, \dots, n_l+1) u_{\tau_\zeta]}(\vec{p}; l) = 0 \end{cases} \quad \square$$

引理2.1.2.

$$\begin{cases} a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) + \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k) \\ \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1+1, n_2, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2+1, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2, \dots, n_l+1) u_{\tau_\zeta]}(\vec{p}; l) = 0 \end{cases}$$

$$\Leftrightarrow a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k) u_{\eta_\zeta}(\vec{p}; k), d(\vec{p}; n_1, n_2, \dots, n_l; k) = 0$$

$$\begin{cases} a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \dots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \dots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \dots, n_l+1; 1) u_{\eta_\zeta}(\vec{p}; l), n_1 \geq 1 \end{cases}$$

$$\begin{cases} a_{\eta_\zeta}(\vec{p}; 0, n_2, \dots, n_l) = c(\vec{p}; 0, n_2, \dots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) \\ + \frac{\sqrt{n_2}}{\sqrt{n_2}} c(\vec{p}; 0, n_2, \dots, n_l; 2) u_{\eta_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3+1, \dots, n_l; 2) u_{\eta_\zeta}(\vec{p}; 3) + \dots \\ + \frac{\sqrt{n_l+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3, \dots, n_l+1; 2) u_{\eta_\zeta}(\vec{p}; l), n_2 \geq 1 \end{cases}$$

$$\begin{cases} a_{\eta_\zeta}(\vec{p}; 0, 0, n_3, \dots, n_l) = c(\vec{p}; 0, 0, n_3, \dots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, 0, n_3, \dots, n_l; 2) u_{\eta_\zeta}(\vec{p}; 2) \\ + \frac{\sqrt{n_3}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3, n_4, \dots, n_l; 3) u_{\eta_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3-1, n_4+1, \dots, n_l; 3) u_{\eta_\zeta}(\vec{p}; 4) + \dots \\ + \frac{\sqrt{n_l+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3-1, \dots, n_l+1; 3) u_{\eta_\zeta}(\vec{p}; l), n_3 \geq 1 \end{cases}$$

$$\dots\dots\dots$$

$$\begin{cases} a_{\eta_\zeta}(\vec{p}; 0, \dots, 0, n_l) = c(\vec{p}; 0, \dots, 0, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, \dots, 0, n_l; 2) u_{\eta_\zeta}(\vec{p}; 2) \\ + \dots + c(\vec{p}; 0, \dots, 0, n_l; l-1) u_{\eta_\zeta}(\vec{p}; l-1) + \frac{\sqrt{n_l}}{\sqrt{n_l}} c(\vec{p}; 0, \dots, 0, n_l; l) u_{\eta_\zeta}(\vec{p}; l), n_l = 2s \geq 1 \end{cases}$$

证明:

$$\begin{cases} a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) + \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k) \\ \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1+1, n_2, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2+1, \dots, n_l) u_{\tau_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2, \dots, n_l+1) u_{\tau_\zeta]}(\vec{p}; l) = 0 \end{cases}$$

 $\Leftrightarrow$ 

$$a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k) u_{\eta_\zeta}(\vec{p}; k), d(\vec{p}; n_1, n_2, \dots, n_l; k) = 0$$

$$\begin{cases} c(\vec{p}; n_1+1, n_2, \dots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2+1, \dots, n_l; 1) \dots \\ c(\vec{p}; n_1+1, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \dots, n_l+1; 1) \\ c(\vec{p}; 0, n_2+1, \dots, n_l; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2+1}} c(\vec{p}; 0, n_2, n_3+1, \dots, n_l; 2) \dots \\ c(\vec{p}; 0, n_2+1, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_2+1}} c(\vec{p}; 0, n_2, \dots, n_l+1; 2) \\ c(\vec{p}; 0, 0, n_3+1, \dots, n_l; 4) = \frac{\sqrt{n_4+1}}{\sqrt{n_3+1}} c(\vec{p}; 0, 0, n_3, n_4+1, \dots, n_l; 3) \dots \\ c(\vec{p}; 0, 0, n_3+1, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_3+1}} c(\vec{p}; 0, 0, n_3, n_4, \dots, n_l+1; 3) \\ \dots \\ c(\vec{p}; 0, \dots, 0, n_{l-1}+1, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}+1}} c(\vec{p}; 0, \dots, 0, n_{l-1}, n_l+1; l-1) \end{cases}$$

 $\Leftrightarrow$ 

$$a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k) u_{\eta_\zeta}(\vec{p}; k), d(\vec{p}; n_1, n_2, \dots, n_l; k) = 0$$

$$\begin{cases} a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \dots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \dots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \dots, n_l+1; 1) u_{\eta_\zeta}(\vec{p}; l), n_1 \geq 1 \end{cases}$$

$$\begin{cases}
a_{\eta_\zeta}(\vec{p}; 0, n_2, \dots, n_l) = c(\vec{p}; 0, n_2, \dots, n_l; 1)u_{\eta_\zeta}(\vec{p}; 1) \\
+ \frac{\sqrt{n_2}}{\sqrt{n_2}}c(\vec{p}; 0, n_2, \dots, n_l; 2)u_{\eta_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_l; 2)u_{\eta_\zeta}(\vec{p}; 3) + \dots \\
+ \frac{\sqrt{n_l+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l + 1; 2)u_{\eta_\zeta}(\vec{p}; l), n_2 \geq 1 \\
a_{\eta_\zeta}(\vec{p}; 0, 0, n_3, \dots, n_l) = c(\vec{p}; 0, 0, n_3, \dots, n_l; 1)u_{\eta_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, 0, n_3, \dots, n_l; 2)u_{\eta_\zeta}(\vec{p}; 2) \\
+ \frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3, n_4, \dots, n_l; 3)u_{\eta_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, n_4 + 1, \dots, n_l; 3)u_{\eta_\zeta}(\vec{p}; 4) + \dots \\
+ \frac{\sqrt{n_l+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, \dots, n_l + 1; 3)u_{\eta_\zeta}(\vec{p}; l), n_3 \geq 1 \\
\dots\dots\dots \\
a_{\eta_\zeta}(\vec{p}; 0, \dots, 0, n_l) = c(\vec{p}; 0, \dots, 0, n_l; 1)u_{\eta_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, \dots, 0, n_l; 2)u_{\eta_\zeta}(\vec{p}; 2) \\
+ \dots + c(\vec{p}; 0, \dots, 0, n_l; l-1)u_{\eta_\zeta}(\vec{p}; l-1) + \frac{\sqrt{n_l}}{\sqrt{n_l}}c(\vec{p}; 0, \dots, 0, n_l; l)u_{\eta_\zeta}(\vec{p}; l), n_l = 2s \geq 1
\end{cases}$$

□

推论2.1.1.  $a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k)u_{\eta_\zeta}(\vec{p}; k)$

$$\begin{cases}
c(\vec{p}; n_1, n_2, \dots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l; 1), n_1 \geq 1 \dots \\
c(\vec{p}; n_1, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1; 1), n_1 \geq 1 \\
c(\vec{p}; 0, n_2, \dots, n_l; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_l; 2), n_2 \geq 1 \dots \\
c(\vec{p}; 0, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l + 1; 2), n_2 \geq 1 \\
\dots \\
c(\vec{p}; 0, \dots, 0, n_{l-1}, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}}}c(\vec{p}; 0, \dots, 0, n_{l-1} - 1, n_l + 1; l), n_{l-1} \geq 1 \\
\Leftrightarrow \\
a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k)u_{\eta_\zeta}(\vec{p}; k) \\
\begin{cases}
a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{n_1}}{\sqrt{n_1}}c(\vec{p}; n_1, n_2, \dots, n_l; 1)u_{\eta_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l; 1)u_{\eta_\zeta}(\vec{p}; 2) \\
+ \dots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1; 1)u_{\eta_\zeta}(\vec{p}; l), n_1 \geq 1 \\
a_{\eta_\zeta}(\vec{p}; 0, n_2, \dots, n_l) = c(\vec{p}; 0, n_2, \dots, n_l; 1)u_{\eta_\zeta}(\vec{p}; 1) \\
+ \frac{\sqrt{n_2}}{\sqrt{n_2}}c(\vec{p}; 0, n_2, \dots, n_l; 2)u_{\eta_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_l; 2)u_{\eta_\zeta}(\vec{p}; 3) + \dots \\
+ \frac{\sqrt{n_l+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l + 1; 2)u_{\eta_\zeta}(\vec{p}; l), n_2 \geq 1 \\
a_{\eta_\zeta}(\vec{p}; 0, 0, n_3, \dots, n_l) = c(\vec{p}; 0, 0, n_3, \dots, n_l; 1)u_{\eta_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, 0, n_3, \dots, n_l; 2)u_{\eta_\zeta}(\vec{p}; 2) \\
+ \frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3, n_4, \dots, n_l; 3)u_{\eta_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, n_4 + 1, \dots, n_l; 3)u_{\eta_\zeta}(\vec{p}; 4) + \dots \\
+ \frac{\sqrt{n_l+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, \dots, n_l + 1; 3)u_{\eta_\zeta}(\vec{p}; l), n_3 \geq 1 \\
\dots\dots\dots \\
a_{\eta_\zeta}(\vec{p}; 0, \dots, 0, n_l) = c(\vec{p}; 0, \dots, 0, n_l; 1)u_{\eta_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, \dots, 0, n_l; 2)u_{\eta_\zeta}(\vec{p}; 2) \\
+ \dots + c(\vec{p}; 0, \dots, 0, n_l; l-1)u_{\eta_\zeta}(\vec{p}; l-1) + \frac{\sqrt{n_l}}{\sqrt{n_l}}c(\vec{p}; 0, \dots, 0, n_l; l)u_{\eta_\zeta}(\vec{p}; l), n_l = 2s \geq 1
\end{cases}
\end{cases}$$

引理2.1.3.

$$\begin{cases}
c(\vec{p}; n_1, n_2, \dots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l; 1), n_1 \geq 1 \dots \\
c(\vec{p}; n_1, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1; 1), n_1 \geq 1 \\
\Leftrightarrow \\
\sum_{n_1+\dots+n_l=2s} \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k)u_{\eta_\zeta}(\vec{p}; k) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) \\
= \sum_{n_1+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}}c(\vec{p}; n_1, n_2, \dots, n_l; 1) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1 + 1, n_2, \dots, n_l)
\end{cases}$$

$$+ \sum_{n_2 \cdots + n_l = 2s} \sum_{k=2}^l c(\vec{p}; 0, n_2, \cdots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \cdots, n_l)$$

证明:

$$\begin{cases} c(\vec{p}; n_1, n_2, \cdots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2 + 1, \cdots, n_l; 1), n_1 \geq 1 \cdots \\ c(\vec{p}; n_1, n_2, \cdots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \cdots, n_l + 1; 1), n_1 \geq 1 \end{cases}$$

 $\Leftrightarrow$ 

$$\begin{aligned} & \sum_{n_1 \cdots + n_l = 2s} \sum_{k=1}^l c(\vec{p}; n_1, n_2, \cdots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \cdots, n_l) \\ &= \sum_{n_1 \cdots + n_l = 2s}^{n_1 \neq 0} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, n_2, \cdots, n_l) \left[ \frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \cdots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) \right. \\ &+ \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2 + 1, \cdots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 2) + \cdots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \cdots, n_l + 1; 1) u_{\eta_\zeta}(\vec{p}; l) \left. \right] \\ &+ \sum_{n_1 \cdots + n_l = 2s}^{n_1=0} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \cdots, n_l) [c(\vec{p}; 0, n_2, \cdots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) \\ &+ c(\vec{p}; 0, n_2, \cdots, n_l; 2) u_{\eta_\zeta}(\vec{p}; 2) + \cdots + c(\vec{p}; 0, n_2, \cdots, n_l; l) u_{\eta_\zeta}(\vec{p}; l)] \\ &= \sum_{n_1 \cdots + n_l = 2s}^{1 \leq n_1 \leq 2s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, n_2, \cdots, n_l) c(\vec{p}; n_1, n_2, \cdots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) \\ &+ \sum_{n_1 \cdots + n_l = 2s}^{0 \leq n_1 \leq 2s-1, 1 \leq n_2 \leq 2s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1 + 1, n_2 - 1, \cdots, n_l) \frac{\sqrt{n_2}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 2) + \cdots \\ &+ \sum_{n_1 \cdots + n_l = 2s}^{0 \leq n_1 \leq 2s-1, 1 \leq n_l \leq 2s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1 + 1, n_2, \cdots, n_l - 1) \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \cdots, n_l; 1) u_{\eta_\zeta}(\vec{p}; l) \\ &+ \sum_{n_1 \cdots + n_l = 2s}^{n_1=0} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \cdots, n_l) [c(\vec{p}; 0, n_2, \cdots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) \\ &+ c(\vec{p}; 0, n_2, \cdots, n_l; 2) u_{\eta_\zeta}(\vec{p}; 2) + \cdots + c(\vec{p}; 0, n_2, \cdots, n_l; l) u_{\eta_\zeta}(\vec{p}; l)] \\ &= \sum_{n_1 \cdots + n_l = 2s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, n_2, \cdots, n_l) c(\vec{p}; n_1, n_2, \cdots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) \\ &+ \sum_{n_1 \cdots + n_l = 2s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1 + 1, n_2 - 1, \cdots, n_l) \frac{\sqrt{n_2}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 2) + \cdots \\ &+ \sum_{n_1 \cdots + n_l = 2s} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1 + 1, n_2, \cdots, n_l - 1) \frac{\sqrt{n_l}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdots, n_l; 1) u_{\eta_\zeta}(\vec{p}; l) \\ &+ \sum_{n_1 \cdots + n_l = 2s}^{n_1=0} \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \cdots, n_l) [c(\vec{p}; 0, n_2, \cdots, n_l; 2) u_{\eta_\zeta}(\vec{p}; 2) + \cdots + c(\vec{p}; 0, n_2, \cdots, n_l; l) u_{\eta_\zeta}(\vec{p}; l)] \\ &= \sum_{n_1 \cdots + n_l = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdots, n_l; 1) \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1 + 1, n_2, \cdots, n_l) \\ &+ \sum_{n_2 \cdots + n_l = 2s} \sum_{k=2}^l c(\vec{p}; 0, n_2, \cdots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \cdots, n_l) \end{aligned}$$

□

推论2.1.2.

$$\begin{cases} c(\vec{p}; n_1, n_2, \cdots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2 + 1, \cdots, n_l; 1), n_1 \geq 1 \cdots \\ c(\vec{p}; n_1, n_2, \cdots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \cdots, n_l + 1; 1), n_1 \geq 1 \\ c(\vec{p}; 0, n_2, \cdots, n_l; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3 + 1, \cdots, n_l; 2), n_2 \geq 1 \cdots \\ c(\vec{p}; 0, n_2, \cdots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \cdots, n_l + 1; 2), n_2 \geq 1 \\ \cdots \end{cases}$$

$$\begin{cases} c(\vec{p}; 0, \cdots, 0, n_{l-1}, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}}} c(\vec{p}; 0, \cdots, 0, n_{l-1} - 1, n_l + 1; l), n_{l-1} \geq 1 \end{cases}$$

 $\Leftrightarrow$ 

$$\sum_{n_1 \cdots + n_l = 2s} \sum_{k=1}^l c(\vec{p}; n_1, n_2, \cdots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) \underbrace{U_{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \cdots, n_l)$$



$$\begin{aligned}
&= \sum_{n_1+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \dots, n_l; 1) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1+1, n_2, \dots, n_l) \\
&+ \sum_{n_2+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} c(\vec{p}; 0, n_2, n_3, \dots, n_l; 2) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, n_2+1, n_3, \dots, n_l) \\
&+ \sum_{n_3+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} c(\vec{p}; 0, 0, n_3, \dots, n_l; 3) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3+1, \dots, n_l) \\
&+ \dots + \sum_{n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_l+1}} c(\vec{p}; 0, \dots, 0, n_l; l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, \dots, 0, n_l+1)
\end{aligned}$$

引理2.1.4.

$$\begin{aligned}
&\sum_{n_1+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \dots, n_l; 1) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1+1, n_2, \dots, n_l) \\
&+ \sum_{n_2+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} c(\vec{p}; 0, n_2, n_3, \dots, n_l; 2) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, n_2+1, n_3, \dots, n_l) \\
&+ \sum_{n_3+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} c(\vec{p}; 0, 0, n_3, \dots, n_l; 3) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3+1, \dots, n_l) \\
&+ \dots + \sum_{n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_l+1}} c(\vec{p}; 0, \dots, 0, n_l; l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, \dots, 0, n_l+1) \\
&= \sum_{n_1+\dots+n_l=2s+1} a(\vec{p}; n_1, n_2, \dots, n_l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_l) \\
&\begin{cases} a(\vec{p}; n_1, n_2, \dots, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \dots, n_l; 1), n_1 \neq 0 \\ a(\vec{p}; 0, n_2, \dots, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3, \dots, n_l; 2), n_2 \neq 0 \\ a(\vec{p}; 0, 0, n_3, \dots, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3-1, \dots, n_l; 3), n_3 \neq 0 \\ \dots \\ a(\vec{p}; 0, 0, \dots, 0, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_l}} c(\vec{p}; 0, \dots, 0, n_l-1; l), n_l \neq 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
\text{证明: } &\sum_{n_1+\dots+n_l=2s} \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) \\
&= \sum_{n_1+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \dots, n_l; 1) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1+1, n_2, \dots, n_l) \\
&+ \sum_{n_2+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} c(\vec{p}; 0, n_2, n_3, \dots, n_l; 2) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, n_2+1, n_3, \dots, n_l) \\
&+ \sum_{n_3+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} c(\vec{p}; 0, 0, n_3, \dots, n_l; 3) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3+1, \dots, n_l) \\
&+ \dots + \sum_{n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_l+1}} c(\vec{p}; 0, \dots, 0, n_l; l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, \dots, 0, n_l+1) \\
&= \sum_{n_1+\dots+n_l=2s+1}^{n_1 \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \dots, n_l; 1) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_l) \\
&+ \sum_{n_2+\dots+n_l=2s+1}^{n_1=0, n_2 \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3, \dots, n_l; 2) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, n_2, n_3, \dots, n_l) \\
&+ \sum_{n_3+\dots+n_l=2s+1}^{n_1=0, n_2=0, n_3 \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3-1, \dots, n_l; 3) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3, \dots, n_l) \\
&+ \dots + \sum_{n_l=2s+1}^{n_1=0, \dots, n_{l-1}=0, n_l \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_l}} c(\vec{p}; 0, \dots, 0, n_l-1; l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, \dots, 0, n_l) \\
&= \sum_{n_1+\dots+n_l=2s+1} a(\vec{p}; n_1, n_2, \dots, n_l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_l)
\end{aligned}$$

$$\begin{cases} a(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_l; 1), n_1 \neq 0 \\ a(\vec{p}; 0, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l; 2), n_2 \neq 0 \\ a(\vec{p}; 0, 0, n_3, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, \dots, n_l; 3), n_3 \neq 0 \\ \dots \\ a(\vec{p}; 0, 0, \dots, 0, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_l}} c(\vec{p}; 0, \dots, 0, n_l - 1; l), n_l \neq 0 \end{cases} \quad \square$$

### 2.1.2 几个推论

$$\begin{aligned} \text{推论2.1.3. } U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \dots, n_l) &= \frac{\sqrt{n_1}}{\sqrt{2s}} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}_{2s-1}}(\vec{p}; n_1 - 1, n_2, \dots, n_l) U_{\tau_\zeta}(\vec{p}; 1) \\ &+ \frac{\sqrt{n_2}}{\sqrt{2s}} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}_{2s-1}}(\vec{p}; n_1, n_2 - 1, \dots, n_l) U_{\tau_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_l}}{\sqrt{2s}} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}_{2s-1}}(\vec{p}; n_1, n_2, \dots, n_l - 1) U_{\tau_\zeta}(\vec{p}; l) \end{aligned}$$

$$\begin{aligned} \text{推论2.1.4. } U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}; 0, n_2, \dots, n_l) &= \frac{\sqrt{n_2}}{\sqrt{2s}} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}_{2s-1}}(\vec{p}; 0, n_2 - 1, \dots, n_l) U_{\tau_\zeta}(\vec{p}; 2) \\ &+ \frac{\sqrt{n_3}}{\sqrt{2s}} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}_{2s-1}}(\vec{p}; 0, n_2, n_3 - 1, \dots, n_l) U_{\tau_\zeta}(\vec{p}; 0, 0, 1, \dots, 0) + \dots + \frac{\sqrt{n_l}}{\sqrt{2s}} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}_{2s-1}}(\vec{p}; 0, n_2, \dots, n_l - 1) U_{\tau_\zeta}(\vec{p}; l) \end{aligned}$$

$$\begin{aligned} \text{推论2.1.5. } U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}_{2s+1}}(\vec{p}; n_1, n_2, \dots, n_l) &= \frac{\sqrt{n_1}}{\sqrt{2s+1}} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}; n_1 - 1, n_2, \dots, n_l) U_{\eta_\zeta}(\vec{p}; 1) \\ &+ \frac{\sqrt{n_2}}{\sqrt{2s+1}} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2 - 1, \dots, n_l) U_{\eta_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_l}}{\sqrt{2s+1}} U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \dots, n_l - 1) U_{\eta_\zeta}(\vec{p}; l) \end{aligned}$$

### 2.1.3 一个重要定理

#### 定理2.1.1.

$$\begin{aligned} \sum_{n_1 + \dots + n_l = 2s} a_{\eta_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}; n_1, \dots, n_l) &= \sum_{n_1 + \dots + n_l = 2s} a_{\tau_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}_{2s}}(\vec{p}; n_1, \dots, n_l) \\ \Leftrightarrow \\ a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) &= \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) \\ \sum_{n_1 + \dots + n_l}^{=2s} a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}; n_1, \dots, n_l) &= \sum_{n_1 + \dots + n_l}^{=2s+1} a(\vec{p}; n_1, n_2, \dots, n_l) U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}_{2s+1}}(\vec{p}; n_1, n_2, \dots, n_l) \\ \begin{cases} a(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_l; 1), n_1 \neq 0 \\ a(\vec{p}; 0, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l; 2), n_2 \neq 0 \\ a(\vec{p}; 0, 0, n_3, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, \dots, n_l; 3), n_3 \neq 0 \\ \dots \\ a(\vec{p}; 0, 0, \dots, 0, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_l}} c(\vec{p}; 0, \dots, 0, n_l - 1; l), n_l \neq 0 \end{cases} \end{aligned}$$

证明:

$$\begin{aligned} \sum_{n_1 + \dots + n_l = 2s} a_{\eta_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}; n_1, \dots, n_l) &= \sum_{n_1 + \dots + n_l = 2s} a_{\tau_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\underbrace{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}_{2s}}(\vec{p}; n_1, \dots, n_l) \\ \Leftrightarrow \\ \begin{cases} \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1 + 1, n_2, \dots, n_l) u_{\tau_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2 + 1, \dots, n_l) u_{\tau_\zeta}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} a_{[\eta_\zeta]}(\vec{p}; n_1, n_2, \dots, n_l + 1) u_{\tau_\zeta}(\vec{p}; l) = 0 \end{cases} \\ \Leftrightarrow \\ a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) &= \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) \\ \begin{cases} a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \dots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l; 1) u_{\eta_\zeta}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1; 1) u_{\eta_\zeta}(\vec{p}; l), n_1 \geq 1 \end{cases} \end{aligned}$$

$$\begin{cases} a_{\eta_\zeta}(\vec{p}; 0, n_2, \dots, n_l) = c(\vec{p}; 0, n_2, \dots, n_l; 1)u_{\eta_\zeta}(\vec{p}; 1) \\ + \frac{\sqrt{n_2}}{\sqrt{n_2}}c(\vec{p}; 0, n_2, \dots, n_l; 2)u_{\eta_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_l; 2)u_{\eta_\zeta}(\vec{p}; 3) + \dots \\ + \frac{\sqrt{n_l+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l + 1; 2)u_{\eta_\zeta}(\vec{p}; l), n_2 \geq 1 \\ a_{\eta_\zeta}(\vec{p}; 0, 0, n_3, \dots, n_l) = c(\vec{p}; 0, 0, n_3, \dots, n_l; 1)u_{\eta_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, 0, n_3, \dots, n_l; 2)u_{\eta_\zeta}(\vec{p}; 2) \\ + \frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3, n_4, \dots, n_l; 3)u_{\eta_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, n_4 + 1, \dots, n_l; 3)u_{\eta_\zeta}(\vec{p}; 4) + \dots \\ + \frac{\sqrt{n_l+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, \dots, n_l + 1; 3)u_{\eta_\zeta}(\vec{p}; l), n_3 \geq 1 \\ \dots \\ a_{\eta_\zeta}(\vec{p}; 0, \dots, 0, n_l) = c(\vec{p}; 0, \dots, 0, n_l; 1)u_{\eta_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, \dots, 0, n_l; 2)u_{\eta_\zeta}(\vec{p}; 2) \\ + \dots + c(\vec{p}; 0, \dots, 0, n_l; l-1)u_{\eta_\zeta}(\vec{p}; l-1) + \frac{\sqrt{n_l}}{\sqrt{n_l}}c(\vec{p}; 0, \dots, 0, n_l; l)u_{\eta_\zeta}(\vec{p}; l), n_l = 2s \geq 1 \end{cases}$$

$$\Leftrightarrow a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k)u_{\eta_\zeta}(\vec{p}; k)$$

$$\sum_{n_1+\dots+n_l}^{=2s} a_{\eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) = \sum_{n_1+\dots+n_l}^{=2s+1} a(\vec{p}; n_1, n_2, \dots, n_l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_l)$$

$$\begin{cases} a(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2, \dots, n_l; 1), n_1 \neq 0 \\ a(\vec{p}; 0, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l; 2), n_2 \neq 0 \\ a(\vec{p}; 0, 0, n_3, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, \dots, n_l; 3), n_3 \neq 0 \\ \dots \\ a(\vec{p}; 0, 0, \dots, 0, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_l}}c(\vec{p}; 0, \dots, 0, n_l - 1; l), n_l \neq 0 \end{cases}$$

□

## 2.2 N+1维时空中Bargmann-Wigner方程V-自旋基的性质

### 2.2.1 关于对称性条件的V-自旋基引理

$$\text{引理2.2.1. } \sum_{n_1+\dots+n_l}^{=2s} b_{\eta_\zeta}^+(\vec{p}; n_1, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) = \sum_{n_1+\dots+n_l}^{=2s} b_{\tau_\zeta}^+(\vec{p}; n_1, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l)$$

$$\Leftrightarrow \begin{cases} \frac{\sqrt{n_1+1}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}; n_1 + 1, n_2, \dots, n_l)v_{\tau_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2 + 1, \dots, n_l)v_{\tau_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}}b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l + 1)v_{\tau_\zeta]}(\vec{p}; l) = 0 \end{cases}$$

$$\text{证明: } \sum_{n_1+\dots+n_l}^{=2s} b_{\eta_\zeta}^+(\vec{p}; n_1, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) = \sum_{n_1+\dots+n_l}^{=2s} b_{\tau_\zeta}^+(\vec{p}; n_1, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l)$$

$$\Leftrightarrow \sum_{n_1+\dots+n_l}^{=2s} b_{\eta_\zeta}^+(\vec{p}; n_1, \dots, n_l) \left[ \frac{\sqrt{n_1}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1 - 1, n_2, \dots, n_l) V_{\tau_\zeta}(\vec{p}; 1) \right.$$

$$+ \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1, n_2 - 1, \dots, n_l) V_{\tau_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_l}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1, n_2, \dots, n_l - 1) V_{\tau_\zeta}(\vec{p}; l) \left. \right]$$

$$= \sum_{n_1+\dots+n_l}^{=2s} b_{\tau_\zeta}^+(\vec{p}; n_1, \dots, n_l) \left[ \frac{\sqrt{n_1}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1 - 1, n_2, \dots, n_l) V_{\eta_\zeta}(\vec{p}; 1) \right.$$

$$+ \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1, n_2 - 1, \dots, n_l) V_{\eta_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_l}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1, n_2, \dots, n_l - 1) V_{\eta_\zeta}(\vec{p}; l) \left. \right]$$

$$\begin{aligned}
& \left\{ \begin{aligned} & \frac{\sqrt{n_1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 2) \\ & + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1) v_{\tau_\zeta}(\vec{p}; l) = 0 \\ & \frac{\sqrt{n_1+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1 + 1, n_2 - 1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 2) \\ & + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2 - 1, \dots, n_l + 1) v_{\tau_\zeta}(\vec{p}; l) = 0 \dots \dots \\ & \frac{\sqrt{n_1+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1 + 1, n_2, \dots, n_l - 1) v_{\tau_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2 + 1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 2) \\ & + \dots + \frac{\sqrt{n_l}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l) v_{\tau_\zeta}(\vec{p}; l) = 0 \end{aligned} \right. \\
& \Leftrightarrow \left\{ \begin{aligned} & \frac{\sqrt{n_1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 2) \\ & + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1) v_{\tau_\zeta}(\vec{p}; l) = 0 \end{aligned} \right. \\
& \Leftrightarrow \left\{ \begin{aligned} & \frac{\sqrt{n_1+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1 + 1, n_2, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2 + 1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 2) \\ & + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l + 1) v_{\tau_\zeta}(\vec{p}; l) = 0 \end{aligned} \right. \quad \square
\end{aligned}$$

引理2.2.2.

$$\begin{aligned}
& \left\{ \begin{aligned} & b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k) + \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) \\ & \frac{\sqrt{n_1+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1 + 1, n_2, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2 + 1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 2) \\ & + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l + 1) v_{\tau_\zeta}(\vec{p}; l) = 0 \end{aligned} \right. \\
& \Leftrightarrow b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k), c(\vec{p}; n_1, n_2, \dots, n_l; k) = 0 \\
& \left\{ \begin{aligned} & b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{n_1}}{\sqrt{n_1}} d(\vec{p}; n_1, n_2, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 2) \\ & + \dots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1; 1) v_{\eta_\zeta}(\vec{p}; l), n_1 \geq 1 \\ & b_{[\eta_\zeta]}^+(\vec{p}; 0, n_2, \dots, n_l) = d(\vec{p}; 0, n_2, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) \\ & + \frac{\sqrt{n_2}}{\sqrt{n_2}} d(\vec{p}; 0, n_2, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 3) + \dots \\ & + \frac{\sqrt{n_l+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l + 1; 2) v_{\eta_\zeta}(\vec{p}; l), n_2 \geq 1 \\ & b_{[\eta_\zeta]}^+(\vec{p}; 0, 0, n_3, \dots, n_l) = d(\vec{p}; 0, 0, n_3, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + d(\vec{p}; 0, 0, n_3, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) \\ & + \frac{\sqrt{n_3}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3, n_4, \dots, n_l; 3) v_{\eta_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3 - 1, n_4 + 1, \dots, n_l; 3) v_{\eta_\zeta}(\vec{p}; 4) + \dots \\ & + \frac{\sqrt{n_l+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3 - 1, \dots, n_l + 1; 3) v_{\eta_\zeta}(\vec{p}; l), n_3 \geq 1 \\ & \dots \dots \dots \\ & b_{[\eta_\zeta]}^+(\vec{p}; 0, \dots, 0, n_l) = d(\vec{p}; 0, \dots, 0, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + d(\vec{p}; 0, \dots, 0, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) \\ & + \dots + d(\vec{p}; 0, \dots, 0, n_l; l - 1) v_{\eta_\zeta}(\vec{p}; l - 1) + \frac{\sqrt{n_l}}{\sqrt{n_l}} d(\vec{p}; 0, \dots, 0, n_l; l) v_{\eta_\zeta}(\vec{p}; l), n_l = 2s \geq 1 \end{aligned} \right.
\end{aligned}$$

证明:

$$\begin{aligned}
& \left\{ \begin{aligned} & b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k) + \sum_{k=1}^l c(\vec{p}; n_1, n_2, \dots, n_l; k) u_{\eta_\zeta}(\vec{p}; k) \\ & \frac{\sqrt{n_1+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1 + 1, n_2, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2 + 1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 2) \\ & + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l + 1) v_{\tau_\zeta}(\vec{p}; l) = 0 \end{aligned} \right. \\
& \Leftrightarrow b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k), c(\vec{p}; n_1, n_2, \dots, n_l; k) = 0 \\
& \left\{ \begin{aligned} & d(\vec{p}; n_1 + 1, n_2, \dots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1+1}} d(\vec{p}; n_1, n_2 + 1, \dots, n_l; 1) \dots \\ & d(\vec{p}; n_1 + 1, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1+1}} d(\vec{p}; n_1, n_2, \dots, n_l + 1; 1) \end{aligned} \right.
\end{aligned}$$

$$\begin{cases}
d(\vec{p}; 0, n_2 + 1, \dots, n_l; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2+1}} d(\vec{p}; 0, n_2, n_3 + 1, \dots, n_l; 2) \cdots \\
d(\vec{p}; 0, n_2 + 1, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_2+1}} d(\vec{p}; 0, n_2, \dots, n_l + 1; 2) \\
d(\vec{p}; 0, 0, n_3 + 1, \dots, n_l; 4) = \frac{\sqrt{n_4+1}}{\sqrt{n_3+1}} d(\vec{p}; 0, 0, n_3, n_4 + 1, \dots, n_l; 3) \cdots \\
d(\vec{p}; 0, 0, n_3 + 1, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_3+1}} d(\vec{p}; 0, 0, n_3, n_4, \dots, n_l + 1; 3) \\
\cdots \\
d(\vec{p}; 0, \dots, 0, n_{l-1} + 1, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}+1}} d(\vec{p}; 0, \dots, 0, n_{l-1}, n_l + 1; l-1) \\
\Leftrightarrow \\
b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k), c(\vec{p}; n_1, n_2, \dots, n_l; k) = 0 \\
\begin{cases}
b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{n_1}}{\sqrt{n_1}} d(\vec{p}; n_1, n_2, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 2) \\
+ \cdots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1; 1) v_{\eta_\zeta}(\vec{p}; l), n_1 \geq 1 \\
b_{\eta_\zeta}^+(\vec{p}; 0, n_2, \dots, n_l) = d(\vec{p}; 0, n_2, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) \\
+ \frac{\sqrt{n_2}}{\sqrt{n_2}} d(\vec{p}; 0, n_2, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 3) + \cdots \\
+ \frac{\sqrt{n_l+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l + 1; 2) v_{\eta_\zeta}(\vec{p}; l), n_2 \geq 1 \\
b_{\eta_\zeta}^+(\vec{p}; 0, 0, n_3, \dots, n_l) = d(\vec{p}; 0, 0, n_3, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + d(\vec{p}; 0, 0, n_3, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) \\
+ \frac{\sqrt{n_3}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3, n_4, \dots, n_l; 3) v_{\eta_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3 - 1, n_4 + 1, \dots, n_l; 3) v_{\eta_\zeta}(\vec{p}; 4) + \cdots \\
+ \frac{\sqrt{n_l+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3 - 1, \dots, n_l + 1; 3) v_{\eta_\zeta}(\vec{p}; l), n_3 \geq 1 \\
\cdots \cdots \cdots \\
b_{\eta_\zeta}^+(\vec{p}; 0, \dots, 0, n_l) = d(\vec{p}; 0, \dots, 0, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + d(\vec{p}; 0, \dots, 0, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) \\
+ \cdots + d(\vec{p}; 0, \dots, 0, n_l; l-1) v_{\eta_\zeta}(\vec{p}; l-1) + \frac{\sqrt{n_l}}{\sqrt{n_l}} d(\vec{p}; 0, \dots, 0, n_l; l) v_{\eta_\zeta}(\vec{p}; l), n_l = 2s \geq 1
\end{cases}
\end{cases}$$

□

推论2.2.1.  $b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k)$

$$\begin{cases}
d(\vec{p}; n_1, n_2, \dots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l; 1), n_1 \geq 1 \cdots \\
d(\vec{p}; n_1, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1; 1), n_1 \geq 1 \\
d(\vec{p}; 0, n_2, \dots, n_l; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_l; 2), n_2 \geq 1 \cdots \\
d(\vec{p}; 0, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l + 1; 2), n_2 \geq 1 \\
\cdots \\
d(\vec{p}; 0, \dots, 0, n_{l-1}, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}}} d(\vec{p}; 0, \dots, 0, n_{l-1} - 1, n_l + 1; l), n_{l-1} \geq 1 \\
\Leftrightarrow \\
b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k) \\
\begin{cases}
b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{n_1}}{\sqrt{n_1}} d(\vec{p}; n_1, n_2, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 2) \\
+ \cdots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1; 1) v_{\eta_\zeta}(\vec{p}; l), n_1 \geq 1 \\
b_{\eta_\zeta}^+(\vec{p}; 0, n_2, \dots, n_l) = d(\vec{p}; 0, n_2, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) \\
+ \frac{\sqrt{n_2}}{\sqrt{n_2}} d(\vec{p}; 0, n_2, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 3) + \cdots \\
+ \frac{\sqrt{n_l+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l + 1; 2) v_{\eta_\zeta}(\vec{p}; l), n_2 \geq 1 \\
b_{\eta_\zeta}^+(\vec{p}; 0, 0, n_3, \dots, n_l) = d(\vec{p}; 0, 0, n_3, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + d(\vec{p}; 0, 0, n_3, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) \\
+ \frac{\sqrt{n_3}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3, n_4, \dots, n_l; 3) v_{\eta_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3 - 1, n_4 + 1, \dots, n_l; 3) v_{\eta_\zeta}(\vec{p}; 4) + \cdots \\
+ \frac{\sqrt{n_l+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3 - 1, \dots, n_l + 1; 3) v_{\eta_\zeta}(\vec{p}; l), n_3 \geq 1 \\
\cdots \cdots \cdots
\end{cases}
\end{cases}$$

$$\begin{cases} b_{\eta_\zeta}^+(\vec{p}; 0, \dots, 0, n_l) = d(\vec{p}; 0, \dots, 0, n_l; 1)v_{\eta_\zeta}(\vec{p}; 1) + d(\vec{p}; 0, \dots, 0, n_l; 2)v_{\eta_\zeta}(\vec{p}; 2) \\ + \dots + d(\vec{p}; 0, \dots, 0, n_l; l-1)v_{\eta_\zeta}(\vec{p}; l-1) + \frac{\sqrt{n_l}}{\sqrt{n_l}}d(\vec{p}; 0, \dots, 0, n_l; l)v_{\eta_\zeta}(\vec{p}; l), n_l = 2s \geq 1 \end{cases}$$

引理2.2.3.

$$\begin{cases} d(\vec{p}; n_1, n_2, \dots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p}; n_1-1, n_2+1, \dots, n_l; 1), n_1 \geq 1 \dots \\ d(\vec{p}; n_1, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p}; n_1-1, n_2, \dots, n_l+1; 1), n_1 \geq 1 \end{cases}$$

⇔

$$\begin{aligned} & \sum_{n_1+\dots+n_l=2s} \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k)v_{\eta_\zeta}(\vec{p}; k)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) \\ &= \sum_{n_1+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}}d(\vec{p}; n_1, n_2, \dots, n_l; 1)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}; n_1+1, n_2, \dots, n_l) \\ &+ \sum_{n_2+\dots+n_l=2s} \sum_{k=2}^l d(\vec{p}; 0, n_2, \dots, n_l; k)v_{\eta_\zeta}(\vec{p}; k)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \dots, n_l) \end{aligned}$$

证明:

$$\begin{cases} d(\vec{p}; n_1, n_2, \dots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p}; n_1-1, n_2+1, \dots, n_l; 1), n_1 \geq 1 \dots \\ d(\vec{p}; n_1, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p}; n_1-1, n_2, \dots, n_l+1; 1), n_1 \geq 1 \end{cases}$$

⇔

$$\begin{aligned} & \sum_{n_1+\dots+n_l=2s} \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k)v_{\eta_\zeta}(\vec{p}; k)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) \\ &= \sum_{n_1+\dots+n_l=2s}^{n_1 \neq 0} \underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; n_1, n_2, \dots, n_l) \left[ \frac{\sqrt{n_1}}{\sqrt{n_1}}d(\vec{p}; n_1, n_2, \dots, n_l; 1)v_{\eta_\zeta}(\vec{p}; 1) \right. \\ &+ \frac{\sqrt{n_2+1}}{\sqrt{n_1}}d(\vec{p}; n_1-1, n_2+1, \dots, n_l; 1)v_{\eta_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p}; n_1-1, n_2, \dots, n_l+1; 1)v_{\eta_\zeta}(\vec{p}; l) \left. \right] \\ &+ \sum_{n_1+\dots+n_l=2s}^{n_1=0} \underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \dots, n_l) [d(\vec{p}; 0, n_2, \dots, n_l; 1)v_{\eta_\zeta}(\vec{p}; 1) \\ &+ d(\vec{p}; 0, n_2, \dots, n_l; 2)v_{\eta_\zeta}(\vec{p}; 2) + \dots + d(\vec{p}; 0, n_2, \dots, n_l; l)v_{\eta_\zeta}(\vec{p}; l)] \\ &= \sum_{n_1+\dots+n_l=2s}^{1 \leq n_1 \leq 2s} \underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; n_1, n_2, \dots, n_l) d(\vec{p}; n_1, n_2, \dots, n_l; 1)v_{\eta_\zeta}(\vec{p}; 1) \\ &+ \sum_{n_1+\dots+n_l=2s}^{0 \leq n_1 \leq 2s-1, 1 \leq n_2 \leq 2s} \underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; n_1+1, n_2-1, \dots, n_l) \frac{\sqrt{n_2}}{\sqrt{n_1+1}}d(\vec{p}; n_1, n_2, \dots, n_l; 1)v_{\eta_\zeta}(\vec{p}; 2) + \dots \\ &+ \sum_{n_1+\dots+n_l=2s}^{0 \leq n_1 \leq 2s-1, 1 \leq n_l \leq 2s} \underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; n_1+1, n_2, \dots, n_l-1) \frac{\sqrt{n_l+1}}{\sqrt{n_1}}d(\vec{p}; n_1, n_2, \dots, n_l; 1)v_{\eta_\zeta}(\vec{p}; l) \\ &+ \sum_{n_1+\dots+n_l=2s}^{n_1=0} \underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \dots, n_l) [d(\vec{p}; 0, n_2, \dots, n_l; 1)v_{\eta_\zeta}(\vec{p}; 1) \\ &+ d(\vec{p}; 0, n_2, \dots, n_l; 2)v_{\eta_\zeta}(\vec{p}; 2) + \dots + d(\vec{p}; 0, n_2, \dots, n_l; l)v_{\eta_\zeta}(\vec{p}; l)] \\ &= \sum_{n_1+\dots+n_l=2s} \underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; n_1, n_2, \dots, n_l) d(\vec{p}; n_1, n_2, \dots, n_l; 1)v_{\eta_\zeta}(\vec{p}; 1) \\ &+ \sum_{n_1+\dots+n_l=2s} \underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; n_1+1, n_2-1, \dots, n_l) \frac{\sqrt{n_2}}{\sqrt{n_1+1}}d(\vec{p}; n_1, n_2, \dots, n_l; 1)v_{\eta_\zeta}(\vec{p}; 2) + \dots \\ &+ \sum_{n_1+\dots+n_l=2s} \underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; n_1+1, n_2, \dots, n_l-1) \frac{\sqrt{n_l}}{\sqrt{n_1+1}}d(\vec{p}; n_1, n_2, \dots, n_l; 1)v_{\eta_\zeta}(\vec{p}; l) \\ &+ \sum_{n_1+\dots+n_l=2s}^{n_1=0} \underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \dots, n_l) [d(\vec{p}; 0, n_2, \dots, n_l; 2)v_{\eta_\zeta}(\vec{p}; 2) + \dots + d(\vec{p}; 0, n_2, \dots, n_l; l)v_{\eta_\zeta}(\vec{p}; l)] \\ &= \sum_{n_1+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}}d(\vec{p}; n_1, n_2, \dots, n_l; 1)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta\eta_\zeta}}_{2s+1}(\vec{p}; n_1+1, n_2, \dots, n_l) \\ &+ \sum_{n_2+\dots+n_l=2s} \sum_{k=2}^l d(\vec{p}; 0, n_2, \dots, n_l; k)v_{\eta_\zeta}(\vec{p}; k)\underbrace{V_{\lambda_\zeta\mu_\zeta \dots \sigma_\zeta\tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \dots, n_l) \end{aligned}$$

□

推论2.2.2.

$$\begin{cases}
d(\vec{p}; n_1, n_2, \dots, n_l; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_l; 1), n_1 \geq 1 \dots \\
d(\vec{p}; n_1, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2, \dots, n_l + 1; 1), n_1 \geq 1 \\
d(\vec{p}; 0, n_2, \dots, n_l; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_l; 2), n_2 \geq 1 \dots \\
d(\vec{p}; 0, n_2, \dots, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l + 1; 2), n_2 \geq 1 \\
\dots \\
d(\vec{p}; 0, \dots, 0, n_{l-1}, n_l; l) = \frac{\sqrt{n_l+1}}{\sqrt{n_{l-1}}} d(\vec{p}; 0, \dots, 0, n_{l-1} - 1, n_l + 1; l), n_{l-1} \geq 1
\end{cases}$$

$$\begin{aligned}
&\Leftrightarrow \sum_{n_1+\dots+n_l=2s} \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) \\
&= \sum_{n_1+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} d(\vec{p}; n_1, n_2, \dots, n_l; 1) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1 + 1, n_2, \dots, n_l) \\
&+ \sum_{n_2+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} d(\vec{p}; 0, n_2, n_3, \dots, n_l; 2) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, n_2 + 1, n_3, \dots, n_l) \\
&+ \sum_{n_3+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} d(\vec{p}; 0, 0, n_3, \dots, n_l; 3) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3 + 1, \dots, n_l) \\
&+ \dots + \sum_{n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_l+1}} d(\vec{p}; 0, \dots, 0, n_l; l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, \dots, 0, n_l + 1)
\end{aligned}$$

引理2.2.4.

$$\begin{aligned}
&\sum_{n_1+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} d(\vec{p}; n_1, n_2, \dots, n_l; 1) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1 + 1, n_2, \dots, n_l) \\
&+ \sum_{n_2+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} d(\vec{p}; 0, n_2, n_3, \dots, n_l; 2) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, n_2 + 1, n_3, \dots, n_l) \\
&+ \sum_{n_3+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} d(\vec{p}; 0, 0, n_3, \dots, n_l; 3) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3 + 1, \dots, n_l) \\
&+ \dots + \sum_{n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_l+1}} d(\vec{p}; 0, \dots, 0, n_l; l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, \dots, 0, n_l + 1) \\
&= \sum_{n_1+\dots+n_l=2s+1} b^+(\vec{p}; n_1, n_2, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_l) \\
&\begin{cases}
b^+(\vec{p}; n_1, n_2, \dots, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2, \dots, n_l; 1), n_1 \neq 0 \\
b^+(\vec{p}; 0, n_2, \dots, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l; 2), n_2 \neq 0 \\
b^+(\vec{p}; 0, 0, n_3, \dots, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3 - 1, \dots, n_l; 3), n_3 \neq 0 \\
\dots \\
b^+(\vec{p}; 0, 0, \dots, 0, n_l) := \frac{\sqrt{2s+1}}{\sqrt{n_l}} d(\vec{p}; 0, \dots, 0, n_l - 1; l), n_l \neq 0
\end{cases}
\end{aligned}$$

$$\begin{aligned}
\text{证明: } &\sum_{n_1+\dots+n_l=2s} \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) \\
&= \sum_{n_1+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} d(\vec{p}; n_1, n_2, \dots, n_l; 1) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1 + 1, n_2, \dots, n_l) \\
&+ \sum_{n_2+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} d(\vec{p}; 0, n_2, n_3, \dots, n_l; 2) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, n_2 + 1, n_3, \dots, n_l) \\
&+ \sum_{n_3+\dots+n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} d(\vec{p}; 0, 0, n_3, \dots, n_l; 3) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3 + 1, \dots, n_l) \\
&+ \dots + \sum_{n_l=2s} \frac{\sqrt{2s+1}}{\sqrt{n_l+1}} d(\vec{p}; 0, \dots, 0, n_l; l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, \dots, 0, n_l + 1)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n_1+\dots+n_l=2s+1}^{n_1 \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2, \dots, n_l; 1) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_l) \\
&+ \sum_{n_2+\dots+n_l=2s+1}^{n_1=0, n_2 \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l; 2) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, n_2, n_3, \dots, n_l) \\
&+ \sum_{n_3+\dots+n_l=2s+1}^{n_1=0, n_2=0, n_3 \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3 - 1, \dots, n_l; 3) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3, \dots, n_l) \\
&+ \dots + \sum_{n_l=2s+1}^{n_1=0, \dots, n_{l-1}=0, n_l \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_l}} d(\vec{p}; 0, \dots, 0, n_l - 1; l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; 0, \dots, 0, n_l) \\
&= \sum_{n_1+\dots+n_l=2s+1} b^+(\vec{p}; n_1, n_2, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_l) \\
&\begin{cases} b^+(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2, \dots, n_l; 1), n_1 \neq 0 \\ b^+(\vec{p}; 0, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l; 2), n_2 \neq 0 \\ b^+(\vec{p}; 0, 0, n_3, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3 - 1, \dots, n_l; 3), n_3 \neq 0 \\ \dots \\ b^+(\vec{p}; 0, 0, \dots, 0, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_l}} d(\vec{p}; 0, \dots, 0, n_l - 1; l), n_l \neq 0 \end{cases} \quad \square
\end{aligned}$$

### 2.2.2 几个推论

**推论2.2.3.** 
$$\underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{n_1}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1 - 1, n_2, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 1)$$

$$+ \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1, n_2 - 1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_l}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; n_1, n_2, \dots, n_l - 1) v_{\tau_\zeta}(\vec{p}; l)$$

**推论2.2.4.** 
$$\underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; 0, n_2, \dots, n_l) = \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; 0, n_2 - 1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 2)$$

$$+ \frac{\sqrt{n_3}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; 0, n_2, n_3 - 1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 0, 0, 1, \dots, 0) + \dots + \frac{\sqrt{n_l}}{\sqrt{2s}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}; 0, n_2, \dots, n_l - 1) v_{\tau_\zeta}(\vec{p}; l)$$

**推论2.2.5.** 
$$\underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{n_1}}{\sqrt{2s+1}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1 - 1, n_2, \dots, n_l) v_{\eta_\zeta}(\vec{p}; 1)$$

$$+ \frac{\sqrt{n_2}}{\sqrt{2s+1}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, n_2 - 1, \dots, n_l) v_{\eta_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_l}}{\sqrt{2s+1}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, n_2, \dots, n_l - 1) v_{\eta_\zeta}(\vec{p}; l)$$

### 2.2.3 一个重要定理

#### 定理2.2.1.

$$\begin{aligned}
\sum_{n_1+\dots+n_l=2s} b_{\eta_\zeta}^+(\vec{p}; n_1, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) &= \sum_{n_1+\dots+n_l=2s} b_{\tau_\zeta}^+(\vec{p}; n_1, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) \\
\Leftrightarrow \\
b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) &= \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k) \\
\sum_{n_1+\dots+n_l}^{=2s} b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) &= \sum_{n_1+\dots+n_l}^{=2s+1} b^+(\vec{p}; n_1, n_2, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_l) \\
\begin{cases} b^+(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_1}} d(\vec{p}; n_1 - 1, n_2, \dots, n_l; 1), n_1 \neq 0 \\ b^+(\vec{p}; 0, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2 - 1, n_3, \dots, n_l; 2), n_2 \neq 0 \\ b^+(\vec{p}; 0, 0, n_3, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3 - 1, \dots, n_l; 3), n_3 \neq 0 \\ \dots \\ b^+(\vec{p}; 0, 0, \dots, 0, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_l}} d(\vec{p}; 0, \dots, 0, n_l - 1; l), n_l \neq 0 \end{cases}
\end{aligned}$$

证明:

$$\sum_{n_1+\dots+n_l=2s} b_{\eta_\zeta}^+(\vec{p}; n_1, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) = \sum_{n_1+\dots+n_l=2s} b_{\tau_\zeta}^+(\vec{p}; n_1, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l)$$



$$\begin{aligned}
& \Leftrightarrow \begin{cases} \frac{\sqrt{n_1+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1+1, n_2, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2+1, \dots, n_l) v_{\tau_\zeta}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{2s}} b_{[\eta_\zeta]}^+(\vec{p}; n_1, n_2, \dots, n_l+1) v_{\tau_\zeta}(\vec{p}; l) = 0 \end{cases} \\
& \Leftrightarrow b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k) \\
& \begin{cases} b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{n_1}}{\sqrt{n_1}} d(\vec{p}; n_1, n_2, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} d(\vec{p}; n_1-1, n_2+1, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_l+1}}{\sqrt{n_1}} d(\vec{p}; n_1-1, n_2, \dots, n_l+1; 1) v_{\eta_\zeta}(\vec{p}; l), n_1 \geq 1 \end{cases} \\
& \begin{cases} b_{\eta_\zeta}^+(\vec{p}; 0, n_2, \dots, n_l) = d(\vec{p}; 0, n_2, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) \\ + \frac{\sqrt{n_2}}{\sqrt{n_2}} d(\vec{p}; 0, n_2, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2-1, n_3+1, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 3) + \dots \\ + \frac{\sqrt{n_l+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2-1, n_3, \dots, n_l+1; 2) v_{\eta_\zeta}(\vec{p}; l), n_2 \geq 1 \end{cases} \\
& \begin{cases} b_{\eta_\zeta}^+(\vec{p}; 0, 0, n_3, \dots, n_l) = d(\vec{p}; 0, 0, n_3, \dots, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + d(\vec{p}; 0, 0, n_3, \dots, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) \\ + \frac{\sqrt{n_3}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3, n_4, \dots, n_l; 3) v_{\eta_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3-1, n_4+1, \dots, n_l; 3) v_{\eta_\zeta}(\vec{p}; 4) + \dots \\ + \frac{\sqrt{n_l+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3-1, \dots, n_l+1; 3) v_{\eta_\zeta}(\vec{p}; l), n_3 \geq 1 \end{cases} \\
& \dots\dots\dots \\
& \begin{cases} b_{\eta_\zeta}^+(\vec{p}; 0, \dots, 0, n_l) = d(\vec{p}; 0, \dots, 0, n_l; 1) v_{\eta_\zeta}(\vec{p}; 1) + d(\vec{p}; 0, \dots, 0, n_l; 2) v_{\eta_\zeta}(\vec{p}; 2) \\ + \dots + d(\vec{p}; 0, \dots, 0, n_l; l-1) v_{\eta_\zeta}(\vec{p}; l-1) + \frac{\sqrt{n_l}}{\sqrt{n_l}} d(\vec{p}; 0, \dots, 0, n_l; l) v_{\eta_\zeta}(\vec{p}; l), n_l = 2s \geq 1 \end{cases} \\
& \Leftrightarrow \\
& b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) = \sum_{k=1}^l d(\vec{p}; n_1, n_2, \dots, n_l; k) v_{\eta_\zeta}(\vec{p}; k) \\
& \sum_{n_1+\dots+n_l=2s}^{=2s} b_{\eta_\zeta}^+(\vec{p}; n_1, n_2, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) = \sum_{n_1+\dots+n_l=2s+1}^{=2s+1} b^+(\vec{p}; n_1, n_2, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_l) \\
& \begin{cases} b^+(\vec{p}; n_1, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_1}} d(\vec{p}; n_1-1, n_2, \dots, n_l; 1), n_1 \neq 0 \\ b^+(\vec{p}; 0, n_2, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_2}} d(\vec{p}; 0, n_2-1, n_3, \dots, n_l; 2), n_2 \neq 0 \\ b^+(\vec{p}; 0, 0, n_3, \dots, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_3}} d(\vec{p}; 0, 0, n_3-1, \dots, n_l; 3), n_3 \neq 0 \\ \dots \\ b^+(\vec{p}; 0, 0, \dots, 0, n_l) = \frac{\sqrt{2s+1}}{\sqrt{n_l}} d(\vec{p}; 0, \dots, 0, n_l-1; l), n_l \neq 0 \end{cases} \quad \square
\end{aligned}$$

## 2.3 用数学归纳法严格求解N+1维时空中Bargmann-Wigner方程的平面波解

### 定理2.3.1.

$$(\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(x) = \frac{1}{(2s)!} \psi_{\{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta\}}(x)$$

$\Leftrightarrow$

$$\begin{aligned}
& \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{n_1+\dots+n_l=2s}^{=2s} \frac{m^s}{\sqrt{E}} \\
& [a(\vec{p}; n_1, \dots, n_l) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) e^{ip \cdot x} + b^+(\vec{p}; n_1, \dots, n_l) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_l) e^{-ip \cdot x}] d^3 \vec{p}
\end{aligned}$$

证明: 采用数学归纳法证明此定理。

第一步:  $s' = 1/2$ 时成立:

$$(\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta}(x) = 0, \psi_{\lambda_\zeta}(x) = \psi_{\lambda_\zeta}(x)$$

$\Leftrightarrow$

$$\psi_{\lambda_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{n_1+\dots+n_l=1}^{=1} \frac{m^{1/2}}{\sqrt{E}} [a(\vec{p}; n_1, \dots, n_l) U_{\lambda_\zeta}(\vec{p}; n_1, \dots, n_l) e^{ip \cdot x} + b^+(\vec{p}, h) V_{\lambda_\zeta}(\vec{p}; n_1, \dots, n_l) e^{-ip \cdot x}] d^3 \vec{p}$$

第二步: 假设  $s' = s$ 时成立:

$$(\gamma^a \partial_a + m)_{\kappa_\zeta} \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{\lambda_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(x) = \frac{1}{(2s)!} \psi_{\{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta\}}(x)$$

$\Leftrightarrow$

$$\psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{n_1+\dots+n_l=2s} \frac{m^s}{\sqrt{E}}$$

$$[a(\vec{p}; n_1, \dots, n_l) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) e^{ip \cdot x} + b^+(\vec{p}; n_1, \dots, n_l) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) e^{-ip \cdot x}] d^3 \vec{p}$$

第三步:  $s' = s + 1/2$ 时:

$$(\gamma^a \partial_a + m)_{\kappa_\zeta} \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}^{\lambda_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}(x) = \frac{1}{(2s+1)!} \psi_{\{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta\}}(x)$$

$\Leftrightarrow$

$$\left\{ \begin{array}{l} \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{n_1+\dots+n_l=2s+1} \frac{m^s}{\sqrt{E}} \\ [a_{\eta_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) e^{ip \cdot x} + b_{\eta_\zeta}^+(\vec{p}; n_1, \dots, n_l) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) e^{-ip \cdot x}] d^3 \vec{p} \\ \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}(x) = \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta \tau_\zeta}(x) \end{array} \right.$$

$\Leftrightarrow$

$$\left\{ \begin{array}{l} \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{n_1+\dots+n_l=2s+1} \frac{m^s}{\sqrt{E}} \\ [a_{\eta_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) e^{ip \cdot x} + b_{\eta_\zeta}^+(\vec{p}; n_1, \dots, n_l) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) e^{-ip \cdot x}] d^3 \vec{p} \\ \int_{\vec{p}=-\infty}^{+\infty} \sum_{n_1+\dots+n_l=2s} \frac{m^s}{\sqrt{E}} \\ [a_{\eta_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) e^{ip \cdot x} + b_{\eta_\zeta}^+(\vec{p}; n_1, \dots, n_l) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) e^{-ip \cdot x}] d^3 \vec{p} \\ = \int_{\vec{p}=-\infty}^{+\infty} \sum_{n_1+\dots+n_l=2s} \frac{m^s}{\sqrt{E}} \\ [a_{\tau_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}(\vec{p}; n_1, \dots, n_l) e^{ip \cdot x} + b_{\tau_\zeta}^+(\vec{p}; n_1, \dots, n_l) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}(\vec{p}; n_1, \dots, n_l) e^{-ip \cdot x}] d^3 \vec{p} \end{array} \right.$$

$\Leftrightarrow$

$$\left\{ \begin{array}{l} \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{n_1+\dots+n_l=2s+1} \frac{m^s}{\sqrt{E}} \\ [a_{\eta_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) e^{ip \cdot x} + b_{\eta_\zeta}^+(\vec{p}; n_1, \dots, n_l) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) e^{-ip \cdot x}] d^3 \vec{p} \\ \sum_{n_1+\dots+n_l=2s} a_{\eta_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) = \sum_{n_1+\dots+n_l=2s} a_{\tau_\zeta}(\vec{p}; n_1, \dots, n_l) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}(\vec{p}; n_1, \dots, n_l) \\ \sum_{n_1+\dots+n_l=2s} b_{\eta_\zeta}^+(\vec{p}; n_1, \dots, n_l) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}; n_1, \dots, n_l) = \sum_{n_1+\dots+n_l=2s} b_{\tau_\zeta}^+(\vec{p}; n_1, \dots, n_l) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \eta_\zeta}(\vec{p}; n_1, \dots, n_l) \end{array} \right.$$

$\Leftrightarrow$

$$\left\{ \begin{array}{l} \psi_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=s}^{-s} \frac{m^{s+1/2}}{\sqrt{E}} \\ [a(\vec{p}; n_1, n_2, \dots, n_l) U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) e^{ip \cdot x} + b^+(\vec{p}; n_1, n_2, \dots, n_l) V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta \eta_\zeta}(\vec{p}; n_1, n_2, \dots, n_l) e^{-ip \cdot x}] d^3 \vec{p} \end{array} \right.$$

此步证明了  $s' = s + 1/2$  时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

### 3 K-G方程平面波解的直观解法

#### 3.1 直观解法的数学基础

引理3.1.1.

$$\begin{cases} \sum_{h=n}^{-n} [a_d(\vec{p}, h) \varepsilon_{ab\dots}(\vec{p}, h)] = \sum_{h=n}^{-n} [a_a(\vec{p}, h) \varepsilon_{db\dots}(\vec{p}, h)] \\ (p^c p_c + m^2) \varepsilon_{ab\dots}(\vec{p}, h) = 0, \delta^{ab} \varepsilon_{ab\dots}(\vec{p}, h) = 0, p^a \varepsilon_{ab\dots}(\vec{p}, h) = 0, \varepsilon_{ab\dots}(\vec{p}, h) = \frac{1}{n!} \varepsilon_{\{ab\dots\}}(\vec{p}, h) \\ a_d(\vec{p}, h) \varepsilon_{ab\dots}(\vec{p}, h) = a_a(\vec{p}, h) \varepsilon_{db\dots}(\vec{p}, h), a_d(\vec{p}, h) := a(\vec{p}, h; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, h; -1) \varepsilon_d(\vec{p}, -1) \end{cases}$$

引理3.1.2.

$$\begin{cases} \varepsilon_{a\dots bc}(\vec{p}, h) = \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a\dots b}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{a\dots b}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a\dots b}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1) \\ \sum_{h=n}^{-n} [a_d(\vec{p}, h) \varepsilon_{ab\dots}(\vec{p}, h)] = \sum_{h=n}^{-n} [a_a(\vec{p}, h) \varepsilon_{db\dots}(\vec{p}, h)], -n \leq h \leq n \\ \varepsilon_{a\dots bc}(\vec{p}, h) = \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a\dots b}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{a\dots b}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a\dots b}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1) \\ a(\vec{p}, h; 1) = \frac{\sqrt{C_{n+h+1}^2}}{\sqrt{C_{n+h}^1 C_{n-h}^1}} a(\vec{p}, h+1; 0), a(\vec{p}, h; -1) = \frac{\sqrt{C_{n-h+1}^2}}{\sqrt{C_{n+h}^1 C_{n-h}^1}} a(\vec{p}, h-1; 0) \\ a(\vec{p}, n; -1) = \frac{1}{\sqrt{C_{2n}^2}} a(\vec{p}, n-2; 1), a(\vec{p}, -n; 1) = \frac{1}{\sqrt{C_{2n}^2}} a(\vec{p}, -n+2; -1) \\ -n+1 \leq h \leq n-1 \end{cases}$$

证明:

$$\begin{aligned} & \left\{ \sum_{h=n}^{-n} [a_d(\vec{p}, h) \varepsilon_{ab\dots}(\vec{p}, h)] = \sum_{h=n}^{-n} [a_a(\vec{p}, h) \varepsilon_{db\dots}(\vec{p}, h)] \right. \\ & \left. a_d(\vec{p}, h) := a(\vec{p}, h; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, h; -1) \varepsilon_d(\vec{p}, -1) \right. \\ & \Leftrightarrow \sum_{h=n}^{-n} [a(\vec{p}, h; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, h; -1) \varepsilon_d(\vec{p}, -1)] \varepsilon_{ab\dots}(\vec{p}, h) \\ & = \sum_{h=n}^{-n} [a(\vec{p}, h; 1) \varepsilon_a(\vec{p}, 1) + a(\vec{p}, h; 0) \varepsilon_a(\vec{p}, 0) + a(\vec{p}, h; -1) \varepsilon_a(\vec{p}, -1)] \varepsilon_{db\dots}(\vec{p}, h) \\ & \Leftrightarrow \sum_{h=n}^{-n} [a(\vec{p}, h; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, h; -1) \varepsilon_d(\vec{p}, -1)] \\ & \left[ \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h-1) \varepsilon_a(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h) \varepsilon_a(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h+1) \varepsilon_a(\vec{p}, -1) \right] \\ & = \sum_{h=n}^{-n} [a(\vec{p}, h; 1) \varepsilon_a(\vec{p}, 1) + a(\vec{p}, h; 0) \varepsilon_a(\vec{p}, 0) + a(\vec{p}, h; -1) \varepsilon_a(\vec{p}, -1)] \\ & \left[ \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h-1) \varepsilon_d(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h) \varepsilon_d(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h+1) \varepsilon_d(\vec{p}, -1) \right] \\ & \Leftrightarrow \begin{cases} \sum_{h=n}^{-n} \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h-1) a(\vec{p}, h; 1) = \sum_{h=n}^{-n} a(\vec{p}, h; 1) \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h-1) \\ \sum_{h=n}^{-n} \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h-1) a(\vec{p}, h; 0) = \sum_{h=n}^{-n} a(\vec{p}, h; 1) \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h) \\ \sum_{h=n}^{-n} \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h-1) a(\vec{p}, h; -1) = \sum_{h=n}^{-n} a(\vec{p}, h; 1) \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h+1) \\ \sum_{h=n}^{-n} \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h) a(\vec{p}, h; 1) = \sum_{h=n}^{-n} a(\vec{p}, h; 0) \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h-1) \\ \sum_{h=n}^{-n} \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h) a(\vec{p}, h; 0) = \sum_{h=n}^{-n} a(\vec{p}, h; 0) \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h) \\ \sum_{h=n}^{-n} \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h) a(\vec{p}, h; -1) = \sum_{h=n}^{-n} a(\vec{p}, h; 0) \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b\dots}(\vec{p}, h+1) \end{cases} \end{aligned}$$

$$\left\{ \begin{aligned} \sum_{h=n}^{-n} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h+1) a(\vec{p}, h; 1) &= \sum_{h=n}^{-n} a(\vec{p}, h; -1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h-1) \\ \sum_{h=n}^{-n} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h+1) a(\vec{p}, h; 0) &= \sum_{h=n}^{-n} a(\vec{p}, h; -1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) \\ \sum_{h=n}^{-n} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h+1) a(\vec{p}, h; -1) &= \sum_{h=n}^{-n} a(\vec{p}, h; -1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h+1) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \sum_{h=n}^{-n} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h-1) a(\vec{p}, h; 0) &= \sum_{h=n}^{-n} a(\vec{p}, h; 1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) \\ \sum_{h=n}^{-n} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h-1) a(\vec{p}, h; -1) &= \sum_{h=n}^{-n} a(\vec{p}, h; 1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h+1) \\ \sum_{h=n}^{-n} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h+1) a(\vec{p}, h; 0) &= \sum_{h=n}^{-n} a(\vec{p}, h; -1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \sum_{h=n-1}^{-n-1} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) a(\vec{p}, h+1; 0) &= \sum_{h=n}^{-n} a(\vec{p}, h; 1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) \\ \sum_{h=n-1}^{-n-1} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) a(\vec{p}, h+1; -1) &= \sum_{h=n+1}^{-n+1} a(\vec{p}, h-1; 1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) \\ \sum_{h=n+1}^{-n+1} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) a(\vec{p}, h-1; 0) &= \sum_{h=n}^{-n} a(\vec{p}, h; -1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \sum_{h=n-1}^{-n+1} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) a(\vec{p}, h+1; 0) &= \sum_{h=n-1}^{-n+1} a(\vec{p}, h; 1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) \\ \sum_{h=n-1}^{-n+1} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) a(\vec{p}, h+1; -1) &= \sum_{h=n-1}^{-n+1} a(\vec{p}, h-1; 1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) \\ \sum_{h=n-1}^{-n+1} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) a(\vec{p}, h-1; 0) &= \sum_{h=n-1}^{-n+1} a(\vec{p}, h; -1) \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{b..}(\vec{p}, h) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} a(\vec{p}, h+1; 0) &= \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} a(\vec{p}, h; 1) \\ \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} a(\vec{p}, h+1; -1) &= \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} a(\vec{p}, h-1; 1) \\ \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} a(\vec{p}, h-1; 0) &= \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} a(\vec{p}, h; -1) \end{aligned} \right.$$

$$-n+1 \leq h \leq n-1$$

$$\left\{ \begin{aligned} a(\vec{p}, h; 1) &= \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} a(\vec{p}, h+1; 0), a(\vec{p}, h; -1) = \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} a(\vec{p}, h-1; 0) \\ a(\vec{p}, n; -1) &= \frac{1}{\sqrt{C_{2n}^2}} a(\vec{p}, n-2; 1), a(\vec{p}, -n; 1) = \frac{1}{\sqrt{C_{2n}^2}} a(\vec{p}, -n+2; -1) \\ -n+1 &\leq h \leq n-1 \end{aligned} \right. \quad \square$$

推论3.1.1.

$$\left\{ \begin{aligned} \varepsilon_{a..bc}(\vec{p}, h) &= \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a..b}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a..b}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a..b}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1) \\ \sum_{h=n}^{-n} [a_d(\vec{p}, h) \varepsilon_{ab..}(\vec{p}, h)] &= \sum_{h=n}^{-n} [a_a(\vec{p}, h) \varepsilon_{db..}(\vec{p}, h)], -n \leq h \leq n \end{aligned} \right.$$

$$\left\{ \begin{aligned} \varepsilon_{a..bc}(\vec{p}, h) &= \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a..b}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a..b}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a..b}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1) \\ a(\vec{p}, h; 1) &= \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} a(\vec{p}, h+1; 0), a(\vec{p}, h; -1) = \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n}^2}} a(\vec{p}, h-1; 0) \\ a(\vec{p}, n; -1) &= \frac{1}{\sqrt{C_{2n}^2}} a(\vec{p}, n-2; 1) = \frac{1}{\sqrt{4n}} a(\vec{p}, n-1; 0), a(\vec{p}, -n; 1) = \frac{1}{\sqrt{C_{2n}^2}} a(\vec{p}, -n+2; -1) = \frac{1}{\sqrt{4n}} a(\vec{p}, -n+1; 0) \\ -n+1 &\leq h \leq n-1 \end{aligned} \right.$$

$$\begin{aligned}
& \Rightarrow \sum_{h=n}^{-n} [a_d(\vec{p}, h) \varepsilon_{ab\dots}(\vec{p}, h)] \\
& = \sum_{h=n-1}^{-n+1} [a_d(\vec{p}, h) \varepsilon_{ab\dots}(\vec{p}, h)] + [a_d(\vec{p}, n) \varepsilon_{ab\dots}(\vec{p}, n)] + [a_d(\vec{p}, -n) \varepsilon_{ab\dots}(\vec{p}, -n)] \\
& = \sum_{h=n-1}^{-n+1} [a(\vec{p}, h; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, h; -1) \varepsilon_d(\vec{p}, -1)] \varepsilon_{ab\dots}(\vec{p}, h) \\
& + [a(\vec{p}, n; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, n; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, n; -1) \varepsilon_d(\vec{p}, -1)] \varepsilon_{ab\dots}(\vec{p}, n) \\
& + [a(\vec{p}, -n; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, -n; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, -n; -1) \varepsilon_d(\vec{p}, -1)] \varepsilon_{ab\dots}(\vec{p}, -n) \\
& = \sum_{h=n-1}^{-n+1} \left[ \frac{\sqrt{C_{n+h+1}^2}}{\sqrt{C_{n+h}^1 C_{n-h}^1}} a(\vec{p}, h+1; 0) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, 0) + \frac{\sqrt{C_{n-h+1}^2}}{\sqrt{C_{n+h}^1 C_{n-h}^1}} a(\vec{p}, h-1; 0) \varepsilon_d(\vec{p}, -1) \right] \varepsilon_{ab\dots}(\vec{p}, h) \\
& + [a(\vec{p}, n; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, n; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, n; -1) \varepsilon_d(\vec{p}, -1)] \varepsilon_{ab\dots}(\vec{p}, n) \\
& + [a(\vec{p}, -n; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, -n; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, -n; -1) \varepsilon_d(\vec{p}, -1)] \varepsilon_{ab\dots}(\vec{p}, -n) \\
& = \left[ \sum_{h=n}^{-n+2} \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{n+h-1}^1 C_{n-h+1}^1}} a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, 1) \varepsilon_{ab\dots}(\vec{p}, h-1) + \sum_{h=n-1}^{-n+1} a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, 0) \varepsilon_{ab\dots}(\vec{p}, h) \right. \\
& + \left. \sum_{h=n-2}^{-n} \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{n+h+1}^1 C_{n-h-1}^1}} a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, -1) \varepsilon_{ab\dots}(\vec{p}, h+1) \right] \\
& + [a(\vec{p}, n; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, n; 0) \varepsilon_d(\vec{p}, 0) + \frac{1}{\sqrt{4n}} a(\vec{p}, n-1; 0) \varepsilon_d(\vec{p}, -1)] \varepsilon_{ab\dots}(\vec{p}, n) \\
& + \left[ \frac{1}{\sqrt{4n}} a(\vec{p}, -n+1; 0) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, -n; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, -n; -1) \varepsilon_d(\vec{p}, -1) \right] \varepsilon_{ab\dots}(\vec{p}, -n) \\
& = \left[ \sum_{h=n}^{-n+2} \frac{\sqrt{C_{n+1+h}^2}}{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}} a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, 1) \varepsilon_{ab\dots}(\vec{p}, h-1) + \sum_{h=n-1}^{-n+1} a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, 0) \varepsilon_{ab\dots}(\vec{p}, h) \right. \\
& + \left. \sum_{h=n-2}^{-n} \frac{\sqrt{C_{n+1-h}^2}}{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}} a(\vec{p}, h; 0) \varepsilon_d(\vec{p}, -1) \varepsilon_{ab\dots}(\vec{p}, h+1) \right] \\
& + [a(\vec{p}, n; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, n; 0) \varepsilon_d(\vec{p}, 0) + \frac{1}{\sqrt{4n}} a(\vec{p}, n-1; 0) \varepsilon_d(\vec{p}, -1)] \varepsilon_{ab\dots}(\vec{p}, n) \\
& + \left[ \frac{1}{\sqrt{4n}} a(\vec{p}, -n+1; 0) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, -n; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, -n; -1) \varepsilon_d(\vec{p}, -1) \right] \varepsilon_{ab\dots}(\vec{p}, -n) \\
& = \sum_{h=n-2}^{-n+2} \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}} a(\vec{p}, h; 0) \\
& \left[ \frac{\sqrt{C_{n+1+h}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 1) \varepsilon_{ab\dots}(\vec{p}, h-1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 0) \varepsilon_{ab\dots}(\vec{p}, h) + \frac{\sqrt{C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, -1) \varepsilon_{ab\dots}(\vec{p}, h+1) \right] \\
& + \sqrt{n} a(\vec{p}, n; 0) \varepsilon_d(\vec{p}, 1) \varepsilon_{ab\dots}(\vec{p}, n-1) + \frac{\sqrt{2n-1}}{2} a(\vec{p}, n-1; 0) \varepsilon_d(\vec{p}, 1) \varepsilon_{ab\dots}(\vec{p}, n-2) \\
& + a(\vec{p}, n-1; 0) \varepsilon_d(\vec{p}, 0) \varepsilon_{ab\dots}(\vec{p}, n-1) + a(\vec{p}, -n+1; 0) \varepsilon_d(\vec{p}, 0) \varepsilon_{ab\dots}(\vec{p}, -n+1) \\
& + \sqrt{n} a(\vec{p}, -n; 0) \varepsilon_d(\vec{p}, -1) \varepsilon_{ab\dots}(\vec{p}, -n+1) + \frac{\sqrt{2n-1}}{2} a(\vec{p}, -n+1; 0) \varepsilon_d(\vec{p}, -1) \varepsilon_{ab\dots}(\vec{p}, -n+2) \\
& + [a(\vec{p}, n; 1) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, n; 0) \varepsilon_d(\vec{p}, 0) + \frac{1}{\sqrt{4n}} a(\vec{p}, n-1; 0) \varepsilon_d(\vec{p}, -1)] \varepsilon_{ab\dots}(\vec{p}, n) \\
& + \left[ \frac{1}{\sqrt{4n}} a(\vec{p}, -n+1; 0) \varepsilon_d(\vec{p}, 1) + a(\vec{p}, -n; 0) \varepsilon_d(\vec{p}, 0) + a(\vec{p}, -n; -1) \varepsilon_d(\vec{p}, -1) \right] \varepsilon_{ab\dots}(\vec{p}, -n) \\
& = \sum_{h=n-2}^{-n+2} \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}} a(\vec{p}, h; 0) \\
& \left[ \frac{\sqrt{C_{n+1+h}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 1) \varepsilon_{ab\dots}(\vec{p}, h-1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 0) \varepsilon_{ab\dots}(\vec{p}, h) + \frac{\sqrt{C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, -1) \varepsilon_{ab\dots}(\vec{p}, h+1) \right] \\
& + a(\vec{p}, n-1; 0) \left[ \frac{\sqrt{2n-1}}{2} \varepsilon_d(\vec{p}, 1) \varepsilon_{ab\dots}(\vec{p}, n-2) + \varepsilon_d(\vec{p}, 0) \varepsilon_{ab\dots}(\vec{p}, n-1) + \frac{1}{\sqrt{4n}} \varepsilon_d(\vec{p}, -1) \varepsilon_{ab\dots}(\vec{p}, n) \right] \\
& + a(\vec{p}, -n+1; 0) \left[ \frac{1}{\sqrt{4n}} \varepsilon_d(\vec{p}, 1) \varepsilon_{ab\dots}(\vec{p}, n) + \varepsilon_d(\vec{p}, 0) \varepsilon_{ab\dots}(\vec{p}, -n+1) + \frac{\sqrt{2n-1}}{2} \varepsilon_d(\vec{p}, -1) \varepsilon_{ab\dots}(\vec{p}, -n+2) \right]
\end{aligned}$$

$$\begin{aligned}
& + a(\vec{p}, n; 0)[\sqrt{n}\varepsilon_d(\vec{p}, 1)\varepsilon_{ab\dots n}(\vec{p}, n-1) + \varepsilon_d(\vec{p}, 0)\varepsilon_{ab\dots n}(\vec{p}, n)] \\
& + a(\vec{p}, -n; 0)[\varepsilon_d(\vec{p}, 0)\varepsilon_{ab\dots n}(\vec{p}, -n) + \sqrt{n}\varepsilon_d(\vec{p}, -1)\varepsilon_{ab\dots n}(\vec{p}, -n+1)] \\
& + a(\vec{p}, n; 1)\varepsilon_d(\vec{p}, 1)\varepsilon_{ab\dots n}(\vec{p}, n) + a(\vec{p}, -n; -1)\varepsilon_d(\vec{p}, -1)\varepsilon_{ab\dots n}(\vec{p}, -n) \\
& = \sum_{h=n-1}^{-n+1} \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}} a(\vec{p}, h; 0) \\
& \left[ \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 1)\varepsilon_{ab\dots n}(\vec{p}, h-1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 0)\varepsilon_{ab\dots n}(\vec{p}, h) + \frac{\sqrt{C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, -1)\varepsilon_{ab\dots n}(\vec{p}, h+1) \right] \\
& + a(\vec{p}, n; 0)[\sqrt{n}\varepsilon_d(\vec{p}, 1)\varepsilon_{ab\dots n}(\vec{p}, n-1) + \varepsilon_d(\vec{p}, 0)\varepsilon_{ab\dots n}(\vec{p}, n)] \\
& + a(\vec{p}, -n; 0)[\varepsilon_d(\vec{p}, 0)\varepsilon_{ab\dots n}(\vec{p}, -n) + \sqrt{n}\varepsilon_d(\vec{p}, -1)\varepsilon_{ab\dots n}(\vec{p}, -n+1)] \\
& + a(\vec{p}, n; 1)\varepsilon_d(\vec{p}, 1)\varepsilon_{ab\dots n}(\vec{p}, n) + a(\vec{p}, -n; -1)\varepsilon_d(\vec{p}, -1)\varepsilon_{ab\dots n}(\vec{p}, -n) \\
& = \sum_{h=n}^{-n} \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}} a(\vec{p}, h; 0) \\
& \left[ \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 1)\varepsilon_{ab\dots n}(\vec{p}, h-1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 0)\varepsilon_{ab\dots n}(\vec{p}, h) + \frac{\sqrt{C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, -1)\varepsilon_{ab\dots n}(\vec{p}, h+1) \right] \\
& + a(\vec{p}, n; 1)\varepsilon_d(\vec{p}, 1)\varepsilon_{ab\dots n}(\vec{p}, n) + a(\vec{p}, -n; -1)\varepsilon_d(\vec{p}, -1)\varepsilon_{ab\dots n}(\vec{p}, -n) \\
& = \sum_{h=n+1}^{-n-1} a(\vec{p}, h) \\
& \left[ \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 1)\varepsilon_{ab\dots n}(\vec{p}, h-1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 0)\varepsilon_{ab\dots n}(\vec{p}, h) + \frac{\sqrt{C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, -1)\varepsilon_{ab\dots n}(\vec{p}, h+1) \right] \\
& a(\vec{p}, h) := \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}} a(\vec{p}, h; 0), -n \leq h \leq n; a(\vec{p}, n+1) := a(\vec{p}, n; 1), a(\vec{p}, -n-1) := a(\vec{p}, -n; -1) \\
& = \sum_{h=n+1}^{-n-1} a(\vec{p}, h)\varepsilon_{ab\dots n+1}(\vec{p}, h) \\
& a(\vec{p}, h) := \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}} a(\vec{p}, h; 0), -n \leq h \leq n; a(\vec{p}, n+1) := a(\vec{p}, n; 1), a(\vec{p}, -n-1) := a(\vec{p}, -n; -1) \\
& \varepsilon_{ab\dots n+1}(\vec{p}, h) := \frac{\sqrt{C_{2n+2}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 1)\varepsilon_{ab\dots n}(\vec{p}, h-1) + \frac{\sqrt{C_{n+1+h}^1 C_{n+1-h}^1}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, 0)\varepsilon_{ab\dots n}(\vec{p}, h) \\
& + \frac{\sqrt{C_{n+1-h}^2}}{\sqrt{C_{2n+2}^2}} \varepsilon_d(\vec{p}, -1)\varepsilon_{ab\dots n}(\vec{p}, h+1), -n-1 \leq h \leq n+1
\end{aligned}$$

### 3.2 K-G方程平面波解的直观解法?

定理3.2.1.

$$(-\partial^c \partial_c + m^2)A_{ab\dots n}(x) = 0, \delta^{ab}A_{ab\dots n}(x) = 0, \partial^a A_{ab\dots n}(x) = 0, A_{ab\dots n}(x) \text{ 全对称} \Leftrightarrow$$

$$\left\{ \begin{aligned}
& A_{ab\dots n}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} [a(\vec{p}, h)\varepsilon_{ab\dots n}(\vec{p}, h)e^{ip \cdot x} + b^+(\vec{p}, h)\tilde{\varepsilon}_{ab\dots n}(\vec{p}, h)e^{-ip \cdot x}] d^3 \vec{p} \\
& (p^c p_c + m^2)\varepsilon_{ab\dots n}(\vec{p}, h) = 0, \delta^{ab}\varepsilon_{ab\dots n}(\vec{p}, h) = 0, p^a \varepsilon_{ab\dots n}(\vec{p}, h) = 0, \varepsilon_{ab\dots n}(\vec{p}, h) \text{ 全对称} \\
& (p^c p_c + m^2)\tilde{\varepsilon}_{ab\dots n}(\vec{p}, h) = 0, \delta^{ab}\tilde{\varepsilon}_{ab\dots n}(\vec{p}, h) = 0, p^a \tilde{\varepsilon}_{ab\dots n}(\vec{p}, h) = 0, \tilde{\varepsilon}_{ab\dots n}(\vec{p}, h) \text{ 全对称}
\end{aligned} \right.$$

证明：采用数学归纳法证明此定理。

第一步： $n' = 1$ 时成立：

$$(-\partial^c \partial_c + m^2)A_a(x) = 0, \partial^a A_{ab\dots n}(x) = 0 \Leftrightarrow$$

$$\left\{ \begin{aligned} A_a(\vec{r}, t) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} [a(\vec{p}, h)\varepsilon_a(\vec{p}, h)e^{ip \cdot x} + b^+(\vec{p}, h)\tilde{\varepsilon}_a(\vec{p}, h)e^{-ip \cdot x}] d^3\vec{p} \\ (p^c p_c + m^2)\varepsilon_a(\vec{p}, h) &= 0, p^a \varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0 \\ (p^c p_c + m^2)\tilde{\varepsilon}_a(\vec{p}, h) &= 0, p^a \tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0 \end{aligned} \right.$$

第二步：假设  $n' = n$  时成立：

$$(-\partial^c \partial_c + m^2)A_{\underbrace{ab \dots}_n}(x) = 0, \delta^{ab}A_{\underbrace{ab \dots}_n}(x) = 0, \partial^a A_{\underbrace{ab \dots}_n}(x) = 0, A_{\underbrace{ab \dots}_n}(x) \text{ 全对称} \Leftrightarrow$$

$$\left\{ \begin{aligned} A_{\underbrace{ab \dots}_n}(\vec{r}, t) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} [a(\vec{p}, h)\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h)e^{ip \cdot x} + b^+(\vec{p}, h)\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h)e^{-ip \cdot x}] d^3\vec{p} \\ (p^c p_c + m^2)\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) &= 0, \delta^{ab}\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, p^a \varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, \varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) \text{ 全对称} \\ (p^c p_c + m^2)\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) &= 0, \delta^{ab}\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, p^a \tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, \tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) \text{ 全对称} \end{aligned} \right.$$

第三步： $n' = n + 1$  时，

$$(-\partial^c \partial_c + m^2)A_{\underbrace{ab \dots}_{n+1}}(x) = 0, \delta^{ab}A_{\underbrace{ab \dots}_{n+1}}(x) = 0, \partial^a A_{\underbrace{ab \dots}_{n+1}}(x) = 0, A_{\underbrace{ab \dots}_{n+1}}(x) \text{ 全对称}$$

$$\Leftrightarrow \left\{ \begin{aligned} (-\partial^c \partial_c + m^2)A_{\underbrace{ab \dots}_n d}(x) &= 0, \delta^{ab}A_{\underbrace{ab \dots}_n d}(x) = 0, \partial^a A_{\underbrace{ab \dots}_n d}(x) = 0 \\ A_{\underbrace{ab \dots}_n d}(x) &= \frac{1}{n!} A_{\underbrace{\{ab \dots\}d}}(x), A_{\underbrace{ab \dots}_n d}(x) = A_{\underbrace{ab \dots}_n a}(x) \end{aligned} \right.$$

$$\Leftrightarrow \left\{ \begin{aligned} A_{\underbrace{ab \dots}_n d}(\vec{r}, t) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} [a_d(\vec{p}, h)\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h)e^{ip \cdot x} + b_d^+(\vec{p}, h)\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h)e^{-ip \cdot x}] d^3\vec{p} \\ (p^c p_c + m^2)\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) &= 0, \delta^{ab}\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, p^a \varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, \varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = \frac{1}{n!} \varepsilon_{\underbrace{\{ab \dots\}}_n}(\vec{p}, h) \\ (p^c p_c + m^2)\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) &= 0, \delta^{ab}\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, p^a \tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, \tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = \frac{1}{n!} \tilde{\varepsilon}_{\underbrace{\{ab \dots\}}_n}(\vec{p}, h) \\ A_{\underbrace{ab \dots}_n d}(x) &= A_{\underbrace{ab \dots}_n a}(x), a_d(\vec{p}, h) := a(\vec{p}, h; 1)\varepsilon_d(\vec{p}, 1) + a(\vec{p}, h; 0)\varepsilon_d(\vec{p}, 0) + a(\vec{p}, h; -1)\varepsilon_d(\vec{p}, -1) \\ b_d^+(\vec{p}, h) &:= b^+(\vec{p}, h; 1)\tilde{\varepsilon}_d(\vec{p}, 1) + b^+(\vec{p}, h; 0)\tilde{\varepsilon}_d(\vec{p}, 0) + b^+(\vec{p}, h; -1)\tilde{\varepsilon}_d(\vec{p}, -1) \end{aligned} \right.$$

$$\Leftrightarrow \left\{ \begin{aligned} A_{\underbrace{ab \dots}_n d}(\vec{r}, t) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=n}^{-n} \frac{1}{\sqrt{2^n E}} [a_d(\vec{p}, h)\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h)e^{ip \cdot x} + b_d^+(\vec{p}, h)\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h)e^{-ip \cdot x}] d^3\vec{p} \\ (p^c p_c + m^2)\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) &= 0, \delta^{ab}\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, p^a \varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, \varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = \frac{1}{n!} \varepsilon_{\underbrace{\{ab \dots\}}_n}(\vec{p}, h) \\ (p^c p_c + m^2)\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) &= 0, \delta^{ab}\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, p^a \tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, \tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = \frac{1}{n!} \tilde{\varepsilon}_{\underbrace{\{ab \dots\}}_n}(\vec{p}, h) \\ a_d(\vec{p}, h)\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) &= a_a(\vec{p}, h)\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h), b_d^+(\vec{p}, h)\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) = b_a^+(\vec{p}, h)\tilde{\varepsilon}_{\underbrace{ab \dots}_n}(\vec{p}, h) \\ a_d(\vec{p}, h) &:= a(\vec{p}, h; 1)\varepsilon_d(\vec{p}, 1) + a(\vec{p}, h; 0)\varepsilon_d(\vec{p}, 0) + a(\vec{p}, h; -1)\varepsilon_d(\vec{p}, -1) \\ b_d^+(\vec{p}, h) &:= b^+(\vec{p}, h; 1)\tilde{\varepsilon}_d(\vec{p}, 1) + b^+(\vec{p}, h; 0)\tilde{\varepsilon}_d(\vec{p}, 0) + b^+(\vec{p}, h; -1)\tilde{\varepsilon}_d(\vec{p}, -1) \end{aligned} \right.$$

此步证明了  $n' = n + 1$  时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

## 4 四维时空中反对称张量场方程的平面波解

### 4.1 自旋-1粒子Klein-Gordon方程的平面波解

定理4.1.1.  $\partial^b F_{ab} + m^2 A_a = 0, F_{ab} = \partial_a A_b - \partial_b A_a \Leftrightarrow (\partial^b \partial_b - m^2) A_a = 0, \partial^a A_a = 0$

$$A_a(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} [a(\vec{p}, h) \varepsilon_a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) \tilde{\varepsilon}_a(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

定理4.1.2.  $\begin{cases} \frac{1}{2!} \partial^{[a} F^{bc]} + m F^{abc} = 0, \partial_a F^{ab} = 0 \\ \partial^{[a} F^{bcd]} = 0, \partial_a F^{abc} + m F^{bc} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_a F^{abc} + m F^{bc} = 0, \frac{1}{2!} \partial^{[a} F^{bc]} + m F^{abc} = 0 \\ \partial^c F_{cab} - m^2 A_{ab} = 0, F_{cab} = \frac{1}{2!} \partial_{[c} A_{ab]}; A_{ab} := \frac{-1}{m} F_{ab} \end{cases}$

定理4.1.3.  $\partial^c F_{cab} - m^2 A_{ab} = 0, F_{cab} = \frac{1}{2!} \partial_{[c} A_{ab]} \Leftrightarrow (\partial^c \partial_c - m^2) A_{ab} = 0, \partial^a A_{ab} = 0, A_{ab} = -A_{ba}$

定理4.1.4.  $(\partial^c \partial_c - m^2) A_{ab} = 0, \partial^a A_{ab} = 0, A_{ab} = -A_{ba}$

$$\Leftrightarrow A_{ab}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} [a(\vec{p}, h) \varepsilon_{ab}(\vec{p}, h) e^{ip\cdot x} + b^+(\vec{p}, h) \tilde{\varepsilon}_{ab}(\vec{p}, h) e^{-ip\cdot x}] d^3\vec{p}$$

证明:  $(\partial^c \partial_c - m^2) A_{ab} = 0, \partial^a A_{ab} = 0, A_{ab} = -A_{ba}$

$$\Leftrightarrow A_{ab}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} [a_b(\vec{p}, h) \varepsilon_a(\vec{p}, h) e^{ip\cdot x} + b_b^+(\vec{p}, h) \tilde{\varepsilon}_a(\vec{p}, h) e^{-ip\cdot x}] d^3\vec{p}, A_{ab} = -A_{ba}$$

$$\Leftrightarrow \sum_{h=1}^{-1} a_{\{a}(\vec{p}, h) \varepsilon_{b\}}(\vec{p}, h) = 0, a_a(\vec{p}, h) = \sum_{h'=1}^{-1} a(\vec{p}, h; h') \varepsilon_a(\vec{p}, h') + c(\vec{p}, h; 0) \frac{p_a}{m}$$

$$\Leftrightarrow \sum_{h, h'=1}^{-1} a(\vec{p}, h; h') \varepsilon_{\{a}(\vec{p}, h') \varepsilon_{b\}}(\vec{p}, h) + c(\vec{p}, h; 0) \frac{1}{m} p_{\{a} \varepsilon_{b\}}(\vec{p}, h) = 0$$

$$\Leftrightarrow \sum_{h, h'=1}^{-1} a(\vec{p}, h; h') \varepsilon_{\{a}(\vec{p}, h') \varepsilon_{b\}}(\vec{p}, h) = 0, c(\vec{p}, h; 0) = 0$$

$$\Leftrightarrow a(\vec{p}, -1; 1) = -a(\vec{p}, 1; -1), a(\vec{p}, 0; 1) = -a(\vec{p}, 1; 0), a(\vec{p}, -1; 0) = -a(\vec{p}, 0; -1), a(\vec{p}, h; h) = 0$$

$$\Leftrightarrow \sum_{h=1}^{-1} a_b(\vec{p}, h) \varepsilon_a(\vec{p}, h) = \sum_{h', h=1}^{-1} a(\vec{p}, h; h') \varepsilon_a(\vec{p}, h) \varepsilon_b(\vec{p}, h')$$

$$= a(\vec{p}, 1; 0) \varepsilon_{[a}(\vec{p}, 1) \varepsilon_{b]}(\vec{p}, 0) + a(\vec{p}, 1; -1) \varepsilon_{[a}(\vec{p}, 1) \varepsilon_{b]}(\vec{p}, -1) + a(\vec{p}, 0; -1) \varepsilon_{[a}(\vec{p}, 0) \varepsilon_{b]}(\vec{p}, -1)$$

$$= a(1) \varepsilon_{ab}(\vec{p}, 1) + a(0) \varepsilon_{ab}(\vec{p}, 0) + a(-1) \varepsilon_{ab}(\vec{p}, -1) = \sum_{h=1}^{-1} a(h) \varepsilon_{ab}(\vec{p}, h)$$

$$a(1) = \sqrt{2} a(\vec{p}, 1; 0), a(0) = \sqrt{2} a(\vec{p}, 1; -1), a(-1) = \sqrt{2} a(\vec{p}, 0; -1)$$

$$\varepsilon_{ab}(\vec{p}, 1) := \frac{1}{\sqrt{2}} \varepsilon_{[a}(\vec{p}, 1) \varepsilon_{b]}(\vec{p}, 0), \varepsilon_{ab}(\vec{p}, 0) := \frac{1}{\sqrt{2}} \varepsilon_{[a}(\vec{p}, 1) \varepsilon_{b]}(\vec{p}, -1), \varepsilon_{ab}(\vec{p}, -1) := \frac{1}{\sqrt{2}} \varepsilon_{[a}(\vec{p}, 0) \varepsilon_{b]}(\vec{p}, -1)$$

$$\Leftrightarrow A_{ab}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} [a(\vec{p}, h) \varepsilon_{ab}(\vec{p}, h) e^{ip\cdot x} + b^+(\vec{p}, h) \tilde{\varepsilon}_{ab}(\vec{p}, h) e^{-ip\cdot x}] d^3\vec{p} \quad \square$$

定理4.1.5.  $\partial^d F_{dabc} - m^2 A_{abc} = 0, F_{dabc} = \frac{1}{3!} \partial_{[d} A_{abc]} \Leftrightarrow (\partial^d \partial_d - m^2) A_{abc} = 0, \partial^a A_{abc} = 0, A_{abc} = \frac{1}{3!} A_{[abc]}$

### 4.2 n=N+1维时空中自旋-1粒子Klein-Gordon方程的平面波解

定理4.2.1.  $\partial^b F_{ab} + m^2 A_a = 0, F_{ab} = \partial_a A_b - \partial_b A_a \Leftrightarrow (\partial^b \partial_b - m^2) A_a = 0, \partial^a A_a = 0$

$$A_a(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^N \frac{1}{\sqrt{2E}} [a(\vec{p}, h) \varepsilon_a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) \tilde{\varepsilon}_a(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\text{定义4.2.1. } L = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}$$

推论4.2.1.  $L_{\vec{p}} := L_{\vec{v}} = e^{-\{ln[\gamma_v(1+v)]\} \hat{v} \cdot L} = 1 - \gamma_v(\vec{v} \cdot L) + \frac{\gamma_v - 1}{v^2} (\vec{v} \cdot L)^2 = 1 - \frac{1}{m} (\vec{p} \cdot L) + \frac{1}{m(E+m)} (\vec{p} \cdot L)^2$

$$\text{推论4.2.2. } L_{\vec{p}} = \frac{1}{m} \begin{bmatrix} m & 0 & 0 & -ip_x \\ 0 & m & 0 & -ip_y \\ 0 & 0 & m & -ip_z \\ ip_x & ip_y & ip_z & E \end{bmatrix} + \frac{1}{m(E+m)} \begin{bmatrix} p_x p_x & p_x p_y & p_x p_z & 0 \\ p_y p_x & p_y p_y & p_y p_z & 0 \\ p_z p_x & p_z p_y & p_z p_z & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{推论4.2.3. } \varepsilon(\vec{p}, h) := L_{\vec{p}} \begin{bmatrix} 0_1 \\ \vdots \\ 1_h \\ \vdots \\ 0_n \end{bmatrix} = \begin{bmatrix} 0_1 \\ \vdots \\ 1_h \\ \vdots \\ 0_n \end{bmatrix} + \frac{p_h}{m(E+m)} \begin{bmatrix} \vec{p} \\ i(E+m) \end{bmatrix}, \varepsilon_a(\vec{p}, n) := L_{\vec{p}} \begin{bmatrix} 0_1 \\ \vdots \\ 0_0 \\ \vdots \\ 1_n \end{bmatrix} = -i \frac{p_a}{m}; h = 1, \dots, N$$



推论4.2.4.  $p^a \varepsilon_a(\vec{p}, h) = 0, \varepsilon_a(\vec{p}, h) \eta^{aa'} \varepsilon_{a'}^+(\vec{p}, h') = \delta_{hh'}, \sum_{h=1}^N \varepsilon_a(\vec{p}, h) \varepsilon_{a'}^+(\vec{p}, h) = \eta_{aa'} + \frac{p_a p_{a'}}{m^2}$

推论4.2.5.  $L_{\vec{v}} \begin{bmatrix} \vec{0} \\ i \end{bmatrix} = e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L} \begin{bmatrix} \vec{0} \\ i \end{bmatrix} = \begin{bmatrix} \gamma_v \vec{v} \\ i\gamma_v \end{bmatrix}, L_{\vec{v}} \begin{bmatrix} \vec{0} \\ im \end{bmatrix} = e^{-ln[\gamma_v(1+v)]\hat{v}\cdot L} \begin{bmatrix} \vec{0} \\ im \end{bmatrix} = \begin{bmatrix} \vec{p} \\ iE \end{bmatrix}$

定理4.2.2.  $-\frac{i}{4}[\Gamma_i, \Gamma_j] = S_{ij} \Rightarrow \Gamma_i = ?$

推论4.2.6.  $\begin{cases} \lambda_m(\hat{p}, 1; 1) = S_m(1)\lambda(\hat{p}, 1; 1) = e^{i\vec{\omega}\cdot\gamma} \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ -1 \end{bmatrix} = \frac{1}{2\hat{p}_-} \begin{bmatrix} i(\hat{p}_x \hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y \hat{p}_z) \\ -2i(\hat{p}_+ \hat{p}_-) \end{bmatrix}, \lambda_m(-\hat{p}, 1; 1) = \frac{\hat{p}_+}{\hat{p}_-} \lambda_m(\hat{p}, -1; 1) \\ \lambda_m(\hat{p}, 0; 1) = S_m(1)\lambda(\hat{p}, 0; 1) = e^{i\vec{\omega}\cdot\gamma} \begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix} = -i \begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{bmatrix} = -i\hat{p}, \lambda_m(-\hat{p}, 0; 1) = -\lambda_m(\hat{p}, 0; 1) \\ \lambda_m(\hat{p}, -1; 1) = S_m(1)\lambda(\hat{p}, -1; 1) = e^{i\vec{\omega}\cdot\gamma} \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ -1 \\ 0 \end{bmatrix} = \frac{1}{2\hat{p}_+} \begin{bmatrix} -i(\hat{p}_x \hat{p}_z + i\hat{p}_y) \\ -1(\hat{p}_x + i\hat{p}_y \hat{p}_z) \\ 2i(\hat{p}_+ \hat{p}_-) \end{bmatrix}, \lambda_m(-\hat{p}, -1; 1) = \frac{\hat{p}_-}{\hat{p}_+} \lambda_m(\hat{p}, 1; 1) \end{cases}$

推论4.2.7.  $\varepsilon(\vec{p}, \pm 1) = [L_{\vec{p}} S_m^+(1) e^{-i\vec{\omega}\cdot R}] \varepsilon(\vec{p}, \pm 1; 1)$

推论4.2.8.  $\frac{1}{2\hat{p}_-} \begin{bmatrix} i(\hat{p}_x \hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y \hat{p}_z) \\ -2i(\hat{p}_+ \hat{p}_-) \end{bmatrix} = \left[ \begin{bmatrix} 0_1 \\ \vdots \\ 1_h \\ \vdots \\ 0_n \end{bmatrix} + \frac{p_x}{m(E+m)} \begin{bmatrix} \vec{p} \\ i(E+m) \end{bmatrix} \right]$

推论4.2.9.  $(\gamma^a \otimes \sigma_y \partial_a + I \otimes \sigma_z \otimes \sigma_y \partial_u + I_4 \otimes \sigma_x \partial_v + m)\psi = 0$

$\Rightarrow (\gamma^a \otimes \sigma_y \partial_a + I \otimes \sigma_z \otimes \sigma_y iM + I_4 \otimes \sigma_x 0 + m)\psi = 0$

$\Rightarrow (\gamma^a \otimes \sigma_z \partial_a + I \otimes \sigma_z \otimes \sigma_z iM + m)\psi' = 0$

### 4.3 n=N+1维时空中自旋-2粒子Klein-Gordon方程的平面波解

定理4.3.1.  $(\partial^c \partial_c - m^2)A_{ab} = 0, \partial^a A_{ab} = 0, A_{ab} = -A_{ba}$

$A_{ab}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h, h'=1}^N \frac{1}{\sqrt{2E}} [a(\vec{p}; h, h') \varepsilon_{[a}(\vec{p}, h) \varepsilon_{b]}(\vec{p}, h') e^{ip\cdot x} + b^+(\vec{p}; h, h') \tilde{\varepsilon}_{[a}(\vec{p}, h) \tilde{\varepsilon}_{b]}(\vec{p}, h') e^{-ip\cdot x}] d^3 \vec{p}$

# 第二十九章 自旋基和CG系数的数学分析

自我评述：本章对于各种自旋基进行了一般的数学分析和逻辑推演，发现自旋基之间变换关系的系数就是自旋耦合系统的CG系数，从而也提供了一种一般求解CG系数的新方法。这个新方法比传统的自旋耦合本征态的求解方法更直观、更具体、更简单，因为新方法完全是构造式的，并且选取的自旋基比传统的更一般、更普遍、更严谨，故而使用起来更方便、更有用，可能对彻底理清量子纠缠能提供一定的帮助。

## 1 Bargmann-Wigner方程自旋基的重新梳理分析

### 1.1 Dirac自旋基是自旋、螺旋度和电荷三个算符的共同本征态

定义1.1.1.  $\hat{Q}(\vec{p}) := \frac{i\gamma^a p_a}{m}$ ,  $\hat{q}(\vec{p}, \kappa) := \frac{-\varsigma E \sigma_x + i\kappa |\vec{p}| \sigma_y}{m}$

性质1.1.1.

$$\begin{cases} \sigma^2(\frac{1}{2}) \otimes I u(\vec{p}, \frac{\kappa}{2}) = \frac{1}{2}(\frac{1}{2} + 1)u(\vec{p}, \frac{\kappa}{2}) & \left\{ \begin{array}{l} \sigma^2(\frac{1}{2}) \otimes I v(\vec{p}, \frac{\kappa}{2}) = \frac{1}{2}(\frac{1}{2} + 1)v(\vec{p}, \frac{\kappa}{2}) \\ \sigma(\frac{1}{2}) \cdot \hat{p} \otimes I u(\vec{p}, \frac{\kappa}{2}) = \frac{\kappa}{2}u(\vec{p}, \frac{\kappa}{2}) \\ \hat{Q}(\vec{p})u(\vec{p}, \frac{\kappa}{2}) = -u(\vec{p}, \frac{\kappa}{2}) \\ \text{描述电子}:(s, h; Q) = (\frac{1}{2}; \frac{\kappa}{2}, -1) \end{array} \right. \\ \sigma(\frac{1}{2}) \cdot \hat{p} \otimes I v(\vec{p}, \frac{\kappa}{2}) = \frac{\kappa}{2}v(\vec{p}, \frac{\kappa}{2}) \\ \hat{Q}(\vec{p})v(\vec{p}, \frac{\kappa}{2}) = v(\vec{p}, \frac{\kappa}{2}) \\ \text{描述正电子}:(s, h; Q) = (\frac{1}{2}; \frac{\kappa}{2}, 1) \end{cases}$$

证明：采用数学归纳法证明此定理。

第一步： $s' = \frac{1}{2}$ 时成立：

第二步：假设 $s' = s - \frac{1}{2}$ 时成立：

第三步： $s' = s$ 时：

此步证明了 $s' = s$ 时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

### 1.2 Dirac方程准投影算子 $\gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x)$

推论1.2.1.  $\mu(\vec{p}, \frac{\kappa}{2})\mu^+(\vec{p}, -\frac{\kappa}{2}) = \frac{1}{2m} \begin{bmatrix} m & \varsigma E - \kappa |\vec{p}| \\ \varsigma E + \kappa |\vec{p}| & m \end{bmatrix} = \frac{1}{2}(I + \varsigma \frac{E}{m} \sigma_x - i\kappa \frac{|\vec{p}|}{m} \sigma_y)$

推论1.2.2.  $\mu(\vec{p}, \frac{\kappa}{2})\mu^+(\vec{p}, \frac{\kappa}{2}) = \frac{\varsigma}{2m} \begin{bmatrix} \varsigma E - \kappa |\vec{p}| & m \\ \varsigma E + \kappa |\vec{p}| & \end{bmatrix} = \frac{\varsigma}{2}(I + \varsigma \frac{E}{m} \sigma_x - i\kappa \frac{|\vec{p}|}{m} \sigma_y)\sigma_x$

推论1.2.3.  $u(\vec{p}, \frac{\kappa}{2}) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2m(E + \kappa \varsigma |\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + \kappa |\vec{p}| \end{bmatrix}$ ,  $v(\vec{p}, \frac{\kappa}{2}) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2m(E + \kappa \varsigma |\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + \kappa |\vec{p}| \end{bmatrix}$

推论1.2.4.  $u(\vec{p}, \frac{\kappa}{2})u^+(\vec{p}, \frac{\kappa}{2}) = \frac{1}{4}[(\kappa \sigma \cdot \hat{p} + I) \otimes (I + \varsigma \frac{E}{m} \sigma_x - i\kappa \frac{|\vec{p}|}{m} \sigma_y)](\varsigma I \otimes \sigma_x)$ ,  $\gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x)$

### 1.3 特殊表象 $\gamma^a = (\sigma \otimes \sigma_y, I \otimes \sigma_x; I \otimes \sigma_z)$ 下相应自旋基

推论1.3.1.  $\gamma^a = (\sigma \otimes \sigma_y, I \otimes \sigma_x)$

$u(\vec{p}, \frac{\kappa}{2}) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2m(E + \kappa |\vec{p}|)}} \otimes \begin{bmatrix} m \\ E + \kappa |\vec{p}| \end{bmatrix}$ ,  $v(\vec{p}, \frac{\kappa}{2}) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2m(E + \kappa |\vec{p}|)}} \otimes \begin{bmatrix} -m \\ E + \kappa |\vec{p}| \end{bmatrix}$

1.4 自旋基组合性质  $(\gamma_a, \gamma_5) := [(\sigma \otimes \sigma_y, I \otimes \sigma_z), -I \otimes \sigma_x]$

推论1.4.1.

$$\begin{cases} \lambda(\hat{p}, \frac{1}{2})\lambda^+(\hat{p}, \frac{1}{2}) = \frac{1}{2}(\sigma \cdot \hat{p} + I) = \frac{1}{2}(\sigma, -i)^a \hat{p}_a, \hat{p}_a := (\hat{p}, i) \\ \lambda(\hat{p}, -\frac{1}{2})\lambda^+(\hat{p}, -\frac{1}{2}) = -\frac{1}{2}(\sigma \cdot \hat{p} - I) = -\frac{1}{2}(\sigma, i)^a \hat{p}_a \\ \lambda(\hat{p}, \frac{1}{2})\lambda^+(\hat{p}, -\frac{1}{2}) = \frac{1}{2}(\sigma \cdot \hat{p} + I)i\sigma_y = \frac{1}{2}(\sigma, i)^a \hat{p}_a i\sigma_y \\ \lambda(\hat{p}, -\frac{1}{2})\lambda^+(\hat{p}, \frac{1}{2}) = -\frac{1}{2}i\sigma_y(\sigma \cdot \hat{p} + I) = -\frac{1}{2}i\sigma_y(\sigma, i)^a \hat{p}_a \end{cases}$$

推论1.4.2.  $u(\vec{p}, \frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p}, \frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix}, u(\vec{p}, -\frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p}, -\frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix}$

证明:  $u(\vec{p}, \frac{1}{2})u^+(\vec{p}, \frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p}, \frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix} \frac{1}{\sqrt{2m}}\lambda^+(\hat{p}, \frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix}^+$   
 $= \frac{1}{2m}[\lambda(\hat{p}, \frac{1}{2})\lambda^+(\hat{p}, \frac{1}{2})] \otimes \left[ \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix} \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix}^+ \right]$   
 $= \frac{1}{4m}(\sigma, -i)^a \hat{p}_a \otimes (E + m\sigma_z + |\vec{p}|\sigma_x)$  □

证明:  $u(\vec{p}, -\frac{1}{2})u^+(\vec{p}, -\frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p}, -\frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix} \frac{1}{\sqrt{2m}}\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix}^+$   
 $= \frac{1}{2m}[\lambda(\hat{p}, -\frac{1}{2})\lambda^+(\hat{p}, -\frac{1}{2})] \otimes \left[ \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix} \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix}^+ \right]$   
 $= -\frac{1}{4m}(\sigma, i)^a \hat{p}_a \otimes (E + m\sigma_z - |\vec{p}|\sigma_x)$  □

证明:  $u(\vec{p}, \frac{1}{2})u^+(\vec{p}, -\frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p}, \frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix} \frac{1}{\sqrt{2m}}\lambda^+(\hat{p}, -\frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix}^+$   
 $= \frac{1}{2m}[\lambda(\hat{p}, \frac{1}{2})\lambda^+(\hat{p}, -\frac{1}{2})] \otimes \left[ \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix} \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix}^+ \right]$   
 $= \frac{1}{4m}[(\sigma, i)^a \hat{p}_a i\sigma_y] \otimes (E\sigma_z + m - i|\vec{p}|\sigma_y)$  □

证明:  $u(\vec{p}, -\frac{1}{2})u^+(\vec{p}, \frac{1}{2}) = \frac{1}{\sqrt{2m}}\lambda(\hat{p}, -\frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix} \frac{1}{\sqrt{2m}}\lambda^+(\hat{p}, \frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix}^+$   
 $= \frac{1}{2m}[\lambda(\hat{p}, -\frac{1}{2})\lambda^+(\hat{p}, \frac{1}{2})] \otimes \left[ \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix} \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix}^+ \right]$   
 $= \frac{1}{2m}[\lambda(\hat{p}, -\frac{1}{2})\lambda^+(\hat{p}, \frac{1}{2})] \otimes \left[ \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix} \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix}^+ \right]$   
 $= -\frac{1}{4m}[i\sigma_y(\sigma, i)^a \hat{p}_a] \otimes (E\sigma_z + m + i|\vec{p}|\sigma_y)$  □

=====

证明:  $u(\vec{p}, \frac{1}{2})u^+(\vec{p}, \frac{1}{2}) + u(\vec{p}, -\frac{1}{2})u^+(\vec{p}, -\frac{1}{2})$   
 $= \frac{1}{4m}(\sigma, -i)^a \hat{p}_a \otimes (E + m\sigma_z + |\vec{p}|\sigma_x) - \frac{1}{4m}(\sigma, i)^a \hat{p}_a \otimes (E + m\sigma_z - |\vec{p}|\sigma_x)$   
 $= \frac{1}{2m}(\sigma \cdot \vec{p} \otimes \sigma_x + E + mI \otimes \sigma_z) = \frac{(m-i\gamma^a p_a)\gamma_4}{2m}$  □

证明:  $u(\vec{p}, \frac{1}{2})u^+(\vec{p}, \frac{1}{2}) - u(\vec{p}, -\frac{1}{2})u^+(\vec{p}, -\frac{1}{2})$   
 $= \frac{1}{4m}(\sigma, -i)^a \hat{p}_a \otimes (E + m\sigma_z + |\vec{p}|\sigma_x) + \frac{1}{4m}(\sigma, i)^a \hat{p}_a \otimes (E + m\sigma_z - |\vec{p}|\sigma_x)$   
 $= \frac{1}{2m}[\sigma \cdot \hat{p} \otimes (E + m\sigma_z) + |\vec{p}|\sigma_x]$   
 $= \frac{1}{2m}[i(\gamma \cdot \hat{p})(E\gamma_4 - m) - |\vec{p}|\gamma_5]$  □

定义1.4.1.  $\Lambda_+(\frac{1}{2}) := \sum_{h=1/2}^{-1/2} u(\vec{p}, h)u^+(\vec{p}, h) = \frac{(m-i\gamma^a p_a)\gamma_4}{2m}, \Lambda_-(\frac{1}{2}) := \sum_{h=1/2}^{-1/2} v(\vec{p}, h)v^+(\vec{p}, h) = \frac{(-m-i\gamma^a p_a)\gamma_4}{2m}$

推论1.4.3.  $u(\vec{p}, \frac{\kappa}{2}) = \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \mu(\vec{p}, \frac{\kappa}{2}), v(\vec{p}, \frac{\kappa}{2}) = \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \nu(\vec{p}, \frac{\kappa}{2})$

1.5 定义-自旋基分解:  $s = (s - \frac{1}{2}) \oplus \frac{1}{2}$

定义1.5.1.  $U_{\underbrace{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = \frac{\sqrt{s+h}}{\sqrt{2s}}U_{\underbrace{\lambda_\zeta \dots \sigma_\zeta}_{2s-1}}(\vec{p}, h - \frac{1}{2})U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}}U_{\underbrace{\lambda_\zeta \dots \sigma_\zeta}_{2s-1}}(\vec{p}, h + \frac{1}{2})U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}), -s \leq h \leq s$

推论1.5.1.  $U_{\underbrace{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}_{2s}}(\vec{p}, h) = \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+h'} C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}} U_{\underbrace{\lambda_\zeta \dots \sigma_\zeta}_{2s-1}}(\vec{p}, h-h')U_{\tau_\zeta}(\vec{p}, h'), -s \leq h \leq s$

$$\begin{aligned} \text{推论1.5.2. } U_{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h) &= \hat{\Gamma}_{\tau_\zeta}(h) U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h) \\ &= \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+h'} C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}} U_{\tau_\zeta}(\vec{p}, h') U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h-h'), -s \leq h \leq s \end{aligned}$$

$$\text{定义1.5.2. } \hat{\Gamma}_{\tau_\zeta}(h) := \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+h'} C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}} U_{\tau_\zeta}(\vec{p}, h') [(\vec{p}, h-h')], -s \leq h \leq s$$

### 1.6 推论- $U_{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)$ 是自旋本征态

$$\text{定义1.6.1. } \Omega(s; \sigma(\frac{1}{2}) \otimes I) := [\sigma(\frac{1}{2}) \otimes I] \otimes I_{4^{2s-1}} + I_4 \otimes [\sigma(\frac{1}{2}) \otimes I] \otimes I_{4^{2s-2}} + \dots + I_{4^{2s-1}} \otimes [\sigma(\frac{1}{2}) \otimes I]$$

$$\text{定理1.6.1. } [\Omega(s; \sigma(\frac{1}{2}) \otimes I) \cdot \hat{p}] U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}(\vec{p}, h) = h U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}(\vec{p}, h), -s \leq h \leq s$$

$$\begin{aligned} \text{证明: } & [\Omega(s; \sigma(\frac{1}{2}) \otimes I) \cdot \hat{p}] U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}(\vec{p}, h) \\ &= \{ \Omega(s - \frac{1}{2}; \sigma(\frac{1}{2}) \otimes I) \otimes I_4 + I_{4^{2s-1}} \otimes [\sigma(\frac{1}{2}) \otimes I] \} \cdot \hat{p} \\ &= \left[ \frac{\sqrt{s+h}}{\sqrt{2s}} U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}(\vec{p}, h - \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}(\vec{p}, h + \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \right], -s \leq h \leq s \\ &= \left[ \frac{\sqrt{s+h}}{\sqrt{2s}} h U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}(\vec{p}, h - \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} h U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}(\vec{p}, h + \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \right], -s \leq h \leq s \\ &= h U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}(\vec{p}, h), -s \leq h \leq s \end{aligned} \quad \square$$

$$\text{定理1.6.2. } \Omega^2(s; \sigma(\frac{1}{2}) \otimes I) U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}(\vec{p}, h) = s(s+1) U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}(\vec{p}, h), -s \leq h \leq s$$

以上定理可以用全对称的表象变换方法可以很容易得到证明？好像不容易，但可以用下面章节的上升和下降算符方法证明( $\hat{p}_z = 1$ 才成立)。由上可知 $U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)$ 是自旋本征态，所以展开系数是CG系数，实际计算结果也表明确实就是对应的CG系数。从而也提供了一种统一、规范、直观和完整的计算CG系数新方法。

### 1.7 特殊表象 $\gamma^a = (\sigma \otimes \sigma_y, I \otimes \sigma_z; -I \otimes \sigma_x)$ 下Dirac自旋基的升降算符

Dirac旋量推动变换:

$$\text{推论1.7.1. } D_{\vec{v}} = e^{-ln[\gamma_v(1+v)]\hat{v} \cdot (\frac{i}{2}\vec{\gamma}\gamma_4)} = \frac{1+\gamma_v - i\gamma_v \vec{v} \cdot \vec{\gamma}\gamma_4}{\sqrt{2(\gamma_v+1)}} = \frac{E+m - i\vec{p} \cdot \vec{\gamma}\gamma_4}{\sqrt{2m(E+m)}} = \frac{m - i\gamma^a p_a \gamma_4}{\sqrt{2m(E+m)}}$$

$$\text{定理1.7.1. } e^{-ln[\gamma_v(1+v)]\hat{v} \cdot (\frac{i}{2}\vec{\gamma}\gamma_4)} (\sigma \otimes I) e^{ln[\gamma_v(1+v)]\hat{v} \cdot (\frac{i}{2}\vec{\gamma}\gamma_4)} = \frac{1}{m(E+m)} \begin{bmatrix} E(E+m)\sigma_i - 2p_i(\sigma \cdot \vec{p}) & (E+m)[(\sigma \cdot \vec{p})\sigma_i - p_i] \\ (E+m)[(\sigma \cdot \vec{p})\sigma_i - p_i] & E(E+m)\sigma_i - 2p_i(\sigma \cdot \vec{p}) \end{bmatrix}$$

$$\begin{aligned} \text{证明: } & e^{-ln[\gamma_v(1+v)]\hat{v} \cdot (\frac{i}{2}\vec{\gamma}\gamma_4)} (\sigma \otimes I) e^{ln[\gamma_v(1+v)]\hat{v} \cdot (\frac{i}{2}\vec{\gamma}\gamma_4)} \\ &= \frac{E+m - i\vec{p} \cdot \vec{\gamma}\gamma_4}{\sqrt{2m(E+m)}} (\sigma \otimes I) \frac{E+m + i\vec{p} \cdot \vec{\gamma}\gamma_4}{\sqrt{2m(E+m)}} \\ &= \frac{1}{\sqrt{2m(E+m)}} \begin{bmatrix} E+m & \sigma \cdot \vec{p} \\ \sigma \cdot \vec{p} & E+m \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \frac{1}{\sqrt{2m(E+m)}} \begin{bmatrix} E+m & -\sigma \cdot \vec{p} \\ -\sigma \cdot \vec{p} & E+m \end{bmatrix} \\ &= \frac{1}{2m(E+m)} \begin{bmatrix} E+m & \sigma \cdot \vec{p} \\ \sigma \cdot \vec{p} & E+m \end{bmatrix} \begin{bmatrix} (E+m)\sigma & -\sigma(\sigma \cdot \vec{p}) \\ -\sigma(\sigma \cdot \vec{p}) & (E+m)\sigma \end{bmatrix} \\ &= \frac{1}{2m(E+m)} \begin{bmatrix} (E+m)^2\sigma - (\sigma \cdot \vec{p})\sigma(\sigma \cdot \vec{p}) & (E+m)[(\sigma \cdot \vec{p})\sigma - \sigma(\sigma \cdot \vec{p})] \\ (E+m)[(\sigma \cdot \vec{p})\sigma - \sigma(\sigma \cdot \vec{p})] & (E+m)^2\sigma - (\sigma \cdot \vec{p})\sigma(\sigma \cdot \vec{p}) \end{bmatrix} \\ &= \frac{1}{2m(E+m)} \begin{bmatrix} (E+m)^2\sigma_i + \vec{p}^2\sigma_i - 2p_i(\sigma \cdot \vec{p}) & 2(E+m)[(\sigma \cdot \vec{p})\sigma_i - p_i] \\ 2(E+m)[(\sigma \cdot \vec{p})\sigma_i - p_i] & (E+m)^2\sigma_i + \vec{p}^2\sigma_i - 2p_i(\sigma \cdot \vec{p}) \end{bmatrix} \\ &= \frac{1}{m(E+m)} \begin{bmatrix} E(E+m)\sigma_i - 2p_i(\sigma \cdot \vec{p}) & (E+m)[(\sigma \cdot \vec{p})\sigma_i - p_i] \\ (E+m)[(\sigma \cdot \vec{p})\sigma_i - p_i] & E(E+m)\sigma_i - 2p_i(\sigma \cdot \vec{p}) \end{bmatrix} \end{aligned} \quad \square$$

$$\text{推论1.7.2. } e^{-ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)} (\sigma \otimes I) e^{ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)} =$$

$$\begin{cases} x : \frac{E}{m}\sigma_x \otimes I + \frac{i|\vec{p}|}{m}\sigma_y \otimes \sigma_x = \frac{E}{m}(-i\gamma_y + \frac{|\vec{p}|}{E}\gamma_x\gamma_5)\gamma_z \\ y : \frac{E}{m}\sigma_y \otimes I - \frac{i|\vec{p}|}{m}\sigma_x \otimes \sigma_x = \frac{1}{m}(i\gamma_x + \frac{|\vec{p}|}{E}\gamma_y\gamma_5)\gamma_z \\ z : \sigma_z \otimes I = -i\gamma_x\gamma_y \end{cases}$$

推论1.7.3.

$$\begin{cases} u\left(\begin{bmatrix} 0 \\ |p| \end{bmatrix}, \frac{1}{2}\right) = e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\tilde{\gamma}_z\gamma_4)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix} \\ u\left(\begin{bmatrix} 0 \\ |p| \end{bmatrix}, -\frac{1}{2}\right) = e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\tilde{\gamma}_z\gamma_4)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix} \\ v\left(\begin{bmatrix} 0 \\ |p| \end{bmatrix}, \frac{1}{2}\right) = e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\tilde{\gamma}_z\gamma_4)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \sqrt{E-m} \\ \sqrt{E+m} \end{bmatrix} \\ v\left(\begin{bmatrix} 0 \\ |p| \end{bmatrix}, -\frac{1}{2}\right) = e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\tilde{\gamma}_z\gamma_4)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} -\sqrt{E-m} \\ \sqrt{E+m} \end{bmatrix} \end{cases}$$

证明:  $u\left(\begin{bmatrix} 0 \\ |p| \end{bmatrix}, \frac{1}{2}\right)$ 

$$\begin{aligned} &= e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\tilde{\gamma}_z\gamma_4)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{E+m-i|p|\tilde{\gamma}_z\gamma_4}{\sqrt{2m(E+m)}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2m(E+m)}} \begin{bmatrix} E+m & \sigma_z|p| \\ \sigma_z|p| & E+m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix} \end{aligned}$$

□

证明:  $u\left(\begin{bmatrix} 0 \\ |p| \end{bmatrix}, -\frac{1}{2}\right)$ 

$$\begin{aligned} &= e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\tilde{\gamma}_z\gamma_4)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{E+m-i|p|\tilde{\gamma}_z\gamma_4}{\sqrt{2m(E+m)}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2m(E+m)}} \begin{bmatrix} E+m & \sigma_z|p| \\ \sigma_z|p| & E+m \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{2m}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix} \end{aligned}$$

□

证明:  $v\left(\begin{bmatrix} 0 \\ |p| \end{bmatrix}, \frac{1}{2}\right)$ 

$$\begin{aligned} &= e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\tilde{\gamma}_z\gamma_4)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{E+m-i|p|\tilde{\gamma}_z\gamma_4}{\sqrt{2m(E+m)}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2m(E+m)}} \begin{bmatrix} E+m & \sigma_z|p| \\ \sigma_z|p| & E+m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \sqrt{E-m} \\ \sqrt{E+m} \end{bmatrix} \end{aligned}$$

□

证明:  $v\left(\begin{bmatrix} 0 \\ |p| \end{bmatrix}, -\frac{1}{2}\right)$ 

$$\begin{aligned} &= e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\tilde{\gamma}_z\gamma_4)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{E+m-i|p|\tilde{\gamma}_z\gamma_4}{\sqrt{2m(E+m)}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2m(E+m)}} \begin{bmatrix} E+m & \sigma_z|p| \\ \sigma_z|p| & E+m \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2m}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} -\sqrt{E-m} \\ \sqrt{E+m} \end{bmatrix} \end{aligned}$$

□

推论1.7.4.  $e^{i\vec{\omega}\cdot\frac{\sigma\otimes I}{2}} e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\tilde{\gamma}_z\gamma_4)} (\sigma \otimes I) e^{\ln[\gamma_v(1+v)](\frac{i}{2}\tilde{\gamma}_z\gamma_4)} e^{-i\vec{\omega}\cdot\frac{\sigma\otimes I}{2}} =$ 

$$\begin{cases} x : \frac{E}{m} [\sigma_x(\frac{1}{2}) - \hat{p}_x \frac{\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})}{1+\hat{p}_z}] \otimes I + \frac{i|p|}{m} [\sigma_y(\frac{1}{2}) - \hat{p}_y \frac{\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})}{1+\hat{p}_z}] \otimes \sigma_x \\ = \frac{iE}{2m} [-\gamma_y\gamma_z + \frac{\hat{p}_x}{1+\hat{p}_z} (\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)] - \frac{|p|}{2m} [-\gamma_z\gamma_x + \frac{\hat{p}_y}{1+\hat{p}_z} (\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)]\gamma_5 \\ y : \frac{E}{m} [\sigma_y(\frac{1}{2}) - \hat{p}_y \frac{\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})}{1+\hat{p}_z}] \otimes I - \frac{i|p|}{m} [\sigma_x(\frac{1}{2}) - \hat{p}_x \frac{\sigma(\frac{1}{2})\cdot\hat{p}+\sigma_z(\frac{1}{2})}{1+\hat{p}_z}] \otimes \sigma_x \\ = \frac{iE}{2m} [-\gamma_z\gamma_x + \frac{\hat{p}_y}{1+\hat{p}_z} (\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)] + \frac{|p|}{2m} [-\gamma_y\gamma_z + \frac{\hat{p}_x}{1+\hat{p}_z} (\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)]\gamma_5 \\ z : [\sigma(\frac{1}{2})\cdot\hat{p}] \otimes I = -\frac{i}{4}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k = \frac{1}{2}\varepsilon^{ijk}\hat{p}_i S_{jk}(e, \frac{1}{2}) \end{cases}$$

定义1.7.1.

$$\begin{cases} \hat{J}_x(\vec{0}, \frac{1}{2}; \gamma_a) := -\frac{i}{2}\gamma_y\gamma_z \\ \hat{J}_y(\vec{0}, \frac{1}{2}; \gamma_a) := -\frac{i}{2}\gamma_z\gamma_x \\ \hat{J}_z(\vec{0}, \frac{1}{2}; \gamma_a) := -\frac{i}{2}\gamma_x\gamma_y \end{cases}$$

定义1.7.2.

$$\begin{cases} x + iy : \hat{J}_+(\vec{p}, \frac{1}{2}; m) := \{[\sigma_x(\frac{1}{2}) + i\sigma_y(\frac{1}{2})] - \frac{(\hat{p}_x + i\hat{p}_y)}{1 + \hat{p}_z}[\sigma(\frac{1}{2}) \cdot \hat{p} + \sigma_z(\frac{1}{2})]\} \otimes I(\frac{E}{m} + \frac{|\vec{p}|}{m} \otimes \sigma_x) \\ = [i(\gamma_x + i\gamma_y)\gamma_z + \frac{(\hat{p}_x + i\hat{p}_y)}{1 + \hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)]\frac{i(E - |\vec{p}|\gamma_5)}{2m} \\ x - iy : \hat{J}_-(\vec{p}, \frac{1}{2}; m) := \{[\sigma_x(\frac{1}{2}) - i\sigma_y(\frac{1}{2})] - \frac{(\hat{p}_x - i\hat{p}_y)}{1 + \hat{p}_z}[\sigma(\frac{1}{2}) \cdot \hat{p} + \sigma_z(\frac{1}{2})]\}(\frac{E}{m} - \frac{|\vec{p}|}{m} \otimes \sigma_x) \\ = [-i(\gamma_x - i\gamma_y)\gamma_z + \frac{(\hat{p}_x - i\hat{p}_y)}{1 + \hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)]\frac{i(E + |\vec{p}|\gamma_5)}{2m} \\ z : \hat{J}_z(\vec{p}, \frac{1}{2}; m) := [\sigma(\frac{1}{2}) \cdot \hat{p}] \otimes I = -\frac{i}{4}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k = \frac{1}{2}\varepsilon^{ijk}\hat{p}_i S_{jk}(e, \frac{1}{2}), \hat{Q}(\vec{p}, \frac{1}{2}; m) := \frac{i\gamma^a p_a}{m} \end{cases}$$

推论1.7.5.

$$\begin{cases} u(\vec{p}, \frac{1}{2}) = e^{i\vec{\omega} \cdot \frac{\sigma \otimes I}{2}} e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2m}}\lambda(\hat{p}, \frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ \sqrt{E-m} \end{bmatrix} \\ u(\vec{p}, -\frac{1}{2}) = e^{i\vec{\omega} \cdot \frac{\sigma \otimes I}{2}} e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2m}}\lambda(\hat{p}, -\frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E+m} \\ -\sqrt{E-m} \end{bmatrix} \\ v(\vec{p}, \frac{1}{2}) = e^{i\vec{\omega} \cdot \frac{\sigma \otimes I}{2}} e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2m}}\lambda(\hat{p}, \frac{1}{2}) \otimes \begin{bmatrix} \sqrt{E-m} \\ \sqrt{E+m} \end{bmatrix} \\ v(\vec{p}, -\frac{1}{2}) = e^{i\vec{\omega} \cdot \frac{\sigma \otimes I}{2}} e^{-\ln[\gamma_v(1+v)](\frac{i}{2}\vec{\gamma}_z\gamma_4)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2m}}\lambda(\hat{p}, -\frac{1}{2}) \otimes \begin{bmatrix} -\sqrt{E-m} \\ \sqrt{E+m} \end{bmatrix} \end{cases}$$

## 1.8 一般表象Dirac自旋基的升降算符

定义1.8.1.

$$\begin{cases} \hat{J}_x(\vec{p}, \frac{1}{2}; \gamma_a) := \frac{iE}{2m}[-\gamma_y\gamma_z + \frac{\hat{p}_x}{1 + \hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)] - \frac{|\vec{p}|}{2m}[-\gamma_z\gamma_x + \frac{\hat{p}_y}{1 + \hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)]\gamma_5 \\ \hat{J}_y(\vec{p}, \frac{1}{2}; \gamma_a) := \frac{iE}{2m}[-\gamma_z\gamma_x + \frac{\hat{p}_y}{1 + \hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)] + \frac{|\vec{p}|}{2m}[-\gamma_y\gamma_z + \frac{\hat{p}_x}{1 + \hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)]\gamma_5 \\ \hat{J}_z(\vec{p}, \frac{1}{2}; \gamma_a) := -\frac{i}{4}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k = \frac{1}{2}\varepsilon^{ijk}\hat{p}_i S_{jk}(e, \frac{1}{2}), \hat{Q}(\vec{p}, \frac{1}{2}; \gamma_a) := \frac{i\gamma^a p_a}{m} \end{cases}$$

推论1.8.1.

$$\begin{cases} \hat{J}_x^2(\vec{p}, \frac{1}{2}; \gamma_a) = \frac{1}{4}, \hat{J}_y^2(\vec{p}, \frac{1}{2}; \gamma_a) = \frac{1}{4}, \hat{J}_z^2(\vec{p}, \frac{1}{2}; \gamma_a) = \frac{1}{4} \\ [\hat{J}_i(\vec{p}, \frac{1}{2}; \gamma_a), \hat{J}_j(\vec{p}, \frac{1}{2}; \gamma_a)] = \varepsilon_{ijk}\hat{J}_k(\vec{p}, \frac{1}{2}; \gamma_a), \hat{J}^2(\vec{p}, \frac{1}{2}; \gamma_a) = \frac{1}{2}(\frac{1}{2} + 1) \end{cases}$$

推论1.8.2.  $\frac{1}{2}\delta_{\lambda'_c}^{\lambda''_c}\frac{1}{2}\delta_{\mu'_c}^{\mu''_c} =$ 

$$\begin{aligned} & [\frac{1}{2}\delta_{\lambda'_c}^{\lambda''_c}\frac{1}{2}\delta_{\mu'_c}^{\mu''_c} + \hat{J}_{x\lambda'_c}^{\lambda''_c}(\vec{p}, \frac{1}{2}; \gamma_a)\hat{J}_{x\mu'_c}^{\mu''_c}(\vec{p}, \frac{1}{2}; \gamma_a) + \hat{J}_{y\lambda'_c}^{\lambda''_c}(\vec{p}, \frac{1}{2}; \gamma_a)\hat{J}_{y\mu'_c}^{\mu''_c}(\vec{p}, \frac{1}{2}; \gamma_a) + \hat{J}_{z\lambda'_c}^{\lambda''_c}(\vec{p}, \frac{1}{2}; \gamma_a)\hat{J}_{z\mu'_c}^{\mu''_c}(\vec{p}, \frac{1}{2}; \gamma_a)] \\ & [\frac{1}{2}\delta_{\lambda'_c}^{\lambda''_c}\frac{1}{2}\delta_{\mu'_c}^{\mu''_c} + \hat{J}_{x\lambda'_c}^{\lambda''_c}(\vec{p}, \frac{1}{2}; \gamma_a)\hat{J}_{x\mu'_c}^{\mu''_c}(\vec{p}, \frac{1}{2}; \gamma_a) + \hat{J}_{y\lambda'_c}^{\lambda''_c}(\vec{p}, \frac{1}{2}; \gamma_a)\hat{J}_{y\mu'_c}^{\mu''_c}(\vec{p}, \frac{1}{2}; \gamma_a) + \hat{J}_{z\lambda'_c}^{\lambda''_c}(\vec{p}, \frac{1}{2}; \gamma_a)\hat{J}_{z\mu'_c}^{\mu''_c}(\vec{p}, \frac{1}{2}; \gamma_a)] \end{aligned}$$

推论1.8.3.

$$\begin{cases} \hat{J}_+(\vec{p}, \frac{1}{2}; \gamma_a) := [i(\gamma_x + i\gamma_y)\gamma_z + \frac{\hat{p}_x + i\hat{p}_y}{1 + \hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)]\frac{i(E - |\vec{p}|\gamma_5)}{2m} \\ \hat{J}_-(\vec{p}, \frac{1}{2}; \gamma_a) := [-i(\gamma_x - i\gamma_y)\gamma_z + \frac{\hat{p}_x - i\hat{p}_y}{1 + \hat{p}_z}(\frac{1}{2}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k + \gamma_x\gamma_y)]\frac{i(E + |\vec{p}|\gamma_5)}{2m} \\ \hat{J}_z(\vec{p}, \frac{1}{2}; \gamma_a) := -\frac{i}{4}\varepsilon^{ijk}\hat{p}_i\gamma_j\gamma_k = \frac{1}{2}\varepsilon^{ijk}\hat{p}_i S_{jk}(e, \frac{1}{2}), \hat{Q}(\vec{p}, \frac{1}{2}; \gamma_a) := \frac{i\gamma^a p_a}{m} \end{cases}$$

推论1.8.4.

$$\begin{cases} \hat{J}_+(\vec{p}, \frac{1}{2}; \gamma_a)u(\vec{p}, -\frac{1}{2}) = u(\vec{p}, \frac{1}{2}); \hat{J}_+(\vec{p}, \frac{1}{2}; \gamma_a)u(\vec{p}, \frac{1}{2}) = 0 \\ \hat{J}_-(\vec{p}, \frac{1}{2}; \gamma_a)u(\vec{p}, \frac{1}{2}) = u(\vec{p}, -\frac{1}{2}); \hat{J}_-(\vec{p}, \frac{1}{2}; \gamma_a)u(\vec{p}, -\frac{1}{2}) = 0 \\ \hat{J}_z(\vec{p}, \frac{1}{2}; \gamma_a)u(\vec{p}, h) = hu(\vec{p}, h), \hat{Q}(\vec{p}, \frac{1}{2}; \gamma_a)u(\vec{p}, h) = u(\vec{p}, h), -\frac{1}{2} \leq h \leq \frac{1}{2} \end{cases}$$

推论1.8.5.

$$\begin{cases} \hat{J}_+(\vec{p}, \frac{1}{2}; \gamma_a)v(\vec{p}, -\frac{1}{2}) = v(\vec{p}, \frac{1}{2}); \hat{J}_+(\vec{p}, \frac{1}{2}; \gamma_a)v(\vec{p}, \frac{1}{2}) = 0 \\ \hat{J}_-(\vec{p}, \frac{1}{2}; \gamma_a)v(\vec{p}, \frac{1}{2}) = v(\vec{p}, -\frac{1}{2}); \hat{J}_-(\vec{p}, \frac{1}{2}; \gamma_a)v(\vec{p}, -\frac{1}{2}) = 0 \\ \hat{J}_z(\vec{p}, \frac{1}{2}; \gamma_a)v(\vec{p}, h) = hv(\vec{p}, h), \hat{Q}(\vec{p}, \frac{1}{2}; \gamma_a)v(\vec{p}, h) = -v(\vec{p}, h), -\frac{1}{2} \leq h \leq \frac{1}{2} \end{cases}$$

## 1.9 Bargmann-Wigner方程自旋基的升降算符

定义1.9.1.

$$\begin{cases} \hat{J}(\vec{p}, s; \gamma_a) := \underbrace{\hat{J}(\vec{p}, \frac{1}{2}; \gamma_a) \otimes I_4 \otimes \cdots \otimes I_4}_{2s} + \underbrace{I_4 \otimes \hat{J}(\vec{p}, \frac{1}{2}; \gamma_a) \otimes \cdots \otimes I_4}_{2s} + \cdots + \underbrace{I_4 \otimes \cdots \otimes I_4 \otimes \hat{J}(\vec{p}, \frac{1}{2}; \gamma_a)}_{2s} \\ \hat{Q}(\vec{p}, s; \gamma_a) := \underbrace{\hat{Q}(\vec{p}, \frac{1}{2}; \gamma_a) \otimes I_4 \otimes \cdots \otimes I_4}_{2s} + \underbrace{I_4 \otimes \hat{Q}(\vec{p}, \frac{1}{2}; \gamma_a) \otimes \cdots \otimes I_4}_{2s} + \cdots + \underbrace{I_4 \otimes \cdots \otimes I_4 \otimes \hat{Q}(\vec{p}, \frac{1}{2}; \gamma_a)}_{2s} \end{cases}$$

推论1.9.1.

$$\begin{cases} \hat{J}_x^2(\vec{p}, \frac{1}{2}; \gamma_a) = \frac{1}{4}, \hat{J}_y^2(\vec{p}, \frac{1}{2}; \gamma_a) = \frac{1}{4}, \hat{J}_z^2(\vec{p}, \frac{1}{2}; \gamma_a) = \frac{1}{4}, \hat{J}^2(\vec{p}, \frac{1}{2}; \gamma_a) = \frac{1}{2}(\frac{1}{2} + 1) \\ [\hat{J}_i(\vec{p}, s; \gamma_a), \hat{J}_j(\vec{p}, s; \gamma_a)] = \varepsilon_{ij}^k \hat{J}_k(\vec{p}, s; \gamma_a) \end{cases}$$

$$\text{定理1.9.1. } \hat{J}_+(\vec{p}, s; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h+1)} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{p}, h+1)$$

证明：采用数学归纳法证明此定理。

第一步：\$s' = \frac{1}{2}\$时成立：

$$\hat{J}_+(\vec{p}, \frac{1}{2}; \gamma_a) U_{\otimes \tau_\zeta}(\vec{p}, h) = \sqrt{\frac{3}{4} - h(h+1)} U_{\otimes \tau_\zeta}(\vec{p}, h+1), -\frac{1}{2} \leq h \leq \frac{1}{2}$$

第二步：假设 \$s' = s - \frac{1}{2}\$ 时成立：

$$\hat{J}_+(\vec{p}, s - \frac{1}{2}; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h) = \sqrt{(s - \frac{1}{2})(s + \frac{1}{2}) - h(h+1)} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h), -s + \frac{1}{2} \leq h \leq s - \frac{1}{2}$$

第三步：\$s' = s\$ 时：\$-s \leq h \leq s\$，\$\hat{J}\_+(\vec{p}, s; \gamma\_a) \underbrace{U\_{\lambda\_\zeta \otimes \cdots \otimes \sigma\_\zeta \otimes \tau\_\zeta}}\_{2s}(\vec{p}, h)\$

$$\begin{aligned} &= \frac{\sqrt{s+h}}{\sqrt{2s}} [\hat{J}_+(\vec{p}, s - \frac{1}{2}; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2})] U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} [\hat{J}_+(\vec{p}, s - \frac{1}{2}; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2})] U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \\ &+ \frac{\sqrt{s+h}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) \hat{J}_+(\vec{p}, \frac{1}{2}; \gamma_a) U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2}) \hat{J}_+(\vec{p}, \frac{1}{2}; \gamma_a) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \\ &= \frac{\sqrt{s+h} \sqrt{(s+\frac{1}{2})(s-\frac{1}{2}) - (h-\frac{1}{2})(h+\frac{1}{2})}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) \\ &+ \frac{\sqrt{s-h} \sqrt{(s+\frac{1}{2})(s-\frac{1}{2}) - (h+1-\frac{1}{2})(h+1+\frac{1}{2})}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h + \frac{3}{2}) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \\ &= \frac{\sqrt{(s-h)(s+h+1)} \sqrt{s+h+1}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{(s-h)(s+h+1)} \sqrt{s-h-1}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h + \frac{3}{2}) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \\ &= \sqrt{s(s+1) - h(h+1)} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{p}, h+1) \end{aligned}$$

此步证明了 \$s' = s\$ 时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 \$\square\$

$$\text{定理1.9.2. } \hat{J}_-(\vec{p}, s; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h-1)} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{p}, h-1)$$

证明：采用数学归纳法证明此定理。

第一步：\$s' = \frac{1}{2}\$ 时成立：

$$\hat{J}_-(\vec{p}, \frac{1}{2}; \gamma_a) U_{\otimes \tau_\zeta}(\vec{p}, h) = \sqrt{\frac{3}{4} - h(h-1)} U_{\otimes \tau_\zeta}(\vec{p}, h-1), -\frac{1}{2} \leq h \leq \frac{1}{2}$$

第二步：假设 \$s' = s - \frac{1}{2}\$ 时成立：

$$\hat{J}_-(\vec{p}, s - \frac{1}{2}; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h) = \sqrt{(s - \frac{1}{2})(s + \frac{1}{2}) - h(h-1)} \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h-1), -s + \frac{1}{2} \leq h \leq s - \frac{1}{2}$$

第三步：\$s' = s\$ 时：\$-s \leq h \leq s\$，\$\hat{J}\_-(\vec{p}, s; \gamma\_a) \underbrace{U\_{\lambda\_\zeta \otimes \cdots \otimes \sigma\_\zeta \otimes \tau\_\zeta}}\_{2s}(\vec{p}, h)\$

$$\begin{aligned} &= \frac{\sqrt{s+h}}{\sqrt{2s}} [\hat{J}_-(\vec{p}, s - \frac{1}{2}; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2})] U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} [\hat{J}_-(\vec{p}, s - \frac{1}{2}; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2})] U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \end{aligned}$$

$$\begin{aligned}
& + \frac{\sqrt{s+h}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) \hat{J}_-(\vec{p}, \frac{1}{2}; \gamma_a) U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \hat{J}_-(\vec{p}, s - \frac{1}{2}; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2}) \hat{J}_-(\vec{p}, \frac{1}{2}; \gamma_a) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
& = \frac{\sqrt{s+h} \sqrt{(s+\frac{1}{2})(s-\frac{1}{2}) - (h-1-\frac{1}{2})(h-1+\frac{1}{2})}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{3}{2}) U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) \\
& + \frac{\sqrt{s-h} \sqrt{(s+\frac{1}{2})(s-\frac{1}{2}) - (h-\frac{1}{2})(h+\frac{1}{2})}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) + \frac{\sqrt{s+h}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
& = \frac{\sqrt{s+h} \sqrt{(s+h-1)(s-h+1)}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{3}{2}) U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) \\
& + \frac{\sqrt{s-h} \sqrt{(s+h)(s-h)}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) + \frac{\sqrt{s+h}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
& = \frac{\sqrt{(s+h)(s-h+1)} \sqrt{(s+h-1)}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{3}{2}) U_{\otimes \tau_\zeta}(\vec{p}, \frac{1}{2}) \\
& + \frac{\sqrt{(s+h)(s-h+1)} \sqrt{s-h+1}}{\sqrt{2s}} \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) U_{\otimes \tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
& = \sqrt{s(s+1) - h(h-1)} \underbrace{U_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{p}, h-1)
\end{aligned}$$

此步证明了  $s' = s$  时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

**推论1.9.2.**

$$\begin{cases}
\hat{J}_+(\vec{p}, s; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h+1)} \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h+1), -s \leq h \leq s \\
\hat{J}_-(\vec{p}, s; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h-1)} \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h-1), -s \leq h \leq s \\
\hat{J}_z(\vec{p}, s; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = h \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h), -s \leq h \leq s \\
\hat{Q}(\vec{p}, s; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = -2s \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h), -s \leq h \leq s
\end{cases}$$

**推论1.9.3.**

$$\begin{cases}
\hat{J}_+(\vec{p}, s; \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h+1)} \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h+1), -s \leq h \leq s \\
\hat{J}_-(\vec{p}, s; \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h-1)} \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h-1), -s \leq h \leq s \\
\hat{J}_z(\vec{p}, s; \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = h \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h), -s \leq h \leq s \\
\hat{Q}(\vec{p}, s; \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = 2s \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h), -s \leq h \leq s
\end{cases}$$

**推论1.9.4.**

$$\begin{cases}
\hat{J}^2(\vec{p}, s; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = s(s+1) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h), \hat{J}^2(\vec{p}, * \frac{1}{2}; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = \frac{3}{4} \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) \\
\hat{J}_z(\vec{p}, s; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = h \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h), \hat{Q}(\vec{p}, * \frac{1}{2}; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = - \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) \\
\underbrace{U_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, h) = \frac{1}{(2s)!} \underbrace{U_{\{\lambda_\zeta \mu_\zeta \dots\}}}_{2s}(\vec{p}, h), \hat{J}^2 = \hat{J}_z^2 + \frac{1}{2}(\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+), -s \leq h \leq s
\end{cases}$$

**推论1.9.5.**



$$\begin{cases} \hat{J}^2(\vec{p}, s; \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = s(s+1) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h), \hat{J}^2(\vec{p}, * \frac{1}{2}; \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = \frac{3}{4} \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) \\ \hat{J}_z(\vec{p}, s; \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = h \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h), \hat{Q}(\vec{p}, * \frac{1}{2}; \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2s}(\vec{p}, h) \\ \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, h) = \frac{1}{(2s)!} \underbrace{V_{\{\lambda_\zeta \mu_\zeta \dots\}}}_{2s}(\vec{p}, h), \hat{J}^2 = \hat{J}_z^2 + \frac{1}{2}(\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+), -s \leq h \leq s \end{cases}$$

### 1.10 推论- $U_{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)$ 的正交性

定义1.10.1.  $\bar{U}^{\tau_\zeta}(\vec{p}, h') U_{\tau_\zeta}(\vec{p}, h) = \delta_{hh'}$ ,  $-\frac{1}{2} \leq h, h' \leq \frac{1}{2}$

定理1.10.1.  $\bar{U}^{\overbrace{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}^{2s}}(\vec{p}, h') \underbrace{U_{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h) = \delta_{hh'}$ ,  $-s \leq h', h \leq s$

证明: 采用数学归纳法证明此定理。

第一步:  $s' = \frac{1}{2}$ 时成立:

$$\bar{U}^{\lambda_\zeta}(\vec{p}, h') U_{\lambda_\zeta}(\vec{p}, h) = \delta_{hh'}, -\frac{1}{2} \leq h, h' \leq \frac{1}{2}$$

第二步: 假设  $s' = s - \frac{1}{2}$ 时成立:

$$\bar{U}^{\overbrace{\lambda_\zeta \dots \sigma_\zeta}^{2s-1}}(\vec{p}, h') \underbrace{U_{\lambda_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}, h) = \delta_{hh'}, -s + \frac{1}{2} \leq h', h \leq s - \frac{1}{2}$$

第三步:  $s' = s$ 时:  $\bar{U}^{\overbrace{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}^{2s}}(\vec{p}, h') \underbrace{U_{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}}_{2s}(\vec{p}, h)$ ,  $-s \leq h', h \leq s$

$$\begin{aligned} &= \left[ \sum_{\bar{h}'=1/2}^{-1/2} \frac{\sqrt{C_{s+h'}^{1/2+\bar{h}'} C_{s-h'}^{1/2-\bar{h}'}}}{\sqrt{C_{2s}^1}} \bar{U}^{\overbrace{\lambda_\zeta \dots \sigma_\zeta}^{2s-1}}(\vec{p}, h' - \bar{h}') \bar{U}^{\tau_\zeta}(\vec{p}, \bar{h}') \right] \left[ \sum_{\bar{h}=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \underbrace{U_{\lambda_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}, h - \bar{h}) U_{\tau_\zeta}(\vec{p}, \bar{h}) \right] \\ &= \sum_{\bar{h}', \bar{h}=1/2}^{-1/2} \left[ \frac{\sqrt{C_{s+h'}^{1/2+\bar{h}'} C_{s-h'}^{1/2-\bar{h}'}}}{\sqrt{C_{2s}^1}} \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \bar{U}^{\overbrace{\lambda_\zeta \dots \sigma_\zeta}^{2s-1}}(\vec{p}, h' - \bar{h}') \underbrace{U_{\lambda_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}, h - \bar{h}) \delta_{\bar{h}\bar{h}'} \right] \\ &= \sum_{\bar{h}=1/2}^{-1/2} \left[ \frac{\sqrt{C_{s+h'}^{1/2+\bar{h}} C_{s-h'}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \bar{U}^{\overbrace{\lambda_\zeta \dots \sigma_\zeta}^{2s-1}}(\vec{p}, h' - \bar{h}) \underbrace{U_{\lambda_\zeta \dots \sigma_\zeta}}_{2s-1}(\vec{p}, h - \bar{h}) \right] \\ &= \sum_{\bar{h}=1/2}^{-1/2} \left[ \frac{\sqrt{C_{s+h'}^{1/2+\bar{h}} C_{s-h'}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \delta_{hh'} \right] \\ &= \sum_{\bar{h}=1/2}^{-1/2} \left[ \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \right] \delta_{hh'} \\ &= \delta_{hh'} \end{aligned}$$

此步证明了  $s' = s$ 时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

### 1.11 推论-自旋基分解: $1 = \frac{1}{2} \oplus \frac{1}{2}$

推论1.11.1.  $U_{\lambda_\zeta \mu_\zeta}(\vec{p}, h) = \frac{\sqrt{1+h}}{\sqrt{2}} U_{\lambda_\zeta}(\vec{p}, h - \frac{1}{2}) U_{\mu_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{1-h}}{\sqrt{2}} U_{\lambda_\zeta}(\vec{p}, h + \frac{1}{2}) U_{\mu_\zeta}(\vec{p}, -\frac{1}{2})$

$$= \begin{cases} U_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) U_{\mu_\zeta}(\vec{p}, \frac{1}{2}), h = 1 \\ \frac{1}{\sqrt{2}} U_{\{\lambda_\zeta}(\vec{p}, \frac{1}{2}) U_{\mu_\zeta}(\vec{p}, -\frac{1}{2})\}}, h = 0 \\ U_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) U_{\mu_\zeta}(\vec{p}, -\frac{1}{2}), h = -1 \end{cases}$$

推论1.11.2.  $U_{\lambda_\zeta \mu_\zeta}(\vec{p}, h) = U_{\mu_\zeta \lambda_\zeta}(\vec{p}, h)$ ,  $-1 \leq h \leq 1$

### 1.12 推论-自旋基分解: $0 = \frac{1}{2} \ominus \frac{1}{2}$

推论1.12.1.  $F_{\lambda_\zeta \mu_\zeta}(\vec{p}, h) = \frac{1}{\sqrt{2}} u_{[\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta]}(\vec{p}, -\frac{1}{2}), h = 0$

推论1.12.2.  $[(\sigma \otimes I) \cdot (I \otimes \sigma)][\lambda(\hat{p}, \frac{\xi}{2}) \otimes \lambda(\hat{p}, \frac{\xi}{2}) - \lambda(\hat{p}, -\frac{\xi}{2}) \otimes \lambda(\hat{p}, \frac{\xi}{2})] = -3[\lambda(\hat{p}, \frac{\xi}{2}) \otimes \lambda(\hat{p}, \frac{\xi}{2}) - \lambda(\hat{p}, -\frac{\xi}{2}) \otimes \lambda(\hat{p}, \frac{\xi}{2})]$

$$\begin{aligned}
& \text{证明: } \sigma \cdot [\lambda(\hat{p}, \frac{\zeta}{2})\lambda^T(\hat{p}, \frac{-\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2})\lambda^T(\hat{p}, \frac{\zeta}{2})]\sigma^T \\
&= \frac{i}{2}\sigma \cdot [(\sigma, -i\zeta)^a \hat{p}_a \sigma_y - (\sigma, i\zeta)^a \hat{p}_a \sigma_y]\sigma^T \\
&= \sigma \cdot (i\zeta \sigma_y)\sigma^T \\
&= \sigma_x(i\zeta \sigma_y)\sigma_x^T + \sigma_y(i\zeta \sigma_y)\sigma_y^T + \sigma_z(i\zeta \sigma_y)\sigma_z^T \\
&= -3(i\zeta \sigma_y) = -3[\lambda(\hat{p}, \frac{\zeta}{2})\lambda^T(\hat{p}, \frac{-\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2})\lambda^T(\hat{p}, \frac{\zeta}{2})]
\end{aligned}$$

□

推论1.12.3.

$$\begin{cases}
[\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]^2 [\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{-\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] = 0[\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{-\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] \\
[\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})] \cdot \hat{p}[\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{-\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] = 0[\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{-\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})]
\end{cases}$$

定理1.12.1.

$$\begin{cases}
F = -[C\phi + im\gamma_a(\zeta)\gamma_5(\zeta)C\mathbf{A}^a + \gamma_5(\zeta)C\Phi] \\
F = \frac{1}{\sqrt{2}}[u(\vec{p}, \frac{1}{2})u^T(\vec{p}, -\frac{1}{2}) - u(\vec{p}, -\frac{1}{2})u^T(\vec{p}, \frac{1}{2})] = -\frac{m-i\gamma^a p_a}{2\sqrt{2}m}\gamma_5 C \\
\varepsilon_a(\vec{p}, 0; 0) := \frac{1}{i\sqrt{2}}(\bar{C}\gamma_a\gamma_5)^{\lambda\zeta\mu\zeta} F_{\lambda\zeta\mu\zeta}(\vec{p}, h) = \frac{p_a}{m}
\end{cases}$$

$$\begin{aligned}
& \text{证明: } -2i\sqrt{2}(-im\mathbf{A}_a) = \varepsilon_a(\vec{p}, 0; 0) = \frac{1}{i\sqrt{2}}(\bar{C}\gamma_a\gamma_5)^{\lambda\zeta\mu\zeta} F_{\lambda\zeta\mu\zeta}(\vec{p}, h) \\
&= \frac{1}{2i}(\bar{C}\gamma_a\gamma_5)^{\lambda\zeta\mu\zeta} u_{[\lambda\zeta}(\vec{p}, \frac{1}{2})u_{\mu\zeta]}(\vec{p}, -\frac{1}{2}) \\
&= -i(\bar{C}\gamma_a\gamma_5)^{\lambda\zeta\mu\zeta} u_{\lambda\zeta}(\vec{p}, \frac{1}{2})u_{\mu\zeta}(\vec{p}, -\frac{1}{2}) \\
&= -iu^T(\vec{p}, \frac{1}{2})(\bar{C}\gamma_a\gamma_5)u(\vec{p}, -\frac{1}{2}) \\
&= iu^T(\vec{p}, \frac{1}{2})\gamma_2\gamma_5\gamma_4\gamma_a u(\vec{p}, -\frac{1}{2}) \\
&= iu^+(\vec{p}, -\frac{1}{2})\gamma_4\gamma_a u(\vec{p}, -\frac{1}{2}) \\
&= \frac{p_a}{m}
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } 2i\sqrt{2}(-\Phi) = \frac{1}{i\sqrt{2}}(\bar{C}\gamma_5)^{\lambda\zeta\mu\zeta} F_{\lambda\zeta\mu\zeta}(\vec{p}, h) \\
&= \frac{1}{2i}(\bar{C}\gamma_5)^{\lambda\zeta\mu\zeta} u_{[\lambda\zeta}(\vec{p}, \frac{1}{2})u_{\mu\zeta]}(\vec{p}, -\frac{1}{2}) \\
&= -i(\bar{C}\gamma_5)^{\lambda\zeta\mu\zeta} u_{\lambda\zeta}(\vec{p}, \frac{1}{2})u_{\mu\zeta}(\vec{p}, -\frac{1}{2}) \\
&= -iu^T(\vec{p}, \frac{1}{2})(\bar{C}\gamma_5)u(\vec{p}, -\frac{1}{2}) \\
&= -iu^T(\vec{p}, \frac{1}{2})\gamma_2\gamma_5\gamma_4 u(\vec{p}, -\frac{1}{2}) \\
&= -iu^+(\vec{p}, -\frac{1}{2})\gamma_4 u(\vec{p}, -\frac{1}{2}) \\
&= -i
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } 2i\sqrt{2}(-\phi) = \frac{1}{i\sqrt{2}}(\bar{C})^{\lambda\zeta\mu\zeta} F_{\lambda\zeta\mu\zeta}(\vec{p}, h) \\
&= \frac{1}{2i}(\bar{C})^{\lambda\zeta\mu\zeta} u_{[\lambda\zeta}(\vec{p}, \frac{1}{2})u_{\mu\zeta]}(\vec{p}, -\frac{1}{2}) \\
&= -i(\bar{C})^{\lambda\zeta\mu\zeta} u_{\lambda\zeta}(\vec{p}, \frac{1}{2})u_{\mu\zeta}(\vec{p}, -\frac{1}{2}) \\
&= -iu^T(\vec{p}, \frac{1}{2})\bar{C}u(\vec{p}, -\frac{1}{2}) \\
&= -iu^T(\vec{p}, \frac{1}{2})\gamma_2\gamma_5\gamma_4\gamma_5 u(\vec{p}, -\frac{1}{2}) \\
&= -iu^+(\vec{p}, -\frac{1}{2})\gamma_4\gamma_5 u(\vec{p}, -\frac{1}{2}) \\
&= 0
\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } F = \frac{1}{\sqrt{2}}[u(\vec{p}, \frac{1}{2})u^T(\vec{p}, -\frac{1}{2}) - u(\vec{p}, -\frac{1}{2})u^T(\vec{p}, \frac{1}{2})] \\
&= \frac{1}{\sqrt{2}}[u(\vec{p}, \frac{1}{2})u^+(\vec{p}, \frac{1}{2}) + u(\vec{p}, -\frac{1}{2})u^+(\vec{p}, -\frac{1}{2})]\gamma_2\gamma_5 = \frac{m-i\gamma^a p_a}{2\sqrt{2}m}\gamma_4\gamma_2\gamma_5 \\
&= \frac{ip^a}{2m\sqrt{2}}\gamma_a\gamma_5 C - \frac{1}{2\sqrt{2}}\gamma_5 C = -\frac{m-i\gamma^a p_a}{2\sqrt{2}m}\gamma_5 C
\end{aligned}$$

□

$$\text{证明: } [u(\vec{p}, \frac{1}{2})u^T(\vec{p}, -\frac{1}{2}) - u(\vec{p}, -\frac{1}{2})u^T(\vec{p}, \frac{1}{2})] = \frac{ip^a}{2m}\gamma_a\gamma_5 C - \frac{1}{2}\gamma_5 C$$

□

1.13 推论-自旋基分解:  $s = (s-1) \oplus 1$ 

$$\text{定理1.13.1. } \underbrace{U_{\lambda\zeta\mu\zeta} \cdots \sigma_\zeta \tau_\zeta}_{2s}(\vec{p}, h) = \sum_{h'=1}^{-1} \frac{\sqrt{C_{s+h}^{1+h'} C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^2}} \underbrace{U_{\lambda\zeta\mu\zeta} \cdots}_{2(s-1)}(\vec{p}, h-h') U_{\sigma_\zeta \tau_\zeta}(\vec{p}, h'), s \geq 1, -s \leq h \leq s$$

证明:  $U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)$

$$\begin{aligned}
&= \frac{\sqrt{s+h}}{\sqrt{2s}} U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}(\vec{p}, h - \frac{1}{2}) U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}(\vec{p}, h + \frac{1}{2}) U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \frac{\sqrt{s+h}}{\sqrt{2s}} \left[ \frac{\sqrt{s+h-1}}{\sqrt{2s-1}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-1) U_{\sigma_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s-1}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) U_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) \right] U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \\
&+ \frac{\sqrt{s-h}}{\sqrt{2s}} \left[ \frac{\sqrt{s+h}}{\sqrt{2s-1}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) U_{\sigma_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h-1}}{\sqrt{2s-1}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h+1) U_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) \right] U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \left[ \frac{\sqrt{s+h}}{\sqrt{2s}} \frac{\sqrt{s+h-1}}{\sqrt{2s-1}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-1) U_{\sigma_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+h}}{\sqrt{2s}} \frac{\sqrt{s-h}}{\sqrt{2s-1}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) U_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \right] \\
&+ \left[ \frac{\sqrt{s-h}}{\sqrt{2s}} \frac{\sqrt{s+h}}{\sqrt{2s-1}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) U_{\sigma_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \frac{\sqrt{s-h-1}}{\sqrt{2s-1}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h+1) U_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \right] \\
&= \frac{\sqrt{C_{s+h}^2 C_{s-h}^0}}{\sqrt{C_{2s}^2}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-1) U_{\sigma_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{C_{s+h}^1 C_{s-h}^1}}{\sqrt{C_{2s}^2}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) \frac{1}{\sqrt{2}} U_{\{\sigma_\zeta(\vec{p}, \frac{1}{2}) U_{\tau_\zeta}(\vec{p}, -\frac{1}{2})\}} \\
&+ \frac{\sqrt{C_{s+h}^0 C_{s-h}^2}}{\sqrt{C_{2s}^2}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h+1) U_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \frac{\sqrt{C_{s+h}^2 C_{s-h}^0}}{\sqrt{C_{2s}^2}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-1) U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 1) + \frac{\sqrt{C_{s+h}^1 C_{s-h}^1}}{\sqrt{C_{2s}^2}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h) U_{\sigma_\zeta \tau_\zeta}(\vec{p}, 0) \\
&+ \frac{\sqrt{C_{s+h}^0 C_{s-h}^2}}{\sqrt{C_{2s}^2}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h+1) U_{\sigma_\zeta \tau_\zeta}(\vec{p}, -1) \\
&= \sum_{h'=1}^{-1} \frac{\sqrt{C_{s+h'}^{1+h'} C_{s-h'}^{1-h'}}}{\sqrt{C_{2s}^2}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-h') U_{\sigma_\zeta \tau_\zeta}(\vec{p}, h') \quad \square
\end{aligned}$$

推论1.13.1.  $U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h) = U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta \sigma_\zeta}(\vec{p}, h), s \geq 1, -s \leq h \leq s$

### 1.14 推论-自旋基分解: $s + s' = s \oplus s'$

定理1.14.1.  $U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta \dots \tau_\zeta}(\vec{p}, h) = \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h'}^{s'+h'} C_{s+s'-h'}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2(s+s')}}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-h') U_{\rho_\zeta \sigma_\zeta \dots \tau_\zeta}(\vec{p}, h'), -s-s' \leq h \leq s+s'$

证明: 对  $s'$  采用数学归纳法证明此定理.

第一步:  $s'' = \frac{1}{2}$  时成立:

$$U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}, h) = \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+1/2+h'}^{1/2+h'} C_{s+1/2-h'}^{1/2-h'}}}{\sqrt{C_{2(s+1/2)}^{2(s+1/2)}}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-h') U_{\tau_\zeta}(\vec{p}, h'), -s - \frac{1}{2} \leq h \leq s + \frac{1}{2}$$

第二步: 假设  $s'' = s' - \frac{1}{2}$  时成立:

$$U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta \dots}(\vec{p}, h) = \sum_{h'=s'-1/2}^{-s'+1/2} \frac{\sqrt{C_{s+s'+1/2+h'}^{s'+1/2+h'} C_{s+s'+1/2-h'}^{s'+1/2-h'}}}{\sqrt{C_{2(s+s')-1}^{2(s+s')-1}}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-h') U_{\rho_\zeta \sigma_\zeta \dots}(\vec{p}, h')$$

$$-s-s'+\frac{1}{2} \leq h \leq s+s'-\frac{1}{2}$$

第三步:  $s'' = s'$  时:  $-s-s' \leq h \leq s+s', U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta \dots \tau_\zeta}(\vec{p}, h)$

$$\begin{aligned}
&= \frac{\sqrt{s+s'+h}}{\sqrt{2(s+s')}} U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta \dots}(\vec{p}, h - \frac{1}{2}) U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s+s'-h}}{\sqrt{2(s+s')}} U_{\lambda_\zeta \mu_\zeta \dots \rho_\zeta \sigma_\zeta \dots}(\vec{p}, h + \frac{1}{2}) U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \frac{\sqrt{s+s'+h}}{\sqrt{2(s+s')}} \left[ \sum_{h'=s'-1/2}^{-s'+1/2} \frac{\sqrt{C_{s+s'+1/2+h'}^{s'+1/2+h'} C_{s+s'+1/2-h'}^{s'+1/2-h'}}}{\sqrt{C_{2(s+s')-1}^{2(s+s')-1}}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h - \frac{1}{2} - h') U_{\rho_\zeta \sigma_\zeta \dots}(\vec{p}, h') \right] U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \\
&+ \left[ \sum_{h'=s'-1/2}^{-s'+1/2} \frac{\sqrt{C_{s+s'+1/2+h'}^{s'+1/2+h'} C_{s+s'+1/2-h'}^{s'+1/2-h'}}}{\sqrt{C_{2(s+s')-1}^{2(s+s')-1}}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h + \frac{1}{2} - h') U_{\rho_\zeta \sigma_\zeta \dots}(\vec{p}, h') \right] U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \left[ \sum_{h'=s'}^{-s'+1} \frac{\sqrt{C_{s+s'+1+h'}^{s'+1+h'} C_{s+s'+1-h'}^{s'+1-h'}}}{\sqrt{C_{2(s+s')}^{2(s+s')}}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, h-h') \frac{\sqrt{s+s'+h}}{\sqrt{2s'}} U_{\rho_\zeta \sigma_\zeta \dots}(\vec{p}, h' - \frac{1}{2}) \right] U_{\tau_\zeta}(\vec{p}, \frac{1}{2})
\end{aligned}$$

$$\begin{aligned}
& + \left[ \sum_{h'=s'-1}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-1-h}^{s'-1-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} U_{\lambda_s \mu_s \dots}(\vec{p}, h-h') \underbrace{\dots}_{2s} \frac{\sqrt{s+s'-h}}{\sqrt{2s'}} U_{\rho_s \sigma_s \dots}(\vec{p}, h'+\frac{1}{2}) \right] U_{\tau_s}(\vec{p}, -\frac{1}{2}) \\
& = \left[ \sum_{h'=s'}^{-s'+1} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} U_{\lambda_s \mu_s \dots}(\vec{p}, h-h') \frac{\sqrt{s'+h'}}{\sqrt{2s'}} U_{\rho_s \sigma_s \dots}(\vec{p}, h'-\frac{1}{2}) \right] U_{\tau_s}(\vec{p}, \frac{1}{2}) \\
& + \left[ \sum_{h'=s'-1}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} U_{\lambda_s \mu_s \dots}(\vec{p}, h-h') \frac{\sqrt{s'-h'}}{\sqrt{2s'}} U_{\rho_s \sigma_s \dots}(\vec{p}, h'+\frac{1}{2}) \right] U_{\tau_s}(\vec{p}, -\frac{1}{2}) \\
& = \left[ \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} U_{\lambda_s \mu_s \dots}(\vec{p}, h-h') \frac{\sqrt{s'+h'}}{\sqrt{2s'}} U_{\rho_s \sigma_s \dots}(\vec{p}, h'-\frac{1}{2}) \right] U_{\tau_s}(\vec{p}, \frac{1}{2}) \\
& + \left[ \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} U_{\lambda_s \mu_s \dots}(\vec{p}, h-h') \frac{\sqrt{s'-h'}}{\sqrt{2s'}} U_{\rho_s \sigma_s \dots}(\vec{p}, h'+\frac{1}{2}) \right] U_{\tau_s}(\vec{p}, -\frac{1}{2}) \\
& = \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} U_{\lambda_s \mu_s \dots}(\vec{p}, h-h') U_{\rho_s \sigma_s \dots \tau_s}(\vec{p}, h'), -s-s' \leq h \leq s+s'
\end{aligned}$$

此步证明了  $s'' = s'$  时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

**推论1.14.1.**  $-s_1 - s_2 \leq h \leq s_1 + s_2$

$$\begin{cases}
U_{\lambda_s \mu_s \dots \rho_s \sigma_s \dots}(\vec{p}, h) = \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \frac{\sqrt{C_{s_1+s_2+h}^{s_2+h} C_{s_1+s_2-h}^{s_2-h}}}{\sqrt{C_{2(s_1+s_2)}^{2s_2}}} U_{\lambda_s \mu_s \dots}(\vec{p}, h_1) U_{\rho_s \sigma_s \dots}(\vec{p}, h_2) \delta(h-h_1-h_2) \\
U_{\lambda_s \mu_s \dots \rho_s \sigma_s \dots}(\vec{p}, h) = \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \frac{\sqrt{C_{s_1+s_2+h}^{s_1+h} C_{s_1+s_2-h}^{s_1-h}}}{\sqrt{C_{2(s_1+s_2)}^{2s_1}}} U_{\lambda_s \mu_s \dots}(\vec{p}, h_1) U_{\rho_s \sigma_s \dots}(\vec{p}, h_2) \delta(h-h_1-h_2)
\end{cases}$$

**推论1.14.2.**  $-s_1 - s_2 \leq h \leq s_1 + s_2, U_{\lambda_s \mu_s \dots \rho_s \sigma_s \dots}(\vec{p}, h)$

$$= \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \left[ \frac{(2s_1)!(2s_2)! (s_1+h_1+s_2+h_2)! (s_1-h_1+s_2-h_2)!}{(2s_1+2s_2)! (s_1+h_1)!(s_2+h_2)! (s_1-h_1)!(s_2-h_2)!} \right]^{1/2} U_{\lambda_s \mu_s \dots}(\vec{p}, h_1) U_{\rho_s \sigma_s \dots}(\vec{p}, h_2) \delta(h-h_1-h_2)$$

## 1.15 推论-自旋基反向合成

**推论1.15.1.**  $\frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} U_{\lambda_s \mu_s \dots}(\vec{p}, h-h') = U_{\lambda_s \mu_s \dots \rho_s \sigma_s \dots \tau_s}(\vec{p}, h) \overline{U_{\rho_s \sigma_s \dots \tau_s}}(\vec{p}, h'), -s-s' \leq h \leq s+s'$

**推论1.15.2.**  $\frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} U_{\lambda_s \mu_s \dots}(\vec{p}, h-h') = \overline{U_{\rho_s \sigma_s \dots \tau_s}}(\vec{p}, h') U_{\rho_s \sigma_s \dots \tau_s \lambda_s \mu_s \dots}(\vec{p}, h), -s-s' \leq h \leq s+s'$

**推论1.15.3.**  $U_{\lambda_s \mu_s \dots}(\vec{p}, h_1) = \frac{\sqrt{C_{2(s_1+s_2)}^{2s_2}}}{\sqrt{C_{s_1+h_1+s_2+h_2}^{s_2+h_2} C_{s_1-h_1+s_2-h_2}^{s_2-h_2}}} U_{\lambda_s \mu_s \dots \rho_s \sigma_s \dots}(\vec{p}, h_1+h_2) \overline{U_{\rho_s \sigma_s \dots}}(\vec{p}, h_2)$

**推论1.15.4.**  $\begin{cases} U_{\lambda_s \dots \sigma_s \tau_s}(\vec{p}, h) \overline{U_{\tau_s}}(\vec{p}, \frac{1}{2}) = \frac{\sqrt{s+h}}{\sqrt{2s}} U_{\lambda_s \dots \sigma_s}(\vec{p}, h-\frac{1}{2}), -s \leq h \leq s \\ U_{\lambda_s \dots \sigma_s \tau_s}(\vec{p}, h) \overline{U_{\tau_s}}(\vec{p}, -\frac{1}{2}) = \frac{\sqrt{s-h}}{\sqrt{2s}} U_{\lambda_s \dots \sigma_s}(\vec{p}, h+\frac{1}{2}), -s \leq h \leq s \end{cases}$

## 1.16 推论-自旋基分解: $s_1 + s_2 + s_3 = s_1 \oplus s_2 \oplus s_3$

**推论1.16.1.**  $-s_1 - s_2 - s_3 \leq h \leq s_1 + s_2 + s_3, U_{\lambda_s \mu_s \dots \eta_s \xi_s \dots \rho_s \sigma_s \dots}(\vec{p}, h)$

$$\begin{aligned}
& = \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \sum_{h_3=s_3}^{-s_3} \left[ \frac{(2s_1)!(2s_2)!(2s_3)! (s_1+h_1+s_2+h_2+s_3+h_3)! (s_1-h_1+s_2-h_2+s_3-h_3)!}{(2s_1+2s_2+2s_3)! (s_1+h_1)!(s_2+h_2)!(s_3+h_3)! (s_1-h_1)!(s_2-h_2)!(s_3-h_3)!} \right]^{1/2} \\
& U_{\lambda_s \mu_s \dots}(\vec{p}, h_1) U_{\eta_s \xi_s \dots}(\vec{p}, h_2) U_{\rho_s \sigma_s \dots}(\vec{p}, h_3) \delta(h-h_1-h_2-h_3)
\end{aligned}$$

证明:  $-s_1 - s_2 - s_3 \leq h \leq s_1 + s_2 + s_3, U_{\lambda_{\zeta}\mu_{\zeta}\dots\eta_{\zeta}\xi_{\zeta}\dots\rho_{\zeta}\sigma_{\zeta}\dots}(\vec{p}, h)$

$$= \sum_{h_1=s_1}^{-s_1} \sum_{h_{23}=s_2+s_3}^{-s_2-s_3} \left[ \frac{(2s_1)!(2s_2+2s_3)! (s_1+h_1+s_2+s_3+h_{23})! (s_1-h_1+s_2+s_3-h_{23})!}{(2s_1+2s_2+2s_3)! (s_1+h_1)!(s_2+s_3+h_{23})! (s_1-h_1)!(s_2+s_3-h_{23})!} \right]^{1/2}$$

$$U_{\lambda_{\zeta}\mu_{\zeta}\dots}(\vec{p}, h_1) U_{\eta_{\zeta}\xi_{\zeta}\dots\rho_{\zeta}\sigma_{\zeta}\dots}(\vec{p}, h_{23}) \delta(h - h_1 - h_{23})$$

$$= \sum_{h_1=s_1}^{-s_1} \sum_{h_{23}=s_2+s_3}^{-s_2-s_3} \left[ \frac{(2s_1)!(2s_2+2s_3)! (s_1+h_1+s_2+s_3+h_{23})! (s_1-h_1+s_2+s_3-h_{23})!}{(2s_1+2s_2+2s_3)! (s_1+h_1)!(s_2+s_3+h_{23})! (s_1-h_1)!(s_2+s_3-h_{23})!} \right]^{1/2} U_{\lambda_{\zeta}\mu_{\zeta}\dots}(\vec{p}, h_1) \delta(h - h_1 - h_{23})$$

$$\sum_{h_2=s_2}^{-s_2} \sum_{h_3=s_3}^{-s_3} \left[ \frac{(2s_2)!(2s_3)! (s_2+h_2+s_3+h_3)! (s_2-h_2+s_3-h_3)!}{(2s_2+2s_3)! (s_2+h_2)!(s_3+h_3)! (s_2-h_2)!(s_3-h_3)!} \right]^{1/2} U_{\eta_{\zeta}\xi_{\zeta}\dots}(\vec{p}, h_2) U_{\rho_{\zeta}\sigma_{\zeta}\dots}(\vec{p}, h_3) \delta(h_{23} - h_2 - h_3)$$

$$= \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \sum_{h_3=s_3}^{-s_3} \left[ \frac{(2s_1)!(2s_2)!(2s_3)! (s_1+h_1+s_2+h_2+s_3+h_3)! (s_1-h_1+s_2-h_2+s_3-h_3)!}{(2s_1+2s_2+2s_3)! (s_1+h_1)!(s_2+h_2)!(s_3+h_3)! (s_1-h_1)!(s_2-h_2)!(s_3-h_3)!} \right]^{1/2}$$

$$U_{\lambda_{\zeta}\mu_{\zeta}\dots}(\vec{p}, h_1) U_{\eta_{\zeta}\xi_{\zeta}\dots}(\vec{p}, h_2) U_{\rho_{\zeta}\sigma_{\zeta}\dots}(\vec{p}, h_3) \delta(h - h_1 - h_2 - h_3) \quad \square$$

**1.17 推论-自旋基分解:**  $s_1 + s_2 \dots + s_n = s_1 \oplus s_2 \dots \oplus s_n$

推论1.17.1.  $-\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, U_{\lambda_{\zeta}\mu_{\zeta}\dots\eta_{\zeta}\xi_{\zeta}\dots\rho_{\zeta}\sigma_{\zeta}\dots}(\vec{p}, h)$

$$= \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \dots \sum_{h_n=s_n}^{-s_n} \left[ \frac{\prod_{i=1}^n (2s_i)! [\sum_{i=1}^n (s_i+h_i)]! [\sum_{i=1}^n (s_i-h_i)]!}{[\sum_{i=1}^n (2s_i)]! \prod_{i=1}^n (s_i+h_i)! \prod_{i=1}^n (s_i-h_i)!} \right]^{1/2} U_{\lambda_{\zeta}\mu_{\zeta}\dots}(\vec{p}, h_1) U_{\eta_{\zeta}\xi_{\zeta}\dots}(\vec{p}, h_2) \dots U_{\rho_{\zeta}\sigma_{\zeta}\dots}(\vec{p}, h_n) \delta(h - \sum_{i=1}^n h_i)$$

**1.18 一个重要的数学推论**

推论1.18.1.  $\sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \dots \sum_{h_n=s_n}^{-s_n} \frac{[\sum_{i=1}^n (s_i+h_i)]! [\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i+h_i)! \prod_{i=1}^n (s_i-h_i)!} \delta(h - \sum_{i=1}^n h_i) = \frac{[\sum_{i=1}^n (2s_i)]!}{\prod_{i=1}^n (2s_i)!}, -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i$

**1.19 推论-自旋基分解:**  $s = \frac{1}{2} \oplus \frac{1}{2} \dots \oplus \frac{1}{2}$

推论1.19.1.  $-s \leq h \leq s$

$$U_{\lambda_{\zeta}\mu_{\zeta}\dots\tau_{\zeta}}(\vec{p}, h) = \sum_{h_1=1/2}^{-1/2} \sum_{h_2=1/2}^{-1/2} \dots \sum_{h_n=1/2}^{-1/2} \left[ \frac{(s+h)!(s-h)!}{(2s)!} \right]^{1/2} U_{\lambda_{\zeta}}(\vec{p}, h_1) U_{\mu_{\zeta}}(\vec{p}, h_2) \dots U_{\tau_{\zeta}}(\vec{p}, h_n) \delta(h - \sum_{i=1}^n h_i)$$

推论1.19.2.  $-s \leq h \leq s$

$$U_{\lambda_{\zeta}\mu_{\zeta}\dots\sigma_{\zeta}\tau_{\zeta}}(\vec{p}, h) = \frac{1}{(2s)!} \sqrt{C_{2s}^{s-h}} \underbrace{u_{\{\lambda_{\zeta}}(\vec{p}, \frac{1}{2}) u_{\mu_{\zeta}}(\vec{p}, \frac{1}{2}) \dots u_{\sigma_{\zeta}}(\vec{p}, -\frac{1}{2}) u_{\tau_{\zeta}}(\vec{p}, -\frac{1}{2})\}}_{s+h} \underbrace{\dots}_{s-h}$$

推论1.19.3.  $-s \leq h \leq s, \forall s$

$$U_{\lambda_{\zeta}\dots\sigma_{\zeta}\tau_{\zeta}}(\vec{p}, h) = \frac{\sqrt{s+h}}{\sqrt{2s}} U_{\lambda_{\zeta}\dots\sigma_{\zeta}}(\vec{p}, h - \frac{1}{2}) U_{\tau_{\zeta}}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} U_{\lambda_{\zeta}\dots\sigma_{\zeta}}(\vec{p}, h + \frac{1}{2}) U_{\tau_{\zeta}}(\vec{p}, -\frac{1}{2})$$

$$\Leftrightarrow U_{\lambda_{\zeta}\mu_{\zeta}\dots\sigma_{\zeta}\tau_{\zeta}}(\vec{p}, h) = \frac{1}{(2s)!} \sqrt{C_{2s}^{s-h}} \underbrace{u_{\{\lambda_{\zeta}}(\vec{p}, \frac{1}{2}) u_{\mu_{\zeta}}(\vec{p}, \frac{1}{2}) \dots u_{\sigma_{\zeta}}(\vec{p}, -\frac{1}{2}) u_{\tau_{\zeta}}(\vec{p}, -\frac{1}{2})\}}_{s+h} \underbrace{\dots}_{s-h}$$

**1.20 推论- $U_{\lambda_{\zeta}\mu_{\zeta}\dots\sigma_{\zeta}\tau_{\zeta}}(\vec{p}, h)$ 的全对称性**

定理1.20.1.  $U_{\lambda_{\zeta}\mu_{\zeta}\dots\sigma_{\zeta}\tau_{\zeta}}(\vec{p}, h) = \frac{1}{(2s)!} U_{\{\lambda_{\zeta}\mu_{\zeta}\dots\tau_{\zeta}\sigma_{\zeta}\}}(\vec{p}, h), -s \leq h \leq s$

证明: 采用数学归纳法证明此定理。

第一步:  $s' = \frac{1}{2}, 1$ 时成立:

$$U_{\lambda_{\zeta}}(\vec{p}, h) = \frac{1}{1!} U_{\lambda_{\zeta}}(\vec{p}, h), -\frac{1}{2} \leq h \leq \frac{1}{2}; U_{\lambda_{\zeta}\mu_{\zeta}}(\vec{p}, h) = \frac{1}{2!} U_{\{\lambda_{\zeta}\mu_{\zeta}\}}(\vec{p}, h), -1 \leq h \leq 1$$

第二步: 假设  $s' = s - \frac{1}{2}$ 时成立:

$$U_{\lambda_{\zeta}\mu_{\zeta}\dots\sigma_{\zeta}}(\vec{p}, h) = \frac{1}{(2s-1)!} U_{\{\lambda_{\zeta}\mu_{\zeta}\dots\tau_{\zeta}\sigma_{\zeta}\}}(\vec{p}, h), -s + \frac{1}{2} \leq h \leq s - \frac{1}{2}$$

第三步:  $1 \leq s' = s$ 时:  $-s \leq h \leq s, U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)$

$$\begin{aligned} &= \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+h'} C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}} U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}(\vec{p}, h-h') U_{\tau_\zeta}(\vec{p}, h') = \sum_{h'=1}^{-1} \frac{\sqrt{C_{s+h}^{1+h'} C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^2}} U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta}(\vec{p}, h-h') U_{\sigma_\zeta \tau_\zeta}(\vec{p}, h') \\ \Rightarrow &U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h) = \frac{1}{(2s-1)!} U_{\{\lambda_\zeta \mu_\zeta \dots \tau_\zeta \sigma_\zeta\}}(\vec{p}, h), U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h) = U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h), -s \leq h \leq s \\ \Leftrightarrow &U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h) = \frac{1}{(2s)!} U_{\{\lambda_\zeta \mu_\zeta \dots \tau_\zeta \sigma_\zeta\}}(\vec{p}, h), -s \leq h \leq s \end{aligned}$$

此步证明了  $s' = s$ 时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

## 1.21 有质量 $s$ -自旋粒子的准投影算子

引理1.21.1.

$$\begin{cases} \hat{J}_+(\vec{p}, s - \frac{1}{2}; \gamma_a) U_{\lambda_\zeta \otimes \mu_\zeta \dots}(\vec{p}, h - \frac{1}{2}) = \sqrt{(s-h)(s+h)} U_{\lambda_\zeta \otimes \mu_\zeta \dots}(\vec{p}, h + \frac{1}{2}), -s \leq h \leq s \\ \hat{J}_-(\vec{p}, s - \frac{1}{2}; \gamma_a) U_{\lambda_\zeta \otimes \mu_\zeta \dots}(\vec{p}, h + \frac{1}{2}) = \sqrt{(s+h)(s-h)} U_{\lambda_\zeta \otimes \mu_\zeta \dots}(\vec{p}, h - \frac{1}{2}), -s \leq h \leq s \\ \hat{J}_z(\vec{p}, s; \gamma_a) U_{\lambda_\zeta \otimes \mu_\zeta \dots}(\vec{p}, h) = h U_{\lambda_\zeta \otimes \mu_\zeta \dots}(\vec{p}, h), -s \leq h \leq s \\ \hat{J}(s) \prec \hat{J}(\vec{p}, s; \gamma_a) \end{cases}$$

定理1.21.1.  $\Lambda_+(s) = [\frac{2s+1}{4s} + \frac{1}{s} \hat{J}_x(s - \frac{1}{2}) \otimes \hat{J}_x(\frac{1}{2}) + \frac{1}{s} \hat{J}_y(s - \frac{1}{2}) \otimes \hat{J}_y(\frac{1}{2}) + \frac{1}{s} \hat{J}_z(s - \frac{1}{2}) \otimes \hat{J}_z(\frac{1}{2})][\Lambda_+(s - \frac{1}{2}) \otimes \Lambda_+(\frac{1}{2})]$

$$\begin{aligned} \text{证明: } \Lambda_+(s) &\prec \sum_{h=s}^{-s} U_{\lambda_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h) U_{\lambda'_\zeta \dots \sigma'_\zeta \tau'_\zeta}^+(\vec{p}, h) \\ &= \sum_{h=s}^{-s} [\frac{\sqrt{s+h}}{\sqrt{2s}} U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h - \frac{1}{2}) U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h + \frac{1}{2}) U_{\tau_\zeta}(\vec{p}, -\frac{1}{2})] \\ &[\frac{\sqrt{s+h}}{\sqrt{2s}} U_{\lambda'_\zeta \dots \sigma'_\zeta}^+(\vec{p}, h - \frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} U_{\lambda'_\zeta \dots \sigma'_\zeta}^+(\vec{p}, h + \frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\ &= \sum_{h=s}^{-s} [\frac{s+h}{2s} U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h - \frac{1}{2}) U_{\lambda'_\zeta \dots \sigma'_\zeta}^+(\vec{p}, h - \frac{1}{2})][U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, \frac{1}{2})] \\ &+ \sum_{h=s}^{-s} [\frac{s-h}{2s} U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h + \frac{1}{2}) U_{\lambda'_\zeta \dots \sigma'_\zeta}^+(\vec{p}, h + \frac{1}{2})][U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\ &+ \sum_{h=s}^{-s} [\frac{\sqrt{(s+h)(s-h)}}{2s} U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h - \frac{1}{2}) U_{\lambda'_\zeta \dots \sigma'_\zeta}^+(\vec{p}, h + \frac{1}{2})][U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\ &+ \sum_{h=s}^{-s} [\frac{\sqrt{(s-h)(s+h)}}{2s} U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h + \frac{1}{2}) U_{\lambda'_\zeta \dots \sigma'_\zeta}^+(\vec{p}, h - \frac{1}{2})][U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, \frac{1}{2})] \\ &= [\frac{s+\frac{1}{2}+\hat{J}_z(s-\frac{1}{2})}{2s} \sum_{h=s}^{-s} U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h - \frac{1}{2}) U_{\lambda'_\zeta \dots \sigma'_\zeta}^+(\vec{p}, h - \frac{1}{2})][U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, \frac{1}{2})] \\ &+ [\frac{s+\frac{1}{2}-\hat{J}_z(s-\frac{1}{2})}{2s} \sum_{h=s}^{-s} U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h + \frac{1}{2}) U_{\lambda'_\zeta \dots \sigma'_\zeta}^+(\vec{p}, h + \frac{1}{2})][U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\ &+ [\frac{\hat{J}_-(s-\frac{1}{2})}{2s} \sum_{h=s}^{-s} U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h + \frac{1}{2}) U_{\lambda'_\zeta \dots \sigma'_\zeta}^+(\vec{p}, h + \frac{1}{2})][U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\ &+ [\frac{\hat{J}_+(s-\frac{1}{2})}{2s} \sum_{h=s}^{-s} U_{\lambda_\zeta \dots \sigma_\zeta}(\vec{p}, h - \frac{1}{2}) U_{\lambda'_\zeta \dots \sigma'_\zeta}^+(\vec{p}, h - \frac{1}{2})][U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, \frac{1}{2})] \end{aligned}$$

$$\begin{aligned}
&= \left[ \frac{s+\frac{1}{2}+\hat{J}_z(s-\frac{1}{2})}{2s} \sum_{h=s-1/2}^{-s+1/2} U_{\underbrace{\lambda_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}}^+(\vec{p}, h) \right] [U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, \frac{1}{2})] \\
&+ \left[ \frac{s+\frac{1}{2}-\hat{J}_z(s-\frac{1}{2})}{2s} \sum_{h=s-1/2}^{-s+1/2} U_{\underbrace{\lambda_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}}^+(\vec{p}, h) \right] [U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
&+ \left[ \frac{\hat{J}_-(s-\frac{1}{2})}{2s} \sum_{h=s-1/2}^{-s+1/2} U_{\underbrace{\lambda_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}}^+(\vec{p}, h) \right] [U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
&+ \left[ \frac{\hat{J}_+(s-\frac{1}{2})}{2s} \sum_{h=s-1/2}^{-s+1/2} U_{\underbrace{\lambda_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}}^+(\vec{p}, h) \right] [U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, \frac{1}{2})] \\
&= \left[ \frac{s+\frac{1}{2}+\hat{J}_z(s-\frac{1}{2})}{2s} \sum_{h=s-1/2}^{-s+1/2} U_{\underbrace{\lambda_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}}^+(\vec{p}, h) \right] [U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, \frac{1}{2})] \\
&+ \left[ \frac{\hat{J}_+(s-\frac{1}{2})}{2s} \sum_{h=s-1/2}^{-s+1/2} U_{\underbrace{\lambda_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}}^+(\vec{p}, h) \right] [\hat{J}_-(\frac{1}{2}) U_{\tau_\zeta}(\vec{p}, \frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, \frac{1}{2})] \\
&+ \left[ \frac{s+\frac{1}{2}-\hat{J}_z(s-\frac{1}{2})}{2s} \sum_{h=s-1/2}^{-s+1/2} U_{\underbrace{\lambda_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}}^+(\vec{p}, h) \right] [U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
&+ \left[ \frac{\hat{J}_-(s-\frac{1}{2})}{2s} \sum_{h=s-1/2}^{-s+1/2} U_{\underbrace{\lambda_\zeta \cdots \sigma_\zeta}_{2s-1}}(\vec{p}, h) U_{\underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}}^+(\vec{p}, h) \right] [\hat{J}_+(\frac{1}{2}) U_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) U_{\tau'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
&\succ \left[ \frac{2s+1}{4s} + \frac{1}{s} \hat{J}_x(s-\frac{1}{2}) \otimes \hat{J}_x(\frac{1}{2}) + \frac{1}{s} \hat{J}_y(s-\frac{1}{2}) \otimes \hat{J}_y(\frac{1}{2}) + \frac{1}{s} \hat{J}_z(s-\frac{1}{2}) \otimes \hat{J}_z(\frac{1}{2}) \right] [\Lambda_+(s-\frac{1}{2}) \otimes \Lambda_+(\frac{1}{2})] \quad \square
\end{aligned}$$

$$\begin{aligned}
&\text{定理1.21.2. } \Lambda_{+\underbrace{\lambda_\zeta \cdots \sigma_\zeta}_{2s} \underbrace{\lambda''_\zeta \cdots \sigma''_\zeta}_{2s}}(s) - \frac{3}{4} \Lambda_{+\underbrace{\lambda_\zeta \cdots \sigma_\zeta}_{2s-1} \underbrace{\lambda''_\zeta \cdots \sigma''_\zeta}_{2s-1}}(s-\frac{1}{2}) \Lambda_{+\tau_\zeta \tau''_\zeta}(\frac{1}{2}) \\
&= \left[ \hat{J}_x \underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}(s-\frac{1}{2}) \hat{J}_x \tau'_\zeta(\frac{1}{2}) + \hat{J}_y \underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}(s-\frac{1}{2}) \hat{J}_y \tau'_\zeta(\frac{1}{2}) + \hat{J}_z \underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1}(s-\frac{1}{2}) \hat{J}_z \tau'_\zeta(\frac{1}{2}) \right] \frac{1}{s} \Lambda_{+\underbrace{\lambda'_\zeta \cdots \sigma'_\zeta}_{2s-1} \underbrace{\lambda''_\zeta \cdots \sigma''_\zeta}_{2s-1}}(s-\frac{1}{2}) \Lambda_{+\tau'_\zeta \tau''_\zeta}(\frac{1}{2})
\end{aligned}$$

## 1.22 有质量1-自旋粒子的准投影算子

$$\text{定理1.22.1. } \Lambda_+(1) = \left[ \frac{3}{4} + \hat{J}_x(\frac{1}{2}) \otimes \hat{J}_x(\frac{1}{2}) + \hat{J}_y(\frac{1}{2}) \otimes \hat{J}_y(\frac{1}{2}) + \hat{J}_z(\frac{1}{2}) \otimes \hat{J}_z(\frac{1}{2}) \right] [\Lambda_+(\frac{1}{2}) \otimes \Lambda_+(\frac{1}{2})]$$

$$\begin{aligned}
&\text{推论1.22.1. } \Lambda_{+\lambda_\zeta \mu_\zeta \lambda'_\zeta \mu'_\zeta}(1) - \frac{1}{2} \Lambda_{+\lambda_\zeta \lambda'_\zeta}(\frac{1}{2}) \Lambda_{+\mu_\zeta \mu'_\zeta}(\frac{1}{2}) = \\
&\frac{1}{2} \Lambda_{+\lambda_\zeta \mu'_\zeta}(\frac{1}{2}) \Lambda_{+\mu_\zeta \lambda'_\zeta}(\frac{1}{2}) = \left[ \frac{1}{2} \delta_{\lambda_\zeta \lambda'_\zeta} \frac{1}{2} \delta_{\mu_\zeta \mu'_\zeta} + \hat{J}_x \lambda'_\zeta(\frac{1}{2}) \hat{J}_x \mu'_\zeta(\frac{1}{2}) + \hat{J}_y \lambda'_\zeta(\frac{1}{2}) \hat{J}_y \mu'_\zeta(\frac{1}{2}) + \hat{J}_z \lambda'_\zeta(\frac{1}{2}) \hat{J}_z \mu'_\zeta(\frac{1}{2}) \right] \Lambda_{+\lambda'_\zeta \lambda'_\zeta}(\frac{1}{2}) \Lambda_{+\mu'_\zeta \mu'_\zeta}(\frac{1}{2})
\end{aligned}$$

## 1.23 Bargmann-Wigner方程平面波解的算子表述

$$\text{定理1.23.1. } (\gamma^a \partial_a + m)_{\kappa_\zeta} \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}}(x) = 0, \psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}}(x) = \frac{1}{(2s)!} \psi_{\{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}\}}(x)$$

$$\psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{m^s}{\sqrt{E}} \sum_{h=s}^{-s} \frac{\hat{J}_-^{s-h}(\vec{p}, s; \gamma_a)}{(s-h)! \sqrt{C_{2s}^{s-h}}} [a(\vec{p}, h) U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}}(\vec{p}, s) e^{ip \cdot x} + b^+(\vec{p}, h) V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}}(\vec{p}, s) e^{-ip \cdot x}] d^3 \vec{p}$$

$$\psi_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{m^s}{\sqrt{E}} \sum_{h=s}^{-s} \frac{\hat{J}_+^{s+h}(\vec{p}, s; \gamma_a)}{(s+h)! \sqrt{C_{2s}^{s+h}}} [a(\vec{p}, h) U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}}(\vec{p}, -s) e^{ip \cdot x} + b^+(\vec{p}, h) V_{\underbrace{\lambda_\zeta \mu_\zeta \cdots}_{2s}}(\vec{p}, -s) e^{-ip \cdot x}] d^3 \vec{p}$$

## 2 Klein-Gordon方程自旋基的重新梳理分析

### 2.1 静止系的有质量矢量自旋基

$$\text{推论2.1.1. } (R \cdot \hat{p}) \varepsilon(\vec{p}, h) = h \varepsilon(\vec{p}, h), (R \cdot \hat{p}) \frac{p_{[a]}^{\alpha]}{m} = 0; R^2 \varepsilon(\vec{p}, h) = 1(1+1) \varepsilon(\vec{p}, h)$$

$$\text{推论2.1.2. } \begin{cases} \lambda_m \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 1 \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ -1 \\ 0 \end{bmatrix} \\ \lambda_m \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, 0 \right) = \begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix} \\ \lambda_m \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, -1 \right) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ -1 \\ 0 \end{bmatrix} \end{cases}$$

$$\text{推论2.1.3. } \varepsilon_a\left(\begin{bmatrix} 0 \\ 0 \\ |\vec{p}| \end{bmatrix}, 1\right) := \frac{1}{\sqrt{2}}[-1, -i, 0, 0]_a, \varepsilon_a\left(\begin{bmatrix} 0 \\ 0 \\ |\vec{p}| \end{bmatrix}, 0\right) := \frac{1}{m}[0, 0, E, i|\vec{p}|]_a, \varepsilon_a\left(\begin{bmatrix} 0 \\ 0 \\ |\vec{p}| \end{bmatrix}, -1\right) := \frac{1}{\sqrt{2}}[1, -i, 0, 0]_a$$

$$\text{推论2.1.4. } \varepsilon_a(\vec{0}, 1) := \frac{1}{\sqrt{2}}[-1, -i, 0, 0]_a, \varepsilon_a(\vec{0}, 0) := [0, 0, 1, 0]_a, \varepsilon_a(\vec{0}, -1) := \frac{1}{\sqrt{2}}[1, -i, 0, 0]_a$$

$$\text{推论2.1.5. } \begin{cases} (R_x + iR_y)\varepsilon_a(\vec{0}, h) = \varepsilon_a(\vec{0}, h+1), -1 \leq h < 1; (R_x + iR_y)\varepsilon_a(\vec{0}, 1) = 0 \\ (R_x - iR_y)\varepsilon_a(\vec{0}, h) = \varepsilon_a(\vec{0}, h-1), -1 < h \leq 1; (R_x - iR_y)\varepsilon_a(\vec{0}, -1) = 0 \end{cases}$$

## 2.2 z轴方向的有质量矢量自旋基

推论2.2.1.

$$L_{\vec{v}} = e^{-\ln[\gamma_v(1+v)]\vec{v}\cdot L} = 1 - \gamma_v(\vec{v}\cdot L) + \frac{\gamma_v-1}{v^2}(\vec{v}\cdot L)^2 = \gamma_v(1 - \vec{v}\cdot L) - \frac{\gamma_v-1}{v^2}(\vec{v}\cdot R)^2, L_{\vec{v}}L_{-\vec{v}} = L_{-\vec{v}}L_{\vec{v}} = I$$

$$\text{定理2.2.1. } \varepsilon\left(\begin{bmatrix} 0 \\ 0 \\ |\vec{p}| \end{bmatrix}, h\right) = e^{-\ln[\gamma_v(1+v)]L_z}\varepsilon(\vec{0}, h)$$

$$\text{证明: } e^{-\ln[\gamma_v(1+v)]L_z}\varepsilon(\vec{0}, 1) = \frac{1}{m} \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix} \frac{1}{\sqrt{2}}[-1, -i, 0, 0]^T = \frac{1}{\sqrt{2}}[-1, -i, 0, 0]^T \quad \square$$

$$\text{证明: } e^{-\ln[\gamma_v(1+v)]L_z}\varepsilon(\vec{0}, 0) = \frac{1}{m} \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix} [0, 0, 1, 0]^T = \frac{1}{m}[0, 0, E, i|\vec{p}|]^T \quad \square$$

$$\text{证明: } e^{-\ln[\gamma_v(1+v)]L_z}\varepsilon(\vec{0}, -1) = \frac{1}{m} \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix} \frac{1}{\sqrt{2}}[1, -i, 0, 0]^T = \frac{1}{\sqrt{2}}[1, -i, 0, 0]^T \quad \square$$

定理2.2.2.

$$\begin{cases} e^{-\ln[\gamma_v(1+v)]L_z}R_x e^{\ln[\gamma_v(1+v)]L_z} = \frac{E}{m}R_x - \frac{i|\vec{p}|}{m}L_y \\ e^{-\ln[\gamma_v(1+v)]L_z}R_y e^{\ln[\gamma_v(1+v)]L_z} = \frac{E}{m}R_y + \frac{i|\vec{p}|}{m}L_x \\ e^{-\ln[\gamma_v(1+v)]L_z}R_z e^{\ln[\gamma_v(1+v)]L_z} = R_z \end{cases}$$

$$\text{证明: } e^{-\ln[\gamma_v(1+v)]L_z}R_z e^{\ln[\gamma_v(1+v)]L_z}$$

$$\begin{aligned} &= [1 - \gamma_v v L_z + (\gamma_v - 1)L_z^2]R_z[1 + \gamma_v v L_z + (\gamma_v - 1)L_z^2] \\ &= \frac{1}{m^2}[m - |\vec{p}|L_z + (E - m)L_z^2]R_z[m + |\vec{p}|L_z + (E - m)L_z^2] \\ &= \frac{1}{m^2} \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix} R_z \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & i|\vec{p}| \\ 0 & 0 & -i|\vec{p}| & E \end{bmatrix} = R_z \quad \square \end{aligned}$$

$$\text{证明: } e^{-\ln[\gamma_v(1+v)]L_z}R_x e^{\ln[\gamma_v(1+v)]L_z}$$

$$\begin{aligned} &= \frac{1}{m^2} \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & i|\vec{p}| \\ 0 & 0 & -i|\vec{p}| & E \end{bmatrix} \\ &= \frac{1}{m^2} \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -iE & |\vec{p}| \\ 0 & im & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \frac{1}{m} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -iE & |\vec{p}| \\ 0 & iE & 0 & 0 \\ 0 & -|\vec{p}| & 0 & 0 \end{bmatrix} = \frac{E}{m}R_x - \frac{i|\vec{p}|}{m}L_y \quad \square \end{aligned}$$

$$\text{证明: } e^{-\ln[\gamma_v(1+v)]L_z}R_y e^{\ln[\gamma_v(1+v)]L_z}$$

$$\begin{aligned} &= \frac{1}{m^2} \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix} \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & i|\vec{p}| \\ 0 & 0 & -i|\vec{p}| & E \end{bmatrix} \\ &= \frac{1}{m^2} \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & E & -i|\vec{p}| \\ 0 & 0 & i|\vec{p}| & E \end{bmatrix} \begin{bmatrix} 0 & 0 & iE & -|\vec{p}| \\ 0 & 0 & 0 & 0 \\ -im & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \frac{1}{m} \begin{bmatrix} 0 & 0 & iE & -|\vec{p}| \\ 0 & 0 & 0 & 0 \\ -iE & 0 & 0 & 0 \\ |\vec{p}| & 0 & 0 & 0 \end{bmatrix} = \frac{E}{m}R_y + \frac{i|\vec{p}|}{m}L_x \quad \square \end{aligned}$$

推论2.2.2.

$$\begin{cases} e^{-\ln[\gamma_v(1+v)]L_z}(R_x + iR_y)e^{\ln[\gamma_v(1+v)]L_z} = \frac{E}{m}(R_x + iR_y) - \frac{|\vec{p}|}{m}(L_x + iL_y) \\ e^{-\ln[\gamma_v(1+v)]L_z}(R_x - iR_y)e^{\ln[\gamma_v(1+v)]L_z} = \frac{E}{m}(R_x - iR_y) + \frac{|\vec{p}|}{m}(L_x - iL_y) \\ e^{-\ln[\gamma_v(1+v)]L_z}R_z e^{\ln[\gamma_v(1+v)]L_z} = R_z \end{cases}$$







推论2.5.1.

$$\begin{cases} e^{-ln[\gamma_v(1+v)]L_z} R_x e^{ln[\gamma_v(1+v)]L_z} e^{-i\vec{\omega}\cdot R} = \frac{E}{m} [R_x - \frac{\hat{p}_x}{1+\hat{p}_z} (R \cdot \hat{p} + R_z)] - \frac{i|\vec{p}|}{m} [L_y - \frac{\hat{p}_y}{1+\hat{p}_z} (L \cdot \hat{p} + L_z)] \\ e^{i\vec{\omega}\cdot R} e^{-ln[\gamma_v(1+v)]L_z} R_y e^{ln[\gamma_v(1+v)]L_z} e^{-i\vec{\omega}\cdot R} = \frac{E}{m} [R_y - \frac{\hat{p}_y}{1+\hat{p}_z} (R \cdot \hat{p} + R_z)] + \frac{i|\vec{p}|}{m} [L_x - \frac{\hat{p}_x}{1+\hat{p}_z} (L \cdot \hat{p} + L_z)] \\ e^{i\vec{\omega}\cdot R} e^{-ln[\gamma_v(1+v)]L_z} R_z e^{ln[\gamma_v(1+v)]L_z} e^{-i\vec{\omega}\cdot R} = R \cdot \hat{p} \end{cases}$$

定义2.5.1.

$$\begin{cases} \hat{J}_x(\vec{p}, 1; R, L) := \frac{E}{m} [R_x - \frac{\hat{p}_x}{1+\hat{p}_z} (R \cdot \hat{p} + R_z)] - \frac{i|\vec{p}|}{m} [L_y - \frac{\hat{p}_y}{1+\hat{p}_z} (L \cdot \hat{p} + L_z)] \\ \hat{J}_y(\vec{p}, 1; R, L) := \frac{E}{m} [R_y - \frac{\hat{p}_y}{1+\hat{p}_z} (R \cdot \hat{p} + R_z)] + \frac{i|\vec{p}|}{m} [L_x - \frac{\hat{p}_x}{1+\hat{p}_z} (L \cdot \hat{p} + L_z)] \\ \hat{J}_z(\vec{p}, 1; R, L) := R \cdot \hat{p} \end{cases}$$

推论2.5.2.  $[\hat{J}_i(\vec{p}, 1; R, L), \hat{J}_j(\vec{p}, 1; R, L)] = \varepsilon_{ij}^k \hat{J}_k(\vec{p}, 1; R, L)$

推论2.5.3.

$$\begin{cases} \hat{J}_+(\vec{p}, 1; R, L) := e^{i\vec{\omega}\cdot R} e^{-ln[\gamma_v(1+v)]L_z} (R_x + iR_y) e^{ln[\gamma_v(1+v)]L_z} e^{-i\vec{\omega}\cdot R} \\ = \frac{E}{m} [(R_x + iR_y) - \frac{\hat{p}_x + i\hat{p}_y}{1+\hat{p}_z} (R \cdot \hat{p} + R_z)] - \frac{i|\vec{p}|}{m} [(L_x + iL_y) - \frac{\hat{p}_x + i\hat{p}_y}{1+\hat{p}_z} (L \cdot \hat{p} + L_z)] \\ \hat{J}_-(\vec{p}, 1; R, L) := e^{i\vec{\omega}\cdot R} e^{-ln[\gamma_v(1+v)]L_z} (R_x - iR_y) e^{ln[\gamma_v(1+v)]L_z} e^{-i\vec{\omega}\cdot R} \\ = \frac{E}{m} [(R_x - iR_y) - \frac{\hat{p}_x - i\hat{p}_y}{1+\hat{p}_z} (R \cdot \hat{p} + R_z)] + \frac{i|\vec{p}|}{m} [(L_x - iL_y) - \frac{\hat{p}_x - i\hat{p}_y}{1+\hat{p}_z} (L \cdot \hat{p} + L_z)] \\ \hat{J}_z(\vec{p}, 1; R, L) := e^{i\vec{\omega}\cdot R} e^{-ln[\gamma_v(1+v)]L_z} R_z e^{ln[\gamma_v(1+v)]L_z} e^{-i\vec{\omega}\cdot R} = R \cdot \hat{p} \end{cases}$$

推论2.5.4.

$$\begin{cases} \hat{J}_+(\vec{p}, 1; R, L) \varepsilon(\vec{p}, h) = \sqrt{2-h(h+1)} \varepsilon(\vec{p}, h+1), -1 \leq h \leq 1 \\ \hat{J}_-(\vec{p}, 1; R, L) \varepsilon(\vec{p}, h) = \sqrt{2-h(h+1)} \varepsilon(\vec{p}, h-1), -1 \leq h \leq 1 \\ \hat{J}_z(\vec{p}, 1; R, L) \varepsilon(\vec{p}, h) = h \varepsilon(\vec{p}, h), -1 \leq h \leq 1 \end{cases}$$

## 2.6 定义-自旋基分解: $n = (n-1) \oplus 1$

定义2.6.1.  $-n \leq h \leq n$

$$\varepsilon_{\underbrace{a \dots bc}_n}(\vec{p}, h) := \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1)$$

定义2.6.2.  $-n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

$$\varepsilon_{\underbrace{ab \dots \tau c}_n}(\vec{p}, h) = \frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots}_{n-1}}(\vec{p}, h + \frac{1}{2}) u_{\tau c}(\vec{p}, -\frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots}_{n-1}}(\vec{p}, h - \frac{1}{2}) u_{\tau c}(\vec{p}, \frac{1}{2})$$

推论2.6.1.

$$\bar{\varepsilon}_{\underbrace{a \dots bc}_n}(\vec{p}, h) = \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \bar{\varepsilon}_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h-1) \bar{\varepsilon}_c(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \bar{\varepsilon}_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h) \bar{\varepsilon}_c(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \bar{\varepsilon}_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h+1) \bar{\varepsilon}_c(\vec{p}, -1)$$

推论2.6.2.

$$\varepsilon_{\underbrace{a \dots bc}_n}(\vec{p}, h) = \sum_{h'=1}^{-1} \frac{\sqrt{C_{n+h}^{1+h'} C_{n-h}^{1-h'}}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h-h') \varepsilon_c(\vec{p}, h'), \bar{\varepsilon}_{\underbrace{a \dots bc}_n}(\vec{p}, h) = \sum_{h'=1}^{-1} \frac{\sqrt{C_{n+h}^{1+h'} C_{n-h}^{1-h'}}}{\sqrt{C_{2n}^2}} \bar{\varepsilon}_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h-h') \bar{\varepsilon}_c(\vec{p}, h')$$

## 2.7 推论- $\varepsilon_{a \dots bc}(\vec{p}, h)$ 是自旋本征态

定义2.7.1.  $\Omega(n; R) := R \otimes I_{4^{n-1}} + I_4 \otimes R \otimes I_{4^{n-2}} + \dots + I_{4^{n-1}} \otimes R$

定理2.7.1.  $[\Omega(n; R) \cdot \hat{p}] \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_n}(\vec{p}, h) = h \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_n}(\vec{p}, h), -n \leq h \leq n$

证明:  $[\Omega(n; R) \cdot \hat{p}] \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_n}(\vec{p}, h)$

$$= [\Omega(n-1; R) \otimes I_4 + I_{4^{n-1}} \otimes R] \cdot \hat{p}$$

$$[\frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots \otimes b}_{n-1}}(\vec{p}, h-1) \varepsilon_{\otimes c}(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots \otimes b}_{n-1}}(\vec{p}, h) \varepsilon_{\otimes c}(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots \otimes b}_{n-1}}(\vec{p}, h+1) \varepsilon_{\otimes c}(\vec{p}, -1)]$$

$$\begin{aligned}
&= \left[ \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} h \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h-1) \varepsilon_{\otimes c}(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} h \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h) \varepsilon_{\otimes c}(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} h \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h+1) \varepsilon_{\otimes c}(\vec{p}, -1) \right] \\
&= h \varepsilon_{a \otimes \dots \otimes b \otimes c}(\vec{p}, h), \quad -n \leq h \leq n \quad \square
\end{aligned}$$

$$\text{定理2.7.2. } \Omega^2(n; R) \varepsilon_{a \otimes \dots \otimes b \otimes c}(\vec{p}, h) = n(n+1) \varepsilon_{a \otimes \dots \otimes b \otimes c}(\vec{p}, h)$$

以上定理可以用全对称的表象变换方法可以很容易得到证明？好像不容易，但可以用下面章节的上升和下降算符方法证明（ $\hat{p}_z = 1$ 才成立）。由上可知  $\varepsilon_{a \dots bc}(\vec{p}, h)$  是自旋本征态，所以展开系数是CG系数，实际计算结果也表明确实就是对应的CG系数。从而也提供了一种规范、统一、直观和完整的计算CG系数新方法。

## 2.8 Klein-Gordon方程自旋基的升降算符

定义2.8.1.

$$\begin{cases} \hat{J}(\vec{p}, n; R, L) := \underbrace{\hat{J}(\vec{p}, 1; R, L) \otimes I_4 \otimes \dots \otimes I_4}_n + \underbrace{I_4 \otimes \hat{J}(\vec{p}, 1; R, L) \otimes \dots \otimes I_4}_n + \dots + \underbrace{I_4 \otimes \dots \otimes I_4 \otimes \hat{J}(\vec{p}, 1; R, L)}_n \\ \hat{Q}(\vec{p}, n; R, L) := \underbrace{\hat{Q}(\vec{p}, 1; R, L) \otimes I_4 \otimes \dots \otimes I_4}_n + \underbrace{I_4 \otimes \hat{Q}(\vec{p}, 1; R, L) \otimes \dots \otimes I_4}_n + \dots + \underbrace{I_4 \otimes \dots \otimes I_4 \otimes \hat{Q}(\vec{p}, 1; R, L)}_n \end{cases}$$

$$\text{推论2.8.1. } [\hat{J}_i(\vec{p}, n; R, L), \hat{J}_j(\vec{p}, n; R, L)] = \varepsilon_{ij}^k \hat{J}_k(\vec{p}, n; R, L)$$

$$\text{定理2.8.1. } \hat{J}_+(\vec{p}, n; R, L) \varepsilon_{a \otimes \dots \otimes b \otimes c}(\vec{p}, h) = \sqrt{n(n+1) - h(h+1)} \varepsilon_{a \otimes \dots \otimes b \otimes c}(\vec{p}, h+1), \quad -n \leq h \leq n$$

证明：采用数学归纳法证明此定理。

第一步： $n' = 1$ 时成立：

$$\hat{J}_+(\vec{p}, 1; R, L) \varepsilon(\vec{p}, h) = \sqrt{2 - h(h+1)} \varepsilon(\vec{p}, h+1), \quad -1 \leq h \leq 1$$

第二步：假设  $n' = n-1$  时成立：

$$\hat{J}_+(\vec{p}, n-1; R, L) \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h) = \sqrt{(n-1)n - h(h+1)} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h+1), \quad -n+1 \leq h \leq n-1$$

第三步： $n' = n$  时： $-n \leq h \leq n$ ,  $\hat{J}_+(\vec{p}, n; R, L) \varepsilon_{a \otimes \dots \otimes b \otimes c}(\vec{p}, h)$

$$\begin{aligned}
&= \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} [\hat{J}_+(\vec{p}, n-1; R, L) \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h-1)] \varepsilon_{\otimes c}(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} [\hat{J}_+(\vec{p}, n-1; R, L) \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h)] \varepsilon_{\otimes c}(\vec{p}, 0) \\
&+ \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} [\hat{J}_+(\vec{p}, n-1; R, L) \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h+1)] \varepsilon_{\otimes c}(\vec{p}, -1) + \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h-1) \hat{J}_+(\vec{p}, \frac{1}{2}; R, L) \varepsilon_{\otimes c}(\vec{p}, 1) \\
&+ \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h) \hat{J}_+(\vec{p}, \frac{1}{2}; R, L) \varepsilon_{\otimes c}(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h+1) \hat{J}_+(\vec{p}, \frac{1}{2}; R, L) \varepsilon_{\otimes c}(\vec{p}, -1) \\
&= \frac{\sqrt{C_{n+h}^2} \sqrt{(n-1)n - (h-1)h}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h) \varepsilon_{\otimes c}(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h) \sqrt{2} \varepsilon_{\otimes c}(\vec{p}, 1) \\
&+ \frac{\sqrt{C_{n+h}^1 C_{n-h}^1} \sqrt{(n-1)n - h(h+1)}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h+1) \varepsilon_{\otimes c}(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h+1) \sqrt{2} \varepsilon_{\otimes c}(\vec{p}, 0) \\
&+ \frac{\sqrt{C_{n-h}^2} \sqrt{(n-1)n - (h+1)(h+2)}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h+2) \varepsilon_{\otimes c}(\vec{p}, -1) \\
&= \frac{\sqrt{n(n+1) - h(h+1)} \sqrt{C_{n+h+1}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h) \varepsilon_{\otimes c}(\vec{p}, 1) + \frac{\sqrt{n(n+1) - h(h+1)} \sqrt{C_{n+h+1}^1 C_{n-h-1}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h+1) \varepsilon_{\otimes c}(\vec{p}, 0) \\
&+ \frac{\sqrt{n(n+1) - h(h+1)} \sqrt{C_{n-h-1}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{a \otimes \dots \otimes b}(\vec{p}, h+2) \varepsilon_{\otimes c}(\vec{p}, -1) \\
&= \sqrt{n(n+1) - h(h+1)} \varepsilon_{a \otimes \dots \otimes b \otimes c}(\vec{p}, h+1)
\end{aligned}$$

此步证明了  $n' = n$  时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

**定理2.8.2.**  $\hat{J}_-(\vec{p}, n; R, L)\varepsilon_{\underbrace{a \otimes \dots \otimes b \otimes c}_n}(\vec{p}, h) = \sqrt{n(n+1) - h(h-1)}\varepsilon_{\underbrace{a \otimes \dots \otimes b \otimes c}_n}(\vec{p}, h-1), -n \leq h \leq n$

**证明:** 采用数学归纳法证明此定理。

第一步:  $n' = 1$ 时成立:

$$\hat{J}_-(\vec{p}, 1; R, L)\varepsilon(\vec{p}, h) = \sqrt{2 - h(h-1)}\varepsilon(\vec{p}, h-1), -1 \leq h \leq 1$$

第二步: 假设 $n' = n-1$ 时成立:

$$\hat{J}_-(\vec{p}, n-1; R, L)\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h) = \sqrt{(n-1)n - h(h-1)}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h-1), -n+1 \leq h \leq n-1$$

第三步:  $n' = n$ 时:  $-n \leq h \leq n, \hat{J}_-(\vec{p}, n; R, L)\varepsilon_{\underbrace{a \otimes \dots \otimes b \otimes c}_n}(\vec{p}, h)$

$$\begin{aligned} &= \frac{\sqrt{C_{2n}^{n+h}}}{\sqrt{C_{2n}^{2n}}}[\hat{J}_-(\vec{p}, n-1; R, L)\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h-1)]\varepsilon_{\otimes c}(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^{2n}}}[\hat{J}_-(\vec{p}, n-1; R, L)\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h)]\varepsilon_{\otimes c}(\vec{p}, 0) \\ &+ \frac{\sqrt{C_{2n}^{n-h}}}{\sqrt{C_{2n}^{2n}}}[\hat{J}_-(\vec{p}, n-1; R, L)\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h+1)]\varepsilon_{\otimes c}(\vec{p}, -1) + \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h-1)\hat{J}_-(\vec{p}, \frac{1}{2}; R, L)\varepsilon_{\otimes c}(\vec{p}, 1) \\ &+ \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h)\hat{J}_-(\vec{p}, \frac{1}{2}; R, L)\varepsilon_{\otimes c}(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h+1)\hat{J}_-(\vec{p}, \frac{1}{2}; R, L)\varepsilon_{\otimes c}(\vec{p}, -1) \\ &= \frac{\sqrt{C_{n+h}^2}\sqrt{(n-1)n - (h-1)(h-2)}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h-2)\varepsilon_{\otimes c}(\vec{p}, 1) \\ &+ \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}\sqrt{(n-1)n - h(h-1)}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h-1)\varepsilon_{\otimes c}(\vec{p}, 0) + \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h-1)\sqrt{2}\varepsilon_{\otimes c}(\vec{p}, 0) \\ &+ \frac{\sqrt{C_{n-h}^2}\sqrt{(n-1)n - (h+1)h}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h)\varepsilon_{\otimes c}(\vec{p}, -1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h)\sqrt{2}\varepsilon_{\otimes c}(\vec{p}, -1) \\ &= \frac{\sqrt{n(n+1) - h(h-1)}\sqrt{C_{n+h-1}^2}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h-2)\varepsilon_{\otimes c}(\vec{p}, 1) + \frac{\sqrt{n(n+1) - h(h-1)}\sqrt{C_{n+h-1}^1 C_{n-h+1}^1}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h-1)\varepsilon_{\otimes c}(\vec{p}, 0) \\ &+ \frac{\sqrt{n(n+1) - h(h-1)}\sqrt{C_{n-h+1}^2}}{\sqrt{C_{2n}^{2n}}}\varepsilon_{\underbrace{a \otimes \dots \otimes b}_n}(\vec{p}, h)\varepsilon_{\otimes c}(\vec{p}, -1) \\ &= \sqrt{n(n+1) - h(h-1)}\varepsilon_{\underbrace{a \otimes \dots \otimes b \otimes c}_n}(\vec{p}, h-1) \end{aligned}$$

此步证明了 $n' = n$ 时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

**推论2.8.2.**  $\hat{J}^2(\vec{p}, n; R, L)\varepsilon_{\underbrace{a \otimes \dots \otimes b \otimes c}_n}(\vec{p}, h) = n(n+1)\varepsilon_{\underbrace{a \otimes \dots \otimes b \otimes c}_n}(\vec{p}, h), -n \leq h \leq n$

**推论2.8.3.**

$$\begin{cases} \hat{J}_+(\vec{p}, n; R, L)\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h) = \sqrt{n(n+1) - h(h+1)}\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h+1), -n \leq h \leq n \\ \hat{J}_-(\vec{p}, n; R, L)\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h) = \sqrt{n(n+1) - h(h-1)}\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h-1), -n \leq h \leq n \\ \hat{J}_z(\vec{p}, n; R, L)\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h) = h\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h), -n \leq h \leq n \end{cases}$$

**推论2.8.4.**

$$\begin{cases} \hat{J}^2(\vec{p}, n; R, L)\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h) = n(n+1)\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h), \hat{J}^2(\vec{p}, *1; R, L)\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h) = 2\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h) \\ \hat{J}_z(\vec{p}, n; R, L)\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h) = h\varepsilon_{\underbrace{a \otimes b \otimes \dots}_n}(\vec{p}, h), \hat{J}^2 = \hat{J}_z^2 + \frac{1}{2}(\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+), -n \leq h \leq n \\ \delta^{ab}\varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, p^a \varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = 0, \varepsilon_{\underbrace{ab \dots}_n}(\vec{p}, h) = \frac{1}{n!}\varepsilon_{\{ab \dots\}}(\vec{p}, h) \end{cases}$$

**引理2.8.1.**  $(\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^b(p) = (\bar{C}\gamma_a)^{\lambda_\zeta \mu_\zeta} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^b(-p) = 4im\delta_a^b$

$$\text{定理2.8.3. } \hat{J}(\vec{p}, s; R, L) = \frac{1}{(i4m)^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \otimes \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \otimes \xi_\zeta} \cdots}^n \hat{J}(\vec{p}, s; \gamma_a) \underbrace{\mathbb{X}_{\lambda'_\zeta \otimes \mu'_\zeta}^{a'}(p) \mathbb{X}_{\eta'_\zeta \otimes \xi'_\zeta}^{b'}(p) \cdots}_n$$

$$\text{证明: } \hat{J}_+(\vec{p}, s; \gamma_a) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta} \cdots}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h+1)} \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta} \cdots}_{2s}(\vec{p}, h+1)$$

$$\Rightarrow \frac{1}{(i\sqrt{2})^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \otimes \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \otimes \xi_\zeta} \cdots}^n \hat{J}_+(\vec{p}, s; \gamma_a) \frac{1}{(2\sqrt{2}m)^n} \underbrace{\mathbb{X}_{\lambda'_\zeta \otimes \mu'_\zeta}^{a'}(p) \mathbb{X}_{\eta'_\zeta \otimes \xi'_\zeta}^{b'}(p) \cdots}_n \underbrace{\varepsilon_{a'b'} \cdots}_n(\vec{p}, h)$$

$$= \sqrt{s(s+1) - h(h+1)} \underbrace{\varepsilon_{ab} \cdots}_n(\vec{p}, h+1)$$

$$\Rightarrow \hat{J}_+(\vec{p}, s; R, L) = \frac{1}{(i4m)^n} \overbrace{(\bar{C}\gamma_a)^{\lambda_\zeta \otimes \mu_\zeta} (\bar{C}\gamma_b)^{\eta_\zeta \otimes \xi_\zeta} \cdots}^n \hat{J}_+(\vec{p}, s; \gamma_a) \underbrace{\mathbb{X}_{\lambda'_\zeta \otimes \mu'_\zeta}^{a'}(p) \mathbb{X}_{\eta'_\zeta \otimes \xi'_\zeta}^{b'}(p) \cdots}_n$$

□

## 2.9 推论 $-\varepsilon_{ab \cdots c}(\vec{p}, h)$ 的正交性

$$\text{定义2.9.1. } \bar{\varepsilon}^a(\vec{p}, h) \varepsilon_a(\vec{p}, h') = \delta_{hh'}, -1 \leq h', h \leq 1$$

$$\text{定义2.9.2. } \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \bar{\varepsilon}_b(\vec{p}, h) = \delta_{ab} + \frac{p_a p_b}{m^2}$$

$$\text{定理2.9.1. } \bar{\varepsilon}^{\overbrace{a \cdots bc}^n}(\vec{p}, h') \varepsilon_{\overbrace{a \cdots bc}^n}(\vec{p}, h) = \delta_{hh'}, -n \leq h', h \leq n$$

证明: 采用数学归纳法证明此定理。

第一步:  $n' = 1$ 时成立:

$$\bar{\varepsilon}^a(\vec{p}, h') \varepsilon_a(\vec{p}, h) = \delta_{hh'}, -1 \leq h', h \leq 1$$

第二步: 假设 $n' = n - 1$ 时成立:

$$\bar{\varepsilon}^{\overbrace{a \cdots b}^{n-1}}(\vec{p}, h') \varepsilon_{\overbrace{a \cdots b}^{n-1}}(\vec{p}, h) = \delta_{hh'}, -n + 1 \leq h', h \leq n - 1$$

$$\text{第三步: } n' = n \text{时: } \bar{\varepsilon}^{\overbrace{a \cdots bc}^n}(\vec{p}, h') \varepsilon_{\overbrace{a \cdots bc}^n}(\vec{p}, h), -n \leq h', h \leq n$$

$$\begin{aligned} &= \left[ \frac{\sqrt{C_{n+h'}^2}}{\sqrt{C_{2n}^2}} \bar{\varepsilon}^{\overbrace{a \cdots b}^{n-1}}(\vec{p}, h' - 1) \bar{\varepsilon}^c(\vec{p}, 1) + \frac{\sqrt{C_{n+h'}^1 C_{n-h'}^1}}{\sqrt{C_{2n}^2}} \bar{\varepsilon}^{\overbrace{a \cdots b}^{n-1}}(\vec{p}, h') \bar{\varepsilon}^c(\vec{p}, 0) + \frac{\sqrt{C_{n-h'}^2}}{\sqrt{C_{2n}^2}} \bar{\varepsilon}^{\overbrace{a \cdots b}^{n-1}}(\vec{p}, h' + 1) \bar{\varepsilon}^c(\vec{p}, -1) \right] \\ & \left[ \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\overbrace{a \cdots b}^{n-1}}(\vec{p}, h - 1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\overbrace{a \cdots b}^{n-1}}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\overbrace{a \cdots b}^{n-1}}(\vec{p}, h + 1) \varepsilon_c(\vec{p}, -1) \right] \\ &= \frac{\sqrt{C_{n+h'}^2}}{\sqrt{C_{2n}^2}} \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \delta_{hh'} + \frac{\sqrt{C_{n+h'}^1 C_{n-h'}^1}}{\sqrt{C_{2n}^2}} \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \delta_{hh'} + \frac{\sqrt{C_{n-h'}^2}}{\sqrt{C_{2n}^2}} \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \delta_{hh'} = \delta_{hh'} \end{aligned}$$

此步证明了 $n' = n$ 时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。

□

## 2.10 推论 $-p^a \varepsilon_{ab \cdots c}(\vec{p}, h)$ 的零性

$$\text{定义2.10.1. } p^a \varepsilon_a(\vec{p}, h) = 0, -1 \leq h', h \leq 1$$

$$\text{定理2.10.1. } p^a \varepsilon_{\overbrace{a \cdots bc}^n}(\vec{p}, h) = 0, -n \leq h \leq n$$

证明: 采用数学归纳法证明此定理。

第一步:  $n' = 1$ 时成立:

$$p^a \varepsilon_a(\vec{p}, h) = \delta_{hh'}, -1 \leq h \leq 1$$

第二步: 假设 $n' = n - 1$ 时成立:

$$p^a \varepsilon_{\overbrace{a \cdots b}^{n-1}}(\vec{p}, h) = 0, -n + 1 \leq h \leq n - 1$$

第三步:  $n' = n$ 时:  $p^a \varepsilon_{\underbrace{a \dots bc}_n}(\vec{p}, h), -n \leq h \leq n$

$$= p^a \left[ \frac{\sqrt{C_{2n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{2n+h}^1 C_{2n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{2n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1) \right]$$

$$= 0 + 0 + 0 = 0$$

此步证明了  $n' = n$ 时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

## 2.11 推论- $\varepsilon_{ab \dots c}(\vec{p}, h)$ 的无迹性

定义2.11.1.  $\bar{\varepsilon}_a(\vec{p}, h) = (-1)^h \varepsilon_a(\vec{p}, -h), -1 \leq h \leq 1$

引理2.11.1.  $\delta^{ab} \varepsilon_{ab}(\vec{p}, h) = 0, -2 \leq h \leq 2$

证明:  $\delta^{ab} \varepsilon_{ab}(\vec{p}, h), -2 \leq h \leq 2$

$$= \delta^{ab} \left[ \frac{\sqrt{C_{2+h}^2}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, h-1) \varepsilon_b(\vec{p}, 1) + \frac{\sqrt{C_{2+h}^1 C_{2-h}^1}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, h) \varepsilon_b(\vec{p}, 0) + \frac{\sqrt{C_{2-h}^2}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, h+1) \varepsilon_b(\vec{p}, -1) \right]$$

$$= \begin{cases} \delta^{ab} \varepsilon_a(\vec{p}, 1) \varepsilon_b(\vec{p}, 1) = 0; h = 2 \\ \delta^{ab} \varepsilon_a(\vec{p}, -1) \varepsilon_b(\vec{p}, -1) = 0; h = -2 \\ \delta^{ab} \left[ \frac{\sqrt{C_3^2}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, 0) \varepsilon_b(\vec{p}, 1) + \frac{\sqrt{C_3^1 C_1^1}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, 1) \varepsilon_b(\vec{p}, 0) \right] = 0; h = 1 \\ \delta^{ab} \left[ \frac{\sqrt{C_3^1 C_1^1}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, 1) \varepsilon_b(\vec{p}, 0) + \frac{\sqrt{C_3^2}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, 0) \varepsilon_b(\vec{p}, -1) \right] = 0; h = -1 \\ \delta^{ab} \left[ \frac{\sqrt{C_2^2}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, -1) \varepsilon_b(\vec{p}, 1) + \frac{\sqrt{C_2^1 C_2^1}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, 0) \varepsilon_b(\vec{p}, 0) + \frac{\sqrt{C_2^2}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, 1) \varepsilon_b(\vec{p}, -1) \right] = 0; h = 0 \end{cases}$$

定理2.11.1.  $\delta^{ab} \varepsilon_{\underbrace{ab \dots c}_n}(\vec{p}, h) = 0, n \geq 2, -n \leq h \leq n$

证明: 采用数学归纳法证明此定理。

第一步:  $n' = 2$ 时成立:

$$\delta^{ab} \varepsilon_{ab}(\vec{p}, h) = 0, -2 \leq h \leq 2$$

第二步: 假设  $2 \leq n' = n-1$ 时成立:

$$\delta^{ab} \varepsilon_{\underbrace{ab \dots c}_{n-1}}(\vec{p}, h) = 0, -n+1 \leq h \leq n-1$$

第三步:  $3 \leq n' = n$ 时:  $\delta^{ab} \varepsilon_{\underbrace{ab \dots c}_n}(\vec{p}, h), -n \leq h \leq n$

$$= \delta^{ab} \left[ \frac{\sqrt{C_{2n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{ab \dots c}_{n-1}}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{2n+h}^1 C_{2n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{ab \dots c}_{n-1}}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{2n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{ab \dots c}_{n-1}}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1) \right]$$

$$= 0 + 0 + 0 = 0$$

此步证明了  $n' = n$ 时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

## 2.12 推论-自旋基分解: $2 = 1 \oplus 1$

推论2.12.1.  $\varepsilon_{ab}(\vec{p}, h) = \sum_{h'=1}^{-1} \frac{\sqrt{C_{2+h}^1 C_{2-h}^1}}{\sqrt{C_4^2}} \varepsilon_a(\vec{p}, h-h') \varepsilon_b(\vec{p}, h')$

$$= \begin{cases} \varepsilon_{ab}(\vec{p}, 2) = \varepsilon_a(\vec{p}, 1) \varepsilon_b(\vec{p}, 1) \\ \varepsilon_{ab}(\vec{p}, 1) = \frac{1}{\sqrt{2}} [\varepsilon_a(\vec{p}, 1) \varepsilon_b(\vec{p}, 0) + \varepsilon_a(\vec{p}, 0) \varepsilon_b(\vec{p}, 1)] \\ \varepsilon_{ab}(\vec{p}, 0) = \frac{1}{\sqrt{6}} [\varepsilon_a(\vec{p}, 1) \varepsilon_b(\vec{p}, -1) + \varepsilon_a(\vec{p}, -1) \varepsilon_b(\vec{p}, 1) + 2\varepsilon_a(\vec{p}, 0) \varepsilon_b(\vec{p}, 0)] \\ \varepsilon_{ab}(\vec{p}, -1) = \frac{1}{\sqrt{2}} [\varepsilon_a(\vec{p}, -1) \varepsilon_b(\vec{p}, 0) + \varepsilon_a(\vec{p}, 0) \varepsilon_b(\vec{p}, -1)] \\ \varepsilon_{ab}(\vec{p}, -2) = \varepsilon_a(\vec{p}, -1) \varepsilon_b(\vec{p}, -1) \end{cases}$$

推论2.12.2.  $\varepsilon_{ab}(\vec{p}, h) = \varepsilon_{ba}(\vec{p}, h), -2 \leq h \leq 2$

2.13 推论-自旋基分解:  $n = (n-2) \oplus 2$ 

$$\text{定理2.13.1. } \varepsilon_{\underbrace{a \dots bc}_n}(\vec{p}, h) = \sum_{h'=2}^{-2} \frac{\sqrt{C_{n+h}^2 C_{n-h}^2}}{\sqrt{C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-h') \varepsilon_{bc}(\vec{p}, h')$$

证明:  $\varepsilon_{\underbrace{a \dots bc}_n}(\vec{p}, h)$

$$\begin{aligned} &= \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1) \\ &= \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_c(\vec{p}, 1) \\ &+ \left[ \frac{\sqrt{C_{n+h-2}^2}}{\sqrt{C_{2n-2}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-2) \varepsilon_b(\vec{p}, 1) + \frac{\sqrt{C_{n+h-2}^1 C_{n-h}^1}}{\sqrt{C_{2n-2}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-1) \varepsilon_b(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n-2}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_b(\vec{p}, -1) \right] \\ &+ \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_c(\vec{p}, 0) \\ &+ \left[ \frac{\sqrt{C_{n+h-1}^2}}{\sqrt{C_{2n-2}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-1) \varepsilon_b(\vec{p}, 1) + \frac{\sqrt{C_{n+h-1}^1 C_{n-h-1}^1}}{\sqrt{C_{2n-2}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_b(\vec{p}, 0) + \frac{\sqrt{C_{n-h-1}^2}}{\sqrt{C_{2n-2}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+1) \varepsilon_b(\vec{p}, -1) \right] \\ &+ \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_c(\vec{p}, -1) \\ &+ \left[ \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n-2}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_b(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h-2}^1}}{\sqrt{C_{2n-2}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+1) \varepsilon_b(\vec{p}, 0) + \frac{\sqrt{C_{n-h-2}^2}}{\sqrt{C_{2n-2}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+2) \varepsilon_b(\vec{p}, -1) \right] \\ &= \left[ \frac{\sqrt{C_{n+h-2}^2}}{\sqrt{C_{2n-2}^2}} \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-2) \varepsilon_b(\vec{p}, 1) \varepsilon_c(\vec{p}, 1) \right. \\ &+ \left[ \frac{\sqrt{C_{n+h-1}^2}}{\sqrt{C_{2n-2}^2}} \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-1) \varepsilon_b(\vec{p}, 1) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{n+h-2}^1 C_{n-h}^1}}{\sqrt{C_{2n-2}^2}} \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-1) \varepsilon_b(\vec{p}, 0) \varepsilon_c(\vec{p}, 1) \right] \\ &+ \left[ \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n-2}^2}} \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_b(\vec{p}, 1) \varepsilon_c(\vec{p}, -1) + \frac{\sqrt{C_{n-h}^1}}{\sqrt{C_{2n-2}^2}} \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_b(\vec{p}, -1) \varepsilon_c(\vec{p}, 1) \right] \\ &+ \frac{\sqrt{C_{n+h-1}^1 C_{n-h-1}^1}}{\sqrt{C_{2n-2}^2}} \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_b(\vec{p}, 0) \varepsilon_c(\vec{p}, 0) \\ &+ \left[ \frac{\sqrt{C_{n+h}^1 C_{n-h-2}^1}}{\sqrt{C_{2n-2}^2}} \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+1) \varepsilon_b(\vec{p}, 0) \varepsilon_c(\vec{p}, -1) + \frac{\sqrt{C_{n-h-1}^2}}{\sqrt{C_{2n-2}^2}} \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+1) \varepsilon_b(\vec{p}, -1) \varepsilon_c(\vec{p}, 0) \right] \\ &+ \frac{\sqrt{C_{n-h-2}^2}}{\sqrt{C_{2n-2}^2}} \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^2}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+2) \varepsilon_b(\vec{p}, -1) \varepsilon_c(\vec{p}, -1) \\ &= \frac{1}{2!} \frac{\sqrt{C_{n+h}^4 C_{n-h}^4}}{\sqrt{C_{2n}^4 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-2) \varepsilon_{\{b(\vec{p}, 1) \varepsilon_{c\}}(\vec{p}, 1) + \frac{1}{2!} \frac{\sqrt{C_{n-h}^4 C_{n-h}^4}}{\sqrt{C_{2n}^4 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+2) \varepsilon_{\{b(\vec{p}, -1) \varepsilon_{c\}}(\vec{p}, -1) \\ &+ \frac{\sqrt{C_{n+h}^3 C_{n-h}^3}}{\sqrt{C_{2n}^4 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-1) \varepsilon_{\{b(\vec{p}, 1) \varepsilon_{c\}}(\vec{p}, 0) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^3 C_{n-h}^3}}{\sqrt{C_{2n}^4 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+1) \varepsilon_{\{b(\vec{p}, 0) \varepsilon_{c\}}(\vec{p}, -1) \\ &+ \frac{\sqrt{C_{n+h}^2 C_{n-h}^2}}{\sqrt{C_{2n}^4 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_{\{b(\vec{p}, 1) \varepsilon_{c\}}(\vec{p}, -1) + \frac{\sqrt{C_{n+h}^2 C_{n-h}^2}}{\sqrt{C_{2n}^4 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_{\{b(\vec{p}, 0) \varepsilon_{c\}}(\vec{p}, 0) \\ &= \frac{\sqrt{C_{n+h}^4 C_{n-h}^4}}{\sqrt{2^2 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-2) \varepsilon_{\{b(\vec{p}, 1) \varepsilon_{c\}}(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^0 C_{n-h}^4}}{\sqrt{2^2 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+2) \varepsilon_{\{b(\vec{p}, -1) \varepsilon_{c\}}(\vec{p}, -1) \\ &+ \frac{\sqrt{C_{n+h}^3 C_{n-h}^3}}{\sqrt{2^1 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-1) \varepsilon_{\{b(\vec{p}, 1) \varepsilon_{c\}}(\vec{p}, 0) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^3}}{\sqrt{2^1 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+1) \varepsilon_{\{b(\vec{p}, 0) \varepsilon_{c\}}(\vec{p}, -1) \\ &+ \frac{\sqrt{C_{n+h}^2 C_{n-h}^2}}{\sqrt{C_{2n}^4 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_{\{b(\vec{p}, 1) \varepsilon_{c\}}(\vec{p}, -1) + \frac{\sqrt{C_{n+h}^2 C_{n-h}^2}}{\sqrt{C_{2n}^4 C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_{\{b(\vec{p}, 0) \varepsilon_{c\}}(\vec{p}, 0) \\ &= \frac{\sqrt{C_{n+h}^4 C_{n-h}^4}}{\sqrt{C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-2) \varepsilon_{bc}(\vec{p}, 2) + \frac{\sqrt{C_{n+h}^0 C_{n-h}^4}}{\sqrt{C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+2) \varepsilon_{bc}(\vec{p}, -2) \\ &+ \frac{\sqrt{C_{n+h}^3 C_{n-h}^3}}{\sqrt{C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h-1) \varepsilon_{bc}(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^3}}{\sqrt{C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h+1) \varepsilon_{bc}(\vec{p}, -1) \\ &+ \frac{\sqrt{C_{n+h}^2 C_{n-h}^2}}{\sqrt{C_{2n}^4}} \varepsilon_{\underbrace{a \dots}_{n-2}}(\vec{p}, h) \varepsilon_{bc}(\vec{p}, 0) \end{aligned}$$



$$= \sum_{h'=2}^{-2} \frac{\sqrt{C_{n+h}^{2+h'} C_{n-h}^{2-h'}}}{\sqrt{C_{2n}^4}} \varepsilon_{\underbrace{a \dots b \dots c}_{n-2}}(\vec{p}, h-h') \varepsilon_{bc}(\vec{p}, h')$$

□

推论2.13.1.  $\varepsilon_{\underbrace{a \dots bc}_n}(\vec{p}, h) = \varepsilon_{\underbrace{a \dots cb}_n}(\vec{p}, h), n \geq 2, -n \leq h \leq n$

## 2.14 推论-自旋基分解: $n + n' = n \oplus n'$

$$\text{定理2.14.1. } \varepsilon_{\underbrace{a \dots b \dots c}_n \underbrace{\dots}_{n'}}(\vec{p}, h) = \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+n'+h}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \varepsilon_{\underbrace{b \dots c}_{n'}}(\vec{p}, h')$$

证明: 对  $n'$  采用数学归纳法证明此定理。

第一步:  $n'' = 1$  时成立:

$$\varepsilon_{\underbrace{a \dots b \dots}_{n}}(\vec{p}, h) = \sum_{h'=1}^{-1} \frac{\sqrt{C_{n+1+h'}^{1+h'} C_{n+1-h'}^{1-h'}}}{\sqrt{C_{2n+2}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \varepsilon_{\underbrace{b \dots}_{1}}(\vec{p}, h'), -n-1 \leq h \leq n+1$$

第二步: 假设  $s'' = s' - \frac{1}{2}$  时成立:

$$\varepsilon_{\underbrace{a \dots}_{n} \underbrace{b \dots}_{n'-1}}(\vec{p}, h) = \sum_{h'=n'-1}^{-n'+1} \frac{\sqrt{C_{n+n'+1+h'}^{n'+1+h'} C_{n+n'-1-h'}^{n'-1-h'}}}{\sqrt{C_{2n+2n'-2}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h'), -n-n'+1 \leq h \leq n+n'+1$$

第三步:  $n'' = n'$  时:  $-n-n' \leq h \leq n+n', \varepsilon_{\underbrace{a \dots b \dots c}_{n} \underbrace{\dots}_{n'}}(\vec{p}, h)$

$$\begin{aligned} &= \frac{\sqrt{C_{n+n'+h}^2}}{\sqrt{C_{2n+2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n} \underbrace{b \dots}_{n'-1}}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+n'+h}^1 C_{n+n'-h}^1}}{\sqrt{C_{2n+2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n} \underbrace{b \dots}_{n'-1}}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{n+n'-h}^2}}{\sqrt{C_{2n+2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n} \underbrace{b \dots}_{n'-1}}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1) \\ &= \frac{\sqrt{C_{n+n'+h}^2}}{\sqrt{C_{2n+2n'}^2}} \sum_{h'=n'-1}^{-n'+1} \frac{\sqrt{C_{n+n'+2+h'}^{n'+2+h'} C_{n+n'-h'}^{n'-h'}}}{\sqrt{C_{2n+2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-1-h') \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h') \varepsilon_c(\vec{p}, 1) \\ &+ \frac{\sqrt{C_{n+n'+h}^1 C_{n+n'-h}^1}}{\sqrt{C_{2n+2n'}^2}} \sum_{h'=n'-1}^{-n'+1} \frac{\sqrt{C_{n+n'+1+h'}^{n'+1+h'} C_{n+n'-1-h'}^{n'-1-h'}}}{\sqrt{C_{2n+2n'-2}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h') \varepsilon_c(\vec{p}, 0) \\ &+ \frac{\sqrt{C_{n+n'-h}^2}}{\sqrt{C_{2n+2n'}^2}} \sum_{h'=n'-1}^{-n'+1} \frac{\sqrt{C_{n+n'+1+h'}^{n'+1+h'} C_{n+n'-2-h'}^{n'-2-h'}}}{\sqrt{C_{2n+2n'-2}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h+1-h') \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h') \varepsilon_c(\vec{p}, -1) \\ &= \sum_{h'=n'-1}^{-n'+1} \frac{\sqrt{C_{n+n'+h'}^{n'+1+h'} C_{n+n'-h'}^{n'-1-h'}}}{\sqrt{C_{2n+2n'}^2}} \frac{\sqrt{C_{n'+1+h'}^2}}{\sqrt{C_{2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-1-h') \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h') \varepsilon_c(\vec{p}, 1) \\ &+ \sum_{h'=n'-1}^{-n'+1} \frac{\sqrt{C_{n+n'+h'}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^2}} \frac{\sqrt{C_{n'+h'}^1 C_{n'-h'}^1}}{\sqrt{C_{2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h') \varepsilon_c(\vec{p}, 0) \\ &+ \sum_{h'=n'-1}^{-n'+1} \frac{\sqrt{C_{n+n'+h'}^{n'+1+h'} C_{n+n'-h}^{n'+1-h'}}}{\sqrt{C_{2n+2n'}^2}} \frac{\sqrt{C_{n'+1-h'}^2}}{\sqrt{C_{2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h+1-h') \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h') \varepsilon_c(\vec{p}, -1) \\ &= \sum_{h'=n'}^{-n'+2} \frac{\sqrt{C_{n+n'+h'}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \frac{\sqrt{C_{n'+h'}^2}}{\sqrt{C_{2n'}^2}} \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h'-1) \varepsilon_c(\vec{p}, 1) \\ &+ \sum_{h'=n'-1}^{-n'+1} \frac{\sqrt{C_{n+n'+h'}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \frac{\sqrt{C_{n'+h'}^1 C_{n'-h'}^1}}{\sqrt{C_{2n'}^2}} \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h') \varepsilon_c(\vec{p}, 0) \\ &+ \sum_{h'=n'-2}^{-n'} \frac{\sqrt{C_{n+n'+h'}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^2}} \frac{\sqrt{C_{n'-h'}^2}}{\sqrt{C_{2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h'+1) \varepsilon_c(\vec{p}, -1) \\ &= \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+n'+h'}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \frac{\sqrt{C_{n'+h'}^2}}{\sqrt{C_{2n'}^2}} \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h'-1) \varepsilon_c(\vec{p}, 1) \\ &+ \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+n'+h'}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \frac{\sqrt{C_{n'+h'}^1 C_{n'-h'}^1}}{\sqrt{C_{2n'}^2}} \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h') \varepsilon_c(\vec{p}, 0) \\ &+ \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+n'+h'}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^2}} \frac{\sqrt{C_{n'-h'}^2}}{\sqrt{C_{2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \varepsilon_{\underbrace{b \dots}_{n'-1}}(\vec{p}, h'+1) \varepsilon_c(\vec{p}, -1) \\ &= \sum_{h'=n'}^{-n'} \frac{\sqrt{C_{n+n'+h'}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^2}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h-h') \varepsilon_{\underbrace{b \dots c}_{n'}}(\vec{p}, h') \end{aligned}$$

此步证明了  $n'' = n'$  时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

**推论2.14.1.**  $-n_1 - n_2 \leq h \leq n_1 + n_2$

$$\begin{cases} \varepsilon_{\underbrace{a \dots b}_{n_1} \dots}_{\underbrace{\phantom{a \dots b}}_{n_2}}(\vec{p}, h) = \sum_{h_1=n_1}^{-n_1} \sum_{h_2=n_2}^{-n_2} \frac{\sqrt{C_{n_1+n_2+h}^{n_2+h_2} C_{n_1+n_2-h}^{n_2-h_2}}}{\sqrt{C_{2n_1+2n_2}^{2n_2}}} \varepsilon_{\underbrace{a \dots}_{n_1}}(\vec{p}, h_1) \varepsilon_{\underbrace{b \dots}_{n_2}}(\vec{p}, h_2) \delta(h - h_1 - h_2) \\ \varepsilon_{\underbrace{a \dots}_{n_1} \dots}_{\underbrace{\phantom{a \dots}}_{n_2}}(\vec{p}, h) = \sum_{h_1=n_1}^{-n_1} \sum_{h_2=n_2}^{-n_2} \frac{\sqrt{C_{n_1+n_2+h}^{n_1+h_1} C_{n_1+n_2-h}^{n_1-h_1}}}{\sqrt{C_{2n_1+2n_2}^{2n_1}}} \varepsilon_{\underbrace{a \dots}_{n_1}}(\vec{p}, h_1) \varepsilon_{\underbrace{b \dots}_{n_2}}(\vec{p}, h_2) \delta(h - h_1 - h_2) \end{cases}$$

**推论2.14.2.**  $-n_1 - n_2 \leq h \leq n_1 + n_2, \varepsilon_{\underbrace{a \dots}_{n_1} \dots}_{\underbrace{\phantom{a \dots}}_{n_2}}(\vec{p}, h)$

$$= \sum_{h_1=n_1}^{-n_1} \sum_{h_2=n_2}^{-n_2} \left[ \frac{(2n_1)!(2n_2)! (n_1+h_1+n_2+h_2)! (n_1-h_1+n_2-h_2)!}{(2n_1+2n_2)! (n_1+h_1)!(n_2+h_2)! (n_1-h_1)!(n_2-h_2)!} \right]^{1/2} \varepsilon_{\underbrace{a \dots}_{n_1}}(\vec{p}, h_1) \varepsilon_{\underbrace{b \dots}_{n_2}}(\vec{p}, h_2) \delta(h - h_1 - h_2)$$

## 2.15 推论-自旋基反向合成

**推论2.15.1.**  $\frac{\sqrt{C_{n+n'+h}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^{2n'}}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h - h') = \varepsilon_{\underbrace{a \dots}_{n} \dots}_{\underbrace{\phantom{a \dots}}_{n'}}(\vec{p}, h) \bar{\varepsilon}^{\underbrace{b \dots c}_{n'}}(\vec{p}, h')$

**推论2.15.2.**  $\frac{\sqrt{C_{n+n'+h}^{n'+h'} C_{n+n'-h}^{n'-h'}}}{\sqrt{C_{2n+2n'}^{2n'}}} \varepsilon_{\underbrace{a \dots}_{n}}(\vec{p}, h - h') = \bar{\varepsilon}^{\underbrace{b \dots c}_{n'}}(\vec{p}, h') \varepsilon_{\underbrace{b \dots c}_{n'} \dots}_{\underbrace{\phantom{b \dots c}}_{n}}(\vec{p}, h)$

## 2.16 推论-自旋基分解: $n_1 + n_2 \dots + n_n = n_1 \oplus n_2 \dots \oplus n_n$

**推论2.16.1.**  $-\sum_{i=1}^n n_i \leq h \leq \sum_{i=1}^n n_i, \varepsilon_{\underbrace{a \dots}_{n_1} \dots}_{\underbrace{\phantom{a \dots}}_{n_2} \dots}_{\underbrace{\phantom{a \dots}}_{n_n}}(\vec{p}, h)$

$$= \sum_{h_1=n_1}^{-n_1} \sum_{h_2=n_2}^{-n_2} \dots \sum_{h_n=n_n}^{-n_n} \left[ \frac{\prod_{i=1}^n (2n_i)!}{\prod_{i=1}^n (2n_i)!} \frac{[\sum_{i=1}^n (n_i+h_i)]!}{\prod_{i=1}^n (n_i+h_i)!} \frac{[\sum_{i=1}^n (n_i-h_i)]!}{\prod_{i=1}^n (n_i-h_i)!} \right]^{1/2} \varepsilon_{\underbrace{a \dots}_{n_1}}(\vec{p}, h_1) \varepsilon_{\underbrace{b \dots}_{n_2}}(\vec{p}, h_2) \dots \varepsilon_{\underbrace{c \dots}_{n_n}}(\vec{p}, h_n) \delta(h - \sum_{i=1}^n h_i)$$

## 2.17 一个重要的数学推论

**推论2.17.1.**  $\sum_{h_1=n_1}^{-n_1} \sum_{h_2=n_2}^{-n_2} \dots \sum_{h_n=n_n}^{-n_n} \frac{[\sum_{i=1}^n (n_i+h_i)]!}{\prod_{i=1}^n (n_i+h_i)!} \frac{[\sum_{i=1}^n (n_i-h_i)]!}{\prod_{i=1}^n (n_i-h_i)!} \delta(h - \sum_{i=1}^n h_i) = \frac{[\sum_{i=1}^n (2n_i)]!}{\prod_{i=1}^n (2n_i)!}, -\sum_{i=1}^n n_i \leq h \leq \sum_{i=1}^n n_i$

## 2.18 推论-自旋基分解: $s = 1 \oplus 1 \dots \oplus 1$

**推论2.18.1.**  $-n \leq h \leq n$

$$\varepsilon_{\underbrace{ab \dots c}_{n}}(\vec{p}, h) = \sum_{h_1=1}^{-1} \sum_{h_2=1}^{-1} \dots \sum_{h_n=1}^{-1} \left[ \frac{2^n (n+h)! (n-h)!}{(2n)! \prod_{i=1}^n (1+h_i)! \prod_{i=1}^n (1-h_i)!} \right]^{1/2} \underbrace{\varepsilon_a(\vec{p}, h_1) \varepsilon_b(\vec{p}, h_2) \dots \varepsilon_c(\vec{p}, h_n)}_n \delta(h - \sum_{i=1}^n h_i)$$

**推论2.18.2.**  $-n \leq h \leq n, \forall n$

$$\begin{cases} \varepsilon_{\underbrace{a \dots b \dots c}_{n}}(\vec{p}, n - 2k) = \frac{1}{\sqrt{C_{2n}^{2k}}} \sum_{i=0}^{\min(k, n-k)} \frac{2^i}{(n-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c(\vec{p}, -1)}_{n-k-i} \\ \varepsilon_{\underbrace{a \dots b \dots c}_{n}}(\vec{p}, n - 2k - 1) = \frac{1}{\sqrt{C_{2n}^{2k+1}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i \sqrt{2}}{(n-k-i-1)!(2i+1)!(k-i)!} \underbrace{\varepsilon_a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c(\vec{p}, -1)}_{n-k-i-1} \end{cases}$$

**推论2.18.3.**  $-n \leq h \leq n, \forall n$

$$\begin{aligned} \varepsilon_{\underbrace{a \dots bc}_{n}}(\vec{p}, h) &= \frac{\sqrt{C_{n+h}^2}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h-1) \varepsilon_c(\vec{p}, 1) + \frac{\sqrt{C_{n+h}^1 C_{n-h}^1}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h) \varepsilon_c(\vec{p}, 0) + \frac{\sqrt{C_{n-h}^2}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{a \dots b}_{n-1}}(\vec{p}, h+1) \varepsilon_c(\vec{p}, -1) \\ \Leftrightarrow \begin{cases} \varepsilon_{\underbrace{a \dots b \dots c}_{n}}(\vec{p}, n - 2k) = \frac{1}{\sqrt{C_{2n}^{2k}}} \sum_{i=0}^{\min(k, n-k)} \frac{2^i}{(n-k-i)!(2i)!(k-i)!} \underbrace{\varepsilon_a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c(\vec{p}, -1)}_{n-k-i} \\ \varepsilon_{\underbrace{a \dots b \dots c}_{n}}(\vec{p}, n - 2k - 1) = \frac{1}{\sqrt{C_{2n}^{2k+1}}} \sum_{i=0}^{\min(k, n-1-k)} \frac{2^i \sqrt{2}}{(n-k-i-1)!(2i+1)!(k-i)!} \underbrace{\varepsilon_a(\vec{p}, 1) \dots \varepsilon_b(\vec{p}, 0) \dots \varepsilon_c(\vec{p}, -1)}_{n-k-i-1} \end{cases} \end{aligned}$$

## 2.19 推论- $\varepsilon_{ab..c}(\vec{p}, h)$ 的全对称性

**定理2.19.1.**  $\varepsilon_{\underbrace{ab..c}_n}(\vec{p}, h) = \frac{1}{n!} \varepsilon_{\underbrace{\{ab..c\}_n}(\vec{p}, h), -n \leq h \leq n$

证明: 采用数学归纳法证明此定理。

第一步:  $n' = 1, 2$ 时成立:

$$\varepsilon_a(\vec{p}, h) = \frac{1}{1!} \varepsilon_a(\vec{p}, h), -1 \leq h \leq 1; \varepsilon_{ab}(\vec{p}, h) = \frac{1}{2!} \varepsilon_{\{ab\}}(\vec{p}, h), -2 \leq h \leq 2$$

第二步: 假设 $n' = n - 1$ 时成立:

$$\varepsilon_{\underbrace{a..b}_{n-1}}(\vec{p}, h) = \frac{1}{(n-1)!} \varepsilon_{\underbrace{\{a..b\}_{n-1}}(\vec{p}, h), -n+1 \leq h \leq n-1$$

第三步:  $2 \leq n' = n$ 时:  $-n \leq h \leq n, \varepsilon_{\underbrace{a..bc}_n}(\vec{p}, h)$

$$= \sum_{h'=1}^{-1} \frac{\sqrt{C_{n+h}^{1+h'} C_{n+h}^{1-h'}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{a..b}_{n-1}}(\vec{p}, h-h') \varepsilon_c(\vec{p}, h') = \sum_{h'=2}^{-2} \frac{\sqrt{C_{n+h}^{2+h'} C_{n-h}^{2-h'}}}{\sqrt{C_{2n}^{4n}}} \varepsilon_{\underbrace{a..}_{n-2}}(\vec{p}, h-h') \varepsilon_{bc}(\vec{p}, h')$$

$$\Rightarrow \varepsilon_{\underbrace{a..bc}_n}(\vec{p}, h) = \frac{1}{(n-1)!} \varepsilon_{\underbrace{\{a..b\}_c}_n}(\vec{p}, h), \varepsilon_{\underbrace{a..bc}_n}(\vec{p}, h) = \varepsilon_{\underbrace{a..cb}_n}(\vec{p}, h), -n \leq h \leq n$$

$$\Leftrightarrow \varepsilon_{\underbrace{a..bc}_n}(\vec{p}, h) = \frac{1}{n!} \varepsilon_{\underbrace{\{a..bc\}_n}(\vec{p}, h), -n \leq h \leq n$$

此步证明了 $n' = n$ 时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

## 2.20 $\varepsilon_{ab..c}(\vec{p}, h)$ 性质的小结

**定理2.20.1.**

$$\begin{cases} \varepsilon_{\underbrace{ab..c}_n}(\vec{p}, h) = \sum_{h'=1}^{-1} \frac{\sqrt{C_{n+h}^{1+h'} C_{n+h}^{1-h'}}}{\sqrt{C_{2n}^{2n}}} \varepsilon_{\underbrace{ab..}_{n-1}}(\vec{p}, h-h') \varepsilon_c(\vec{p}, h') \\ \bar{\varepsilon}_a(\vec{p}, h) = (-1)^h \varepsilon_a(\vec{p}, -h), p^a \varepsilon_a(\vec{p}, h) = 0, \bar{\varepsilon}^a(\vec{p}, h) \varepsilon_a(\vec{p}, h') = \delta_{hh'}, -1 \leq h', h \leq 1 \\ \varepsilon_{\underbrace{ab..c}_n}(\vec{p}, h) = \frac{1}{n!} \varepsilon_{\underbrace{\{ab..c\}_n}(\vec{p}, h), \delta^{ab} \varepsilon_{\underbrace{ab..c}_n}(\vec{p}, h) = 0, p^a \varepsilon_{\underbrace{ab..c}_n}(\vec{p}, h) = 0 \\ \bar{\varepsilon}^{ab..c}(\vec{p}, h') \varepsilon_{\underbrace{ab..c}_n}(\vec{p}, h) = \delta_{hh'}, -n \leq h', h \leq n \end{cases}$$

## 2.21 Klein-Gordon方程平面波解的算子表述

**定理2.21.1.**  $(-\partial^c \partial_c + m^2) A_{\underbrace{ab..}_n}(x) = 0, \delta^{ab} A_{\underbrace{ab..}_n}(x) = 0, \partial^a A_{\underbrace{ab..}_n}(x) = 0, A_{\underbrace{ab..}_n}(x) = \frac{1}{n!} A_{\underbrace{\{ab..}_n}(x)$

$$A_{\underbrace{ab..}_n}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2^n E}} \sum_{h=n}^{-n} \frac{j_{-}^{n-h}(\vec{p}, n; R, L)}{(n-h)! \sqrt{C_{2n}^{n-h}}} \varepsilon_{\underbrace{ab..}_n}(\vec{p}, n) [a(\vec{p}, h) e^{ip \cdot x} + (-1)^n b^+(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p}$$

$$A_{\underbrace{ab..}_n}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2^n E}} \sum_{h=n}^{-n} \frac{j_{+}^{n+h}(\vec{p}, n; R, L)}{(n+h)! \sqrt{C_{2n}^{n+h}}} \varepsilon_{\underbrace{ab..}_n}(\vec{p}, -n) [a(\vec{p}, h) e^{ip \cdot x} + (-1)^n b^+(\vec{p}, h) e^{-ip \cdot x}] d^3 \vec{p}$$

## 3 Rarita-Schwinger方程自旋基的重新梳理分析

**3.1 定义-自旋基分解:**  $n + \frac{1}{2} = n \oplus \frac{1}{2}$

**定义3.1.1.**  $-n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

$$\varepsilon_{\underbrace{ab..}_{n-1} \tau_c}(\vec{p}, h) = \frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab..}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_c}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab..}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_c}(\vec{p}, -\frac{1}{2})$$

**推论3.1.1.**  $\bar{\varepsilon}^{\underbrace{ab..}_{n-1} \tau_c}(\vec{p}, h') \varepsilon_{\underbrace{ab..}_{n-1} \tau_c}(\vec{p}, h) = \delta_{hh'}, -n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

**推论3.1.2.**  $p^a \varepsilon_{\underbrace{ab..}_{n-1} \tau_c}(\vec{p}, h) = 0, -n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

推论3.1.3.  $\delta^{ab} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h) = 0, -\frac{5}{2} \leq h \leq \frac{5}{2}$

### 3.2 推论- $\varepsilon_{ab \dots c \tau_c}(\vec{p}, h)$ 是自旋本征态

定理3.2.1.  $[\Omega(n; R) \otimes I_4 + I_{4^n} \otimes \hat{J}(\vec{p}, \frac{1}{2}; \gamma_a)] \cdot \hat{p} \varepsilon_{\underbrace{a \otimes \dots \otimes b \otimes \tau_c}_n}(\vec{p}, h) = h \varepsilon_{\underbrace{a \otimes \dots \otimes b \otimes \tau_c}_n}(\vec{p}, h), -n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

猜想3.2.1.  $[\Omega(n; R) \otimes I_4 + I_{4^n} \otimes \hat{J}(\vec{p}, \frac{1}{2}; \gamma_a)]^2 \varepsilon_{\underbrace{a \otimes \dots \otimes b \otimes \tau_c}_n}(\vec{p}, h) = (n + \frac{1}{2})(n + \frac{3}{2}) \varepsilon_{\underbrace{a \otimes \dots \otimes b \otimes \tau_c}_n}(\vec{p}, h)$

### 3.3 Rarita-Schwinger方程自旋基的升降算符

定义3.3.1.  $\hat{J}(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a) := [\hat{J}(\vec{p}, n; R, L) \otimes I_4 + I_{4^n} \otimes \hat{J}(\vec{p}, \frac{1}{2}; \gamma_a)]$

推论3.3.1.  $[\hat{J}_i(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a), \hat{J}_j(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a)] = \varepsilon_{ij}^k \hat{J}_k(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a)$

定理3.3.1.

$$\begin{cases} \hat{J}_+(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a) \varepsilon_{\underbrace{a \otimes \dots \otimes \tau_c}_n}(\vec{p}, h) = \sqrt{(n + \frac{1}{2})(n + \frac{3}{2}) - h(h + 1)} \varepsilon_{\underbrace{a \otimes \dots \otimes \tau_c}_n}(\vec{p}, h + 1), -n - \frac{1}{2} \leq h \leq n + \frac{1}{2} \\ \hat{J}_-(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a) \varepsilon_{\underbrace{a \otimes \dots \otimes \tau_c}_n}(\vec{p}, h) = \sqrt{(n + \frac{1}{2})(n + \frac{3}{2}) - h(h - 1)} \varepsilon_{\underbrace{a \otimes \dots \otimes \tau_c}_n}(\vec{p}, h - 1), -n - \frac{1}{2} \leq h \leq n + \frac{1}{2} \\ \hat{J}_z(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a) \varepsilon_{\underbrace{a \otimes \dots \otimes \tau_c}_n}(\vec{p}, h) = h \varepsilon_{\underbrace{a \otimes \dots \otimes \tau_c}_n}(\vec{p}, h - 1), -n - \frac{1}{2} \leq h \leq n + \frac{1}{2} \end{cases}$$

证明:  $-n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

$$\begin{aligned} & \hat{J}_+(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a) \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h) \\ &= [\hat{J}_+(\vec{p}, n; R, L) \otimes I_4 + I_{4^n} \otimes \hat{J}_+(\vec{p}, \frac{1}{2}; \gamma_a)] [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_c}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_c}(\vec{p}, -\frac{1}{2})] \\ &= [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \hat{J}_+(\vec{p}, n; R, L, \gamma_a) \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_c}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \hat{J}_+(\vec{p}, n; R, L, \gamma_a) \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_c}(\vec{p}, -\frac{1}{2})] \\ &+ \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{1}{2}) \hat{J}_+(\vec{p}, \frac{1}{2}; R, L, \gamma_a) u_{\tau_c}(\vec{p}, -\frac{1}{2})] \\ &= \frac{\sqrt{n+1/2+h} \sqrt{n(n+1) - (h - \frac{1}{2})(h + \frac{1}{2})}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_c}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_c}(\vec{p}, \frac{1}{2}) \\ &+ \frac{\sqrt{n+1/2-h} \sqrt{n(n+1) - (h + \frac{1}{2})(h + \frac{3}{2})}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{3}{2}) u_{\tau_c}(\vec{p}, -\frac{1}{2}) \\ &= \frac{\sqrt{n+1/2+(h+1)} \sqrt{(n + \frac{1}{2})(n + \frac{3}{2}) - h(h+1)}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_c}(\vec{p}, \frac{1}{2}) \\ &+ \frac{\sqrt{n+1/2-(h+1)} \sqrt{(n + \frac{1}{2})(n + \frac{3}{2}) - h(h+1)}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{3}{2}) u_{\tau_c}(\vec{p}, -\frac{1}{2}) \\ &= \sqrt{(n + \frac{1}{2})(n + \frac{3}{2}) - h(h + 1)} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + 1) \end{aligned} \quad \square$$

证明:  $-n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

$$\begin{aligned} & \hat{J}_-(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a) \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h) \\ &= [\hat{J}_-(\vec{p}, n; R, L) \otimes I_4 + I_{4^n} \otimes \hat{J}_-(\vec{p}, \frac{1}{2}; \gamma_a)] [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_c}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_c}(\vec{p}, -\frac{1}{2})] \\ &= [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \hat{J}_-(\vec{p}, n; R, L, \gamma_a) \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_c}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \hat{J}_-(\vec{p}, n; R, L, \gamma_a) \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_c}(\vec{p}, -\frac{1}{2})] \\ &+ \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab \dots \tau_c}_n}(\vec{p}, h + \frac{1}{2}) \hat{J}_-(\vec{p}, \frac{1}{2}; R, L, \gamma_a) u_{\tau_c}(\vec{p}, -\frac{1}{2})] \end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{n+1/2+h}\sqrt{n(n+1)-(h-\frac{1}{2})(h-\frac{3}{2})}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{3}{2}) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \\
&+ \frac{\sqrt{n+1/2-h}\sqrt{n(n+1)-(h+\frac{1}{2})(h-\frac{1}{2})}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) + \frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \frac{\sqrt{n+1/2+(h-1)}\sqrt{(n+\frac{1}{2})(n+\frac{3}{2})-h(h-1)}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{3}{2}) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \\
&+ \frac{\sqrt{n+1/2-(h-1)}\sqrt{(n+\frac{1}{2})(n+\frac{3}{2})-h(h-1)}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\
&= \sqrt{(n+\frac{1}{2})(n+\frac{3}{2})-h(h-1)} \varepsilon_{\underbrace{ab\dots}_n} \tau_\zeta(\vec{p}, h-1) \quad \square
\end{aligned}$$

证明:  $-n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

$$\begin{aligned}
&\hat{J}_z(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a) \varepsilon_{\underbrace{ab\dots}_n} \tau_\zeta(\vec{p}, h) \\
&= [\hat{J}_z(\vec{p}, n; R, L) \otimes I_4 + I_4 \otimes \hat{J}_z(\vec{p}, \frac{1}{2}; \gamma_a)] [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})] \\
&= [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \hat{J}_z(\vec{p}, n; R, L) \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \hat{J}_z(\vec{p}, n; R, L) \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})] \\
&+ [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{1}{2}) \hat{J}_z(\vec{p}, \frac{1}{2}; \gamma_a) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h + \frac{1}{2}) \hat{J}_z(\vec{p}, \frac{1}{2}; \gamma_a) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})] \\
&= [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} (h - \frac{1}{2}) \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} (h + \frac{1}{2}) \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})] \\
&+ [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{1}{2}) (\frac{1}{2}) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h + \frac{1}{2}) (-\frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})] \\
&= h [\frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h - \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{ab\dots}_n}(\vec{p}, h + \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})] \\
&= h \varepsilon_{\underbrace{ab\dots}_n} \tau_\zeta(\vec{p}, h) \quad \square
\end{aligned}$$

推论3.3.2.  $\hat{J}^2 = \hat{J}_z^2 + \frac{1}{2}(\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+)$

$$\begin{cases} \hat{J}^2(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a) \varepsilon_{\underbrace{a \otimes b \otimes \dots}_n} \tau_\zeta(\vec{p}, h) = (n + \frac{1}{2})(n + \frac{3}{2}) \varepsilon_{\underbrace{a \otimes b \otimes \dots}_n} \tau_\zeta(\vec{p}, h), -n - \frac{1}{2} \leq h \leq n + \frac{1}{2} \\ \hat{J}_z(\vec{p}, n + \frac{1}{2}; R, L, \gamma_a) \varepsilon_{\underbrace{a \otimes \dots}_n} \tau_\zeta(\vec{p}, h) = h \varepsilon_{\underbrace{a \otimes \dots}_n} \tau_\zeta(\vec{p}, h - 1), -n - \frac{1}{2} \leq h \leq n + \frac{1}{2} \end{cases}$$

### 3.4 推论- $\varepsilon_{ab\dots c\tau_\zeta}(\vec{p}, h)$ 的正交性

定理3.4.1.  $\varepsilon_{\underbrace{a \dots bc}_n} \tau_\zeta(\vec{p}, h') \varepsilon_{\underbrace{a \dots bc}_n} \tau_\zeta(\vec{p}, h) = \delta_{hh'}, -n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

### 3.5 推论- $p^a \varepsilon_{ab\dots c\tau_\zeta}(\vec{p}, h)$ 的零性

定理3.5.1.  $p^a \varepsilon_{\underbrace{a \dots bc}_n} \tau_\zeta(\vec{p}, h) = 0, \gamma^a \varepsilon_{\underbrace{a \dots bc}_{[\tau_\zeta]}}(\vec{p}, h) = 0, -n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

### 3.6 推论- $\varepsilon_{ab\dots c\tau_\zeta}(\vec{p}, h)$ 的无迹性

定理3.6.1.  $\delta^{ab} \varepsilon_{\underbrace{ab\dots c}_n} \tau_\zeta(\vec{p}, h) = 0, n \geq 2, -n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

### 3.7 推论- $\varepsilon_{ab\cdots c\tau_\zeta}(\vec{p}, h)$ 的全对称性

定理3.7.1.  $\varepsilon_{\underbrace{ab\cdots c}_{n}\tau_\zeta}(\vec{p}, h) = \frac{1}{n!}\varepsilon_{\underbrace{\{ab\cdots c\}_{\tau_\zeta}}_n}(\vec{p}, h), -n - \frac{1}{2} \leq h \leq n + \frac{1}{2}$

### 3.8 推论-自旋基分解: $n + n' + \frac{1}{2} = n \oplus n' + \frac{1}{2}$

定理3.8.1.  $\varepsilon_{\underbrace{a\cdots b}_{n}\underbrace{\cdots}_{n'}\tau_\zeta}(\vec{p}, h) = \sum_{h'=n'+1/2}^{-n'-1/2} \frac{\sqrt{C_{n+n'+1/2+h}^{n'+1/2+h'} C_{n+n'+1/2-h}^{n'+1/2-h'}}}{\sqrt{C_{2n+2n'+1}^{2n'+1}}} \varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p}, h-h') \varepsilon_{\underbrace{b\cdots}_{n'}\tau_\zeta}(\vec{p}, h')$

推论3.8.1.  $-n_1 - n_2 - \frac{1}{2} \leq h \leq n_1 + n_2 + \frac{1}{2}$

$$\begin{cases} \varepsilon_{\underbrace{a\cdots b}_{n_1}\underbrace{\cdots}_{n_2}\tau_\zeta}(\vec{p}, h) = \sum_{h_1=n_1}^{-n_1} \sum_{h_2=n_2+1/2}^{-n_2-1/2} \frac{\sqrt{C_{n_1+n_2+1/2+h_2}^{n_2+1/2+h_2} C_{n_1+n_2+1/2-h_2}^{n_2+1/2-h_2}}}{\sqrt{C_{2n_1+2n_2+1}^{2n_2+1}}} \varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p}, h_1) \varepsilon_{\underbrace{b\cdots}_{n_2}\tau_\zeta}(\vec{p}, h_2) \delta(h - h_1 - h_2) \\ \varepsilon_{\underbrace{a\cdots b}_{n_1}\underbrace{\cdots}_{n_2}\tau_\zeta}(\vec{p}, h) = \sum_{h_1=n_1}^{-n_1} \sum_{h_2=n_2+1/2}^{-n_2-1/2} \frac{\sqrt{C_{n_1+h_1}^{n_1+h_1} C_{n_1+n_2+1/2-h}^{n_1-h_1}}}{\sqrt{C_{2n_1+2n_2+1}^{2n_1}}} \varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p}, h_1) \varepsilon_{\underbrace{b\cdots}_{n_2}\tau_\zeta}(\vec{p}, h_2) \delta(h - h_1 - h_2) \end{cases}$$

推论3.8.2.  $-n_1 - n_2 - \frac{1}{2} \leq h \leq n_1 + n_2 + \frac{1}{2}, \varepsilon_{\underbrace{a\cdots b}_{n_1}\underbrace{\cdots}_{n_2}\tau_\zeta}(\vec{p}, h)$

$$= \sum_{h_1=n_1}^{-n_1} \sum_{h_2=n_2+1/2}^{-n_2-1/2} \left[ \frac{(2n_1)!(2n_2+1)!}{(2n_1+2n_2+1)!} \frac{(n_1+h_1+n_2+1/2+h_2)!}{(n_1+h_1)!(n_2+1/2+h_2)!} \frac{(n_1-h_1+n_2+1/2-h_2)!}{(n_1-h_1)!(n_2+1/2-h_2)!} \right]^{1/2} \varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p}, h_1) \varepsilon_{\underbrace{b\cdots}_{n_2}\tau_\zeta}(\vec{p}, h_2) \delta(h - h_1 - h_2)$$

### 3.9 推论-自旋基反向合成

推论3.9.1.  $\varepsilon_{\underbrace{a\cdots}_{n}}(\vec{p}, h-h') = \frac{\sqrt{C_{2n+2n'+1}^{2n'+1}}}{\sqrt{C_{n+n'+1/2+h}^{n'+1/2+h'} C_{n+n'+1/2-h}^{n'+1/2-h}}} \varepsilon_{\underbrace{a\cdots b}_{n}\underbrace{\cdots}_{n'}\tau_\zeta}(\vec{p}, h) \varepsilon_{\underbrace{b\cdots}_{n'}\tau_\zeta}(\vec{p}, h')$

### 3.10 推论-自旋基分解: $n_1 + n_2 \cdots + n_n + \frac{1}{2} = n_1 \oplus n_2 \cdots \oplus n_n \oplus \frac{1}{2}$

推论3.10.1.  $-n - \frac{1}{2} \leq h \leq n + \frac{1}{2}, n_0 = \frac{1}{2}; \varepsilon_{\underbrace{a\cdots b}_{n_1}\underbrace{\cdots}_{n_2}\underbrace{\cdots}_{n_n}\tau_\zeta}(\vec{p}, h)$

$$= \sum_{h_0=n_0}^{-n_0} \sum_{h_1=n_1}^{-n_1} \cdots \sum_{h_n=n_n}^{-n_n} \left[ \frac{\prod_{i=0}^n (2n_i)!}{[\sum_{i=0}^n (2n_i)!]} \frac{[\sum_{i=0}^n (n_i+h_i)]!}{\prod_{i=0}^n (n_i+h_i)!} \frac{[\sum_{i=0}^n (n_i-h_i)]!}{\prod_{i=0}^n (n_i-h_i)!} \right]^{1/2} \varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p}, h_1) \cdots \varepsilon_{\underbrace{c\cdots}_{n_n}}(\vec{p}, h_n) \delta(h - \sum_{i=0}^n h_i) u_{\tau_\zeta}(\vec{p}, h_0)$$

证明:  $-n - \frac{1}{2} \leq h \leq n + \frac{1}{2}, n_0 = \frac{1}{2}$

$$\begin{aligned} \varepsilon_{\underbrace{a\cdots b}_{n_1}\underbrace{\cdots}_{n_2}\underbrace{\cdots}_{n_n}\tau_\zeta}(\vec{p}, h) &= \frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{a\cdots b}_{n_1}\underbrace{\cdots}_{n_2}\underbrace{\cdots}_{n_n}}(\vec{p}, h - \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \varepsilon_{\underbrace{a\cdots b}_{n_1}\underbrace{\cdots}_{n_2}\underbrace{\cdots}_{n_n}}(\vec{p}, h + \frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\ &= \frac{\sqrt{n+1/2+h}}{\sqrt{2n+1}} \sum_{h_1=n_1}^{-n_1} \cdots \sum_{h_n=n_n}^{-n_n} \left[ \frac{\prod_{i=1}^n (2n_i)!}{[\sum_{i=1}^n (2n_i)!]} \frac{[\sum_{i=1}^n (n_i+h_i)]!}{\prod_{i=1}^n (n_i+h_i)!} \frac{[\sum_{i=1}^n (n_i-h_i)]!}{\prod_{i=1}^n (n_i-h_i)!} \right]^{1/2} \varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p}, h_1) \cdots \varepsilon_{\underbrace{c\cdots}_{n_n}}(\vec{p}, h_n) \delta(h - \frac{1}{2} - \sum_{i=1}^n h_i) u_{\tau_\zeta}(\vec{p}, \frac{1}{2}) \\ &+ \frac{\sqrt{n+1/2-h}}{\sqrt{2n+1}} \sum_{h_1=n_1}^{-n_1} \cdots \sum_{h_n=n_n}^{-n_n} \left[ \frac{\prod_{i=1}^n (2n_i)!}{[\sum_{i=1}^n (2n_i)!]} \frac{[\sum_{i=1}^n (n_i+h_i)]!}{\prod_{i=1}^n (n_i+h_i)!} \frac{[\sum_{i=1}^n (n_i-h_i)]!}{\prod_{i=1}^n (n_i-h_i)!} \right]^{1/2} \varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p}, h_1) \cdots \varepsilon_{\underbrace{c\cdots}_{n_n}}(\vec{p}, h_n) \delta(h + \frac{1}{2} - \sum_{i=1}^n h_i) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2}) \\ &= \sum_{h_0=n_0}^{-n_0} \sum_{h_1=n_1}^{-n_1} \cdots \sum_{h_n=n_n}^{-n_n} \left[ \frac{\prod_{i=0}^n (2n_i)!}{[\sum_{i=0}^n (2n_i)!]} \frac{[\sum_{i=0}^n (n_i+h_i)]!}{\prod_{i=0}^n (n_i+h_i)!} \frac{[\sum_{i=0}^n (n_i-h_i)]!}{\prod_{i=0}^n (n_i-h_i)!} \right]^{1/2} \varepsilon_{\underbrace{a\cdots}_{n_1}}(\vec{p}, h_1) \cdots \varepsilon_{\underbrace{c\cdots}_{n_n}}(\vec{p}, h_n) \delta(h - \sum_{i=0}^n h_i) u_{\tau_\zeta}(\vec{p}, h_0) \quad \square \end{aligned}$$

### 3.11 Rarita-Schwinger方程平面波解的算子表述

定理3.11.1.  $s := n + \frac{1}{2}$

$$(\gamma^c \partial_c + m) A_{\underbrace{ab\cdots}_{n}\tau_\zeta}(x) = 0, \delta^{ab} A_{\underbrace{ab\cdots}_{n}\tau_\zeta}(x) = 0, \gamma^a A_{\underbrace{ab\cdots}_{n}\tau_\zeta}(x) = 0, A_{\underbrace{ab\cdots}_{n}\tau_\zeta}(x) = \frac{1}{n!} A_{\underbrace{\{ab\cdots\}_{\tau_\zeta}}_n}(x)$$

$$A_{\underbrace{ab\cdots}_{n}}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{\sqrt{m}}{\sqrt{2^s E}} \sum_{h=s}^{-s} \frac{j_{s-h}(\vec{p}, s; R, L, \gamma_a)}{(s-h)! \sqrt{C_{2s}^{s-h}}} [a(\vec{p}, h) \varepsilon_{\underbrace{ab\cdots}_{n}\tau_\zeta}(\vec{p}, s) e^{ip \cdot x} + b^+(\vec{p}, h) \tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\zeta}(\vec{p}, s) e^{-ip \cdot x}] d^3 \vec{p}$$

$$A_{\underbrace{ab\cdots}_{n}}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{\sqrt{m}}{\sqrt{2^s E}} \sum_{h=s}^{-s} \frac{j_{s+h}(\vec{p}, s; R, L, \gamma_a)}{(s+h)! \sqrt{C_{2s}^{s+h}}} [a(\vec{p}, h) \varepsilon_{\underbrace{ab\cdots}_{n}\tau_\zeta}(\vec{p}, -s) e^{ip \cdot x} + b^+(\vec{p}, h) \tilde{\varepsilon}_{\underbrace{ab\cdots}_{n}\tau_\zeta}(\vec{p}, -s) e^{-ip \cdot x}] d^3 \vec{p}$$

## 4 Penrose方程自旋基的重新梳理分析

### 4.1 中微子自旋基是自旋、螺旋度的共同本征态

$$\text{性质4.1.1.} \quad \begin{cases} \sigma^2(\frac{1}{2})\lambda(\hat{p}, \frac{\zeta}{2}) = \frac{1}{2}(\frac{1}{2} + 1)\lambda(\hat{p}, \frac{\zeta}{2}) \\ \sigma(\frac{1}{2}) \cdot \hat{p}\lambda(\hat{p}, \frac{\zeta}{2}) = \frac{\zeta}{2}\lambda(\hat{p}, \frac{\zeta}{2}) \end{cases}$$

### 4.2 定义-自旋基分解: $s = (s - \frac{1}{2}) \oplus \frac{1}{2}$

$$\text{定义4.2.1.} \quad \lambda_{\underbrace{A_\zeta \cdots B_\zeta C_\zeta}_{2s}}(\hat{p}, h) = \frac{\sqrt{s+h}}{\sqrt{2s}} \lambda_{\underbrace{A_\zeta \cdots B_\zeta}_{2s-1}}(\hat{p}, h - \frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \lambda_{\underbrace{A_\zeta \cdots B_\zeta}_{2s-1}}(\hat{p}, h + \frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}), -s \leq h \leq s$$

$$\text{推论4.2.1.} \quad \lambda_{\underbrace{A_\zeta \cdots B_\zeta C_\zeta}_{2s}}(\hat{p}, h) = \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+h'} C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}} \lambda_{\underbrace{A_\zeta \cdots B_\zeta}_{2s-1}}(\hat{p}, h - h') \lambda_{C_\zeta}(\hat{p}, h'), -s \leq h \leq s$$

### 4.3 推论- $\lambda_{A_\zeta \cdots B_\zeta C_\zeta}(\hat{p}, h)$ 是自旋本征态

$$\text{定理4.3.1.} \quad [\Omega(s) \cdot \hat{p}] \lambda_{\underbrace{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}_{2s}}(\hat{p}, h) = h \lambda_{\underbrace{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}_{2s}}(\hat{p}, h), -s \leq h \leq s$$

$$\text{证明:} \quad [\Omega(s) \cdot \hat{p}] \lambda_{\underbrace{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}_{2s}}(\hat{p}, h)$$

$$= [\Omega(s - \frac{1}{2}) \otimes I + I_{2^{2s-1}} \otimes \sigma(\frac{1}{2})] \cdot \hat{p}$$

$$[\frac{\sqrt{s+h}}{\sqrt{2s}} \lambda_{\underbrace{A_\zeta \otimes \cdots \otimes B_\zeta}_{2s-1}}(\hat{p}, h - \frac{1}{2}) \lambda_{\otimes C_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \lambda_{\underbrace{A_\zeta \otimes \cdots \otimes B_\zeta}_{2s-1}}(\hat{p}, h + \frac{1}{2}) \lambda_{\otimes C_\zeta}(\hat{p}, -\frac{1}{2})], -s \leq h \leq s$$

$$= [\frac{\sqrt{s+h}}{\sqrt{2s}} h \lambda_{\underbrace{A_\zeta \otimes \cdots \otimes B_\zeta}_{2s-1}}(\hat{p}, h - \frac{1}{2}) \lambda_{\otimes C_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} h \lambda_{\underbrace{A_\zeta \otimes \cdots \otimes B_\zeta}_{2s-1}}(\hat{p}, h + \frac{1}{2}) \lambda_{\otimes C_\zeta}(\hat{p}, -\frac{1}{2})], -s \leq h \leq s$$

$$= h \lambda_{\underbrace{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}_{2s}}(\hat{p}, h), -s \leq h \leq s \quad \square$$

$$\text{定理4.3.2.} \quad \Omega^2(s) \lambda_{\underbrace{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}_{2s}}(\hat{p}, h) = s(s+1) \lambda_{\underbrace{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}_{2s}}(\hat{p}, h), -s \leq h \leq s$$

以上定理可以用全对称的表象变换方法可以很容易得到证明。由上可知 $\lambda_{\underbrace{A_\zeta \cdots B_\zeta C_\zeta}_{2s}}(\hat{p}, h)$ 是自旋本征态，所以展开系数是CG系数，实际计算结果也表明确实就是对应的CG系数。从而也提供了一种规范、统一、直观和完整的计算CG系数新方法。

### 4.4 Penrose方程自旋基的升降算符

定理4.4.1.

$$\begin{cases} e^{i\vec{\omega} \cdot \Omega(s)} \Omega_x(s) e^{-i\vec{\omega} \cdot \Omega(s)} = \Omega_x(s) - \hat{p}_x \frac{\Omega(s) \cdot \hat{p} + \Omega_z(s)}{(1+\hat{p}_z)} \\ e^{i\vec{\omega} \cdot \Omega(s)} \Omega_y(s) e^{-i\vec{\omega} \cdot \Omega(s)} = \Omega_y(s) - \hat{p}_y \frac{\Omega(s) \cdot \hat{p} + \Omega_z(s)}{(1+\hat{p}_z)} \\ e^{i\vec{\omega} \cdot \Omega(s)} \Omega_z(s) e^{-i\vec{\omega} \cdot \Omega(s)} = \Omega(s) \cdot \hat{p} \end{cases}$$

定义4.4.1.

$$\begin{cases} \hat{J}_x(\hat{p}, \Omega(s)) := \{\Omega_x(s) - \frac{\hat{p}_x}{(1+\hat{p}_z)} [\Omega(s) \cdot \hat{p} + \Omega_z(s)]\} \\ \hat{J}_y(\hat{p}, \Omega(s)) := \{\Omega_y(s) - \frac{\hat{p}_y}{(1+\hat{p}_z)} [\Omega(s) \cdot \hat{p} + \Omega_z(s)]\} \\ \hat{J}_z(\hat{p}, \Omega(s)) := \Omega(s) \cdot \hat{p} \end{cases}$$

推论4.4.1.

$$\begin{cases} \hat{J}_x^2(\hat{p}, \Omega(\frac{1}{2})) = \frac{1}{4}, \hat{J}_y^2(\hat{p}, \Omega(\frac{1}{2})) = \frac{1}{4}, \hat{J}_z^2(\hat{p}, \Omega(\frac{1}{2})) = \frac{1}{4} \\ [\hat{J}_i(\hat{p}, \Omega(s)), \hat{J}_j(\hat{p}, \Omega(s))] = \varepsilon_{ij}^k \hat{J}_k(\hat{p}, \Omega(s)) \end{cases}$$

推论4.4.2.

$$\begin{cases} \hat{J}_+(\hat{p}, \Omega(s)) := \{[\Omega_x(s) + i\Omega_y(s)] - \frac{(\hat{p}_x + i\hat{p}_y)}{(1+\hat{p}_z)} [\Omega(s) \cdot \hat{p} + \Omega_z(s)]\} \\ \hat{J}_-(\hat{p}, \Omega(s)) := \{[\Omega_x(s) - i\Omega_y(s)] - \frac{(\hat{p}_x - i\hat{p}_y)}{(1+\hat{p}_z)} [\Omega(s) \cdot \hat{p} + \Omega_z(s)]\} \\ \hat{J}_z(\hat{p}, \Omega(s)) := \Omega(s) \cdot \hat{p} \end{cases}$$

$$\text{推论4.4.3. } \hat{J}(\hat{p}, \Omega(s)) := \underbrace{\hat{J}(\vec{p}, \sigma(\frac{1}{2})) \otimes I_4 \otimes \cdots \otimes I_4}_{2s} + \underbrace{I_4 \otimes \hat{J}(\vec{p}, \sigma(\frac{1}{2})) \otimes \cdots \otimes I_4}_{2s} + \cdots + \underbrace{I_4 \otimes \cdots \otimes I_4 \otimes \hat{J}(\vec{p}, \sigma(\frac{1}{2}))}_{2s}$$

$$\text{定理4.4.2. } \hat{J}_+(\hat{p}, \Omega(s)) \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h+1)} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}}_{2s}(\vec{p}, h+1)$$

证明: 采用数学归纳法证明此定理。

第一步:  $s' = \frac{1}{2}$ 时成立:

$$\hat{J}_+(\vec{p}, \sigma(\frac{1}{2})) \lambda_{\otimes C_\zeta}(\vec{p}, h) = \sqrt{\frac{3}{4} - h(h+1)} \lambda_{\otimes C_\zeta}(\vec{p}, h+1), -\frac{1}{2} \leq h \leq \frac{1}{2}$$

第二步: 假设  $s' = s - \frac{1}{2}$ 时成立:

$$\hat{J}_+(\vec{p}, s - \frac{1}{2}; \sigma) \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h) = \sqrt{(s - \frac{1}{2})(s + \frac{1}{2}) - h(h+1)} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h), -s + \frac{1}{2} \leq h \leq s - \frac{1}{2}$$

第三步:  $s' = s$ 时:  $-s \leq h \leq s$ ,  $\hat{J}_+(\hat{p}, \Omega(s)) \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}}_{2s}(\vec{p}, h)$

$$\begin{aligned} &= \frac{\sqrt{s+h}}{\sqrt{2s}} [\hat{J}_+(\vec{p}, s - \frac{1}{2}; \sigma) \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2})] \lambda_{\otimes C_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} [\hat{J}_+(\vec{p}, s - \frac{1}{2}; \sigma) \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2})] \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) \\ &+ \frac{\sqrt{s+h}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) \hat{J}_+(\vec{p}, \sigma(\frac{1}{2})) \lambda_{\otimes C_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2}) \hat{J}_+(\vec{p}, \sigma(\frac{1}{2})) \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) \\ &= \frac{\sqrt{s+h} \sqrt{(s+\frac{1}{2})(s-\frac{1}{2}) - (h-\frac{1}{2})(h+\frac{1}{2})}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, \frac{1}{2}) \\ &+ \frac{\sqrt{s-h} \sqrt{(s+\frac{1}{2})(s-\frac{1}{2}) - (h+1-\frac{1}{2})(h+1+\frac{1}{2})}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h + \frac{3}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) \\ &= \frac{\sqrt{(s-h)(s+h+1)} \sqrt{s+h+1}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{(s-h)(s+h+1)} \sqrt{s-h-1}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h + \frac{3}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) \\ &= \sqrt{s(s+1) - h(h+1)} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}}_{2s}(\vec{p}, h+1) \end{aligned}$$

此步证明了  $s' = s$ 时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

$$\text{定理4.4.3. } \hat{J}_-(\hat{p}, \Omega(s)) \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h-1)} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}}_{2s}(\vec{p}, h-1)$$

证明: 采用数学归纳法证明此定理。

第一步:  $s' = \frac{1}{2}$ 时成立:

$$\hat{J}_-(\vec{p}, \sigma(\frac{1}{2})) \lambda_{\otimes C_\zeta}(\vec{p}, h) = \sqrt{\frac{3}{4} - h(h-1)} \lambda_{\otimes C_\zeta}(\vec{p}, h-1), -\frac{1}{2} \leq h \leq \frac{1}{2}$$

第二步: 假设  $s' = s - \frac{1}{2}$ 时成立:

$$\hat{J}_-(\vec{p}, s - \frac{1}{2}; \sigma) \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h) = \sqrt{(s - \frac{1}{2})(s + \frac{1}{2}) - h(h-1)} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h-1), -s + \frac{1}{2} \leq h \leq s - \frac{1}{2}$$

第三步:  $s' = s$ 时:  $-s \leq h \leq s$ ,  $\hat{J}_-(\hat{p}, \Omega(s)) \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta \otimes C_\zeta}}_{2s}(\vec{p}, h)$

$$\begin{aligned} &= \frac{\sqrt{s+h}}{\sqrt{2s}} [\hat{J}_-(\vec{p}, s - \frac{1}{2}; \sigma) \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2})] \lambda_{\otimes C_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} [\hat{J}_-(\vec{p}, s - \frac{1}{2}; \sigma) \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2})] \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) \\ &+ \frac{\sqrt{s+h}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) \hat{J}_-(\vec{p}, \sigma(\frac{1}{2})) \lambda_{\otimes C_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h + \frac{1}{2}) \hat{J}_-(\vec{p}, \sigma(\frac{1}{2})) \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) \\ &= \frac{\sqrt{s+h} \sqrt{(s+\frac{1}{2})(s-\frac{1}{2}) - (h-1-\frac{1}{2})(h-1+\frac{1}{2})}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{3}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, \frac{1}{2}) \\ &+ \frac{\sqrt{s-h} \sqrt{(s+\frac{1}{2})(s-\frac{1}{2}) - (h-\frac{1}{2})(h+\frac{1}{2})}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) + \frac{\sqrt{s+h}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) \\ &= \frac{\sqrt{s+h} \sqrt{(s+h-1)(s-h+1)}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \cdots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{3}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, \frac{1}{2}) \end{aligned}$$



$$\begin{aligned}
& + \frac{\sqrt{s-h}\sqrt{(s+h)(s-h)}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \dots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) + \frac{\sqrt{s+h}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \dots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) \\
& = \frac{\sqrt{(s+h)(s-h+1)}\sqrt{(s+h-1)}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \dots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{3}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, \frac{1}{2}) + \frac{\sqrt{(s+h)(s-h+1)}\sqrt{s-h+1}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta \otimes \dots \otimes B_\zeta}}_{2s-1}(\vec{p}, h - \frac{1}{2}) \lambda_{\otimes C_\zeta}(\vec{p}, -\frac{1}{2}) \\
& = \sqrt{s(s+1) - h(h-1)} \underbrace{\lambda_{A_\zeta \otimes \dots \otimes B_\zeta \otimes C_\zeta}}_{2s}(\vec{p}, h - 1)
\end{aligned}$$

此步证明了  $s' = s$  时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。  $\square$

推论4.4.4.

$$\begin{cases} \hat{J}_+(\hat{p}, \Omega(s)) \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h+1)} \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h+1), -s \leq h \leq s \\ \hat{J}_-(\hat{p}, \Omega(s)) \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = \sqrt{s(s+1) - h(h-1)} \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h-1), -s \leq h \leq s \\ \hat{J}_z(\hat{p}, \Omega(s)) \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = h \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h), -s \leq h \leq s \end{cases}$$

推论4.4.5.  $\hat{J}^2 = \hat{J}_z^2 + \frac{1}{2}(\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+)$

$$\begin{cases} \hat{J}^2(\hat{p}, \Omega(s)) \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = s(s+1) \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h), \hat{J}_z(\hat{p}, \Omega(s)) \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = h \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h) \\ \hat{J}^2(\hat{p}, * \sigma(\frac{1}{2})) \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h) = \frac{3}{4} \underbrace{\lambda_{A_\zeta \otimes B_\zeta \otimes \dots}}_{2s}(\vec{p}, h), \lambda_{A_\zeta B_\zeta \dots}(\vec{p}, h) = \frac{1}{(2s)!} \underbrace{\lambda_{\{A_\zeta B_\zeta \dots\}}}_{2s}(\vec{p}, h), -s \leq h \leq s \end{cases}$$

## 4.5 推论- $\lambda_{A_\zeta \dots B_\zeta C_\zeta}(\hat{p}, h)$ 的正交性

定义4.5.1.  $\lambda^{+A_\zeta}(\hat{p}, h') \lambda_{A_\zeta}(\hat{p}, h) = \delta_{hh'}, -\frac{1}{2} \leq h, h' \leq \frac{1}{2}$

定理4.5.1.  $\lambda^{+A_\zeta \dots B_\zeta C_\zeta}(\hat{p}, h') \underbrace{\lambda_{A_\zeta \dots B_\zeta C_\zeta}}_{2s}(\hat{p}, h) = \delta_{hh'}, -s \leq h', h \leq s$

证明: 采用数学归纳法证明此定理。

第一步:  $s' = \frac{1}{2}$  时成立:

$$\lambda^{+A_\zeta}(\hat{p}, h') \lambda_{A_\zeta}(\hat{p}, h) = \delta_{hh'}, -\frac{1}{2} \leq h, h' \leq \frac{1}{2}$$

第二步: 假设  $s' = s - \frac{1}{2}$  时成立:

$$\lambda^{+A_\zeta \dots B_\zeta}(\hat{p}, h') \underbrace{\lambda_{A_\zeta \dots B_\zeta}}_{2s-1}(\hat{p}, h) = \delta_{hh'}, -s + \frac{1}{2} \leq h', h \leq s - \frac{1}{2}$$

第三步:  $s' = s$  时:  $\lambda^{+A_\zeta \dots B_\zeta C_\zeta}(\hat{p}, h') \underbrace{\lambda_{A_\zeta \dots B_\zeta C_\zeta}}_{2s}(\hat{p}, h), -s \leq h', h \leq s$

$$\begin{aligned}
& = \left[ \sum_{\bar{h}'=1/2}^{-1/2} \frac{\sqrt{C_{s+h'}^{1/2+\bar{h}'} C_{s-h'}^{1/2-\bar{h}'}}}{\sqrt{C_{2s}^1}} \lambda^{+A_\zeta \dots B_\zeta}(\hat{p}, h' - \bar{h}') \lambda^{+C_\zeta}(\hat{p}, \bar{h}') \right] \left[ \sum_{\bar{h}=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \underbrace{\lambda_{A_\zeta \dots B_\zeta}}_{2s-1}(\hat{p}, h - \bar{h}) \lambda_{C_\zeta}(\hat{p}, \bar{h}) \right] \\
& = \sum_{\bar{h}', \bar{h}=1/2}^{-1/2} \left[ \frac{\sqrt{C_{s+h'}^{1/2+\bar{h}'} C_{s-h'}^{1/2-\bar{h}'}}}{\sqrt{C_{2s}^1}} \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \lambda^{+A_\zeta \dots B_\zeta}(\hat{p}, h' - \bar{h}') \underbrace{\lambda_{A_\zeta \dots B_\zeta}}_{2s-1}(\hat{p}, h - \bar{h}) \delta_{\bar{h}\bar{h}'} \right] \\
& = \sum_{\bar{h}=1/2}^{-1/2} \left[ \frac{\sqrt{C_{s+h'}^{1/2+\bar{h}} C_{s-h'}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \lambda^{+A_\zeta \dots B_\zeta}(\hat{p}, h' - \bar{h}) \underbrace{\lambda_{A_\zeta \dots B_\zeta}}_{2s-1}(\hat{p}, h - \bar{h}) \right] \\
& = \sum_{\bar{h}=1/2}^{-1/2} \left[ \frac{\sqrt{C_{s+h'}^{1/2+\bar{h}} C_{s-h'}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \delta_{hh'} \right] \\
& = \sum_{\bar{h}=1/2}^{-1/2} \left[ \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \frac{\sqrt{C_{s+h}^{1/2+\bar{h}} C_{s-h}^{1/2-\bar{h}}}}{\sqrt{C_{2s}^1}} \right] \delta_{hh'} \\
& = \delta_{hh'}
\end{aligned}$$

此步证明了  $s' = s$  时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。  $\square$

#### 4.6 推论-自旋基分解: $1 = \frac{1}{2} \oplus \frac{1}{2}$

$$\begin{aligned} \text{推论4.6.1. } \lambda_{A_\zeta B_\zeta}(\hat{p}, h) &= \frac{\sqrt{1+h}}{\sqrt{2}} \lambda_{A_\zeta}(\hat{p}, h - \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{1-h}}{\sqrt{2}} \lambda_{A_\zeta}(\hat{p}, h + \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}) \\ &= \begin{cases} \lambda_{A_\zeta}(\hat{p}, \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}), h = 1 \\ \frac{1}{\sqrt{2}} \lambda_{\{A_\zeta, B_\zeta\}}(\hat{p}, \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}), h = 0 \\ \lambda_{A_\zeta}(\hat{p}, -\frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}), h = -1 \end{cases} \end{aligned}$$

$$\text{推论4.6.2. } \lambda_{A_\zeta B_\zeta}(\hat{p}, h) = \lambda_{B_\zeta A_\zeta}(\hat{p}, h), -1 \leq h \leq 1$$

$$\text{性质4.6.1. } \lambda(\hat{p}, -\frac{\zeta}{2}) \lambda^T(\hat{p}, \frac{\zeta}{2}) = \frac{i}{2} (\sigma, i\zeta)^a \hat{p}_a \sigma_y$$

$$\text{推论4.6.3. } [(\sigma \otimes I) \cdot (I \otimes \sigma)] [\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) + \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] = [\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) + \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})]$$

$$\begin{aligned} \text{证明: } & \sigma \cdot [\lambda(\hat{p}, \frac{\zeta}{2}) \lambda^T(\hat{p}, \frac{\zeta}{2}) + \lambda(\hat{p}, -\frac{\zeta}{2}) \lambda^T(\hat{p}, \frac{\zeta}{2})] \sigma^T \\ &= \frac{i}{2} \sigma \cdot [(\sigma, -i\zeta)^a \hat{p}_a \sigma_y + (\sigma, i\zeta)^a \hat{p}_a \sigma_y] \sigma^T \\ &= \sigma \cdot [i(\sigma \cdot \hat{p}) \sigma_y] \sigma^T \\ &= [\sigma_x i(\sigma \cdot \hat{p}) \sigma_y \sigma_x^T + \sigma_y i(\sigma \cdot \hat{p}) \sigma_y \sigma_y^T + \sigma_z i(\sigma \cdot \hat{p}) \sigma_y \sigma_z^T] \\ &= [\sigma_x i(\sigma \cdot \hat{p}) \sigma_y \sigma_x^T + \sigma_y i(\sigma \cdot \hat{p}) \sigma_y \sigma_y^T + \sigma_z i(\sigma \cdot \hat{p}) \sigma_y \sigma_z^T] \\ &= i(\sigma \cdot \hat{p}) \sigma_y = [\lambda(\hat{p}, \frac{\zeta}{2}) \lambda^T(\hat{p}, \frac{\zeta}{2}) + \lambda(\hat{p}, -\frac{\zeta}{2}) \lambda^T(\hat{p}, \frac{\zeta}{2})] \end{aligned} \quad \square$$

推论4.6.4.

$$\begin{cases} [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]^2 [\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) + \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] = 2[\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) + \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] \\ [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})] \cdot \hat{p} [\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) + \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] = 0[\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) + \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] \end{cases}$$

#### 4.7 推论-自旋基分解: $0 = \frac{1}{2} \ominus \frac{1}{2}$

$$\text{推论4.7.1. } F_{A_\zeta B_\zeta}(\hat{p}, h) = \frac{1}{\sqrt{2}} \lambda_{[A_\zeta]}(\hat{p}, \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}), h = 0$$

$$\text{推论4.7.2. } [(\sigma \otimes I) \cdot (I \otimes \sigma)] [\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] = -3[\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})]$$

$$\begin{aligned} \text{证明: } & \sigma \cdot [\lambda(\hat{p}, \frac{\zeta}{2}) \lambda^T(\hat{p}, \frac{\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \lambda^T(\hat{p}, \frac{\zeta}{2})] \sigma^T \\ &= \frac{i}{2} \sigma \cdot [(\sigma, -i\zeta)^a \hat{p}_a \sigma_y - (\sigma, i\zeta)^a \hat{p}_a \sigma_y] \sigma^T \\ &= \sigma \cdot (i\zeta \sigma_y) \sigma^T \\ &= \sigma_x (i\zeta \sigma_y) \sigma_x^T + \sigma_y (i\zeta \sigma_y) \sigma_y^T + \sigma_z (i\zeta \sigma_y) \sigma_z^T \\ &= -3(i\zeta \sigma_y) = -3[\lambda(\hat{p}, \frac{\zeta}{2}) \lambda^T(\hat{p}, \frac{\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \lambda^T(\hat{p}, \frac{\zeta}{2})] \end{aligned} \quad \square$$

推论4.7.3.

$$\begin{cases} [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]^2 [\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] = 0[\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] \\ [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})] \cdot \hat{p} [\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] = 0[\lambda(\hat{p}, \frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2}) - \lambda(\hat{p}, -\frac{\zeta}{2}) \otimes \lambda(\hat{p}, \frac{\zeta}{2})] \end{cases}$$

$$\text{推论4.7.4. } u(\hat{p}, \frac{\kappa}{2}) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2m(E+\kappa|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \zeta E + \kappa|\vec{p}| \end{bmatrix}, v(\hat{p}, \frac{\kappa}{2}) = \frac{\lambda(\hat{p}, \frac{\kappa}{2})}{\sqrt{2m(E+\kappa|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \zeta E + \kappa|\vec{p}| \end{bmatrix}$$

#### 4.8 推论-自旋基分解: $s = (s-1) \oplus 1$

$$\text{定理4.8.1. } \lambda_{\underbrace{A_\zeta \dots B_\zeta C_\zeta}_{2s}}(\hat{p}, h) = \sum_{h'=1}^{-1} \frac{\sqrt{C_{s+h}^{1+h'} C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^2}} \lambda_{\underbrace{A_\zeta \dots}_{2(s-1)}}(\hat{p}, h-h') \lambda_{B_\zeta C_\zeta}(\hat{p}, h'), s \geq 1, -s \leq h \leq s$$

证明:  $\lambda_{A_\zeta \dots B_\zeta C_\zeta}(\hat{p}, h)$

$$\begin{aligned}
 &= \frac{\sqrt{s+h}}{\sqrt{2s}} \lambda_{A_\zeta \dots B_\zeta}(\hat{p}, h - \frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \lambda_{A_\zeta \dots B_\zeta}(\hat{p}, h + \frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}) \\
 &= \frac{\sqrt{s+h}}{\sqrt{2s}} \left[ \frac{\sqrt{s+h-1}}{\sqrt{2s-1}} \lambda_{A_\zeta \dots}(\hat{p}, h-1) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s-1}} \lambda_{A_\zeta \dots}(\hat{p}, h) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}) \right] \lambda_{C_\zeta}(\hat{p}, \frac{1}{2}) \\
 &+ \frac{\sqrt{s-h}}{\sqrt{2s}} \left[ \frac{\sqrt{s+h}}{\sqrt{2s-1}} \lambda_{A_\zeta \dots}(\hat{p}, h) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{s-h-1}}{\sqrt{2s-1}} \lambda_{A_\zeta \dots}(\hat{p}, h+1) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}) \right] \lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}) \\
 &= \left[ \frac{\sqrt{s+h}}{\sqrt{2s}} \frac{\sqrt{s+h-1}}{\sqrt{2s-1}} \lambda_{A_\zeta \dots}(\hat{p}, h-1) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{s+h}}{\sqrt{2s}} \frac{\sqrt{s-h}}{\sqrt{2s-1}} \lambda_{A_\zeta \dots}(\hat{p}, h) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, \frac{1}{2}) \right] \\
 &+ \left[ \frac{\sqrt{s-h}}{\sqrt{2s}} \frac{\sqrt{s+h}}{\sqrt{2s-1}} \lambda_{A_\zeta \dots}(\hat{p}, h) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \frac{\sqrt{s-h-1}}{\sqrt{2s-1}} \lambda_{A_\zeta \dots}(\hat{p}, h+1) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}) \right] \\
 &= \frac{\sqrt{C_{s+h}^{2s} C_{s-h}^0}}{\sqrt{C_{2s}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h-1) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{C_{s+h}^{2s} C_{s-h}^0}}{\sqrt{C_{2s}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h) \frac{1}{\sqrt{2}} \lambda_{\{B_\zeta, C_\zeta\}}(\hat{p}, \frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}) \\
 &+ \frac{\sqrt{C_{s+h}^0 C_{s-h}^{2s}}}{\sqrt{C_{2s}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h+1) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}) \\
 &= \frac{\sqrt{C_{s+h}^{2s} C_{s-h}^0}}{\sqrt{C_{2s}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h-1) \lambda_{B_\zeta C_\zeta}(\hat{p}, 1) + \frac{\sqrt{C_{s+h}^{2s} C_{s-h}^0}}{\sqrt{C_{2s}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h) \lambda_{B_\zeta C_\zeta}(\hat{p}, 0) \\
 &+ \frac{\sqrt{C_{s+h}^0 C_{s-h}^{2s}}}{\sqrt{C_{2s}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h+1) \lambda_{B_\zeta C_\zeta}(\hat{p}, -1) \\
 &= \sum_{h'=1}^{-1} \frac{\sqrt{C_{s+h}^{1+h'} C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h-h') \lambda_{B_\zeta C_\zeta}(\hat{p}, h')
 \end{aligned}$$

□

推论4.8.1.  $\lambda_{A_\zeta \dots B_\zeta C_\zeta}(\hat{p}, h) = \lambda_{A_\zeta \dots C_\zeta B_\zeta}(\hat{p}, h), s \geq 1, -s \leq h \leq s$

### 4.9 推论-自旋基分解: $s + s' = s \oplus s'$

定理4.9.1.  $\lambda_{A_\zeta \dots B_\zeta \dots C_\zeta}(\hat{p}, h) = \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h'}^{s'+h'} C_{s+s'-h'}^{s'-h'}}}{\sqrt{C_{2(s+s')}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h-h') \lambda_{B_\zeta \dots C_\zeta}(\hat{p}, h'), -s-s' \leq h \leq s+s'$

证明: 对  $s'$  采用数学归纳法证明此定理。

第一步:  $s'' = \frac{1}{2}$  时成立:

$$\lambda_{A_\zeta \dots C_\zeta}(\hat{p}, h) = \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+1/2+h'}^{1/2+h'} C_{s+1/2-h'}^{1/2-h'}}}{\sqrt{C_{2(s+1/2)}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h-h') \lambda_{C_\zeta}(\hat{p}, h'), -s - \frac{1}{2} \leq h \leq s + \frac{1}{2}$$

第二步: 假设  $s'' = s' - \frac{1}{2}$  时成立:

$$\begin{aligned}
 \lambda_{A_\zeta \dots B_\zeta \dots}(\hat{p}, h) &= \sum_{h'=s'-1/2}^{-s'+1/2} \frac{\sqrt{C_{s+s'+1/2+h'}^{s'+1/2+h'} C_{s+s'-1/2-h'}^{s'-1/2-h'}}}{\sqrt{C_{2(s+s')-1}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h-h') \lambda_{B_\zeta \dots}(\hat{p}, h') \\
 -s-s'+\frac{1}{2} &\leq h \leq s+s'-\frac{1}{2}
 \end{aligned}$$

第三步:  $s'' = s'$  时:  $-s-s' \leq h \leq s+s', \lambda_{A_\zeta \dots B_\zeta \dots C_\zeta}(\hat{p}, h)$

$$\begin{aligned}
 &= \frac{\sqrt{s+s'+h}}{\sqrt{2(s+s')}} \lambda_{A_\zeta \dots B_\zeta \dots}(\hat{p}, h - \frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{s+s'-h}}{\sqrt{2(s+s')}} \lambda_{A_\zeta \dots B_\zeta \dots}(\hat{p}, h + \frac{1}{2}) \lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}) \\
 &= \frac{\sqrt{s+s'+h}}{\sqrt{2(s+s')}} \left[ \sum_{h'=s'-1/2}^{-s'+1/2} \frac{\sqrt{C_{s+s'+1/2+h'}^{s'+1/2+h'} C_{s+s'-1/2-h'}^{s'-1/2-h'}}}{\sqrt{C_{2(s+s')-1}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h - \frac{1}{2} - h') \lambda_{B_\zeta \dots}(\hat{p}, h') \right] \lambda_{C_\zeta}(\hat{p}, \frac{1}{2}) \\
 &+ \left[ \sum_{h'=s'-1/2}^{-s'+1/2} \frac{\sqrt{C_{s+s'+1/2+h'}^{s'+1/2+h'} C_{s+s'-1/2-h'}^{s'-1/2-h'}}}{\sqrt{C_{2(s+s')-1}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h + \frac{1}{2} - h') \lambda_{B_\zeta \dots}(\hat{p}, h') \right] \lambda_{C_\zeta}(\hat{p}, -\frac{1}{2}) \\
 &= \left[ \sum_{h'=s'}^{-s'+1} \frac{\sqrt{C_{s+s'+1+h'}^{s'+1+h'} C_{s+s'-1-h'}^{s'-1-h'}}}{\sqrt{C_{2(s+s')}^2}} \lambda_{A_\zeta \dots}(\hat{p}, h-h') \frac{\sqrt{s+s'+h}}{\sqrt{2s'}} \lambda_{B_\zeta \dots}(\hat{p}, h' - \frac{1}{2}) \right] \lambda_{C_\zeta}(\hat{p}, \frac{1}{2})
 \end{aligned}$$

$$\begin{aligned}
 &+ \left[ \sum_{h'=s'-1}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-1-h}^{s'-1-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{A_{\zeta}} \dots (\hat{p}, h-h') \lambda_{B_{\zeta}} \dots (\hat{p}, h'+\frac{1}{2}) \right] \lambda_{C_{\zeta}} (\hat{p}, -\frac{1}{2}) \\
 &= \left[ \sum_{h'=s'}^{-s'+1} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{A_{\zeta}} \dots (\hat{p}, h-h') \lambda_{B_{\zeta}} \dots (\hat{p}, h'-\frac{1}{2}) \right] \lambda_{C_{\zeta}} (\hat{p}, \frac{1}{2}) \\
 &+ \left[ \sum_{h'=s'-1}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{A_{\zeta}} \dots (\hat{p}, h-h') \lambda_{B_{\zeta}} \dots (\hat{p}, h'+\frac{1}{2}) \right] \lambda_{C_{\zeta}} (\hat{p}, -\frac{1}{2}) \\
 &= \left[ \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{A_{\zeta}} \dots (\hat{p}, h-h') \lambda_{B_{\zeta}} \dots (\hat{p}, h'-\frac{1}{2}) \right] \lambda_{C_{\zeta}} (\hat{p}, \frac{1}{2}) \\
 &+ \left[ \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{A_{\zeta}} \dots (\hat{p}, h-h') \lambda_{B_{\zeta}} \dots (\hat{p}, h'+\frac{1}{2}) \right] \lambda_{C_{\zeta}} (\hat{p}, -\frac{1}{2}) \\
 &= \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{A_{\zeta}} \dots (\hat{p}, h-h') \lambda_{B_{\zeta}} \dots C_{\zeta} (\hat{p}, h'), -s-s' \leq h \leq s+s'
 \end{aligned}$$

此步证明了  $s'' = s'$  时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。 □

**推论4.9.1.**  $-s_1 - s_2 \leq h \leq s_1 + s_2$

$$\begin{cases}
 \lambda_{A_{\zeta}} \dots B_{\zeta} \dots (\hat{p}, h) = \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \frac{\sqrt{C_{s_1+s_2+h}^{s_1+h_1} C_{s_1+s_2-h}^{s_2-h_2}}}{\sqrt{C_{2(s_1+s_2)}^{2s_1+2s_2}}} \lambda_{A_{\zeta}} \dots (\hat{p}, h_1) \lambda_{B_{\zeta}} \dots (\hat{p}, h_2) \delta(h-h_1-h_2) \\
 \lambda_{A_{\zeta}} \dots B_{\zeta} \dots (\hat{p}, h) = \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \frac{\sqrt{C_{s_1+s_2+h}^{s_1+h_1} C_{s_1+s_2-h}^{s_2-h_2}}}{\sqrt{C_{2(s_1+s_2)}^{2s_1+2s_2}}} \lambda_{A_{\zeta}} \dots (\hat{p}, h_1) \lambda_{B_{\zeta}} \dots (\hat{p}, h_2) \delta(h-h_1-h_2)
 \end{cases}$$

**推论4.9.2.**  $-s_1 - s_2 \leq h \leq s_1 + s_2, \lambda_{A_{\zeta}} \dots B_{\zeta} \dots (\hat{p}, h)$

$$= \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \left[ \frac{(2s_1)!(2s_2)! (s_1+h_1+s_2+h_2)! (s_1-h_1+s_2-h_2)!}{(2s_1+2s_2)! (s_1+h_1)!(s_2+h_2)! (s_1-h_1)!(s_2-h_2)!} \right]^{1/2} \lambda_{A_{\zeta}} \dots (\hat{p}, h_1) \lambda_{B_{\zeta}} \dots (\hat{p}, h_2) \delta(h-h_1-h_2)$$

### 4.10 推论-自旋基反向合成

**推论4.10.1.**  $\lambda_{A_{\zeta}} \dots (\hat{p}, h-h') = \frac{\sqrt{C_{2(s+s')}^{2s'}}}{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}} \lambda_{A_{\zeta}} \dots B_{\zeta} \dots C_{\zeta} (\hat{p}, h) \lambda^{+B_{\zeta} \dots C_{\zeta}} (\hat{p}, h'), -s-s' \leq h \leq s+s'$

**推论4.10.2.**  $\lambda_{A_{\zeta}} \dots (\hat{p}, h-h') = \frac{\sqrt{C_{2(s+s')}^{2s'}}}{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}} \lambda^{+B_{\zeta} \dots C_{\zeta}} (\hat{p}, h') \lambda_{B_{\zeta}} \dots C_{\zeta} A_{\zeta} \dots (\hat{p}, h), -s-s' \leq h \leq s+s'$

### 4.11 推论-自旋基分解: $s_1 + s_2 + s_3 = s_1 \oplus s_2 \oplus s_3$

**推论4.11.1.**  $-s_1 - s_2 - s_3 \leq h \leq s_1 + s_2 + s_3, \lambda_{A_{\zeta}} \dots B_{\zeta} \dots C_{\zeta} \dots (\hat{p}, h)$

$$= \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \sum_{h_3=s_3}^{-s_3} \left[ \frac{(2s_1)!(2s_2)!(2s_3)! (s_1+h_1+s_2+h_2+s_3+h_3)! (s_1-h_1+s_2-h_2+s_3-h_3)!}{(2s_1+2s_2+2s_3)! (s_1+h_1)!(s_2+h_2)!(s_3+h_3)! (s_1-h_1)!(s_2-h_2)!(s_3-h_3)!} \right]^{1/2} \lambda_{A_{\zeta}} \dots (\hat{p}, h_1) \lambda_{B_{\zeta}} \dots (\hat{p}, h_2) \lambda_{C_{\zeta}} \dots (\hat{p}, h_3) \delta(h-h_1-h_2-h_3)$$

**证明:**  $-s_1 - s_2 - s_3 \leq h \leq s_1 + s_2 + s_3, \lambda_{A_{\zeta}} \dots B_{\zeta} \dots C_{\zeta} \dots (\hat{p}, h)$

$$\begin{aligned}
 &= \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \sum_{h_3=s_3}^{-s_3} \left[ \frac{(2s_1)!(2s_2+2s_3)! (s_1+h_1+s_2+s_3+h_23)! (s_1-h_1+s_2+s_3-h_23)!}{(2s_1+2s_2+2s_3)! (s_1+h_1)!(s_2+s_3+h_23)! (s_1-h_1)!(s_2+s_3-h_23)!} \right]^{1/2} \\
 &\lambda_{A_{\zeta}} \dots (\hat{p}, h_1) \lambda_{B_{\zeta}} \dots C_{\zeta} \dots (\hat{p}, h_23) \delta(h-h_1-h_23) \\
 &= \sum_{h_1=s_1}^{-s_1} \sum_{h_23=s_2+s_3}^{-s_2-s_3} \left[ \frac{(2s_1)!(2s_2+2s_3)! (s_1+h_1+s_2+s_3+h_23)! (s_1-h_1+s_2+s_3-h_23)!}{(2s_1+2s_2+2s_3)! (s_1+h_1)!(s_2+s_3+h_23)! (s_1-h_1)!(s_2+s_3-h_23)!} \right]^{1/2} \lambda_{A_{\zeta}} \dots (\hat{p}, h_1) \delta(h-h_1-h_23)
 \end{aligned}$$

$$\begin{aligned} & \sum_{h_2=s_2}^{-s_2} \sum_{h_3=s_3}^{-s_3} \left[ \frac{(2s_2)!(2s_3)!}{(2s_2+2s_3)!} \frac{(s_2+h_2+s_3+h_3)!}{(s_2+h_2)!(s_3+h_3)!} \frac{(s_2-h_2+s_3-h_3)!}{(s_2-h_2)!(s_3-h_3)!} \right]^{1/2} \underbrace{\lambda_{B_{s_2}} \dots (\hat{p}, h_2)}_{2s_2} \underbrace{\lambda_{C_{s_3}} \dots (\hat{p}, h_3)}_{2s_3} \delta(h_{23} - h_2 - h_3) \\ &= \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \sum_{h_3=s_3}^{-s_3} \left[ \frac{(2s_1)!(2s_2)!(2s_3)!}{(2s_1+2s_2+2s_3)!} \frac{(s_1+h_1+s_2+h_2+s_3+h_3)!}{(s_1+h_1)!(s_2+h_2)!(s_3+h_3)!} \frac{(s_1-h_1+s_2-h_2+s_3-h_3)!}{(s_1-h_1)!(s_2-h_2)!(s_3-h_3)!} \right]^{1/2} \\ & \underbrace{\lambda_{A_{s_1}} \dots (\hat{p}, h_1)}_{2s_1} \underbrace{\lambda_{B_{s_2}} \dots (\hat{p}, h_2)}_{2s_2} \underbrace{\lambda_{C_{s_3}} \dots (\hat{p}, h_3)}_{2s_3} \delta(h - h_1 - h_2 - h_3) \end{aligned}$$

□

#### 4.12 推论-自旋基分解: $s_1 + s_2 \dots + s_n = s_1 \oplus s_2 \dots \oplus s_n$

$$\begin{aligned} \text{推论4.12.1. } & -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, \underbrace{\lambda_{A_{s_1}} \dots}_{2s_1} \underbrace{\lambda_{B_{s_2}} \dots}_{2s_2} \dots \underbrace{\lambda_{C_{s_n}} \dots}_{2s_n} (\hat{p}, h) \\ &= \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \dots \sum_{h_n=s_n}^{-s_n} \left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{1/2} \underbrace{\lambda_{A_{s_1}} \dots (\hat{p}, h_1)}_{2s_1} \underbrace{\lambda_{B_{s_2}} \dots (\hat{p}, h_2)}_{2s_2} \dots \underbrace{\lambda_{C_{s_n}} \dots (\hat{p}, h_n)}_{2s_n} \delta(h - \sum_{i=1}^n h_i) \end{aligned}$$

#### 4.13 一个重要的数学推论

$$\text{推论4.13.1. } \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \dots \sum_{h_n=s_n}^{-s_n} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \delta(h - \sum_{i=1}^n h_i) = \frac{[\sum_{i=1}^n (2s_i)]!}{\prod_{i=1}^n (2s_i)!}, -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i$$

#### 4.14 推论-自旋基分解: $s = \frac{1}{2} \oplus \frac{1}{2} \dots \oplus \frac{1}{2}$

$$\begin{aligned} \text{推论4.14.1. } & -s \leq h \leq s \\ \underbrace{\lambda_{A_{s/2} B_{s/2} \dots D_{s/2}}}_{2s} (\vec{p}, h) &= \sum_{h_1=1/2}^{-1/2} \sum_{h_2=1/2}^{-1/2} \dots \sum_{h_n=1/2}^{-1/2} \left[ \frac{(s+h)! (s-h)!}{(2s)!} \right]^{1/2} \underbrace{\lambda_{A_{s/2}} (\vec{p}, h_1) \lambda_{B_{s/2}} (\vec{p}, h_2) \dots \lambda_{D_{s/2}} (\vec{p}, h_n)}_{2s} \delta(h - \sum_{i=1}^n h_i) \end{aligned}$$

#### 推论4.14.2. $-s \leq h \leq s$

$$\underbrace{\lambda_{A_{s/2} B_{s/2} \dots C_{s/2} D_{s/2}}}_{2s} (\vec{p}, h) = \frac{1}{(2s)!} \sqrt{C_{2s}^{s-h}} \underbrace{\lambda_{\{A_{s/2} (\vec{p}, \frac{1}{2}) \lambda_{B_{s/2}} (\vec{p}, \frac{1}{2}) \dots \lambda_{C_{s/2}} (\vec{p}, -\frac{1}{2}) \lambda_{D_{s/2}} (\vec{p}, -\frac{1}{2})\}}}_{s+h} (\vec{p}, -\frac{1}{2})$$

#### 推论4.14.3. $-s \leq h \leq s, \forall s$

$$\begin{aligned} \underbrace{\lambda_{A_{s/2} \dots C_{s/2} D_{s/2}}}_{2s} (\vec{p}, h) &= \frac{\sqrt{s+h}}{\sqrt{2s}} \underbrace{\lambda_{A_{s/2} \dots C_{s/2}}}_{2s-1} (\vec{p}, h - \frac{1}{2}) \lambda_{D_{s/2}} (\vec{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \underbrace{\lambda_{A_{s/2} \dots C_{s/2}}}_{2s-1} (\vec{p}, h + \frac{1}{2}) \lambda_{D_{s/2}} (\vec{p}, -\frac{1}{2}) \\ \Leftrightarrow \underbrace{\lambda_{A_{s/2} B_{s/2} \dots C_{s/2} D_{s/2}}}_{2s} (\vec{p}, h) &= \frac{1}{(2s)!} \sqrt{C_{2s}^{s-h}} \underbrace{\lambda_{\{A_{s/2} (\vec{p}, \frac{1}{2}) \lambda_{B_{s/2}} (\vec{p}, \frac{1}{2}) \dots \lambda_{C_{s/2}} (\vec{p}, -\frac{1}{2}) \lambda_{D_{s/2}} (\vec{p}, -\frac{1}{2})\}}}_{s+h} (\vec{p}, -\frac{1}{2}) \end{aligned}$$

#### 4.15 推论- $\lambda_{A_{s_1} \dots B_{s_2} C_{s_3}} (\hat{p}, h)$ 的全对称性

$$\text{定理4.15.1. } \underbrace{\lambda_{A_{s_1} \dots B_{s_2} C_{s_3}}}_{2s} (\hat{p}, h) = \frac{1}{(2s)!} \lambda_{\{A_{s_1} \dots B_{s_2} C_{s_3}\}} (\hat{p}, h), -s \leq h \leq s$$

证明: 采用数学归纳法证明此定理。

第一步:  $s' = \frac{1}{2}, 1$ 时成立:

$$\lambda_{A_{s'}} (\hat{p}, h) = \frac{1}{1!} \lambda_{A_{s'}} (\hat{p}, h), -\frac{1}{2} \leq h \leq \frac{1}{2}; \lambda_{A_{s'} B_{s'}} (\hat{p}, h) = \frac{1}{2!} \lambda_{\{A_{s'} B_{s'}\}} (\hat{p}, h), -1 \leq h \leq 1$$

第二步: 假设  $s' = s - \frac{1}{2}$ 时成立:

$$\underbrace{\lambda_{A_{s'} \dots B_{s'}}}_{2s-1} (\hat{p}, h) = \frac{1}{(2s-1)!} \lambda_{\{A_{s'} \dots B_{s'}\}} (\hat{p}, h), -s + \frac{1}{2} \leq h \leq s - \frac{1}{2}$$

第三步:  $1 \leq s' = s$ 时:  $-s \leq h \leq s, \lambda_{A_{s'} \dots B_{s'} C_{s'}} (\hat{p}, h)$

$$= \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+h'} C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}} \underbrace{\lambda_{A_{s'} \dots B_{s'}}}_{2s-1} (\hat{p}, h-h') \lambda_{C_{s'}} (\hat{p}, h') = \sum_{h'=1}^{-1} \frac{\sqrt{C_{s+h}^{1+h'} C_{s-h}^{1-h'}}}{\sqrt{C_{2s}^2}} \underbrace{\lambda_{A_{s'} \dots B_{s'}}}_{2(s-1)} (\hat{p}, h-h') \lambda_{B_{s'} C_{s'}} (\hat{p}, h')$$

$$\Rightarrow \underbrace{\lambda_{A_{s'} \dots B_{s'} C_{s'}}}_{2s} (\hat{p}, h) = \frac{1}{(2s-1)!} \lambda_{\{A_{s'} \dots B_{s'}\}} (\hat{p}, h), \lambda_{A_{s'} \dots B_{s'} C_{s'}} (\hat{p}, h) = \lambda_{A_{s'} \dots C_{s'} B_{s'}} (\hat{p}, h), -s \leq h \leq s$$

$$\Leftrightarrow \underbrace{\lambda_{A_{s'} \dots B_{s'} C_{s'}}}_{2s} (\hat{p}, h) = \frac{1}{(2s)!} \lambda_{\{A_{s'} \dots C_{s'} B_{s'}\}} (\hat{p}, h), -s \leq h \leq s$$

此步证明了  $s' = s$  时命题成立。

第四步: 根据以上归纳法推理, 命题成立, 定理得证。 □

## 5 自旋方程自旋基的重新梳理分析

### 5.1 定义-自旋基分解: $s = (s - \frac{1}{2}) \oplus \frac{1}{2}$

定义5.1.1.  $-s \leq h \leq s$

$$\lambda_{k_\zeta}(\hat{p}, h; s) = \Gamma_{k_\zeta}^{A_\zeta \cdots B_\zeta} \Gamma_{A_\zeta}^{l_\zeta} \cdot \underbrace{\Gamma_{A_\zeta}^{l_\zeta}}_{2s} \cdot \left[ \frac{\sqrt{s+h}}{\sqrt{2s}} \lambda_{l_\zeta}(\hat{p}, h - \frac{1}{2}; s - \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{s-h}}{\sqrt{2s}} \lambda_{l_\zeta}(\hat{p}, h + \frac{1}{2}; s - \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}) \right]$$

$$\text{推论5.1.1. } \lambda(\hat{p}, h; s) = \Gamma_{k_\zeta}^{A_\zeta \cdots B_\zeta} \Gamma_{A_\zeta}^{l_\zeta} \cdot \underbrace{\Gamma_{A_\zeta}^{l_\zeta}}_{2s} \cdot \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+h'} C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}} \lambda(\hat{p}, h - h'; s - \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, h'), -s \leq h \leq s$$

$$\text{推论5.1.2. } \lambda(\hat{p}, h; s) = N^{A_\zeta}(s) \sum_{h'=1/2}^{-1/2} \frac{\sqrt{C_{s+h}^{1/2+h'} C_{s-h}^{1/2-h'}}}{\sqrt{C_{2s}^1}} \lambda(\hat{p}, h - h'; s - \frac{1}{2}) \lambda_{A_\zeta}(\hat{p}, h'), -s \leq h \leq s$$

### 5.2 推论- $\lambda(\hat{p}, h; s)$ 是自旋本征态

定理5.2.1.  $[\sigma(s) \cdot \hat{p}] \lambda(\hat{p}, h; s) = h \lambda(\hat{p}, h; s), -s \leq h \leq s$

定理5.2.2.  $\sigma^2(s) \lambda(\hat{p}, h; s) = s(s+1) \lambda(\hat{p}, h; s), -s \leq h \leq s$

所以  $\lambda(\hat{p}, h; s)$  是自旋本征态, 所以展开系数是CG系数。

### 5.3 推论-自旋本征态 $\lambda(\hat{p}, h; s)$ 的升降算符

定理5.3.1.

$$\begin{cases} e^{i\vec{\omega} \cdot \sigma(s)} \sigma_x(s) e^{-i\vec{\omega} \cdot \sigma(s)} = \sigma_x(s) - \hat{p}_x \frac{\sigma(s) \cdot \hat{p} + \sigma_z(s)}{(1 + \hat{p}_z)} \\ e^{i\vec{\omega} \cdot \sigma(s)} \sigma_y(s) e^{-i\vec{\omega} \cdot \sigma(s)} = \sigma_y(s) - \hat{p}_y \frac{\sigma(s) \cdot \hat{p} + \sigma_z(s)}{(1 + \hat{p}_z)} \\ e^{i\vec{\omega} \cdot \sigma(s)} \sigma_z(s) e^{-i\vec{\omega} \cdot \sigma(s)} = \sigma(s) \cdot \hat{p} \end{cases}$$

证明:  $e^{i\vec{\omega} \cdot \sigma(s)} \sigma_i(s) e^{-i\vec{\omega} \cdot \sigma(s)} = (e^{-i\vec{\omega} \cdot \gamma})_i^j \sigma_j(s)$

$$\begin{aligned} &= [1 - i(\gamma \times \hat{p})_z - (\gamma \times \hat{p})_z^2 / (1 + \hat{p}_z)]_i^j \sigma_j(s) \\ &= [1 - i(\gamma_x \hat{p}_y - \gamma_y \hat{p}_x) - (\gamma_x \hat{p}_y - \gamma_y \hat{p}_x)^2 / (1 + \hat{p}_z)]_i^j \sigma_j(s) \\ &= [1 - i \begin{bmatrix} 0 & 0 & -i\hat{p}_x \\ 0 & 0 & -i\hat{p}_y \\ i\hat{p}_x & i\hat{p}_y & 0 \end{bmatrix} - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x \hat{p}_y & 0 \\ \hat{p}_x \hat{p}_y & \hat{p}_y^2 & 0 \\ 0 & 0 & \hat{p}_x^2 + \hat{p}_y^2 \end{bmatrix}] / (1 + \hat{p}_z)]_i^j \sigma_j(s) \\ &= [1 - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x \hat{p}_y & \hat{p}_x(1 + \hat{p}_z) \\ \hat{p}_x \hat{p}_y & \hat{p}_y^2 & \hat{p}_y(1 + \hat{p}_z) \\ -\hat{p}_x(1 + \hat{p}_z) & -\hat{p}_y(1 + \hat{p}_z) & \hat{p}_x^2 + \hat{p}_y^2 \end{bmatrix}] / (1 + \hat{p}_z)]_i^j \sigma_j(s) \\ &= \sigma_i(s) - \begin{bmatrix} \hat{p}_x^2 & \hat{p}_x \hat{p}_y & \hat{p}_x \hat{p}_z + \hat{p}_x \\ \hat{p}_x \hat{p}_y & \hat{p}_y^2 & \hat{p}_y \hat{p}_z + \hat{p}_y \\ -\hat{p}_x(1 + \hat{p}_z) & -\hat{p}_y(1 + \hat{p}_z) & (1 - \hat{p}_z)(1 + \hat{p}_z) \end{bmatrix} / (1 + \hat{p}_z)]_i^j \sigma_j(s) \\ &= \begin{bmatrix} \sigma_x(s) - \hat{p}_x[\sigma_z(s) + \sigma(s) \cdot \hat{p}] / (1 + \hat{p}_z) \\ \sigma_y(s) - \hat{p}_y[\sigma_z(s) + \sigma(s) \cdot \hat{p}] / (1 + \hat{p}_z) \\ \sigma(s) \cdot \hat{p} \end{bmatrix}_i \end{aligned}$$

定义5.3.1.

$$\begin{cases} \hat{J}_x(\hat{p}, \sigma(s)) := \bar{\Gamma}(s) \hat{J}_x(\hat{p}, \Omega(s)) \Gamma(s) = \{\sigma_x(s) - \frac{\hat{p}_x}{(1 + \hat{p}_z)} [\sigma(s) \cdot \hat{p} + \sigma_z(s)]\} \\ \hat{J}_y(\hat{p}, \sigma(s)) := \bar{\Gamma}(s) \hat{J}_y(\hat{p}, \Omega(s)) \Gamma(s) = \{\sigma_y(s) - \frac{\hat{p}_y}{(1 + \hat{p}_z)} [\sigma(s) \cdot \hat{p} + \sigma_z(s)]\} \\ \hat{J}_z(\hat{p}, \sigma(s)) := \bar{\Gamma}(s) \hat{J}_z(\hat{p}, \Omega(s)) \Gamma(s) = \sigma(s) \cdot \hat{p} \end{cases}$$

推论5.3.1.

$$\begin{cases} \hat{J}_x^2(\hat{p}, \sigma(\frac{1}{2})) = \frac{1}{4}, \hat{J}_y^2(\hat{p}, \sigma(\frac{1}{2})) = \frac{1}{4}, \hat{J}_z^2(\hat{p}, \sigma(\frac{1}{2})) = \frac{1}{4} \\ [\hat{J}_i(\hat{p}, \sigma(s)), \hat{J}_j(\hat{p}, \sigma(s))] = \varepsilon_{ij}^k \hat{J}_k(\hat{p}, \sigma(s)), \hat{J}^2(\hat{p}, \sigma(s)) = s(s+1) \end{cases}$$

推论5.3.2.

$$\begin{cases} \hat{J}_+(\hat{p}, \sigma(s)) := \bar{\Gamma}(s)\hat{J}_+(\hat{p}, \Omega(s))\Gamma(s) = \{[\sigma_x(s) + i\sigma_y(s)] - \frac{(\hat{p}_x + i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(s) \cdot \hat{p} + \sigma_z(s)]\} \\ \hat{J}_-(\hat{p}, \sigma(s)) := \bar{\Gamma}(s)\hat{J}_-(\hat{p}, \Omega(s))\Gamma(s) = \{[\sigma_x(s) - i\sigma_y(s)] - \frac{(\hat{p}_x - i\hat{p}_y)}{(1+\hat{p}_z)}[\sigma(s) \cdot \hat{p} + \sigma_z(s)]\} \\ \hat{J}_z(\hat{p}, \sigma(s)) := \bar{\Gamma}(s)\hat{J}_z(\hat{p}, \Omega(s))\Gamma(s) = \sigma(s) \cdot \hat{p} \end{cases}$$

推论5.3.3.

$$\begin{cases} \hat{J}_+(\hat{p}, \sigma(s))\lambda(\vec{p}, h; s) = \sqrt{s(s+1) - h(h+1)}\lambda(\vec{p}, h+1; s), -s \leq h \leq s \\ \hat{J}_-(\hat{p}, \sigma(s))\lambda(\vec{p}, h; s) = \sqrt{s(s+1) - h(h-1)}\lambda(\vec{p}, h-1; s), -s \leq h \leq s \\ \hat{J}_z(\hat{p}, \sigma(s))\lambda(\vec{p}, h; s) = h\lambda(\vec{p}, h; s), -s \leq h \leq s \end{cases}$$

推论5.3.4.  $\hat{J}^2 = \hat{J}_z^2 + \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+)$

$$\begin{cases} \hat{J}^2(\hat{p}, \sigma(s))\lambda(\vec{p}, h; s) = s(s+1)\lambda(\vec{p}, h; s), -s \leq h \leq s \\ \hat{J}_z(\hat{p}, \sigma(s))\lambda(\vec{p}, h; s) = h\lambda(\vec{p}, h; s), -s \leq h \leq s \end{cases}$$

## 5.4 推论- $\lambda_{A_\zeta \dots B_\zeta C_\zeta}(\hat{p}, h)$ 的正交性

定理5.4.1.  $\lambda^+(\hat{p}, h'; s)\lambda(\hat{p}, h; s) = \delta_{hh'}$ ,  $-s \leq h \leq s$

## 5.5 推论-自旋基分解: $1 = \frac{1}{2} \oplus \frac{1}{2}$

推论5.5.1.

$$\lambda_{k_\zeta}(\hat{p}, h; 1) = \Gamma_{k_\zeta}^{A_\zeta B_\zeta} \left[ \frac{\sqrt{1+h}}{\sqrt{2}} \lambda_{A_\zeta}(\hat{p}, h - \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{1-h}}{\sqrt{2}} \lambda_{A_\zeta}(\hat{p}, h + \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}) \right], -1 \leq h \leq 1$$

## 5.6 推论-自旋基分解: $s + s' = s \oplus s'$

定理5.6.1.

$$\lambda_{k_\zeta}(\hat{p}, h; s + s') = \Gamma_{k_\zeta}^{\overbrace{A_\zeta \dots B_\zeta}^{2s+2s'}} \Gamma_{A_\zeta}^{l_\zeta} \dots \Gamma_{B_\zeta}^{m_\zeta} \dots \sum_{h'=s'}^{-s'} \frac{\sqrt{C_{s+s'+h}^{s'+h'} C_{s+s'-h}^{s'-h'}}}{\sqrt{C_{2(s+s')}^{2s'}}} \lambda_{l_\zeta}(\hat{p}, h - h'; s) \lambda_{m_\zeta}(\hat{p}, h'; s'), -s - s' \leq h \leq s + s'$$

## 5.7 推论-自旋基分解: $s_1 + s_2 \dots + s_n = s_1 \oplus s_2 \dots \oplus s_n$

$$\text{推论5.7.1. } -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, \lambda_{k_\zeta}(\hat{p}, h; \sum_{i=1}^n s_i) = \Gamma_{k_\zeta}^{\overbrace{A_\zeta \dots B_\zeta}^{2s_1} \dots \overbrace{C_\zeta \dots}^{2s_n}} \Gamma_{A_\zeta}^{l_\zeta} \dots \Gamma_{B_\zeta}^{m_\zeta} \dots \Gamma_{C_\zeta}^{n_\zeta} \dots$$

$$\sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \dots \sum_{h_n=s_n}^{-s_n} \left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right] \frac{1}{2} \lambda_{l_\zeta}(\hat{p}, h_1; s_1) \lambda_{m_\zeta}(\hat{p}, h_2; s_2) \dots \lambda_{n_\zeta}(\hat{p}, h_n; s_n) \delta(h - \sum_{i=1}^n h_i)$$

## 5.8 引入一个新的常数不变张量

$$\text{定义5.8.1. } \Gamma_{k_\zeta}^{l_\zeta m_\zeta \dots n_\zeta} := \Gamma_{k_\zeta}^{\overbrace{A_\zeta \dots B_\zeta}^{2s_1} \dots \overbrace{C_\zeta \dots}^{2s_n}} \Gamma_{A_\zeta}^{l_\zeta} \dots \Gamma_{B_\zeta}^{m_\zeta} \dots \Gamma_{C_\zeta}^{n_\zeta} \dots$$

$$\text{推论5.8.1. } -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, \lambda_{k_\zeta}(\hat{p}, h; \sum_{i=1}^n s_i) = \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \dots \sum_{h_n=s_n}^{-s_n}$$

$$\left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right] \frac{1}{2} \Gamma_{k_\zeta}^{l_\zeta m_\zeta \dots n_\zeta} \lambda_{l_\zeta}(\hat{p}, h_1; s_1) \lambda_{m_\zeta}(\hat{p}, h_2; s_2) \dots \lambda_{n_\zeta}(\hat{p}, h_n; s_n) \delta(h - \sum_{i=1}^n h_i)$$

## 6 关于自旋基的么正变换

### 6.1 常数不变张量: 动量变换等价于么正变换

$$\text{定义6.1.1. } \hat{p}' = A(\hat{p} \rightarrow \hat{p}')\hat{p}, A(\hat{p} \rightarrow \hat{p}') := \exp\{i \frac{(\gamma \times \hat{p}')_z}{\sqrt{1-\hat{p}'^2}} \arccos \hat{p}'_z\} \exp\{-i \frac{(\gamma \times \hat{p})_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\}$$

$$= [1 + i(\gamma \times \hat{p}')_z - (\gamma \times \hat{p}')_z^2 / (1 + \hat{p}'_z)] [1 - i(\gamma \times \hat{p})_z - (\gamma \times \hat{p})_z^2 / (1 + \hat{p}_z)]$$

推论6.1.1.

$$\begin{cases} [\sigma(s) \cdot \hat{p}'] = [\sigma(s) \cdot A(\hat{p} \rightarrow \hat{p}')\hat{p}] = [A(\hat{p}' \rightarrow \hat{p})\sigma(s) \cdot \hat{p}] \\ = \exp\left\{i \frac{[\sigma(s) \times \hat{p}']_z}{\sqrt{1-\hat{p}'^2}} \arccos \hat{p}'_z\right\} \exp\left\{-i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} [\sigma(s) \cdot \hat{p}] \exp\left\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \exp\left\{i \frac{[\sigma(s) \times \hat{p}']_z}{\sqrt{1-\hat{p}'^2}} \arccos \hat{p}'_z\right\} \\ [\Omega(s) \cdot \hat{p}'] = [\Omega(s) \cdot A(\hat{p} \rightarrow \hat{p}')\hat{p}] = [A(\hat{p}' \rightarrow \hat{p})\Omega(s) \cdot \hat{p}] \\ = \exp\left\{i \frac{[\Omega(s) \times \hat{p}']_z}{\sqrt{1-\hat{p}'^2}} \arccos \hat{p}'_z\right\} \exp\left\{-i \frac{[\Omega(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} [\Omega(s) \cdot \hat{p}] \exp\left\{i \frac{[\Omega(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \exp\left\{i \frac{[\Omega(s) \times \hat{p}']_z}{\sqrt{1-\hat{p}'^2}} \arccos \hat{p}'_z\right\} \end{cases}$$

推论6.1.2.

$$\begin{cases} [\Omega(s; \sigma(\frac{1}{2}) \otimes I) \cdot \hat{p}'] = [\Omega(s; \sigma(\frac{1}{2}) \otimes I) \cdot A(\hat{p} \rightarrow \hat{p}')\hat{p}] = [A(\hat{p}' \rightarrow \hat{p})\Omega(s; \sigma(\frac{1}{2}) \otimes I) \cdot \hat{p}] \\ = \exp\left\{i \frac{[\Omega(s; \sigma(\frac{1}{2}) \otimes I) \times \hat{p}']_z}{\sqrt{1-\hat{p}'^2}} \arccos \hat{p}'_z\right\} \exp\left\{-i \frac{[\Omega(s; \sigma(\frac{1}{2}) \otimes I) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} [\Omega(s; \sigma(\frac{1}{2}) \otimes I) \cdot \hat{p}] \\ \exp\left\{i \frac{[\Omega(s; \sigma(\frac{1}{2}) \otimes I) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \exp\left\{i \frac{[\Omega(s; \sigma(\frac{1}{2}) \otimes I) \times \hat{p}']_z}{\sqrt{1-\hat{p}'^2}} \arccos \hat{p}'_z\right\} \\ [\Omega(s; R) \cdot \hat{p}'] = [\Omega(s; R) \cdot A(\hat{p} \rightarrow \hat{p}')\hat{p}] = [A(\hat{p}' \rightarrow \hat{p})\Omega(s; R) \cdot \hat{p}] \\ = \exp\left\{i \frac{[\Omega(s; R) \times \hat{p}']_z}{\sqrt{1-\hat{p}'^2}} \arccos \hat{p}'_z\right\} \exp\left\{-i \frac{[\Omega(s; R) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} [\Omega(s; R) \cdot \hat{p}] \\ \exp\left\{i \frac{[\Omega(s; R) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \exp\left\{i \frac{[\Omega(s; R) \times \hat{p}']_z}{\sqrt{1-\hat{p}'^2}} \arccos \hat{p}'_z\right\} \end{cases}$$

推论6.1.3.

$$\begin{cases} \sigma(s) \cdot \hat{p} = e^{i\vec{\omega} \cdot \sigma(s)} \sigma_z e^{-i\vec{\omega} \cdot \sigma(s)}, \Omega(s) \cdot \hat{p} = e^{i\vec{\omega} \cdot \Omega(s)} \Omega_z(s) e^{-i\vec{\omega} \cdot \Omega(s)} \\ \Omega(s; \sigma(\frac{1}{2}) \otimes I) \cdot \hat{p} = e^{i\vec{\omega} \cdot \Omega(s; \sigma(\frac{1}{2}) \otimes I)} \Omega_z(s; \sigma(\frac{1}{2}) \otimes I) e^{-i\vec{\omega} \cdot \Omega(s; \sigma(\frac{1}{2}) \otimes I)} \\ \Omega(s; R) \cdot \hat{p} = e^{i\vec{\omega} \cdot \Omega(s; R)} \Omega_z(s; R) e^{-i\vec{\omega} \cdot \Omega(s; R)} \end{cases}$$

推论6.1.4.

$$\begin{cases} \sigma(s) \cdot \hat{p} = \exp\left\{i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \sigma_z \exp\left\{-i \frac{[\sigma(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \\ \Omega(s) \cdot \hat{p} = \exp\left\{i \frac{[\Omega(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \Omega_z(s) \exp\left\{-i \frac{[\Omega(s) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \\ \Omega(s; \sigma(\frac{1}{2}) \otimes I) \cdot \hat{p} = \exp\left\{i \frac{[\Omega(s; \sigma(\frac{1}{2}) \otimes I) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \Omega_z(s; \sigma(\frac{1}{2}) \otimes I) \exp\left\{-i \frac{[\Omega(s; \sigma(\frac{1}{2}) \otimes I) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \\ \Omega(s; R) \cdot \hat{p} = \exp\left\{i \frac{[\Omega(s; R) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \Omega_z(s; R) \exp\left\{-i \frac{[\Omega(s; R) \times \hat{p}]_z}{\sqrt{1-\hat{p}^2}} \arccos \hat{p}_z\right\} \end{cases}$$

## 6.2 物理自旋基分解: $1 = \frac{1}{2} \oplus \frac{1}{2}$

$$\text{推论6.2.1. } \lambda_{A_\zeta B_\zeta}(\hat{p}', \hat{p}, h) = \frac{\sqrt{1+h}}{\sqrt{2}} \lambda_{A_\zeta}(\hat{p}', h - \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}) + \frac{\sqrt{1-h}}{\sqrt{2}} \lambda_{A_\zeta}(\hat{p}', h + \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2})$$

$$\text{推论6.2.2. } \lambda_{A_\zeta B_\zeta}(\hat{p}', \hat{p}, h) = \begin{cases} \lambda_{A_\zeta}(\hat{p}', \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, \frac{1}{2}), h = 1 \\ \frac{1}{\sqrt{2}} \lambda_{\{A_\zeta, \frac{1}{2}\}}(\hat{p}', \frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}), h = 0 \\ \lambda_{A_\zeta}(\hat{p}', -\frac{1}{2}) \lambda_{B_\zeta}(\hat{p}, -\frac{1}{2}), h = -1 \end{cases}$$

$$\text{推论6.2.3. } \lambda(\hat{p}', \hat{p}, h) \text{ 对} = \begin{cases} \lambda(\hat{p}', \frac{1}{2}) \otimes \lambda(\hat{p}, \frac{1}{2}), h = 1 \\ \frac{1}{\sqrt{2}} [\lambda(\hat{p}', \frac{1}{2}) \otimes \lambda(\hat{p}, -\frac{1}{2}) + \lambda(\hat{p}', -\frac{1}{2}) \otimes \lambda(\hat{p}, \frac{1}{2})], h = 0 \\ \lambda(\hat{p}', -\frac{1}{2}) \otimes \lambda(\hat{p}, -\frac{1}{2}), h = -1 \end{cases}$$

$$\text{推论6.2.4. } \lambda^+(\hat{p}', \hat{p}, h') \lambda(\hat{p}', \hat{p}, h) = \delta_{hh'}$$

$$\text{推论6.2.5. } \tilde{\lambda}(\hat{p}', \hat{p}, h) \text{ 不对} = \begin{cases} \lambda(\hat{p}', \frac{1}{2}) \otimes \lambda(\hat{p}, \frac{1}{2}) + \lambda(\hat{p}, \frac{1}{2}) \otimes \lambda(\hat{p}', \frac{1}{2}), h = 1 \\ \lambda(\hat{p}', \frac{1}{2}) \otimes \lambda(\hat{p}, -\frac{1}{2}) + \lambda(\hat{p}, -\frac{1}{2}) \otimes \lambda(\hat{p}', \frac{1}{2}), h = 0 \\ \lambda(\hat{p}', -\frac{1}{2}) \otimes \lambda(\hat{p}, -\frac{1}{2}) + \lambda(\hat{p}, -\frac{1}{2}) \otimes \lambda(\hat{p}', -\frac{1}{2}), h = -1 \end{cases}$$

推论6.2.6.

$$\begin{cases} \{[\sigma(\frac{1}{2}) \cdot \hat{p}'] \otimes I + I \otimes [\sigma(\frac{1}{2}) \cdot \hat{p}]\} \lambda(\hat{p}', \hat{p}, h) = h \lambda(\hat{p}', \hat{p}, h) \\ \{[A(\hat{p}' \rightarrow \hat{p})\sigma(\frac{1}{2}) \cdot \hat{p}] \otimes I + I \otimes [\sigma(\frac{1}{2}) \cdot \hat{p}]\} \lambda(\hat{p}', \hat{p}, h) = h \lambda(\hat{p}', \hat{p}, h) \end{cases}$$



推论6.2.7.

$$\begin{cases} [A(\hat{p}' \rightarrow \hat{p})\sigma(\frac{1}{2})] \cdot \hat{p}\lambda(\hat{p}', h) = h\lambda(\hat{p}', h), -\frac{1}{2} \leq h \leq \frac{1}{2} & \left\{ \begin{aligned} [\sigma(\frac{1}{2}) \cdot \hat{p}]\lambda(\hat{p}, h) &= h\lambda(\hat{p}, h), -\frac{1}{2} \leq h \leq \frac{1}{2} \\ [\sigma(\frac{1}{2})]^2\lambda(\hat{p}, h) &= \frac{1}{2}(\frac{1}{2} + 1)\lambda(\hat{p}, h) \end{aligned} \right. \\ [A(\hat{p}' \rightarrow \hat{p})\sigma(\frac{1}{2})]^2\lambda(\hat{p}', h) = \frac{1}{2}(\frac{1}{2} + 1)\lambda(\hat{p}', h) & \\ \left\{ \begin{aligned} \{[A(\hat{p}' \rightarrow \hat{p})\sigma(\frac{1}{2})] \otimes I\} \cdot \hat{p}\lambda(\hat{p}', \hat{p}, h) &= \frac{h}{2}\lambda(\hat{p}', \hat{p}, h) \\ \{[A(\hat{p}' \rightarrow \hat{p})\sigma(\frac{1}{2})] \otimes I\}^2\lambda(\hat{p}', \hat{p}, h) &= \frac{1}{2}(\frac{1}{2} + 1)\lambda(\hat{p}', \hat{p}, h) \end{aligned} \right. & \left\{ \begin{aligned} \{I \otimes \sigma(\frac{1}{2})\} \cdot \hat{p}\lambda(\hat{p}', \hat{p}, h) &= \frac{h}{2}\lambda(\hat{p}', \hat{p}, h) \\ \{I \otimes \sigma(\frac{1}{2})\}^2\lambda(\hat{p}', \hat{p}, h) &= \frac{1}{2}(\frac{1}{2} + 1)\lambda(\hat{p}', \hat{p}, h) \end{aligned} \right. \\ \{[A(\hat{p}' \rightarrow \hat{p})\sigma(\frac{1}{2})] \otimes I + I \otimes \sigma(\frac{1}{2})\} \cdot \hat{p}\lambda(\hat{p}', \hat{p}, h) &= h\lambda(\hat{p}', \hat{p}, h), -1 \leq h \leq 1 \\ \{[A(\hat{p}' \rightarrow \hat{p})\sigma(\frac{1}{2})] \otimes I + I \otimes \sigma(\frac{1}{2})\}^2\lambda(\hat{p}', \hat{p}, h) &= 1(1 + 1)\lambda(\hat{p}', \hat{p}, h) \end{cases}$$

推论6.2.8.  $\lambda^+(\hat{p}', \hat{p}, h)\{[A(\hat{p}' \rightarrow \hat{p})\sigma(\frac{1}{2})] \otimes I\} \cdot \{I \otimes \sigma(\frac{1}{2})\}\lambda(\hat{p}', \hat{p}, h) = \frac{1}{4}\lambda(\hat{p}', \hat{p}, h)$

推论6.2.9.  $\{[A(\hat{p}' \rightarrow \hat{p})\sigma(\frac{1}{2})] \otimes I\} \cdot \{I \otimes \sigma(\frac{1}{2})\}\lambda(\hat{p}', \hat{p}, h) = ?$

推论6.2.10.  $H = -\sum_{i,j} k_{ij}[A(\hat{p}_i)\sigma_i(\frac{1}{2})] \cdot [A(\hat{p}_j)\sigma_j(\frac{1}{2})]$

6.3 么正变换 $\rightarrow$ B-W方程自旋基分解的z-方向表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$

定义6.3.1.  $-\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, U_{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta \cdots \rho_\zeta \sigma_\zeta} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right) := \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \cdots \sum_{h_n=s_n}^{-s_n}$

$$\left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{\frac{1}{2}} U_{\lambda_\zeta \mu_\zeta \cdots} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h_1\right) U_{\eta_\zeta \xi_\zeta \cdots} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h_2\right) \cdots U_{\rho_\zeta \sigma_\zeta \cdots} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h_n\right) \delta\left(h - \sum_{i=1}^n h_i\right)$$

$$\text{推论6.3.1. } \begin{cases} \Omega^2(s; \sigma(\frac{1}{2}) \otimes I) U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right) = s(s+1) U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right), -s \leq h \leq s \\ \Omega_z(s; \sigma(\frac{1}{2}) \otimes I) U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right) = h U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right), -s \leq h \leq s \end{cases}$$

定义6.3.2.  $-\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, V_{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta \cdots \rho_\zeta \sigma_\zeta} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right) := \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \cdots \sum_{h_n=s_n}^{-s_n}$

$$\left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{\frac{1}{2}} V_{\lambda_\zeta \mu_\zeta \cdots} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h_1\right) V_{\eta_\zeta \xi_\zeta \cdots} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h_2\right) \cdots V_{\rho_\zeta \sigma_\zeta \cdots} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h_n\right) \delta\left(h - \sum_{i=1}^n h_i\right)$$

$$\text{推论6.3.2. } \begin{cases} \Omega^2(s; \sigma(\frac{1}{2}) \otimes I) V_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right) = s(s+1) V_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right), -s \leq h \leq s \\ \Omega_z(s; \sigma(\frac{1}{2}) \otimes I) V_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right) = h V_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot \left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right), -s \leq h \leq s \end{cases}$$

6.4 么正变换 $\rightarrow$ B-W方程自旋基分解的静止表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$

定义6.4.1.  $-\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, U_{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta \cdots \rho_\zeta \sigma_\zeta} \cdot (\vec{0}, h) := \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \cdots \sum_{h_n=s_n}^{-s_n}$

$$\left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{\frac{1}{2}} U_{\lambda_\zeta \mu_\zeta \cdots} \cdot (\vec{0}, h_1) U_{\eta_\zeta \xi_\zeta \cdots} \cdot (\vec{0}, h_2) \cdots U_{\rho_\zeta \sigma_\zeta \cdots} \cdot (\vec{0}, h_n) \delta\left(h - \sum_{i=1}^n h_i\right)$$

$$\text{推论6.4.1. } \begin{cases} \Omega^2(s; \sigma(\frac{1}{2}) \otimes I) U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot (\vec{0}, h) = s(s+1) U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot (\vec{0}, h), -s \leq h \leq s \\ \Omega_z(s; \sigma(\frac{1}{2}) \otimes I) U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot (\vec{0}, h) = h U_{\lambda_\zeta \otimes \cdots \otimes \sigma_\zeta \otimes \tau_\zeta} \cdot (\vec{0}, h), -s \leq h \leq s \end{cases}$$

定义6.4.2.  $-\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, V_{\lambda_\zeta \mu_\zeta \cdots \eta_\zeta \xi_\zeta \cdots \rho_\zeta \sigma_\zeta} \cdot (\vec{0}, h) := \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \cdots \sum_{h_n=s_n}^{-s_n}$

$$\left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{\frac{1}{2}} V_{\lambda_\zeta \mu_\zeta \cdots} \cdot (\vec{0}, h_1) V_{\eta_\zeta \xi_\zeta \cdots} \cdot (\vec{0}, h_2) \cdots V_{\rho_\zeta \sigma_\zeta \cdots} \cdot (\vec{0}, h_n) \delta\left(h - \sum_{i=1}^n h_i\right)$$

$$\text{推论6.4.2. } \begin{cases} \Omega^2(s; \sigma(\frac{1}{2}) \otimes I) \underbrace{V_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{0}, h) = s(s+1) \underbrace{V_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{0}, h), -s \leq h \leq s \\ \Omega_z(s; \sigma(\frac{1}{2}) \otimes I) \underbrace{V_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{0}, h) = h \underbrace{V_{\lambda_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{0}, h), -s \leq h \leq s \end{cases}$$

6.5 么正变换→K-G方程自旋基分解的z-方向表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$

$$\text{定义6.5.1. } -\sum_{i=1}^n n_i \leq h \leq \sum_{i=1}^n n_i, \varepsilon_{\underbrace{a \dots b \dots c \dots}_{2n_1 \ 2n_2 \ 2n_n}}\left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right) \\ := \sum_{h_1=n_1}^{-n_1} \sum_{h_2=n_2}^{-n_2} \cdots \sum_{h_n=n_n}^{-n_n} \left[ \frac{\prod_{i=1}^n (2n_i)!}{[\sum_{i=1}^n (2n_i)]!} \frac{[\sum_{i=1}^n (n_i+h_i)]!}{\prod_{i=1}^n (n_i+h_i)!} \frac{[\sum_{i=1}^n (n_i-h_i)]!}{\prod_{i=1}^n (n_i-h_i)!} \right]^{\frac{1}{2}} \varepsilon_{\underbrace{a \dots}_{2n_1}}\left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h_1\right) \varepsilon_{\underbrace{b \dots}_{2n_2}}\left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h_2\right) \cdots \varepsilon_{\underbrace{c \dots}_{2n_n}}\left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h_n\right) \delta\left(h - \sum_{i=1}^n h_i\right)$$

$$\text{推论6.5.1. } \begin{cases} \Omega^2(n; R) \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_{n}}\left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right) = n(n+1) \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_{n}}\left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right), -n \leq h \leq n \\ \Omega_z(n; R) \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_{n}}\left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right) = h \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_{n}}\left(\begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, h\right), -n \leq h \leq n \end{cases}$$

6.6 么正变换→K-G方程自旋基分解的静止表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$

$$\text{定义6.6.1. } -\sum_{i=1}^n n_i \leq h \leq \sum_{i=1}^n n_i, \varepsilon_{\underbrace{a \dots b \dots c \dots}_{2n_1 \ 2n_2 \ 2n_n}}(\vec{0}, h) \\ := \sum_{h_1=n_1}^{-n_1} \sum_{h_2=n_2}^{-n_2} \cdots \sum_{h_n=n_n}^{-n_n} \left[ \frac{\prod_{i=1}^n (2n_i)!}{[\sum_{i=1}^n (2n_i)]!} \frac{[\sum_{i=1}^n (n_i+h_i)]!}{\prod_{i=1}^n (n_i+h_i)!} \frac{[\sum_{i=1}^n (n_i-h_i)]!}{\prod_{i=1}^n (n_i-h_i)!} \right]^{\frac{1}{2}} \varepsilon_{\underbrace{a \dots}_{2n_1}}(\vec{0}, h_1) \varepsilon_{\underbrace{b \dots}_{2n_2}}(\vec{0}, h_2) \cdots \varepsilon_{\underbrace{c \dots}_{2n_n}}(\vec{0}, h_n) \delta\left(h - \sum_{i=1}^n h_i\right)$$

$$\text{推论6.6.1. } \begin{cases} \Omega^2(n; R) \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_{n}}(\vec{0}, h) = n(n+1) \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_{n}}(\vec{0}, h), -n \leq h \leq n \\ \Omega_z(n; R) \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_{n}}(\vec{0}, h) = h \varepsilon_{\underbrace{a \dots \otimes b \otimes c}_{n}}(\vec{0}, h), -n \leq h \leq n \end{cases}$$

6.7 么正变换→Penrose方程自旋基分解的z-方向表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$

$$\text{定义6.7.1. } -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, \lambda_{\underbrace{A_\zeta \dots B_\zeta \dots C_\zeta \dots}_{2s_1 \ 2s_2 \ 2s_n}}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h\right) := \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \cdots \sum_{h_n=s_n}^{-s_n} \\ \left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{\frac{1}{2}} \lambda_{\underbrace{A_\zeta \dots}_{2s_1}}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h_1\right) \lambda_{\underbrace{B_\zeta \dots}_{2s_2}}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h_2\right) \cdots \lambda_{\underbrace{C_\zeta \dots}_{2s_n}}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h_n\right) \delta\left(h - \sum_{i=1}^n h_i\right)$$

$$\text{推论6.7.1. } \begin{cases} \Omega^2(s) \lambda_{\underbrace{A_\zeta \otimes \dots \otimes B_\zeta \otimes C_\zeta}_{2s}}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h\right) = s(s+1) \lambda_{\underbrace{A_\zeta \otimes \dots \otimes B_\zeta \otimes C_\zeta}_{2s}}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h\right), -s \leq h \leq s \\ \Omega_z(s) \lambda_{\underbrace{A_\zeta \otimes \dots \otimes B_\zeta \otimes C_\zeta}_{2s}}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h\right) = h \lambda_{\underbrace{A_\zeta \otimes \dots \otimes B_\zeta \otimes C_\zeta}_{2s}}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h\right), -s \leq h \leq s \end{cases}$$

6.8 么正变换→自旋方程之自旋基分解的z-方向表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$

$$\text{定义6.8.1. } -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, \lambda_{k_\zeta}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; \sum_{i=1}^n s_i\right) := \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \cdots \sum_{h_n=s_n}^{-s_n} \\ \left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{\frac{1}{2}} \Gamma_{k_\zeta}^{l_\zeta m_\zeta \dots n_\zeta} \lambda_{l_\zeta}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h_1; s_1\right) \lambda_{m_\zeta}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h_2; s_2\right) \cdots \lambda_{n_\zeta}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h_n; s_n\right) \delta\left(h - \sum_{i=1}^n h_i\right)$$

$$\text{推论6.8.1. } \begin{cases} \sigma^2(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) = s(s+1) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right), -s \leq h \leq s \\ \sigma_z(s) \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right) = h \lambda\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h; s\right), -s \leq h \leq s \end{cases}$$

## 7 么正变换→自旋基分解的真实物理表象: $s_1 + s_2 \cdots + s_n = s_1 \oplus s_2 \cdots \oplus s_n$

### 7.1 有质量任意自旋粒子情形

定义7.1.1.

$$\begin{cases} U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) := \prod_{i=1}^n \left[ \otimes \exp \left\{ i \frac{[(\sigma(\frac{1}{2}) \otimes I) \times \hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}} \arccos \hat{p}_{iz} \right\} \frac{E+m-i|\vec{p}_i| \gamma_z \gamma_4}{\sqrt{2m(E+m)}} \right]^{2s_i} U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \vec{0}, \sum_{i=1}^n s_i; h \right) \\ \hat{J} \left( \prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a \right) := \prod_{i=1}^n \left[ \otimes \exp \left\{ i \frac{[(\sigma(\frac{1}{2}) \otimes I) \times \hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}} \arccos \hat{p}_{iz} \right\} \frac{E+m-i|\vec{p}_i| \gamma_z \gamma_4}{\sqrt{2m(E+m)}} \right]^{2s_i} \\ \Omega \left( \sum_{i=1}^n s_i; \sigma(\frac{1}{2}) \otimes I \right) \prod_{i=1}^n \left[ \otimes \frac{E+m+i|\vec{p}_i| \gamma_z \gamma_4}{\sqrt{2m(E+m)}} \exp \left\{ -i \frac{[(\sigma(\frac{1}{2}) \otimes I) \times \hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}} \arccos \hat{p}_{iz} \right\} \right]^{2s_i} \end{cases}$$

推论7.1.1.  $-\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, U_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta \dots \rho_\zeta \sigma_\zeta} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) := \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \cdots \sum_{h_n=s_n}^{-s_n}$

$$\left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{\frac{1}{2}} U_{\lambda_\zeta \mu_\zeta \dots} (\vec{p}_1, h_1) U_{\eta_\zeta \xi_\zeta \dots} (\vec{p}_2, h_2) \cdots U_{\rho_\zeta \sigma_\zeta \dots} (\vec{p}_n, h_n) \delta(h - \sum_{i=1}^n h_i)$$

推论7.1.2.

$$\begin{cases} \hat{J}_+ \left( \prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a \right) U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) = \sqrt{\left( \sum_{i=1}^n s_i \right) \left( \sum_{i=1}^n s_i + 1 \right) - h(h+1)} U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h+1 \right) \\ \hat{J}_- \left( \prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a \right) U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) = \sqrt{\left( \sum_{i=1}^n s_i \right) \left( \sum_{i=1}^n s_i + 1 \right) - h(h-1)} U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h-1 \right) \\ \hat{J}_z \left( \prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a \right) U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) = h U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right), -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i \\ \hat{Q} \left( \prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a \right) U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) = -2s U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) \end{cases}$$

推论7.1.3.

$$\begin{cases} \hat{J}^2 \left( \prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a \right) U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) = \left( \sum_{i=1}^n s_i \right) \left( \sum_{i=1}^n s_i + 1 \right) U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) \\ \hat{J}^2 \left( \vec{p}_i, * \frac{1}{2}; \gamma_a \right) U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) = \frac{1}{2} \left( \frac{1}{2} + 1 \right) U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) \\ \hat{J}_z \left( \prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a \right) U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) = h U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) \\ \hat{Q} \left( \vec{p}_i, * \frac{1}{2}; \gamma_a \right) U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) = -U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) \\ \left[ \hat{J}_\alpha \left( \prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a \right), \hat{J}_\beta \left( \prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a \right) \right] = \varepsilon_{\alpha\beta\gamma} \hat{J}_\gamma \left( \prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a \right), -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i \end{cases}$$

推论7.1.4.  $-\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i, V_{\lambda_\zeta \mu_\zeta \dots \eta_\zeta \xi_\zeta \dots \rho_\zeta \sigma_\zeta} \left( \prod_{i=1}^n (\vec{p}_i, s_i); h \right) := \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \cdots \sum_{h_n=s_n}^{-s_n}$

$$\left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{\frac{1}{2}} V_{\lambda_\zeta \mu_\zeta \dots} (\vec{p}_1, h_1) V_{\eta_\zeta \xi_\zeta \dots} (\vec{p}_2, h_2) \cdots V_{\rho_\zeta \sigma_\zeta \dots} (\vec{p}_n, h_n) \delta(h - \sum_{i=1}^n h_i)$$

推论7.1.5.

$$\begin{cases} \hat{J}_+(\prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) = \sqrt{(\sum_{i=1}^n s_i)(\sum_{i=1}^n s_i + 1) - h(h+1)} \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h+1) \\ \hat{J}_-(\prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) = \sqrt{(\sum_{i=1}^n s_i)(\sum_{i=1}^n s_i + 1) - h(h-1)} \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h-1) \\ \hat{J}_z(\prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) = h \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h), -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i \\ \hat{Q}(\prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) = 2s \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) \end{cases}$$

推论7.1.6.

$$\begin{cases} \hat{J}^2(\prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) = (\sum_{i=1}^n s_i)(\sum_{i=1}^n s_i + 1) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) \\ \hat{J}^2(\vec{p}_i, * \frac{1}{2}; \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) = \frac{1}{2}(\frac{1}{2} + 1) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) \\ \hat{J}_z(\prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) = h \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) \\ \hat{Q}(\vec{p}_i, * \frac{1}{2}; \gamma_a) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) = \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots}}_{2(s_1+\dots+s_n)} \dots (\prod_{i=1}^n (\vec{p}_i, s_i); h) \\ [\hat{J}_\alpha(\prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a), \hat{J}_\beta(\prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a)] = \varepsilon_{\alpha\beta\gamma} \hat{J}_\gamma(\prod_{i=1}^n (\vec{p}_i, s_i); \gamma_a), -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i \end{cases}$$

## 7.2 有质量整数自旋粒子情形

定义7.2.1.

$$\begin{cases} \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h) := \prod_{i=1}^n [\otimes \exp\{i \frac{[R \times \hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}} \arccos \hat{p}_{iz}\} \frac{m-|\vec{p}_i|L_z+(E_i-m)L_z^2}{m}]^{l_i} \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}}(\vec{0}, h) \\ \hat{J}(\prod_{i=1}^n (\vec{p}_i, l_i); R) := \prod_{i=1}^n [\otimes \exp\{i \frac{[R \times \hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}} \arccos \hat{p}_{iz}\} \frac{m-|\vec{p}_i|L_z+(E_i-m)L_z^2}{m}]^{l_i} \\ \Omega(\sum_{i=1}^n l_i; R) \prod_{i=1}^n [\otimes \frac{m+|\vec{p}_i|L_z+(E_i-m)L_z^2}{m} \exp\{-i \frac{[R \times \hat{p}_i]_z}{\sqrt{1-\hat{p}_{iz}^2}} \arccos \hat{p}_{iz}\}]^{l_i} \end{cases}$$

推论7.2.1.  $-\sum_{i=1}^n l_i \leq h \leq \sum_{i=1}^n l_i, \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h)$

$$:= \sum_{h_1=l_1}^{-l_1} \sum_{h_2=l_2}^{-l_2} \dots \sum_{h_n=l_n}^{-l_n} \left[ \frac{\prod_{i=1}^n (2l_i)!}{[\sum_{i=1}^n (2l_i)!]} \frac{[\sum_{i=1}^n (l_i+h_i)]!}{\prod_{i=1}^n (l_i+h_i)!} \frac{[\sum_{i=1}^n (l_i-h_i)]!}{\prod_{i=1}^n (l_i-h_i)!} \right]^{\frac{1}{2}} \varepsilon_{\underbrace{a \dots}_{l_1}}(\vec{p}_1, h_1) \varepsilon_{\underbrace{b \dots}_{l_2}}(\vec{p}_2, h_2) \dots \varepsilon_{\underbrace{c \dots}_{l_n}}(\vec{p}_n, h_n) \delta(h - \sum_{i=1}^n h_i)$$

推论7.2.2.

$$\begin{cases} \hat{J}_+(\prod_{i=1}^n (\vec{p}_i, l_i); R) \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h) = \sqrt{(\sum_{i=1}^n l_i)(\sum_{i=1}^n l_i + 1) - h(h+1)} \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h+1) \\ \hat{J}_-(\prod_{i=1}^n (\vec{p}_i, l_i); R) \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h) = \sqrt{(\sum_{i=1}^n l_i)(\sum_{i=1}^n l_i + 1) - h(h-1)} \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h-1) \\ \hat{J}_z(\prod_{i=1}^n (\vec{p}_i, l_i); R) \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h) = h \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h), -\sum_{i=1}^n l_i \leq h \leq \sum_{i=1}^n l_i \end{cases}$$

推论7.2.3.

$$\begin{cases} \hat{J}^2(\prod_{i=1}^n (\vec{p}_i, l_i); R) \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h) = (\sum_{i=1}^n l_i)(\sum_{i=1}^n l_i + 1) \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h) \\ \hat{J}^2(\vec{p}_i, * 1; R, L) \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h) = 1(1+1) \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h) \\ \hat{J}_z(\prod_{i=1}^n (\vec{p}_i, l_i); R) \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h) = h \varepsilon_{\underbrace{a \dots b \dots c \dots}_{l_1 \quad l_2 \quad l_n}} \dots (\prod_{i=1}^n (\vec{p}_i, l_i); h) \\ [\hat{J}_\alpha(\prod_{i=1}^n (\vec{p}_i, l_i); R), \hat{J}_\beta(\prod_{i=1}^n (\vec{p}_i, l_i); R)] = \varepsilon_{\alpha\beta\gamma} \hat{J}_\gamma(\prod_{i=1}^n (\vec{p}_i, l_i); R), -\sum_{i=1}^n l_i \leq h \leq \sum_{i=1}^n l_i \end{cases}$$

### 7.3 无质量任意自旋粒子情形

定义7.3.1.

$$\left\{ \begin{aligned} \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right) &:= \prod_{i=1}^n \otimes \exp \left\{ i \frac{[\sigma(\frac{1}{2}) \times \hat{p}_i]_z}{\sqrt{1-\hat{p}_i^2}} \arccos \hat{p}_{iz} \right\} \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, h \right) \\ \hat{J} \left( \prod_{i=1}^n (\hat{p}_i, s_i); \sigma(\frac{1}{2}) \right) &:= \prod_{i=1}^n \otimes \exp \left\{ i \frac{[\sigma(\frac{1}{2}) \times \hat{p}_i]_z}{\sqrt{1-\hat{p}_i^2}} \arccos \hat{p}_{iz} \right\} \Omega(s) \prod_{i=1}^n \otimes \exp \left\{ -i \frac{[\sigma(\frac{1}{2}) \times \hat{p}_i]_z}{\sqrt{1-\hat{p}_i^2}} \arccos \hat{p}_{iz} \right\} \end{aligned} \right.$$

推论7.3.1.  $-\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i$ ,  $\lambda_{A_\zeta \otimes B_\zeta \otimes \dots \otimes C_\zeta} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right)$

$$:= \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \dots \sum_{h_n=s_n}^{-s_n} \left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{\frac{1}{2}} \lambda_{A_\zeta}(\hat{p}_1, h_1) \lambda_{B_\zeta}(\hat{p}_2, h_2) \dots \lambda_{C_\zeta}(\hat{p}_n, h_n) \delta(h - \sum_{i=1}^n h_i)$$

推论7.3.2.

$$\left\{ \begin{aligned} \hat{J}_+ \left( \prod_{i=1}^n (\hat{p}_i, s_i); \sigma(\frac{1}{2}) \right) \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right) &= \sqrt{\left( \sum_{i=1}^n s_i \right) \left( \sum_{i=1}^n s_i + 1 \right) - h(h+1)} \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h+1 \right) \\ \hat{J}_- \left( \prod_{i=1}^n (\hat{p}_i, s_i); \sigma(\frac{1}{2}) \right) \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right) &= \sqrt{\left( \sum_{i=1}^n s_i \right) \left( \sum_{i=1}^n s_i + 1 \right) - h(h-1)} \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h-1 \right) \\ \hat{J}_z \left( \prod_{i=1}^n (\hat{p}_i, s_i); \sigma(\frac{1}{2}) \right) \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right) &= h \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right), -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i \end{aligned} \right.$$

推论7.3.3.

$$\left\{ \begin{aligned} \hat{J}^2 \left( \prod_{i=1}^n (\hat{p}_i, s_i); \sigma(\frac{1}{2}) \right) \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right) &= \left( \sum_{i=1}^n s_i \right) \left( \sum_{i=1}^n s_i + 1 \right) \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right) \\ \hat{J}^2 \left( \hat{p}_i, * \sigma(\frac{1}{2}) \right) \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right) &= \frac{1}{2} \left( \frac{1}{2} + 1 \right) \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right) \\ \hat{J}_z \left( \prod_{i=1}^n (\hat{p}_i, s_i); \sigma(\frac{1}{2}) \right) \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right) &= h \lambda_{A_\zeta \otimes B_\zeta \otimes \dots} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h \right), -\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i \\ \left[ \hat{J}_\alpha \left( \prod_{i=1}^n (\hat{p}_i, s_i); \sigma(\frac{1}{2}) \right), \hat{J}_\beta \left( \prod_{i=1}^n (\hat{p}_i, s_i); \sigma(\frac{1}{2}) \right) \right] &= \varepsilon_{\alpha\beta\gamma} \hat{J}_\gamma \left( \prod_{i=1}^n (\hat{p}_i, s_i); \sigma(\frac{1}{2}) \right) \end{aligned} \right.$$

推论7.3.4.  $-\sum_{i=1}^n s_i \leq h \leq \sum_{i=1}^n s_i$ ,  $\lambda_{k_\zeta} \left( \prod_{i=1}^n (\hat{p}_i, s_i); h; \sum_{i=1}^n s_i \right) := \sum_{h_1=s_1}^{-s_1} \sum_{h_2=s_2}^{-s_2} \dots \sum_{h_n=s_n}^{-s_n}$

$$\left[ \frac{\prod_{i=1}^n (2s_i)!}{[\sum_{i=1}^n (2s_i)]!} \frac{[\sum_{i=1}^n (s_i+h_i)]!}{\prod_{i=1}^n (s_i+h_i)!} \frac{[\sum_{i=1}^n (s_i-h_i)]!}{\prod_{i=1}^n (s_i-h_i)!} \right]^{\frac{1}{2}} \Gamma_{k_\zeta}^{l_\zeta m_\zeta \dots n_\zeta} \lambda_{l_\zeta}(\hat{p}_1, h_1; s_1) \lambda_{m_\zeta}(\hat{p}_2, h_2; s_2) \dots \lambda_{n_\zeta}(\hat{p}_n, h_n; s_n) \delta(h - \sum_{i=1}^n h_i)$$

### 7.4 自我评述

以上各种自旋基是对应的等动量情形通过么正变换而得来的，除了全对称性破缺之外，其余性质仍全部满足。事实上其全对称性也仍然存在，只是通过么正变换后被隐藏起来了而已。本质上仍满足全对称性，只是表面上破缺而已。以上自旋基也可以通过反变换回到等动量情形（特别是z-轴方向），所以这也表明多粒子系统总自旋与各个粒子速度无关，完全由粒子内部自由度决定。么正变换后只有几率是不变、真实的，但物理图像不一定是真实的，比如动量被改变了；也可以换种观点等价认为动量不变，表象改变了，但这样不直观、不自然。我更倾向于前者，更自然更符合物理直观，可以认为此时表达式中动量就是物理、真实的动量，所以此时的自旋基也是物理、真实的自旋基，这些物理自旋基是接下来纠缠态分析的一个有力的数学工具。以上章节的自旋变换关系表达了如下的一个物理系统：多个不同自旋粒子复合成最大总自旋的物理体系，如果局域在足够小的空间，那么它们完全等价于一个高自旋粒子。反之就是纠缠多粒子系统，故纠缠系统是处于完全自由系统和完全束缚系统中间的一个状态，也许各种基本粒子就是通过量子纠缠合成的。这样的自旋基已不再满足原来的方程，而满足新的方程，但新的方程是什么样子的呢？可以进一步研究，也许隐含新的物理内容。

## 8 自旋耦合和CG系数的一般理论 [34, 41–43]

### 8.1 单自旋本征态

$$\text{公理8.1.1. } \begin{cases} \sigma(s) \times \sigma(s) = i\sigma(s), \sigma^2(s) = s(s+1) \\ \sigma^2(s)|s, m = s, \dots, -s\rangle = s(s+1)|s, m = s, \dots, -s\rangle \\ \sigma_z(s)|s, m = s, \dots, -s\rangle = m|s, m = s, \dots, -s\rangle \end{cases}$$

$$\text{公理8.1.2. } \begin{cases} \hat{J}_k \times \hat{J}_k = i\hat{J}_k \\ \hat{J}_k^2|(j_k, m_k)\rangle = j_k(j_k+1)|(j_k, m_k)\rangle \\ J_{kz}|(j_k, m_k)\rangle = m_k|(j_k, m_k)\rangle \end{cases}, \begin{cases} \hat{J}_k = \sigma(j_k) \\ |(j_k, m_k)\rangle \sim e^{(i\omega + \varsigma\varepsilon) \cdot \sigma(j_k)} \end{cases}$$

$$\text{公理8.1.3. } \langle(j_k, m'_k)|(j_k, m_k)\rangle = \delta_{m'_k m_k}, \sum_{m_k} |(j_k, m_k)\rangle \langle(j_k, m_k)| = 1$$

### 8.2 多自旋耦合本征态

$$\text{定义8.2.1. } |(j_1, m_1); \dots; (j_n, m_n)\rangle := |j_1, m_1\rangle \otimes \dots \otimes |j_n, m_n\rangle$$

$$\text{定义8.2.2. } \hat{J}_k := I_{2j_1+1} \otimes \dots \otimes I_{2j_{k-1}+1} \otimes \sigma(j_k) \otimes I_{2j_{k+1}+1} \otimes \dots \otimes I_{2j_n+1}, \hat{J} = \sum_{k=1}^n \hat{J}_k$$

$$\text{定义8.2.3. } \begin{cases} \hat{J}_k^2|(j_1, m_1); \dots; (j_n, m_n)\rangle = j_k(j_k+1)|(j_1, m_1); \dots; (j_n, m_n)\rangle \\ J_{kz}|(j_1, m_1); \dots; (j_n, m_n)\rangle = m_k|(j_1, m_1); \dots; (j_n, m_n)\rangle \end{cases}$$

$$\text{定义8.2.4. } \begin{cases} \hat{J}_k^2|j_1, j_2 \dots j_n; (j, m)\rangle = j_k(j_k+1)|j_1, j_2 \dots j_n; (j, m)\rangle \\ \hat{J}^2|j_1, j_2 \dots j_n; (j, m)\rangle = j(j+1)|j_1, j_2 \dots j_n; (j, m)\rangle \\ J_z|j_1, j_2 \dots j_n; (j, m)\rangle = m|j_1, j_2 \dots j_n; (j, m)\rangle \end{cases}$$

### 8.3 自旋本征态展开

$$\text{推论8.3.1. } \begin{cases} |j_1, j_2 \dots j_n; (j, m)\rangle = \sum_{m_k} |(j_1, m_1); \dots; (j_n, m_n)\rangle \langle(j_1, m_1); \dots; (j_n, m_n)|j_1, j_2 \dots j_n; (j, m)\rangle \\ |j_1, j_2; (j, m)\rangle = \sum_{m_k} |(j_1, m_1); (j_2, m_2)\rangle \langle(j_1, m_1); (j_2, m_2)|j_1, j_2; (j, m)\rangle \end{cases}$$

[⇔]

$$\text{推论8.3.2. } \begin{cases} |(j_1, m_1); \dots; (j_n, m_n)\rangle = \sum_{m_k} |j_1, j_2 \dots j_n; (j, m)\rangle \langle j_1, j_2 \dots j_n; (j, m)|(j_1, m_1); \dots; (j_n, m_n)\rangle \\ |(j_1, m_1); (j_2, m_2)\rangle = \sum_{m_k} |j_1, j_2; (j, m)\rangle \langle j_1, j_2; (j, m)|(j_1, m_1); (j_2, m_2)\rangle \end{cases}$$

### 8.4 自旋本征态的矩阵变换

$$\text{推论8.4.1. } \begin{cases} |j_1, j_2 \dots j_n; (j, m)\rangle = S_{1\dots n}|(j_1, m_1); \dots; (j_n, m_n)\rangle \\ |j_1, j_2; (j, m)\rangle = S_{12}|(j_1, m_1); (j_2, m_2)\rangle \end{cases}$$

[⇔]

$$\text{推论8.4.2. } \begin{cases} |(j_1, m_1); \dots; (j_n, m_n)\rangle = S_{1\dots n}^{-1}|j_1, j_2 \dots j_n; (j, m)\rangle \\ |(j_1, m_1); (j_2, m_2)\rangle = S_{12}^{-1}|j_1, j_2; (j, m)\rangle \end{cases}$$

## 8.5 自旋本征态的正交性

$$\text{推论8.5.1. } \begin{cases} \langle (j_1, m'_1); \cdots; (j_n, m'_n) | (j_1, m_1); \cdots; (j_n, m_n) \rangle = \delta_{m'_1 m_1} \cdots \delta_{m'_n m_n} \\ \langle (j_1, m'_1); (j_2, m'_2) | (j_1, m_1); (j_2, m_2) \rangle = \delta_{m'_1 m_1} \delta_{m'_2 m_2} \end{cases}$$

$$[\Leftrightarrow]$$

$$\text{推论8.5.2. } \begin{cases} \langle j_1, j_2 \cdots j_n; (j, m') | j_1, j_2 \cdots j_n; (j, m) \rangle = \delta_{m' m} \\ \langle j_1, j_2; (j, m') | j_1, j_2; (j, m) \rangle = \delta_{m' m} \end{cases}$$

## 8.6 自旋本征态的完备性

$$\text{推论8.6.1. } \begin{cases} \sum_{m_k} |(j_1, m_1); \cdots; (j_n, m_n)\rangle \langle (j_1, m_1); \cdots; (j_n, m_n)| = 1 \\ \sum_{m_k} |(j_1, m_1); (j_2, m_2)\rangle \langle (j_1, m_1); (j_2, m_2)| = 1 \end{cases}$$

$$[\Leftrightarrow]$$

$$\text{推论8.6.2. } \begin{cases} \sum_{m_k} |j_1, j_2 \cdots j_n; (j, m)\rangle \langle j_1, j_2 \cdots j_n; (j, m)| = 1 \\ \sum_{m_k} |j_1, j_2; (j, m)\rangle \langle j_1, j_2; (j, m)| = 1 \end{cases}$$

## 8.7 两个角动量耦合CG系数的一般Racah公式

$$\text{定理8.7.1. } |j_1, j_2; (j_3, m_3)\rangle = \sum_{m_1=j_1}^{-j_1} \sum_{m_2=j_2}^{-j_2} |(j_1, m_1); (j_2, m_2)\rangle \langle (j_1, m_1); (j_2, m_2) | j_1, j_2; (j_3, m_3)\rangle$$

$$CG_{Racah} = \langle (j_1, m_1); (j_2, m_2) | j_1, j_2; (j_3, m_3)\rangle = \delta(m_1 + m_2 - m_3)$$

$$\{ (2j_3 + 1) \frac{(j_1 + j_2 - j_3)! (j_1 - j_2 + j_3)! (-j_1 + j_2 + j_3)!}{(j_1 + j_2 + j_3 + 1)!} (j_1 + m_1)! (j_1 - m_1)! (j_2 + m_2)! (j_2 - m_2)! (j_3 + m_3)! (j_3 - m_3)! \}^{1/2}$$

$$[\sum_r (-1)^r r! (j_1 + j_2 - j_3 - r)! (j_1 - m_1 - r)! (j_3 - j_1 - m_2 + r)! (j_2 + m_2 - r)! (j_3 - j_2 + m_1 + r)!]^{-1}$$

## 8.8 两个粒子合成一个粒子CG系数的Racah公式

$$\text{定理8.8.1. } |j_1, j_2; (j_1 + j_2, m_3)\rangle = \sum_{m_1=j_1}^{-j_1} \sum_{m_2=j_2}^{-j_2} |(j_1, m_1); (j_2, m_2)\rangle \langle (j_1, m_1); (j_2, m_2) | j_1, j_2; (j_1 + j_2, m_3)\rangle$$

$$\langle (j_1, m_1); (j_2, m_2) | j_1, j_2; (j_1 + j_2, m_3)\rangle = \delta(m_1 + m_2 - m_3) \left\{ \frac{(2j_1)!(2j_2)!(j_1 + j_2 + m_3)!(j_1 + j_2 - m_3)!}{(2j_1 + 2j_2)!(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!} \right\}^{1/2}$$

$$\text{证明: } \langle (j_1, m_1); (j_2, m_2) | j_1, j_2; (j_1 + j_2, m_3)\rangle$$

$$= \delta(m_1 + m_2 - m_3) \left\{ \frac{(2j_1)!(2j_2)!}{(2j_1 + 2j_2)!} (j_1 + m_1)! (j_1 - m_1)! (j_2 + m_2)! (j_2 - m_2)! (j_3 + m_3)! (j_3 - m_3)! \right\}^{1/2}$$

$$[(j_1 - m_1)! (j_2 - m_2)! (j_2 + m_2)! (j_1 + m_1)!]^{-1}$$

$$= \delta(m_1 + m_2 - m_3) \left\{ \frac{(2j_1)!(2j_2)!(j_1 + j_2 + m_3)!(j_1 + j_2 - m_3)!}{(2j_1 + 2j_2)!(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!} \right\}^{1/2} \quad \square$$

$$\text{推论8.8.1. } |n, 1; (n + 1, m_3)\rangle = \sum_{m_1=n}^{-n} \sum_{m_2=1}^{-1} |(n, m_1); (1, m_2)\rangle \langle (n, m_1); (1, m_2) | n, 1; (n + 1, m_3)\rangle$$

$$\langle (n, m_1); (1, m_2) | n, 1; (n + 1, m_3)\rangle$$

$$= \delta(m_1 + m_2 - m_3) \left\{ \frac{(2n)! 2! (n+1+m_3)! (n+1-m_3)!}{(2n+2)!(n+m_1)!(n-m_1)!(1+m_2)!(1-m_2)!} \right\}^{1/2} = \delta(m_1 + m_2 - m_3) \left\{ \frac{2! C_{2n}^{n-m_1}}{(1+m_2)!(1-m_2)! C_{2n+2}^{n+1-m_3}} \right\}^{1/2}$$

推论8.8.2.

$$\begin{cases} |n, 1; (n + 1, n + 1)\rangle = \frac{\sqrt{C_{2n}^0}}{\sqrt{C_{2n+2}^0}} |(n, n); (1, 1)\rangle \\ |n, 1; (n + 1, n)\rangle = \frac{\sqrt{C_{2n}^1}}{\sqrt{C_{2n+2}^1}} |(n, n - 1); (1, 1)\rangle + \frac{\sqrt{2C_{2n}^0}}{\sqrt{C_{2n+2}^1}} |(n, n); (1, 0)\rangle \\ |n, 1; (n + 1, n - 1)\rangle = \frac{\sqrt{C_{2n}^2}}{\sqrt{C_{2n+2}^2}} |(n, n - 2); (1, 1)\rangle + \frac{\sqrt{2C_{2n}^1}}{\sqrt{C_{2n+2}^2}} |(n, n - 1); (1, 0)\rangle + \frac{\sqrt{C_{2n}^0}}{\sqrt{C_{2n+2}^2}} |(n, n); (1, -1)\rangle \\ |n, 1; (n + 1, n - 2)\rangle = \frac{\sqrt{C_{2n}^3}}{\sqrt{C_{2n+2}^3}} |(n, n - 3); (1, 1)\rangle + \frac{\sqrt{2C_{2n}^2}}{\sqrt{C_{2n+2}^3}} |(n, n - 2); (1, 0)\rangle + \frac{\sqrt{C_{2n}^1}}{\sqrt{C_{2n+2}^3}} |(n, n - 1); (1, -1)\rangle \\ |n, 1; (n + 1, n + 1 - l)\rangle = \frac{\sqrt{C_{2n}^l}}{\sqrt{C_{2n+2}^l}} |(n, n - l); (1, 1)\rangle + \frac{\sqrt{2C_{2n}^{l-1}}}{\sqrt{C_{2n+2}^l}} |(n, n + 1 - l); (1, 0)\rangle + \frac{\sqrt{C_{2n}^{l-2}}}{\sqrt{C_{2n+2}^l}} |(n, n + 2 - l); (1, -1)\rangle \end{cases}$$

## 8.9 两个粒子合成一个粒子CG系数的具体表述

$$\text{定理8.9.1. } \langle (j_1, m_1); (j_2, m_2) | j_1, j_2; (j_1 + j_2, m_3) \rangle = \delta(m_1 + m_2 - m_3) \left\{ \frac{(2j_1)!(2j_2)!(j_1 + j_2 + m_3)!(j_1 + j_2 - m_3)!}{(2j_1 + 2j_2)!(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!} \right\}^{1/2}$$

$$= \delta(m_1 + m_2 - m_3) \left\{ \frac{1}{C_{2j_1+2j_2}^{2j_1}} \right\}^{1/2}$$

推论8.9.1.

$$\left\{ \begin{aligned} &\langle (j_1, j_1); (j_2, j_2) | j_1, j_2; (j_1 + j_2, j_1 + j_2) \rangle = 1 \\ &\langle (j_1, j_1); (j_2, j_2 - 1) | j_1, j_2; (j_1 + j_2, j_1 + j_2 - 1) \rangle = \frac{\sqrt{2j_2}}{\sqrt{2j_1 + 2j_2}} \\ &\langle (j_1, j_1 - 1); (j_2, j_2) | j_1, j_2; (j_1 + j_2, j_1 + j_2 - 1) \rangle = \frac{\sqrt{2j_1}}{\sqrt{2j_1 + 2j_2}} \\ &\langle (j_1, j_1); (j_2, j_2 - 2) | j_1, j_2; (j_1 + j_2, j_1 + j_2 - 2) \rangle = \frac{\sqrt{2j_2(2j_2 - 1)}}{\sqrt{(2j_1 + 2j_2)(2j_1 + 2j_2 - 1)}} \\ &\langle (j_1, j_1 - 1); (j_2, j_2 - 1) | j_1, j_2; (j_1 + j_2, j_1 + j_2 - 2) \rangle = \frac{2\sqrt{2j_1 j_2}}{\sqrt{(2j_1 + 2j_2)(2j_1 + 2j_2 - 1)}} \\ &\langle (j_1, j_1 - 2); (j_2, j_2) | j_1, j_2; (j_1 + j_2, j_1 + j_2 - 2) \rangle = \frac{\sqrt{2j_1(2j_1 - 1)}}{\sqrt{(2j_1 + 2j_2)(2j_1 + 2j_2 - 1)}} \\ &\dots\dots\dots \\ &\langle (j_1, -j_1); (j_2, -j_2 + 2) | j_1, j_2; (j_1 + j_2, -j_1 - j_2 + 2) \rangle = \frac{\sqrt{2j_2(2j_2 - 1)}}{\sqrt{(2j_1 + 2j_2)(2j_1 + 2j_2 - 1)}} \\ &\langle (j_1, -j_1 + 1); (j_2, -j_2 + 1) | j_1, j_2; (j_1 + j_2, -j_1 - j_2 + 2) \rangle = \frac{2\sqrt{2j_1 j_2}}{\sqrt{(2j_1 + 2j_2)(2j_1 + 2j_2 - 1)}} \\ &\langle (j_1, -j_1 + 2); (j_2, -j_2) | j_1, j_2; (j_1 + j_2, -j_1 - j_2 + 2) \rangle = \frac{\sqrt{2j_1(2j_1 - 1)}}{\sqrt{(2j_1 + 2j_2)(2j_1 + 2j_2 - 1)}} \\ &\langle (j_1, -j_1); (j_2, -j_2 + 1) | j_1, j_2; (j_1 + j_2, -j_1 - j_2 + 1) \rangle = \frac{\sqrt{2j_2}}{\sqrt{2j_1 + 2j_2}} \\ &\langle (j_1, -j_1 + 1); (j_2, -j_2) | j_1, j_2; (j_1 + j_2, -j_1 - j_2 + 1) \rangle = \frac{\sqrt{2j_1}}{\sqrt{2j_1 + 2j_2}} \\ &\langle (j_1, -j_1); (j_2, -j_2) | j_1, j_2; (j_1 + j_2, -j_1 - j_2) \rangle = 1 \end{aligned} \right.$$

$$\text{推论8.9.2. } \langle (1, m_1); (1, m_2) | 1, 1; (2, m_3) \rangle = \delta(m_1 + m_2 - m_3) \left\{ \frac{2!2!(2+m_3)!(2-m_3)!}{4!(1+m_1)!(1-m_1)!(1+m_2)!(1-m_2)!} \right\}^{1/2}$$

推论8.9.3.

$$\left\{ \begin{aligned} &\langle (1, 1); (1, 1) | 1, 1; (2, 2) \rangle = 1 \\ &\langle (1, 1); (1, 0) | 1, 1; (2, 1) \rangle = \frac{1}{\sqrt{2}}, \langle (1, 0); (1, 1) | 1, 1; (2, 1) \rangle = \frac{1}{\sqrt{2}} \\ &\langle (1, 1); (1, -1) | 1, 1; (2, 0) \rangle = \frac{1}{\sqrt{6}}, \langle (1, 0); (1, 0) | 1, 1; (2, 0) \rangle = \frac{2}{\sqrt{6}}, \langle (1, -1); (1, 1) | 1, 1; (2, 0) \rangle = \frac{1}{\sqrt{6}} \\ &\langle (1, -1); (1, 0) | 1, 1; (2, -1) \rangle = \frac{1}{\sqrt{2}}, \langle (1, 0); (1, -1) | 1, 1; (2, -1) \rangle = \frac{1}{\sqrt{2}} \\ &\langle (1, -1); (1, -1) | 1, 1; (2, -2) \rangle = 1 \end{aligned} \right.$$

$$\text{推论8.9.4. } \langle (2, m_1); (1, m_2) | 2, 1; (3, m_3) \rangle = \delta(m_1 + m_2 - m_3) \left\{ \frac{4!2!(3+m_3)!(3-m_3)!}{6!(2+m_1)!(2-m_1)!(1+m_2)!(1-m_2)!} \right\}^{1/2}$$

推论8.9.5.

$$\left\{ \begin{aligned} &\langle (2, 2); (1, 1) | 2, 1; (3, 3) \rangle = 1 \\ &\langle (2, 2); (1, 0) | 2, 1; (3, 2) \rangle = \frac{1}{\sqrt{3}}, \langle (2, 1); (1, 1) | 2, 1; (3, 2) \rangle = \frac{\sqrt{2}}{\sqrt{3}} \\ &\langle (2, 2); (1, -1) | 2, 1; (3, 1) \rangle = \frac{1}{\sqrt{15}}, \langle (2, 1); (1, 0) | 2, 1; (3, 1) \rangle = \frac{\sqrt{8}}{\sqrt{15}}, \langle (2, 0); (1, 1) | 2, 1; (3, 1) \rangle = \frac{\sqrt{6}}{\sqrt{15}} \\ &\langle (2, 1); (1, -1) | 2, 1; (3, 0) \rangle = \frac{1}{\sqrt{5}}, \langle (2, 0); (1, 0) | 2, 1; (3, 0) \rangle = \frac{\sqrt{3}}{\sqrt{15}}, \langle (2, -1); (1, 1) | 2, 1; (3, 0) \rangle = \frac{1}{\sqrt{5}} \\ &\langle (2, -2); (1, 1) | 2, 1; (3, -1) \rangle = \frac{1}{\sqrt{15}}, \langle (2, -1); (1, 0) | 2, 1; (3, -1) \rangle = \frac{\sqrt{8}}{\sqrt{15}}, \langle (2, 0); (1, -1) | 2, 1; (3, -1) \rangle = \frac{\sqrt{6}}{\sqrt{15}} \\ &\langle (2, -2); (1, 0) | 2, 1; (3, -2) \rangle = \frac{1}{\sqrt{3}}, \langle (2, -1); (1, -1) | 2, 1; (3, -2) \rangle = \frac{\sqrt{2}}{\sqrt{3}} \\ &\langle (2, -2); (1, -1) | 2, 1; (3, -3) \rangle = 1 \end{aligned} \right.$$



## 9 多个粒子合成一个粒子的CG系数公式 [34, 41-43]

### 9.1 多个光子合成一个粒子的CG系数公式

$$\begin{aligned} \text{引理9.1.1. } |(n+1, m_3)\rangle &= |n, 1; (n+1, m_3)\rangle = \sum_{m_1=n}^{-n} \sum_{m_2=1}^{-1} |(n, m_1); (1, m_2)\rangle \langle (n, m_1); (1, m_2)|n, 1; (n+1, m_3)\rangle \\ &= \sum_{m_1=n}^{-n} \sum_{m_2=1}^{-1} \delta(m_1 + m_2 - m_3) \left\{ \frac{(2n)!2!(n+1+m_3)!(n+1-m_3)!}{(2n+2)!(n+m_1)!(n-m_1)!(1+m_2)!(1-m_2)!} \right\}^{1/2} |(n, m_1); (1, m_2)\rangle \end{aligned}$$

$$\begin{aligned} \text{定理9.1.1. } |\overbrace{1, \dots, 1}^{n+1}; (n+1, m_{n+1})\rangle &= \sum_{m_n=n}^{-n} \cdots \sum_{m_1=1}^{-1} \sum_{l_1, \dots, l_n=1}^{-1} |(1, m_1); (1, l_n); \dots; (1, l_1)\rangle \\ \delta(m_n + l_1 - m_{n+1}) \cdots \delta(m_1 + l_n - m_2) &\left\{ \frac{2^{n+1}}{(2n+2)!} \right\}^{1/2} \left\{ \frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)! \cdots (1+l_n)!(1-l_n)!} \right\}^{1/2} \end{aligned}$$

$$\begin{aligned} \text{证明: } |\overbrace{1, \dots, 1}^{n+1}; (n+1, m_{n+1})\rangle &= \sum_{m_n=n}^{-n} \sum_{l_1=1}^{-1} |(n, m_n); (1, l_1)\rangle \langle (n, m_n); (1, l_1)|n, 1; (n+1, m_{n+1})\rangle \\ &= \sum_{m_n=n}^{-n} \sum_{l_1=1}^{-1} \delta(m_n + l_1 - m_{n+1}) \left\{ \frac{(2n)!2!(n+1+m_{n+1})!(n+1-m_{n+1})!}{(2n+2)!(n+m_n)!(n-m_n)!(1+l_1)!(1-l_1)!} \right\}^{1/2} |(n, m_n); (1, l_1)\rangle \\ &= \sum_{m_n=n}^{-n} \sum_{m_{n-1}=n-1}^{-(n-1)} \sum_{l_1, l_2=1}^{-1} |(n-1, m_{n-1}); (1, l_2); (1, l_1)\rangle \\ &\delta(m_n + l_1 - m_{n+1}) \delta(m_{n-1} + l_2 - m_n) \\ &\left\{ \frac{(2n)!2!(n+1+m_{n+1})!(n+1-m_{n+1})!}{(2n+2)!(n+m_n)!(n-m_n)!(1+l_1)!(1-l_1)!} \right\}^{1/2} \left\{ \frac{(2n-2)!2!(n+m_n)!(n-m_n)!}{(2n)!(n-1+m_{n-1})!(n-1-m_{n-1})!(1+l_2)!(1-l_2)!} \right\}^{1/2} \\ &= \sum_{m_n=n}^{-n} \sum_{m_{n-1}=n-1}^{-(n-1)} \sum_{m_{n-2}=n-2}^{-1} \sum_{l_1, l_2, l_3=1}^{-1} |(n-1, m_{n-1}); (1, l_2); (1, l_1)\rangle \delta(m_n + l_1 - m_{n+1}) \delta(m_{n-1} + l_2 - m_n) \delta(m_{n-2} + l_3 - m_{n-1}) \\ &\left\{ \frac{(2n)!2!(n+1+m_{n+1})!(n+1-m_{n+1})!}{(2n+2)!(n+m_n)!(n-m_n)!(1+l_1)!(1-l_1)!} \right\}^{1/2} \left\{ \frac{(2n-2)!2!(n+m_n)!(n-m_n)!}{(2n)!(n-1+m_{n-1})!(n-1-m_{n-1})!(1+l_2)!(1-l_2)!} \right\}^{1/2} \\ &\left\{ \frac{(2n-4)!2!(n-1+m_{n-1})!(n-1-m_{n-1})!}{(2n-2)!(n-2+m_{n-2})!(n-2-m_{n-2})!(1+l_3)!(1-l_3)!} \right\}^{1/2} \\ &= \sum_{m_n=n}^{-n} \sum_{m_{n-1}=n-1}^{-(n-1)} \sum_{m_{n-2}=n-2}^{-1} \sum_{l_1, l_2, l_3=1}^{-1} |(n-2, m_{n-2}); (1, l_3); (1, l_2); (1, l_1)\rangle \\ &\delta(m_n + l_1 - m_{n+1}) \delta(m_{n-1} + l_2 - m_n) \delta(m_{n-2} + l_3 - m_{n-1}) \\ &\left\{ \frac{(2n-4)!2!^3}{(2n+2)!} \right\}^{1/2} \left\{ \frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(n-2+m_{n-2})!(n-2-m_{n-2})!} \right\}^{1/2} \left\{ \frac{1}{(1+l_1)!(1-l_1)!(1+l_2)!(1-l_2)!(1+l_3)!(1-l_3)!} \right\}^{1/2} \\ &= \sum_{m_n=n}^{-n} \sum_{m_{n-1}=n-1}^{-(n-1)} \cdots \sum_{m_1=1}^{-1} \sum_{l_1, \dots, l_n=1}^{-1} |(1, m_1); (1, l_n); \dots; (1, l_1)\rangle \\ &\delta(m_n + l_1 - m_{n+1}) \delta(m_{n-1} + l_2 - m_n) \cdots \delta(m_1 + l_n - m_2) \left\{ \frac{2^{n+1}}{(2n+2)!} \right\}^{1/2} \left\{ \frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)! \cdots (1+l_n)!(1-l_n)!} \right\}^{1/2} \\ &= \sum_{m_1=1}^{-1} \sum_{l_1, \dots, l_n=1}^{-1} |(1, m_1); (1, l_n); \dots; (1, l_1)\rangle \\ &\delta(m_{n+1} - m_1 - l_1 \cdots - l_n) \left\{ \frac{2^{n+1}}{(2n+2)!} \right\}^{1/2} \left\{ \frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)! \cdots (1+l_n)!(1-l_n)!} \right\}^{1/2} \quad \square \end{aligned}$$

$$\begin{aligned} \text{推论9.1.1. } |\overbrace{1, \dots, 1}^{n+1}; (n+1, m_{n+1})\rangle &= \sum_{m_n=n}^{-n} \cdots \sum_{m_1=1}^{-1} \sum_{l_1, \dots, l_n=1}^{-1} |(1, m_1); (1, l_n); \dots; (1, l_1)\rangle \\ \delta(m_n + l_1 - m_{n+1}) \cdots \delta(m_1 + l_n - m_2) &\left\{ \frac{2^{n+1}}{(2n+2)!} \right\}^{1/2} \left\{ \frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)! \cdots (1+l_n)!(1-l_n)!} \right\}^{1/2} \\ &= \sum_{l_1, \dots, l_n=1}^{-1} |(1, m_1); (1, l_1); \dots; (1, l_n)\rangle \left\{ \frac{2^{n+1}}{(2n+2)!} \right\}^{1/2} \left\{ \frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)! \cdots (1+l_n)!(1-l_n)!} \right\}^{1/2}; m_1 = m_{n+1} - \sum_{i=1}^n l_i \\ &= \frac{\sqrt{(2)^{n+1}}}{\sqrt{(2n+2)!}} \sum_{h_2=1}^{-1} \cdots \sum_{h_{n+1}=1}^{-1} |(1, h_1); \dots; (1, h_{n+1})\rangle \frac{\sqrt{(n+1+h)!}}{\sqrt{(1+h)! \cdots (1+h_{n+1})!}} \frac{\sqrt{(n+1-h)!}}{\sqrt{(1-h)! \cdots (1-h_{n+1})!}}; h = m_{n+1}, h_1 = h - \sum_{i=2}^{n+1} h_i \end{aligned}$$

$$\begin{aligned} \text{证明: } |\overbrace{1, \dots, 1}^{n+1}; (n+1, m_{n+1})\rangle &= \sum_{m_n=n}^{-n} \cdots \sum_{m_1=1}^{-1} \sum_{l_1, \dots, l_n=1}^{-1} |(1, m_1); (1, l_n); \dots; (1, l_1)\rangle \\ \delta(m_n + l_1 - m_{n+1}) \cdots \delta(m_1 + l_n - m_2) &\left\{ \frac{2^{n+1}}{(2n+2)!} \right\}^{1/2} \left\{ \frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)! \cdots (1+l_n)!(1-l_n)!} \right\}^{1/2} \\ &= \sum_{l_1, \dots, l_n=1}^{-1} |(1, m_1); (1, l_1); \dots; (1, l_n)\rangle \left\{ \frac{2^{n+1}}{(2n+2)!} \right\}^{1/2} \left\{ \frac{(n+1+m_{n+1})!(n+1-m_{n+1})!}{(1+m_1)!(1-m_1)!(1+l_1)!(1-l_1)! \cdots (1+l_n)!(1-l_n)!} \right\}^{1/2}; m_1 = m_{n+1} - \sum_{i=1}^n l_i \\ &= \frac{\sqrt{(2)^{n+1}}}{\sqrt{(2n+2)!}} \sum_{h_2=1}^{-1} \cdots \sum_{h_{n+1}=1}^{-1} |(1, h_1); \dots; (1, h_{n+1})\rangle \frac{\sqrt{(n+1+h)!}}{\sqrt{(1+h)! \cdots (1+h_{n+1})!}} \frac{\sqrt{(n+1-h)!}}{\sqrt{(1-h)! \cdots (1-h_{n+1})!}}; h = m_{n+1}, h_1 = h - \sum_{i=2}^{n+1} h_i \end{aligned}$$

$$\begin{cases} m_n + l_1 - m_{n+1} = 0 \\ m_{n-1} + l_2 - m_n = 0 \\ \dots \\ m_2 + l_{n-1} - m_3 = 0 \\ m_1 + l_n - m_2 = 0 \end{cases} \Leftrightarrow \begin{cases} m_n = m_{n+1} - \sum_{i=1}^1 l_i \\ \dots \\ m_3 = m_{n+1} - \sum_{i=1}^{n-2} l_i \\ m_2 = m_{n+1} - \sum_{i=1}^{n-1} l_i \\ m_1 = m_{n+1} - \sum_{i=1}^n l_i \end{cases} \quad \square$$

以上的证明开始主要通过头脑思考和具体验证解决的，排列组合的东西有的时候虽然可以想明白，但很难清清楚楚，明明白白地写出来。后来找到了以上比较严格的方法证明了它。

## 9.2 特殊情形下多个角动量耦合的CG系数公式

多个角动量耦合的CG系数原则上可以通过Racah公式的反复使用得到吗？好像只有特殊的单粒子情形下才可行。最一般的情形Racah和Wigner早已经解决了，数学上属于中等难度，只是繁复而已，公式很复杂，物理上使用起来很不方便，所以有些情况下重新获得便利的表述方式还是很有必要的。

**定义9.2.1.**  $j_{k+1} = n_1 + \dots + n_k + n_{k+1}$ ,  $j_k = n_2 + \dots + n_k + n_{k+1}$ ,  $\dots$ ,  $j_2 = n_k + n_{k+1}$ ,  $j_1 = n_{k+1}$ ,

**引理9.2.1.**

$$\langle (j_{k-1}, m_{k-1}); (n_2, l_2) | j_{k-1}, n_2; (j_k, m_k) \rangle = \delta(m_{k-1} + l_k - m_k) \left\{ \frac{(2j_{k-1})!(2n_2)!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_{k-1}+m_{k-1})!(j_{k-1}-m_{k-1})!(n_2+l_2)!(n_2-l_2)!} \right\}^{1/2}$$

**定理9.2.1.**  $|j_1, n_k, \dots, n_2, n_1; (j, m)\rangle = |j_k, n_1; (j, m)\rangle = |(j, m)\rangle$

$$\begin{aligned} &= \sum_{m_k=j_k}^{-j_k} \dots \sum_{m_1=j_1}^{-j_1} \sum_{l_1=n_1}^{-n_1} \dots \sum_{l_k=n_k}^{-n_k} |(j_1, m_1); (n_k, l_k); \dots; (n_2, l_2); (n_1, l_1)\rangle \\ &\delta(m_k + l_1 - m) \delta(m_{k-1} + l_2 - m_k) \cdot \delta(m_1 + l_k - m_2) \\ &\{ (2j+1) \frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!} (j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)! \}^{1/2} \\ &[\sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)!]^{-1} \\ &\left\{ \frac{(2j_1)!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_1+m_1)!(j_1-m_1)!} \frac{(2n_2)! \cdot (2n_k)!}{(n_2+l_2)!(n_2-l_2)! \cdot (n_k+l_k)!(n_k-l_k)!} \right\}^{1/2} \end{aligned}$$

**证明:**  $|j_k, n_1; (j, m)\rangle = \sum_{m_k=j_k}^{-j_k} \sum_{l_1=n_1}^{-n_1} |(j_k, m_k); (n_1, l_1)\rangle \langle (j_k, m_k); (n_1, l_1) | j_k, n_1; (j, m)\rangle$

$$\begin{aligned} &= \sum_{m_k=j_k}^{-j_k} \sum_{l_1=n_1}^{-n_1} |(j_k, m_k); (n_1, l_1)\rangle \delta(m_k + l_1 - m) \\ &\{ (2j+1) \frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!} (j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)! \}^{1/2} \\ &[\sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)!]^{-1} \\ &= \sum_{m_k=j_k}^{-j_k} \sum_{m_{k-1}=j_{k-1}}^{-j_{k-1}} \sum_{l_1=n_1}^{-n_1} \sum_{l_2=n_2}^{-n_2} |(j_{k-1}, m_{k-1}); (n_2, l_2); (n_1, l_1)\rangle \delta(m_k + l_1 - m) \delta(m_{k-1} + l_2 - m_k) \\ &\{ (2j+1) \frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!} (j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)! \}^{1/2} \\ &[\sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)!]^{-1} \\ &\left\{ \frac{(2j_{k-1})!(2n_2)!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_{k-1}+m_{k-1})!(j_{k-1}-m_{k-1})!(n_2+l_2)!(n_2-l_2)!} \right\}^{1/2} \\ &= \sum_{m_k=j_k}^{-j_k} \sum_{m_{k-1}=j_{k-1}}^{-j_{k-1}} \sum_{m_{k-2}=j_{k-2}}^{-j_{k-2}} \sum_{l_1=n_1}^{-n_1} \sum_{l_2=n_2}^{-n_2} \sum_{l_3=n_3}^{-n_3} |j_{k-2}, m_{k-2}; (n_3, l_3); (n_2, l_2); (n_1, l_1)\rangle \\ &\delta(m_k + l_1 - m) \delta(m_{k-1} + l_2 - m_k) \delta(m_{k-2} + l_3 - m_{k-1}) \\ &\{ (2j+1) \frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!} (j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)! \}^{1/2} \\ &[\sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)!]^{-1} \\ &\left\{ \frac{(2j_{k-1})!(2n_2)!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_{k-1}+m_{k-1})!(j_{k-1}-m_{k-1})!(n_2+l_2)!(n_2-l_2)!} \right\}^{1/2} \left\{ \frac{(2j_{k-2})!(2n_3)!(j_{k-1}+m_{k-1})!(j_{k-1}-m_{k-1})!}{(2j_{k-1})!(j_{k-2}+m_{k-2})!(j_{k-2}-m_{k-2})!(n_3+l_3)!(n_3-l_3)!} \right\}^{1/2} \\ &= \sum_{m_k=j_k}^{-j_k} \sum_{m_{k-1}=j_{k-1}}^{-j_{k-1}} \sum_{m_{k-2}=j_{k-2}}^{-j_{k-2}} \sum_{l_1=n_1}^{-n_1} \sum_{l_2=n_2}^{-n_2} \sum_{l_3=n_3}^{-n_3} |j_{k-2}, m_{k-2}; (n_3, l_3); (n_2, l_2); (n_1, l_1)\rangle \end{aligned}$$

$$\begin{aligned}
& \delta(m_k + l_1 - m) \delta(m_{k-1} + l_2 - m_k) \delta(m_{k-2} + l_3 - m_{k-1}) \\
& \left\{ (2j+1) \frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!} (j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)! \right\}^{1/2} \\
& \left[ \sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)! \right]^{-1} \\
& \left\{ \frac{(2j_{k-2})!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_{k-2}+m_{k-2})!(j_{k-2}-m_{k-2})!} \frac{(2n_2)!(2n_3)!}{(n_2+l_2)!(n_2-l_2)!(n_3+l_3)!(n_3-l_3)!} \right\}^{1/2} \\
& = \sum_{m_k=j_k}^{-j_k} \cdots \sum_{m_1=j_1}^{-j_1} \sum_{l_1=n_1}^{-n_1} \cdots \sum_{l_k=n_k}^{-n_k} |(j_1, m_1); (n_k, l_k); \cdots; (n_2, l_2); (n_1, l_1)\rangle \\
& \delta(m_k + l_1 - m) \delta(m_{k-1} + l_2 - m_k) \cdots \delta(m_1 + l_k - m_2) \\
& \left\{ (2j+1) \frac{(j_k+n_1-j)!(j_k-n_1+j)!(-j_k+n_1+j)!}{(j_k+n_1+j+1)!} (j_k+m_k)!(j_k-m_k)!(n_1+l_1)!(n_1-l_1)!(j+m)!(j-m)! \right\}^{1/2} \\
& \left[ \sum_r (-1)^r (j_k+n_1-j-r)!(j_k-m_k-r)!(j-j_k-l_1+r)!(n_1+l_1-r)!(j-n_1+m_k-r)! \right]^{-1} \\
& \left\{ \frac{(2j_1)!(j_k+m_k)!(j_k-m_k)!}{(2j_k)!(j_1+m_1)!(j_1-m_1)!} \frac{(2n_2)! \cdots (2n_k)!}{(n_2+l_2)!(n_2-l_2)! \cdots (n_k+l_k)!(n_k-l_k)!} \right\}^{1/2} \quad \square
\end{aligned}$$

### 9.3 多个粒子合成一个粒子的CG系数公式

**定理9.3.1.**  $|j_1, n_k, \cdots, n_1; (j_{k+1}, m_{k+1})\rangle = |(j_{k+1}, m_{k+1})\rangle$

$$\begin{aligned}
& = \sum_{m_k=j_k}^{-j_k} \cdots \sum_{m_1=j_1}^{-j_1} \sum_{l_1=n_1}^{-n_1} \cdots \sum_{l_k=n_k}^{-n_k} |(j_1, m_1); (n_k, l_k); \cdots; (n_1, l_1)\rangle \\
& \delta(m_k + l_1 - m_{k+1}) \delta(m_{k-1} + l_2 - m_k) \cdots \delta(m_1 + l_k - m_2) \left\{ \frac{(2j_1)!(j_{k+1}+m_{k+1})!(j_{k+1}-m_{k+1})!}{(2j_{k+1})!(j_1+m_1)!(j_1-m_1)!} \frac{(2n_1)! \cdots (2n_k)!}{(n_1+l_1)!(n_1-l_1)! \cdots (n_k+l_k)!(n_k-l_k)!} \right\}^{1/2} \\
& = \sum_{l_1=n_1}^{-n_1} \cdots \sum_{l_k=n_k}^{-n_k} |(j_1, m_1); (n_k, l_k); \cdots; (n_1, l_1)\rangle \left\{ \frac{(2j_1)!(2n_1)! \cdots (2n_k)!}{(2j_{k+1})!(j_1+m_1)!(j_1-m_1)!} \frac{(j_{k+1}+m_{k+1})!(j_{k+1}-m_{k+1})!}{(n_1+l_1)!(n_1-l_1)! \cdots (n_k+l_k)!(n_k-l_k)!} \right\}^{1/2} \\
& ; m_1 = m_{k+1} - \sum_{i=1}^k l_i
\end{aligned}$$

$$\begin{cases} m_k + l_1 - m_{k+1} = 0 \\ m_{k-1} + l_2 - m_k = 0 \\ \dots \\ m_2 + l_{k-1} - m_3 = 0 \\ m_1 + l_k - m_2 = 0 \end{cases} \Leftrightarrow \begin{cases} m_k = m_{k+1} - \sum_{i=1}^1 l_i \\ \dots \\ m_3 = m_{k+1} - \sum_{i=1}^{k-2} l_i \\ m_2 = m_{k+1} - \sum_{i=1}^{k-1} l_i \\ m_1 = m_{k+1} - \sum_{i=1}^k l_i \end{cases}$$

多个光子合成一个粒子的CG系数公式只是它的一个特例。

## 10 两个角动量耦合CG系数的一些特例

### 10.1 特例: $\frac{1}{2} \oplus \frac{1}{2} = (0, 0)$

**定理10.1.1.**  $|j_1, j_2; (j_3, m_3)\rangle = \sum_{m_1=j_1}^{-j_1} \sum_{m_2=j_2}^{-j_2} |(j_1, m_1); (j_2, m_2)\rangle \langle (j_1, m_1); (j_2, m_2) | j_1, j_2; (j_3, m_3)\rangle$

$$\begin{aligned}
CG_{Racah} & = \langle (j_1, m_1); (j_2, m_2) | j_1, j_2; (j_3, m_3)\rangle = \delta(m_1 + m_2 - m_3) \\
& \left\{ (2j_3+1) \frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!} (j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)! \right\}^{1/2} \\
& \left[ \sum_r (-1)^r r!(j_1+j_2-j_3-r)!(j_1-m_1-r)!(j_3-j_1-m_2+r)!(j_2+m_2-r)!(j_3-j_2+m_1+r)! \right]^{-1}
\end{aligned}$$

**推论10.1.1.**  $|j_1, \frac{1}{2}; (j_3, m_3)\rangle = \sum_{m_1=j_1}^{-j_1} \sum_{m_2=\frac{1}{2}}^{-\frac{1}{2}} |(j_1, m_1); (\frac{1}{2}, m_2)\rangle \langle (j_1, m_1); (\frac{1}{2}, m_2) | j_1, \frac{1}{2}; (j_3, m_3)\rangle$

$$\begin{aligned}
CG_{Racah} & = \langle (j_1, m_1); (\frac{1}{2}, m_2) | j_1, \frac{1}{2}; (j_3, m_3)\rangle = \delta(m_1 + m_2 - m_3) \\
& \left\{ (2j_3+1) \frac{(j_1+\frac{1}{2}-j_3)!(j_1-\frac{1}{2}+j_3)!(-j_1+\frac{1}{2}+j_3)!}{(j_1+\frac{1}{2}+j_3+1)!} (j_1+m_1)!(j_1-m_1)!(j_3+m_3)!(j_3-m_3)! \right\}^{1/2} \\
& \left[ \sum_r (-1)^r r!(j_1+\frac{1}{2}-j_3-r)!(j_1-m_1-r)!(j_3-j_1-m_2+r)!(\frac{1}{2}+m_2-r)!(j_3-\frac{1}{2}+m_1+r)! \right]^{-1}
\end{aligned}$$

**推论10.1.2.**

$$\begin{cases} CG_{Racah} = \langle (j_1, m_1); (\frac{1}{2}, \frac{1}{2}) | j_1, \frac{1}{2}; (j_1 + \frac{1}{2}, m_3) \rangle = \delta(m_1 + \frac{1}{2} - m_3) \left\{ \frac{j_1 + \frac{1}{2} + m_3}{2j_1 + 1} \right\}^{1/2} \\ CG_{Racah} = \langle (j_1, m_1); (\frac{1}{2}, -\frac{1}{2}) | j_1, \frac{1}{2}; (j_1 + \frac{1}{2}, m_3) \rangle = \delta(m_1 - \frac{1}{2} - m_3) \left\{ \frac{j_1 + \frac{1}{2} - m_3}{2j_1 + 1} \right\}^{1/2} \\ CG_{Racah} = \langle (j_1, m_1); (\frac{1}{2}, \frac{1}{2}) | j_1, \frac{1}{2}; (j_1 - \frac{1}{2}, m_3) \rangle = -\delta(m_1 + \frac{1}{2} - m_3) \left\{ \frac{j_1 + \frac{1}{2} - m_3}{2j_1 + 1} \right\}^{1/2} \\ CG_{Racah} = \langle (j_1, m_1); (\frac{1}{2}, -\frac{1}{2}) | j_1, \frac{1}{2}; (j_1 - \frac{1}{2}, m_3) \rangle = \delta(m_1 - \frac{1}{2} - m_3) \left\{ \frac{j_1 + \frac{1}{2} + m_3}{2j_1 + 1} \right\}^{1/2} \end{cases}$$

$$\text{定理10.1.2. } |\frac{1}{2}, \frac{1}{2}; (0, 0)\rangle = \sum_{m_1=1/2}^{-1/2} \sum_{m_2=1/2}^{-1/2} |(\frac{1}{2}, m_1); (\frac{1}{2}, m_2)\rangle \langle (\frac{1}{2}, m_1); (\frac{1}{2}, m_2) | \frac{1}{2}, \frac{1}{2}; (0, 0)\rangle$$

$$\begin{aligned} CG_{Racah} &= \langle (\frac{1}{2}, m_1); (\frac{1}{2}, m_2) | \frac{1}{2}, \frac{1}{2}; (0, 0)\rangle = \delta(m_1 + m_2 - 0) \\ &\{(2 \cdot 0 + 1) \frac{(\frac{1}{2} + \frac{1}{2} - 0)! (\frac{1}{2} - \frac{1}{2} + 0)! (-\frac{1}{2} + \frac{1}{2} + 0)!}{(\frac{1}{2} + \frac{1}{2} + 0 + 1)!} (\frac{1}{2} + m_1)! (\frac{1}{2} - m_1)! (\frac{1}{2} + m_2)! (\frac{1}{2} - m_2)! (0 + 0)! (0 - 0)!\}^{1/2} \\ &[\sum_r (-1)^r r! (\frac{1}{2} + \frac{1}{2} - 0 - r)! (\frac{1}{2} - m_1 - r)! (0 - \frac{1}{2} - m_2 + r)! (\frac{1}{2} + m_2 - r)! (0 - \frac{1}{2} + m_1 + r)!]^{-1} \\ &= \delta(m_1 + m_2) \left\{ \frac{1}{2!} (\frac{1}{2} + m_1)! (\frac{1}{2} - m_1)! (\frac{1}{2} + m_2)! (\frac{1}{2} - m_2)! \right\}^{1/2} \\ &[\sum_r (-1)^r r! (1 - r)! (\frac{1}{2} - m_1 - r)! (-\frac{1}{2} - m_2 + r)! (\frac{1}{2} + m_2 - r)! (-\frac{1}{2} + m_1 + r)!]^{-1} \\ &= \delta(m_1 + m_2) \frac{1}{\sqrt{2}!} (\frac{1}{2} + m_1)! (\frac{1}{2} - m_1)! [\sum_r (-1)^r r! (1 - r)! (\frac{1}{2} - m_1 - r)!]^2 [(-\frac{1}{2} + m_1 + r)!]^2]^{-1} \end{aligned}$$

推论10.1.3.

$$\begin{cases} \langle (\frac{1}{2}, \frac{1}{2}); (\frac{1}{2}, -\frac{1}{2}) | \frac{1}{2}, \frac{1}{2}; (0, 0)\rangle = \frac{1}{\sqrt{2}} & |\frac{1}{2}, \frac{1}{2}; (0, 0)\rangle = \frac{1}{\sqrt{2}} |(\frac{1}{2}, \frac{1}{2}); (\frac{1}{2}, -\frac{1}{2})\rangle - \frac{1}{\sqrt{2}} |(\frac{1}{2}, -\frac{1}{2}); (\frac{1}{2}, \frac{1}{2})\rangle \\ \langle (\frac{1}{2}, -\frac{1}{2}); (\frac{1}{2}, \frac{1}{2}) | \frac{1}{2}, \frac{1}{2}; (0, 0)\rangle = -\frac{1}{\sqrt{2}} \end{cases}$$

## 10.2 特例: $1 \oplus 1 = (0, 0)$

$$\begin{aligned} \text{定理10.2.1. } |1, 1; (0, 0)\rangle &= \sum_{m_1=1}^{-1} \sum_{m_2=1}^{-1} |(1, m_1); (1, m_2)\rangle \langle (1, m_1); (1, m_2) | 1, 1; (0, 0)\rangle \\ CG_{Racah} &= \langle (1, m_1); (1, m_2) | 1, 1; (0, 0)\rangle = \delta(m_1 + m_2 - 0) \\ &\{(2 \cdot 0 + 1) \frac{(1+1-0)! (1-1+0)! (-1+1+0)!}{(1+1+0+1)!} (1 + m_1)! (1 - m_1)! (1 + m_2)! (1 - m_2)! (0 + 0)! (0 - 0)!\}^{1/2} \\ &[\sum_r (-1)^r r! (1 + 1 - 0 - r)! (1 - m_1 - r)! (0 - 1 - m_2 + r)! (1 + m_2 - r)! (0 - 1 + m_1 + r)!]^{-1} \\ &= \delta(m_1 + m_2) \frac{1}{\sqrt{3}} (1 + m_1)! (1 - m_1)! [\sum_r (-1)^r r! (2 - r)! (1 - m_1 - r)!]^2 [(-1 + m_1 + r)!]^2]^{-1} \end{aligned}$$

推论10.2.1.

$$\begin{cases} \langle (1, 1); (1, -1) | 1, 1; (0, 0)\rangle = \frac{1}{\sqrt{3}} \\ \langle (1, 0); (1, 0) | 1, 1; (0, 0)\rangle = -\frac{1}{\sqrt{3}} & |1, 1; (0, 0)\rangle = \frac{1}{\sqrt{3}} |(1, 1); (1, -1)\rangle - \frac{1}{\sqrt{3}} |(1, 0); (1, 0)\rangle + \frac{1}{\sqrt{3}} |(1, -1); (1, 1)\rangle \\ \langle (1, -1); (1, 1) | 1, 1; (0, 0)\rangle = \frac{1}{\sqrt{3}} \end{cases}$$

## 10.3 特例: $1 \oplus 1 = (1, 0)$

$$\begin{aligned} \text{定理10.3.1. } |1, 1; (1, m_3)\rangle &= \sum_{m_1=1}^{-1} \sum_{m_2=1}^{-1} |(1, m_1); (1, m_2)\rangle \langle (1, m_1); (1, m_2) | 1, 1; (1, m_3)\rangle \\ CG_{Racah} &= \langle (1, m_1); (1, m_2) | 1, 1; (1, m_3)\rangle = \delta(m_1 + m_2 - m_3) \\ &\{(2 \cdot 1 + 1) \frac{(1+1-1)! (1-1+1)! (-1+1+1)!}{(1+1+1+1)!} (1 + m_1)! (1 - m_1)! (1 + m_2)! (1 - m_2)! (1 + m_3)! (1 - m_3)!\}^{1/2} \\ &[\sum_r (-1)^r r! (1 + 1 - 1 - r)! (1 - m_1 - r)! (1 - 1 - m_2 + r)! (1 + m_2 - r)! (1 - 1 + m_1 + r)!]^{-1} \\ &= \delta(m_1 + m_2 - m_3) \left\{ \frac{3}{4!} (1 + m_1)! (1 - m_1)! (1 + m_2)! (1 - m_2)! (1 + m_3)! (1 - m_3)! \right\}^{1/2} \\ &[\sum_r (-1)^r r! (1 - r)! (1 - m_1 - r)! (-m_2 + r)! (1 + m_2 - r)! (m_1 + r)!]^{-1} \end{aligned}$$

推论10.3.1.  $\langle (1, m_1); (1, m_2) | 1, 1; (1, m_3)\rangle$

$$= \delta(m_1 + m_2 - m_3) \left\{ \frac{3}{4!} (1 + m_1)! (1 - m_1)! (1 + m_2)! (1 - m_2)! (1 + m_3)! (1 - m_3)! \right\}^{1/2}$$

$$[\sum_r (-1)^r r! (1 - r)! (1 - m_1 - r)! (-m_2 + r)! (1 + m_2 - r)! (m_1 + r)!]^{-1}$$

推论10.3.2.  $\langle (1, m_1); (1, m_2) | 1, 1; (1, 0)\rangle$

$$= \delta(m_1 + m_2) \frac{1}{\sqrt{8}} (1 + m_1)! (1 - m_1)! [\sum_r (-1)^r r! (1 - r)! (1 - m_1 - r)! (m_1 + r)!]^2]^{-1}$$

推论10.3.3.

$$\begin{cases} \langle (1, 1); (1, -1) | 1, 1; (1, 0) \rangle = \frac{1}{\sqrt{2}} \\ \langle (1, 0); (1, 0) | 1, 1; (1, 0) \rangle = 0 & |1, 1; (1, 0) \rangle = \frac{1}{\sqrt{2}} |(1, 1); (1, -1) \rangle - \frac{1}{\sqrt{2}} |(1, -1); (1, 1) \rangle \\ \langle (1, -1); (1, 1) | 1, 1; (1, 0) \rangle = -\frac{1}{\sqrt{2}} \end{cases}$$

10.4 特例:  $1 \oplus 1 = (1, 1)$

推论10.4.1.  $\langle (1, m_1); (1, m_2) | 1, 1; (1, m_3) \rangle$   
 $= \delta(m_1 + m_2 - m_3) \left\{ \frac{3}{4!} (1 + m_1)! (1 - m_1)! (1 + m_2)! (1 - m_2)! (1 + m_3)! (1 - m_3)! \right\}^{1/2}$   
 $\left[ \sum_r (-1)^r r! (1 - r)! (1 - m_1 - r)! (-m_2 + r)! (1 + m_2 - r)! (m_1 + r)! \right]^{-1}$

推论10.4.2.  $\langle (1, m_1); (1, m_2) | 1, 1; (1, 1) \rangle$   
 $= \delta(m_1 + m_2 - 1) \left\{ \frac{1}{4} (1 + m_1)! (1 - m_1)! (1 + m_2)! (1 - m_2)! \right\}^{1/2}$   
 $\left[ \sum_r (-1)^r r! (1 - r)! (1 - m_1 - r)! (-m_2 + r)! (1 + m_2 - r)! (m_1 + r)! \right]^{-1}$

推论10.4.3.

$$\begin{cases} \langle (1, 1); (1, 0) | 1, 1; (1, 1) \rangle = \frac{1}{\sqrt{2}} & |1, 1; (1, 1) \rangle = \frac{1}{\sqrt{2}} |(1, 1); (1, 0) \rangle - \frac{1}{\sqrt{2}} |(1, 0); (1, 1) \rangle \\ \langle (1, 0); (1, 1) | 1, 1; (1, 1) \rangle = -\frac{1}{\sqrt{2}} \end{cases}$$

10.5 特例:  $1 \oplus 1 = (1, -1)$

推论10.5.1.  $\langle (1, m_1); (1, m_2) | 1, 1; (1, m_3) \rangle$   
 $= \delta(m_1 + m_2 - m_3) \left\{ \frac{3}{4!} (1 + m_1)! (1 - m_1)! (1 + m_2)! (1 - m_2)! (1 + m_3)! (1 - m_3)! \right\}^{1/2}$   
 $\left[ \sum_r (-1)^r r! (1 - r)! (1 - m_1 - r)! (-m_2 + r)! (1 + m_2 - r)! (m_1 + r)! \right]^{-1}$

推论10.5.2.  $\langle (1, m_1); (1, m_2) | 1, 1; (1, -1) \rangle$   
 $= \delta(m_1 + m_2 + 1) \left\{ \frac{1}{4} (1 + m_1)! (1 - m_1)! (1 + m_2)! (1 - m_2)! \right\}^{1/2}$   
 $\left[ \sum_r (-1)^r r! (1 - r)! (1 - m_1 - r)! (-m_2 + r)! (1 + m_2 - r)! (m_1 + r)! \right]^{-1}$

推论10.5.3.

$$\begin{cases} \langle (1, -1); (1, 0) | 1, 1; (1, -1) \rangle = -\frac{1}{\sqrt{2}} & |1, 1; (1, -1) \rangle = \frac{1}{\sqrt{2}} |(1, 0); (1, -1) \rangle - \frac{1}{\sqrt{2}} |(1, -1); (1, 0) \rangle \\ \langle (1, 0); (1, -1) | 1, 1; (1, -1) \rangle = \frac{1}{\sqrt{2}} \end{cases}$$

10.6 特例:  $1 \oplus 1 = (1, -1)$

推论10.6.1.

$$\begin{cases} \hat{J}_+ |s, h\rangle = \sqrt{s(s+1) - h(h+1)} |s, h+1\rangle = \sqrt{(s-h)(s+h+1)} |s, h+1\rangle, -s \leq h \leq s \\ \hat{J}_- |s, h\rangle = \sqrt{s(s+1) - h(h-1)} |s, h-1\rangle = \sqrt{(s-h+1)(s+h)} |s, h-1\rangle, -s \leq h \leq s \\ \hat{J}_z |s, h\rangle = h |s, h-1\rangle, -s \leq h \leq s \end{cases}$$

## 11 不变张量算符

### 11.1 不变张量算符

推论11.1.1.

$$\begin{cases} \hat{J}^2 = U(\omega) \hat{J}^2 U^+(\omega) \Leftrightarrow U(-\omega) \hat{J}^2 U^+(-\omega) = \hat{J}^2 \\ \hat{J}_i = e^{i\omega R} |_{i,j} U(\omega) \hat{J}_j U^+(\omega) \Leftrightarrow U(-\omega) \hat{J}_i U^+(-\omega) = e^{i\omega R} |_{i,j} \hat{J}_j \\ T_{ij} = e^{i\omega R} |_{i,k} e^{i\omega R} |_{j,l} U(\omega) T_{kl} U^+(\omega) \Leftrightarrow U(-\omega) T_{ij} U^+(-\omega) = e^{i\omega R} |_{i,k} e^{i\omega R} |_{j,l} T_{kl} \\ T_{i..j} = e^{i\omega R} |_{i,k} \dots e^{i\omega R} |_{j,l} U(\omega) T_{k..l} U^+(\omega) \Leftrightarrow U(-\omega) T_{i..j} U^+(-\omega) = e^{i\omega R} |_{i,k} e^{i\omega R} |_{j,l} T_{k..l} \end{cases}$$

## 12 球谐函数

### 12.1 数学准备

$$\text{定义12.1.1. } \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \varphi = \arctan \frac{y}{x} \end{cases} \Leftrightarrow \begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

$$\text{定义12.1.2. } \begin{cases} \hat{L}_x = -i\hbar(y\partial_z - z\partial_y) = i\hbar(\sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi) \\ \hat{L}_y = -i\hbar(z\partial_x - x\partial_z) = -i\hbar(\cos \varphi \partial_\theta - \cot \theta \sin \varphi \partial_\varphi) \\ \hat{L}_z = -i\hbar(x\partial_y - y\partial_x) = -i\hbar \partial_\varphi \end{cases} \begin{cases} \nabla_{\theta, \varphi}^2 = \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\varphi^2 \\ \nabla^2 = \frac{1}{r} \partial_r^2 r + \frac{1}{r^2} \nabla_{\theta, \varphi}^2 \\ \hat{L}^2 = -\hbar^2 \nabla_{\theta, \varphi}^2, \hat{L}_z = -i\hbar \partial_\varphi \end{cases}$$

$$\text{定义12.1.3. } \begin{cases} P_l(\xi) = \frac{1}{2^l l!} \frac{d^l}{d\xi^l} (\xi^2 - 1)^l, -1 \leq \xi \leq 1 \\ P_l^{|m|}(\xi) = (1 - \xi^2)^{|m|/2} \left[ \frac{d^{|m|}}{d\xi^{|m|}} P_l(\xi) - \frac{1}{2^l l!} \frac{d^{l+|m|}}{d\xi^{l+|m|}} (\xi^2 - 1)^l \right], -1 \leq \xi \leq 1 \end{cases}$$

### 12.2 球谐函数

定义12.2.1.

$$\begin{cases} \hat{L}^2 \psi = L^2 \psi \\ \hat{L}_z \psi = L_z \psi \end{cases} \Leftrightarrow \begin{cases} \hat{L}^2 Y_{lm}(\theta, \varphi) = l(l+1) Y_{lm}(\theta, \varphi), \hat{L}_z Y_{lm}(\theta, \varphi) = m Y_{lm}(\theta, \varphi) \\ \psi(\theta, \varphi) = Y_{lm}(\theta, \varphi) := A_{lm} P_l^{|m|}(\cos \theta) e^{im\varphi} \end{cases}$$

推论12.2.1.

$$\begin{cases} \hat{L}_x Y_{lm}(\theta, \varphi) = \frac{1}{2} [\sqrt{l(l+1) - m(m+1)} Y_{l, m+1}(\theta, \varphi) + \sqrt{l(l+1) - m(m-1)} Y_{l, m-1}(\theta, \varphi)] \\ \hat{L}_y Y_{lm}(\theta, \varphi) = -\frac{i}{2} [\sqrt{l(l+1) - m(m+1)} Y_{l, m+1}(\theta, \varphi) - \sqrt{l(l+1) - m(m-1)} Y_{l, m-1}(\theta, \varphi)] \end{cases}$$

推论12.2.2.

$$\begin{cases} \hat{L}_+ Y_{lm}(\theta, \varphi) = \sqrt{l(l+1) - m(m+1)} Y_{l, m+1}(\theta, \varphi) = \sqrt{(l-m)(l+m+1)} Y_{l, m+1}(\theta, \varphi), \hat{L}_+ := \hat{L}_x + i\hat{L}_y \\ \hat{L}_- Y_{lm}(\theta, \varphi) = \sqrt{l(l+1) - m(m-1)} Y_{l, m-1}(\theta, \varphi) = \sqrt{(l+m)(l-m+1)} Y_{l, m-1}(\theta, \varphi), \hat{L}_- := \hat{L}_x - i\hat{L}_y \end{cases}$$

推论12.2.3.

$$\begin{cases} \int Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) d\Omega = \delta_{l'l} \delta_{m'm} \\ \int d\Omega' \sum_{lm} Y_{lm}(\theta, \varphi) Y_{l'm'}^*(\theta', \varphi') |f(\theta', \varphi')| = \int d\Omega' \delta(\theta - \theta') \delta(\varphi - \varphi') |f(\theta', \varphi')| \end{cases}$$

推论12.2.4.

$$\begin{cases} [\hat{L}^2]_{m'm} = \int Y_{l'm'}^*(\theta, \varphi) \hat{L}^2 Y_{lm}(\theta, \varphi) d\Omega = \int l(l+1) Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) d\Omega = l(l+1) \delta_{m'm} \\ [\hat{L}_z]_{m'm} = \int Y_{l'm'}^*(\theta, \varphi) \hat{L}_z Y_{lm}(\theta, \varphi) d\Omega = \int m Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) d\Omega = m \delta_{m'm} \\ [\hat{L}_x + i\hat{L}_y]_{m'm} = \int Y_{l'm'}^*(\theta, \varphi) [\hat{L}_x + i\hat{L}_y] Y_{lm}(\theta, \varphi) d\Omega = \sqrt{(l-m)(l+m+1)} \delta_{m'(m+1)} \\ [\hat{L}_x - i\hat{L}_y]_{m'm} = \int Y_{l'm'}^*(\theta, \varphi) [\hat{L}_x - i\hat{L}_y] Y_{lm}(\theta, \varphi) d\Omega = \sqrt{(l+m)(l-m+1)} \delta_{m'(m-1)} \end{cases}$$

推论12.2.5.

$$\begin{cases} [\hat{L}_x]_{m'm} = \frac{1}{2} [\sqrt{(l+m'+1)(l-m')} \delta_{m'(m-1)} + \sqrt{(l-m'+1)(l+m')} \delta_{m'(m+1)}] \\ [\hat{L}_y]_{m'm} = \frac{i}{2} [\sqrt{(l+m'+1)(l-m')} \delta_{m'(m-1)} - \sqrt{(l-m'+1)(l+m')} \delta_{m'(m+1)}] \\ [\hat{L}_z]_{m'm} = m' \delta_{m'm} \end{cases}$$

推论12.2.6.  $A_n = \sqrt{n} \cdot \sqrt{2l+1-n}, n = 1, 2, \dots, 2l$

$$[\hat{L}] = \left( \frac{1}{2} \begin{bmatrix} 0 & A_1 & 0 & 0 & 0 \\ A_1 & 0 & A_2 & 0 & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & A_{2l} \\ 0 & 0 & 0 & A_{2l} & 0 \end{bmatrix}, \frac{i}{2} \begin{bmatrix} 0 & -A_1 & 0 & 0 & 0 \\ A_1 & 0 & -A_2 & 0 & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -A_{2l} \\ 0 & 0 & 0 & A_{2l} & 0 \end{bmatrix}, \begin{bmatrix} l & 0 & 0 & 0 & 0 \\ 0 & l-1 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -(l-1) & 0 \\ 0 & 0 & 0 & 0 & -l \end{bmatrix} \right)$$

## 12.3 一般算子方法 [34]

定义12.3.1.  $\vec{J} \times \vec{J} = i\vec{J}$ ,  $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ ;  $[J_i, J_j] = \varepsilon_{ij}^k J_k$ ,  $[\vec{J}^2, J_i] = 0$

定义12.3.2. 
$$\begin{cases} J_+ := J_x + iJ_y, J_- := J_x - iJ_y, \hat{J}^2 = \frac{1}{2}(J_+J_- + J_+J_-) + J_z^2 \\ [J_z, J_+] = J_+, [J_z, J_-] = -J_-, [J_+, J_-] = 2J_z \\ J_+J_- = \hat{J}^2 - J_z(J_z - 1), J_-J_+ = \hat{J}^2 - J_z(J_z + 1) \end{cases}$$

性质12.3.1.  $\hat{J}^2|j, m\rangle = j(j+1)|j, m\rangle$ ,  $J_z|j, m\rangle = m|j, m\rangle$

$J_+|j, m\rangle = \sqrt{l(l+1) - m(m+1)}|j, m+1\rangle$ ,  $J_-|j, m\rangle = \sqrt{l(l+1) - m(m-1)}|j, m-1\rangle$

解1-薛定谔氢原子:  $\{\hat{L}; |l, m\rangle\} = \{-i\vec{r} \times \nabla; Y_{lm}(\theta, \varphi)\}$

解2-自旋单态:  $\{\sigma(s); |s, h\rangle\} = \{\sigma(s); \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}, h\}$

解3-狄拉克氢原子:  $\{\hat{J}; |j, m\rangle\} = \{\hat{L} + \frac{1}{2}\sigma; \begin{bmatrix} g(\vec{r})\phi_{jI_A}^{m_j}(\theta, \varphi) \\ if(\vec{r})\phi_{jI_B}^{m_j}(\theta, \varphi) \end{bmatrix}\}$

解4-狄拉克s原子:  $\{\hat{J}; |l, s, m\rangle\} = \{\hat{L} + \hat{s}; Y_{lsm}(\theta, \varphi)\}$

# 第三十章 B-F公式和投影算子猜想

自我评述：在本章中我推广发展了全对称指标的多项式定理，为上一步按新的程式统一量子化有质量粒子提供了数学支持，同时基于历史上Behrends和Fronsdal构造出来的公式，并结合自己新的结论，得到了一个十分有意义的投影算子猜想。

## 1 多项式定理及其全对称指标推广

### 1.1 零阶全对称指标二项式展开

性质1.1.1.

$$\begin{cases} (A+B)^2 = A^2 + 2AB + B^2 \\ (A-B)^2 = A^2 - 2AB + B^2 \\ (A+B)(A-B) = A^2 - B^2 \\ (A-B)(A+B) = A^2 - B^2 \end{cases}$$

定理1.1.1.  $(A+B)^n = \sum_{i=0}^n C_n^i A^i B^{n-i}$

### 1.2 零阶全对称指标多项式展开

定理1.2.1.  $(A_1 + \dots + A_l)^n = \sum_{n_1 n_2 \dots n_l} \frac{n!}{n_1! n_2! \dots n_l!} A_1^{n_1} A_2^{n_2} \dots A_l^{n_l}, n_1 + n_2 + \dots + n_l = n$

### 1.3 一阶全对称指标二项式展开

性质1.3.1.

$$\begin{cases} [A_{\{a_1\}} + B_{\{a_1\}}][A_{\{a_2\}} + B_{\{a_2\}}] = A_{\{a_1\}a_2} + 2A_{\{a_1\}B_{\{a_2\}}} + B_{\{a_1\}B_{\{a_2\}}} \\ [A_{\{a_1\}} - B_{\{a_1\}}][A_{\{a_2\}} - B_{\{a_2\}}] = A_{\{a_1\}a_2} - 2A_{\{a_1\}B_{\{a_2\}}} + B_{\{a_1\}B_{\{a_2\}}} \\ [A_{\{a_1\}} + B_{\{a_1\}}][A_{\{a_2\}} - B_{\{a_2\}}] = A_{\{a_1\}a_2} - B_{\{a_1\}B_{\{a_2\}}} \\ [A_{\{a_1\}} - B_{\{a_1\}}][A_{\{a_2\}} + B_{\{a_2\}}] = A_{\{a_1\}a_2} - B_{\{a_1\}B_{\{a_2\}}} \\ A_{\{a_1\}B_{\{a_2\}}} = A_{\{a_2\}B_{\{a_1\}}}, A_{\{\dots a_i \dots a_j \dots\}} = A_{\{a_j \dots a_i \dots\}} \end{cases}$$

定理1.3.1.  $[A_{\{a_1\}} + B_{\{a_1\}}] \dots [A_{\{a_n\}} + B_{\{a_n\}}] = \sum_{i=0}^n C_n^i [A_{\{a_1\}} \dots A_{\{a_i\}}][B_{\{a_{i+1}\}} \dots B_{\{a_n\}}]$

推论1.3.1.  $(A_a + B_a)^n = \sum_{i=0}^n C_n^i A_a^i B_a^{n-i}$

### 1.4 一阶全对称指标多项式展开

定理1.4.1.  $[A_{1\{a_1\}} + \dots + A_{l\{a_1\}}] \dots [A_{1a_n} + \dots + A_{la_n}], n_1 + n_2 + \dots + n_l = n$   
 $= \sum_{n_1 n_2 \dots n_l} \frac{n!}{n_1! n_2! \dots n_l!} [A_{1\{a_1\}} \dots A_{1a_{n_1}}][A_{2a_{n_1+1}} \dots A_{2a_{n_1+n_2}}] \dots [A_{la_{n_1+\dots+n_{l-1}+1}} \dots A_{la_n}]$

推论1.4.1.  $(A_{1a} + \dots + A_{la})^n = \sum_{n_1 n_2 \dots n_l} \frac{n!}{n_1! n_2! \dots n_l!} A_{1a}^{n_1} A_{2a}^{n_2} \dots A_{la}^{n_l}, n_1 + n_2 + \dots + n_l = n$

### 1.5 二阶全对称指标二项式展开

性质1.5.1.

$$\begin{cases} [A_{\{a_1(b_1\}} + B_{\{a_1(b_1\}}][A_{\{a_2\}b_2} + B_{\{a_2\}b_2}] = A_{\{a_1(b_1\}A_{\{a_2\}b_2\}} + 2A_{\{a_1(b_1\}B_{\{a_2\}b_2\}} + B_{\{a_1(b_1\}B_{\{a_2\}b_2\}} \\ [A_{\{a_1(b_1\}} - B_{\{a_1(b_1\}}][A_{\{a_2\}b_2} - B_{\{a_2\}b_2}] = A_{\{a_1(b_1\}A_{\{a_2\}b_2\}} - 2A_{\{a_1(b_1\}B_{\{a_2\}b_2\}} + B_{\{a_1(b_1\}B_{\{a_2\}b_2\}} \\ [A_{\{a_1(b_1\}} + B_{\{a_1(b_1\}}][A_{\{a_2\}b_2} - B_{\{a_2\}b_2}] = A_{\{a_1(b_1\}A_{\{a_2\}b_2\}} - B_{\{a_1(b_1\}B_{\{a_2\}b_2\}} \\ [A_{\{a_1(b_1\}} - B_{\{a_1(b_1\}}][A_{\{a_2\}b_2} + B_{\{a_2\}b_2}] = A_{\{a_1(b_1\}A_{\{a_2\}b_2\}} - B_{\{a_1(b_1\}B_{\{a_2\}b_2\}} \end{cases}$$



$$\text{定理1.5.1. } [A_{\{a_1(b_1 + B_{\{a_1(b_1)} \cdot \cdot [A_{a_n\}b_n\} + B_{a_n\}b_n\})} = \sum_{i=0}^n C_n^i [A_{\{a_1(b_1} \cdot \cdot A_{a_i b_i\}} [B_{a_{i+1} b_{i+1}} \cdot \cdot B_{a_n\}b_n\}]$$

$$\text{推论1.5.1. } (A_{ab} + B_{ab})^n = \sum_{i=0}^n C_n^i A_{ab}^i B_{ab}^{n-i}$$

## 1.6 二阶全对称指标多项式展开

$$\text{定理1.6.1. } [A_{1\{a_1(b_1} + \cdot \cdot + A_{l\{a_1(b_1)} \cdot \cdot [A_{1a_n\}b_n\} + \cdot \cdot + A_{la_n\}b_n\}], n_1 + n_2 + \cdot \cdot + n_l = n$$

$$= \sum_{n_1 n_2 \cdot \cdot n_l} \frac{n!}{n_1! n_2! \cdot \cdot n_l!} [A_{1\{a_1(b_1} \cdot \cdot A_{1a_{n_1} b_{n_1}} [A_{2a_{n_1+1} b_{n_1+1}} \cdot \cdot A_{2a_{n_1+n_2} b_{n_1+n_2}}] \cdot \cdot [A_{la_{n_1+\cdot \cdot + n_{l-1}+1} b_{n_1+\cdot \cdot + n_{l-1}+1}} \cdot \cdot A_{la_n\}b_n\}]$$

$$\text{推论1.6.1. } (A_{1ab} + \cdot \cdot + A_{lab})^n = \sum_{n_1 n_2 \cdot \cdot n_l} \frac{n!}{n_1! n_2! \cdot \cdot n_l!} A_{1ab}^{n_1} A_{2ab}^{n_2} \cdot \cdot A_{lab}^{n_l}, n_1 + n_2 + \cdot \cdot + n_l = n$$

## 1.7 多阶全对称指标多项式展开

$$\text{定理1.7.1. } [A_{1\{a_1(b_1} + \cdot \cdot + A_{l\{a_1(b_1)} \cdot \cdot [A_{1a_n\}b_n\} + \cdot \cdot + A_{la_n\}b_n\}], n_1 + n_2 + \cdot \cdot + n_l = n$$

$$= \sum_{n_1 n_2 \cdot \cdot n_l} \frac{n!}{n_1! n_2! \cdot \cdot n_l!} [A_{1\{a_1(b_1} \cdot \cdot A_{1a_{n_1} b_{n_1}} [A_{2a_{n_1+1} b_{n_1+1}} \cdot \cdot A_{2a_{n_1+n_2} b_{n_1+n_2}}] \cdot \cdot [A_{la_{n_1+\cdot \cdot + n_{l-1}+1} b_{n_1+\cdot \cdot + n_{l-1}+1}} \cdot \cdot A_{la_n\}b_n\}]$$

$$\text{推论1.7.1. } (A_{1a \cdot b} + \cdot \cdot + A_{la \cdot b})^n = \sum_{n_1 n_2 \cdot \cdot n_l} \frac{n!}{n_1! n_2! \cdot \cdot n_l!} A_{1ab}^{n_1} A_{2a \cdot b}^{n_2} \cdot \cdot A_{la \cdot b}^{n_l}, n_1 + n_2 + \cdot \cdot + n_l = n$$

## 1.8 全对称指标多项式性质

推论1.8.1.

$$[A_{\{a_1 B_{(b_1} + B_{\{a_1 A_{(b_1}} [A_{a_2\} B_{b_2\}} + B_{a_2\} A_{b_2\}})] = A_{\{a_1 A_{a_2\}} B_{(b_1 B_{b_2\}}) + 2A_{\{a_1 B_{a_2\}} A_{(b_1 B_{b_2\}}) + B_{\{a_1 B_{a_2\}} A_{(b_1 A_{b_2\}})$$

推论1.8.2.

$$[A_{\{a_1 B_{(b_1} - B_{\{a_1 A_{(b_1}} [A_{a_2\} B_{b_2\}} - B_{a_2\} A_{b_2\}})] = A_{\{a_1 A_{a_2\}} B_{(b_1 B_{b_2\}}) - 2A_{\{a_1 B_{a_2\}} A_{(b_1 B_{b_2\}}) + B_{\{a_1 B_{a_2\}} A_{(b_1 A_{b_2\}})$$

$$\text{推论1.8.3. } [A_{\{a_1 B_{(b_1} + B_{\{a_1 A_{(b_1}} [A_{a_2\} B_{b_2\}} - B_{a_2\} A_{b_2\}})] = A_{\{a_1 A_{a_2\}} B_{(b_1 B_{b_2\}}) - B_{\{a_1 B_{a_2\}} A_{(b_1 A_{b_2\}})$$

$$\text{推论1.8.4. } [A_{\{a_1 B_{(b_1} - B_{\{a_1 A_{(b_1}} [A_{a_2\} B_{b_2\}} + B_{a_2\} A_{b_2\}})] = A_{\{a_1 A_{a_2\}} B_{(b_1 B_{b_2\}}) - B_{\{a_1 B_{a_2\}} A_{(b_1 A_{b_2\}})$$

$$\text{定理1.8.1. } [A_{\{a_1 B_{(b_1} + B_{\{a_1 A_{(b_1}} \cdot \cdot [A_{a_n\} B_{b_n\}} + B_{a_n\} A_{b_n\}})] = \sum_{i=0}^n C_n^i [A_{\{a_1} \cdot \cdot A_{a_i B_{a_{i+1}}} \cdot \cdot B_{a_n\}}] [B_{(b_1} \cdot \cdot B_{b_i A_{b_{i+1}}} \cdot \cdot A_{b_n\}}]$$

$$\text{定理1.8.2. } [A_{\{a_1 B_{(b_1} - B_{\{a_1 A_{(b_1}} \cdot \cdot [A_{a_n\} B_{b_n\}} - B_{a_n\} A_{b_n\}})]$$

$$= \sum_{i=0}^n (-1)^{n-i} C_n^i [A_{\{a_1} \cdot \cdot A_{a_i B_{a_{i+1}}} \cdot \cdot B_{a_n\}}] [B_{(b_1} \cdot \cdot B_{b_i A_{b_{i+1}}} \cdot \cdot A_{b_n\}}]$$

$$\text{定理1.8.3. } [A_{\{a_1 B_{(b_1} + B_{\{a_1 A_{(b_1} + C_{\{a_1} C_{(b_1}} \cdot \cdot [A_{a_n\} B_{b_n\}} + B_{a_n\} A_{b_n\}} + C_{a_n\} C_{b_n\}})]$$

$$= \sum_{n_1 n_2 \cdot \cdot n_l} \frac{n!}{n_1! n_2! \cdot \cdot n_l!} [A_{\{a_1} \cdot \cdot A_{a_{n_1} B_{a_{n_1+1}}} \cdot \cdot B_{n_1+n_2} C_{a_{n_1+n_2+1}} \cdot \cdot C_{a_n\}}] [B_{(b_1} \cdot \cdot B_{b_{n_1} A_{b_{n_1+1}}} \cdot \cdot A_{n_1+n_2} C_{b_{n_1+n_2+1}} \cdot \cdot C_{b_n\}}]$$

$$\text{推论1.8.5. } [A_{\{a_1 B_{(b_1} + B_{\{a_1 A_{(b_1} + C_{\{a_1} C_{(b_1}} [A_{a_2\} B_{b_2\}} + B_{a_2\} A_{b_2\}} + C_{a_2\} C_{b_2\}})]$$

$$= [A_{\{a_1 B_{(b_1} + B_{\{a_1 A_{(b_1}} [A_{a_2\} B_{b_2\}} + B_{a_2\} A_{b_2\}})] + 2[A_{\{a_1 B_{(b_1} + B_{\{a_1 A_{(b_1}} C_{a_2\} C_{b_2\}}] + [C_{\{a_1} C_{a_2\}}] [C_{(b_1} C_{b_2\}})]$$

$$= A_{\{a_1 A_{a_2\}} B_{(b_1 B_{b_2\}}) + B_{\{a_1 B_{a_2\}} A_{(b_1 A_{b_2\}}) + C_{\{a_1} C_{a_2\}} C_{(b_1} C_{b_2\}})$$

$$+ 2A_{\{a_1 B_{a_2\}} A_{(b_1 B_{b_2\}}) + 2A_{\{a_1} C_{a_2\}} B_{(b_1} C_{b_2\}} + 2B_{\{a_1} C_{a_2\}} A_{(b_1} C_{b_2\}})$$

## 2 多项式定理及其反对称指标推广

### 2.1 一阶全对称指标二项式展开

性质2.1.1.

$$\begin{cases} [A_{[a_1} + B_{[a_1}][A_{a_2\}} + B_{a_2\}}] = 0 \\ [A_{[a_1} - B_{[a_1}][A_{a_2\}} - B_{a_2\}}] = 0 \\ [A_{[a_1} + B_{[a_1}][A_{a_2\}} - B_{a_2\}}] = 2B_{[a_1} A_{a_2\}} \\ [A_{[a_1} - B_{[a_1}][A_{a_2\}} + B_{a_2\}}] = 2A_{[a_1} B_{a_2\}} \end{cases}$$

$$\text{定理2.1.1. } [A_{[a_1} + B_{[a_1} \cdot \cdot [A_{a_n\}} + B_{a_n\}}] = 0$$

## 2.2 一阶全对称指标多项式展开

定理2.2.1.  $[A_{1[a_1]} + \cdots + A_{l[a_1]}] \cdots [A_{1[a_n]} + \cdots + A_{l[a_n]}] = 0$

## 2.3 反对称指标二项式展开

性质2.3.1.

$$\begin{cases} [A_{[a_1 \langle b_1} + B_{[a_1 \langle b_1}][A_{a_2 \rangle b_2} + B_{a_2 \rangle b_2}] = A_{[a_1 \langle b_1} A_{a_2 \rangle b_2} + 2A_{[a_1 \langle b_1} B_{a_2 \rangle b_2} + B_{[a_1 \langle b_1} B_{a_2 \rangle b_2}] \\ [A_{[a_1 \langle b_1} - B_{[a_1 \langle b_1}][A_{a_2 \rangle b_2} - B_{a_2 \rangle b_2}] = A_{[a_1 \langle b_1} A_{a_2 \rangle b_2} - 2A_{[a_1 \langle b_1} B_{a_2 \rangle b_2} + B_{[a_1 \langle b_1} B_{a_2 \rangle b_2}] \\ [A_{[a_1 \langle b_1} + B_{[a_1 \langle b_1}][A_{a_2 \rangle b_2} - B_{a_2 \rangle b_2}] = A_{[a_1 \langle b_1} A_{a_2 \rangle b_2} - B_{[a_1 \langle b_1} B_{a_2 \rangle b_2}] \\ [A_{[a_1 \langle b_1} - B_{[a_1 \langle b_1}][A_{a_2 \rangle b_2} + B_{a_2 \rangle b_2}] = A_{[a_1 \langle b_1} A_{a_2 \rangle b_2} - B_{[a_1 \langle b_1} B_{a_2 \rangle b_2}] \end{cases}$$

定理2.3.1.  $[A_{[a_1 \langle b_1} + B_{[a_1 \langle b_1}][A_{a_n \rangle b_n} + B_{a_n \rangle b_n}] = \sum_{i=1}^n C_n^i [A_{[a_1 \langle b_1} \cdots A_{a_i \rangle b_i}][B_{a_{i+1} \rangle b_{i+1}} \cdots B_{a_n \rangle b_n}]$

推论2.3.1.  $[A_{[a_1} B_{\langle b_1} + B_{[a_1} A_{\langle b_1} + C_{[a_1} C_{\langle b_1}][A_{a_2} B_{\rangle b_2} + B_{a_2} A_{\rangle b_2} + C_{a_2} C_{\rangle b_2}]$   
 $= 2A_{[a_1} B_{a_2} A_{\langle b_1} B_{\rangle b_2} + 2A_{[a_1} C_{a_2} B_{\langle b_1} C_{\rangle b_2} + 2B_{[a_1} C_{a_2} A_{\langle b_1} C_{\rangle b_2}$

## 2.4 反对称指标多项式展开(错的?)

定理2.4.1.  $[A_{1[a_1 \langle b_1} + \cdots + A_{l[a_1 \langle b_1}][A_{1[a_n \rangle b_n} + \cdots + A_{l[a_n \rangle b_n}], n_1 + n_2 + \cdots + n_l = n$

$$= \sum_{n_1 n_2 \cdots n_l} \frac{n!}{n_1! n_2! \cdots n_l!} [A_{1[a_1 \langle b_1} \cdots A_{1[a_{n_1} \rangle b_{n_1}]] [A_{2[a_{n_1+1} \rangle b_{n_1+1}} \cdots A_{2[a_{n_1+n_2} \rangle b_{n_1+n_2}]] \cdots [A_{l[a_{n_1+\cdots+n_{l-1}+1} \rangle b_{n_1+\cdots+n_{l-1}+1}} \cdots A_{l[a_n \rangle b_n}]]$$

定理2.4.2. 偶数指标:  $[A_{1[a_1 \langle b_1} + \cdots + A_{l[a_1 \langle b_1}][A_{1[a_n \rangle b_n} + \cdots + A_{l[a_n \rangle b_n}], n_1 + n_2 + \cdots + n_l = n$

$$= \sum_{n_1 n_2 \cdots n_l} \frac{n!}{n_1! n_2! \cdots n_l!} [A_{1[a_1 \langle b_1} \cdots A_{1[a_{n_1} \rangle b_{n_1}]] [A_{2[a_{n_1+1} \rangle b_{n_1+1}} \cdots A_{2[a_{n_1+n_2} \rangle b_{n_1+n_2}]] \cdots [A_{l[a_{n_1+\cdots+n_{l-1}+1} \rangle b_{n_1+\cdots+n_{l-1}+1}} \cdots A_{l[a_n \rangle b_n}]]$$

## 3 自旋-n粒子Klein-Gordon方程 [22] 投影算子

### 3.1 自旋-n粒子Klein-Gordon方程投影算子的经典表述

推论3.1.1.  $\sum_{h=2}^{-2} (-1)^h \varepsilon_{\{a_1 a_2\}}(\vec{p}, h) \varepsilon_{(b_1 b_2)}(\vec{p}, -h)$   
 $= [\sum_{h=1}^{-1} (-1)^h \varepsilon_{\{a_1}(\vec{p}, h) \varepsilon_{(b_1}(\vec{p}, -h))] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{a_2}(\vec{p}, h) \varepsilon_{b_2)}(\vec{p}, -h)]$   
 $- \frac{1}{3} [\sum_{h=1}^{-1} (-1)^h \varepsilon_{\{a_1}(\vec{p}, h) \varepsilon_{a_2\}}(\vec{p}, -h)] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{(b_1}(\vec{p}, h) \varepsilon_{b_2)}(\vec{p}, -h)]$

推论3.1.2.  $\sum_{h=3}^{-3} (-1)^h \varepsilon_{\{a_1 a_2 a_3\}}(\vec{p}, h) \varepsilon_{(b_1 b_2 b_3)}(\vec{p}, -h)$   
 $= [\sum_{h=1}^{-1} (-1)^h \varepsilon_{\{a_1}(\vec{p}, h) \varepsilon_{(b_1}(\vec{p}, -h))] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{a_2}(\vec{p}, h) \varepsilon_{b_2)}(\vec{p}, -h)] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{a_3}(\vec{p}, h) \varepsilon_{b_3)}(\vec{p}, -h)]$   
 $- \frac{3}{5} [\sum_{h=1}^{-1} (-1)^h \varepsilon_{\{a_1}(\vec{p}, h) \varepsilon_{a_2\}}(\vec{p}, -h)] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{(b_1}(\vec{p}, h) \varepsilon_{b_2)}(\vec{p}, -h)] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{a_3}(\vec{p}, h) \varepsilon_{b_3)}(\vec{p}, -h)]$

推论3.1.3.  $\sum_{h=4}^{-4} (-1)^h \varepsilon_{\{a_1 a_2 a_3 a_4\}}(\vec{p}, h) \varepsilon_{(b_1 b_2 b_3 b_4)}(\vec{p}, -h)$   
 $= [\sum_{h=1}^{-1} (-1)^h \varepsilon_{\{a_1}(\vec{p}, h) \varepsilon_{(b_1}(\vec{p}, -h))] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{a_2}(\vec{p}, h) \varepsilon_{b_2)}(\vec{p}, -h)] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{a_3}(\vec{p}, h) \varepsilon_{b_3)}(\vec{p}, -h)]$   
 $[\sum_{h=1}^{-1} (-1)^h \varepsilon_{a_4}(\vec{p}, h) \varepsilon_{b_4)}(\vec{p}, -h)] - \frac{6}{7} [\sum_{h=1}^{-1} (-1)^h \varepsilon_{\{a_1}(\vec{p}, h) \varepsilon_{a_2\}}(\vec{p}, -h)] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{(b_1}(\vec{p}, h) \varepsilon_{b_2)}(\vec{p}, -h)]$   
 $[\sum_{h=1}^{-1} (-1)^h \varepsilon_{a_3}(\vec{p}, h) \varepsilon_{b_3)}(\vec{p}, -h)] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{a_4}(\vec{p}, h) \varepsilon_{b_4)}(\vec{p}, -h)] + \frac{3}{35} [\sum_{h=1}^{-1} (-1)^h \varepsilon_{\{a_1}(\vec{p}, h) \varepsilon_{a_2\}}(\vec{p}, -h)]$   
 $[\sum_{h=1}^{-1} (-1)^h \varepsilon_{(b_1}(\vec{p}, h) \varepsilon_{b_2)}(\vec{p}, -h)] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{a_3}(\vec{p}, h) \varepsilon_{a_4\}}(\vec{p}, -h)] [\sum_{h=1}^{-1} (-1)^h \varepsilon_{b_3}(\vec{p}, h) \varepsilon_{b_4)}(\vec{p}, -h)]$

$$\begin{aligned}
\text{猜想3.1.1. } & \sum_{h=n}^{-n} (-1)^h \varepsilon_{\{a_1 a_2 \dots a_n\}}(\vec{p}, h) \varepsilon_{\{b_1 b_2 \dots b_n\}}(\vec{p}, -h) = \sum_{r=0}^{[n/2]} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} \\
& \left\{ \left[ \sum_{h=1}^{-1} (-1)^h \varepsilon_{\{a_1\}}(\vec{p}, h) \varepsilon_{a_2}(\vec{p}, -h) \right] \left[ \sum_{h=1}^{-1} (-1)^h \varepsilon_{\{b_1\}}(\vec{p}, h) \varepsilon_{b_2}(\vec{p}, -h) \right] \cdot \right. \\
& \left. \left[ \sum_{h=1}^{-1} (-1)^h \varepsilon_{a_{2r-1}}(\vec{p}, h) \varepsilon_{a_{2r}}(\vec{p}, -h) \right] \left[ \sum_{h=1}^{-1} (-1)^h \varepsilon_{b_{2r-1}}(\vec{p}, h) \varepsilon_{b_{2r}}(\vec{p}, -h) \right] \right\} \\
& \left[ \sum_{h=1}^{-1} (-1)^h \varepsilon_{a_{2r+1}}(\vec{p}, h) \varepsilon_{b_{2r+1}}(\vec{p}, -h) \right] \cdot \left[ \sum_{h=1}^{-1} (-1)^h \varepsilon_{a_n}(\vec{p}, h) \varepsilon_{b_n}(\vec{p}, -h) \right] \\
& = \sum_{r=0}^{[n/2]} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} \\
& \{ [C_{\{a_1\}} C_{a_2} - A_{\{a_1\}} B_{a_2} - B_{\{a_1\}} A_{a_2}] [C_{\{b_1\}} C_{b_2} - A_{\{b_1\}} B_{b_2} - B_{\{b_1\}} A_{b_2}] \cdot \\
& [C_{a_{2r-1}} C_{a_{2r}} - A_{a_{2r-1}} B_{a_{2r}} - B_{a_{2r-1}} A_{a_{2r}}] [C_{b_{2r-1}} C_{b_{2r}} - A_{b_{2r-1}} B_{b_{2r}} - B_{b_{2r-1}} A_{b_{2r}}] \} \\
& [C_{a_{2r+1}} C_{b_{2r+1}} - A_{a_{2r+1}} B_{b_{2r+1}} - B_{a_{2r+1}} A_{b_{2r+1}}] \cdot [C_{a_n} C_{b_n} - A_{a_n} B_{b_n} - B_{a_n} A_{b_n}] \}
\end{aligned}$$

以上猜想是从Behrends和Fronsdal构造出来的公式<sup>[52, 53]</sup>等价变形过来的，并没有严格地被证明，本质上还是一个猜想，它是后面很多重要结论的前提条件。

### 3.2 自旋-n粒子Klein-Gordon方程投影算子的定义表述

特别声明一点，本节使用了前面章节的结论，然后利用它们导出了重要猜想。

定义3.2.1.  $A = \varepsilon(\vec{p}, 1)$ ,  $B = \varepsilon(\vec{p}, -1)$ ,  $C = \varepsilon(\vec{p}, 0)$

推论3.2.1.  $A_{r,n} = (-\frac{1}{2})^r \frac{n!(2n-2r-1)!!}{r!(n-2r)!(2n-1)!!} = (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$

定理3.2.1.  $\varepsilon_{\{a_1 a_2 \dots a_n\}}(\vec{p}, n-2k)$   
 $= \frac{1}{\sqrt{C_{2n}^{2k}}} \sum_{i=0}^{k|(n-k)} \frac{(\sqrt{2})^{2i} n!}{(n-k-i)!(k-i)!(2i)!} [A_{\{a_1 \dots a_{n-k-i}\}} [B_{a_{n+1-k-i}} \dots B_{a_{n-2i}}] [C_{a_{n+1-2i}} \dots C_{a_n}]]$

定理3.2.2.  $\varepsilon_{\{a_1 a_2 \dots a_n\}}(\vec{p}, n-2k-1)$   
 $= \frac{1}{\sqrt{C_{2n}^{2k+1}}} \sum_{i=0}^{k|(n-1-k)} \frac{(\sqrt{2})^{2i+1} n!}{(n-k-i-1)!(k-i)!(2i+1)!} [A_{\{a_1 \dots a_{n-k-i-1}\}} [B_{a_{n-k-i}} \dots B_{a_{n-2i-1}}] [C_{a_{n-2i}} \dots C_{a_n}]]$

定理3.2.3.  $\varepsilon_{\{a_1 a_2 \dots a_n\}}(\vec{p}, n-2k) \varepsilon_{\{b_1 b_2 \dots b_n\}}(\vec{p}, n-2(n-k))$   
 $= \frac{1}{C_{2n}^{2k}} \sum_{i,j=0}^{k|(n-k)} \frac{2^{i+j} n!n!}{(n-k-i)!(n-k-j)!(k-i)!(k-j)!(2i)!(2j)!}$   
 $\{ [A_{\{a_1 \dots a_{n-k-i}\}} [B_{a_{n+1-k-i}} \dots B_{a_{n-2i}}] [C_{a_{n+1-2i}} \dots C_{a_n}]] \} \{ [B_{\{b_1 \dots b_{n-k-j}\}} [A_{b_{n+1-k-j}} \dots A_{b_{n-2j}}] [C_{b_{n+1-2j}} \dots C_{b_n}]] \}$

定理3.2.4.  $\varepsilon_{\{a_1 a_2 \dots a_n\}}(\vec{p}, n-2k-1) \varepsilon_{\{b_1 b_2 \dots b_n\}}(\vec{p}, n-2(n-1-k)-1)$   
 $= \frac{1}{C_{2n}^{2k+1}} \sum_{i,j=0}^{k|(n-1-k)} \frac{2^{i+j+1} n!n!}{(n-k-i-1)!(n-k-j-1)!(k-i)!(k-j)!(2i+1)!(2j+1)!}$   
 $\{ [A_{\{a_1 \dots a_{n-k-i-1}\}} [B_{a_{n-k-i}} \dots B_{a_{n-2i-1}}] [C_{a_{n-2i}} \dots C_{a_n}]] \} \{ [B_{\{b_1 \dots b_{n-k-j-1}\}} [A_{b_{n-k-j}} \dots A_{b_{n-2j-1}}] [C_{b_{n-2j}} \dots C_{b_n}]] \}$

定理3.2.5.  $\sum_{h=n}^{-n} (-1)^h \varepsilon_{\{a_1 a_2 \dots a_n\}}(\vec{p}, h) \varepsilon_{\{b_1 b_2 \dots b_n\}}(\vec{p}, -h)$   
 $= \sum_{k=0}^n \frac{(-1)^k}{C_{2n}^{2k}} \sum_{i,j=0}^{k|(n-k)} \frac{2^{i+j} n!n!}{(n-k-i)!(n-k-j)!(k-i)!(k-j)!(2i)!(2j)!}$   
 $\{ [A_{\{a_1 \dots a_{n-k-i}\}} [B_{a_{n+1-k-i}} \dots B_{a_{n-2i}}] [C_{a_{n+1-2i}} \dots C_{a_n}]] \} \{ [B_{\{b_1 \dots b_{n-k-j}\}} [A_{b_{n+1-k-j}} \dots A_{b_{n-2j}}] [C_{b_{n+1-2j}} \dots C_{b_n}]] \}$   
 $- \sum_{k=0}^{n-1} \frac{(-1)^k}{C_{2n}^{2k+1}} \sum_{i,j=0}^{k|(n-1-k)} \frac{2^{i+j+1} n!n!}{(n-k-i-1)!(n-k-j-1)!(k-i)!(k-j)!(2i+1)!(2j+1)!}$   
 $\{ [A_{\{a_1 \dots a_{n-k-i-1}\}} [B_{a_{n-k-i}} \dots B_{a_{n-2i-1}}] [C_{a_{n-2i}} \dots C_{a_n}]] \} \{ [B_{\{b_1 \dots b_{n-k-j-1}\}} [A_{b_{n-k-j}} \dots A_{b_{n-2j-1}}] [C_{b_{n-2j}} \dots C_{b_n}]] \}$

以上这些定理是我本人通过归纳试探提出来的，并已经在前面章节严格证明了它们。

### 3.3 自旋-n粒子Klein-Gordon方程投影算子的重要猜想

猜想3.3.1.  $\sum_{r=0}^{[n/2]} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$   
 $\{ [C_{\{a_1\}} C_{a_2} - A_{\{a_1\}} B_{a_2} - B_{\{a_1\}} A_{a_2}] [C_{\{b_1\}} C_{b_2} - A_{\{b_1\}} B_{b_2} - B_{\{b_1\}} A_{b_2}] \cdot$

$$\begin{aligned}
& [C_{a_{2r-1}}C_{a_{2r}} - A_{a_{2r-1}}B_{a_{2r}} - B_{a_{2r-1}}A_{a_{2r}}][C_{b_{2r-1}}C_{b_{2r}} - A_{b_{2r-1}}B_{b_{2r}} - B_{b_{2r-1}}A_{b_{2r}}] \\
& [C_{a_{2r+1}}C_{b_{2r+1}} - A_{a_{2r+1}}B_{b_{2r+1}} - B_{a_{2r+1}}A_{b_{2r+1}}] \cdot [C_{a_n}C_{b_n} - A_{a_n}B_{b_n} - B_{a_n}A_{b_n}] \\
& = \sum_{k=0}^n \frac{(-1)^n}{C_{2n}^{2k}} \sum_{i,j=0}^{k|(n-k)} \frac{2^{i+j}n!n!}{(n-k-i)!(n-k-j)!(k-i)!(k-j)!(2i)!(2j)!} \\
& \{[A_{\{a_1 \cdots A_{a_{n-k-i}}\}}][B_{a_{n+1-k-i}} \cdots B_{a_{n-2i}}][C_{a_{n+1-2i}} \cdots C_{a_n}]\} \{[B_{\{b_1 \cdots B_{b_{n-k-j}}\}}][A_{b_{n+1-k-j}} \cdots A_{b_{n-2j}}][C_{b_{n+1-2j}} \cdots C_{b_n}]\} \\
& - \sum_{k=0}^{n-1} \frac{(-1)^n}{C_{2n}^{2k+1}} \sum_{i,j=0}^{k|(n-1-k)} \frac{2^{i+j+1}n!n!}{(n-k-i-1)!(n-k-j-1)!(k-i)!(k-j)!(2i+1)!(2j+1)!} \\
& \{[A_{\{a_1 \cdots A_{a_{n-k-i-1}}\}}][B_{a_{n-k-i}} \cdots B_{a_{n-2i-1}}][C_{a_{n-2i}} \cdots C_{a_n}]\} \{[B_{\{b_1 \cdots B_{b_{n-k-j-1}}\}}][A_{b_{n-k-j}} \cdots A_{b_{n-2j-1}}][C_{b_{n-2j}} \cdots C_{b_n}]\}
\end{aligned}$$

以上猜想是从Behrends和Fronsdal构造出来的公式(猜想)<sup>[52, 53]</sup>和我本人上节提出的定理相结合而得到的,它是后面很多重要结论的前提条件,很多情形下验证都是正确的,目前还没遇到反例。但如果要严格证明它,却需要先证明Behrends和Fronsdal构造出来的公式,目前我还做不到这一点。当然如果此猜想通过其他方法得到证明,那么Behrends和Fronsdal构造出来的公式就可以得到严格的证明。

## 4 从物理到数学抽象: 投影算子猜想

本节A,B,C不再特指,泛指一般对易变量。

### 4.1 从自旋-n粒子Klein-Gordon方程投影算子引出的数学猜想

$$\begin{aligned}
\text{猜想4.1.1. } & \sum_{r=0}^{[n/2]} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} \\
& \{[A_{\{a_1 B_{a_2} + B_{\{a_1} A_{a_2} + C_{\{a_1} C_{a_2}}[A_{\{b_1} B_{b_2} + B_{\{b_1} A_{b_2} + C_{\{b_1} C_{b_2}}] \cdots} \\
& [A_{a_{2r-1}}B_{a_{2r}} + B_{a_{2r-1}}A_{a_{2r}} + C_{a_{2r-1}}C_{a_{2r}}][A_{b_{2r-1}}B_{b_{2r}} + B_{b_{2r-1}}A_{b_{2r}} + C_{b_{2r-1}}C_{b_{2r}}] \\
& [A_{a_{2r+1}}B_{b_{2r+1}} + B_{a_{2r+1}}A_{b_{2r+1}} + C_{a_{2r+1}}C_{b_{2r+1}}] \cdot [A_{a_n}B_{b_n} + B_{a_n}A_{b_n} + C_{a_n}C_{b_n}]\} \\
& = \sum_{k=0}^n \frac{1}{C_{2n}^{2k}} \sum_{i,j=0}^{k|(n-k)} \frac{(-2)^{i+j}n!n!}{(n-k-i)!(n-k-j)!(k-i)!(k-j)!(2i)!(2j)!} \\
& \{[A_{\{a_1 \cdots A_{a_{n-k-i}}\}}][B_{a_{n+1-k-i}} \cdots B_{a_{n-2i}}][C_{a_{n+1-2i}} \cdots C_{a_n}]\} \{[B_{\{b_1 \cdots B_{b_{n-k-j}}\}}][A_{b_{n+1-k-j}} \cdots A_{b_{n-2j}}][C_{b_{n+1-2j}} \cdots C_{b_n}]\} \\
& - \sum_{k=0}^{n-1} \frac{1}{C_{2n}^{2k+1}} \sum_{i,j=0}^{k|(n-1-k)} \frac{(-2)^{i+j+1}n!n!}{(n-k-i-1)!(n-k-j-1)!(k-i)!(k-j)!(2i+1)!(2j+1)!} \\
& \{[A_{\{a_1 \cdots A_{a_{n-k-i-1}}\}}][B_{a_{n-k-i}} \cdots B_{a_{n-2i-1}}][C_{a_{n-2i}} \cdots C_{a_n}]\} \{[B_{\{b_1 \cdots B_{b_{n-k-i-1}}\}}][A_{b_{n-k-i}} \cdots A_{b_{n-2i-1}}][C_{b_{n-2i}} \cdots C_{b_n}]\}
\end{aligned}$$

以上是更一般的猜想,如果它成立,则上节的猜想自然成立。

推论4.1.1.  $C = 0 \Rightarrow$

$$\begin{aligned}
& \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} \\
& \{[A_{\{a_1 B_{a_2} + B_{\{a_1} A_{a_2}}[A_{\{b_1} B_{b_2} + B_{\{b_1} A_{b_2}}] \cdots [A_{a_{2r-1}}B_{a_{2r}} + B_{a_{2r-1}}A_{a_{2r}}][A_{b_{2r-1}}B_{b_{2r}} + B_{b_{2r-1}}A_{b_{2r}}] \\
& [A_{a_{2r+1}}B_{b_{2r+1}} + B_{a_{2r+1}}A_{b_{2r+1}}] \cdot [A_{a_n}B_{b_n} + B_{a_n}A_{b_n}]\} \\
& = \sum_{k=0}^n \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} [A_{\{a_1 \cdots A_{a_{n-k}} B_{a_{n+1-k}} \cdots B_{a_n}\}}][B_{\{b_1 \cdots B_{b_{n-k}} A_{b_{n+1-k}} \cdots A_{b_n}\}}]
\end{aligned}$$

### 4.2 投影算子猜想得到的新组合学恒等式

$$\text{推论4.2.1. } A = 0, B = 0, C = \pm 1 \Rightarrow \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} = 2^n$$

$$\text{推论4.2.2. } A = \pm 1, B = \pm 1, C = 0 \Rightarrow \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} = \sum_{k=0}^n \frac{1}{2^n} \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} = 2^n$$

$$\text{推论4.2.3. } \sum_{k=0}^n \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} = \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} = 2^{2n}, \sum_{k=0}^n \frac{(2k-1)!!}{(2k)!!} \frac{(2n-2k-1)!!}{(2n-2k)!!} = 1$$

### 4.3 以上组合学恒等式的推论

$$\text{推论4.3.1. } \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} = 2^{2n} \Leftrightarrow \sum_{k=0}^n \frac{(C_n^k)^2}{C_{2n}^{2k}} = \frac{2^{2n}}{C_{2n}^n}$$

$$\text{推论4.3.2. } \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} = 2^{2n} \Leftrightarrow \sum_{k=0}^n \frac{C_{2k}^k C_{2n-2k}^{n-k}}{n-k+1} = \frac{1}{2} C_{2n+2}^{n+1} \Leftrightarrow \sum_{k=0}^n \frac{C_{2k}^k C_{2n-2k}^{n-k}}{k+1} = \frac{1}{2} C_{2n+2}^{n+1}$$

$$\begin{aligned} \text{证明: } & \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} = 2^{2n} \\ \Leftrightarrow & 2^{2n+2} = \sum_{k=0}^{n+1} C_{2k}^k C_{2n+2-2k}^{n+1-k} \\ = & C_{2n+2}^{n+1} C_0^0 + \sum_{k=0}^n C_{2k}^k C_{2n+2-2k}^{n+1-k} \\ = & C_{2n+2}^{n+1} C_0^0 + \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} \frac{(2n+2-2k)(2n+1-2k)}{(n+1-k)^2} \\ = & C_{2n+2}^{n+1} C_0^0 + \sum_{k=0}^n 2C_{2k}^k C_{2n-2k}^{n-k} \left(2 - \frac{1}{n+1-k}\right) \\ = & C_{2n+2}^{n+1} C_0^0 + 2^{2n+2} - \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} \frac{2}{n+1-k} \\ \Leftrightarrow & \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} \frac{2}{n+1-k} = C_{2n+2}^{n+1} \Leftrightarrow \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} \frac{2}{k+1} = C_{2n+2}^{n+1} \\ \Leftrightarrow & \sum_{k=0}^n \frac{C_{2k}^k C_{2n-2k}^{n-k}}{n-k+1} = \frac{1}{2} C_{2n+2}^{n+1} \Leftrightarrow \sum_{k=0}^n \frac{C_{2k}^k C_{2n-2k}^{n-k}}{k+1} = \frac{1}{2} C_{2n+2}^{n+1} \end{aligned}$$

□

$$\text{推论4.3.3. } \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} = 2^{2n} \Leftrightarrow \sum_{k=0}^n \frac{C_{2k}^k C_{2n-2k}^{n-k}}{k+1} = \frac{1}{2} C_{2n+2}^{n+1} \Leftrightarrow \sum_{k=0}^n \frac{C_{2k}^k C_{2n-2k}^{n-k}}{(k+1)(n-k+1)} = \frac{1}{n+2} C_{2n+2}^{n+1}$$

$$\begin{aligned} \text{证明: } & C_{2n+4}^{n+2} = \sum_{k=0}^{n+1} C_{2k}^k C_{2n+2-2k}^{n+1-k} \frac{2}{k+1} \\ = & C_{2n+2}^{n+1} C_0^0 \frac{2}{n+2} + \sum_{k=0}^n C_{2k}^k C_{2n+2-2k}^{n+1-k} \frac{2}{k+1} \\ = & C_{2n+2}^{n+1} C_0^0 \frac{2}{n+2} + \sum_{k=0}^n 2C_{2k}^k C_{2n-2k}^{n-k} \left(2 - \frac{1}{n+1-k}\right) \frac{1}{k+1} \\ = & C_{2n+2}^{n+1} C_0^0 \frac{2}{n+2} + \sum_{k=0}^n 4C_{2k}^k C_{2n-2k}^{n-k} \frac{2}{k+1} - \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} \frac{2}{n+1-k} \frac{2}{k+1} \\ = & C_{2n+2}^{n+1} \frac{2}{n+2} + 4C_{2n+2}^{n+1} - \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} \frac{2}{n+1-k} \frac{2}{k+1} \\ \Leftrightarrow & \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} \frac{2}{n+1-k} \frac{2}{k+1} = C_{2n+2}^{n+1} \frac{2}{n+2} + 4C_{2n+2}^{n+1} - C_{2n+4}^{n+2} \\ \Leftrightarrow & \sum_{k=0}^n C_{2k}^k C_{2n-2k}^{n-k} \frac{2}{k+1} \frac{2}{n-k+1} = \frac{4}{n+2} C_{2n+2}^{n+1} \\ \Leftrightarrow & \sum_{k=0}^n \frac{C_{2k}^k C_{2n-2k}^{n-k}}{(k+1)(n-k+1)} = \frac{1}{n+2} C_{2n+2}^{n+1} \end{aligned}$$

□

$$\begin{aligned} \text{推论4.3.4. } & \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} = 2^n \Leftrightarrow \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n+1-2r)!} = (-1)^{[n/2]} \frac{(n-2[n/2])!}{2^n} C_{2[n/2]+2}^{[n/2]+1} \\ \Leftrightarrow & \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n+2-2r)!} = (-1)^{[n/2]} \frac{(n-2[n/2])!}{2^n} C_{2[n/2]+2}^{[n/2]+1} + (-1)^{[(n+1)/2]} \frac{(n+1-2[(n+1)/2])!}{(2n+2)(2n-2)} C_{2[(n+1)/2]+2}^{[(n+1)/2]+1} \end{aligned}$$

$$\begin{aligned} \text{证明: } & 2^{n+1} = \sum_{r=0}^{[(n+1)/2]} (-1)^r \frac{(2n+2-2r)!}{r!(n+1-r)!(n+1-2r)!} \\ = & \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n+2-2r)!}{r!(n+1-r)!(n+1-2r)!} + \sum_{r=[n/2]+1}^{[(n+1)/2]} (-1)^r \frac{(2n+2-2r)!}{r!(n+1-r)!(n+1-2r)!} \\ = & \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n+2-2r)(2n+1-2r)(2n-2r)!}{r!(n+1-r)(n-r)!(n+1-2r)(n-2r)!} + (-1)^{[n/2]+1} \frac{(n-2[n/2])!(2[n/2]+2)!}{([n/2]+1)!([n/2]+1)!} \\ = & \sum_{r=0}^{[n/2]} (-1)^r \frac{2(2n+1-2r)}{(n+1-2r)} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} + (-1)^{[n/2]+1} \frac{(n-2[n/2])!(2[n/2]+2)!}{([n/2]+1)!([n/2]+1)!} \\ = & \sum_{r=0}^{[n/2]} (-1)^r \left[2 + \frac{2n}{(n+1-2r)}\right] \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} + (-1)^{[n/2]+1} \frac{(n-2[n/2])!(2[n/2]+2)!}{([n/2]+1)!([n/2]+1)!} \end{aligned}$$

$$\begin{aligned}
 &= 2^{n+1} + \sum_{r=0}^{[n/2]} (-1)^r \frac{2n}{(n+1-2r)} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} + (-1)^{[n/2]+1} \frac{(n-2[n/2])(2[n/2]+2)!}{([n/2]+1)!([n/2]+1)!} \\
 &\Leftrightarrow \sum_{r=0}^{[n/2]} (-1)^r \frac{2n}{(n+1-2r)} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} = (-1)^{[n/2]} \frac{(n-2[n/2])(2[n/2]+2)!}{([n/2]+1)!([n/2]+1)!} \\
 &\Leftrightarrow \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n+1-2r)!} = (-1)^{[n/2]} \frac{(n-2[n/2])(2[n/2]+2)!}{2n([n/2]+1)!([n/2]+1)!} \\
 &\Leftrightarrow \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n+1-2r)!} = (-1)^{[n/2]} \frac{(n-2[n/2])}{2n} C_{2[n/2]+2}^{[n/2]+1}
 \end{aligned}$$

□

**证明:**  $(-1)^{[(n+1)/2]} \frac{(n+1-2[(n+1)/2])}{2n+2} C_{2[(n+1)/2]+2}^{[(n+1)/2]+1}$

$$\begin{aligned}
 &= \sum_{r=0}^{[(n+1)/2]} (-1)^r \frac{(2n+2-2r)!}{r!(n+1-r)!(n+2-2r)!} \\
 &= \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n+2-2r)!}{r!(n+1-r)!(n+2-2r)!} + \sum_{r=[n/2]+1}^{[(n+1)/2]} (-1)^r \frac{(2n+2-2r)!}{r!(n+1-r)!(n+2-2r)!} \\
 &= \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n+2-2r)!}{r!(n+1-r)!(n+2-2r)!} + (-1)^{[n/2]+1} (n-2[n/2]) C_{2[n/2]+2}^{[n/2]+1} \\
 &= \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n+2-2r)(2n+1-2r)(2n-2r)!}{r!(n+1-r)(n-r)!(n+2-2r)(n+1-2r)!} + (-1)^{[n/2]+1} (n-2[n/2]) C_{2[n/2]+2}^{[n/2]+1} \\
 &= \sum_{r=0}^{[n/2]} (-1)^r \left[ 2 + \frac{2(n-1)}{(n+2-2r)} \right] \frac{(2n-2r)!}{r!(n-r)!(n+1-2r)!} + (-1)^{[n/2]+1} (n-2[n/2]) C_{2[n/2]+2}^{[n/2]+1} \\
 &= \sum_{r=0}^{[n/2]} (-1)^r \frac{2(n-1)(2n-2r)!}{r!(n-r)!(n+2-2r)!} + (-1)^{[n/2]} \frac{(n-2[n/2])}{n} C_{2[n/2]+2}^{[n/2]+1} + (-1)^{[n/2]+1} (n-2[n/2]) C_{2[n/2]+2}^{[n/2]+1} \\
 &= \sum_{r=0}^{[n/2]} (-1)^r \frac{2(n-1)(2n-2r)!}{r!(n-r)!(n+2-2r)!} - (-1)^{[n/2]} \frac{(n-1)(n-2[n/2])}{n} C_{2[n/2]+2}^{[n/2]+1} \\
 &\Leftrightarrow \sum_{r=0}^{[n/2]} (-1)^r \frac{2(n-1)(2n-2r)!}{r!(n-r)!(n+2-2r)!} = (-1)^{[n/2]} \frac{(n-1)(n-2[n/2])}{n} C_{2[n/2]+2}^{[n/2]+1} + (-1)^{[(n+1)/2]} \frac{(n+1-2[(n+1)/2])}{2n+2} C_{2[(n+1)/2]+2}^{[(n+1)/2]+1} \\
 &\Leftrightarrow \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n+2-2r)!} = (-1)^{[n/2]} \frac{(n-2[n/2])}{2n} C_{2[n/2]+2}^{[n/2]+1} + (-1)^{[(n+1)/2]} \frac{(n+1-2[(n+1)/2])}{(2n+2)(2n-2)} C_{2[(n+1)/2]+2}^{[(n+1)/2]+1}
 \end{aligned}$$

□

#### 4.4 投影算子猜想的C(2i,2j)项情形

**定理4.4.1.**  $\sum_{r=0}^{[n/2]} \sum_{l \geq 0, i|j-r}^{i,j,n/2-r} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$

$$\begin{aligned}
 &C_r^{i-l} \{ [C_{a_1} C_{a_2}] \cdot [C_{a_{2i-2l-1}} C_{a_{2i-2l}}] [A_{a_{2i-2l+1}} B_{a_{2i-2l+2}} + B_{a_{2i-2l+1}} A_{a_{2i-2l+2}}] \cdot [A_{a_{2r-1}} B_{a_{2r}} + B_{a_{2r-1}} A_{a_{2r}}] \\
 &C_r^{j-l} [C_{b_1} C_{b_2}] \cdot [C_{b_{2j-2l-1}} C_{b_{2j-2l}}] [A_{b_{2j-2l+1}} B_{b_{2j-2l+2}} + B_{b_{2j-2l+1}} A_{b_{2j-2l+2}}] \cdot [A_{b_{2r-1}} B_{b_{2r}} + B_{b_{2r-1}} A_{b_{2r}}] \} \\
 &C_{n-2r}^{2l} [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot [A_{a_{n-2l}} B_{b_{n-2l}} + B_{a_{n-2l}} A_{b_{n-2l}}] [C_{a_{n-2l+1}} C_{b_{n-2l+1}}] \cdot [C_{a_n} C_{b_n}] \\
 &= \frac{n!n!}{(2n)!} \sum_{k \geq i|j}^{\leq n-i|j} \frac{(-2)^{i+j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} \\
 &\{ [A_{a_1} \cdots A_{a_{n-k-i}}] [B_{a_{n+1-k-i}} \cdots B_{a_{n-2i}}] [C_{a_{n+1-2i}} \cdots C_{a_n}] \} \{ [B_{b_1} \cdots B_{b_{n-k-j}}] [A_{b_{n+1-k-j}} \cdots A_{b_{n-2j}}] [C_{b_{n+1-2j}} \cdots C_{b_n}] \}
 \end{aligned}$$

以上是关于 $(AB + BA)$ ,  $CC$ 型的二项式展开后分母含 $(2i)!(2j)!$ 项的恒等式。

**推论4.4.1.**  $\sum_{r=0}^{[n/2]} \sum_{l \geq 0, i|j-r}^{i,j,n/2-r} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} 2^{r+l-i} C_r^{j-l} 2^{r+l-j} C_{n-2r}^{2l}$

$$\begin{aligned}
 &[C_{a_1} C_{a_2}] \cdot [C_{a_{2i-2l-1}} C_{a_{2i-2l}}] [A_{a_{2i-2l+1}} B_{a_{2i-2l+2}}] \cdot [A_{a_{2r-1}} B_{a_{2r}}] \\
 &[C_{b_1} C_{b_2}] \cdot [C_{b_{2j-2l-1}} C_{b_{2j-2l}}] [A_{b_{2j-2l+1}} B_{b_{2j-2l+2}}] \cdot [A_{b_{2r-1}} B_{b_{2r}}] \\
 &[A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot [A_{a_{n-2l}} B_{b_{n-2l}} + B_{a_{n-2l}} A_{b_{n-2l}}] [C_{a_{n-2l+1}} C_{b_{n-2l+1}}] \cdot [C_{a_n} C_{b_n}] \\
 &= \frac{n!n!}{(2n)!} \sum_{k \geq i|j}^{\leq n-i|j} \frac{(-2)^{i+j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} \\
 &\{ [A_{a_1} \cdots A_{a_{n-k-i}}] [B_{a_{n+1-k-i}} \cdots B_{a_{n-2i}}] [C_{a_{n+1-2i}} \cdots C_{a_n}] \} \{ [B_{b_1} \cdots B_{b_{n-k-j}}] [A_{b_{n+1-k-j}} \cdots A_{b_{n-2j}}] [C_{b_{n+1-2j}} \cdots C_{b_n}] \}
 \end{aligned}$$

**推论4.4.2.**  $A = 1, B = 1, C = 1 \Rightarrow$

$$\sum_{r=0}^{[n/2]} \sum_{l \geq 0, i|j-r}^{i,j,n/2-r} (-1)^{r+i+j} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} C_r^{j-l} C_{n-2r}^{2l} = \sum_{k \geq i|j}^{\leq n-i|j} \frac{2^{2i+2j-n}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$$

$$\begin{aligned}
& \text{推论4.4.3. } \sum_{r=0}^{[n/2]} \left[ \sum_{l \geq 0, i|j-r}^{\leq i, j, k-r, n-r-k} \right] (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} 2^{r+l-i} C_r^{j-l} 2^{r+l-j} C_{n-2r}^{2l} C_{n-2r-2l}^{k-r-l} \\
& [C_{\{a_1 C_{a_2}\}} \cdot [C_{a_{2i-2l-1}} C_{a_{2i-2l}}] [A_{a_{2i-2l+1}} B_{a_{2i-2l+2}}] \cdot [A_{a_{2r-1}} B_{a_{2r}}] \\
& [C_{\{b_1 C_{b_2}\}} \cdot [C_{b_{2j-2l-1}} C_{b_{2j-2l}}] [A_{b_{2j-2l+1}} B_{b_{2j-2l+2}}] \cdot [A_{b_{2r-1}} B_{b_{2r}}] \\
& [A_{a_{2r+1}} B_{b_{2r+1}}] \cdot [A_{a_{n+r-l-k}} B_{b_{n+r-l-k-1}}] [B_{a_{n+r-l-k+1}} A_{b_{n+r-l-k+1}}] \cdot [B_{a_{n-2l}} A_{b_{n-2l}}] [C_{a_{n-2l+1}} C_{b_{n-2l+1}}] \cdot [C_{a_n} C_{b_n}] \\
& = \frac{n!n!}{(2n)!} \sum_{k=0}^n \frac{(-2)^{i+j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} \\
& \{[A_{\{a_1 \cdots A_{a_{n-k-i}}\}} [B_{a_{n+1-k-i}} \cdots B_{a_{n-2i}}] [C_{a_{n+1-2i}} \cdots C_{a_n}]]\} \{[B_{\{b_1 \cdots B_{b_{n-k-j}}\}} [A_{b_{n+1-k-j}} \cdots A_{b_{n-2j}}] [C_{b_{n+1-2j}} \cdots C_{b_n}]]\} \\
& \Leftrightarrow \sum_{r=0}^{[n/2]} \left[ \sum_{l \geq 0, i|j-r}^{\leq i, j, k-r, n-r-k} \right] (-1)^{r+i+j} 2^{2r+2l} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} C_r^{j-l} C_{n-2r}^{2l} C_{n-2r-2l}^{k-r-l} = \frac{(-2)^{2i+2j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}
\end{aligned}$$

以上最后一式是关于第k项的恒等式。

#### 4.5 投影算子猜想的C(2i+1, 2j+1)项情形

$$\begin{aligned}
& \text{定理4.5.1. } \sum_{r=0}^{[n/2]} \sum_{l \geq 0, i|j-r}^{\leq i, j, (n-1)/2-r} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} \\
& C_r^{i-l} \{[C_{\{a_1 C_{a_2}\}} \cdot [C_{a_{2i-2l-1}} C_{a_{2i-2l}}] [A_{a_{2i-2l+1}} B_{a_{2i-2l+2}} + B_{a_{2i-2l+1}} A_{a_{2i-2l+2}}] \cdot [A_{a_{2r-1}} B_{a_{2r}} + B_{a_{2r-1}} A_{a_{2r}}] \\
& C_r^{j-l} [C_{\{b_1 C_{b_2}\}} \cdot [C_{b_{2j-2l-1}} C_{b_{2j-2l}}] [A_{b_{2j-2l+1}} B_{b_{2j-2l+2}} + B_{b_{2j-2l+1}} A_{b_{2j-2l+2}}] \cdot [A_{b_{2r-1}} B_{b_{2r}} + B_{b_{2r-1}} A_{b_{2r}}]\} \\
& C_{n-2r}^{2l+1} [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot [A_{a_{n-2l-1}} B_{b_{n-2l-1}} + B_{a_{n-2l-1}} A_{b_{n-2l-1}}] [C_{a_{n-2l}} C_{b_{n-2l}}] \cdot [C_{a_n} C_{b_n}] \\
& = - \sum_{k \geq i|j}^{\leq n-1-i|j} \frac{n!n!}{(2n)!} \frac{(-2)^{i+j+1}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!} \\
& \{[A_{\{a_1 \cdots A_{a_{n-k-i-1}}\}} [B_{a_{n-k-i}} \cdots B_{a_{n-2i-1}}] [C_{a_{n-2i}} \cdots C_{a_n}]]\} \{[B_{\{b_1 \cdots B_{b_{n-k-j-1}}\}} [A_{b_{n-k-j}} \cdots A_{b_{n-2j-1}}] [C_{b_{n-2j}} \cdots C_{b_n}]]\}
\end{aligned}$$

以上是关于(AB + BA), CC型的二项式展开后分母含(2i + 1)!(2j + 1)!项的恒等式。

$$\begin{aligned}
& \text{推论4.5.1. } \sum_{r=0}^{[n/2]} \sum_{l \geq 0, i|j-r}^{\leq i, j, (n-1)/2-r} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} 2^{r+l-i} C_r^{j-l} 2^{r+l-j} C_{n-2r}^{2l+1} \\
& [C_{\{a_1 C_{a_2}\}} \cdot [C_{a_{2i-2l-1}} C_{a_{2i-2l}}] [A_{a_{2i-2l+1}} B_{a_{2i-2l+2}}] \cdot [A_{a_{2r-1}} B_{a_{2r}}] \\
& [C_{\{b_1 C_{b_2}\}} \cdot [C_{b_{2j-2l-1}} C_{b_{2j-2l}}] [A_{b_{2j-2l+1}} B_{b_{2j-2l+2}}] \cdot [A_{b_{2r-1}} B_{b_{2r}}] \\
& [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot [A_{a_{n-2l-1}} B_{b_{n-2l-1}} + B_{a_{n-2l-1}} A_{b_{n-2l-1}}] [C_{a_{n-2l}} C_{b_{n-2l}}] \cdot [C_{a_n} C_{b_n}] \\
& = - \sum_{k \geq i|j}^{\leq n-1-i|j} \frac{n!n!}{(2n)!} \frac{(-2)^{i+j+1}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!} \\
& \{[A_{\{a_1 \cdots A_{a_{n-k-i-1}}\}} [B_{a_{n-k-i}} \cdots B_{a_{n-2i-1}}] [C_{a_{n-2i}} \cdots C_{a_n}]]\} \{[B_{\{b_1 \cdots B_{b_{n-k-j-1}}\}} [A_{b_{n-k-j}} \cdots A_{b_{n-2j-1}}] [C_{b_{n-2j}} \cdots C_{b_n}]]\}
\end{aligned}$$

推论4.5.2. A = 1, B = 1, C = 1 ⇒

$$\sum_{r=0}^{[n/2]} \sum_{l \geq 0, i|j-r}^{\leq i, j, (n-1)/2-r} (-1)^{r+i+j} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} C_r^{j-l} C_{n-2r}^{2l+1} = \sum_{k \geq i|j}^{\leq n-1-i|j} \frac{2^{2i+2j+2-n}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}$$

$$\begin{aligned}
& \text{推论4.5.3. } \sum_{r=0}^{[n/2]} \left[ \sum_{l \geq 0, i|j-r}^{\leq i, j, k-r, n-1-r-k} \right] (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} 2^{r+l-i} C_r^{j-l} 2^{r+l-j} C_{n-2r}^{2l+1} C_{n-2r-2l-1}^{k-r-l} \\
& [C_{\{a_1 C_{a_2}\}} \cdot [C_{a_{2i-2l-1}} C_{a_{2i-2l}}] [A_{a_{2i-2l+1}} B_{a_{2i-2l+2}}] \cdot [A_{a_{2r-1}} B_{a_{2r}}] \\
& [C_{\{b_1 C_{b_2}\}} \cdot [C_{b_{2j-2l-1}} C_{b_{2j-2l}}] [A_{b_{2j-2l+1}} B_{b_{2j-2l+2}}] \cdot [A_{b_{2r-1}} B_{b_{2r}}] \\
& [A_{a_{2r+1}} B_{b_{2r+1}}] \cdot [A_{a_{n+r-l-k-1}} B_{b_{n+r-l-k-1}}] [B_{a_{n+r-l-k}} A_{b_{n+r-l-k}}] \cdot [B_{a_{n-2l-1}} A_{b_{n-2l-1}}] [C_{a_{n-2l}} C_{b_{n-2l}}] \cdot [C_{a_n} C_{b_n}] \\
& = - \sum_{k=0}^{n-1} \frac{n!n!}{(2n)!} \frac{(-2)^{i+j+1}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!} \\
& \{[A_{\{a_1 \cdots A_{a_{n-k-i-1}}\}} [B_{a_{n-k-i}} \cdots B_{a_{n-2i-1}}] [C_{a_{n-2i}} \cdots C_{a_n}]]\} \{[B_{\{b_1 \cdots B_{b_{n-k-j-1}}\}} [A_{b_{n-k-j}} \cdots A_{b_{n-2j-1}}] [C_{b_{n-2j}} \cdots C_{b_n}]]\} \\
& \Leftrightarrow \sum_{r=0}^{[n/2]} \left[ \sum_{l \geq 0, i|j-r}^{\leq i, j, k-r, n-1-r-k} \right] (-1)^{r+i+j} 2^{2r+2l} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} C_r^{j-l} C_{n-2r}^{2l+1} C_{n-2r-2l-1}^{k-r-l} \\
& = - \frac{(-2)^{2i+2j+1}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}
\end{aligned}$$

以上最后一式是关于第k项的恒等式。

### 4.6 与投影算子猜想等价的组合学恒等式

**定理4.6.1.** 
$$\sum_{r=0}^{[n/2]} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$$

$$\{[C_{\{a_1} C_{a_2} - A_{\{a_1} B_{a_2} - B_{\{a_1} A_{a_2}}][C_{(b_1} C_{b_2} - A_{(b_1} B_{b_2} - B_{(b_1} A_{b_2})] \cdot \cdot$$

$$[C_{a_{2r-1}} C_{a_{2r}} - A_{a_{2r-1}} B_{a_{2r}} - B_{a_{2r-1}} A_{a_{2r}}][C_{b_{2r-1}} C_{b_{2r}} - A_{b_{2r-1}} B_{b_{2r}} - B_{b_{2r-1}} A_{b_{2r}}]\}$$

$$[C_{a_{2r+1}} C_{b_{2r+1}} - A_{a_{2r+1}} B_{b_{2r+1}} - B_{a_{2r+1}} A_{b_{2r+1}}] \cdot \cdot [C_{a_n} C_{b_n} - A_{a_n} B_{b_n} - B_{a_n} A_{b_n}]\}$$

$$= \sum_{k=0}^n \frac{(-1)^n}{C_{2n}^{2k}} \sum_{i,j=0}^{k \binom{n-k}{i,j}} \frac{2^{i+j} n!n!}{(n-k-i)!(n-k-j)!(k-i)!(k-j)!(2i)!(2j)!}$$

$$\{[A_{\{a_1} \cdot \cdot A_{a_{n-k-i}}][B_{a_{n+1-k-i}} \cdot \cdot B_{a_{n-2i}}][C_{a_{n+1-2i}} \cdot \cdot C_{a_n}]\}\{[B_{(b_1} \cdot \cdot B_{b_{n-k-j}}][A_{b_{n+1-k-j}} \cdot \cdot A_{b_{n-2j}}][C_{b_{n+1-2j}} \cdot \cdot C_{b_n}]\}\}$$

$$- \sum_{k=0}^{n-1} \frac{(-1)^n}{C_{2n}^{2k+1}} \sum_{i,j=0}^{k \binom{n-1-k}{i,j}} \frac{2^{i+j+1} n!n!}{(n-k-i-1)!(n-k-j-1)!(k-i)!(k-j)!(2i+1)!(2j+1)!}$$

$$\{[A_{\{a_1} \cdot \cdot A_{a_{n-k-i-1}}][B_{a_{n-k-i}} \cdot \cdot B_{a_{n-2i-1}}][C_{a_{n-2i}} \cdot \cdot C_{a_n}]\}\{[B_{(b_1} \cdot \cdot B_{b_{n-k-j-1}}][A_{b_{n-k-j}} \cdot \cdot A_{b_{n-2j-1}}][C_{b_{n-2j}} \cdot \cdot C_{b_n}]\}\}$$

$$\Leftrightarrow \begin{cases} \sum_{r=0}^{[n/2]} \left[ \sum_{l \geq 0, i|j-r}^{\leq i,j,k-r,n-r-k} \right] (-1)^{r+i+j} 2^{2r+2l} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} C_r^{j-l} C_{n-2r}^{2l} C_{n-2r-2l}^{k-r-l} \\ = \frac{2^{2i+2j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} \\ \sum_{r=0}^{[n/2]} \left[ \sum_{l \geq 0, i|j-r}^{\leq i,j,k-r,n-1-r-k} \right] (-1)^{r+i+j} 2^{2r+2l} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} C_r^{j-l} C_{n-2r}^{2l+1} C_{n-2r-2l-1}^{k-r-l} \\ = \frac{2^{2i+2j+1}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!} \end{cases}$$

如果以上两个组合学恒等式成立，则投影算子猜想和前节的重要猜想都自然成立。

### 4.7 投影算子组合学恒等式的等价分析

**推论4.7.1.**

$$\begin{cases} \sum_{r=0}^{[n/2]} \left[ \sum_{l \geq 0, i|j-r}^{\leq i,j,k-r,n-r-k} \right] (-1)^{r+i+j} \frac{2^{2r+2l} (2n-2r)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l)!} \frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} \\ = \frac{2^{2i+2j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} \\ \sum_{r=0}^{[n/2]} \left[ \sum_{l \geq 0, i|j-r}^{\leq i,j,k-r,n-1-r-k} \right] (-1)^{r+i+j} \frac{2^{2r+2l} (2n-2r)!}{(n-r)!(n-k-r-l-1)!(k-r-l)!(2l+1)!} \frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} \\ = \frac{2^{2i+2j+1}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!} \end{cases}$$

$$\Leftrightarrow \begin{cases} \sum_{r=0}^{k \binom{n-k}{i,j}} \left[ \sum_{l \geq 0, i|j-r}^{\leq i,j,k-r,n-r-k} \right] (-4)^r \frac{4^l (2n-2r)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l)!} \frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} \\ = \frac{(-4)^{i+j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} \\ \sum_{r=0}^{k \binom{n-k}{i,j}} \left[ \sum_{l \geq 0, i|j-r}^{\leq i,j,k-r,n-r-k} \right] (-4)^r \frac{4^l (2n-2r+1)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l+1)!} \frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} \\ = \frac{(-4)^{i+j}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k+1)!}{(n-k-i)!(n-k-j)!} \end{cases}$$

如果以上两个组合学恒等式成立，则投影算子猜想和前节的重要猜想都自然成立。这样猜想明显得到了极大简化，下一步就是要努力证明以上两个组合学恒等式，等我有时间再来做这件事。

### 4.8 投影算子组合学恒等式的严格证明???

**定理4.8.1.**

$$\sum_{r=0}^{k \binom{n-k}{i,j}} \left[ \sum_{l \geq 0, (i|j)-r}^{\leq i,j,k-r,n-r-k} \right] \frac{(-4)^r 4^l (2n-2r)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l)!} \frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} = \frac{(-4)^{i+j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$$

**证明:**

$$\sum_{r=0}^{k \binom{n-k}{i,j}} \sum_{l=0}^{i|j} (-4)^r 4^l \frac{(2n-2r)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l)!} \frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} = \frac{(-4)^{i+j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} \quad \square$$

**定理4.8.2.**

$$\sum_{r=0}^{k \binom{n-k}{i,j}} \left[ \sum_{l \geq 0, (i|j)-r}^{\leq i,j,k-r,n-r-k} \right] \frac{(-4)^r 4^l (2n-2r+1)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l+1)!} \frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} = \frac{(-4)^{i+j}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k+1)!}{(n-k-i)!(n-k-j)!}$$



证明:

$$\sum_{r=0}^{k|(n-k)} \sum_{i|j} (-4)^r 4^l \frac{(2n-2r+1)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l+1)!} \frac{r!}{(i-l)!(j-l)!(r+l-i)!(r+l-j)!} = \frac{(-4)^{i+j}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k+1)!}{(n-k-i)!(n-k-j)!} \quad \square$$

### 4.9 投影算子组合学恒等式特例的试探

证明:  $\sum_{k=0}^n \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k+1)!}{(n-k-i)!(n-k-j)!} = \frac{(2n+2)!}{(n-2i)!(n-2j)!} \sum_{k=0}^n \frac{C_{n-2i}^{k-i} C_{n-2j}^{k-j}}{C_{2n+2}^{2k}}$   
 $= \sum_{k=0}^n (2k+1)(2n-2k+1) \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} = ? \quad \square$

证明:  $\sum_{k=0}^n \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!} = \frac{(2n)!}{(n-2i)!(n-2j)!} \sum_{k=0}^n \frac{C_{n-2i}^{k-i} C_{n-2j}^{k-j}}{C_{2n}^{2k}} \quad \square$

证明:  $\sum_{k=0}^n \frac{(2k)!}{(k-i)!(k-i)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-i)!} = \frac{(2n)!}{(n-2i)!(n-2i)!} \sum_{k=0}^n \frac{(C_{n-2i}^{k-i})^2}{C_{2n}^{2k}} \quad \square$

证明:  $\sum_{k=0}^{2l} \frac{(2k)!}{(k-i)!(k-i)!} \frac{(4l-2k)!}{(2l-k-i)!(2l-k-i)!} = ? \quad \square$

证明:  $\sum_{k=0}^{2l} \frac{(2k)!}{(k-l)!(k-l)!} \frac{(4l-2k)!}{(2l-k-l)!(2l-k-l)!}$   
 $= \sum_{k=0}^{2l} \frac{(2k)!}{(k-l)!(k-l)!} \frac{(4l-2k)!}{(l-k)!(l-k)!} = \sum_{k=0}^{2l} \frac{(C_{4l}^{k-l})^2 (4l)!}{C_{4l}^{2k} 0!0!}$   
 $= \frac{(2l)!}{(l-l)!(l-l)!} \frac{(4l-2l)!}{(l-l)!(l-l)!} = [(2l)!]^2 \quad \square$

证明:  $\sum_{k=0}^{2l} \frac{(2k)!}{(k-l+1)!(k-l+1)!} \frac{(4l-2k)!}{(2l-k-l+1)!(2l-k-l+1)!}$   
 $= \sum_{k=0}^{2l} \frac{(2k)!}{(k-l+1)!(k-l+1)!} \frac{(4l-2k)!}{(l+1-k)!(l+1-k)!} = \sum_{k=0}^{2l} \frac{(C_{4l}^{k-l+1})^2 (4l)!}{C_{4l}^{2k} 2!2!}$   
 $= \frac{(2l-2)!}{0!0!} \frac{(2l+2)!}{2!2!} + \frac{(2l)!}{1!1!} \frac{(2l)!}{1!1!} + \frac{(2l+2)!}{2!2!} \frac{(2l-2)!}{0!0!} \quad \square$

证明:  $\sum_{k=0}^{2l} \frac{(2k)!}{(k-l+2)!(k-l+2)!} \frac{(4l-2k)!}{(2l-k-l+2)!(2l-k-l+2)!}$   
 $= \sum_{k=0}^{2l} \frac{(2k)!}{(k-l+2)!(k-l+2)!} \frac{(4l-2k)!}{(l+2-k)!(l+2-k)!} = \sum_{k=0}^{2l} \frac{(C_{4l}^{k-l+2})^2 (4l)!}{C_{4l}^{2k} 4!4!}$   
 $= \frac{(2l-4)!}{0!0!} \frac{(2l+4)!}{4!4!} + \frac{(2l-2)!}{1!1!} \frac{(2l+2)!}{3!3!} + \frac{(2l)!}{2!2!} \frac{(2l)!}{2!2!} + \frac{(2l+2)!}{3!3!} \frac{(2l-2)!}{1!1!} + \frac{(2l+4)!}{4!4!} \frac{(2l-4)!}{0!0!} \quad \square$

## 5 投影算子猜想具体情形表述和验证

### 5.1 取值范围分析

推论5.1.1.

$$\sum_{r=0}^{[n/2]} \left[ \sum_{\substack{i,j,k-r,n-r-k \\ l \geq 0, i|j-r}} (-1)^{r+i+j} 2^{2r+2l} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} C_r^{j-l} C_{n-2r}^{2l} C_{n-2r-2l}^{k-r-l} \right] = \frac{2^{2i+2j}}{(2i)!(2j)!} \frac{(2k)!}{(k-i)!(k-j)!} \frac{(2n-2k)!}{(n-k-i)!(n-k-j)!}$$

推论5.1.2.  $0, i|j-r \leq l \leq i, j, k-r, n-r-k \Rightarrow |i-j| \leq r \leq k|(n-k), i|j \leq k \leq n-i|j$

推论5.1.3.  $\sum_{r=0}^{[n/2]} \left[ \sum_{\substack{i,j,k-r,n-1-r-k \\ l \geq 0, i|j-r}} (-1)^{r+i+j} 2^{2r+2l} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} C_r^{i-l} C_r^{j-l} C_{n-2r}^{2l+1} C_{n-2r-2l-1}^{k-r-l} \right]$   
 $= \frac{2^{2i+2j+1}}{(2i+1)!(2j+1)!} \frac{(2k+1)!}{(k-i)!(k-j)!} \frac{(2n-2k-1)!}{(n-k-i-1)!(n-k-j-1)!}$

推论5.1.4.  $0, i|j-r \leq l \leq i, j, k-r, n-1-r-k \Rightarrow |i-j| \leq r \leq k|(n-1-k), i|j \leq k \leq n-1-i|j$

## 5.2 $i=0, j=0$ 情形

### 5.2.1 $C(2i, 2j)=(0, 0)$ 项情形

推论5.2.1.  $i = 0, j = 0 \Rightarrow l = 0, r \leq k | (n - k), 0 \leq k \leq n$

$$\text{推论5.2.2. } \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!}$$

$$\begin{aligned} \text{推论5.2.3. } & \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} \\ & \{ [A_{a_1} B_{a_2} + B_{a_1} A_{a_2}] [A_{b_1} B_{b_2} + B_{b_1} A_{b_2}] \cdot [A_{a_{2r-1}} B_{a_{2r}} + B_{a_{2r-1}} A_{a_{2r}}] [A_{b_{2r-1}} B_{b_{2r}} + B_{b_{2r-1}} A_{b_{2r}}] \} \\ & [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot [A_{a_n} B_{b_n} + B_{a_n} A_{b_n}] \\ & = \sum_{k=0}^n \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} [A_{a_1} \cdots A_{a_{n-k}} B_{a_{n+1-k}} \cdots B_{a_n}] [B_{b_1} \cdots B_{b_{n-k}} A_{b_{n+1-k}} \cdots A_{b_n}] \\ \Leftrightarrow & \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} \end{aligned}$$

$$\text{推论5.2.4. } \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)! 2^n}{r!(n-r)!(n-2r)!} = \sum_{k=0}^n \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!} = 2^{2n}$$

$$\begin{aligned} \text{证明: } & \sum_{r=0}^{(k+1)|(n-k-1)} (-4)^r \frac{(2n-2r)!}{r!(n-r)!(k+1-r)!(n-k-1-r)!} \\ & = \sum_{r=0}^{(k+1)|(n-k-1)} (-4)^r \frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} \frac{n-k-r}{k+1-r} \end{aligned}$$

□

### 5.2.2 $C(2i+1, 2j+1)=(1, 1)$ 项情形

推论5.2.5.  $i = 0, j = 0 \Rightarrow l = 0, r \leq k | (n - 1 - k), 0 \leq k \leq n - 1$

$$\text{推论5.2.6. } \sum_{r=0}^{k|(n-1-k)} (-4)^r \frac{(2n-2r-1)!}{r!(n-1-r)!(k-r)!(n-1-k-r)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k-1)!}{(n-1-k)!(n-1-k)!}$$

$$\begin{aligned} \text{定理5.2.1. } & \sum_{r=0}^{[n/2]} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} \\ & \{ [A_{a_1} B_{a_2} + B_{a_1} A_{a_2}] [A_{b_1} B_{b_2} + B_{b_1} A_{b_2}] \cdot [A_{a_{2r-1}} B_{a_{2r}} + B_{a_{2r-1}} A_{a_{2r}}] [A_{b_{2r-1}} B_{b_{2r}} + B_{b_{2r-1}} A_{b_{2r}}] \} \\ & C_{n-2r}^1 [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot [A_{a_{n-1}} B_{b_{n-1}} + B_{a_{n-1}} A_{b_{n-1}}] [C_{a_n} C_{b_n}] \\ & = \sum_{k=0}^{n-1} \frac{2}{C_{2n}^{2k+1}} \frac{n!n!}{(n-k-1)!(n-k-1)!k!k!1!1!} \{ [A_{a_1} \cdots A_{a_{n-k-1}}] [B_{a_{n-k}} \cdots B_{a_{n-1}}] [C_{a_n}] \} \{ [B_{b_1} \cdots B_{b_{n-k-1}}] [A_{b_{n-k}} \cdots A_{b_{n-1}}] [C_{b_n}] \} \\ \Leftrightarrow & \sum_{r=0}^{k|(n-1-k)} (-4)^r \frac{(2n-2r-1)!}{r!(n-1-r)!(k-r)!(n-1-k-r)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k-1)!}{(n-1-k)!(n-1-k)!} \end{aligned}$$

$$\text{推论5.2.7. } \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!(n-2r)2^{n-2}}{r!(n-r)!(n-2r)!} = \sum_{k=0}^{n-1} \frac{(2k+1)!}{k!k!} \frac{(2n-2k-1)!}{(n-1-k)!(n-1-k)!} = 2^{2n-3} n(n+1)$$

### 5.2.3 $(1, 1) \Rightarrow (0, 0)$

定理5.2.2.

$$\sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r+1)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k+1)!}{(n-k)!(n-k)!} \Rightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!}$$

$n \geq 0, 0 \leq k \leq n$

$$\text{证明: } \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r+1)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k+1)!}{(n-k)!(n-k)!}, n \geq 0, 0 \leq k \leq n$$

$$\Leftrightarrow \sum_{r=0}^{k|(n-k-2)} (-4)^r \frac{(2n-2r-3)!}{r!(n-r-2)!(k-r)!(n-k-r-2)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k-3)!}{(n-k-2)!(n-k-2)!}, n \geq 2, 0 \leq k \leq n-2$$

$$\Leftrightarrow \sum_{r=1}^{(k+1)|(n-k-1)} (-4)^{r-1} \frac{(2n-2r-1)!}{(r-1)!(n-r-1)!(k-r+1)!(n-k-r-1)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k-3)!}{(n-k-2)!(n-k-2)!}, n \geq 2, 0 \leq k \leq n-2$$

$$\Leftrightarrow \sum_{r=1}^{k|(n-k)} (-4)^{r-1} \frac{(2n-2r-1)!}{(r-1)!(n-r-1)!(k-r)!(n-k-r)!} = \frac{(2k-1)!}{(k-1)!(k-1)!} \frac{(2n-2k-1)!}{(n-k-1)!(n-k-1)!}, n \geq 2, 1 \leq k \leq n-1$$

$$\begin{aligned}
&\Leftrightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2r)(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = -\frac{(2n-2k)(2k)!}{k!k!} \frac{(2k)(2n-2k)!}{(n-k)!(n-k)!}, n \geq 2, 1 \leq k \leq n-1 \\
&\Leftrightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2r)(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = -\frac{(2n-2k)(2k)!}{k!k!} \frac{(2k)(2n-2k)!}{(n-k)!(n-k)!}, n \geq 2, 0 \leq k \leq n \\
&\Leftrightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2r)(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = -\frac{(2n-2k)(2k)!}{k!k!} \frac{(2k)(2n-2k)!}{(n-k)!(n-k)!}, n \geq 0, 0 \leq k \leq n \\
&\Leftrightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r+1)!(2n+1)(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2n-2k)(2k)!}{k!k!} \frac{(2k)(2n-2k)!}{(n-k)!(n-k)!}, n \geq 0, 0 \leq k \leq n \\
&\Rightarrow \frac{(2k+1)!}{k!k!} \frac{(2n-2k+1)!}{(n-k)!(n-k)!} - \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n+1)(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2n-2k)(2k)!}{k!k!} \frac{(2k)(2n-2k)!}{(n-k)!(n-k)!}, n \geq 0, 0 \leq k \leq n \\
&\Leftrightarrow \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!}, n \geq 0, 0 \leq k \leq n
\end{aligned}$$

□

### 5.2.4 小结

推论5.2.8.  $\sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} = 2^n, \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!(n-2r)}{r!(n-r)!(n-2r)!} = n(n+1)2^{n-1}$

推论5.2.9.  $\sum_{r=0}^{[n/2]} (-1)^r \frac{r(2n-2r)!}{r!(n-r)!(n-2r)!} = -n(n-1)2^{n-2}, \sum_{r=0}^{[n/2]} (-1)^r \frac{(n-r)(2n-2r)!}{r!(n-r)!(n-2r)!} = n(n+3)2^{n-2}$

推论5.2.10.  $\sum_{k=0}^n \frac{(2k-1)!!}{(2k)!!} \frac{(2n-2k-1)!!}{(2n-2k)!!} = 1, \sum_{k=0}^n \frac{(2k+1)!!}{(2k)!!} \frac{(2n-2k+1)!!}{(2n-2k)!!} = C_{n+2}^2$

推论5.2.11.

$$\sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{k!k!} \frac{(2n-2k)!}{(n-k)!(n-k)!}, \sum_{r=0}^{k|(n-k)} (-4)^r \frac{(2n-2r+1)!}{r!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k+1)!}{k!k!} \frac{(2n-2k+1)!}{(n-k)!(n-k)!}$$

## 5.3 i=1, j=0和i=0, j=1情形

### 5.3.1 C(2i, 2j)=(2, 0)项和C(2i, 2j)=(0, 2)项情形

推论5.3.1.  $i=1, j=0 | i=0, j=1 \Rightarrow l=0, 1 \leq r \leq k|(n-k), 1 \leq k \leq n-1$

推论5.3.2.  $\sum_{r=1}^{k|(n-k)} (-1)^{r+1} 2^{2r-1} \frac{(2n-2r)!}{(r-1)!(n-r)!(k-r)!(n-k-r)!} = \frac{(2k)!}{(k-1)!k!} \frac{(2n-2k)!}{(n-1-k)!(n-k)!}$

定理5.3.1.  $\sum_{r=0}^{[n/2]} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$

$$\begin{aligned}
&C_r^1 C_r^0 \{ [A_{\{a_1\}} B_{a_2} + B_{\{a_1\}} A_{a_2}] [A_{\{b_1\}} B_{b_2} + B_{\{b_1\}} A_{b_2}] \cdot \\
&[A_{a_{2r-3}} B_{a_{2r-1}} + B_{a_{2r-3}} A_{a_{2r-1}}] [A_{b_{2r-3}} B_{b_{2r-1}} + B_{b_{2r-3}} A_{b_{2r-1}}] [C_{a_{2r-1}} C_{a_{2r}}] [A_{b_{2r-1}} B_{b_{2r}} + B_{b_{2r-1}} A_{b_{2r}}] \} \\
&C_{n-2r}^0 [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot [A_{a_n} B_{b_n} + B_{a_n} A_{b_n}] \\
&= - \sum_{k=0}^n \frac{n!n!}{(2n)!} \frac{(2n-2k)!(2k)!}{(n-k-1)!(n-k)!(k-1)!k!} \\
&\{ [A_{\{a_1\}} \cdot A_{a_{n-k-1}}] [B_{a_{n-k}} \cdot B_{a_{n-2}}] [C_{a_{n-1}} \cdot C_{a_n}] \} \{ [B_{\{b_1\}} \cdot B_{b_{n-k}}] [A_{b_{n+1-k}} \cdot A_{b_n}] \}
\end{aligned}$$

定理5.3.2.  $\sum_{r=0}^{[n/2]} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$

$$\begin{aligned}
&C_r^0 C_r^1 \{ [A_{\{a_1\}} B_{a_2} + B_{\{a_1\}} A_{a_2}] [A_{\{b_1\}} B_{b_2} + B_{\{b_1\}} A_{b_2}] \cdot \\
&[A_{a_{2r-3}} B_{a_{2r-1}} + B_{a_{2r-3}} A_{a_{2r-1}}] [A_{b_{2r-3}} B_{b_{2r-1}} + B_{b_{2r-3}} A_{b_{2r-1}}] [A_{a_{2r-1}} B_{a_{2r}} + B_{a_{2r-1}} A_{a_{2r}}] [C_{b_{2r-1}} C_{b_{2r}}] \} \\
&C_{n-2r}^0 [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot [A_{a_n} B_{b_n} + B_{a_n} A_{b_n}] \\
&= - \sum_{k=0}^n \frac{n!n!}{(2n)!} \frac{(2n-2k)!(2k)!}{(n-k)!(n-k-1)!k!(k-1)!} \\
&\{ [A_{\{a_1\}} \cdot A_{a_{n-k}}] [B_{a_{n-k+1}} \cdot B_{a_n}] \} \{ [B_{\{b_1\}} \cdot B_{b_{n-k-1}}] [A_{b_{n-k}} \cdot A_{b_{n-2}}] [C_{b_{n-1}} \cdot C_{b_n}] \}
\end{aligned}$$

推论5.3.3.  $-\sum_{r=0}^{[n/2]} (-1)^r \frac{(2r)(2n-2r)!2^n}{r!(n-r)!(n-2r)!} = \sum_{k=0}^n \frac{(2k)!(2k)}{k!k!} \frac{(2n-2k)!(2n-2k)}{(n-k)!(n-k)!} = n(n-1)2^{2n-1}$

推论5.3.4.  $\sum_{k=0}^n \frac{(2k)!}{k!(k-1)!} \frac{(2n-2k)!}{(n-k)!(n-k-1)!} = n(n-1)2^{2n-3}$

5.3.2  $C(2i+1, 2j+1)=(3, 1)$ 项和 $C(2i+1, 2j+1)=(1, 3)$ 项情形

推论5.3.5.

$$\sum_{r=0}^{k(n-k)} (-4)^r \frac{(2n-2r+1)!}{(n-r)!(n-k-r)!(k-r)!1!1!0!(r-1)!r!} = \frac{(-4)^1}{3!1!} \frac{(2k+1)!}{(k-1)!k!} \frac{(2n-2k+1)!}{(n-k-1)!(n-k)!}$$

$$\sum_{r=0}^{k(n-k)} (-4)^r \frac{(2n-2r+1)!}{(r-1)!(n-r)!(n-k-r)!(k-r)!} = -\frac{2}{3} \frac{(2k+1)!}{(k-1)!k!} \frac{(2n-2k+1)!}{(n-k-1)!(n-k)!}$$

$$\text{定理5.3.3. } \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$$

$$C_r^1 [C_{\{a_1 C_{a_2}\}} [A_{a_3} B_{a_4} + B_{a_3} A_{a_4}] \cdot \cdot [A_{a_{2r-1}} B_{a_{2r}} + B_{a_{2r-1}} A_{a_{2r}}] C_r^0 [A_{b_1} B_{b_2} + B_{b_1} A_{b_2}] \cdot \cdot [A_{b_{2r-1}} B_{b_{2r}} + B_{b_{2r-1}} A_{b_{2r}}]$$

$$C_{n-2r}^1 [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot \cdot [A_{a_{n-1}} B_{b_{n-1}} + B_{a_{n-1}} A_{b_{n-1}}] [C_{a_n} C_{b_n}]$$

$$= -\sum_{k=1}^{n-2} \frac{(-2)^2}{3!1!} \frac{(2k+1)!}{(k-1)!k!} \frac{(2n-2k-1)!}{(n-k-2)!(n-k-1)!}$$

$$\{[A_{\{a_1 \cdot \cdot A_{a_{n-k-2}}\}} [B_{a_{n-k-1}} \cdot \cdot B_{a_{n-3}}] [C_{a_{n-2}} \cdot \cdot C_{a_n}]]\} \{[B_{(b_1 \cdot \cdot B_{b_{n-k-1}})}] [A_{b_{n-k}} \cdot \cdot A_{b_{n-1}}] [C_{b_n}]\}$$

$$\text{定理5.3.4. } \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$$

$$C_r^0 [A_{a_1} B_{a_2} + B_{a_1} A_{a_2}] \cdot \cdot [A_{a_{2r-1}} B_{a_{2r}} + B_{a_{2r-1}} A_{a_{2r}}] C_r^1 [C_{\{a_1 C_{a_2}\}} [A_{b_3} B_{b_4} + B_{b_3} A_{b_4}] \cdot \cdot [A_{b_{2r-1}} B_{b_{2r}} + B_{b_{2r-1}} A_{b_{2r}}]$$

$$C_{n-2r}^1 [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot \cdot [A_{a_{n-1}} B_{b_{n-1}} + B_{a_{n-1}} A_{b_{n-1}}] [C_{a_n} C_{b_n}]$$

$$= -\sum_{k=1}^{n-2} \frac{(-2)^2}{1!3!} \frac{(2k+1)!}{k!(k-1)!} \frac{(2n-2k-1)!}{(n-k-1)!(n-k-2)!}$$

$$\{[A_{\{a_1 \cdot \cdot A_{a_{n-k-1}}\}} [B_{a_{n-k}} \cdot \cdot B_{a_{n-1}}] [C_{a_n}]]\} \{[B_{(b_1 \cdot \cdot B_{b_{n-k-2}})}] [A_{b_{n-k-1}} \cdot \cdot A_{b_{n-3}}] [C_{b_{n-2}} \cdot \cdot C_{b_n}]\}$$

$$\text{推论5.3.6. } \frac{3}{8} \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} r(n-2r)2^n = -\sum_{k=1}^{n-2} \frac{(2k+1)!}{(k-1)!k!} \frac{(2n-2k-1)!}{(n-k-2)!(n-k-1)!}$$

5.4  $i=1, j=1$ 情形5.4.1  $C(2i, 2j)=(2, 2)$ 项情形推论5.4.1.  $0, i|j-r \leq l \leq i, j, k-r, n-r-k \Rightarrow |i-j| \leq r \leq k|(n-k), 1 \leq k \leq n-1$ 

$$\text{定理5.4.1. } \sum_{r=0}^{[n/2]} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$$

$$\{(C_r^1)^2 \{[A_{\{a_1 B_{a_2} + B_{\{a_1} A_{a_2}\}} [A_{(b_1 B_{b_2} + B_{(b_1} A_{b_2})} \cdot \cdot [A_{a_{2r-3}} B_{a_{2r-2}} + B_{a_{2r-3}} A_{a_{2r-2}}] [A_{b_{2r-3}} B_{b_{2r-2}} + B_{b_{2r-3}} A_{b_{2r-2}}]$$

$$[C_{a_{2r-1}} C_{a_{2r}}] [C_{b_{2r-1}} C_{b_{2r}}]\} [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot \cdot [A_{a_n} B_{b_n}] + B_{a_n} A_{b_n}\}$$

$$+ \{[A_{\{a_1 B_{a_2} + B_{\{a_1} A_{a_2}\}} [A_{(b_1 B_{b_2} + B_{(b_1} A_{b_2})} \cdot \cdot [A_{a_{2r-1}} B_{a_{2r}} + B_{a_{2r-1}} A_{a_{2r}}] [A_{b_{2r-1}} B_{b_{2r}} + B_{b_{2r-1}} A_{b_{2r}}]\}$$

$$C_{n-2r}^2 [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot \cdot [A_{a_{n-2}} B_{b_{n-2}} + B_{a_{n-2}} A_{b_{n-2}}] [C_{a_{n-1}} C_{b_{n-1}} C_{a_n} C_{b_n}]\}$$

$$= \sum_{k=0}^n \frac{n!n!}{(2n)!} \frac{(2n-2k)!(2k)!}{(n-k-1)!(n-k-1)!(k-1)!(k-1)!}$$

$$\{[A_{\{a_1 \cdot \cdot A_{a_{n-k-1}}\}} [B_{a_{n-k}} \cdot \cdot B_{a_{n-2}}] [C_{a_{n-1}} \cdot \cdot C_{a_n}]]\} \{[B_{(b_1 \cdot \cdot B_{b_{n-k-1}})}] [A_{b_{n-k}} \cdot \cdot A_{b_{n-2}}] [C_{b_{n-1}} \cdot \cdot C_{b_n}]\}$$

$$\text{推论5.4.2. } \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} [3r^2 - (2n-1)r + \frac{n(n-1)}{2}] 2^{n-2} = \sum_{k=0}^n \frac{(2n-2k)!(2k)!}{(n-k-1)!(n-k-1)!(k-1)!(k-1)!}$$

$$\text{推论5.4.3. } \sum_{k=0}^n \frac{(2n-2k)!(2k)!}{(n-k-1)!(n-k-1)!(k-1)!(k-1)!}$$

5.4.2  $C(2i+1, 2j+1)=(3, 3)$ 项情形

$$\text{推论5.4.4. } \sum_{r=0}^{k(n-k) \leq 1, k-r, n-r-k} \sum_{l \geq 0, 1-r} (-4)^r \frac{4^l (2n-2r+1)!}{(n-r)!(n-k-r-l)!(k-r-l)!(2l+1)!} \frac{r!}{(1-l)!(1-l)!(r+l-1)!(r+l-1)!}$$

$$= \frac{(-4)^2}{3!3!} \frac{(2k+1)!}{(k-1)!(k-1)!} \frac{(2n-2k+1)!}{(n-k-1)!(n-k-1)!}$$

$$\text{推论5.4.5. } \sum_{r=0}^{k(n-k)} (-4)^r \frac{4^0 (2n-2r+1)!}{(n-r)!(n-k-r)!(k-r)!1!1!1!(r-1)!(r-1)!} + (-4)^r \frac{4^1 (2n-2r+1)!}{(n-r)!(n-k-r-1)!(k-r-1)!3!} \frac{r!}{0!0!r!r!}$$

$$= \frac{(-4)^2}{3!3!} \frac{(2k+1)!}{(k-1)!(k-1)!} \frac{(2n-2k+1)!}{(n-k-1)!(n-k-1)!}$$

$$\begin{aligned} & \sum_{r=0}^{k|n-k} (-4)^r \frac{(2n-2r+1)!}{(n-r)!(n-k-r)!(k-r)!} \frac{r^2}{r!} + (-4)^r \frac{4^1(2n-2r+1)!}{(n-r)!(n-k-r-1)!(k-r-1)!3!} \frac{1}{r!} = \frac{(-4)^2}{3!3!} \frac{(2k+1)!}{(k-1)!(k-1)!} \frac{(2n-2k+1)!}{(n-k-1)!(n-k-1)!} \\ & \sum_{r=0}^{k|n-k} (-4)^r \frac{(2n-2r+1)!}{(n-r)!(n-k-r)!(k-r)!} \frac{r^2}{r!} + (-4)^r \frac{4^1(2n-2r+1)!}{(n-r)!(n-k-r)!(k-r)!3!} \frac{(n-k-r)(k-r)}{r!} = \frac{(-4)^2}{3!3!} \frac{(2k+1)!}{(k-1)!(k-1)!} \frac{(2n-2k+1)!}{(n-k-1)!(n-k-1)!} \\ & \sum_{r=0}^{k|n-k} -(-4)^{r-1} \frac{(2n-2r+1)!}{r!(n-r)!(n-k-r)!(k-r)!} [9r^2 + 6(n-k-r)(k-r)] = \frac{(2k+1)!}{(k-1)!(k-1)!} \frac{(2n-2k+1)!}{(n-k-1)!(n-k-1)!} \end{aligned}$$

**定理5.4.2.**  $\sum_{r=0}^{\lfloor n/2 \rfloor} \sum_{l \geq 0, 1-r}^{(n-1)/2-r} (-1)^r \frac{n!n!}{(2n)!} \frac{(2n-2r)!}{r!(n-r)!(n-2r)!}$

$$\begin{aligned} & C_r^{1-l} [C_{a_1} C_{a_2}] \cdot [C_{a_{1-2l}} C_{a_{2-2l}}] 2^{r+l-1} [A_{a_{3-2l}} B_{a_{4-2l}}] \cdot [A_{a_{2r-1}} B_{a_{2r}}] \\ & C_r^{1-l} [C_{b_1} C_{b_2}] \cdot [C_{b_{1-2l}} C_{b_{2-2l}}] 2^{r+l-1} [A_{b_{3-2l}} B_{b_{4-2l}}] \cdot [A_{b_{2r-1}} B_{b_{2r}}] \\ & C_{n-2r}^{2l+1} [A_{a_{2r+1}} B_{b_{2r+1}} + B_{a_{2r+1}} A_{b_{2r+1}}] \cdot [A_{a_{n-2l-1}} B_{b_{n-2l-1}} + B_{a_{n-2l-1}} A_{b_{n-2l-1}}] [C_{a_{n-2l}} C_{b_{n-2l}}] \cdot [C_{a_n} C_{b_n}] \\ & = - \sum_{k \geq 1}^{\leq n-2} \frac{n!n!}{(2n)!} \frac{(-2)^3}{3!3!} \frac{(2k+1)!}{(k-1)!(k-1)!} \frac{(2n-2k-1)!}{(n-k-2)!(n-k-2)!} \\ & \{ [A_{a_1} \cdots A_{a_{n-k-2}}] [B_{a_{n-k-1}} \cdots B_{a_{n-3}}] [C_{a_{n-2}} \cdots C_{a_n}] \} \{ [B_{b_1} \cdots B_{b_{n-k-2}}] [A_{b_{n-k-1}} \cdots A_{b_{n-3}}] [C_{b_{n-2}} \cdots C_{b_n}] \} \end{aligned}$$

**推论5.4.6.**  $\frac{9}{16} \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} 2^n [r^2(n-2r) + C_{n-2r}^3] = \sum_{k=1}^{n-2} \frac{(2k+1)!}{(k-1)!(k-1)!} \frac{(2n-2k-1)!}{(n-k-2)!(n-k-2)!}$

## 6 Klein-Gordon方程投影算子递推关系的分析

利用Klein-Gordon  $n+1$ -投影算子按1-投影算子展开的唯一性和等于 $n$ -投影算子乘1-投影算子所有可能的组合(含待定系数), 便可以求出 $n+1$ -投影算子= $\sum$  $n$ -投影算子乘1-投影算子的展开系数, 但解不唯一, 一般有无穷多解, 故无明确的物理意义。

**定义6.0.1.** 
$$\begin{cases} \hat{P}_{a_1 \cdots a_n b_1 \cdots b_n}(n) = \frac{1}{(n!)^2} \sum_{P(a)}^{P(b)} \sum_{r=0}^{\lfloor n/2 \rfloor} k_r \hat{P}_{a_1 a_2} \hat{P}_{b_1 b_2} \cdots \hat{P}_{a_{2r-1} a_{2r}} \hat{P}_{b_{2r-1} b_{2r}} \prod_{i=2r+1}^n \hat{P}_{a_i b_i} \\ \hat{P}_{a_1 \cdots a_n b_1 \cdots b_n}(n) := \eta_{b_1}^{a'_1} \eta_{b_2}^{a'_2} \cdots \eta_{b_n}^{a'_n} \hat{P}_{a_1 \cdots a_n a'_1 \cdots a'_n}(n) \end{cases}$$

### 6.1 自旋-1粒子Klein-Gordon方程投影算子的基本性质

**推论6.1.1.**  $P_{a_1 b_1} = P_{b_1 a_1}, p^{a_1} P_{a_1 b_1} = 0, P_{a_1 c_1} \delta^{c_1 d_1} P_{d_1 b_1} = P_{a_1 b_1}$

**推论6.1.2.** 
$$\begin{cases} P_{a_1 b_1}, P_{a_1 a_2}; P_{b_1 b_2}, P_{a_2 b_2}; \\ P_{a_1 b_1} P_{a_2 b_2}, P_{a_1 a_2} P_{b_1 b_2}; \end{cases}$$

### 6.2 自旋-2粒子Klein-Gordon方程投影算子的基本性质

**推论6.2.1.** 
$$\begin{cases} P_{a_1 b_1; a_2 b_2}, P_{a_1 a_2; b_1 b_2}; P_{a_1 a_2; a_3 b_1}, P_{a_1 b_1; b_2 b_3}; \\ P_{a_1 b_1; a_2 b_2} P_{a_3 b_3}, P_{a_1 a_2; b_1 b_2} P_{a_3 b_3}; P_{a_1 a_2; a_3 b_1} P_{b_2 b_3}; \end{cases}$$

**推论6.2.2.**

$$\begin{cases} P_{a_1 a_2; b_1 b_2}(2) = \frac{1}{(2!)^2} \{ [P_{\{a_1(b_1 P_{a_2} b_2)\}}] - \frac{1}{3} [P_{\{a_1 a_2\}} P_{(b_1 b_2)}] \} = \frac{2}{(2!)^2} \{ P_{a_1 b_1} P_{a_2 b_2} + P_{a_1 b_2} P_{a_2 b_1} - \frac{2}{3} P_{a_1 a_2} P_{b_1 b_2} \} \\ P_{a_1 b_1; a_2 b_2}(2) = \frac{2}{(2!)^2} \{ P_{a_1 a_2} P_{b_1 b_2} + P_{a_1 b_2} P_{a_2 b_1} - \frac{2}{3} P_{a_1 b_1} P_{a_2 b_2} \} \\ P_{a_1 a_2; a_3 b_1}(2) = \frac{2}{(2!)^2} \{ P_{a_1 b_1} P_{a_2 a_3} + P_{a_1 a_3} P_{a_2 b_1} - \frac{2}{3} P_{a_1 a_2} P_{b_1 a_3} \} \\ P_{b_1 b_2; b_3 a_1}(2) = \frac{2}{(2!)^2} \{ P_{b_1 a_1} P_{b_2 b_3} + P_{b_1 b_3} P_{b_2 a_1} - \frac{2}{3} P_{b_1 b_2} P_{a_1 b_3} \} \end{cases}$$

### 6.3 自旋-3粒子Klein-Gordon方程投影算子的基本性质

**推论6.3.1.** 
$$\begin{cases} P_{a_1 a_2 b_3; b_1 b_2 a_3}, P_{a_1 a_2 a_3; b_1 b_2 b_3}; P_{a_1 a_2 b_1; a_3 a_4 b_2}, P_{b_1 b_2 a_1; b_3 b_4 a_2}; P_{a_1 a_2 a_3; a_4 b_1 b_2}, P_{a_1 a_2 b_4; b_1 b_2 b_3}; \\ P_{a_1 a_2 b_3; b_1 b_2 a_3} P_{a_4 b_4}, P_{a_1 a_2 a_3; b_1 b_2 b_3} P_{a_4 b_4}; P_{a_1 a_2 b_1; a_3 a_4 b_2} P_{b_3 b_4}, P_{a_1 a_2 a_3; a_4 b_1 b_2} P_{b_3 b_4}; \end{cases}$$

## 6.4 自旋-n粒子Klein-Gordon方程投影算子的基本性质

推论6.4.1.

$$\begin{cases} P_{a_1 \cdots a_k b_{k+1} \cdots b_n; b_1 \cdots b_k a_{k+1} \cdots a_n}; P_{a_1 \cdots a_k b_{k+1} \cdots b_{n-1} a_{n+1}; b_1 \cdots b_k a_{k+1} \cdots a_n}; \\ P_{a_1 \cdots a_k b_{k+1} \cdots b_n; b_1 \cdots b_k a_{k+1} \cdots a_n} P_{a_{n+1} b_{n+1}} : (n+1) - [(n+1)/2]; \\ P_{a_1 \cdots a_l b_l \cdots b_{n-1}; b_1 \cdots b_{l-1} a_{l+1} \cdots a_{n+1}} P_{b_n b_{n+1}} : n - [n/2]; \\ k = n, \cdots, [(n+1)/2], l = n, \cdots, [n/2] + 1 \\ (n+1) - [(n+1)/2] + n - [n/2] = n+1 \end{cases}$$

推论6.4.2.  $P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}}$

$$= \frac{1}{[(n+1)!]^2} \left\{ \sum_{k=n}^{[(n+1)/2]} B_k P_{\{a_1 \cdots a_k b_{k+1} \cdots b_n; (b_1 \cdots b_k a_{k+1} \cdots a_n) P_{a_{n+1}}\} b_{n+1}} \right\} + \sum_{l=n}^{[n/2]+1} C_l P_{\{a_1 \cdots a_l b_l \cdots b_{n-1}; (b_1 \cdots b_{l-1} a_{l+1} \cdots a_{n+1}) P_{b_n b_{n+1}}\}}$$

推论6.4.3.

$$\begin{cases} P_{a_1 \cdots a_k b_{k+1} \cdots b_n; b_1 \cdots b_k a_{k+1} \cdots a_n}; P_{a_1 \cdots a_k b_{k+1} \cdots b_{n-1} a_{n+1}; b_1 \cdots b_k a_{k+1} \cdots a_n}; \\ P_{a_1 \cdots a_k b_{k+1} \cdots b_n; b_1 \cdots b_k a_{k+1} \cdots a_n} P_{a_{n+1} b_{n+1}} : (n+1) - [(n+1)/2]; \\ P_{a_1 \cdots a_l b_l \cdots b_{n-1}; b_1 \cdots b_{l-1} a_{l+1} \cdots a_{n+1}} P_{b_n b_{n+1}} : n - [n/2]; \\ k = 0, \cdots, [n/2], l = 1, \cdots, [(n+1)/2] \\ [n/2] + [(n-1)/2] + 2 = n+1 \end{cases}$$

推论6.4.4.  $P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}}$

$$= \frac{1}{[(n+1)!]^2} \left\{ \sum_{k=0}^{[n/2]} B_k P_{\{a_1 \cdots a_k b_{k+1} \cdots b_n; (b_1 \cdots b_k a_{k+1} \cdots a_n) P_{a_{n+1}}\} b_{n+1}} \right\} \\ + \sum_{j=1}^{[(n+1)/2]} A_j P_{\{a_1 \cdots a_{j-1} b_{j+1} \cdots b_{n+1}; (b_1 \cdots b_j a_j \cdots a_{n-1}) P_{a_n a_{n+1}}\}} + \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1 \cdots a_l b_l \cdots b_{n-1}; (b_1 \cdots b_{l-1} a_{l+1} \cdots a_{n+1}) P_{b_n b_{n+1}}\}}$$

推论6.4.5.  $P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}}$

$$= \frac{1}{[(n+1)!]^2} \left\{ \sum_{k=0}^{[n/2]} B_k P_{\{a_1 \cdots a_k b_{k+1} \cdots b_n; (b_1 \cdots b_k a_{k+1} \cdots a_n) P_{a_{n+1}}\} b_{n+1}} \right\} \\ + \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1 \cdots a_{l-1} b_{l+1} \cdots b_{n+1}; (b_1 \cdots b_l a_l \cdots a_{n-1}) P_{a_n a_{n+1}}\}} + \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1 \cdots a_l b_l \cdots b_{n-1}; (b_1 \cdots b_{l-1} a_{l+1} \cdots a_{n+1}) P_{b_n b_{n+1}}\}}$$

推论6.4.6.  $P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}}$

$$= \frac{1}{[(n+1)!]^2} \left\{ \sum_{k=0}^0 B_k P_{\{a_1 \cdots a_k b_{k+1} \cdots b_n; (b_1 \cdots b_k a_{k+1} \cdots a_n) P_{a_{n+1}}\} b_{n+1}} \right\} \\ + \sum_{l=1}^1 C_l P_{\{a_1 \cdots a_{l-1} b_{l+1} \cdots b_{n+1}; (b_1 \cdots b_l a_l \cdots a_{n-1}) P_{a_n a_{n+1}}\}} + \sum_{l=1}^1 C_l P_{\{a_1 \cdots a_l b_l \cdots b_{n-1}; (b_1 \cdots b_{l-1} a_{l+1} \cdots a_{n+1}) P_{b_n b_{n+1}}\}}$$

## 6.5 自旋-2粒子Klein-Gordon方程投影算子的展开(唯一)

推论6.5.1.  $P_{a_1 a_2 b_1 b_2}(2) = \frac{1}{(2!)^2} \{ [P_{\{a_1(b_1 P_{a_2} b_2)\}}] - \frac{1}{3} [P_{\{a_1 a_2\}} P_{(b_1 b_2)}] \}$

证明:  $P_{a_1 a_2; b_1 b_2}$

$$= \frac{1}{(2!)^2} \left\{ \sum_{k=0}^0 B_k(2) P_{\{a_1 \cdots a_k b_{k+1} \cdots b_2; (b_1; b_1 \cdots b_k a_{k+1} \cdots a_2) P_{a_2}\} b_2} \right\} + \sum_{l=1}^1 C_l(2) P_{\{a_1 \cdots a_l b_l \cdots b_0; (b_1 \cdots b_{l-1} a_{l+1} \cdots a_2) P_{b_1 b_2}\}} \\ = \frac{1}{(2!)^2} \{ B_0(2) P_{(b_1 \{a_1 P_{a_2}\} b_2)} + C_1(2) P_{\{a_1 a_2\}} P_{(b_1 b_2)} \} \\ \Rightarrow B_0(2) = 1, C_1(2) = -\frac{1}{3}$$

□

## 6.6 自旋-3粒子Klein-Gordon方程投影算子的展开(不唯一, 故无大的物理意义)

推论6.6.1.  $P_{a_1 a_2 a_3 b_1 b_2 b_3}(3) = \frac{1}{(3!)^2} \{ [P_{\{a_1(b_1 P_{a_2} b_2) P_{a_3}\} b_3}] - \frac{3}{5} [P_{\{a_1 a_2\}} P_{(b_1 b_2)} [P_{a_3} b_3]] \}$

引理6.6.1.

$$\begin{cases} P_{\{a_1 a_2 (b_1 b_2) (2) P_{a_3} b_3\}} = \{P_{\{a_1 (b_1 P_{a_2 b_2} - \frac{1}{3}[P_{\{a_1 a_2} P_{(b_1 b_2)}\}}] P_{a_3} b_3\}} \\ P_{\{a_1 (b_1; a_2 b_2) (2) P_{a_3} b_3\}} = \frac{1}{2} \{P_{\{a_1 a_2} P_{(b_1 b_2)} + \frac{1}{3}[P_{\{a_1 (b_1 P_{a_2 b_2})\}}] P_{a_3} b_3\}} \\ P_{\{a_1 a_2; a_3\} (b_1) (2) P_{b_2 b_3}} = \frac{2}{3} P_{\{a_1 a_2} P_{a_3\} (b_1 P_{b_2 b_3})} \end{cases}$$

证明:

$$\begin{cases} P_{\{a_1 a_2; (b_1 b_2) (2) P_{a_3} b_3\}} = \frac{2}{(2!)^2} \{P_{\{a_1 (b_1 P_{a_2 (b_2)} + P_{\{a_1 (b_2) P_{a_2 b_1}} - \frac{2}{3} P_{a_1 a_2} P_{b_1 b_2}\}}] P_{a_3} b_3\}} \\ = \{P_{\{a_1 (b_1 P_{a_2 b_2} - \frac{1}{3}[P_{\{a_1 a_2} P_{(b_1 b_2)}\}}] P_{a_3} b_3\}} \\ P_{\{a_1 (b_1; a_2 b_2) (2) P_{a_3} b_3\}} = \frac{2}{(2!)^2} \{P_{\{a_1 a_2} P_{(b_1 b_2)} + P_{\{a_1 (b_2) P_{a_2 b_1}} - \frac{2}{3} P_{\{a_1 (b_1) P_{a_2 b_2}\}}] P_{a_3} b_3\}} \\ = \frac{1}{2} \{P_{\{a_1 a_2} P_{(b_1 b_2)} + \frac{1}{3}[P_{\{a_1 (b_1) P_{a_2 b_2}\}}] P_{a_3} b_3\}} \\ P_{\{a_1 a_2; a_3\} (b_1) (2) P_{b_2 b_3}} = \frac{2}{(2!)^2} \{P_{\{a_1 (b_1) P_{a_2 a_3}} + P_{\{a_1 a_3} P_{a_2 (b_1)} - \frac{2}{3} P_{\{a_1 a_2} P_{(b_1 a_3)}\}}] P_{b_2 b_3}\} = \frac{2}{3} P_{\{a_1 a_2} P_{(b_1 b_2) P_{a_3} b_3\}} \\ P_{(b_1 b_2; b_3) \{a_1\} (2) P_{a_2 a_3}} = \frac{2}{(2!)^2} \{P_{(b_1 \{a_1} P_{b_2 b_3} + P_{(b_1 b_3) P_{b_2 \{a_1}} - \frac{2}{3} P_{(b_1 b_2) P_{\{a_1 b_3}\}}] P_{a_2 a_3}\}} = \frac{2}{3} P_{\{a_1 a_2} P_{(b_1 b_2) P_{a_3} b_3\}} \end{cases}$$

□

定理6.6.1.  $P_{a_1 a_2 a_3; b_1 b_2 b_3} (3)$ 

$$\begin{aligned} &= \frac{1}{(3!)^2} \{P_{\{a_1 a_2; (b_1 b_2) (2) P_{a_3} b_3\}} - \frac{2}{5} P_{\{a_1 a_2; a_3\} (b_1) (2) P_{b_2 b_3}\}} \\ &= \frac{1}{(3!)^2} \{6 P_{\{a_1 (b_1; a_2 b_2) (2) P_{a_3} b_3\}} - \frac{27}{5} P_{\{a_1 a_2; a_3\} (b_1) (2) P_{b_2 b_3}\}} \\ &= \frac{1}{(3!)^2} \{ \frac{27}{25} P_{\{a_1 a_2; (b_1 b_2) (2) P_{a_3} b_3\}} - \frac{12}{25} P_{\{a_1 (b_1; a_2 b_2) (2) P_{a_3} b_3\}} \} \\ &= \frac{1}{(3!)^2} \{ \frac{6}{7} P_{\{a_1 a_2; (b_1 b_2) (2) P_{a_3} b_3\}} + \frac{6}{7} P_{\{a_1 (b_1; a_2 b_2) (2) P_{a_3} b_3\}} - \frac{39}{35} P_{\{a_1 a_2; a_3\} (b_1) (2) P_{b_2 b_3}\}} \\ &= \frac{1}{(3!)^2} \{ \frac{6}{5} P_{\{a_1 a_2; (b_1 b_2) (2) P_{a_3} b_3\}} - \frac{6}{5} P_{\{a_1 (b_1; a_2 b_2) (2) P_{a_3} b_3\}} + \frac{3}{5} P_{\{a_1 a_2; a_3\} (b_1) (2) P_{b_2 b_3}\}} \end{aligned}$$

证明:  $P_{a_1 a_2 a_3; b_1 b_2 b_3}$ 

$$\begin{aligned} &= \frac{1}{(3!)^2} \{ \sum_{k=0}^1 B_k P_{\{a_1 \dots a_k b_{k+1} \dots b_2; (b_1 \dots b_k a_{k+1} \dots a_2) P_{a_3} b_3\}} + \sum_{l=1}^1 C_l P_{\{a_1 \dots a_l (b_1 \dots b_l; b_1 \dots b_{l-1} a_{l+1} \dots a_3) P_{b_2 b_3}\}} \} \\ &= \frac{1}{(3!)^2} \{ B_1(3) P_{\{a_1 (b_2; b_1 a_2) P_{a_3} b_3\}} + B_0(3) P_{(b_1 b_2; \{a_1 a_2\} P_{a_3} b_3)} + C_1(3) P_{\{a_1 (b_1; a_2 a_3) P_{b_2 b_3}\}} \} \\ &= \frac{1}{(3!)^2} \{ B_1(3) P_{\{a_1 (b_1; a_2 b_2) P_{a_3} b_3\}} + B_0(3) P_{\{a_1 a_2; (b_1 b_2) P_{a_3} b_3\}} + C_1(3) P_{\{a_1 a_2; a_3\} (b_1) P_{b_2 b_3}\}} \} \\ &= \frac{1}{(3!)^2} \{ B_1(3) \frac{1}{2} \{P_{\{a_1 a_2} P_{(b_1 b_2)} + \frac{1}{3}[P_{\{a_1 (b_1) P_{a_2 b_2}}] P_{a_3} b_3\}} + B_0(3) \{P_{\{a_1 (b_1) P_{a_2 b_2}} - \frac{1}{3}[P_{\{a_1 a_2} P_{(b_1 b_2)}\}}] P_{a_3} b_3\}} \\ &\quad + C_1(3) \frac{2}{3} P_{\{a_1 a_2} P_{a_3\} (b_1) P_{b_2 b_3}\}} \} \\ &= \frac{1}{(3!)^2} \{ [\frac{1}{6} B_1(3) + B_0(3)] P_{\{a_1 (b_1) P_{a_2 b_2} P_{a_3} b_3\}} + [\frac{1}{2} B_1(3) - \frac{1}{3} B_0(3) + \frac{2}{3} C_1(3)] P_{\{a_1 a_2} P_{(b_1 b_2) P_{a_3} b_3\}} \} \\ &\Leftrightarrow [\frac{1}{6} B_1(3) + B_0(3)] = 1, [\frac{1}{2} B_1(3) - \frac{1}{3} B_0(3) + \frac{2}{3} C_1(3)] = -\frac{3}{5} \\ &\Leftrightarrow B_0(3) = 1, B_1(3) = 0, C_1(3) = -\frac{2}{5} \\ &B_0(3) = 0, B_1(3) = 6, C_1(3) = -\frac{27}{5} \\ &B_0(3) = \frac{27}{25}, B_1(3) = -\frac{12}{25}, C_1(3) = 0 \\ &B_0(3) = \frac{6}{7}, B_1(3) = \frac{6}{7}, C_1(3) = -\frac{39}{35} \\ &B_0(3) = \frac{6}{5}, B_1(3) = -\frac{6}{5}, C_1(3) = \frac{3}{5} \end{aligned}$$

□

## 6.7 自旋-4粒子Klein-Gordon方程投影算子的展开(不唯一且复杂, 暂时放放)

$$\text{推论6.7.1. } \begin{cases} P_{a_1 a_2 b_3; b_1 b_2 a_3}, P_{a_1 a_2 a_3; b_1 b_2 b_3}; P_{a_1 a_2 b_1; a_3 a_4 b_2}, P_{b_1 b_2 a_1; b_3 b_4 a_2}; P_{a_1 a_2 a_3; a_4 b_1 b_2}, P_{a_1 a_2 b_4; b_1 b_2 b_3}; \\ P_{a_1 a_2 b_3; b_1 b_2 a_3} P_{a_4 b_4}, P_{a_1 a_2 a_3; b_1 b_2 b_3} P_{a_4 b_4}; P_{a_1 a_2 b_1; a_3 a_4 b_2} P_{b_3 b_4}, P_{a_1 a_2 a_3; a_4 b_1 b_2} P_{b_3 b_4}; \end{cases}$$

$$\text{推论6.7.2. } P_{a_1 a_2 a_3 b_1 b_2 b_3} (3) = \frac{1}{(3!)^2} \{ [P_{\{a_1 (b_1) P_{a_2 b_2} P_{a_3} b_3\}}] - \frac{3}{5} [P_{\{a_1 a_2} P_{(b_1 b_2)}] [P_{a_3} b_3] \}$$

推论6.7.3.  $P_{a_1 a_2 a_3 a_4 b_1 b_2 b_3 b_4} (4)$ 

$$= \frac{1}{(4!)^2} \{ [P_{\{a_1 (b_1) P_{a_2 b_2} P_{a_3 b_3} P_{a_4} b_4\}}] - \frac{6}{7} [P_{\{a_1 a_2} P_{(b_1 b_2)}] [P_{a_3 b_3} P_{a_4} b_4] \} + \frac{3}{35} [P_{\{a_1 a_2} P_{(b_1 b_2) P_{a_3 a_4}}] P_{b_3 b_4} \} \}$$

## 6.8 自旋-n粒子Klein-Gordon方程投影算子的展开(不唯一且复杂)

推论6.8.1. 
$$P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = \frac{1}{[(n+1)!]^2} \left\{ \sum_{k=0}^{[n/2]} B_k P_{\{a_1 \cdots a_k b_{k+1} \cdots b_n; (b_1 \cdots b_k a_{k+1} \cdots a_n) P_{a_{n+1}}\} b_{n+1}} \right.$$

$$+ \left. \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1 \cdots a_{l-1} b_{l+1} \cdots b_{n+1}; (b_1 \cdots b_l a_l \cdots a_{n-1}) P_{a_n a_{n+1}}\}} + \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1 \cdots a_l b_l \cdots b_{n-1}; (b_1 \cdots b_{l-1} a_{l+1} \cdots a_{n+1}) P_{b_n b_{n+1}}\}} \right\}$$

$$P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = \frac{1}{(n+1)!} P_{\{a_1 \cdots a_{n+1}\}; b_1 \cdots b_{n+1}}, P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = \frac{1}{(n+1)!} P_{a_1 \cdots a_{n+1}; (b_1 \cdots b_{n+1})}$$

$$P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = P_{b_1 \cdots b_{n+1}; a_1 \cdots a_{n+1}}, p^{a_1} P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = 0$$

$$\delta^{a_1 a_2} P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = 0, \delta^{a_{n+1} b_{n+1}} P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = \frac{2n+3}{2n+1} P_{a_1 \cdots a_n; b_1 \cdots b_n}$$

$$\Leftrightarrow P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = \frac{1}{[(n+1)!]^2} \left\{ \sum_{k=0}^{[n/2]} B_k P_{\{a_1 \cdots a_k b_{k+1} \cdots b_n; (b_1 \cdots b_k a_{k+1} \cdots a_n) P_{a_{n+1}}\} b_{n+1}} \right.$$

$$+ \left. \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1 \cdots a_{l-1} b_{l+1} \cdots b_{n+1}; (b_1 \cdots b_l a_l \cdots a_{n-1}) P_{a_n a_{n+1}}\}} + \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1 \cdots a_l b_l \cdots b_{n-1}; (b_1 \cdots b_{l-1} a_{l+1} \cdots a_{n+1}) P_{b_n b_{n+1}}\}} \right\}$$

$$\delta^{a_1 a_2} P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = 0, \delta^{a_{n+1} b_{n+1}} P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = \frac{2n+3}{2n+1} P_{a_1 \cdots a_n; b_1 \cdots b_n}$$

$$\Leftrightarrow P_{a_1 \cdots a_{n+1}; b_1 \cdots b_{n+1}} = \frac{1}{[(n+1)!]^2} \left\{ \sum_{k=0}^{[n/2]} B_k P_{\{a_1 \cdots a_k b_{k+1} \cdots b_n; (b_1 \cdots b_k a_{k+1} \cdots a_n) P_{a_{n+1}}\} b_{n+1}} \right.$$

$$+ \left. \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1 \cdots a_{l-1} b_{l+1} \cdots b_{n+1}; (b_1 \cdots b_l a_l \cdots a_{n-1}) P_{a_n a_{n+1}}\}} + \sum_{l=1}^{[(n+1)/2]} C_l P_{\{a_1 \cdots a_l b_l \cdots b_{n-1}; (b_1 \cdots b_{l-1} a_{l+1} \cdots a_{n+1}) P_{b_n b_{n+1}}\}} \right\}$$

## 7 平移准投影算子

### 7.1 自旋-1基的完备性

定义7.1.1.  $\varepsilon_a(\vec{p}, \kappa) := [i\lambda_m(\vec{p}, \kappa), 0]_a, \varepsilon_a(\vec{p}, 0) := \frac{1}{m} [iE\lambda_m(\vec{p}, 0), i|\vec{p}|]_a, \varepsilon_a(\vec{p}, 0; 0) := \frac{p_a}{m}$

推论7.1.1.

$$\begin{cases} \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \bar{\varepsilon}_b(\vec{p}, h) - \varepsilon_a(\vec{p}, 0; 0) \bar{\varepsilon}_b(\vec{p}, 0; 0) = \delta_{ab}, \bar{\varepsilon}_a(\vec{p}, h; s) := \varepsilon_{a'}^+(\vec{p}, h; s) \eta_a^{a'} \\ \bar{\varepsilon}^a(\vec{p}, h') \varepsilon_a(\vec{p}, h) = \delta_{h'h}, \bar{\varepsilon}^a(\vec{p}, 0; 0) \varepsilon_a(\vec{p}, 0; 0) = -1, \bar{\varepsilon}^a(\vec{p}, 0; 0) \varepsilon_a(\vec{p}, h) = 0, \bar{\varepsilon}^a(\vec{p}, h) \varepsilon_a(\vec{p}, 0; 0) = 0 \end{cases}$$

推论7.1.2.

$$\bar{\varepsilon}_a(\vec{p}, h'; s') \varepsilon_b(\vec{p}, h; s) = \eta_{s's} \delta_{h'h}, \sum_{s=1}^0 \sum_{h=s}^{-s} \eta_{ss} \varepsilon_a(\vec{p}, h; s) \bar{\varepsilon}_b(\vec{p}, h; s) = \delta_{ab}; \eta_{11} := 1, \eta_{00} := -1, \eta_{10} := 0, \eta_{01} := 0$$

### 7.2 一般自旋基的完备性猜想

猜想7.2.1.

$$\begin{cases} \bar{\varepsilon}^{\overbrace{a \cdots b}^n}(\vec{p}, h'; s') \varepsilon_{\overbrace{a \cdots b}^n}(\vec{p}, h; s) = \eta_{s's} \delta_{h'h}, \sum_{s=n}^0 \sum_{h=s}^{-s} \eta_{ss} \varepsilon_{\overbrace{a \cdots b}^n}(\vec{p}, h; s) \bar{\varepsilon}_{\overbrace{a' \cdots b'}^n}(\vec{p}, h; s) = \frac{1}{(2n)!^2} \overbrace{\delta_{\{a(a' \cdots b_b)\} b'}}^n \\ \bar{U}_{\overbrace{\lambda_\zeta \cdots \mu_\zeta}^{2sm}}(\vec{p}, h'; s') U_{\overbrace{\lambda_\zeta \cdots \mu_\zeta}^{2sm}}(\vec{p}, h; s) = \eta_{s's} \delta_{h'h}, \sum_{s=sm}^{-sm} \sum_{h=s}^{-s} U_{\overbrace{\lambda_\zeta \cdots \mu_\zeta}^{2sm}}(\vec{p}, h; s) \bar{U}_{\overbrace{\lambda'_\zeta \cdots \mu'_\zeta}^{2sm}}(\vec{p}, h; s) = \frac{1}{(2sm)!^2} \overbrace{\delta_{\{\lambda_\zeta(\lambda'_\zeta \cdots \mu_\zeta)\} \mu'_\zeta}}^{2sm} \\ U_{\overbrace{\lambda_\zeta \cdots \mu_\zeta}^{2sm}}(\vec{p}, h; -s) := V_{\overbrace{\lambda_\zeta \cdots \mu_\zeta}^{2sm}}(\vec{p}, h; s), 0 \leq s \leq s_m \end{cases}$$

### 7.3 质量猜想(高维时空中的运动)

猜想7.3.1.  $E^2 = \vec{p}^2 + \vec{p}_m^2, (\gamma_a, \gamma_5) := [(\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x), \zeta I \otimes \sigma_z]$

### 7.4 数学准备(以前章节的结论)

推论7.4.1.

$$\begin{cases} \lambda(\hat{p}, \frac{\kappa}{2}) \lambda^+(\hat{p}, \frac{\kappa}{2}) = \frac{1}{2} (\kappa \sigma \cdot \hat{p} + I) = \frac{1}{2} (\kappa \sigma, -i) \hat{p}_a, \hat{p}_a := (\hat{p}, i) \\ \lambda(\hat{p}, -\frac{1}{2}) \lambda^+(\hat{p}, -\frac{1}{2}) = -\frac{1}{2} (\sigma \cdot \hat{p} - I) = -\frac{1}{2} (\sigma, i) \hat{p}_a \\ \lambda(\hat{p}, \frac{\kappa}{2}) \lambda^+(\hat{p}, -\frac{\kappa}{2}) = \frac{\kappa}{2} (\sigma \cdot \hat{p} + I) i \sigma_y = \frac{\kappa}{2} (\sigma, i) \hat{p}_a i \sigma_y \end{cases}$$



推论7.4.2.

$$\begin{cases} \mu(\vec{p}, \frac{\kappa}{2})\mu^+(\vec{p}, -\frac{\kappa}{2}) = \frac{1}{2}(I + \varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y) \\ \mu(\vec{p}, \frac{\kappa}{2})\mu^+(\vec{p}, \frac{\kappa}{2}) = \frac{\varsigma}{2}(I + \varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y)\sigma_x \end{cases}$$

推论7.4.3.  $u(\vec{p}, \frac{\kappa}{2}) = \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \mu(\vec{p}, \frac{\kappa}{2}), v(\vec{p}, \frac{\kappa}{2}) = \lambda(\hat{p}, \frac{\kappa}{2}) \otimes \nu(\vec{p}, \frac{\kappa}{2})$

推论7.4.4.  $u(\vec{p}, \frac{\kappa}{2})u^+(\vec{p}, \frac{\kappa}{2}) = \frac{1}{4}[(\kappa\sigma \cdot \hat{p} + I) \otimes (I + \varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y)](\varsigma I \otimes \sigma_x)$   
 $= \frac{1}{4}(i\kappa\vec{\gamma} \cdot \hat{p}\gamma_4\gamma_5 + I_4)(I_4 + \frac{E}{m}\gamma_4 + \kappa\frac{|\vec{p}|}{m}\gamma_4\gamma_5)\gamma_4$   
 $= \frac{1}{4}(i\kappa\vec{\gamma} \cdot \hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 - \kappa\frac{|\vec{p}|}{m})\gamma_4$

推论7.4.5.  $u(\vec{p}, \frac{\kappa}{2})u^+(\vec{p}, -\frac{\kappa}{2}) = \frac{1}{4}[\kappa(\sigma \cdot \hat{p} + I)i\sigma_y] \otimes (I + \varsigma\frac{E}{m}\sigma_x - i\kappa\frac{|\vec{p}|}{m}\sigma_y)$   
 $= \frac{1}{4}\kappa(i\vec{\gamma} \cdot \hat{p}\gamma_2 - \gamma_2\gamma_4\gamma_5)(I_4 + \frac{E}{m}\gamma_4 + \kappa\frac{|\vec{p}|}{m}\gamma_4\gamma_5)$   
 $= \frac{1}{4}\kappa(i\vec{\gamma} \cdot \hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 - \kappa\frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5$

推论7.4.6.

$$\begin{cases} U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, h) = \frac{1}{2\sqrt{2m}}\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(p)\varepsilon_a(\vec{p}, h), V_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, h) = -\frac{1}{2\sqrt{2m}}\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(-p)\varepsilon_a(\vec{p}, h) \\ \varepsilon_a(\vec{p}, h) = -\frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}U_{\lambda_\varsigma\mu_\varsigma}(\hat{p}, h) = \frac{i}{\sqrt{2}}(\bar{C}\gamma_a)^{\lambda_\varsigma\mu_\varsigma}V_{\lambda_\varsigma\mu_\varsigma}(\hat{p}, h) \\ \varepsilon_a^+(\vec{p}, h) = \frac{i}{\sqrt{2}}(\gamma_a C)^{\lambda'_\varsigma\mu'_\varsigma}U_{\lambda'_\varsigma\mu'_\varsigma}^+(\hat{p}, h) = -\frac{i}{\sqrt{2}}(\gamma_a C)^{\lambda'_\varsigma\mu'_\varsigma}V_{\lambda'_\varsigma\mu'_\varsigma}^+(\hat{p}, h) \end{cases}$$

推论7.4.7.  $\lambda_m(\hat{p}, -1) = \lambda_m^*(\hat{p}, 1), \lambda_m(\hat{p}, 0) = -\lambda_m^*(\hat{p}, 0), \lambda_m(\hat{p}, 1) = \lambda_m^*(\hat{p}, -1)$

定义7.4.1.  $\varepsilon_a(\vec{p}, \kappa) := [i\lambda_m(\vec{p}, \kappa), 0]_a, \varepsilon_a(\vec{p}, 0) := \frac{1}{m}[iE\lambda_m(\vec{p}, 0), i|\vec{p}|]_a, \bar{\varepsilon}_a(\vec{p}, h) := \varepsilon_a^+(\vec{p}, h)\eta_a^a$

推论7.4.8.

$$\begin{cases} \lambda_m(\hat{p}, 1; 1) = \frac{1}{2\hat{p}_-} \begin{bmatrix} i(\hat{p}_x\hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ -2i(\hat{p}_+\hat{p}_-) \end{bmatrix}, \lambda_m(-\hat{p}, 1; 1) = \frac{\hat{p}_+}{\hat{p}_-} \lambda_m(\hat{p}, -1; 1) \\ \lambda_m(\hat{p}, 0; 1) = -i \begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{bmatrix} = -i\hat{p}, \lambda_m(-\hat{p}, 0; 1) = -\lambda_m(\hat{p}, 0; 1) \\ \lambda_m(\hat{p}, -1; 1) = \frac{1}{2\hat{p}_+} \begin{bmatrix} -i(\hat{p}_x\hat{p}_z + i\hat{p}_y) \\ -1(\hat{p}_x + i\hat{p}_y\hat{p}_z) \\ 2i(\hat{p}_+\hat{p}_-) \end{bmatrix}, \lambda_m(-\hat{p}, -1; 1) = \frac{\hat{p}_-}{\hat{p}_+} \lambda_m(\hat{p}, 1; 1) \end{cases}$$

证明:  $\lambda_m(\hat{p}, 1)\lambda_m^+(\hat{p}, -1) = \frac{1}{2\hat{p}_-} \begin{bmatrix} i(\hat{p}_x\hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ -2i(\hat{p}_+\hat{p}_-) \end{bmatrix} \frac{1}{2\hat{p}_-} \begin{bmatrix} i(\hat{p}_x\hat{p}_z - i\hat{p}_y) \\ -1(\hat{p}_x - i\hat{p}_y\hat{p}_z) \\ -2i(\hat{p}_+\hat{p}_-) \end{bmatrix}^T$  □

## 7.5 二阶平移准投影算子

定义7.5.1.  $\sum_{h=1}^{-1} U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, h)U_{\lambda'_\varsigma\mu'_\varsigma}^+(\vec{p}, h - h')$

$$\begin{cases} U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, 1) = u_{\lambda_\varsigma}(\vec{p}, \frac{1}{2})u_{\mu_\varsigma}(\vec{p}, \frac{1}{2}) \\ U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, 0) = \frac{1}{\sqrt{2}}[u_{\lambda_\varsigma}(\vec{p}, \frac{1}{2})u_{\mu_\varsigma}(\vec{p}, -\frac{1}{2}) + u_{\mu_\varsigma}(\vec{p}, \frac{1}{2})u_{\lambda_\varsigma}(\vec{p}, -\frac{1}{2})] \\ U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, -1) = u_{\lambda_\varsigma}(\vec{p}, -\frac{1}{2})u_{\mu_\varsigma}(\vec{p}, -\frac{1}{2}) \end{cases}$$

证明:  $\sum_{h=1}^{-1} U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, h)U_{\lambda'_\varsigma\mu'_\varsigma}^+(\vec{p}, h - 2) = U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, 1)U_{\lambda'_\varsigma\mu'_\varsigma}^+(\vec{p}, -1)$   
 $= u_{\lambda_\varsigma}(\vec{p}, \frac{1}{2})u_{\mu_\varsigma}(\vec{p}, \frac{1}{2})u_{\lambda'_\varsigma}^+(\vec{p}, -\frac{1}{2})u_{\mu'_\varsigma}^+(\vec{p}, -\frac{1}{2})$   
 $= [\frac{1}{4}(i\vec{\gamma} \cdot \hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 - \frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]_{\lambda_\varsigma\lambda'_\varsigma} [\frac{1}{4}(i\vec{\gamma} \cdot \hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 - \frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]_{\mu_\varsigma\mu'_\varsigma}$   
 $= \frac{1}{16}\{[(i\vec{\gamma} \cdot \hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 - \frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5] \otimes [(i\vec{\gamma} \cdot \hat{p} - \gamma_4\gamma_5)(\gamma_4\gamma_5 - \frac{E}{m}\gamma_5 - \frac{|\vec{p}|}{m})\gamma_4\gamma_2\gamma_5]\}_{\lambda_\varsigma\lambda'_\varsigma\mu_\varsigma\mu'_\varsigma}$   
 $= \frac{1}{16}\{[(i\vec{\gamma} \cdot \hat{p} - \gamma_4\gamma_5) \otimes (i\vec{\gamma} \cdot \hat{p} - \gamma_4\gamma_5)][(I_4 - \frac{E}{m}\gamma_4 + \frac{|\vec{p}|}{m}\gamma_4\gamma_5) \otimes (I_4 - \frac{E}{m}\gamma_4 + \frac{|\vec{p}|}{m}\gamma_4\gamma_5)]\}_{\lambda_\varsigma\lambda'_\varsigma\mu_\varsigma\mu'_\varsigma}$  □

证明:  $\sum_{h=1}^{-1} U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, h)U_{\lambda'_\varsigma\mu'_\varsigma}^+(\vec{p}, h - 1) = U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, 1)U_{\lambda'_\varsigma\mu'_\varsigma}^+(\vec{p}, 0) + U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, 0)U_{\lambda'_\varsigma\mu'_\varsigma}^+(\vec{p}, -1)$   
 $= \frac{1}{\sqrt{2}}u_{\lambda_\varsigma}(\vec{p}, \frac{1}{2})u_{\mu_\varsigma}(\vec{p}, \frac{1}{2})[u_{\lambda'_\varsigma}^+(\vec{p}, \frac{1}{2})u_{\mu'_\varsigma}^+(\vec{p}, -\frac{1}{2}) + u_{\mu'_\varsigma}^+(\vec{p}, \frac{1}{2})u_{\lambda'_\varsigma}^+(\vec{p}, -\frac{1}{2})]$   
 $+ \frac{1}{\sqrt{2}}[u_{\lambda_\varsigma}(\vec{p}, \frac{1}{2})u_{\mu_\varsigma}(\vec{p}, -\frac{1}{2}) + u_{\mu_\varsigma}(\vec{p}, \frac{1}{2})u_{\lambda_\varsigma}(\vec{p}, -\frac{1}{2})]u_{\lambda'_\varsigma}^+(\vec{p}, -\frac{1}{2})u_{\mu'_\varsigma}^+(\vec{p}, -\frac{1}{2})$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2}) [u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2}) + u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
&+ \frac{1}{\sqrt{2}} u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2}) [u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
&= \frac{1}{\sqrt{2}} [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5 - \frac{|\vec{p}|}{m}) \gamma_4 \gamma_2 \gamma_5]_{\lambda_\zeta \lambda'_\zeta} \Lambda_{+\mu_\zeta \mu'_\zeta}(\vec{p}, \frac{1}{2}) \\
&+ \frac{1}{\sqrt{2}} \Lambda_{+\lambda_\zeta \lambda'_\zeta}(\vec{p}, \frac{1}{2}) [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5 - \frac{|\vec{p}|}{m}) \gamma_4 \gamma_2 \gamma_5]_{\mu_\zeta \mu'_\zeta}
\end{aligned} \quad \square$$

$$\begin{aligned}
\text{证明: } &\sum_{h=1}^{-1} U_{\lambda_\zeta \mu_\zeta}(\vec{p}, h) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, h) = U_{\lambda_\zeta \mu_\zeta}(\vec{p}, 1) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, 1) + U_{\lambda_\zeta \mu_\zeta}(\vec{p}, 0) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, 0) + U_{\lambda_\zeta \mu_\zeta}(\vec{p}, -1) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, -1) \\
&= u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2}) \\
&+ \frac{1}{2} [u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) + u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})] [u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2}) + u_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
&+ u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2}) \\
&= \frac{1}{(2!)^2} \Lambda_{+\{\lambda_\zeta \lambda'_\zeta, \mu_\zeta \mu'_\zeta\}}(\vec{p}, \frac{1}{2})
\end{aligned} \quad \square$$

$$\begin{aligned}
\text{证明: } &\sum_{h=1}^{-1} U_{\lambda_\zeta \mu_\zeta}(\vec{p}, h) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, h+1) = U_{\lambda_\zeta \mu_\zeta}(\vec{p}, -1) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, 0) + U_{\lambda_\zeta \mu_\zeta}(\vec{p}, 0) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, 1) \\
&= \frac{1}{\sqrt{2}} u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) [u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2}) + u_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
&+ \frac{1}{\sqrt{2}} [u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) + u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})] u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2}) \\
&= \frac{1}{\sqrt{2}} u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2}) [u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2}) + u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
&+ \frac{1}{\sqrt{2}} u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2}) [u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
&= \frac{1}{\sqrt{2}} [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5 + \frac{|\vec{p}|}{m}) \gamma_4 \gamma_2 \gamma_5]_{\lambda_\zeta \lambda'_\zeta} \Lambda_{+\mu_\zeta \mu'_\zeta}(\vec{p}, \frac{1}{2}) \\
&+ \frac{1}{\sqrt{2}} \Lambda_{+\lambda_\zeta \lambda'_\zeta}(\vec{p}, \frac{1}{2}) [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 + \frac{E}{m} \gamma_5 - \frac{|\vec{p}|}{m}) \gamma_4 \gamma_2 \gamma_5]_{\mu_\zeta \mu'_\zeta}
\end{aligned} \quad \square$$

$$\begin{aligned}
\text{证明: } &\sum_{h=1}^{-1} U_{\lambda_\zeta \mu_\zeta}(\vec{p}, h) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, h+2) = U_{\lambda_\zeta \mu_\zeta}(\vec{p}, -1) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, 1) \\
&= u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2}) \\
&= [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5 + \frac{|\vec{p}|}{m}) \gamma_4 \gamma_2 \gamma_5]_{\lambda_\zeta \lambda'_\zeta} [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5 + \frac{|\vec{p}|}{m}) \gamma_4 \gamma_2 \gamma_5]_{\mu_\zeta \mu'_\zeta} \\
&= \frac{1}{16} \{ [(i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5 + \frac{|\vec{p}|}{m}) \gamma_4 \gamma_2 \gamma_5] \otimes [(i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5 + \frac{|\vec{p}|}{m}) \gamma_4 \gamma_2 \gamma_5] \}_{\lambda_\zeta \lambda'_\zeta \mu_\zeta \mu'_\zeta} \\
&= \frac{1}{16} \{ [(i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) \otimes (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5)] [(I_4 - \frac{E}{m} \gamma_4 - \frac{|\vec{p}|}{m} \gamma_4 \gamma_5) \otimes (I_4 - \frac{E}{m} \gamma_4 - \frac{|\vec{p}|}{m} \gamma_4 \gamma_5)] [\gamma_2 \otimes \gamma_2] \}_{\lambda_\zeta \lambda'_\zeta \mu_\zeta \mu'_\zeta}
\end{aligned} \quad \square$$

$$\begin{aligned}
\text{证明: } &U_{\lambda_\zeta \mu_\zeta}(\vec{p}, 1) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, -1) + U_{\lambda_\zeta \mu_\zeta}(\vec{p}, -1) U_{\lambda'_\zeta \mu'_\zeta}^+(\vec{p}, 1) \\
&= 2 [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5) \gamma_4 \gamma_2 \gamma_5]_{\lambda_\zeta \lambda'_\zeta} [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5) \gamma_4 \gamma_2 \gamma_5]_{\mu_\zeta \mu'_\zeta} \\
&+ 2 [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) \frac{|\vec{p}|}{m} \gamma_4 \gamma_2 \gamma_5]_{\lambda_\zeta \lambda'_\zeta} [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) \frac{|\vec{p}|}{m} \gamma_4 \gamma_2 \gamma_5]_{\mu_\zeta \mu'_\zeta} \\
&= 2 [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5) \gamma_4 \gamma_2 \gamma_5] \otimes [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (\gamma_4 \gamma_5 - \frac{E}{m} \gamma_5) \gamma_4 \gamma_2 \gamma_5]_{\lambda_\zeta \lambda'_\zeta \mu_\zeta \mu'_\zeta} \\
&+ 2 [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) \frac{|\vec{p}|}{m} \gamma_4 \gamma_2 \gamma_5] \otimes [\frac{1}{4} (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) \frac{|\vec{p}|}{m} \gamma_4 \gamma_2 \gamma_5]_{\lambda_\zeta \lambda'_\zeta \mu_\zeta \mu'_\zeta} \\
&= \frac{1}{8} \{ [(i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (I_4 - \frac{E}{m} \gamma_4) \gamma_2] \otimes [(i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) (I_4 - \frac{E}{m} \gamma_4) \gamma_2] \\
&+ [(i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) \frac{|\vec{p}|}{m} \gamma_4 \gamma_5 \gamma_2] \otimes [(i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) \frac{|\vec{p}|}{m} \gamma_4 \gamma_5 \gamma_2] \}_{\lambda_\zeta \lambda'_\zeta \mu_\zeta \mu'_\zeta} \\
&= \frac{1}{8} \{ [(i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5) \otimes (i\vec{\gamma} \cdot \hat{p} - \gamma_4 \gamma_5)] [(I_4 - \frac{E}{m} \gamma_4) \otimes (I_4 - \frac{E}{m} \gamma_4) + \frac{\vec{p}^2}{m^2} (\gamma_4 \gamma_5) \otimes (\gamma_4 \gamma_5)] [\gamma_2 \otimes \gamma_2] \}_{\lambda_\zeta \lambda'_\zeta \mu_\zeta \mu'_\zeta}
\end{aligned} \quad \square$$

# 第三十一章 有质量矢量粒子的协变量子化方案

自我评述：对于Bargmann-Wigner方程描述的粒子，一般来说既可以描述带荷的复粒子，也可以描述不带荷的马约拉纳粒子。两种情形的主对易规则形式一致，但其余对易或反对易括号，对带荷的复粒子一般为零；对不带荷的马约拉纳粒子，其余对易或反对易括号由主对易规则和马约拉纳条件自然得到，一般不为零。在本章节既讨论了复粒子情形，也讨论了马约拉纳粒子情形，完整给出了两种情形下的全部对易规则。但以后章节一般不再求全，一般只讨论复粒子情形和主对易规则，不再专门讨论马约拉纳粒子情形，若要得到马约拉纳粒子情形的量子场论，只需在复粒子情形加上马约拉纳条件即可自然得到。

## 1 有质量矢量粒子两种等价描述的相互转换

### 1.1 自旋-1粒子的B-W全对称方程和K-G矢量方程两种等价描述<sup>[17, 19, 22]</sup>

定义1.1.1.  $\mathbb{X}_a(x) := [im\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)\partial^b]C$ ,  $\mathbb{X}_a(p) := i[m\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)p^b]C$ ,  $C = \gamma_2\gamma_4$   
 $\gamma_a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_x)$

定理1.1.1. 
$$\begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi_{[\lambda_\varsigma\mu_\varsigma]} = 0, \psi_{\lambda_\varsigma\mu_\varsigma} = \psi_{\mu_\varsigma\lambda_\varsigma} \\ im\frac{A_a}{2} = \frac{1}{4}\text{tr}[\bar{C}\gamma_a(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}], C = \gamma_y(\varsigma)\gamma_4(\varsigma) \end{cases} \Leftrightarrow \begin{cases} \partial^b F_{ab} + m^2 A_a = 0, F_{ab} = \partial_a A_b - \partial_b A_a \\ \psi_{\lambda_\varsigma\mu_\varsigma} = \mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(x)\frac{A_a}{2} \end{cases}$$

定理1.1.2.  $\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(p)(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})\mathbb{X}_{\lambda'_\varsigma\mu'_\varsigma}^{+a'}(p) = \frac{1}{2}[(m - i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma[(m - i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma})}$

### 1.2 自旋-1粒子Bargmann-Wigner方程<sup>[17]</sup>的平面波解

定理1.2.1.  $(\gamma^a\partial_a + m)_{\kappa_\varsigma} \psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r}, t) = 0, \psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r}, t) = \frac{1}{2!}\psi_{\{\lambda_\varsigma\mu_\varsigma\}}(\vec{r}, t)$

$$\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \sqrt{\frac{m^2}{E}} [a(\vec{p}, h)U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)V_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\begin{cases} U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, 1) = u_{\lambda_\varsigma}(\vec{p}, \frac{1}{2})u_{\mu_\varsigma}(\vec{p}, \frac{1}{2}), U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, -1) = u_{\lambda_\varsigma}(\vec{p}, -\frac{1}{2})u_{\mu_\varsigma}(\vec{p}, -\frac{1}{2}) \\ U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, 0) = \frac{1}{\sqrt{2}}[u_{\lambda_\varsigma}(\vec{p}, \frac{1}{2})u_{\mu_\varsigma}(\vec{p}, -\frac{1}{2}) + u_{\lambda_\varsigma}(\vec{p}, -\frac{1}{2})u_{\mu_\varsigma}(\vec{p}, \frac{1}{2})] \end{cases}$$

$$\begin{cases} V_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, 1) = v_{\lambda_\varsigma}(\vec{p}, \frac{1}{2})v_{\mu_\varsigma}(\vec{p}, \frac{1}{2}), V_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, -1) = v_{\lambda_\varsigma}(\vec{p}, -\frac{1}{2})v_{\mu_\varsigma}(\vec{p}, -\frac{1}{2}) \\ V_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, 0) = \frac{1}{\sqrt{2}}[v_{\lambda_\varsigma}(\vec{p}, \frac{1}{2})v_{\mu_\varsigma}(\vec{p}, -\frac{1}{2}) + v_{\lambda_\varsigma}(\vec{p}, -\frac{1}{2})v_{\mu_\varsigma}(\vec{p}, \frac{1}{2})] \end{cases}$$

$$\begin{cases} a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{r}=-\infty}^{+\infty} \sqrt{\frac{m^2}{E^3}} U^{+\lambda_\varsigma\mu_\varsigma}(\vec{p}, h)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r}, t)e^{-i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \\ b^+(\vec{p}, s) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{r}=-\infty}^{+\infty} \sqrt{\frac{m^2}{E^3}} V^{+\lambda_\varsigma\mu_\varsigma}(\vec{p}, h)\psi_{\lambda_\varsigma\mu_\varsigma}(\vec{r}, t)e^{i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \end{cases}$$

定理1.2.2.  $[\psi_{\lambda_\varsigma\mu_\varsigma}(x), \psi_{\lambda'_\varsigma\mu'_\varsigma}^+(x')]$

$$= \frac{i}{8}[(m - \gamma^a\partial_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma[(m - \gamma^b\partial_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma})}\Delta(x - x') = \frac{i}{4}\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(x)\mathbb{X}_{\lambda'_\varsigma\mu'_\varsigma}^{+a'}(x')[\eta_{aa'} - \frac{\partial_a\partial_{a'}}{m^2}]\Delta(x - x')$$

定义1.2.1.

$$\begin{cases} \Lambda_{+\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p}, 1) := \sum_{h=1}^{-1} U_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, h)U_{\lambda'_\varsigma\mu'_\varsigma}^+(\vec{p}, h) \\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p}, 1) := \sum_{h=1}^{-1} V_{\lambda_\varsigma\mu_\varsigma}(\vec{p}, h)V_{\lambda'_\varsigma\mu'_\varsigma}^+(\vec{p}, h) \end{cases}$$

定理1.2.3.

$$\begin{cases} \Lambda_{+\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p}, 1) = \frac{1}{8m^2}\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(p)\Lambda_{maa'}(\vec{p}, 1)\mathbb{X}_{\lambda'_\varsigma\mu'_\varsigma}^{+a'}(p) = \frac{1}{(2!)^2}\Lambda_{\{\lambda_\varsigma(\lambda'_\varsigma(\frac{1}{2})\Lambda_{+\mu_\varsigma\mu'_\varsigma})\}}(\vec{p}, \frac{1}{2}) \\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p}, 1) = \frac{1}{8m^2}\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(-p)\Lambda_{maa'}(\vec{p}, 1)\mathbb{X}_{\lambda'_\varsigma\mu'_\varsigma}^{+a'}(-p) = \frac{1}{(2!)^2}\Lambda_{\{\lambda_\varsigma(\lambda'_\varsigma(\frac{1}{2})\Lambda_{-\mu_\varsigma\mu'_\varsigma})\}}(\vec{p}, \frac{1}{2}) \end{cases}$$

定理1.2.4.

$$\begin{cases} \Lambda_{+\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p}, 1) = \frac{1}{8m^2}\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(p)(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})\mathbb{X}_{\lambda'_\varsigma\mu'_\varsigma}^{+a'}(p) = \frac{1}{16m^2}[(m - i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma[(m - i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma})} \\ \Lambda_{-\lambda_\varsigma\mu_\varsigma\lambda'_\varsigma\mu'_\varsigma}(\vec{p}, 1) = \frac{1}{8m^2}\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(-p)(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})\mathbb{X}_{\lambda'_\varsigma\mu'_\varsigma}^{+a'}(-p) = \frac{1}{16m^2}[(m + i\gamma^a p_a)\gamma^4]_{\{\lambda_\varsigma(\lambda'_\varsigma[(m + i\gamma^b p_b)\gamma^4]_{\mu_\varsigma\mu'_\varsigma})} \end{cases}$$

↓

### 1.3 推导到自旋-1粒子Klein-Gordon方程 [26, 39, 40]的平面波解

定理1.3.1.  $\partial^b F_{ab} + m^2 A_a = 0, F_{ab} = \partial_a A_b - \partial_b A_a, A_a = \frac{1}{2im} (\bar{C}\gamma_a)^{\lambda\mu\kappa} \psi_{\lambda\mu\kappa}$

$$A_a(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} [a(\vec{p}, h) \varepsilon_a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) \tilde{\varepsilon}_a(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\begin{cases} \varepsilon_a(\vec{p}, 1) = \frac{1}{i\sqrt{2}} u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_a u(\vec{p}, \frac{1}{2}), \varepsilon_a(\vec{p}, -1) = \frac{1}{i\sqrt{2}} u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_a u(\vec{p}, -\frac{1}{2}) \\ \varepsilon_a(\vec{p}, 0) = \frac{1}{i\sqrt{2}} \frac{1}{\sqrt{2}} [u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_a u(\vec{p}, -\frac{1}{2}) + u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_a u(\vec{p}, \frac{1}{2})] \\ \tilde{\varepsilon}_a(\vec{p}, 1) = \frac{1}{i\sqrt{2}} v^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_a v(\vec{p}, \frac{1}{2}), \tilde{\varepsilon}_a(\vec{p}, -1) = \frac{1}{i\sqrt{2}} v^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_a v(\vec{p}, -\frac{1}{2}) \\ \tilde{\varepsilon}_a(\vec{p}, 0) = \frac{1}{i\sqrt{2}} \frac{1}{\sqrt{2}} [v^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_a v(\vec{p}, -\frac{1}{2}) + v^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_a v(\vec{p}, \frac{1}{2})] \end{cases}$$

定理1.3.2.  $\varepsilon^+(\vec{p}, h) \varepsilon(\vec{p}, h') = (\frac{E^2+p^2}{m^2})^{1-|h|} \delta_{hh'}, \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \varepsilon_a^+(\vec{p}, h) = \eta_{aa'} + \frac{p_a p_a'}{m^2}, \sum_{h=1}^{-1} h \varepsilon(\vec{p}, h) \varepsilon^+(\vec{p}, h) = R \cdot \hat{p}$

引理1.3.1.  $\gamma^a(\zeta) C = \begin{bmatrix} 0 & (\sigma, i\zeta) \sigma_y \\ (\sigma, -i\zeta) \sigma_y & 0 \end{bmatrix}, \bar{C} \gamma^a(\zeta) = \begin{bmatrix} 0 & \sigma_y (\sigma, i\zeta)^a \\ \sigma_y (\sigma, -i\zeta)^a & 0 \end{bmatrix}$

定义1.3.1.  $u^+(\vec{p}, \frac{1}{2}) = -i\zeta u^T(\vec{p}, \frac{1}{2}) \sigma_y \otimes \sigma_x, u^+(\vec{p}, -\frac{1}{2}) = i\zeta u^T(\vec{p}, -\frac{1}{2}) \sigma_y \otimes \sigma_x$

性质1.3.1.

$$\begin{cases} \varepsilon_a(\vec{p}, 1) = -\frac{i}{\sqrt{2}} u^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_a u(\vec{p}, \frac{1}{2}) = -\frac{i}{\sqrt{2}} \lambda^T(\vec{p}, \frac{1}{2}) \sigma_y(\sigma, 0)_a \lambda(\vec{p}, \frac{1}{2}) = [i\lambda_m(\vec{p}, 1), 0]_a \\ \varepsilon_a(\vec{p}, 0) = -iu^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_a u(\vec{p}, -\frac{1}{2}) = -\frac{i}{m} \lambda^T(\vec{p}, \frac{1}{2}) \sigma_y(E\sigma, i|\vec{p}|)_a \lambda(\vec{p}, -\frac{1}{2}) = \frac{1}{m} [iE\lambda_m(\vec{p}, 0), i|\vec{p}|]_a \\ \varepsilon_a(\vec{p}, -1) = -\frac{i}{\sqrt{2}} u^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_a u(\vec{p}, -\frac{1}{2}) = -\frac{i}{\sqrt{2}} \lambda^T(\vec{p}, -\frac{1}{2}) \sigma_y(\sigma, 0)_a \lambda(\vec{p}, -\frac{1}{2}) = [i\lambda_m(\vec{p}, -1), 0]_a \end{cases}$$

性质1.3.2.

$$\begin{cases} \tilde{\varepsilon}_a(\vec{p}, 1) = -\frac{i}{\sqrt{2}} v^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_a v(\vec{p}, \frac{1}{2}) = \frac{i}{\sqrt{2}} \lambda^T(\vec{p}, \frac{1}{2}) \sigma_y(\sigma, 0)_a \lambda(\vec{p}, \frac{1}{2}) = -[i\lambda_m(\vec{p}, 1), 0]_a \\ \tilde{\varepsilon}_a(\vec{p}, 0) = -iv^T(\vec{p}, \frac{1}{2}) \bar{C} \gamma_a v(\vec{p}, -\frac{1}{2}) = \frac{i}{m} \lambda^T(\vec{p}, \frac{1}{2}) \sigma_y(E\sigma, i|\vec{p}|)_a \lambda(\vec{p}, -\frac{1}{2}) = -\frac{1}{m} [iE\lambda_m(\vec{p}, 0), i|\vec{p}|]_a \\ \tilde{\varepsilon}_a(\vec{p}, -1) = -\frac{i}{\sqrt{2}} v^T(\vec{p}, -\frac{1}{2}) \bar{C} \gamma_a v(\vec{p}, -\frac{1}{2}) = \frac{i}{\sqrt{2}} \lambda^T(\vec{p}, -\frac{1}{2}) \sigma_y(\sigma, 0)_a \lambda(\vec{p}, -\frac{1}{2}) = -[i\lambda_m(\vec{p}, -1), 0]_a \end{cases}$$

推论1.3.1.  $\tilde{\varepsilon}_a(\vec{p}, 1) = -\varepsilon_a(\vec{p}, 1), \tilde{\varepsilon}_a(\vec{p}, 0) = -\varepsilon_a(\vec{p}, 0), \tilde{\varepsilon}_a(\vec{p}, -1) = -\varepsilon_a(\vec{p}, -1)$

推论1.3.2.  $\partial^b F_{ab} + m^2 A_a = 0, F_{ab} = \partial_a A_b - \partial_b A_a, A_a = \frac{1}{2im} (\bar{C}\gamma_a)^{\lambda\mu\kappa} \psi_{\lambda\mu\kappa}$

$$A_a(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \varepsilon_a(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\varepsilon_a(\vec{p}, 1) = [i\lambda_m(\vec{p}, 1), 0]_a, \varepsilon_a(\vec{p}, 0) = \frac{1}{m} [iE\lambda_m(\vec{p}, 0), i|\vec{p}|]_a, \varepsilon_a(\vec{p}, -1) = [i\lambda_m(\vec{p}, -1), 0]_a$$

定理1.3.3.  $[A_a(x), A_a^+(x')] = i(\eta_{aa'} - \frac{\partial_a \partial_a'}{m^2}) \Delta(x - x')$

定理1.3.4.  $\Lambda_{maa'}(\vec{p}, 1) := \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \varepsilon_a^+(\vec{p}, h) = \eta_{aa'} + \frac{p_a p_a'}{m^2}$

定理1.3.5.  $\Lambda_{\pm\tau\tau'}(\vec{p}, \frac{1}{2}) = \frac{1}{3} \Lambda_{maa'}(\vec{p}, 1) \gamma^a \Lambda_{\mp}(\vec{p}, \frac{1}{2}) \gamma^{a'}$

↓

### 1.4 再回到自旋-1粒子Bargmann-Wigner方程 [17]的平面波解

定理1.4.1.  $(\gamma^a \partial_a + m)_{\kappa\zeta} \lambda_{\lambda\mu\kappa}(\vec{r}, t) = 0, \psi_{\lambda\mu\kappa}(\vec{r}, t) = [im\gamma^a(\zeta) C - 2S^{ab}(e, \zeta) C \partial_b] \frac{A_a(\vec{r}, t)}{2}$

$$\psi_{\lambda\mu\kappa}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \sqrt{\frac{m^2}{E}} [a(\vec{p}, h) U_{\lambda\mu\kappa}(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) V_{\lambda\mu\kappa}(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$U_{\lambda\mu\kappa}(\vec{p}, h) = \frac{1}{2\sqrt{2m}} \mathbb{X}_{\lambda\mu\kappa}^a(p) \varepsilon_a(\vec{p}, h), V_{\lambda\mu\kappa}(\vec{p}, h) = \frac{1}{2\sqrt{2m}} \mathbb{X}_{\lambda\mu\kappa}^a(-p) \tilde{\varepsilon}_a(\vec{p}, h)$$

$$\begin{cases} a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m^2}{E^3}} U^{+\lambda\mu\kappa}(\vec{p}, h) \psi_{\lambda\mu\kappa}(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \\ b^+(\vec{p}, s) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m^2}{E^3}} V^{+\lambda\mu\kappa}(\vec{p}, h) \psi_{\lambda\mu\kappa}(\vec{r}, t) e^{i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \end{cases}$$

## 2 有质量矢量粒子方程的第三种等价描述

### 2.1 有质量矢量粒子自旋方程等价描述

定理2.1.1.  $(\partial_a + iS_{ab}\partial^b)_{\beta\gamma}\alpha_\gamma\psi_{\alpha\gamma} = \frac{i}{\sqrt{2}}im^2\sigma_{\beta\gamma}^{ab}A_b, \psi_{\alpha\gamma} := \frac{i}{\sqrt{2}}\frac{i}{2}\sigma_{\beta\gamma}^{ab}F_{ab}, S_{ab} := i\sigma_{\beta\gamma}^{ab}\gamma_{\alpha\gamma}$

$$A_a(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \varepsilon_a(\vec{p}, h) [a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$F_{ab}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} [ip_a\varepsilon_b(\vec{p}, h) - ip_b\varepsilon_a(\vec{p}, h)] [a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\psi_{\alpha\gamma}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \frac{-i}{\sqrt{2}} \sigma_{\beta\gamma}^{ab} p_a \varepsilon_b(\vec{p}, h) [a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

### 2.2 有质量矢量粒子场 $F_{ab}$ 的平面波解和投影算子

定义2.2.1.  $\lambda_{ab}(\vec{p}, h) := [ip_a\varepsilon_b(\vec{p}, h) - ip_b\varepsilon_a(\vec{p}, h)]$

$$\text{推论2.2.1. } F_{ab}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \lambda_{ab}(\vec{p}, h) [a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

定理2.2.1.  $\sum_{h=1}^{-1} \lambda_{ab}(\vec{p}, h) \lambda_{a'b'}^+(\vec{p}, h) = p_{[a} p_{[a'}^+ \eta_{b]b'}$

证明:  $\sum_{h=1}^{-1} \lambda_{ab}(\vec{p}, h) \lambda_{a'b'}^+(\vec{p}, h)$

$$= \sum_{h=1}^{-1} [ip_a\varepsilon_b(\vec{p}, h) - ip_b\varepsilon_a(\vec{p}, h)] [ip_{a'}\varepsilon_{b'}(\vec{p}, h) - ip_{b'}\varepsilon_{a'}(\vec{p}, h)]^+$$

$$= p_a p_{a'}^+ \sum_{h=1}^{-1} \varepsilon_b(\vec{p}, h) \varepsilon_{b'}^+(\vec{p}, h) + p_b p_{b'}^+ \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \varepsilon_{a'}^+(\vec{p}, h) - p_a p_{b'}^+ \sum_{h=1}^{-1} \varepsilon_b(\vec{p}, h) \varepsilon_{a'}^+(\vec{p}, h) - p_b p_{a'}^+ \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \varepsilon_{b'}^+(\vec{p}, h)$$

$$= p_a p_{a'}^+ (\eta_{bb'} + \frac{p_b p_{b'}^+}{m^2}) + p_b p_{b'}^+ (\eta_{aa'} + \frac{p_a p_{a'}^+}{m^2}) - p_a p_{b'}^+ (\eta_{ba'} + \frac{p_b p_{a'}^+}{m^2}) - p_b p_{a'}^+ (\eta_{ab'} + \frac{p_a p_{b'}^+}{m^2})$$

$$= p_a p_{a'}^+ \eta_{bb'} + p_b p_{b'}^+ \eta_{aa'} - p_a p_{b'}^+ \eta_{ba'} - p_b p_{a'}^+ \eta_{ab'}$$

$$= p_{[a} p_{[a'}^+ \eta_{b]b'}$$

□

定理2.2.2.  $[F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a[a'}\partial_{b]}\partial_{b']}^+ \Delta(x-x')$

### 2.3 有质量矢量粒子场 $\Psi_{\alpha\gamma}$ 的平面波解和投影算子

定义2.3.1.  $\lambda_{\alpha\gamma}(\vec{p}, h) := \frac{-i}{\sqrt{2}} \sigma_{\beta\gamma}^{ab} p_a \varepsilon_b(\vec{p}, h)$

$$\text{推论2.3.1. } \psi_{\alpha\gamma}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \lambda_{\alpha\gamma}(\vec{p}, h) [a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

推论2.3.2.  $\lambda_{\alpha\gamma}(\vec{p}, h) = \frac{-\zeta}{\sqrt{2}} h |\vec{p}| \lambda_{m\alpha\gamma}(\vec{p}, h) - \frac{-\zeta}{\sqrt{2}} p_{\alpha\gamma} \varepsilon_4(\vec{p}, h) + \frac{-i\zeta}{\sqrt{2}} E \varepsilon_{\alpha\gamma}(\vec{p}, h)$

证明:  $-\zeta \lambda_{\alpha\gamma}(\vec{p}, h) := \frac{i\zeta}{\sqrt{2}} \sigma_{\beta\gamma}^{ab} p_a \varepsilon_b(\vec{p}, h)$

$$= \frac{i\zeta}{\sqrt{2}} \sigma_{\beta\gamma}^{ij} p_i \varepsilon_j(\vec{p}, h) + \frac{i\zeta}{\sqrt{2}} \sigma_{\beta\gamma}^{i4} p_i \varepsilon_4(\vec{p}, h) + \frac{i\zeta}{\sqrt{2}} \sigma_{\beta\gamma}^{4j} p_4 \varepsilon_j(\vec{p}, h)$$

$$= \frac{1}{\sqrt{2}} h |\vec{p}| \lambda_{\alpha\gamma}(\vec{p}, h) - \frac{\zeta}{\sqrt{2}} p_{\alpha\gamma} \varepsilon_4(\vec{p}, h) + \frac{\zeta}{\sqrt{2}} p_4 \varepsilon_{\alpha\gamma}(\vec{p}, h)$$

$$= \frac{1}{\sqrt{2}} h |\vec{p}| \lambda_{m\alpha\gamma}(\vec{p}, h) - \frac{1}{\sqrt{2}} p_{\alpha\gamma} \varepsilon_4(\vec{p}, h) + \frac{i}{\sqrt{2}} E \varepsilon_{\alpha\gamma}(\vec{p}, h)$$

□

推论2.3.3.  $\lambda_{\alpha\gamma}(\vec{p}, \kappa) = \frac{1}{\sqrt{2}} (E - \zeta \kappa |\vec{p}|) \lambda_{m\alpha\gamma}(\vec{p}, \kappa), \lambda_{\alpha\gamma}(\vec{p}, 0) = \frac{1}{\sqrt{2}} m \lambda_{m\alpha\gamma}(\vec{p}, 0)$

定理2.3.1.  $\sum_{h=1}^{-1} \lambda_{\alpha\gamma}(\vec{p}, h) \lambda_{\alpha'\gamma'}^+(\vec{p}, h) = -\sigma_{\alpha\alpha'}^{ab} p_a p_b$

证明:  $\sum_{h=1}^{-1} \lambda_{\alpha\gamma}(\vec{p}, h) \lambda_{\alpha'\gamma'}^+(\vec{p}, h)$

$$= \frac{1}{2} (E - \zeta |\vec{p}|)^2 \lambda_{m\alpha\gamma}(\vec{p}, 1) \lambda_{m\alpha'\gamma'}^+(\vec{p}, 1) + \frac{1}{2} (E + \zeta |\vec{p}|)^2 \lambda_{m\alpha\gamma}(\vec{p}, -1) \lambda_{m\alpha'\gamma'}^+(\vec{p}, -1) + \frac{1}{2} m^2 \hat{p}_{\alpha\gamma} \hat{p}_{\alpha'\gamma'}$$

$$= -\frac{1}{4} (E - \zeta |\vec{p}|)^2 (\hat{p}_{\alpha\gamma} \hat{p}_{\alpha'\gamma'} - \delta_{\alpha\alpha'} \delta_{\gamma\gamma'}) + i \varepsilon^k_{\alpha\alpha'} \hat{p}_k - \frac{1}{4} (E + \zeta |\vec{p}|)^2 (\hat{p}_{\alpha\gamma} \hat{p}_{\alpha'\gamma'} - \delta_{\alpha\alpha'} \delta_{\gamma\gamma'}) + i \varepsilon^k_{\alpha\alpha'} \hat{p}_k + \frac{1}{2} m^2 \hat{p}_{\alpha\gamma} \hat{p}_{\alpha'\gamma'}$$

$$= -p_{\alpha\gamma} p_{\alpha'\gamma'} + \frac{1}{2} (E^2 + \vec{p}^2) \delta_{\alpha\alpha'} \delta_{\gamma\gamma'} - i \zeta E \varepsilon^k_{\alpha\alpha'} p_k$$

$$= -\sigma_{\alpha\alpha'}^{ab} p_a p_b$$

□

$$\begin{aligned}
& \text{证明: } \sum_{h=1}^{-1} \lambda_{\alpha_\zeta}(\vec{p}, h) \lambda_{\alpha'_\zeta}^+(\vec{p}, h) \\
&= \sum_{h=1}^{-1} \frac{-i}{\sqrt{2}} \sigma_{\zeta\alpha_\zeta}^{ab} p_a \varepsilon_b(\vec{p}, h) \frac{-i}{\sqrt{2}} \sigma_{\zeta\alpha'_\zeta}^{a'b'} p_{a'}^+ \varepsilon_{b'}^+(\vec{p}, h) \\
&= -\frac{1}{2} \sigma_{\zeta\alpha_\zeta}^{ab} \sigma_{\zeta\alpha'_\zeta}^{a'b'} p_a p_{a'}^+ \sum_{h=1}^{-1} \varepsilon_b(\vec{p}, h) \varepsilon_{b'}^+(\vec{p}, h) \\
&= -\frac{1}{2} \sigma_{\zeta\alpha_\zeta}^{ab} \sigma_{\zeta\alpha'_\zeta}^{a'b'} p_a p_{a'}^+ (\eta_{bb'} + \frac{p_b p_{b'}^+}{m^2}) \\
&= -\frac{1}{2} \sigma_{\zeta\alpha_\zeta}^{ab} \sigma_{-\zeta\alpha'_\zeta}^{a'b'} p_a p_{a'}^+ \delta_{bb'} \\
&= -\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} p_a p_b
\end{aligned}$$

□

$$\text{定理2.3.2. } [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x')$$

## 2.4 有质量矢量粒子方程的第三种等价描述小结

$$\text{定理2.4.1. } (\partial_a + iS_{ab}\partial^b)_{\beta_\zeta\alpha_\zeta} \psi_{\alpha_\zeta} = \frac{i}{\sqrt{2}} im^2 \sigma_{\zeta\beta_\zeta}^{ab} A_b, \psi_{\alpha_\zeta} := \frac{i}{\sqrt{2}} \frac{i}{2} \sigma_{\zeta\alpha_\zeta}^{ab} F_{ab}, S_{ab} := i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta}$$

$$A_a(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \varepsilon_a(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$F_{ab}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \lambda_{ab}(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\psi_{\alpha_\zeta}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \lambda_{\alpha_\zeta}(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\text{定理2.4.2. } \begin{cases} \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \varepsilon_{a'}^+(\vec{p}, h) = \eta_{aa'} + \frac{p_a p_{a'}^+}{m^2} \\ \sum_{h=1}^{-1} \lambda_{ab}(\vec{p}, h) \lambda_{a'b'}^+(\vec{p}, h) = p_{[a} p_{a']}^+ \eta_{b]b'} \\ \sum_{h=1}^{-1} \lambda_{\alpha_\zeta}(\vec{p}, h) \lambda_{\alpha'_\zeta}^+(\vec{p}, h) = -\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} p_a p_b \end{cases} \begin{cases} [A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a \partial_{a'}^+}{m^2}) \Delta(x - x') \\ [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a[a'] \partial_b] \partial_{b'}^+} \Delta(x - x') \\ [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \end{cases}$$

## 2.5 $m \rightarrow 0$ 形式推得光子情形

$$\text{定理2.5.1. } (\partial_a + iS_{ab}\partial^b)_{\beta_\zeta\alpha_\zeta} \psi_{\alpha_\zeta} \rightarrow 0, \psi_{\alpha_\zeta} := \frac{i}{\sqrt{2}} \frac{i}{2} \sigma_{\zeta\alpha_\zeta}^{ab} F_{ab}, S_{ab} := i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta}$$

$$A_a(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \varepsilon_a(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p} \rightarrow \infty$$

$$F_{ab}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \lambda_{ab}(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p} \rightarrow \infty$$

$$\psi_{\alpha_\zeta}(\vec{r}, t) \rightarrow \frac{-\zeta}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{|\vec{p}|} \lambda_{m\alpha_\zeta}(\vec{p}, -\zeta) [a(\vec{p}, -\zeta) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, -\zeta) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\text{推论2.5.1. } \varepsilon_a(\vec{p}, 1) = [i\lambda_m(\vec{p}, 1), 0]_a, \varepsilon_a(\vec{p}, 0) = \frac{1}{m} [iE\lambda_m(\vec{p}, 0), i|\vec{p}|]_a \rightarrow \infty, \varepsilon_a(\vec{p}, -1) = [i\lambda_m(\vec{p}, -1), 0]_a$$

$$\text{推论2.5.2. } \lambda_{\alpha_\zeta}(\vec{p}, -\zeta) \rightarrow \sqrt{2} |\vec{p}| \lambda_{m\alpha_\zeta}(\vec{p}, -\zeta), \lambda_{\alpha_\zeta}(\vec{p}, \zeta) \rightarrow 0, \lambda_{\alpha_\zeta}(\vec{p}, 0) \rightarrow 0$$

$$\text{推论2.5.3. } \begin{cases} [A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a \partial_{a'}^+}{m^2}) \Delta(x - x') \rightarrow \infty \\ [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a[a'] \partial_b] \partial_{b'}^+} \Delta(x - x') \\ [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \end{cases}$$

从上面可以看出，在 $m \rightarrow 0$ 时， $A_a, F_{ab} \rightarrow \infty$ 变得没意义，但 $\psi_{\alpha_\zeta}$ 仍然有意义，可以自然过渡，重新写在下面。当然严格的做法仍要采用无质量粒子的方法，这里只是形式的推导。

$$\text{推论2.5.4. } (\partial_a + iS_{ab}\partial^b)_{\beta_\zeta\alpha_\zeta} \psi_{\alpha_\zeta} \rightarrow 0$$

$$\psi_{\alpha_\zeta}(\vec{r}, t) \rightarrow \frac{-\zeta}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{|\vec{p}|} \lambda_{m\alpha_\zeta}(\vec{p}, -\zeta) [a(\vec{p}, -\zeta) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, -\zeta) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\lambda_{\alpha_\zeta}(\vec{p}, -\zeta) \lambda_{\alpha'_\zeta}^+(\vec{p}, -\zeta) = -\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} p_a p_b, [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x')$$

### 3 有质量矢量场对易规则两种表述等价性的直观证明方法

自我评述：简单巧妙的解析证法前一章已经给出，那为什么还要给出其它的证法呢？原因有三个：一是我最先按这复杂直观的方法证明的；二是在这个复杂的证明过程中，可以顺便得到有质量矢量场分解的全部对易规则；三是可以得到一批有用的恒等式。我已经多次遇到这样类似的情况，不同证法往往需要完全不同的数学技巧。有些抽象证明很简洁，但看不清细节，有的时候明明证出来了，总有些怀疑，因为不直观、太抽象；而构造性的证明，有时简洁，有时很繁琐，但每一步都可以看得清清楚楚。

#### 3.1 数学准备

##### 3.1.1 常数不变张量 $\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}$ 的引入

$$\text{定义3.1.1. } \Gamma_{\lambda_\zeta \mu_\zeta}^a := [\gamma^a(\zeta)C]_{\lambda_\zeta \mu_\zeta} = \begin{bmatrix} 0_{A_\zeta B_\zeta} & [(\sigma, i\zeta)\sigma_y]_{A_\zeta B_\zeta}^a \\ [(\sigma, -i\zeta)\sigma_y]_{A'_\zeta B'_\zeta} & 0_{A'_\zeta B'_\zeta} \end{bmatrix}$$

$$\text{定义3.1.2. } \Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}^{ab} := \Gamma_{\lambda_\zeta \mu_\zeta}^a \Gamma_{\eta_\zeta \xi_\zeta}^b, \Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} := \Gamma_{\lambda_\zeta \mu_\zeta}^a \delta_{ab} \Gamma_{\eta_\zeta \xi_\zeta}^b$$

性质3.1.1.

$$\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \Gamma_{\mu_\zeta \lambda_\zeta \eta_\zeta \xi_\zeta}, \Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \Gamma_{\lambda_\zeta \mu_\zeta \xi_\zeta \eta_\zeta}, \Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \Gamma_{\eta_\zeta \xi_\zeta \lambda_\zeta \mu_\zeta}$$

$$\Gamma_{1_\zeta 3_\zeta 2_\zeta 4_\zeta} = \Gamma_{3_\zeta 1_\zeta 2_\zeta 4_\zeta} = \Gamma_{1_\zeta 3_\zeta 4_\zeta 2_\zeta} = \Gamma_{3_\zeta 1_\zeta 4_\zeta 2_\zeta} = 1,$$

$$\Gamma_{2_\zeta 4_\zeta 1_\zeta 3_\zeta} = \Gamma_{2_\zeta 4_\zeta 3_\zeta 1_\zeta} = \Gamma_{4_\zeta 2_\zeta 1_\zeta 3_\zeta} = \Gamma_{4_\zeta 2_\zeta 3_\zeta 1_\zeta} = 1,$$

$$\Gamma_{1_\zeta 4_\zeta 2_\zeta 3_\zeta} = \Gamma_{4_\zeta 1_\zeta 2_\zeta 3_\zeta} = \Gamma_{1_\zeta 4_\zeta 3_\zeta 2_\zeta} = \Gamma_{4_\zeta 1_\zeta 3_\zeta 2_\zeta} = -1,$$

$$\Gamma_{2_\zeta 3_\zeta 1_\zeta 4_\zeta} = \Gamma_{2_\zeta 3_\zeta 4_\zeta 1_\zeta} = \Gamma_{3_\zeta 2_\zeta 1_\zeta 4_\zeta} = \Gamma_{3_\zeta 2_\zeta 4_\zeta 1_\zeta} = -1$$

$$\Gamma_{rest} = 0, \Gamma_{\{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta\}} = 0$$

$$\text{性质3.1.2. } \Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = (-1)^{\lambda_\zeta + \mu_\zeta} u(\lambda_\zeta + \mu_\zeta - 3) u(\eta_\zeta + \xi_\zeta - 3) |\varepsilon_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}|$$

##### 3.1.2 各种量的矩阵展开

$$\text{引理3.1.1. } \gamma^a = (\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x) = i \begin{bmatrix} 0 & -(\sigma, i\zeta) \\ (\sigma, -i\zeta) & 0 \end{bmatrix}, S_{ab}(\zeta) = \frac{i}{2} \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\alpha\zeta} = -\frac{i}{4} (\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]}$$

$$\text{引理3.1.2. } im\gamma^a(\zeta)C = \begin{bmatrix} 0 & im(\sigma, i\zeta)\sigma_y \\ im(\sigma, -i\zeta)\sigma_y & 0 \end{bmatrix}$$

$$\text{证明: } im\gamma^a(\zeta)C = im(\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x) \zeta \gamma_y(\zeta) \gamma_4(\zeta) = im(\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x) (\sigma_y \otimes \sigma_y) (I \otimes \sigma_x)$$

$$= im(\sigma \otimes \sigma_y \otimes \sigma_x, -\zeta \sigma_y \otimes \sigma_y) = \begin{bmatrix} 0 & im(\sigma, i\zeta)\sigma_y \\ im(\sigma, -i\zeta)\sigma_y & 0 \end{bmatrix} \quad \square$$

$$\text{引理3.1.3. } S_{ab}(e, \zeta) = \begin{bmatrix} S_{ab}(\zeta) & 0 \\ 0 & S_{ab}(-\zeta) \end{bmatrix}$$

$$\text{证明: } S_{ab}(e, \zeta) = -\frac{i}{4} [\gamma_a(\zeta), \gamma_b(\zeta)]$$

$$= -\frac{i}{4} \left\{ \begin{bmatrix} 0 & -i(\sigma, i\zeta)_a \\ i(\sigma, -i\zeta)_a & 0 \end{bmatrix} \begin{bmatrix} 0 & -i(\sigma, i\zeta)_b \\ i(\sigma, -i\zeta)_b & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i(\sigma, i\zeta)_b \\ i(\sigma, -i\zeta)_b & 0 \end{bmatrix} \begin{bmatrix} 0 & -i(\sigma, i\zeta)_a \\ i(\sigma, -i\zeta)_a & 0 \end{bmatrix} \right\}$$

$$= -\frac{i}{4} \begin{bmatrix} (\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]} & 0 \\ 0 & (\sigma, -i\zeta)_{[a} (\sigma, i\zeta)_{b]} \end{bmatrix} = \begin{bmatrix} S_{ab}(\zeta) & 0 \\ 0 & S_{ab}(-\zeta) \end{bmatrix} \quad \square$$

$$\text{引理3.1.4. } -2S^{ab}(e, \zeta)C\partial_b = \begin{bmatrix} \frac{1}{2}(\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]} \sigma_y \partial^b & 0 \\ 0 & -\frac{1}{2}(\sigma, -i\zeta)_{[a} (\sigma, i\zeta)_{b]} \sigma_y \partial^b \end{bmatrix}$$

$$\text{证明: } -2S^{ab}(e, \zeta)C\partial_b = -2 \begin{bmatrix} S^{ab}(\zeta)\partial_b & 0 \\ 0 & S^{ab}(-\zeta)\partial_b \end{bmatrix} (\sigma_y \otimes \sigma_y) (I \otimes \sigma_x) = 2i \begin{bmatrix} S^{ab}(\zeta)\partial_b \sigma_y & 0 \\ 0 & -S^{ab}(-\zeta)\partial_b \sigma_y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]} \sigma_y \partial^b & 0 \\ 0 & -\frac{1}{2}(\sigma, -i\zeta)_{[a} (\sigma, i\zeta)_{b]} \sigma_y \partial^b \end{bmatrix} \quad \square$$

$$\text{推论3.1.1. } \mathbb{X}_a = \begin{bmatrix} 2iS_{ab}(\zeta)\sigma_y \partial^b & im(\sigma, i\zeta)_a \sigma_y \\ im(\sigma, -i\zeta)_a \sigma_y & -2iS_{ab}(-\zeta)\sigma_y \partial^b \end{bmatrix} = \begin{bmatrix} 2\zeta S^{ab}{}_{A_\zeta B_\zeta} \partial_b & im(\sigma, i\zeta)_a \sigma_y \\ im(\sigma, -i\zeta)_a \sigma_y & 2\zeta S_{ab}{}^{A'_\zeta B'_\zeta} \partial^b \end{bmatrix}$$

$$\text{推论3.1.2. } \mathbb{X}_a = \begin{bmatrix} \frac{1}{2}(\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]} \sigma_y \partial^b & im(\sigma, i\zeta)_a \sigma_y \\ im(\sigma, -i\zeta)_a \sigma_y & -\frac{1}{2}(\sigma, -i\zeta)_{[a} (\sigma, i\zeta)_{b]} \sigma_y \partial^b \end{bmatrix}, \mathbb{X}_a^+ = \begin{bmatrix} -\frac{1}{2}\sigma_y (\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]} \partial^{+b} & -im\sigma_y (\sigma, i\zeta)_a \\ -im\sigma_y (\sigma, -i\zeta)_a & \frac{1}{2}\sigma_y (\sigma, -i\zeta)_{[a} (\sigma, i\zeta)_{b]} \partial^{+b} \end{bmatrix}$$

$$\begin{aligned}
\text{推论3.1.3. } \psi_{\lambda_\zeta \mu_\zeta} &= \mathbb{X}_a \frac{A^a}{2} (\sigma, i\zeta)_{A_\zeta A'_\zeta} = [im\gamma_a(\zeta)C - 2S_{ab}(e, \zeta)\partial^b C]_{\lambda_\zeta \mu_\zeta} \frac{A^a}{2} \\
&= \frac{1}{2} \begin{bmatrix} 2\zeta S^{ab}{}_{A_\zeta B_\zeta} \partial_b A_a & im[(\sigma, i\zeta)_a \sigma_y]_{A_\zeta B'_\zeta} A_a \\ im[(\sigma, -i\zeta)_a \sigma_y]_{A'_\zeta B'_\zeta} A^a & 2\zeta S_{ab}{}^{A'_\zeta B'_\zeta} \partial^b A_a \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -\zeta S^{ab}{}_{A_\zeta B_\zeta} F_{ab} & im[(\sigma, i\zeta)_a \sigma_y]_{A_\zeta B'_\zeta} A_a \\ im[(\sigma, -i\zeta)_a \sigma_y]_{A'_\zeta B'_\zeta} A^a & -\zeta S_{ab}{}^{A'_\zeta B'_\zeta} F^{ab} \end{bmatrix} \\
&= \begin{bmatrix} \psi_{A_\zeta B_\zeta} & \psi_{A_\zeta B'_\zeta} \\ \psi_{A'_\zeta B'_\zeta} & \psi_{A'_\zeta B_\zeta} \end{bmatrix}_{B-G} = -\zeta \begin{bmatrix} \Psi_{A_\zeta B_\zeta} & \Psi_{A_\zeta B'_\zeta} \\ \Psi_{A'_\zeta B'_\zeta} & \Psi_{A'_\zeta B_\zeta} \end{bmatrix}_{T_{wo}} = \frac{i}{\sqrt{2}} \begin{bmatrix} \psi_{A_\zeta B_\zeta} & \psi_{A_\zeta B'_\zeta} \\ \psi_{A'_\zeta B'_\zeta} & \psi_{A'_\zeta B_\zeta} \end{bmatrix}_{One}
\end{aligned}$$

### 3.1.3 一个重要引理

$$\begin{aligned}
\text{引理3.1.5. } [\psi_{\lambda_\zeta \mu_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta}^+(x')] &= \frac{i}{4} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x) \mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x') (\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x - x') \\
&= -\frac{i}{4} \begin{bmatrix} 2\zeta S^{ab}{}_{A_\zeta B_\zeta} \partial_b & im[(\sigma, i\zeta)_a \sigma_y]_{A_\zeta B'_\zeta} \\ im[(\sigma, -i\zeta)_a \sigma_y]_{A'_\zeta B'_\zeta} & 2\zeta S_{ab}{}^{A'_\zeta B'_\zeta} \partial^b \end{bmatrix}_{\lambda_\zeta \mu_\zeta} \begin{bmatrix} 2\zeta S^{a'b'}{}_{A'_\zeta B'_\zeta} \partial_{b'} & im[\sigma_y(\sigma, -i\zeta)]_{A'_\zeta B_\zeta} \\ im[\sigma_y(\sigma, i\zeta)]_{A'_\zeta B'_\zeta} & 2\zeta S_{a'b'}{}^{A_\zeta B_\zeta} \partial^{b'} \end{bmatrix}_{\lambda'_\zeta \mu'_\zeta} (\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x - x')
\end{aligned}$$

$$\begin{aligned}
\text{证明: } [\psi_{\lambda_\zeta \mu_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta}^+(x')] &= \frac{i}{4} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a [\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'} \Delta(x - x') \\
&= \frac{i}{4} \{ [im\gamma^a(\zeta) - 2S^{ab}(e, \zeta)\partial_b] C \}_{\lambda_\zeta \mu_\zeta} [\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \{ [im\gamma^{a'}(\zeta) + 2S^{a'b'}(e, \zeta)\partial_{b'}] C \}_{\lambda'_\zeta \mu'_\zeta}^+ \\
&\Delta(x - x') \\
&= \frac{i}{4} \{ [im\gamma^a(\zeta) - 2S^{ab}(e, \zeta)\partial_b] C \}_{\lambda_\zeta \mu_\zeta} [\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] \{ C^+ [-im\gamma^{a'}(\zeta) - 2S^{a'b'}(e, \zeta)\partial_{b'}] \}_{\lambda'_\zeta \mu'_\zeta} \Delta(x - x') \\
&= \frac{i}{4} \{ [im\gamma^a(\zeta) - 2S^{ab}(e, \zeta)\partial_b] C \}_{\lambda_\zeta \mu_\zeta} (\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \{ C^+ [-im\gamma^{a'}(-\zeta) - 2S^{a'b'}(e, -\zeta)\partial_{b'}] \}_{\lambda'_\zeta \mu'_\zeta} \Delta(x - x') \\
&= \frac{i}{4} \{ [im\gamma^a(\zeta) - 2S^{ab}(e, \zeta)\partial_b] \zeta \gamma_y(\zeta) \gamma_4(\zeta) \}_{\lambda_\zeta \mu_\zeta} \\
&(\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \{ \zeta \gamma_4(\zeta) \gamma_y(\zeta) [-im\gamma^{a'}(-\zeta) - 2S^{a'b'}(e, -\zeta)\partial_{b'}] \}_{\lambda'_\zeta \mu'_\zeta} \Delta(x - x') \\
&= \frac{i}{4} \{ [im\gamma^a(\zeta) - 2S^{ab}(e, \zeta)\partial_b] [-\zeta \gamma_4(\zeta) \gamma_y(\zeta)] \}_{\lambda_\zeta \mu_\zeta} \\
&(\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \{ \zeta \gamma_y(-\zeta) \gamma_4(-\zeta) [-im\gamma^{a'}(-\zeta) - 2S^{a'b'}(e, -\zeta)\partial_{b'}] \}_{\lambda'_\zeta \mu'_\zeta} \Delta(x - x') \\
&= \frac{i}{4} \{ im \begin{bmatrix} 0 & (\sigma, i\zeta)_a \sigma_y \\ (\sigma, -i\zeta)_a \sigma_y & 0 \end{bmatrix} + 2i \begin{bmatrix} S^{ab}(\zeta) \sigma_y & 0 \\ 0 & -S^{ab}(-\zeta) \sigma_y \end{bmatrix} \partial_b \}_{\lambda_\zeta \mu_\zeta} (\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \\
&\{ -im \begin{bmatrix} 0 & \sigma_y(\sigma, -i\zeta) \\ \sigma_y(\sigma, i\zeta) & 0 \end{bmatrix} - 2i \begin{bmatrix} \sigma_y S^{ab}(-\zeta) & 0 \\ 0 & -\sigma_y S^{ab}(\zeta) \end{bmatrix} \partial_{b'} \}_{\lambda'_\zeta \mu'_\zeta} \Delta(x - x') \\
&= \frac{i}{4} \left[ \begin{matrix} 2iS^{ab}(\zeta) \sigma_y \partial_b & im(\sigma, i\zeta)_a \sigma_y \\ im(\sigma, -i\zeta)_a \sigma_y & -2iS^{ab}(-\zeta) \sigma_y \partial_b \end{matrix} \right]_{\lambda_\zeta \mu_\zeta} (\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \left[ \begin{matrix} -2i\sigma_y S^{a'b'}(-\zeta) \partial_{b'} & -im\sigma_y(\sigma, -i\zeta) \\ -im\sigma_y(\sigma, i\zeta) & 2i\sigma_y S^{a'b'}(\zeta) \partial_{b'} \end{matrix} \right]_{\lambda'_\zeta \mu'_\zeta} \Delta(x - x') \\
&= -\frac{i}{4} \begin{bmatrix} 2\zeta S^{ab}{}_{A_\zeta B_\zeta} \partial_b & im[(\sigma, i\zeta)_a \sigma_y]_{A_\zeta B'_\zeta} \\ im[(\sigma, -i\zeta)_a \sigma_y]_{A'_\zeta B'_\zeta} & 2\zeta S_{ab}{}^{A'_\zeta B'_\zeta} \partial^b \end{bmatrix}_{\lambda_\zeta \mu_\zeta} \begin{bmatrix} 2\zeta S^{a'b'}{}_{A'_\zeta B'_\zeta} \partial_{b'} & im[\sigma_y(\sigma, -i\zeta)]_{A'_\zeta B_\zeta} \\ im[\sigma_y(\sigma, i\zeta)]_{A'_\zeta B'_\zeta} & 2\zeta S_{a'b'}{}^{A_\zeta B_\zeta} \partial^{b'} \end{bmatrix}_{\lambda'_\zeta \mu'_\zeta} (\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x - x') \quad \square
\end{aligned}$$

### 3.2 有质量矢量场对易规则两种表述等价性的第一种直观证明方法

$$\begin{aligned}
\text{引理3.2.1. } [(m - \gamma^a \partial_a) \gamma^4]_{\lambda_\zeta \lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \Delta(x - x') \\
= -[-imI \otimes \sigma_x + (\sigma \otimes \sigma_z, i\zeta)^a \partial_a]_{\lambda_\zeta \lambda'_\zeta} [-imI \otimes \sigma_x + (\sigma \otimes \sigma_z, i\zeta)^b \partial_b]_{\mu_\zeta \mu'_\zeta} \Delta(x - x'), \gamma^a = (\sigma \otimes \sigma_y, \zeta I \otimes \sigma_x)
\end{aligned}$$

$$\begin{aligned}
\text{定理3.2.1. } [\psi_{\lambda_\zeta \mu_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta}^+(x')] &= \frac{i}{8} [(m - \gamma^a \partial_a) \gamma^4]_{\lambda_\zeta \lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} \Delta(x - x') \\
&= -\frac{i}{8} \begin{bmatrix} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \partial_a & -im\delta_{A_\zeta B_\zeta} \\ -im\delta_{A'_\zeta B'_\zeta} & -(\sigma, -i\zeta)_{A'_\zeta A_\zeta}^a \partial_a \end{bmatrix}_{\lambda_\zeta \lambda'_\zeta} \begin{bmatrix} (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_b & -im\delta_{A_\zeta B_\zeta} \\ -im\delta_{A'_\zeta B'_\zeta} & -(\sigma, -i\zeta)_{B'_\zeta B_\zeta}^b \partial^b \end{bmatrix}_{\mu_\zeta \mu'_\zeta} \Delta(x - x') \\
\Leftrightarrow [\psi_{\lambda_\zeta \mu_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta}^+(x')] &= \frac{i}{4} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x) \mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x') (\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x - x') \\
&= -\frac{i}{4} \begin{bmatrix} 2\zeta S^{ab}{}_{A_\zeta B_\zeta} \partial_b & im[(\sigma, i\zeta)_a \sigma_y]_{A_\zeta B'_\zeta} \\ im[(\sigma, -i\zeta)_a \sigma_y]_{A'_\zeta B'_\zeta} & 2\zeta S_{ab}{}^{A'_\zeta B'_\zeta} \partial^b \end{bmatrix}_{\lambda_\zeta \mu_\zeta} \begin{bmatrix} 2\zeta S^{a'b'}{}_{A'_\zeta B'_\zeta} \partial_{b'} & im[\sigma_y(\sigma, -i\zeta)]_{A'_\zeta B_\zeta} \\ im[\sigma_y(\sigma, i\zeta)]_{A'_\zeta B'_\zeta} & 2\zeta S_{a'b'}{}^{A_\zeta B_\zeta} \partial^{b'} \end{bmatrix}_{\lambda'_\zeta \mu'_\zeta} (\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x - x')
\end{aligned}$$

证明:

$$\begin{cases} [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] = -\frac{i}{8} (\sigma, i\zeta)_a^a \{A_\zeta A'_\zeta\} (\sigma, i\zeta)_b^b \{B_\zeta B'_\zeta\} \partial_a \partial_b \Delta(x - x') = iS^{ac}{}_{A_\zeta B_\zeta} \delta_{cd} S^{db}{}_{A'_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \\ [\psi_{A'_\zeta B'_\zeta}(x), \psi_{A_\zeta B_\zeta}^+(x')] = -\frac{i}{8} (\sigma, -i\zeta)_a^a \{A'_\zeta A_\zeta\} (\sigma, -i\zeta)_b^b \{B'_\zeta B_\zeta\} \partial^a \partial^b \Delta(x - x') = iS_{ac}{}^{A'_\zeta B'_\zeta} \delta^{cd} S_{db}{}^{A_\zeta B_\zeta} \partial^a \partial^b \Delta(x - x') \\ [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^{C_\zeta D_\zeta}(x')] = \frac{i}{8} m^2 \delta_{\{A_\zeta B_\zeta\} \{A'_\zeta B'_\zeta\}}^{C_\zeta D_\zeta} \Delta(x - x') = iS^{ac}{}_{A_\zeta B_\zeta} \delta_c^d S_{db}{}^{C_\zeta D_\zeta} \partial_a \partial^b \Delta(x - x') \\ [\psi_{A'_\zeta B'_\zeta}(x), \psi_{A_\zeta B_\zeta}^{C'_\zeta D'_\zeta}(x')] = \frac{i}{8} m^2 \delta_{\{C'_\zeta D'_\zeta\} \{C_\zeta D_\zeta\}}^{A'_\zeta B'_\zeta} \Delta(x - x') = iS_{ac}{}^{A'_\zeta B'_\zeta} \delta_a^c S^{db}{}_{C'_\zeta D'_\zeta} \partial_a \partial^b \Delta(x - x') \\ [\psi_{A'_\zeta}^{B'_\zeta}(x), \psi_{A'_\zeta}^{B'_\zeta}(x')] = \frac{i}{4} [(\sigma, i\zeta)_a^a \{A'_\zeta A'_\zeta\} \partial_a (\sigma, -i\zeta)_b^b \{B'_\zeta B'_\zeta\} \partial^b + m^2 \delta_{A'_\zeta A'_\zeta}^{B'_\zeta B'_\zeta}] \Delta(x - x') \\ = -\frac{i}{4} \{ [(\sigma, i\zeta)_a^a \{A'_\zeta A'_\zeta\} \{B'_\zeta B'_\zeta\} \sigma_y(\sigma, -i\zeta)_b^b \partial_a \partial_b \Delta(x - x') - 2m^2 \delta_{A'_\zeta A'_\zeta}^{B'_\zeta B'_\zeta} \} \\ [\psi_{B'_\zeta}^{A'_\zeta}(x), \psi_{B'_\zeta}^{A'_\zeta}(x')] = \frac{i}{4} [(\sigma, -i\zeta)_a^a \{B'_\zeta B'_\zeta\} \partial^a (\sigma, i\zeta)_b^b \{A'_\zeta A'_\zeta\} \partial_b + m^2 \delta_{B'_\zeta B'_\zeta}^{A'_\zeta A'_\zeta}] \Delta(x - x') \\ = -\frac{i}{4} \{ [(\sigma, i\zeta)_a^a \{B'_\zeta B'_\zeta\} \{A'_\zeta A'_\zeta\} \sigma_y(\sigma, -i\zeta)_b^b \partial_a \partial_b \Delta(x - x') - 2m^2 \delta_{B'_\zeta B'_\zeta}^{A'_\zeta A'_\zeta} \} \end{cases}$$



$$\begin{cases}
[\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta}^{C_\zeta}(x')] = -\frac{1}{4}m(\sigma, i\zeta)_{\{A_\zeta A'_\zeta\}}^a \delta_{B_\zeta}^{C_\zeta} \partial_a \Delta(x-x') = \frac{\zeta}{2}mS^{ab}_{A_\zeta B_\zeta} \partial_b \delta_{aa'} [\sigma_y(\sigma, -i\zeta)]_{a' A'_\zeta}^{C_\zeta} \Delta(x-x') \\
[\psi_{A'_\zeta B'_\zeta}(x), \psi_{C'_\zeta}^{A_\zeta}(x')] = \frac{1}{4}m(\sigma, -i\zeta)_a^{\{A'_\zeta A_\zeta\}} \delta_{C'_\zeta}^{B'_\zeta} \partial^a \Delta(x-x') = \frac{\zeta}{2}mS_{ab}^{A'_\zeta B'_\zeta} \partial^b \delta^{aa'} [\sigma_y(\sigma, i\zeta)]_{a' A'_\zeta}^{C'_\zeta} \Delta(x-x') \\
[\psi_{A'_\zeta}^{C'_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] = -\frac{1}{4}m(\sigma, i\zeta)_{A_\zeta \{A'_\zeta\}}^a \delta_{B'_\zeta}^{C'_\zeta} \partial_a \Delta(x-x') = \frac{\zeta}{2}mS^{ab}_{A'_\zeta B'_\zeta} \partial_b \delta_{aa'} [(\sigma, -i\zeta)\sigma_y]_{a' A'_\zeta}^{C'_\zeta} \Delta(x-x') \\
[\psi_{C'_\zeta}^{A'_\zeta}(x), \psi_{+}^{A_\zeta B_\zeta}(x')] = \frac{1}{4}m(\sigma, -i\zeta)_a^{\{A'_\zeta A_\zeta\}} \delta_{C'_\zeta}^{B'_\zeta} \partial_a \Delta(x-x') = \frac{\zeta}{2}mS_{ab}^{A_\zeta B_\zeta} \partial^b \delta^{aa'} [(\sigma, i\zeta)\sigma_y]_{a' C'_\zeta}^{A'_\zeta} \Delta(x-x') \\
[\sigma_y(\sigma, -i\zeta)]_{a' A'_\zeta}^{B_\zeta} = [\sigma_y(\sigma, i\zeta)]_{a' B_\zeta}^{A'_\zeta}, [(\sigma, i\zeta)\sigma_y]_{a' A'_\zeta}^{B_\zeta} = [(\sigma, -i\zeta)\sigma_y]_{a' B_\zeta}^{A'_\zeta} \\
\{[\sigma_y(\sigma, -i\zeta)]_{a' A'_\zeta}^{B_\zeta}\}^* = [(\sigma, i\zeta)\sigma_y]_{a' A'_\zeta}^{B_\zeta}, \{[\sigma_y(\sigma, i\zeta)]_{a' B_\zeta}^{A'_\zeta}\}^* = [(\sigma, -i\zeta)\sigma_y]_{a' B_\zeta}^{A'_\zeta}
\end{cases} \quad \square$$

推论3.2.1.  $2\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x) \mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x') (\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x-x') = [(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta} \mu'_\zeta})} \Delta(x-x')$

### 3.3 有质量矢量场对易规则两种表述等价性的第二种直观证明方法

$$\text{引理3.3.1. } \begin{cases} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \delta_{ab} (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b = -2\varepsilon_{A_\zeta B_\zeta} \varepsilon_{A'_\zeta B'_\zeta} \\ (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \delta^{ab} (\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} = -2\varepsilon^{A_\zeta B_\zeta} \varepsilon^{A'_\zeta B'_\zeta} \end{cases} \quad \begin{cases} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \delta_a^b (\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} = 2\delta_{A_\zeta}^{B_\zeta} \delta_{A'_\zeta}^{B'_\zeta} \\ (\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \delta_b^a (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b = 2\delta_{B_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{A'_\zeta} \end{cases}$$

$$\begin{aligned}
\text{定理3.3.1. } & [\psi_{\lambda_\zeta \mu_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x')] = \frac{i}{4} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x) (\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x') \Delta(x-x') \\
& = \begin{bmatrix} \frac{1}{2}(\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]} \sigma_y \partial^b & im(\sigma, i\zeta)_a \sigma_y \\ im(\sigma, -i\zeta)_a \sigma_y & -\frac{1}{2}(\sigma, -i\zeta)_{[a} (\sigma, i\zeta)_{b]} \sigma_y \partial^b \end{bmatrix}_{\lambda_\zeta \mu_\zeta} \begin{bmatrix} \frac{1}{2}\sigma_y (\sigma, -i\zeta)_{[a'} (\sigma, i\zeta)_{b']} \partial^{b'} & -im\sigma_y (\sigma, -i\zeta)_{a'} \\ -im\sigma_y (\sigma, i\zeta)_{a'} & -\frac{1}{2}\sigma_y (\sigma, i\zeta)_{[a'} (\sigma, -i\zeta)_{b']} \partial^{b'} \end{bmatrix}_{\lambda'_\zeta \mu'_\zeta} \\
& \frac{i}{4} (\delta^{aa'} - \frac{\partial^a \partial_{a'}}{m^2}) \Delta(x-x') \\
\Rightarrow & [\psi_{\lambda_\zeta \mu_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x')] = \frac{i}{8} [(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta} \mu'_\zeta})} \Delta(x-x') \\
& = -\frac{i}{8} \begin{bmatrix} (\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \partial_a & -im\delta_{A_\zeta}^{B_\zeta} \\ -im\delta_{A'_\zeta}^{B'_\zeta} & -(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \end{bmatrix}_{\{\lambda_\zeta(\lambda'_\zeta} \begin{bmatrix} (\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_b & -im\delta_{A_\zeta}^{B_\zeta} \\ -im\delta_{A'_\zeta}^{B'_\zeta} & -(\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} \partial^b \end{bmatrix}_{\mu_\zeta \mu'_\zeta}} \Delta(x-x')
\end{aligned}$$

证明:

$$\begin{cases}
[\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] = \frac{i}{16} \{(\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]} \sigma_y \partial^b\}_{A_\zeta B_\zeta} \delta^{aa'} \{\sigma_y (\sigma, -i\zeta)_{[a'} (\sigma, i\zeta)_{b']} \partial^{b'}\}_{A'_\zeta B'_\zeta} \Delta(x-x') \\
= -\frac{i}{8} (\sigma, i\zeta)_{\{A_\zeta(A'_\zeta}^a (\sigma, i\zeta)_{B_\zeta\}^b B'_\zeta\}} \partial_a \partial_b \Delta(x-x') \\
[\psi_{A'_\zeta B'_\zeta}(x), \psi_{+}^{A_\zeta B_\zeta}(x')] = \frac{i}{16} \{(\sigma, -i\zeta)_{[a} (\sigma, i\zeta)_{b]} \sigma_y \partial^b\}_{A'_\zeta B'_\zeta} \delta^{aa'} \{\sigma_y (\sigma, i\zeta)_{[a'} (\sigma, -i\zeta)_{b']} \partial^{b'}\}_{A_\zeta B_\zeta} \Delta(x-x') \\
= -\frac{i}{8} (\sigma, -i\zeta)_a^{\{A'_\zeta(A_\zeta} (\sigma, -i\zeta)_b^{B'_\zeta\} B_\zeta\}} \partial^a \partial^b \Delta(x-x') \\
[\psi_{A_\zeta B_\zeta}(x), \psi_{+}^{C_\zeta D_\zeta}(x')] = -\frac{i}{16} \{(\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]} \sigma_y \partial^b\}_{A_\zeta B_\zeta} \delta^{aa'} \{\sigma_y (\sigma, i\zeta)_{[a'} (\sigma, -i\zeta)_{b']} \partial^{b'}\}_{C_\zeta D_\zeta} \Delta(x-x') \\
= \frac{i}{8} \delta_{\{A_\zeta}^{C_\zeta} \delta_{B_\zeta}^{D_\zeta\}} \partial^a \partial_a \Delta(x-x') \\
[\psi_{A'_\zeta B'_\zeta}(x), \psi_{C'_\zeta D'_\zeta}^+(x')] = -\frac{i}{16} \{(\sigma, -i\zeta)_{[a} (\sigma, i\zeta)_{b]} \sigma_y \partial^b\}_{A'_\zeta B'_\zeta} \delta^{aa'} \{\sigma_y (\sigma, -i\zeta)_{[a'} (\sigma, i\zeta)_{b']} \partial^{b'}\}_{C'_\zeta D'_\zeta} \Delta(x-x') \\
= \frac{i}{8} \delta_{\{C'_\zeta}^{A'_\zeta} \delta_{D'_\zeta}^{B'_\zeta\}} \partial^a \partial_a \Delta(x-x') \\
[\psi_{A'_\zeta}^{B'_\zeta}(x), \psi_{A'_\zeta}^{B'_\zeta}(x')] = -\frac{i}{4} \{[(\sigma, i\zeta)\sigma_y]_{A_\zeta}^{B'_\zeta} [\sigma_y (\sigma, -i\zeta)]_{A'_\zeta}^{B_\zeta} \partial_a \partial_b \Delta(x-x') - 2m^2 \delta_{A_\zeta}^{B_\zeta} \delta_{A'_\zeta}^{B'_\zeta}\} \\
= \frac{i}{4} [(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a \partial_a (\sigma, -i\zeta)_b^{B'_\zeta B_\zeta} \partial^b + m^2 \delta_{A_\zeta}^{B_\zeta} \delta_{A'_\zeta}^{B'_\zeta}] \Delta(x-x') \\
[\psi_{B_\zeta}^{A'_\zeta}(x), \psi_{B'_\zeta}^{A'_\zeta}(x')] = -\frac{i}{4} \{[(\sigma, i\zeta)\sigma_y]_{B_\zeta}^{A'_\zeta} [\sigma_y (\sigma, -i\zeta)]_{B'_\zeta}^{A'_\zeta} \partial_a \partial_b \Delta(x-x') - 2m^2 \delta_{B_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{A'_\zeta}\} \\
= \frac{i}{4} [(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a (\sigma, i\zeta)_b^{B'_\zeta B_\zeta} \partial_b + m^2 \delta_{B_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{A'_\zeta}] \Delta(x-x') \\
[\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta}^{C'_\zeta}(x')] = \frac{1}{8} m \{(\sigma, i\zeta)_{[a} (\sigma, -i\zeta)_{b]} \sigma_y \partial^b\}_{A_\zeta B_\zeta} \delta_{a'}^a [\sigma_y (\sigma, -i\zeta)]_{a' A'_\zeta}^{C'_\zeta} \Delta(x-x') \\
= -\frac{1}{4} m (\sigma, i\zeta)_{\{A_\zeta A'_\zeta}^a \delta_{B_\zeta}^{C'_\zeta} \partial_a \Delta(x-x') \\
[\psi_{A'_\zeta B'_\zeta}(x), \psi_{C'_\zeta}^{A'_\zeta}(x')] = -\frac{1}{8} m \{(\sigma, -i\zeta)_{[a} (\sigma, i\zeta)_{b]} \sigma_y \partial^b\}_{A'_\zeta B'_\zeta} \delta_{a'}^a [\sigma_y (\sigma, i\zeta)]_{a' A'_\zeta}^{C'_\zeta} \Delta(x-x') \\
= \frac{1}{4} m (\sigma, -i\zeta)_a^{\{A'_\zeta A_\zeta} \delta_{C'_\zeta}^{B'_\zeta} \partial^a \Delta(x-x')
\end{cases}$$

$$\begin{cases} [\psi_{A'_\zeta}^{C'_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] = -\frac{1}{8}m\{\sigma_y(\sigma, -i\zeta)_{[a}(\sigma, i\zeta)_{b]}\partial^b\}_{A'_\zeta B'_\zeta} \delta_{a'}^a [(\sigma, -i\zeta)\sigma_y]^{a' C'_\zeta}_{A'_\zeta} \Delta(x-x') \\ = -\frac{1}{4}m(\sigma, i\zeta)_{A'_\zeta \{A'_\zeta} \delta_{B'_\zeta}^{C'_\zeta} \partial_a \Delta(x-x') \\ [\psi_{C'_\zeta}^{A'_\zeta}(x), \psi_{+}^{A_\zeta B_\zeta}(x')] = \frac{1}{8}m\{\sigma_y(\sigma, i\zeta)_{[a}(\sigma, -i\zeta)_{b]}\partial^b\}_{A_\zeta B_\zeta} \delta^{aa'} [(\sigma, i\zeta)\sigma_y]_{a' C'_\zeta}^{A'_\zeta} \Delta(x-x') \\ = \frac{1}{4}m(\sigma, -i\zeta)_{a' \{A'_\zeta} \delta_{C'_\zeta}^{B_\zeta} \partial_a \Delta(x-x') \\ \left\{ \begin{aligned} [\sigma_y(\sigma, -i\zeta)]_{A'_\zeta}^{A_\zeta B_\zeta} &= [\sigma_y(\sigma, i\zeta)]^{a B_\zeta}_{A'_\zeta}, [(\sigma, i\zeta)\sigma_y]_{A'_\zeta}^{A_\zeta B_\zeta} = [(\sigma, -i\zeta)\sigma_y]^{a B_\zeta}_{A'_\zeta} \\ \{[\sigma_y(\sigma, -i\zeta)]_{A'_\zeta}^{A_\zeta B_\zeta}\}^* &= [(\sigma, i\zeta)\sigma_y]_{A'_\zeta}^{A_\zeta B_\zeta}, \{[\sigma_y(\sigma, i\zeta)]_{A'_\zeta}^{A_\zeta B_\zeta}\}^* = [(\sigma, -i\zeta)\sigma_y]_{A'_\zeta}^{A_\zeta B_\zeta} \end{aligned} \right. \end{cases} \quad \square$$

## 4 自旋-1粒子Bargmann-Wigner方程各种量子算符的提取

### 4.1 自旋-1粒子Bargmann-Wigner方程<sup>[17]</sup>的等时对易规则

定理4.1.1.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t) = 0, \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t) = \frac{1}{2!} \psi_{\{\lambda_\zeta \mu_\zeta\}}(\vec{r}, t)$

$$\begin{aligned} \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \sqrt{\frac{m^2}{E}} [a(\vec{p}, h) U_{\lambda_\zeta \mu_\zeta}(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) V_{\lambda_\zeta \mu_\zeta}(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p} \\ \left\{ \begin{aligned} a(\vec{p}, h) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sqrt{\frac{m^2}{E^3}} U^{+\lambda_\zeta \mu_\zeta}(\vec{p}', h) \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}', t) e^{-i(\vec{p}'\cdot\vec{r}-Et)} d^3\vec{r}' \\ b^+(\vec{p}, h) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sqrt{\frac{m^2}{E^3}} V^{+\lambda_\zeta \mu_\zeta}(\vec{p}', h) \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}', t) e^{i(\vec{p}'\cdot\vec{r}-Et)} d^3\vec{r}' \end{aligned} \right. \end{aligned}$$

定理4.1.2.  $[\psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta \mu'_\zeta}^+(\vec{r}', t)] = \frac{1}{4} [(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_\zeta(\lambda'_\zeta \delta_{\mu_\zeta \mu'_\zeta})\}}] \delta^3(\vec{r} - \vec{r}')$

证明:  $[\psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t), \psi_{\lambda'_\zeta \mu'_\zeta}^+(\vec{r}', t)]$

$$\begin{aligned} &= \frac{i}{8} [(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta})\}} \Delta(x-x')|_{t=t'} \\ &= \frac{i}{8} [(m\gamma^4 - \vec{\gamma}\gamma^4 \cdot \nabla) + i\partial_t]_{\{\lambda_\zeta(\lambda'_\zeta [(m\gamma^4 - \vec{\gamma}\gamma^4 \cdot \nabla) + i\partial_t]_{\mu_\zeta \mu'_\zeta})\}} \Delta(x-x')|_{t=t'} \\ &= \frac{i}{8} [(m\gamma^4 - \vec{\gamma}\gamma^4 \cdot \nabla)_{\{\lambda_\zeta(\lambda'_\zeta i\partial_t \delta_{\mu_\zeta \mu'_\zeta})\}} + i\partial_t \delta_{\{\lambda_\zeta(\lambda'_\zeta (m\gamma^4 - \vec{\gamma}\gamma^4 \cdot \nabla)_{\mu_\zeta \mu'_\zeta})\}}] \Delta(x-x')|_{t=t'} \\ &= \frac{1}{4} [(m\gamma^4 - \vec{\gamma}\gamma^4 \cdot \nabla)_{\{\lambda_\zeta(\lambda'_\zeta \delta_{\mu_\zeta \mu'_\zeta})\}}] \delta^3(\vec{r} - \vec{r}') \\ &= \frac{1}{4} [(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)_{\{\lambda_\zeta(\lambda'_\zeta \delta_{\mu_\zeta \mu'_\zeta})\}}] \delta^3(\vec{r} - \vec{r}') \end{aligned} \quad \square$$

### 4.2 自旋-1粒子Bargmann-Wigner方程能量算符的提取

定理4.2.1.  $H = \int_{h=1}^{-1} E [a^+(\vec{p}, h) a(\vec{p}, h) + b(\vec{p}, h) b^+(\vec{p}, h)] d^3\vec{p} = \int \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t) d^3\vec{p}$

证明:  $\int \sum_{h=1}^{-1} E [a^+(\vec{p}, h) a(\vec{p}, h) + b(\vec{p}, h) b^+(\vec{p}, h)] d^3\vec{p}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \frac{m^2}{E^2} \psi_{\lambda_\zeta \mu_\zeta}^+(\vec{r}, t) \psi_{\lambda'_\zeta \mu'_\zeta}(\vec{r}', t) \\ &\quad \sum_{h=1}^{-1} [U^{\lambda_\zeta \mu_\zeta}(\vec{p}, h) U^{+\lambda'_\zeta \mu'_\zeta}(\vec{p}, h) e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} + V^{\lambda_\zeta \mu_\zeta}(\vec{p}, h) V^{+\lambda'_\zeta \mu'_\zeta}(\vec{p}, h) e^{-i\vec{p}\cdot(\vec{r}-\vec{r}')}] d^3\vec{r} d^3\vec{r}' d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{m^2}{E^2} \psi_{\lambda_\zeta \mu_\zeta}^+(\vec{r}, t) \psi_{\lambda'_\zeta \mu'_\zeta}(\vec{r}', t) \\ &\quad \sum_{h=1}^{-1} [U^{\lambda_\zeta \mu_\zeta}(\vec{p}, h) U^{+\lambda'_\zeta \mu'_\zeta}(\vec{p}, h) + V^{\lambda_\zeta \mu_\zeta}(-\vec{p}, h) V^{+\lambda'_\zeta \mu'_\zeta}(-\vec{p}, h)] e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{r} d^3\vec{r}' d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{m^2}{E^2} \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi^{\lambda'_\zeta \mu'_\zeta}(\vec{r}', t) \\ &\quad \frac{1}{16m^2} [(m - i\gamma^a p_a) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - i\gamma^b p_b) \gamma^4]_{\mu_\zeta \mu'_\zeta})\}} + [(m - i\gamma^a p_a^+) \gamma^4]_{\{\lambda_\zeta(\lambda'_\zeta [(m - i\gamma^b p_b^+) \gamma^4]_{\mu_\zeta \mu'_\zeta})\}} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{r} d^3\vec{r}' d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \frac{m^2}{E^2} \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi^{\lambda'_\zeta \mu'_\zeta}(\vec{r}', t) \\ &\quad \frac{1}{16m^2} [(m\gamma^4 - i\vec{\gamma}\gamma^4 \cdot \vec{p}) + E]_{\{\lambda_\zeta(\lambda'_\zeta [(m\gamma^4 - i\vec{\gamma}\gamma^4 \cdot \vec{p}) + E]_{\mu_\zeta \mu'_\zeta})\}} \\ &\quad + [(m\gamma^4 - i\vec{\gamma}\gamma^4 \cdot \vec{p}) - E]_{\{\lambda_\zeta(\lambda'_\zeta [(m\gamma^4 - i\vec{\gamma}\gamma^4 \cdot \vec{p}) - E]_{\mu_\zeta \mu'_\zeta})\}} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{r} d^3\vec{r}' d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi^{\lambda'_\zeta \mu'_\zeta}(\vec{r}', t) \\ &\quad \frac{1}{16E^2} \{[(m\gamma^4 - i\vec{\gamma}\gamma^4 \cdot \vec{p}) + E]_{\{\lambda_\zeta(\lambda'_\zeta [(m\gamma^4 - i\vec{\gamma}\gamma^4 \cdot \vec{p}) + E]_{\mu_\zeta \mu'_\zeta})\}} \\ &\quad + [(m\gamma^4 - i\vec{\gamma}\gamma^4 \cdot \vec{p}) - E]_{\{\lambda_\zeta(\lambda'_\zeta [(m\gamma^4 - i\vec{\gamma}\gamma^4 \cdot \vec{p}) - E]_{\mu_\zeta \mu'_\zeta})\}}\} e^{i\vec{p}\cdot(\vec{r}-\vec{r}')} d^3\vec{r} d^3\vec{r}' d^3\vec{p} \\ &= \frac{1}{(2\pi)^3} \int \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi^{\lambda'_\zeta \mu'_\zeta}(\vec{r}', t) \end{aligned}$$

$$\begin{aligned}
& \frac{1}{8E^2} \{[(m\gamma^4 - i\vec{\gamma}\gamma^4 \cdot \vec{p})]_{\{\lambda_\zeta(\lambda'_\zeta)\} \{\mu_\zeta\} \mu'_\zeta} [(m\gamma^4 - i\vec{\gamma}\gamma^4 \cdot \vec{p})]_{\mu_\zeta \mu'_\zeta} + E^2 \delta_{\{\lambda_\zeta(\lambda'_\zeta)\} \{\mu_\zeta\} \mu'_\zeta}\} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3\vec{r} d^3\vec{r}' d^3\vec{p} \\
&= \frac{1}{(2\pi)^3} \int \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi^{\lambda'_\zeta \mu'_\zeta}(\vec{r}', t) \\
& \{ \frac{1}{8E^2} [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p})]_{\{\lambda_\zeta(\lambda'_\zeta)\} \{\mu_\zeta\} \mu'_\zeta} + \frac{1}{2} \delta_{\{\lambda_\zeta(\lambda'_\zeta)\} \{\mu_\zeta\} \mu'_\zeta} \} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} d^3\vec{r} d^3\vec{r}' d^3\vec{p} \\
&= \frac{1}{2} \frac{1}{(2\pi)^3} \int \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi^{\lambda'_\zeta \mu'_\zeta}(\vec{r}', t) \\
& \{ [(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)]_{\{\lambda_\zeta(\lambda'_\zeta)\} \{\mu_\zeta\} \mu'_\zeta} \frac{1}{E^2} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} + \delta_{\{\lambda_\zeta(\lambda'_\zeta)\} \{\mu_\zeta\} \mu'_\zeta} e^{i\vec{p} \cdot (\vec{r} - \vec{r}')} \} d^3\vec{r} d^3\vec{r}' d^3\vec{p} \\
&= \frac{1}{8} \int \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi^{\lambda'_\zeta \mu'_\zeta}(\vec{r}', t) \{ \frac{[(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)]_{\{\lambda_\zeta(\lambda'_\zeta)\} \{\mu_\zeta\} \mu'_\zeta}}{m^2 - \nabla^2} + \delta_{\{\lambda_\zeta(\lambda'_\zeta)\} \{\mu_\zeta\} \mu'_\zeta} \} \delta^3(\vec{r} - \vec{r}') d^3\vec{r} d^3\vec{r}' \\
&= \frac{1}{8} \int \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \{ \frac{[(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)]_{\{\lambda_\zeta(\lambda'_\zeta)\} \{\mu_\zeta\} \mu'_\zeta}}{m^2 - \nabla^2} + \delta_{\{\lambda_\zeta(\lambda'_\zeta)\} \{\mu_\zeta\} \mu'_\zeta} \} \psi^{\lambda'_\zeta \mu'_\zeta}(\vec{r}, t) d^3\vec{r}' \\
&= \frac{1}{2} \int \{ \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \frac{[(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)]_{\lambda_\zeta \lambda'_\zeta} [(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla)]_{\mu_\zeta \mu'_\zeta}}{m^2 - \nabla^2} \psi^{\lambda'_\zeta \mu'_\zeta}(\vec{r}, t) + \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t) \} d^3\vec{r}' \\
&= \frac{1}{2} \int \{ \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \frac{(i\partial_t)^2}{m^2 - \nabla^2} \psi^{\lambda'_\zeta \mu'_\zeta}(\vec{r}, t) + \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t) \} d^3\vec{r}' \\
&= \int \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t) d^3\vec{r}'
\end{aligned}$$

□

$$\text{定理4.2.2. } H = \int \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t) d^3\vec{r} = \int \{ \frac{1}{2} F^{+ab} F_{ab} + m^2 A^{+a}(\vec{r}, t) A_a(\vec{r}, t) \} d^3\vec{r}$$

$$\text{证明: } H = \int \psi^{+\lambda_\zeta \mu_\zeta}(\vec{r}, t) \psi_{\lambda_\zeta \mu_\zeta}(\vec{r}, t) d^3\vec{r}$$

$$\begin{aligned}
& tr[S_{ab}(e, \varsigma) S_{cd}(e, \varsigma)] = S_{abcd} = \delta_{ac} \delta_{db} - \delta_{ad} \delta_{cb} \\
&= \int \{ \bar{C}[-im\gamma^{a'}(\varsigma) - 2S^{a'b'}(e, \varsigma) \partial_{b'}^+] \}_{\lambda_\zeta \mu_\zeta} \frac{A_{a'}^+(\vec{r}, t)}{2} [im\gamma^a(\varsigma) C - 2S^{ab}(e, \varsigma) C \partial_b]_{\lambda_\zeta \mu_\zeta} \frac{A_a(\vec{r}, t)}{2} d^3\vec{r}' \\
&= \frac{1}{4} \int tr \{ \bar{C}[-im\gamma^{a'}(\varsigma) - 2S^{a'b'}(e, \varsigma) \partial_{b'}^+] [im\gamma^a(\varsigma) - 2S^{ab}(e, \varsigma) \partial_b] C \} A_{a'}^+(\vec{r}, t) A_a(\vec{r}, t) d^3\vec{r}' \\
&= \frac{1}{4} \int \{ m^2 tr[\gamma^{a'}(\varsigma) \gamma^a(\varsigma)] A_{a'}^+(\vec{r}, t) A_a(\vec{r}, t) + 4tr[S^{a'b'}(e, \varsigma) S^{ab}(e, \varsigma)] \partial_{b'}^+ A_{a'}^+(\vec{r}, t) \partial_b A_a(\vec{r}, t) \} d^3\vec{r}' \\
&= \int \{ m^2 \delta^{a'a} A_{a'}^+(\vec{r}, t) A_a(\vec{r}, t) + S^{a'b'ab} \partial_{b'}^+ A_{a'}^+(\vec{r}, t) \partial_b A_a(\vec{r}, t) \} d^3\vec{r}' \\
&= \int \{ m^2 \delta^{a'a} A_{a'}^+(\vec{r}, t) A_a(\vec{r}, t) + (\delta^{a'a} \delta^{b'b} - \delta^{a'b} \delta^{b'a}) \partial_{b'}^+ A_{a'}^+(\vec{r}, t) \partial_b A_a(\vec{r}, t) \} d^3\vec{r}' \\
&= \int \{ m^2 A^{+a}(\vec{r}, t) A_a(\vec{r}, t) + \partial^{+b} A^{+a}(\vec{r}, t) \partial_b A_a(\vec{r}, t) - \partial^{+a} A^{+b}(\vec{r}, t) \partial_b A_a(\vec{r}, t) \} d^3\vec{r}' \\
&= \int \{ \frac{1}{2} F^{+ab} F_{ab} + m^2 A^{+a}(\vec{r}, t) A_a(\vec{r}, t) \} d^3\vec{r}'
\end{aligned}$$

□

## 5 有质量Majorana矢量场(取 $\theta = 0$ )

### 5.1 有质量Majorana矢量场与有质量矢量场的对比

本章有质量矢量场的以上结论对于有质量Majorana矢量场一样相应成立，但要加上下面的附加条件。

$$\text{定理5.1.1. } \psi = \gamma_2 \otimes \gamma_2 \psi^*, A_a = A_a^+, \eta_a^{a'} (F_{ab} = F_{a'b}^+, \eta_a^{a'} \eta_b^{b'}), b^+(\vec{p}, h) = (-1)^{s+h} a^+(\vec{p}, -h)$$

下面就有质量Majorana矢量场与有质量复矢量场不一样的情形进行详细对比讨论。

### 5.2 不依赖方程导出自旋-1粒子各种物理量之间等价对易规则

#### 5.2.1 重新定义

定义5.2.1. 电磁场矢量第三种定义

$$\begin{cases} F_{ab} := \partial_a A_b - \partial_b A_a \\ \psi_{\alpha_\zeta} := -\frac{1}{2\sqrt{2}} \sigma_{\zeta\alpha_\zeta}^{ab} F_{ab} = -\frac{\varsigma}{\sqrt{2}} (E - i\varsigma B)_{\alpha_\zeta} \\ \psi_{A_\zeta B_\zeta} := \frac{i\varsigma}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \psi_{\alpha_\zeta} \Leftrightarrow \psi_{\alpha_\zeta} = \frac{i\varsigma}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta} \end{cases} \Rightarrow \begin{cases} \psi_{A_\zeta B_\zeta} = -\varsigma S^{ab}{}_{A_\zeta B_\zeta} F_{ab} \\ F_{ab} = \frac{1}{\sqrt{2}} (\sigma_{-sab}^{\alpha_\zeta} \psi_{\alpha_\zeta} + \sigma_{sab}^{\alpha_\zeta} \psi_{\alpha_\zeta}) \\ *F_{ab} = \frac{\varsigma}{\sqrt{2}} (\sigma_{-sab}^{\alpha_\zeta} \psi_{\alpha_\zeta} - \sigma_{sab}^{\alpha_\zeta} \psi_{\alpha_\zeta}) \end{cases}$$

### 5.3 $\psi_{\lambda_\zeta \mu_\zeta}$ 与 $(A_a, F_{ab})$ 的等价对易规则

#### 5.3.1 复场与实场的共同对易规则

定理5.3.1.

$$\begin{cases} [\psi_{\lambda_\zeta \mu_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta}(x')] = \frac{i}{4} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x) \mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x') (\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x - x'), \psi_{\lambda_\zeta \mu_\zeta} = \psi_{\mu_\zeta \lambda_\zeta} \\ im \frac{A_a}{2}(x) = \frac{1}{4} tr \{ \bar{C} \gamma_a(\varsigma) \psi_{[\lambda_\zeta \mu_\zeta]}(x) \}, i \frac{F_{ab}}{2}(x) = \frac{1}{2} tr \{ \bar{C} S_{ab}(e, \varsigma) \psi_{[\lambda_\zeta \mu_\zeta]}(x) \} \\ \Leftrightarrow [A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x - x'), [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a < a'} \partial_b] \partial_{b']^+} \Delta(x - x') \\ \psi_{[\lambda_\zeta \mu_\zeta]} = [im\gamma^a(\varsigma) C \frac{A_a}{2} + S^{ab}(e, \varsigma) C \frac{F_{ab}}{2}] \end{cases}$$

$$\begin{aligned}
& \text{证明: } m^2[A_a(x), A_{a'}^+(x')] = [imA_a(x), -imA_{a'}^+(x')] \\
& = [\frac{1}{2}tr[\bar{C}\gamma_a(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)], \frac{1}{2}tr^+[\bar{C}\gamma_a(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x')]] \\
& = [\frac{1}{2}[\bar{C}\gamma_a(\varsigma)]^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}(x), \frac{1}{2}[\gamma_{a'}(\varsigma)C]^{\lambda'_\varsigma\mu'_\varsigma}\psi_{\lambda'_\varsigma\mu'_\varsigma}^+(x')] \\
& = \frac{1}{4}[\bar{C}\gamma_a(\varsigma)]^{\lambda_\varsigma\mu_\varsigma}[\gamma_{a'}(\varsigma)C]^{\lambda'_\varsigma\mu'_\varsigma}[\psi_{\lambda_\varsigma\mu_\varsigma}(x), \psi_{\lambda'_\varsigma\mu'_\varsigma}(x')] \\
& = \frac{i}{16}[\bar{C}\gamma_a(\varsigma)]^{\lambda_\varsigma\mu_\varsigma}[\gamma_{a'}(\varsigma)C]^{\lambda'_\varsigma\mu'_\varsigma}\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^b(x)\mathbb{X}_{\lambda'_\varsigma\mu'_\varsigma}^{+b'}(x')(\eta_{bb'} - \frac{\partial_b\partial_{b'}^+}{m^2})\Delta(x-x') \\
& = \frac{i}{16}tr[\bar{C}\gamma_a(\varsigma)\mathbb{X}^b(x)]tr[\mathbb{X}^{+b'}(x')\gamma_{a'}(\varsigma)C](\eta_{bb'} - \frac{\partial_b\partial_{b'}^+}{m^2})\Delta(x-x') \\
& = \frac{i}{16}tr\{\bar{C}\gamma_a(\varsigma)[im\gamma^b(\varsigma) - 2S^{bc}(e, \varsigma)\partial_c]C\}tr\{\bar{C}[-im\gamma^{b'}(\varsigma) - 2S^{b'c'}(e, \varsigma)\partial_{c'}^+]\gamma_{a'}(\varsigma)C\}(\eta_{bb'} - \frac{\partial_b\partial_{b'}^+}{m^2})\Delta(x-x') \\
& = \frac{i}{16}tr\{\gamma_a(\varsigma)[im\gamma^b(\varsigma) - 2S^{bc}(e, \varsigma)\partial_c]\}tr\{[-im\gamma^{b'}(\varsigma) - 2S^{b'c'}(e, \varsigma)\partial_{c'}^+]\gamma_{a'}(\varsigma)\}(\eta_{bb'} - \frac{\partial_b\partial_{b'}^+}{m^2})\Delta(x-x') \\
& = \frac{i}{16}m^2tr[\gamma_a(\varsigma)\gamma^b(\varsigma)]tr[\gamma^{b'}(\varsigma)\gamma_{a'}(\varsigma)](\eta_{bb'} - \frac{\partial_b\partial_{b'}^+}{m^2})\Delta(x-x') \\
& = im^2\delta_a^b\delta_{a'}^{b'}(\eta_{bb'} - \frac{\partial_b\partial_{b'}^+}{m^2})\Delta(x-x') \\
& = i(m^2\eta_{aa'} - \partial_a\partial_{a'}^+)\Delta(x-x') \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } [F_{ab}(x), F_{a'b'}^+(x')] = [iF_{ab}(x), -iF_{a'b'}^+(x')] \\
& = [tr[\bar{C}S_{ab}(e, \varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)], tr^+[\bar{C}S_{a'b'}(e, \varsigma)(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x')]] \\
& = [[\bar{C}S_{ab}(e, \varsigma)]^{\lambda_\varsigma\mu_\varsigma}\psi_{\lambda_\varsigma\mu_\varsigma}(x), [S_{a'b'}(e, \varsigma)(\varsigma)C]^{\lambda'_\varsigma\mu'_\varsigma}\psi_{\lambda'_\varsigma\mu'_\varsigma}^+(x')] \\
& = [\bar{C}S_{ab}(e, \varsigma)]^{\lambda_\varsigma\mu_\varsigma}[S_{a'b'}(e, \varsigma)C]^{\lambda'_\varsigma\mu'_\varsigma}[\psi_{\lambda_\varsigma\mu_\varsigma}(x), \psi_{\lambda'_\varsigma\mu'_\varsigma}(x')] \\
& = \frac{i}{4}[\bar{C}S_{ab}(e, \varsigma)]^{\lambda_\varsigma\mu_\varsigma}[S_{a'b'}(e, \varsigma)C]^{\lambda'_\varsigma\mu'_\varsigma}\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^c(x)\mathbb{X}_{\lambda'_\varsigma\mu'_\varsigma}^{+c'}(x')(\eta_{cc'} - \frac{\partial_c\partial_{c'}^+}{m^2})\Delta(x-x') \\
& = \frac{i}{4}tr[\bar{C}S_{ab}(e, \varsigma)\mathbb{X}^c(x)]tr[\mathbb{X}^{+c'}(x')S_{a'b'}(e, \varsigma)C](\eta_{cc'} - \frac{\partial_c\partial_{c'}^+}{m^2})\Delta(x-x') \\
& = \frac{i}{4}tr\{\bar{C}S_{ab}(e, \varsigma)[im\gamma^c(\varsigma) - 2S^{cd}(e, \varsigma)\partial_d]C\}tr\{\bar{C}[-im\gamma^{c'}(\varsigma) - 2S^{c'd'}(e, \varsigma)\partial_{d'}^+]S_{a'b'}(e, \varsigma)C\} \\
& (\eta_{cc'} - \frac{\partial_c\partial_{c'}^+}{m^2})\Delta(x-x') \\
& = \frac{i}{4}tr\{S_{ab}(e, \varsigma)[im\gamma^c(\varsigma) - 2S^{cd}(e, \varsigma)\partial_d]\}tr\{[-im\gamma^{c'}(\varsigma) - 2S^{c'd'}(e, \varsigma)\partial_{d'}^+]S_{a'b'}(e, \varsigma)\}(\eta_{cc'} - \frac{\partial_c\partial_{c'}^+}{m^2})\Delta(x-x') \\
& = -itr[S_{ab}(e, \varsigma)S^{cd}(e, \varsigma)]tr[S^{c'd'}(e, \varsigma)S_{a'b'}(e, \varsigma)]\partial_d\partial_{d'}^+(\eta_{cc'} - \frac{\partial_c\partial_{c'}^+}{m^2})\Delta(x-x') \\
& = -itr[S_{ab}(e, \varsigma)S^{cd}(e, \varsigma)]tr[S^{c'd'}(e, \varsigma)S_{a'b'}(e, \varsigma)]\eta_{cc'}\partial_d\partial_{d'}^+\Delta(x-x') \\
& = -iS_{abcd}S_{a'b'c'd'}\eta^{cc'}\partial^d\partial_{d'}^+\Delta(x-x') \\
& = -i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})(\delta_{a'e'}\delta_{b'd'} - \delta_{a'd'}\delta_{b'e'})\eta^{cc'}\partial^d\partial_{d'}^+\Delta(x-x') \\
& = -i\eta_{[a<a'}\partial_{b]}\partial_{b'}^+\Delta(x-x') \quad \square
\end{aligned}$$

### 5.3.2 复场条件

定理5.3.2.

$$\begin{cases} [\psi_{\lambda_\varsigma\mu_\varsigma}(x), \psi_{\eta_\varsigma\xi_\varsigma}(x')] = 0, [\psi_{\lambda'_\varsigma\mu'_\varsigma}^+(x), \psi_{\eta'_\varsigma\xi'_\varsigma}^+(x')] = 0, \psi_{\lambda_\varsigma\mu_\varsigma} = \psi_{\mu_\varsigma\lambda_\varsigma} \\ im\frac{A_a}{2}(x) = \frac{1}{4}tr\{\bar{C}\gamma_a(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\}, i\frac{F_{ab}}{2}(x) = \frac{1}{2}tr\{\bar{C}S_{ab}(e, \varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\} \\ \Leftrightarrow \begin{cases} [A_a(x), A_b(x')] = 0, [F_{ab}(x), F_{cd}(x')] = 0; [A_{a'}^+(x), A_{b'}^+(x')] = 0, [F_{a'b'}^+(x), F_{c'd'}^+(x')] = 0 \\ \psi_{[\lambda_\varsigma\mu_\varsigma]} = [im\gamma^a(\varsigma)C\frac{A_a}{2} + S^{ab}(e, \varsigma)C\frac{F_{ab}}{2}] \end{cases} \end{cases}$$

### 5.3.3 复场完整的对易规则

定理5.3.3.

$$\begin{cases} [\psi_{\lambda_\varsigma\mu_\varsigma}(x), \psi_{\lambda'_\varsigma\mu'_\varsigma}^+(x')] = \frac{i}{4}\mathbb{X}_{\lambda_\varsigma\mu_\varsigma}^a(x)\mathbb{X}_{\lambda'_\varsigma\mu'_\varsigma}^{+a'}(x')(\eta_{aa'} - \frac{\partial_a\partial_{a'}^+}{m^2})\Delta(x-x') \\ \begin{cases} [\psi_{\lambda_\varsigma\mu_\varsigma}(x), \psi_{\eta_\varsigma\xi_\varsigma}(x')] = 0, [\psi_{\lambda'_\varsigma\mu'_\varsigma}^+(x), \psi_{\eta'_\varsigma\xi'_\varsigma}^+(x')] = 0, \psi_{\lambda_\varsigma\mu_\varsigma} = \psi_{\mu_\varsigma\lambda_\varsigma} \\ im\frac{A_a}{2}(x) = \frac{1}{4}tr\{\bar{C}\gamma_a(\varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\}, i\frac{F_{ab}}{2}(x) = \frac{1}{2}tr\{\bar{C}S_{ab}(e, \varsigma)\psi_{[\lambda_\varsigma\mu_\varsigma]}(x)\} \end{cases} \\ \Leftrightarrow \begin{cases} [A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a\partial_{a'}^+}{m^2})\Delta(x-x'), [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a<a'}\partial_{b]}\partial_{b'}^+\Delta(x-x') \\ [A_a(x), A_b(x')] = 0, [F_{ab}(x), F_{cd}(x')] = 0; [A_{a'}^+(x), A_{b'}^+(x')] = 0, [F_{a'b'}^+(x), F_{c'd'}^+(x')] = 0 \\ \psi_{[\lambda_\varsigma\mu_\varsigma]} = [im\gamma^a(\varsigma)C\frac{A_a}{2} + S^{ab}(e, \varsigma)C\frac{F_{ab}}{2}] \end{cases} \end{cases}$$

### 5.3.4 Majorana实场条件

定理5.3.4.

$$\begin{cases} \psi = \gamma_2 \psi^+ \gamma_2, \psi_{\lambda\zeta\mu\varsigma} = \psi_{\mu\varsigma\lambda\zeta} \\ im \frac{A_a}{2}(x) = \frac{1}{4} tr \{ \bar{C} \gamma_a(\varsigma) \psi_{[\lambda\zeta\mu\varsigma]}(x) \} \\ i \frac{F_{ab}}{2}(x) = \frac{1}{2} tr \{ \bar{C} S_{ab}(e, \varsigma) \psi_{[\lambda\zeta\mu\varsigma]}(x) \} \end{cases} \Leftrightarrow \begin{cases} A_a = A_a^+ \eta_a^{a'}, F_{ab} = F_{a'b'}^+ \eta_a^{a'} \eta_b^{b'} \\ \psi_{[\lambda\zeta\mu\varsigma]} = [im \gamma^a(\varsigma) C \frac{A_a}{2} + S^{ab}(e, \varsigma) C \frac{F_{ab}}{2}] \end{cases}$$

### 5.3.5 Majorana实场完整的对易规则

定理5.3.5.

$$\begin{cases} [\psi_{\lambda\zeta\mu\varsigma}(x), \psi_{\lambda'\zeta'\mu'\varsigma'}^+(x')] = \frac{i}{4} \mathbb{X}_{\lambda\zeta\mu\varsigma}^a(x) \mathbb{X}_{\lambda'\zeta'\mu'\varsigma'}^{+a'}(x') (\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x-x') \\ [\psi_{\lambda\zeta\mu\varsigma}(x), \psi_{\lambda'\zeta'\mu'\varsigma'}^+(x')] = \frac{i}{4} \mathbb{X}_{\lambda\zeta\mu\varsigma}^a(x) \mathbb{X}_{\lambda'\zeta'\mu'\varsigma'}^{+a'}(x') (\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x-x') \\ [\psi_{\lambda\zeta\mu\varsigma}^+(x), \psi_{\lambda'\zeta'\mu'\varsigma'}^+(x')] = \frac{i}{4} \mathbb{X}_{\lambda\zeta\mu\varsigma}^{+a}(x) \mathbb{X}_{\lambda'\zeta'\mu'\varsigma'}^{+a'}(x') (\delta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x-x') \\ \psi = \gamma_2 \psi^+ \gamma_2, \psi_{\lambda\zeta\mu\varsigma} = \psi_{\mu\varsigma\lambda\zeta} \\ im \frac{A_a}{2}(x) = \frac{1}{4} tr \{ \bar{C} \gamma_a(\varsigma) \psi_{[\lambda\zeta\mu\varsigma]}(x) \}, i \frac{F_{ab}}{2}(x) = \frac{1}{2} tr \{ \bar{C} S_{ab}(e, \varsigma) \psi_{[\lambda\zeta\mu\varsigma]}(x) \} \\ \Leftrightarrow \begin{cases} [A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x-x'), [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a<a'} \partial_b] \partial_{b'}^+ \Delta(x-x') \\ [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\partial_a \partial_b}{m^2}) \Delta(x-x'), [F_{ab}(x), F_{cd}(x')] = -i\delta_{[a<c} \partial_b] \partial_{d>} \Delta(x-x') \\ [A_{a'}^+(x), A_{b'}^+(x')] = i(\delta_{a'b'} - \frac{\partial_{a'} \partial_{b'}}{m^2}) \Delta(x-x'), [F_{a'b'}^+(x), F_{c'd'}^+(x')] = -i\delta_{[a'<c'} \partial_{b'}^+] \partial_{d'}^+ \Delta(x-x') \\ A_a = A_a^+ \eta_a^{a'}, F_{ab} = F_{a'b'}^+ \eta_a^{a'} \eta_b^{b'} \\ \psi_{[\lambda\zeta\mu\varsigma]} = [im \gamma^a(\varsigma) C \frac{A_a}{2} + S^{ab}(e, \varsigma) C \frac{F_{ab}}{2}] \end{cases} \end{cases}$$

## 5.4 从 $A_a$ 导出 $F_{ab}$ 的对易关系

### 5.4.1 复场与实场的共同对易规则

$$\text{定理5.4.1. } \begin{cases} [A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x-x') \\ F_{ab} := \partial_a A_b - \partial_b A_a \end{cases} \Rightarrow [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a<a'} \partial_b] \partial_{b'}^+ \Delta(x-x')$$

### 5.4.2 复场条件

$$\text{定理5.4.2. } \begin{cases} [A_a(x), A_b(x')] = 0, [A_{a'}^+(x), A_{b'}^+(x')] = 0 \\ F_{ab} := \partial_a A_b - \partial_b A_a \end{cases} \Rightarrow \begin{cases} [F_{ab}(x), F_{cd}(x')] = 0 \\ [F_{a'b'}^+(x), F_{c'd'}^+(x')] = 0 \end{cases}$$

### 5.4.3 复场完整的对易规则

$$\text{定理5.4.3. } \begin{cases} [A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x-x') \\ [A_a(x), A_b(x')] = 0, [A_{a'}^+(x), A_{b'}^+(x')] = 0 \\ F_{ab} := \partial_a A_b - \partial_b A_a \end{cases} \Rightarrow \begin{cases} [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a<a'} \partial_b] \partial_{b'}^+ \Delta(x-x') \\ [F_{ab}(x), F_{cd}(x')] = 0, [F_{a'b'}^+(x), F_{c'd'}^+(x')] = 0 \end{cases}$$

### 5.4.4 Majorana实场条件

$$\text{定理5.4.4. } A_a = A_a^+ \eta_a^{a'}, F_{ab} := \partial_a A_b - \partial_b A_a \Rightarrow F_{ab} = F_{a'b'}^+ \eta_a^{a'} \eta_b^{b'}$$

### 5.4.5 Majorana实场完整的对易规则

$$\text{定理5.4.5. } \begin{cases} [A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x-x') \\ [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\partial_a \partial_b}{m^2}) \Delta(x-x') \\ [A_{a'}^+(x), A_{b'}^+(x')] = i(\delta_{a'b'} - \frac{\partial_{a'} \partial_{b'}}{m^2}) \Delta(x-x') \\ A_a = A_a^+ \eta_a^{a'}, F_{ab} := \partial_a A_b - \partial_b A_a \end{cases} \Rightarrow \begin{cases} [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a<a'} \partial_b] \partial_{b'}^+ \Delta(x-x') \\ [F_{ab}(x), F_{cd}(x')] = -i\delta_{[a<c} \partial_b] \partial_{d>} \Delta(x-x') \\ [F_{a'b'}^+(x), F_{c'd'}^+(x')] = -i\delta_{[a'<c'} \partial_{b'}^+] \partial_{d'}^+ \Delta(x-x') \\ F_{ab} = F_{a'b'}^+ \eta_a^{a'} \eta_b^{b'} \end{cases}$$

## 5.5 $\psi_{\alpha_\zeta}$ 与 $F_{ab}$ 的等价对易关系

### 5.5.1 复场与实场的共同对易规则

定理5.5.1.

$$\begin{cases} [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a<a'}\partial_b]\partial_{b'}^+\Delta(x-x') \\ \psi_{\alpha_\zeta} := -\frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha_\zeta}^{ab}F_{ab} \end{cases} \Leftrightarrow \begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_{-\zeta}}^+(x')] = -\frac{i}{2}m^2\delta_{\alpha_\zeta\alpha'_{-\zeta}}\Delta(x-x') \\ F_{ab} = \frac{1}{\sqrt{2}}(\sigma_{\zeta ab}^{\alpha_\zeta}\psi_{\alpha_\zeta} + \sigma_{-\zeta ab}^{\alpha_{-\zeta}}\psi_{\alpha_{-\zeta}}) \end{cases}$$

证明:  $[\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')]$

$$\begin{aligned} &= [-\frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha_\zeta}^{ab}F_{ab}(x), \frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha'_\zeta}^{a'b'}F_{a'b'}^+(x')] \\ &= -\frac{1}{8}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{\zeta\alpha'_\zeta}^{a'b'}[F_{ab}(x), F_{a'b'}^+(x')] \\ &= \frac{i}{8}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{\zeta\alpha'_\zeta}^{a'b'}\eta_{[a<a'}\partial_b]\partial_{b'}^+\Delta(x-x') \\ &= \frac{i}{2}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{\zeta\alpha'_\zeta}^{a'b'}\eta_{aa'}\partial_b\partial_{b'}^+\Delta(x-x') \\ &= \frac{i}{2}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{-\zeta\alpha'_\zeta}^{a'b'}\delta_{aa'}\partial_b\partial_{b'}\Delta(x-x') \\ &= i\sigma_{\alpha_\zeta\alpha'_\zeta}^{bb'}\partial_b\partial_{b'}\Delta(x-x') \\ &= i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \end{aligned}$$

□

### 5.5.2 复场条件

定理5.5.2.

$$\begin{cases} [F_{ab}(x), F_{cd}(x')] = 0, [F_{a'b'}^+(x), F_{c'd'}^+(x')] = 0 \\ \psi_{\alpha_\zeta} := -\frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha_\zeta}^{ab}F_{ab} \end{cases} \Leftrightarrow \begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\beta_\kappa}(x')] = 0, [\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\kappa}^+(x')] = 0 \\ F_{ab} = \frac{1}{\sqrt{2}}(\sigma_{\zeta ab}^{\alpha_\zeta}\psi_{\alpha_\zeta} + \sigma_{-\zeta ab}^{\alpha_{-\zeta}}\psi_{\alpha_{-\zeta}}) \end{cases}$$

### 5.5.3 复场完整的对易规则

定理5.5.3.

$$\begin{cases} [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a<a'}\partial_b]\partial_{b'}^+\Delta(x-x') \\ [F_{ab}(x), F_{cd}(x')] = 0, [F_{a'b'}^+(x), F_{c'd'}^+(x')] = 0 \\ \psi_{\alpha_\zeta} := -\frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha_\zeta}^{ab}F_{ab} \end{cases} \Leftrightarrow \begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_{-\zeta}}^+(x')] = -\frac{i}{2}m^2\delta_{\alpha_\zeta\alpha'_{-\zeta}}\Delta(x-x') \\ [\psi_{\alpha_\zeta}(x), \psi_{\beta_\kappa}(x')] = 0, [\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\kappa}^+(x')] = 0 \\ F_{ab} = \frac{1}{\sqrt{2}}(\sigma_{\zeta ab}^{\alpha_\zeta}\psi_{\alpha_\zeta} + \sigma_{-\zeta ab}^{\alpha_{-\zeta}}\psi_{\alpha_{-\zeta}}) \end{cases}$$

定理5.5.4.

$$\begin{cases} [F_{ab}(x), A_{c'}^+(x')] = -i\eta_{c'[a}\partial_b]\Delta(x-x') \\ [F_{a'b'}^+(x), A_c(x')] = -i\eta_{c[a'}\partial_{b'}^+\Delta(x-x') \\ [F_{ab}(x), A_c(x')] = 0, [F_{a'b'}^+(x), A_{c'}^+(x')] = 0 \end{cases} \Leftrightarrow \begin{cases} [\psi_{\alpha_\zeta}(x), A_{c'}^+(x')] = -\frac{i}{\sqrt{2}}(\sigma_{+\zeta}, -i\zeta)^b|_{c'\alpha_\zeta}\partial_b\Delta(x-x') \\ [\psi_{\alpha'_\zeta}^+(x), A_c(x')] = -\frac{i}{\sqrt{2}}(\sigma_{+\zeta}, -i\zeta)^b|_{c\alpha'_\zeta}\partial_b\Delta(x-x') \\ [\psi_{\alpha_\zeta}(x), A_c(x')] = 0, [\psi_{\alpha'_\zeta}^+(x), A_{c'}^+(x')] = 0 \end{cases}$$

证明:  $[F_{ab}(x), A_{c'}^+(x')] = [\partial_a A_b(x) - \partial_b A_a(x), A_{c'}^+(x')]$

$$\begin{aligned} &= i(\eta_{bc} - \frac{\partial_b\partial_c^+}{m^2})\partial_a\Delta(x-x') - i(\eta_{ac} - \frac{\partial_a\partial_c^+}{m^2})\partial_b\Delta(x-x') \\ &= -i\eta_{c[a}\partial_b]\Delta(x-x') \end{aligned}$$

□

证明:  $[\psi_{\alpha_\zeta}(x), A_{c'}^+(x')] = \frac{i}{\sqrt{2}}\frac{i}{2}\sigma_{\zeta\alpha_\zeta}^{ab}[F_{ab}(x), A_{c'}^+(x')]$

$$\begin{aligned} &= -\frac{i}{\sqrt{2}}\frac{i}{2}\sigma_{\zeta\alpha_\zeta}^{ab}\eta_{c[a}\partial_b]\Delta(x-x') \\ &= \frac{i}{\sqrt{2}}\sigma_{\zeta\alpha_\zeta}^{ab}\eta_{ca}\partial_b\Delta(x-x') \\ &= -\frac{i}{\sqrt{2}}(\sigma_\zeta)_{\alpha_\zeta}|^{ab}\eta_{ca}\partial_b\Delta(x-x'), [(\sigma_\zeta, -i\zeta)^{\alpha_\zeta}|_{ab} = (\sigma_{-\zeta}, -i\zeta)_a|_{b\alpha_\zeta}] \\ &= -\frac{i}{\sqrt{2}}(\sigma_{-\zeta}, -i\zeta)^b|_{\alpha_\zeta}\eta_{ca}\partial_b\Delta(x-x') \\ &= -\frac{i}{\sqrt{2}}(\sigma_{+\zeta}, -i\zeta)^b|_{c\alpha_\zeta}\partial_b\Delta(x-x') \end{aligned}$$

□

## 5.5.4 复场的部分等时对易规则

推论5.5.1.

$$\begin{cases} [E_i(\vec{r}, t), A_c^+(\vec{r}', t)] = i\eta_{ic} \delta^3(\vec{r} - \vec{r}'), [E_i^+(\vec{r}, t), A_c(\vec{r}', t)] = i\eta_{i'c} \delta^3(\vec{r} - \vec{r}') \\ [E_i(\vec{r}, t), A_c(\vec{r}', t)] = 0, [E_i^+(\vec{r}, t), A_c^+(\vec{r}', t)] = 0 \\ [B_i(\vec{r}, t), A_c^+(\vec{r}', t)] = 0, [B_i^+(\vec{r}, t), A_c(\vec{r}', t)] = 0 \\ [B_i(\vec{r}, t), A_c(\vec{r}', t)] = 0, [B_i^+(\vec{r}, t), A_c^+(\vec{r}', t)] = 0 \end{cases}$$

推论5.5.2.

$$\begin{cases} [E_i(\vec{r}, t), A_j^+(\vec{r}', t)] = i\delta_{ij} \delta^3(\vec{r} - \vec{r}'), [E_i^+(\vec{r}, t), A_j(\vec{r}', t)] = i\delta_{ij} \delta^3(\vec{r} - \vec{r}') \\ [E_i(\vec{r}, t), A_j(\vec{r}', t)] = 0, [E_i^+(\vec{r}, t), A_j^+(\vec{r}', t)] = 0 \\ [E_i(\vec{r}, t), E_j^+(\vec{r}', t)] = 0, [E_i^+(\vec{r}, t), E_j(\vec{r}', t)] = 0, [E_i(\vec{r}, t), E_j(\vec{r}', t)] = 0, [E_i^+(\vec{r}, t), E_j^+(\vec{r}', t)] = 0 \\ [A_i(\vec{r}, t), A_j^+(\vec{r}', t)] = 0, [A_i^+(\vec{r}, t), A_j(\vec{r}', t)] = 0, [A_i(\vec{r}, t), A_j(\vec{r}', t)] = 0, [A_i^+(\vec{r}, t), A_j^+(\vec{r}', t)] = 0 \end{cases}$$

[↓] [↓]

推论5.5.3.

$$\begin{cases} [E_i(\vec{r}, t), B_j^+(\vec{r}', t)] = -i\varepsilon_{ij}^k \partial_k \delta^3(\vec{r} - \vec{r}'), [E_i^+(\vec{r}, t), B_j(\vec{r}', t)] = -i\varepsilon_{ij}^k \partial_k \delta^3(\vec{r} - \vec{r}') \\ [E_i(\vec{r}, t), B_j(\vec{r}', t)] = 0, [E_i^+(\vec{r}, t), B_j^+(\vec{r}', t)] = 0 \\ [B_i(\vec{r}, t), B_j^+(\vec{r}', t)] = 0, [B_i^+(\vec{r}, t), B_j(\vec{r}', t)] = 0, [B_i(\vec{r}, t), B_j(\vec{r}', t)] = 0, [B_i^+(\vec{r}, t), B_j^+(\vec{r}', t)] = 0 \\ [B_i(\vec{r}, t), A_j^+(\vec{r}', t)] = 0, [B_i^+(\vec{r}, t), A_j(\vec{r}', t)] = 0, [B_i(\vec{r}, t), A_j(\vec{r}', t)] = 0, [B_i^+(\vec{r}, t), A_j^+(\vec{r}', t)] = 0 \end{cases}$$

## 5.5.5 Majorana实场条件

定理5.5.5.

$$\begin{cases} F_{ab} = F_{a'b'}^+ \eta_a^+ \eta_b^+ \\ \psi_{\alpha_\zeta} := -\frac{1}{2\sqrt{2}} \sigma_{\alpha_\zeta}^{ab} F_{ab} \end{cases} \Leftrightarrow \begin{cases} \psi_{\alpha_{-\zeta}}(x) = -\psi_{\alpha_\zeta}^+(x) \\ F_{ab} = \frac{1}{\sqrt{2}} (\sigma_{\zeta ab} \psi_{\alpha_\zeta} - \sigma_{-\zeta ab} \psi_{\alpha_\zeta}^+) \end{cases}$$

## 5.5.6 Majorana实场完整的对易规则

引理5.5.1.

$$2\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta a'b'}^{\alpha'_\zeta} \sigma_{\alpha_\zeta \alpha'_\zeta}^{cc'} \partial_c \partial_{c'} = \sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta \alpha_\zeta cd} \sigma_{\zeta a'b'}^{\alpha'_\zeta} \sigma_{\zeta \alpha'_\zeta c'd'} \eta^{cc'} \partial^d \partial^{d'} = (S_{abcd} - \zeta \varepsilon_{abcd})(S_{a'b'c'd'} - \zeta \varepsilon_{a'b'c'd'}) \eta^{cc'} \partial^d \partial^{d'}$$

$$2\sigma_{-\zeta ab}^{\alpha'_\zeta} \sigma_{-\zeta a'b'}^{\alpha_\zeta} \sigma_{\alpha_\zeta \alpha'_\zeta}^{cc'} \partial_c \partial_{c'} = \sigma_{-\zeta ab}^{\alpha'_\zeta} \sigma_{-\zeta \alpha_\zeta cd} \sigma_{-\zeta a'b'}^{\alpha'_\zeta} \sigma_{-\zeta \alpha'_\zeta c'd'} \eta^{cc'} \partial^d \partial^{d'} = (S_{abcd} + \zeta \varepsilon_{abcd})(S_{a'b'c'd'} + \zeta \varepsilon_{a'b'c'd'}) \eta^{cc'} \partial^d \partial^{d'}$$

$$\begin{aligned} \text{证明:} &= -\frac{i}{2} \{ (\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta a'b'}^{\alpha'_\zeta} \sigma_{\alpha_\zeta \alpha'_\zeta}^{cc'} + \sigma_{-\zeta a'b'}^{\alpha'_\zeta} \sigma_{-\zeta ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta \alpha'_\zeta}^{c'c}) \partial_c \partial_{c'} \} \Delta(x - x') \\ &= -\frac{i}{4} [ (\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta a'b'}^{\alpha'_\zeta} \sigma_{\zeta \alpha_\zeta cd} \sigma_{-\zeta \alpha'_\zeta c'd'} \delta^{dd'} + \sigma_{-\zeta a'b'}^{\alpha'_\zeta} \sigma_{-\zeta ab}^{\alpha_\zeta} \sigma_{\zeta \alpha_\zeta c'd'} \sigma_{-\zeta \alpha'_\zeta cd} \delta^{dd'}) \partial^c \partial^{c'} ] \Delta(x - x') \\ &= -\frac{i}{4} [ (\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta a'b'}^{\alpha'_\zeta} \sigma_{\zeta \alpha_\zeta cd} \sigma_{\zeta \alpha'_\zeta c'd'} + \sigma_{-\zeta a'b'}^{\alpha'_\zeta} \sigma_{-\zeta ab}^{\alpha_\zeta} \sigma_{-\zeta \alpha_\zeta c'd'} \sigma_{-\zeta \alpha'_\zeta cd}) \eta^{dd'} \partial^c \partial^{c'} ] \Delta(x - x') \\ &= -\frac{i}{4} \{ [(-S_{abcd} + \zeta \varepsilon_{abcd})(-S_{a'b'c'd'} + \zeta \varepsilon_{a'b'c'd'}) + (-S_{abcd} - \zeta \varepsilon_{abcd})(-S_{a'b'c'd'} - \zeta \varepsilon_{a'b'c'd'})] \eta^{dd'} \partial^c \partial^{c'} \} \\ &\Delta(x - x') \\ &= -\frac{i}{2} [ (S_{abcd} S_{a'b'c'd'} + \varepsilon_{abcd} \varepsilon_{a'b'c'd'}) \eta^{dd'} \partial^c \partial^{c'} ] \Delta(x - x') \\ &= -\frac{i}{2} \{ [(\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc})(\delta_{a'c'} \delta_{b'd'} - \delta_{a'd'} \delta_{b'c'}) + \varepsilon_{abcd} \varepsilon_{a'b'c'd'}] \eta^{dd'} \partial^c \partial^{c'} \} \Delta(x - x') \\ &= -\frac{i}{2} [ (\delta_{a[c} \delta_{b]d} \delta_{a'[c'} \delta_{b']d'} + \delta_{a[a'} \delta_{b]b'} \delta_{c'c} \delta_{d'd'}) \eta^{dd'} \partial^c \partial^{c'} + m^2 \delta_{a[c} \delta_{b]d} \eta_a^c \eta_b^d ] \Delta(x - x') \\ &= -i \eta_{a < a'} \partial_{b]} \partial_{b']^+ \Delta(x - x') \end{aligned}$$

□

$$\begin{aligned} \text{证明:} &= -\frac{i}{2} \{ (\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta a'b'}^{\alpha'_\zeta} \sigma_{\alpha_\zeta \alpha'_\zeta}^{cc'} + \sigma_{-\zeta a'b'}^{\alpha'_\zeta} \sigma_{-\zeta ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta \alpha'_\zeta}^{c'c}) \partial_c \partial_{c'} \} \Delta(x - x') \\ &= -\frac{i}{4} [ (\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta a'b'}^{\alpha'_\zeta} \sigma_{\zeta \alpha_\zeta cd} \sigma_{-\zeta \alpha'_\zeta c'd'} \delta^{dd'} + \sigma_{-\zeta a'b'}^{\alpha'_\zeta} \sigma_{-\zeta ab}^{\alpha_\zeta} \sigma_{\zeta \alpha_\zeta c'd'} \sigma_{-\zeta \alpha'_\zeta cd} \delta^{dd'}) \partial^c \partial^{c'} ] \Delta(x - x') \\ &= -\frac{i}{4} [ (\sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\zeta a'b'}^{\alpha'_\zeta} \sigma_{\zeta \alpha_\zeta cd} \sigma_{\zeta \alpha'_\zeta c'd'} + \sigma_{-\zeta a'b'}^{\alpha'_\zeta} \sigma_{-\zeta ab}^{\alpha_\zeta} \sigma_{-\zeta \alpha_\zeta c'd'} \sigma_{-\zeta \alpha'_\zeta cd}) \eta^{dd'} \partial^c \partial^{c'} ] \Delta(x - x') \\ &= -\frac{i}{4} \{ [(-S_{abcd} + \zeta \varepsilon_{abcd})(-S_{a'b'c'd'} + \zeta \varepsilon_{a'b'c'd'}) + (-S_{abcd} - \zeta \varepsilon_{abcd})(-S_{a'b'c'd'} - \zeta \varepsilon_{a'b'c'd'})] \eta^{dd'} \partial^c \partial^{c'} \} \\ &\Delta(x - x') \end{aligned}$$

$$\begin{aligned}
&= -\frac{i}{2}[(S_{abcd}S_{a'b'c'd'} + \varepsilon_{abcd}\varepsilon_{a'b'c'd'})\eta^{dd'}\partial^c\partial^{+c'}]\Delta(x-x') \\
&= -\frac{i}{2}\{[(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})(\delta_{a'c'}\delta_{b'd'} - \delta_{a'd'}\delta_{b'c'}) + \varepsilon_{abcd}\varepsilon_{a'b'c'd'}]\eta^{dd'}\partial^c\partial^{+c'}\}\Delta(x-x') \\
&= -\frac{i}{2}[(\delta_{a[c}\delta_{bd]}\delta_{a'[c'}\delta_{b'd']}) + \delta_{a'[a'}\delta_{bb'}\delta_{c'c}\delta_{dd'}]\eta^{dd'}\partial^c\partial^{+c'} + m^2\delta_{a[c}\delta_{bd]}\eta_a^c\eta_{b'}^d]\Delta(x-x') \\
&= -i\eta_{[a<a'}\partial_{b]}\partial_{b'}^+\Delta(x-x')
\end{aligned}$$

□

定理5.5.6.

$$\begin{cases} [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a<a'}\partial_{b]}\partial_{b'}^+\Delta(x-x') \\ [F_{ab}(x), F_{cd}(x')] = -i\delta_{[a<c}\partial_{b]}\partial_{d'}\Delta(x-x') \\ [F_{a'b'}^+(x), F_{c'd'}^+(x')] = -i\delta_{[a'<c'}\partial_{b']}\partial_{d'}^+\Delta(x-x') \\ F_{ab} = F_{a'b'}^+\eta_a^a'\eta_b^b', \psi_{\alpha_\zeta} := -\frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha_\zeta}^{ab}F_{ab} \end{cases} \Leftrightarrow \begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\psi_{\alpha_\zeta}(x), \psi_{\beta_\zeta}(x')] = \frac{i}{2}m^2\delta_{\alpha_\zeta\beta_\zeta}\Delta(x-x') \\ [\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\zeta}^+(x')] = \frac{i}{2}m^2\delta_{\alpha'_\zeta\beta'_\zeta}\Delta(x-x') \\ \psi_{\alpha_{-\zeta}}(x) = -\psi_{\alpha_\zeta}^+(x), F_{ab} = \frac{1}{\sqrt{2}}(\sigma_{\zeta ab}^{\alpha_\zeta}\psi_{\alpha_\zeta} - \sigma_{-\zeta ab}^{\alpha'_\zeta}\psi_{\alpha'_\zeta}^+) \end{cases}$$

证明:  $[\psi_{\alpha_\zeta}(x), \psi_{\beta_\zeta}(x')]$ 

$$\begin{aligned}
&= [-\frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha_\zeta}^{ab}F_{ab}(x), -\frac{1}{2\sqrt{2}}\sigma_{\zeta\beta_\zeta}^{cd}F_{cd}(x')] \\
&= \frac{1}{8}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{\zeta\beta_\zeta}^{cd}[F_{ab}(x), F_{cd}(x')] \\
&= -\frac{i}{8}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{\zeta\beta_\zeta}^{cd}\delta_{[a<c}\partial_{b]}\partial_{d'}\Delta(x-x') \\
&= -\frac{i}{2}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{\zeta\beta_\zeta}^{cd}\delta_{ac}\partial_b\partial_{d'}\Delta(x-x') \\
&= \frac{i}{2}[\delta^{bd}\delta_{\alpha_\zeta\beta_\zeta} - \sigma_{\zeta\gamma_\zeta}^{bd}\gamma^{\gamma_\zeta}_{\alpha_\zeta\beta_\zeta}]\partial_b\partial_{d'}\Delta(x-x') \\
&= \frac{i}{2}\delta_{\alpha_\zeta\beta_\zeta}\partial_a\partial^a\Delta(x-x') \\
&= \frac{i}{2}m^2\delta_{\alpha_\zeta\beta_\zeta}\Delta(x-x')
\end{aligned}$$

□

证明:  $[\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\zeta}^+(x')]$ 

$$\begin{aligned}
&= [\frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha'_\zeta}^{ab}F_{ab}^+(x), \frac{1}{2\sqrt{2}}\sigma_{\zeta\beta'_\zeta}^{cd}F_{cd}^+(x')] \\
&= [\frac{1}{2\sqrt{2}}\sigma_{-\zeta\alpha'_\zeta}^{ab}F_{ab}(x), \frac{1}{2\sqrt{2}}\sigma_{-\zeta\beta'_\zeta}^{cd}F_{cd}(x')] \\
&= \frac{1}{8}\sigma_{-\zeta\alpha'_\zeta}^{ab}\sigma_{-\zeta\beta'_\zeta}^{cd}[F_{ab}(x), F_{cd}(x')] \\
&= -\frac{i}{8}\sigma_{-\zeta\alpha'_\zeta}^{ab}\sigma_{-\zeta\beta'_\zeta}^{cd}\delta_{[a<c}\partial_{b]}\partial_{d'}\Delta(x-x') \\
&= -\frac{i}{2}\sigma_{-\zeta\alpha'_\zeta}^{ab}\sigma_{-\zeta\beta'_\zeta}^{cd}\delta_{ac}\partial_b\partial_{d'}\Delta(x-x') \\
&= \frac{i}{2}[\delta^{bd}\delta_{\alpha'_\zeta\beta'_\zeta} - \sigma_{-\zeta\gamma'_\zeta}^{bd}\gamma^{\gamma'_\zeta}_{\alpha'_\zeta\beta'_\zeta}]\partial_b\partial_{d'}\Delta(x-x') \\
&= \frac{i}{2}\delta_{\alpha'_\zeta\beta'_\zeta}\partial_a\partial^a\Delta(x-x') \\
&= \frac{i}{2}m^2\delta_{\alpha'_\zeta\beta'_\zeta}\Delta(x-x')
\end{aligned}$$

□

证明:  $[\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')]$ 

$$\begin{aligned}
&= [-\frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha_\zeta}^{ab}F_{ab}(x), \frac{1}{2\sqrt{2}}\sigma_{\zeta\alpha'_\zeta}^{a'b'}F_{a'b'}^+(x')] \\
&= -\frac{1}{8}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{\zeta\alpha'_\zeta}^{a'b'}[F_{ab}(x), F_{a'b'}^+(x')] \\
&= \frac{i}{8}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{\zeta\alpha'_\zeta}^{a'b'}\eta_{[a<a'}\partial_{b]}\partial_{b'}^+\Delta(x-x') \\
&= \frac{i}{2}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{\zeta\alpha'_\zeta}^{a'b'}\eta_{aa'}\partial_b\partial_{b'}^+\Delta(x-x') \\
&= \frac{i}{2}\sigma_{\zeta\alpha_\zeta}^{ab}\sigma_{\zeta\alpha'_\zeta}^{a'b'}\delta_{aa'}\partial_b\partial_{b'}\Delta(x-x') \\
&= i\sigma_{\alpha_\zeta\alpha'_\zeta}^{bb'}\partial_b\partial_{b'}\Delta(x-x') \\
&= i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x')
\end{aligned}$$

□

证明:  $[F_{ab}(x), F_{a'b'}^+(x')]$ 

$$\begin{aligned}
&= [\frac{1}{\sqrt{2}}(\sigma_{\zeta ab}^{\alpha_\zeta}\psi_{\alpha_\zeta}(x) - \sigma_{-\zeta ab}^{\alpha'_\zeta}\psi_{\alpha'_\zeta}^+(x)), -\frac{1}{\sqrt{2}}(\sigma_{\zeta a'b'}^{\alpha'_\zeta}\psi_{\alpha'_\zeta}^+(x') - \sigma_{-\zeta a'b'}^{\alpha_\zeta}\psi_{\alpha_\zeta}(x'))] \\
&= -\frac{1}{2}\{\sigma_{\zeta ab}^{\alpha_\zeta}\sigma_{\zeta a'b'}^{\alpha'_\zeta}[\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] + \sigma_{-\zeta ab}^{\alpha'_\zeta}\sigma_{-\zeta a'b'}^{\alpha_\zeta}[\psi_{\alpha'_\zeta}^+(x), \psi_{\alpha_\zeta}(x')] \\
&\quad - \sigma_{\zeta ab}^{\alpha_\zeta}\sigma_{-\zeta a'b'}^{\beta_\zeta}[\psi_{\alpha_\zeta}(x), \psi_{\beta_\zeta}(x')] - \sigma_{-\zeta ab}^{\alpha'_\zeta}\sigma_{\zeta a'b'}^{\beta'_\zeta}[\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\zeta}^+(x')]\} \\
&= -\frac{1}{2}\{\sigma_{\zeta ab}^{\alpha_\zeta}\sigma_{\zeta a'b'}^{\alpha'_\zeta}[\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] - \sigma_{-\zeta ab}^{\alpha_\zeta}\sigma_{-\zeta a'b'}^{\alpha'_\zeta}[\psi_{\alpha_\zeta}(x'), \psi_{\alpha'_\zeta}^+(x)] \\
&\quad - \sigma_{\zeta ab}^{\alpha_\zeta}\sigma_{-\zeta a'b'}^{\beta_\zeta}[\psi_{\alpha_\zeta}(x), \psi_{\beta_\zeta}(x')] - \sigma_{-\zeta ab}^{\alpha'_\zeta}\sigma_{\zeta a'b'}^{\beta'_\zeta}[\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\zeta}^+(x')]\} \\
&= -\frac{1}{2}\{\sigma_{\zeta ab}^{\alpha_\zeta}\sigma_{\zeta a'b'}^{\alpha'_\zeta}i\sigma_{\alpha_\zeta\alpha'_\zeta}^{cd}\partial_c\partial_d + \sigma_{-\zeta ab}^{\alpha_\zeta}\sigma_{-\zeta a'b'}^{\alpha'_\zeta}i\sigma_{\alpha_\zeta\alpha'_\zeta}^{cd}\partial'_c\partial'_d - \sigma_{\zeta ab}^{\alpha_\zeta}\sigma_{-\zeta a'b'}^{\beta_\zeta}\frac{i}{2}m^2\delta_{\alpha_\zeta\beta_\zeta} - \sigma_{-\zeta ab}^{\alpha'_\zeta}\sigma_{\zeta a'b'}^{\beta'_\zeta}\frac{i}{2}m^2\delta_{\alpha'_\zeta\beta'_\zeta}\}\Delta(x-x')
\end{aligned}$$



$$\begin{aligned}
&= -\frac{i}{2}\{(\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta}^{\zeta c'c} + \sigma_{-\zeta a'b'}^{\alpha\zeta}\sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta}^{\zeta c'c})\partial_c\partial_{c'} - \frac{1}{2}m^2(\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta cd}^{\beta\zeta} + \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta cd}^{\beta\zeta})\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\
&= -\frac{i}{4}\{(\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta cd}\sigma_{-\zeta\alpha\zeta c'd'}\delta^{dd'} + \sigma_{-\zeta a'b'}^{\alpha\zeta}\sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta c'd'}\sigma_{-\zeta\alpha\zeta cd}\delta^{dd'})\partial^c\partial^{c'} + 2m^2 S_{abcd}\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\
&= -\frac{i}{4}\{(\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta cd}\sigma_{\zeta\alpha\zeta c'd'} + \sigma_{-\zeta a'b'}^{\alpha\zeta}\sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta\alpha\zeta c'd'}\sigma_{-\zeta\alpha\zeta cd})\eta^{dd'}\partial^c\partial^{c'} + 2m^2 S_{abcd}\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\
&= -\frac{i}{4}\{(-S_{abcd} + \zeta\varepsilon_{abcd})(-S_{a'b'c'd'} + \zeta\varepsilon_{a'b'c'd'}) + (-S_{abcd} - \zeta\varepsilon_{abcd})(-S_{a'b'c'd'} - \zeta\varepsilon_{a'b'c'd'})\}\eta^{dd'}\partial^c\partial^{c'} + 2m^2 S_{abcd}\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\
&= -\frac{i}{2}\{(S_{abcd}S_{a'b'c'd'} + \varepsilon_{abcd}\varepsilon_{a'b'c'd'})\eta^{dd'}\partial^c\partial^{c'} + m^2 S_{abcd}\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\
&= -\frac{i}{2}\{[(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})(\delta_{a'c'}\delta_{b'd'} - \delta_{a'd'}\delta_{b'c'}) + \varepsilon_{abcd}\varepsilon_{a'b'c'd'}]\eta^{dd'}\partial^c\partial^{c'} + m^2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\
&= -\frac{i}{2}\{[\delta_{[a}\delta_{b]d}\delta_{a'}[c']\delta_{b'd'}] + \delta_{a[a'}\delta_{b]b'}\delta_{c'c}\delta_{d'd']}\eta^{dd'}\partial^c\partial^{c'} + m^2\delta_{a[c}\delta_{b]d}\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\
&= -i\eta_{[a<a'}\partial_{b]}\partial_{b'}^+\Delta(x-x') \quad \square
\end{aligned}$$

定理5.5.7.

$$\begin{cases} [F_{ab}(x), A_{c'}^+(x')] = -i\eta_{c'[a}\partial_{b]}\Delta(x-x') \\ [F_{a'b'}^+(x), A_c(x')] = -i\eta_{c[a'}\partial_{b']}^+\Delta(x-x') \\ [F_{ab}(x), A_c(x')] = -i\delta_{c[a}\partial_{b]}\Delta(x-x') \\ [F_{a'b'}^+(x), A_{c'}^+(x')] = -i\delta_{c'[a'}\partial_{b']}^+\Delta(x-x') \end{cases} \quad \begin{cases} [\psi_{\alpha\zeta}(x), A_{c'}^+(x')] = -\frac{i}{\sqrt{2}}(\sigma_{+\zeta}, -i\zeta)^b|_{c'\alpha\zeta}\partial_b\Delta(x-x') \\ [\psi_{\alpha\zeta}^+(x), A_c(x')] = -\frac{i}{\sqrt{2}}(\sigma_{+\zeta}, -i\zeta)^b|_{c\alpha\zeta}\partial_b\Delta(x-x') \\ [\psi_{\alpha\zeta}(x), A_c(x')] = -\frac{i}{\sqrt{2}}(\sigma_{-\zeta}, -i\zeta)^b|_{c\alpha\zeta}\partial_b\Delta(x-x') \\ [\psi_{\alpha\zeta}^+(x), A_{c'}^+(x')] = -\frac{i}{\sqrt{2}}(\sigma_{-\zeta}, -i\zeta)^b|_{c'\alpha\zeta}\partial_b\Delta(x-x') \end{cases}$$

$$\begin{aligned}
\text{证明: } [F_{ab}(x), A_c(x')] &= [\partial_a A_b(x) - \partial_b A_a(x), A_c(x')] \\
&= i(\delta_{bc} - \frac{\partial_b\partial_c}{m^2})\partial_a\Delta(x-x') - i(\delta_{ac} - \frac{\partial_a\partial_c}{m^2})\partial_b\Delta(x-x') \\
&= -i\delta_{c[a}\partial_{b]}\Delta(x-x') \quad \square
\end{aligned}$$

$$\begin{aligned}
\text{证明: } [\psi_{\alpha\zeta}(x), A_c(x')] &= \frac{i}{\sqrt{2}}\frac{i}{2}\sigma_{\zeta\alpha\zeta}^{ab}[F_{ab}(x), A_c(x')] \\
&= -i\frac{i}{\sqrt{2}}\frac{i}{2}\sigma_{\zeta\alpha\zeta}^{ab}\delta_{c[a}\partial_{b]}\Delta(x-x') \\
&= \frac{i}{\sqrt{2}}\sigma_{\zeta\alpha\zeta}^{ab}\delta_{ca}\partial_b\Delta(x-x') \\
&= -\frac{i}{\sqrt{2}}(\sigma_{\zeta})_{\alpha\zeta}|^b{}_c\partial_b\Delta(x-x'), [(\sigma_{\zeta}, -i\zeta)^{\alpha\zeta}|_{ab} = (\sigma_{-\zeta}, -i\zeta)_a|_b{}^{\alpha\zeta}] \\
&= -\frac{i}{\sqrt{2}}(\sigma_{-\zeta}, -i\zeta)^b|_{c\alpha\zeta}\partial_b\Delta(x-x') \quad \square
\end{aligned}$$

### 5.5.7 Majorana实场的部分等时对易规则

推论5.5.4.

$$\begin{cases} [E_i(\vec{r}, t), A_{c'}^+(\vec{r}', t)] = i\eta_{ic'}\delta^3(\vec{r} - \vec{r}'), [E_i(\vec{r}, t), A_c(\vec{r}', t)] = i\delta_{ic}\delta^3(\vec{r} - \vec{r}') \\ [B_i(\vec{r}, t), A_{c'}^+(\vec{r}', t)] = 0, [B_i(\vec{r}, t), A_c(\vec{r}', t)] = 0 \end{cases}$$

推论5.5.5.

$$\begin{cases} [E_i(\vec{r}, t), A_j(\vec{r}', t)] = i\delta_{ij}\delta^3(\vec{r} - \vec{r}') \\ [E_i(\vec{r}, t), E_j(\vec{r}', t)] = 0, [A_i(\vec{r}, t), A_j(\vec{r}', t)] = 0 \end{cases} \quad \Leftrightarrow \begin{cases} [E_i(\vec{r}, t), B_j(\vec{r}', t)] = -i\varepsilon_{ij}{}^k\partial_k\delta^3(\vec{r} - \vec{r}') \\ [B_i(\vec{r}, t), A_j(\vec{r}', t)] = 0 \end{cases}$$

## 5.6 $\psi_{\alpha\zeta}$ 与 $\psi_{A_\zeta B_\zeta}$ 的等价对易关系

### 5.6.1 复场与实场的共同对易规则

定理5.6.1.

$$\begin{cases} [\psi_{\alpha\zeta}(x), \psi_{\alpha\zeta}^+(x')] = i\sigma_{\alpha\zeta\alpha\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ \psi_{A_\zeta B_\zeta} := \frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\psi_{\alpha\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] \\ = -\frac{i}{2}(\sigma, i\zeta)^a{}_{A_\zeta A'_\zeta}(\sigma, i\zeta)^b{}_{B_\zeta B'_\zeta}\partial_a\partial_b\Delta(x-x') \\ \psi_{\alpha\zeta} = \frac{i\zeta}{\sqrt{2}}\sigma_{\alpha\zeta}^{A_\zeta B_\zeta}\psi_{A_\zeta B_\zeta} \end{cases}$$

$$\begin{aligned}
\text{证明: } [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] &= [\frac{i\zeta}{\sqrt{2}}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\psi_{\alpha\zeta}(x), \frac{-i\zeta}{\sqrt{2}}\sigma_{A'_\zeta B'_\zeta}^{\alpha\zeta}\psi_{\alpha\zeta}^+(x')] \\
&= \frac{1}{2}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\sigma_{A'_\zeta B'_\zeta}^{\alpha\zeta}[\psi_{\alpha\zeta}(x), \psi_{\alpha\zeta}^+(x')] \\
&= \frac{1}{2}\sigma_{A_\zeta B_\zeta}^{\alpha\zeta}\sigma_{A'_\zeta B'_\zeta}^{\alpha\zeta}i\sigma_{\alpha\zeta\alpha\zeta}^{ab}\partial_a\partial_b\Delta(x-x')
\end{aligned}$$

$$\begin{aligned}
&= \frac{i}{2} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)^a_{C_\zeta C'_\zeta} \frac{-i\zeta}{\sqrt{2}} (\sigma, i\zeta)^b_{D_\zeta D'_\zeta} \frac{-i\zeta}{\sqrt{2}} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} \partial_a \partial_b \Delta(x - x') \\
&= -\frac{i}{8} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{\alpha_\zeta}^{C_\zeta D_\zeta} \sigma_{A'_\zeta B'_\zeta}^{\alpha'_\zeta} \sigma_{\alpha'_\zeta}^{C'_\zeta D'_\zeta} (\sigma, i\zeta)^a_{C_\zeta C'_\zeta} (\sigma, i\zeta)^b_{D_\zeta D'_\zeta} \partial_a \partial_b \Delta(x - x') \\
&= -\frac{i}{8} \delta_{\{A_\zeta B_\zeta\}}^{C_\zeta D_\zeta} \delta_{\{A'_\zeta B'_\zeta\}}^{C'_\zeta D'_\zeta} (\sigma, i\zeta)^a_{C_\zeta C'_\zeta} (\sigma, i\zeta)^b_{D_\zeta D'_\zeta} \partial_a \partial_b \Delta(x - x') \\
&= -\frac{i}{8} (\sigma, i\zeta)^a_{\{A_\zeta(A'_\zeta} (\sigma, i\zeta)^b_{B_\zeta\} B'_\zeta)} \partial_a \partial_b \Delta(x - x') \\
&= -\frac{i}{2} (\sigma, i\zeta)^a_{A_\zeta A'_\zeta} (\sigma, i\zeta)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \quad \square
\end{aligned}$$

证明:  $[\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')]$

$$\begin{aligned}
&= \left[ \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta}(x), \frac{-i\zeta}{\sqrt{2}} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} \psi_{A'_\zeta B'_\zeta}^+(x') \right] \\
&= \frac{1}{2} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] \\
&= -\frac{i}{4} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta} (\sigma, i\zeta)^a_{A_\zeta A'_\zeta} (\sigma, i\zeta)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \\
&= i \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \quad \square
\end{aligned}$$

### 5.6.2 复场条件

定理5.6.2.

$$\begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\beta_\zeta}(x')] = 0, [\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\zeta}^+(x')] = 0 \\ \psi_{A_\zeta B_\zeta} := \frac{i\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \psi_{\alpha_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{A_\zeta B_\zeta}(x), \psi_{C_\zeta D_\zeta}(x')] = 0, [\psi_{A'_\zeta B'_\zeta}^+(x), \psi_{C'_\zeta D'_\zeta}^+(x')] = 0 \\ \psi_{\alpha_\zeta} = \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta} \end{cases}$$

### 5.6.3 复场完整的对易规则

定理5.6.3.

$$\begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\psi_{\alpha_\zeta}(x), \psi_{\beta_\zeta}(x')] = 0, [\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\zeta}^+(x')] = 0 \\ \psi_{A_\zeta B_\zeta} := \frac{i\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \psi_{\alpha_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] \\ = -\frac{i}{2} (\sigma, i\zeta)^a_{A_\zeta A'_\zeta} (\sigma, i\zeta)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \\ [\psi_{A_\zeta B_\zeta}(x), \psi_{C_\zeta D_\zeta}(x')] = 0, [\psi_{A'_\zeta B'_\zeta}^+(x), \psi_{C'_\zeta D'_\zeta}^+(x')] = 0 \\ \psi_{\alpha_\zeta} = \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta} \end{cases}$$

### 5.6.4 Majorana实场条件

定理5.6.4.

$$\begin{cases} \psi_{\alpha_{-\zeta}}(x) = -\psi_{\alpha_\zeta}^+(x) \\ \psi_{A_\zeta B_\zeta} := \frac{i\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \psi_{\alpha_\zeta} \end{cases} \Leftrightarrow \begin{cases} \psi^{??} = -\sigma_y \psi^+ \sigma_y \\ \psi_{\alpha_\zeta} = \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta} \end{cases}$$

### 5.6.5 Majorana实场完整的对易规则

定理5.6.5.

$$\begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ [\psi_{\alpha_\zeta}(x), \psi_{\beta_\zeta}(x')] = \frac{i}{2} m^2 \delta_{\alpha_\zeta \beta_\zeta} \Delta(x - x') \\ [\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\zeta}^+(x')] = \frac{i}{2} m^2 \delta_{\alpha'_\zeta \beta'_\zeta} \Delta(x - x') \\ \psi_{\alpha_{-\zeta}}(x) = -\psi_{\alpha_\zeta}^+(x), \psi_{A_\zeta B_\zeta} := \frac{i\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \psi_{\alpha_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] \\ = -\frac{i}{2} (\sigma, i\zeta)^a_{A_\zeta A'_\zeta} (\sigma, i\zeta)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \\ [\psi_{A_\zeta B_\zeta}(x), \psi_{C_\zeta D_\zeta}(x')] = \frac{i}{8} m^2 \varepsilon_{\{A_\zeta(C_\zeta \varepsilon_{B_\zeta\} D_\zeta)} \Delta(x - x') \\ [\psi_{A'_\zeta B'_\zeta}^+(x), \psi_{C'_\zeta D'_\zeta}^+(x')] = \frac{i}{8} m^2 \varepsilon_{\{A'_\zeta(C'_\zeta \varepsilon_{B'_\zeta\} D'_\zeta)} \Delta(x - x') \\ \psi_{\alpha_\zeta} = \frac{i\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \psi_{A_\zeta B_\zeta} \end{cases}$$

## 5.7 $\psi_{k_\zeta}$ 与 $\psi_{A_\zeta B_\zeta}$ 的等价对易关系

### 5.7.1 复场与实场的共同对易规则

定理5.7.1.

$$\begin{cases} [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] \\ = -\frac{i}{2} (\sigma, i\zeta)^a_{A_\zeta A'_\zeta} (\sigma, i\zeta)^b_{B_\zeta B'_\zeta} \partial_a \partial_b \Delta(x - x') \\ \psi_{k_\zeta} = \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1) \psi_{A_\zeta B_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i \Gamma_{k_\zeta k'_\zeta}^{ab} \partial_a \partial_b \Delta(x - x') \\ \psi_{A_\zeta B_\zeta} = \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1) \psi_{k_\zeta} \end{cases}$$

证明:  $[\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')]$

$$\begin{aligned} &= [\Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1)\psi_{A_\zeta B_\zeta}(x), \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta}(1)\psi_{A'_\zeta B'_\zeta}^+(x')] \\ &= \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1)\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta}(1)[\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] \\ &= -\frac{i}{2}\Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1)\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta}(1)(\sigma, i\zeta)^a_{A_\zeta A'_\zeta}(\sigma, i\zeta)^b_{B_\zeta B'_\zeta}\partial_a\partial_b\Delta(x-x') \\ &= i\Gamma_{k_\zeta k'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \end{aligned} \quad \square$$

证明:  $[\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')]$

$$\begin{aligned} &= [\Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\psi_{k_\zeta}(x), \Gamma_{A'_\zeta B'_\zeta}^{k'_\zeta}(1)\psi_{k'_\zeta}^+(x')] \\ &= \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\Gamma_{A'_\zeta B'_\zeta}^{k'_\zeta}(1)[\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] \\ &= \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\Gamma_{A'_\zeta B'_\zeta}^{k'_\zeta}(1)i\Gamma_{k_\zeta k'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ &= -\frac{i}{2}\Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\Gamma_{A'_\zeta B'_\zeta}^{k'_\zeta}(1)\Gamma_{k_\zeta}^{C_\zeta D_\zeta}(1)\Gamma_{k'_\zeta}^{C'_\zeta D'_\zeta}(1)(\sigma, i\zeta)^a_{C_\zeta C'_\zeta}(\sigma, i\zeta)^b_{D_\zeta D'_\zeta}\partial_a\partial_b\Delta(x-x') \\ &= -\frac{i}{8}\delta_{A_\zeta}^{[C_\zeta}\delta_{B_\zeta}^{D_\zeta]}\delta_{A'_\zeta}^{[C'_\zeta}\delta_{B'_\zeta}^{D'_\zeta]}(\sigma, i\zeta)^a_{C_\zeta C'_\zeta}(\sigma, i\zeta)^b_{D_\zeta D'_\zeta}\partial_a\partial_b\Delta(x-x') \\ &= -\frac{i}{8}(\sigma, i\zeta)^a_{\{A_\zeta(A'_\zeta}\}(\sigma, i\zeta)^b_{B_\zeta(B'_\zeta)}\partial_a\partial_b\Delta(x-x') \\ &= -\frac{i}{2}(\sigma, i\zeta)^a_{A_\zeta A'_\zeta}(\sigma, i\zeta)^b_{B_\zeta B'_\zeta}\partial_a\partial_b\Delta(x-x') \end{aligned} \quad \square$$

### 5.7.2 复场条件

定理5.7.2.

$$\begin{cases} [\psi_{A_\zeta B_\zeta}(x), \psi_{C_\zeta D_\zeta}(x')] = 0, [\psi_{A'_\zeta B'_\zeta}^+(x), \psi_{C'_\zeta D'_\zeta}^+(x')] = 0 \\ \psi_{k_\zeta} = \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1)\psi_{A_\zeta B_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0, [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \\ \psi_{A_\zeta B_\zeta} = \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\psi_{k_\zeta} \end{cases}$$

### 5.7.3 复场完整的对易规则

定理5.7.3.

$$\begin{cases} [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] \\ = -\frac{i}{2}(\sigma, i\zeta)^a_{A_\zeta A'_\zeta}(\sigma, i\zeta)^b_{B_\zeta B'_\zeta}\partial_a\partial_b\Delta(x-x') \\ [\psi_{A_\zeta B_\zeta}(x), \psi_{C_\zeta D_\zeta}(x')] = 0, [\psi_{A'_\zeta B'_\zeta}^+(x), \psi_{C'_\zeta D'_\zeta}^+(x')] = 0 \\ \psi_{k_\zeta} = \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1)\psi_{A_\zeta B_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0, [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \\ \psi_{A_\zeta B_\zeta} = \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\psi_{k_\zeta} \end{cases}$$

### 5.7.4 Majorana实场条件

### 5.7.5 Majorana实场完整的对易规则

定理5.7.4.

$$\begin{cases} [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] \\ = -\frac{i}{2}(\sigma, i\zeta)^a_{A_\zeta A'_\zeta}(\sigma, i\zeta)^b_{B_\zeta B'_\zeta}\partial_a\partial_b\Delta(x-x') \\ [\psi_{A_\zeta B_\zeta}(x), \psi_{C_\zeta D_\zeta}(x')] = \frac{i}{8}m^2\varepsilon_{\{A_\zeta(C_\zeta\varepsilon_{B_\zeta}D_\zeta\}}\Delta(x-x') \\ [\psi_{A'_\zeta B'_\zeta}^+(x), \psi_{C'_\zeta D'_\zeta}^+(x')] = \frac{i}{8}m^2\varepsilon_{\{A'_\zeta(C'_\zeta\varepsilon_{B'_\zeta}D'_\zeta\}}\Delta(x-x') \\ \psi_{k_\zeta} = \Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1)\psi_{A_\zeta B_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = \frac{i}{2}m^2\varepsilon_{k_\zeta l_\zeta}(1)\Delta(x-x') \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = \frac{i}{2}m^2\varepsilon_{k'_\zeta l'_\zeta}(1)\Delta(x-x') \\ \psi_{A_\zeta B_\zeta} = \Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1)\psi_{k_\zeta} \end{cases}$$

## 5.8 $\psi_{\alpha_\zeta}$ 与 $\psi_{k_\zeta}$ 的等价对易关系

### 5.8.1 复场与实场的共同对易规则

引理5.8.1.  $\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab} = \Gamma_{\alpha_\zeta}^{k_\zeta}(1)\Gamma_{\alpha'_\zeta}^{k'_\zeta}(1)\Gamma_{k_\zeta k'_\zeta}^{ab}, \Gamma_{k_\zeta k'_\zeta}^{ab} = \Gamma_{k_\zeta}^{\alpha_\zeta}(1)\Gamma_{k'_\zeta}^{\alpha'_\zeta}(1)\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}$

定理5.8.1.

$$\begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ \psi_{k_\zeta} = \Gamma_{k_\zeta}^{\alpha_\zeta}(1)\psi_{\alpha_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ \psi_{\alpha_\zeta} = \Gamma_{\alpha_\zeta}^{k_\zeta}(1)\psi_{k_\zeta} \end{cases}$$

证明:  $[\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')]$

$$\begin{aligned} &= [\Gamma_{k_\zeta}^{\alpha_\zeta}(1)\psi_{\alpha_\zeta}(x), \Gamma_{k'_\zeta}^{\alpha'_\zeta}(1)\psi_{\alpha'_\zeta}^+(x')] \\ &= \Gamma_{k_\zeta}^{\alpha_\zeta}(1)\Gamma_{k'_\zeta}^{\alpha'_\zeta}(1)[\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] \\ &= \Gamma_{k_\zeta}^{\alpha_\zeta}(1)\Gamma_{k'_\zeta}^{\alpha'_\zeta}(1)i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ &= i\Gamma_{k_\zeta k'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \end{aligned} \quad \square$$

证明:  $[\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')]$

$$\begin{aligned} &= [\Gamma_{\alpha_\zeta}^{k_\zeta}(1)\psi_{k_\zeta}(x), \Gamma_{\alpha'_\zeta}^{k'_\zeta}(1)\psi_{k'_\zeta}^+(x')] \\ &= \Gamma_{\alpha_\zeta}^{k_\zeta}(1)\Gamma_{\alpha'_\zeta}^{k'_\zeta}(1)[\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] \\ &= \Gamma_{\alpha_\zeta}^{k_\zeta}(1)\Gamma_{\alpha'_\zeta}^{k'_\zeta}(1)i\Gamma_{k_\zeta k'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ &= i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \end{aligned} \quad \square$$

### 5.8.2 复场条件

定理5.8.2.

$$\begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\beta_\zeta}(x')] = 0, [\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\zeta}^+(x')] = 0 \\ \psi_{k_\zeta} = \Gamma_{k_\zeta}^{\alpha_\zeta}(1)\psi_{\alpha_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0, [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \\ \psi_{\alpha_\zeta} = \Gamma_{\alpha_\zeta}^{k_\zeta}(1)\psi_{k_\zeta} \end{cases}$$

### 5.8.3 复场完整的对易规则

定理5.8.3.

$$\begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\psi_{\alpha_\zeta}(x), \psi_{\beta_\zeta}(x')] = 0, [\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\zeta}^+(x')] = 0 \\ \psi_{k_\zeta} = \Gamma_{k_\zeta}^{\alpha_\zeta}(1)\psi_{\alpha_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = 0, [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = 0 \\ \psi_{\alpha_\zeta} = \Gamma_{\alpha_\zeta}^{k_\zeta}(1)\psi_{k_\zeta} \end{cases}$$

### 5.8.4 Majorana实场条件

### 5.8.5 Majorana实场完整的对易规则

定理5.8.4.

$$\begin{cases} [\psi_{\alpha_\zeta}(x), \psi_{\alpha'_\zeta}^+(x')] = i\sigma_{\alpha_\zeta\alpha'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\psi_{\alpha_\zeta}(x), \psi_{\beta_\zeta}(x')] = \frac{i}{2}m^2\delta_{\alpha_\zeta\beta_\zeta}\Delta(x-x') \\ [\psi_{\alpha'_\zeta}^+(x), \psi_{\beta'_\zeta}^+(x')] = \frac{i}{2}m^2\delta_{\alpha'_\zeta\beta'_\zeta}\Delta(x-x') \\ \psi_{\alpha_{-\zeta}}(x) = -\psi_{\alpha_\zeta}^+(x), \psi_{k_\zeta} = \Gamma_{k_\zeta}^{\alpha_\zeta}(1)\psi_{\alpha_\zeta} \end{cases} \Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')] = i\Gamma_{k_\zeta k'_\zeta}^{ab}\partial_a\partial_b\Delta(x-x') \\ [\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')] = \frac{i}{2}m^2\varepsilon_{k_\zeta l_\zeta}(1)\Delta(x-x') \\ [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')] = \frac{i}{2}m^2\varepsilon_{k'_\zeta l'_\zeta}(1)\Delta(x-x') \\ \psi_{\alpha_\zeta} = \Gamma_{\alpha_\zeta}^{k_\zeta}(1)\psi_{k_\zeta} \end{cases}$$

## 5.9 有质量复矢量场 $\psi_{\lambda_\zeta\mu_\zeta}$ 和复势 $A_a$ 的等价对易关系

$$\begin{aligned} &\text{定理5.9.1. } [\psi_{A_\zeta}^{B'_\zeta}(x), \psi_{A'_\zeta}^{B_\zeta}(x')] = \frac{i}{4}[(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a(\sigma, -i\zeta)_{B'_\zeta B_\zeta}^{b'}\partial_a\partial_b + m^2\delta_{A_\zeta}^{B_\zeta}\delta_{A'_\zeta}^{B'_\zeta}]\Delta(x-x') \\ &\Leftrightarrow [\psi_{A_\zeta B'_\zeta}(x), \psi_{A'_\zeta B_\zeta}^+(x')] = \frac{i}{4}[-(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b\partial_a\partial_b + m^2\varepsilon_{A'_\zeta B'_\zeta}\varepsilon_{A_\zeta B_\zeta}]\Delta(x-x') \\ &\psi_{[\lambda_\zeta\mu_\zeta]} = [im\gamma^a(\zeta)C\frac{A_a}{2} + S^{ab}(e, \zeta)C\frac{F_{ab}}{2}] = [im\gamma^a(\zeta)C - 2S^{ab}(e, \zeta)C\partial_b]\frac{A_a}{2} \end{aligned}$$

定理5.9.2.

$$\begin{aligned} &[A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a\partial_{a'}}{m^2})\Delta(x-x'), \psi_{[\lambda_\zeta\mu_\zeta]} = [im\gamma^a(\zeta)C - 2S^{ab}(e, \zeta)C\partial_b]\frac{A_a}{2} \\ &\Leftrightarrow [\psi_{\lambda_\zeta\mu_\zeta}(x), \psi_{\lambda'_\zeta\mu'_\zeta}^+(x')] = \frac{i}{8}[(m - \gamma^a\partial_a)\gamma^A]_{\{\lambda_\zeta\lambda'_\zeta\}}[(m - \gamma^b\partial_b)\gamma^A]_{\{\mu_\zeta\mu'_\zeta\}}\Delta(x-x') \\ &\Rightarrow [\psi_{A_\zeta B_\zeta}(x), \psi_{A'_\zeta B'_\zeta}^+(x')] = -\frac{i}{2}(\sigma, i\zeta)_{A_\zeta A'_\zeta}^a(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b\partial_a\partial_b\Delta(x-x') \end{aligned}$$

## 5.10 自旋-1 Majorana粒子从实势出发导出各种物理量对易规则

以下对实场情形才是对的。

$$\text{引理5.10.1. } F_{ab} = \frac{1}{\sqrt{2}}(-\sigma_{-cab}^{\alpha'_\zeta}\psi_{\alpha'_\zeta}^+ + \sigma_{cab}^{\alpha_\zeta}\psi_{\alpha_\zeta}), *F_{ab} = \frac{\zeta}{\sqrt{2}}(-\sigma_{-cab}^{\alpha'_\zeta}\psi_{\alpha'_\zeta}^+ - \sigma_{cab}^{\alpha_\zeta}\psi_{\alpha_\zeta})$$

定理5.10.1.

$$[A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2})\Delta(x-x'), A_{a'}^+ = A_a \eta_{a'}^a \Leftrightarrow [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\partial_a \partial_b}{m^2})\Delta(x-x')$$

$$\text{定理5.10.2. } [A_a(x), A_b(x')] = i(\delta_{ab} - \frac{\partial_a \partial_b}{m^2}) \Rightarrow [F_{ab}(x), F_{cd}(x')] = -i\delta_{[a<c}\partial_b]\partial_d \Delta(x-x')$$

$$\text{定理5.10.3. } [F_{ab}(x), F_{cd}(x')] = -i\delta_{[a<c}\partial_b]\partial_d \Delta(x-x') \Leftrightarrow [F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a<a'}\partial_b]\partial_{b'} \Delta(x-x')$$

$$\text{定理5.10.4. } [*F_{ab}(x), *F_{a'b'}^+(x')] = -i\eta_{[a<a'}(\partial_b)\partial_{b'}^+ - \frac{1}{2}m^2\eta_{b|b'}\rangle)\Delta(x-x')$$

证明:  $[*F_{ab}(x), *F_{a'b'}^+(x')]$

$$\begin{aligned} &= [\frac{1}{\sqrt{2}}(-\sigma_{\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}(x) - \sigma_{-\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}^+(x)), -\frac{1}{\sqrt{2}}(-\sigma_{\zeta a'b'}^{\alpha\zeta}\psi_{\alpha\zeta}^+(x') - \sigma_{-\zeta a'b'}^{\alpha\zeta}\psi_{\alpha\zeta}(x'))] \\ &= -\frac{1}{2}\{\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\alpha\zeta}[\psi_{\alpha\zeta}(x), \psi_{\alpha\zeta}^+(x')] + \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\alpha\zeta}[\psi_{\alpha\zeta}^+(x), \psi_{\alpha\zeta}(x')]\} \\ &+ \sigma_{\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\beta\zeta}[\psi_{\alpha\zeta}(x), \psi_{\beta\zeta}(x')] + \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\beta\zeta}[\psi_{\alpha\zeta}^+(x), \psi_{\beta\zeta}^+(x')] \\ &= -\frac{i}{2}\{[(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})(\delta_{a'c'}\delta_{b'd'} - \delta_{a'd'}\delta_{b'c'}) + \varepsilon_{abcd}\varepsilon_{a'b'c'd'}]\eta^{dd'}\partial^c\partial^{+c'} - m^2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\ &= -\frac{i}{2}\{[\delta_{a[c}\delta_{b|d}]\delta_{a'}[c'\delta_{b'd'}] + \delta_{a[a'}\delta_{b|b'}\delta_{c'c'}\delta_{d|d'}]\eta^{dd'}\partial^c\partial^{+c'} + m^2\delta_{a[c}\delta_{b|d}]\eta_{a'}^c\eta_{b'}^d - 2m^2\delta_{a[c}\delta_{b|d}]\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\ &= -i\eta_{[a<a'}(\partial_b)\partial_{b'}^+ - \frac{1}{2}m^2\eta_{b|b'}\rangle)\Delta(x-x') \quad \square \end{aligned}$$

$$\text{定理5.10.5. } [*F_{ab}(x), *F_{a'b'}^+(x')] = -i\eta_{[a<a'}(\partial_b)\partial_{b'}^+ - \frac{1}{2}m^2\eta_{b|b'}\rangle)\Delta(x-x')$$

$$\Leftrightarrow [*F_{ab}(x), *F_{cd}(x')] = -i\delta_{[a<c}(\partial_b)\partial_d - \frac{1}{2}m^2\delta_{b|d}\rangle)\Delta(x-x')$$

证明:  $[F_{ab}(x), *F_{a'b'}^+(x')]$

$$\begin{aligned} &= [\frac{1}{\sqrt{2}}(\sigma_{\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}(x) - \sigma_{-\zeta ab}^{\alpha\zeta}\psi_{\alpha\zeta}^+(x)), \frac{1}{\sqrt{2}}(\sigma_{\zeta a'b'}^{\alpha\zeta}\psi_{\alpha\zeta}^+(x') + \sigma_{-\zeta a'b'}^{\alpha\zeta}\psi_{\alpha\zeta}(x'))] \\ &= \frac{1}{2}\{\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\alpha\zeta}[\psi_{\alpha\zeta}(x), \psi_{\alpha\zeta}^+(x')] - \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\alpha\zeta}[\psi_{\alpha\zeta}^+(x), \psi_{\alpha\zeta}(x')]\} \\ &+ \sigma_{\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\beta\zeta}[\psi_{\alpha\zeta}(x), \psi_{\beta\zeta}(x')] - \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\beta\zeta}[\psi_{\alpha\zeta}^+(x), \psi_{\beta\zeta}^+(x')] \\ &= \frac{1}{2}\{\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\alpha\zeta}[\psi_{\alpha\zeta}(x), \psi_{\alpha\zeta}^+(x')] + \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\alpha\zeta}[\psi_{\alpha\zeta}(x'), \psi_{\alpha\zeta}^+(x)]\} \\ &+ \sigma_{\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\beta\zeta}[\psi_{\alpha\zeta}(x), \psi_{\beta\zeta}(x')] - \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\beta\zeta}[\psi_{\alpha\zeta}^+(x), \psi_{\beta\zeta}^+(x')] \\ &= \frac{1}{2}\{\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\alpha\zeta}i\sigma_{\alpha\zeta\alpha'}^{cd}\partial_c\partial_d - \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\alpha\zeta}i\sigma_{\alpha\zeta\alpha'}^{cd}\partial_c\partial_d + \sigma_{\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\beta\zeta}\frac{i}{2}m^2\delta_{\alpha\zeta\beta\zeta} - \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\beta\zeta}\frac{i}{2}m^2\delta_{\alpha\zeta\beta\zeta}\}\Delta(x-x') \\ &= \frac{i}{2}\{(\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\alpha\zeta}\sigma_{\alpha\zeta\alpha'}^{cc'} - \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\alpha\zeta}\sigma_{\alpha\zeta\alpha'}^{c'c})\partial_c\partial_{c'} + \frac{1}{2}m^2(\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta cd}^{\beta\zeta} - \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta cd}^{\beta\zeta})\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\ &= \frac{i}{4}\{(\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta cd}\sigma_{-\zeta\alpha'\zeta c'd'}\delta^{dd'} - \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta c'd'}\sigma_{-\zeta\alpha'\zeta cd}\delta^{dd'})\partial^c\partial^{c'} + 2m^2\zeta\varepsilon_{abcd}\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\ &= \frac{i}{4}\{(\sigma_{\zeta ab}^{\alpha\zeta}\sigma_{\zeta a'b'}^{\alpha\zeta}\sigma_{\zeta\alpha\zeta cd}\sigma_{\zeta\alpha'\zeta c'd'} - \sigma_{-\zeta ab}^{\alpha\zeta}\sigma_{-\zeta a'b'}^{\alpha\zeta}\sigma_{-\zeta\alpha\zeta c'd'}\sigma_{-\zeta\alpha'\zeta cd})\eta^{dd'}\partial^c\partial^{+c'} + 2m^2\zeta\varepsilon_{abcd}\eta_{a'}^c\eta_{b'}^d\}\Delta(x-x') \\ &= \frac{i}{4}\{(-S_{abcd} + \zeta\varepsilon_{abcd})(-S_{a'b'c'd'} + \zeta\varepsilon_{a'b'c'd'}) - (-S_{abcd} - \zeta\varepsilon_{abcd})(-S_{a'b'c'd'} - \zeta\varepsilon_{a'b'c'd'})\}\eta^{dd'}\partial^c\partial^{+c'} + 2m^2\zeta\varepsilon_{abcd}\eta_{a'}^c\eta_{b'}^d\} \\ &\Delta(x-x') \\ &= \frac{-i\zeta}{2}[(S_{abcd}\varepsilon_{a'b'c'd'} + \varepsilon_{abcd}S_{a'b'c'd'})\eta^{dd'}\partial^c\partial^{+c'} - m^2\varepsilon_{abcd}\eta_{a'}^c\eta_{b'}^d]\Delta(x-x') \quad \square \end{aligned}$$

## 5.11 有质量Majorana矢量场对易关系

$$\text{定义5.11.1. } \psi := \begin{bmatrix} \lambda & \xi \\ \eta & \varphi \end{bmatrix} = \begin{bmatrix} \lambda_{A_\zeta B_\zeta} & \xi_{A_\zeta B'_\zeta} \\ \eta_{A'_\zeta B_\zeta} & \varphi_{A'_\zeta B'_\zeta} \end{bmatrix}$$

定理5.11.1.

$$\psi = \gamma_2\psi^+\gamma_2, \psi = \psi^T \Leftrightarrow \begin{bmatrix} \lambda & \xi \\ \eta & \varphi \end{bmatrix} = \begin{bmatrix} \lambda^T & \eta^T \\ \xi^T & \varphi^T \end{bmatrix}, \begin{bmatrix} \lambda^* & \xi^* \\ \eta^* & \varphi^* \end{bmatrix} = \begin{bmatrix} \sigma_y\varphi\sigma_y & -\sigma_y\eta\sigma_y \\ -\sigma_y\xi\sigma_y & \sigma_y\lambda\sigma_y \end{bmatrix}, \begin{bmatrix} \lambda^+ & \eta^+ \\ \xi^+ & \varphi^+ \end{bmatrix} = \begin{bmatrix} \sigma_y\varphi\sigma_y & -\sigma_y\eta\sigma_y \\ -\sigma_y\xi\sigma_y & \sigma_y\lambda\sigma_y \end{bmatrix}$$

$$\text{定理5.11.2. } \lambda^+ = \sigma_y\varphi\sigma_y, \varphi^+ = \sigma_y\lambda\sigma_y, \eta^+ = -\sigma_y\eta\sigma_y, \xi^+ = -\sigma_y\xi\sigma_y, \eta^T = \xi, \lambda^T = \lambda$$

$$\text{定理5.11.3. } \psi = \gamma_2\psi^+\gamma_2, \psi = \psi^T \Leftrightarrow \lambda^+ = \sigma_y\varphi\sigma_y, \eta^+ = -\sigma_y\eta\sigma_y, \eta^T = \xi, \lambda^T = \lambda$$

$$\text{定理5.11.4. } \psi := \begin{bmatrix} \lambda & \eta^T \\ \eta & \sigma_y\lambda^*\sigma_y \end{bmatrix} = \begin{bmatrix} \lambda_{A_\zeta B_\zeta} & \eta_{A_\zeta B'_\zeta}^* \\ \eta_{A'_\zeta B_\zeta} & \lambda_{A'_\zeta B'_\zeta} \end{bmatrix}$$

$$\lambda_{A'_\zeta B'_\zeta} = (\zeta\varepsilon_{A'_\zeta C'_\zeta})(\zeta\varepsilon_{B'_\zeta D'_\zeta})\lambda_{C'_\zeta D'_\zeta}^*, \eta_{A'_\zeta B'_\zeta}^* = \eta_{A_\zeta B_\zeta}^T = \eta_{A_\zeta B'_\zeta}^* := (-\zeta\varepsilon_{A_\zeta C_\zeta})(\zeta\varepsilon_{B'_\zeta D'_\zeta})\eta_{C_\zeta D'_\zeta}^*$$

证明：

$$\begin{cases}
[\lambda_{A_\zeta B_\zeta}(x), \lambda_{A'_\zeta B'_\zeta}(x')] = -\frac{i}{8}(\sigma, i\zeta)_{\{A_\zeta(A'_\zeta)\}_{B_\zeta\}B'_\zeta}^a(\sigma, i\zeta)_{B_\zeta\}^{b} \partial_a \partial_b \Delta(x-x') \\
[\lambda_{A_\zeta B_\zeta}(x), \lambda_{C_\zeta D_\zeta}(x')] = \frac{i}{8} m^2 \varepsilon_{\{A_\zeta(C_\zeta \varepsilon_{B_\zeta})D_\zeta\}} \Delta(x-x') \\
[\lambda_{A'_\zeta B'_\zeta}(x), \lambda_{C'_\zeta D'_\zeta}(x')] = \frac{i}{8} m^2 \varepsilon_{\{A'_\zeta(C'_\zeta \varepsilon_{B'_\zeta})D'_\zeta\}} \Delta(x-x') \\
[\eta_{A'_\zeta B_\zeta}(x), \eta^{+A_\zeta B'_\zeta}(x')] = \frac{i}{8} [(\sigma, -i\zeta)_a^{A'_\zeta A_\zeta}(\sigma, i\zeta)_{B_\zeta B'_\zeta}^b \partial^a \partial_b + m^2 \delta_{B_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{A'_\zeta}] \Delta(x-x') \\
[\eta_{A'_\zeta B_\zeta}(x), \eta^{B'_\zeta A_\zeta}(x')] = \frac{i}{8} \{[(\sigma, -i\zeta)\sigma_y]_a^{A'_\zeta A_\zeta} [(\sigma, -i\zeta)\sigma_y]_b^{B'_\zeta B_\zeta} \partial^a \partial^b + m^2 \varepsilon^{A'_\zeta B'_\zeta \varepsilon_{A_\zeta B_\zeta}}\} \Delta(x-x') \\
[\eta^{+A_\zeta B'_\zeta}(x), \eta^{+B_\zeta A'_\zeta}(x')] = \frac{i}{8} \{[\sigma_y(\sigma, -i\zeta)]_{a A'_\zeta A_\zeta} [\sigma_y(\sigma, -i\zeta)]_{b B'_\zeta B_\zeta} \partial^a \partial^b + m^2 \varepsilon^{A_\zeta B_\zeta \varepsilon_{A'_\zeta B'_\zeta}}\} \Delta(x-x') \\
[\lambda_{A_\zeta B_\zeta}(x), \eta^{+C_\zeta A'_\zeta}(x')] = -\frac{1}{4} m(\sigma, i\zeta)_{\{A_\zeta A'_\zeta\}_{B_\zeta\}}^{C_\zeta} \partial_a \Delta(x-x') \\
[\lambda_{A'_\zeta B'_\zeta}(x), \eta^{C'_\zeta A_\zeta}(x')] = \frac{1}{4} m(\sigma, i\zeta)_{\{A_\zeta A'_\zeta\}_{B'_\zeta\}}^{C'_\zeta} \partial_a \Delta(x-x') \\
[\lambda_{A_\zeta B_\zeta}(x), \eta^{A'_\zeta C_\zeta}(x')] = \frac{i}{4} m [(\sigma, -i\zeta)\sigma_y]_a^{A'_\zeta} \varepsilon_{\{A_\zeta \varepsilon_{B_\zeta}\} C_\zeta} \partial^a \Delta(x-x') \\
[\lambda_{A'_\zeta B'_\zeta}(x), \eta^{+A_\zeta C'_\zeta}(x')] = \frac{i}{4} m [\sigma_y(\sigma, i\zeta)]_a^{A_\zeta} \varepsilon_{\{A'_\zeta \varepsilon_{B'_\zeta}\} C'_\zeta} \partial^a \Delta(x-x')
\end{cases}$$

□

# 第三十二章 有质量引力微子的协变量子化方案

自我评述：对于Bargmann-Wigner方程描述的粒子，一般来说既可以描述带荷的复粒子，也可以描述不带荷的马约拉纳粒子。两种情形的主对易规则形式一致，但其余对易或反对易括号，对带荷的复粒子一般为零；对不带荷的马约拉纳粒子，其余对易或反对易括号由主对易规则和马约拉纳条件自然得到，一般不为零。在本章只讨论复粒子情形，一般也只给出主对易规则，不再专门讨论马约拉纳粒子情形，若要得到马约拉纳粒子情形的量子场论，只需在复粒子情形加上马约拉纳条件即可自然得到。

## 1 有质量引力微子两种等价描述的相互转换

### 1.1 自旋- $\frac{3}{2}$ 粒子的B-W全对称方程和R-S矢量方程两种等价描述 [17, 19, 20]

$$\text{定理1.1.1.} \quad \begin{cases} (\gamma^a \partial_a + m)_{\kappa\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = 0 \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{1}{3!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta\}} \\ im \frac{A_{a\eta_\zeta}}{2} = \frac{1}{4} \text{tr}[\bar{C} \gamma_a(\zeta) \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}] \end{cases} \Leftrightarrow \begin{cases} [\gamma^b(\zeta) \partial_b + m] A_{a[\eta_\zeta]} = 0 \\ \gamma^a(\zeta) A_{a[\eta_\zeta]} = 0 \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x) \frac{A_{a\eta_\zeta}}{2} \end{cases}$$

$$\text{定理1.1.2.} \quad \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(p) (\eta_{aa'} + \frac{p_a p_{a'}}{m^2}) \mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(p) = \frac{1}{2} [(m - i\gamma^a p_a) \gamma^4]_{\{\lambda_\zeta \lambda'_\zeta [(m - i\gamma^b p_b) \gamma^4]_{\mu_\zeta \mu'_\zeta}\}}$$

### 1.2 自旋- $\frac{3}{2}$ 粒子Bargmann-Wigner方程 [17]的平面波解

$$\text{定理1.2.1.} \quad (\gamma^a \partial_a + m)_{\kappa\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) = 0, \quad \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) = \frac{1}{3!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta\}}(\vec{r}, t)$$

$$\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=3/2}^{-3/2} \sqrt{\frac{m^3}{E}} [a(\vec{p}, h) U_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) V_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3 \vec{p}$$

$$\begin{cases} U_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, \frac{3}{2}) = u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) \\ U_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, \frac{1}{2}) \\ = \frac{1}{\sqrt{3}} [u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) u_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) u_{\eta_\zeta}(\vec{p}, -\frac{1}{2})] \\ U_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, -\frac{1}{2}) \\ = \frac{1}{\sqrt{3}} [u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) + u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) + u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) u_{\eta_\zeta}(\vec{p}, \frac{1}{2})] \\ U_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, -\frac{3}{2}) = u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) u_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) u_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) \\ V_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, \frac{3}{2}) = v_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) v_{\mu_\zeta}(\vec{p}, \frac{1}{2}) v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) \\ V_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, \frac{1}{2}) \\ = \frac{1}{\sqrt{3}} [v_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) v_{\mu_\zeta}(\vec{p}, \frac{1}{2}) v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + v_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) v_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + v_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) v_{\mu_\zeta}(\vec{p}, \frac{1}{2}) v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})] \\ V_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, -\frac{1}{2}) \\ = \frac{1}{\sqrt{3}} [v_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) v_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) + v_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) v_{\mu_\zeta}(\vec{p}, \frac{1}{2}) v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) + v_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) v_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) v_{\eta_\zeta}(\vec{p}, \frac{1}{2})] \\ V_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, -\frac{3}{2}) = v_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) v_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) \\ a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m^3}{E^5}} U^{+\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, h) \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r}-Et)} d^3 \vec{r} \\ b^+(\vec{p}, s) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m^3}{E^5}} V^{+\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, h) \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) e^{i(\vec{p}\cdot\vec{r}-Et)} d^3 \vec{r} \end{cases}$$

$$\begin{aligned} \text{定理1.2.2.} & [\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \eta'_\zeta}^{+a'}(x')] \\ &= \frac{i}{4} \frac{1}{(3!)^2} [(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda_\zeta \lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^4]_{\mu_\zeta \mu'_\zeta} [(m - \gamma^c \partial_c) \gamma^4]_{\eta_\zeta \eta'_\zeta}\}} \Delta(x - x') \\ &= \frac{i}{8} \frac{1}{(3!)^2} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x) \mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x') [(m - \gamma^c \partial_c) \gamma^4]_{\eta_\zeta \eta'_\zeta} (\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x - x') \end{aligned}$$

定义1.2.1.

$$\begin{cases} \Lambda_{+\lambda_s\mu_s\eta_s\lambda'_s\mu'_s\eta'_s}(\vec{p}, \frac{3}{2}) := \sum_{h=1}^{-1} U_{\lambda_s\mu_s\eta_s}(\vec{p}, h) U_{\lambda'_s\mu'_s\eta'_s}^+(\vec{p}, h) \\ \Lambda_{-\lambda_s\mu_s\eta_s\lambda'_s\mu'_s\eta'_s}(\vec{p}, \frac{3}{2}) := \sum_{h=1}^{-1} V_{\lambda_s\mu_s\eta_s}(\vec{p}, h) V_{\lambda'_s\mu'_s\eta'_s}^+(\vec{p}, h) \end{cases}$$

定理1.2.3.

$$\begin{cases} \Lambda_{+\lambda_s\mu_s\eta_s\lambda'_s\mu'_s\eta'_s}(\vec{p}, \frac{3}{2}) \\ = \frac{1}{8m^2} \frac{1}{(3!)^2} \mathbb{X}_{\{\lambda_s\mu_s\}}^a(p) \Lambda_{maa'}(\vec{p}, 1) \mathbb{X}_{(\lambda'_s\mu'_s)}^{+a'}(p) \Lambda_{+\eta_s\eta'_s}(\vec{p}, \frac{1}{2}) = \frac{1}{(3!)^2} \Lambda_{+\{\lambda_s(\lambda'_s(\vec{p}, \frac{1}{2})\Lambda_{+\mu_s\mu'_s}(\vec{p}, \frac{1}{2})\Lambda_{+\eta_s\eta'_s}\}(\vec{p}, \frac{1}{2})} \\ \Lambda_{-\lambda_s\mu_s\eta_s\lambda'_s\mu'_s\eta'_s}(\vec{p}, \frac{3}{2}) \\ = \frac{1}{8m^2} \frac{1}{(3!)^2} \mathbb{X}_{\{\lambda_s\mu_s\}}^a(-p) \Lambda_{maa'}(\vec{p}, 1) \mathbb{X}_{(\lambda'_s\mu'_s)}^{+a'}(-p) \Lambda_{-\eta_s\eta'_s}(\vec{p}, \frac{1}{2}) = \frac{1}{(3!)^2} \Lambda_{-\{\lambda_s(\lambda'_s(\vec{p}, \frac{1}{2})\Lambda_{-\mu_s\mu'_s}(\vec{p}, \frac{1}{2})\Lambda_{-\eta_s\eta'_s}\}(\vec{p}, \frac{1}{2})} \end{cases}$$

定理1.2.4.

$$\begin{cases} \Lambda_{+\lambda_s\mu_s\eta_s\lambda'_s\mu'_s\eta'_s}(\vec{p}, \frac{3}{2}) = \frac{1}{16m^3} \frac{1}{(3!)^2} \mathbb{X}_{\{\lambda_s\mu_s\}}^a(p) (\eta_{aa'} + \frac{p_a p_{a'}}{m^2}) \mathbb{X}_{(\lambda'_s\mu'_s)}^{+a'}(p) [(m - i\gamma^b p_b) \gamma^4]_{\eta_s\eta'_s} \\ = \frac{1}{8m^3} \frac{1}{(3!)^2} [(m - i\gamma^a p_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m - i\gamma^b p_b) \gamma^4]_{\mu_s\mu'_s} [(m - i\gamma^b p_b) \gamma^4]_{\eta_s\eta'_s}\}} \\ \Lambda_{-\lambda_s\mu_s\eta_s\lambda'_s\mu'_s\eta'_s}(\vec{p}, \frac{3}{2}) = -\frac{1}{16m^3} \frac{1}{(3!)^2} \mathbb{X}_{\{\lambda_s\mu_s\}}^a(-p) (\eta_{aa'} + \frac{p_a p_{a'}}{m^2}) \mathbb{X}_{(\lambda'_s\mu'_s)}^{+a'}(-p) [(m + i\gamma^b p_b) \gamma^4]_{\eta_s\eta'_s} \\ = -\frac{1}{8m^3} \frac{1}{(3!)^2} [(m + i\gamma^a p_a) \gamma^4]_{\{\lambda_s(\lambda'_s[(m + i\gamma^b p_b) \gamma^4]_{\mu_s\mu'_s} [(m + i\gamma^b p_b) \gamma^4]_{\eta_s\eta'_s}\}} \end{cases}$$

↓

### 1.3 推导到自旋- $\frac{3}{2}$ 粒子R-S方程的平面波解

#### 1.3.1 推导到自旋- $\frac{3}{2}$ 粒子R-S方程<sup>[17]</sup>的平面波解

定理1.3.1.  $[\gamma^b(\zeta)\partial_b + m]A_{a[\eta_s]} = 0, \gamma^a(\zeta)A_{a[\eta_s]} = 0, A_a = \frac{1}{2im}(\bar{C}\gamma_a)^{\lambda_s\mu_s}\psi_{\lambda_s\mu_s\eta_s}$

$$A_{a\eta_s}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \sqrt{\frac{m}{2E}} [a(\vec{p}, h)\varepsilon_a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)\tilde{\varepsilon}_a(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\begin{cases} \varepsilon_{a\eta_s}(\vec{p}, \frac{3}{2}) = \frac{1}{i\sqrt{2}} u^T(\vec{p}, \frac{1}{2}) \bar{C}\gamma_a u(\vec{p}, \frac{1}{2}) u_{\eta_s}(\vec{p}, \frac{1}{2}) \\ \varepsilon_{a\eta_s}(\vec{p}, \frac{1}{2}) \\ = \frac{1}{i\sqrt{6}} [u^T(\vec{p}, -\frac{1}{2}) \bar{C}\gamma_a u(\vec{p}, \frac{1}{2}) u_{\eta_s}(\vec{p}, \frac{1}{2}) + u^T(\vec{p}, \frac{1}{2}) \bar{C}\gamma_a u(\vec{p}, -\frac{1}{2}) u_{\eta_s}(\vec{p}, \frac{1}{2}) + u^T(\vec{p}, \frac{1}{2}) \bar{C}\gamma_a u(\vec{p}, \frac{1}{2}) u_{\eta_s}(\vec{p}, -\frac{1}{2})] \\ \varepsilon_{a\eta_s}(\vec{p}, -\frac{1}{2}) \\ = \frac{1}{i\sqrt{6}} [u^T(\vec{p}, \frac{1}{2}) \bar{C}\gamma_a u(\vec{p}, -\frac{1}{2}) u_{\eta_s}(\vec{p}, -\frac{1}{2}) + u^T(\vec{p}, -\frac{1}{2}) \bar{C}\gamma_a u(\vec{p}, \frac{1}{2}) u_{\eta_s}(\vec{p}, -\frac{1}{2}) + u^T(\vec{p}, -\frac{1}{2}) \bar{C}\gamma_a u(\vec{p}, -\frac{1}{2}) u_{\eta_s}(\vec{p}, \frac{1}{2})] \\ \varepsilon_{a\eta_s}(\vec{p}, -\frac{3}{2}) = \frac{1}{i\sqrt{2}} u^T(\vec{p}, -\frac{1}{2}) \bar{C}\gamma_a u(\vec{p}, -\frac{1}{2}) u_{\eta_s}(\vec{p}, -\frac{1}{2}) \\ \tilde{\varepsilon}_{a\eta_s}(\vec{p}, \frac{3}{2}) = \frac{1}{i\sqrt{2}} v^T(\vec{p}, \frac{1}{2}) \bar{C}\gamma_a v(\vec{p}, \frac{1}{2}) v_{\eta_s}(\vec{p}, \frac{1}{2}) \\ \tilde{\varepsilon}_{a\eta_s}(\vec{p}, \frac{1}{2}) \\ = \frac{1}{i\sqrt{6}} [v^T(\vec{p}, -\frac{1}{2}) \bar{C}\gamma_a v(\vec{p}, \frac{1}{2}) v_{\eta_s}(\vec{p}, \frac{1}{2}) + v^T(\vec{p}, \frac{1}{2}) \bar{C}\gamma_a v(\vec{p}, -\frac{1}{2}) v_{\eta_s}(\vec{p}, \frac{1}{2}) + v^T(\vec{p}, \frac{1}{2}) \bar{C}\gamma_a v(\vec{p}, \frac{1}{2}) v_{\eta_s}(\vec{p}, -\frac{1}{2})] \\ \tilde{\varepsilon}_{a\eta_s}(\vec{p}, -\frac{1}{2}) \\ = \frac{1}{i\sqrt{6}} [v^T(\vec{p}, \frac{1}{2}) \bar{C}\gamma_a v(\vec{p}, -\frac{1}{2}) v_{\eta_s}(\vec{p}, -\frac{1}{2}) + v^T(\vec{p}, -\frac{1}{2}) \bar{C}\gamma_a v(\vec{p}, \frac{1}{2}) v_{\eta_s}(\vec{p}, -\frac{1}{2}) + v^T(\vec{p}, -\frac{1}{2}) \bar{C}\gamma_a v(\vec{p}, -\frac{1}{2}) v_{\eta_s}(\vec{p}, \frac{1}{2})] \\ \tilde{\varepsilon}_{a\eta_s}(\vec{p}, -\frac{3}{2}) = \frac{1}{i\sqrt{2}} v^T(\vec{p}, -\frac{1}{2}) \bar{C}\gamma_a v(\vec{p}, -\frac{1}{2}) v_{\eta_s}(\vec{p}, -\frac{1}{2}) \end{cases}$$

引理1.3.1.  $\frac{1}{2\sqrt{2}m} U^{\lambda_s\mu_s\eta_s}(\hat{p}, h) \mathbb{X}_{\lambda_s\mu_s}^{+a}(p) = \frac{1}{i\sqrt{2}} U^{\lambda_s\mu_s\eta_s}(\hat{p}, h) (\bar{C}\gamma_a)_{\lambda_s\mu_s} = \varepsilon_{a\eta_s}(\vec{p}, h)$

推论1.3.1.  $[\gamma^b(\zeta)\partial_b + m]A_{a[\eta_s]} = 0, \gamma^a(\zeta)A_{a[\eta_s]} = 0, A_a = \frac{1}{2im}(\bar{C}\gamma_a)^{\lambda_s\mu_s}\psi_{\lambda_s\mu_s\eta_s}$

$$A_{a\eta_s}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \sqrt{\frac{m}{E}} [a(\vec{p}, h)\varepsilon_a(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)\tilde{\varepsilon}_a(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\begin{cases} \varepsilon_{a\eta_s}(\vec{p}, \frac{3}{2}) = \varepsilon_a(\vec{p}, 1) u_{\eta_s}(\vec{p}, \frac{1}{2}) \\ \varepsilon_{a\eta_s}(\vec{p}, \frac{1}{2}) = \frac{1}{\sqrt{3}} [\sqrt{2}\varepsilon_a(\vec{p}, 0) u_{\eta_s}(\vec{p}, \frac{1}{2}) + \varepsilon_a(\vec{p}, 1) u_{\eta_s}(\vec{p}, -\frac{1}{2})] \\ \varepsilon_{a\eta_s}(\vec{p}, -\frac{1}{2}) = \frac{1}{\sqrt{3}} [\sqrt{2}\varepsilon_a(\vec{p}, 0) u_{\eta_s}(\vec{p}, -\frac{1}{2}) + \varepsilon_a(\vec{p}, -1) u_{\eta_s}(\vec{p}, \frac{1}{2})] \\ \varepsilon_{a\eta_s}(\vec{p}, -\frac{3}{2}) = \varepsilon_a(\vec{p}, -1) u_{\eta_s}(\vec{p}, -\frac{1}{2}) \end{cases}$$



$$\begin{cases} \tilde{\varepsilon}_{a\eta_\zeta}(\vec{p}, \frac{3}{2}) = -\varepsilon_a(\vec{p}, 1)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) \\ \tilde{\varepsilon}_{a\eta_\zeta}(\vec{p}, \frac{1}{2}) = -\frac{1}{\sqrt{3}}[\sqrt{2}\varepsilon_a(\vec{p}, 0)v_{\eta_\zeta}(\vec{p}, \frac{1}{2}) + \varepsilon_a(\vec{p}, 1)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2})] \\ \tilde{\varepsilon}_{a\eta_\zeta}(\vec{p}, -\frac{1}{2}) = -\frac{1}{\sqrt{3}}[\sqrt{2}\varepsilon_a(\vec{p}, 0)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) + \varepsilon_a(\vec{p}, -1)v_{\eta_\zeta}(\vec{p}, \frac{1}{2})] \\ \tilde{\varepsilon}_{a\eta_\zeta}(\vec{p}, -\frac{3}{2}) = -\varepsilon_a(\vec{p}, -1)v_{\eta_\zeta}(\vec{p}, -\frac{1}{2}) \end{cases}$$

### 1.3.2 经典约定下自旋- $\frac{3}{2}$ 粒子R-S方程反对易关系的推导

**定理1.3.2.**  $\{A_{a_1 a_2 \dots a_n \tau_\zeta}(x), \bar{A}_{b_1 b_2 \dots b_n \tau'_\zeta}(x')\} = i\hat{P}_{a_1 \dots a_n \tau_\zeta b_1 \dots b_n \tau'_\zeta}(n + \frac{1}{2})\Delta(x - x')$

**引理1.3.2.**

$$\begin{cases} (m - \kappa\gamma^b\partial_b)(\gamma_a + \kappa\frac{\partial_a}{m}) = (\gamma_a - \kappa\frac{\partial_a}{m})(m + \kappa\gamma^b\partial_b) \\ (m - \gamma^b\partial_b)(\gamma_a + \frac{\partial_a}{m}) = (\gamma_a - \frac{\partial_a}{m})(m + \gamma^b\partial_b) \\ (m + \gamma^b\partial_b)(\gamma_a - \frac{\partial_a}{m}) = (\gamma_a + \frac{\partial_a}{m})(m - \gamma^b\partial_b) \end{cases}$$

**证明:**  $(m - \kappa\gamma^b\partial_b)(\gamma_a + \kappa\frac{\partial_a}{m})$

$$\begin{aligned} &= m(\gamma_a + \kappa\frac{\partial_a}{m}) - \kappa\gamma^b\gamma_a\partial_b - \frac{\kappa\kappa\gamma^b\partial_b\partial_a}{m} \\ &= m(\gamma_a + \kappa\frac{\partial_a}{m}) - \kappa\{\gamma^b, \gamma_a\}\partial_b + \kappa\gamma_a\gamma^b\partial_b - \kappa\kappa\frac{\partial_a}{m}\gamma^b\partial_b \\ &= m(\gamma_a + \kappa\frac{\partial_a}{m}) - 2\kappa\delta_a^b\partial_b + \kappa(\gamma_a - \kappa\frac{\partial_a}{m})\gamma^b\partial_b \\ &= m(\gamma_a - \kappa\frac{\partial_a}{m}) + (\gamma_a - \kappa\frac{\partial_a}{m})\kappa\gamma^b\partial_b \\ &= (\gamma_a - \kappa\frac{\partial_a}{m})(m + \kappa\gamma^b\partial_b) \end{aligned}$$

□

**定理1.3.3.**

$$\begin{aligned} \left\{ \hat{P}_{a_1\tau_\zeta b_1\tau'_\zeta}(\frac{3}{2}) = \frac{2}{5}\hat{P}_{aa_1bb_1}(2)[(m - \gamma^c\partial_c)\gamma^a\gamma^b\gamma^4]_{\tau_\zeta\tau'_\zeta} \right. \\ \left. \hat{P}_{aa_1bb_1}(2) = \frac{1}{8}\{[\delta_{\{a\{b - \frac{\partial_{\{a}\partial_{\{b}}}{m^2}}\}}[\delta_{a_1\}b_1] - \frac{\partial_{a_1}\partial_{b_1}}{m^2}] - \frac{1}{3}[\delta_{\{aa_1} - \frac{\partial_{\{a}\partial_{a_1}}}{m^2}}\}}[\delta_{bb_1} - \frac{\partial_{\{b}\partial_{b_1}}}{m^2}]\}\Delta(x - x') \right. \\ \Rightarrow \\ \hat{P}_{a\tau_\zeta b\tau'_\zeta}(\frac{3}{2}) = \frac{1}{2}\{[(\delta_{a_1b_1} - \frac{\partial_{a_1}\partial_{b_1}}{m^2}) - \frac{1}{3}(\gamma_{a_1} - \frac{\partial_{a_1}}{m})(\gamma_{b_1} + \frac{\partial_{b_1}}{m})](m - \gamma^c\partial_c)\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \end{aligned}$$

**证明:**  $\hat{P}_{a_1\tau_\zeta b_1\tau'_\zeta}(\frac{3}{2}) = \frac{2}{5}\hat{P}_{aa_1bb_1}(2)[(m - \gamma^c\partial_c)\gamma^a\gamma^b\gamma^4]_{\tau_\zeta\tau'_\zeta}$

$$\begin{aligned} &= \frac{2}{5}\frac{1}{8}\{[\delta_{\{a\{b - \frac{\partial_{\{a}\partial_{\{b}}}{m^2}}\}}[\delta_{a_1\}b_1] - \frac{\partial_{a_1}\partial_{b_1}}{m^2}] - \frac{1}{3}[\delta_{\{aa_1} - \frac{\partial_{\{a}\partial_{a_1}}}{m^2}}\}}[\delta_{bb_1} - \frac{\partial_{\{b}\partial_{b_1}}}{m^2}]\}\{(m - \gamma^c\partial_c)\gamma^a\gamma^b\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ &= \frac{1}{10}\{[\delta_{ab} - \frac{\partial_a\partial_b}{m^2}][\delta_{a_1b_1} - \frac{\partial_{a_1}\partial_{b_1}}{m^2}] + [\delta_{ab_1} - \frac{\partial_a\partial_{b_1}}{m^2}][\delta_{a_1b} - \frac{\partial_{a_1}\partial_b}{m^2}] - \frac{2}{3}[\delta_{aa_1} - \frac{\partial_a\partial_{a_1}}{m^2}][\delta_{bb_1} - \frac{\partial_b\partial_{b_1}}{m^2}]\} \\ &\{ (m - \gamma^c\partial_c)\gamma^a\gamma^b\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ &= \frac{1}{10}\{(m - \gamma^c\partial_c)[3(\delta_{a_1b_1} - \frac{\partial_{a_1}\partial_{b_1}}{m^2}) + (\gamma_{b_1} + \frac{\partial_{b_1}}{m})(\gamma_{a_1} - \frac{\gamma^c\partial_c\partial_{a_1}}{m^2}) - \frac{2}{3}(\gamma_{a_1} + \frac{\partial_{a_1}}{m})(\gamma_{b_1} - \frac{\gamma^c\partial_c\partial_{b_1}}{m^2})]\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ &= \frac{1}{10}\{[3(\delta_{a_1b_1} - \frac{\partial_{a_1}\partial_{b_1}}{m^2})(m - \gamma^c\partial_c) \\ &+ (\gamma_{b_1} - \frac{\partial_{b_1}}{m})(m + \gamma^c\partial_c)(\gamma_{a_1} - \frac{\partial_{a_1}}{m}) - \frac{2}{3}(\gamma_{a_1} - \frac{\partial_{a_1}}{m})(m + \gamma^c\partial_c)(\gamma_{b_1} - \frac{\partial_{b_1}}{m})]\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ &= \frac{1}{10}\{[3(\delta_{a_1b_1} - \frac{\partial_{a_1}\partial_{b_1}}{m^2})(m - \gamma^c\partial_c) \\ &+ (\gamma_{b_1} - \frac{\partial_{b_1}}{m})(\gamma_{a_1} + \frac{\partial_{a_1}}{m})(m - \gamma^c\partial_c) - \frac{2}{3}(\gamma_{a_1} - \frac{\partial_{a_1}}{m})(\gamma_{b_1} + \frac{\partial_{b_1}}{m})(m - \gamma^c\partial_c)]\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ &= \frac{1}{10}\{[3(\delta_{a_1b_1} - \frac{\partial_{a_1}\partial_{b_1}}{m^2}) + (\gamma_{b_1} - \frac{\partial_{b_1}}{m})(\gamma_{a_1} + \frac{\partial_{a_1}}{m}) - \frac{2}{3}(\gamma_{a_1} - \frac{\partial_{a_1}}{m})(\gamma_{b_1} + \frac{\partial_{b_1}}{m})](m - \gamma^c\partial_c)\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ &= \frac{1}{10}\{[3\delta_{a_1b_1} - \frac{10}{3}\frac{\partial_{a_1}\partial_{b_1}}{m^2} + \{\gamma_{b_1}, \gamma_{a_1}\} - \frac{5}{3}\gamma_{a_1}\gamma_{b_1} - \frac{5}{3}(\gamma_{a_1}\frac{\partial_{b_1}}{m} - \gamma_{b_1}\frac{\partial_{a_1}}{m})](m - \gamma^c\partial_c)\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ &= \frac{1}{2}\{[\delta_{a_1b_1} - \frac{2}{3}\frac{\partial_{a_1}\partial_{b_1}}{m^2} - \frac{1}{3}\gamma_{a_1}\gamma_{b_1} - \frac{1}{3}(\gamma_{a_1}\frac{\partial_{b_1}}{m} - \gamma_{b_1}\frac{\partial_{a_1}}{m})](m - \gamma^c\partial_c)\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ &= \frac{1}{2}\{[(\delta_{a_1b_1} - \frac{\partial_{a_1}\partial_{b_1}}{m^2}) - \frac{1}{3}(\gamma_{a_1} - \frac{\partial_{a_1}}{m})(\gamma_{b_1} + \frac{\partial_{b_1}}{m})](m - \gamma^c\partial_c)\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ &= \frac{1}{2}\{[(\delta_{a_1b_1} - \frac{\partial_{a_1}\partial_{b_1}}{m^2})](m - \gamma^c\partial_c)\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') - \frac{1}{3}[(\gamma_{a_1} - \frac{\partial_{a_1}}{m})(m + \gamma^c\partial_c)(\gamma_{b_1} - \frac{\partial_{b_1}}{m})\gamma^4]_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \end{aligned}$$

□

**推论1.3.2.**  $\hat{P}_{a\tau_\zeta b\tau'_\zeta}(\frac{3}{2}) = \frac{1}{2}\{[(\eta_{ab'} - \frac{\partial_a\partial_{b'}}{m^2}) - \frac{1}{3}(\gamma_a - \frac{\partial_a}{m})(\gamma_b\eta_{b'}^b + \frac{\partial_{b'}}{m})](m - \gamma^c\partial_c)\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x')$

**推论1.3.3.**

$$\begin{cases} \{A_{a\tau_\zeta}(x), \bar{A}_{b\tau'_\zeta}(x')\} = \frac{i}{2}\{[(\delta_{ab} - \frac{\partial_a\partial_b}{m^2}) - \frac{1}{3}(\gamma_a - \frac{\partial_a}{m})(\gamma_b + \frac{\partial_b}{m})](m - \gamma^c\partial_c)\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ \{A_{a\tau_\zeta}(x), A_{a'\tau'_\zeta}^+(x')\} = \frac{i}{2}\{[(\eta_{aa'} - \frac{\partial_a\partial_{a'}}{m^2}) - \frac{1}{3}(\gamma_a - \frac{\partial_a}{m})(\gamma_b\eta_{a'}^b + \frac{\partial_{a'}}{m})](m - \gamma^c\partial_c)\gamma^4\}_{\tau_\zeta\tau'_\zeta}\Delta(x - x') \\ [A_{ab}(x), A_{a'b'}^+(x')] = \frac{i}{8}\{[\eta_{\{a\{a' - \frac{\partial_{\{a}\partial_{\{a'}}}{m^2}}\}}[\eta_{b\}b'] - \frac{\partial_{b_1}\partial_{b'_1}}{m^2}] - \frac{1}{3}[\delta_{\{ab\} - \frac{\partial_{\{a}\partial_{\{b}}}{m^2}}\}}[\delta_{\{a'b'\} - \frac{\partial_{\{a'}\partial_{\{b'}}}{m^2}}\}]\}\Delta(x - x') \end{cases}$$

### 1.3.3 准投影算子间相互关系的对比

定理1.3.4.

$$\begin{cases} \Lambda_{+ma\tau_\zeta a'\tau'_\zeta}(\vec{p}, \frac{3}{2}) := \sum_{h=3/2}^{-3/2} \varepsilon_{a\tau_\zeta}(\vec{p}, h) \varepsilon_{a'\tau'_\zeta}^+(\vec{p}, h) = \frac{2}{5} \sum_{h=2}^{-2} \varepsilon_{ab}(\vec{p}, h) \varepsilon_{a'b'}^+(\vec{p}, h) \gamma^b \Lambda_-(\vec{p}, \frac{1}{2}) \gamma^{b'} \\ \Lambda_{-ma\tau_\zeta a'\tau'_\zeta}(\vec{p}, \frac{3}{2}) := \sum_{h=3/2}^{-3/2} \tilde{\varepsilon}_{a\tau_\zeta}(\vec{p}, h) \tilde{\varepsilon}_{a'\tau'_\zeta}^+(\vec{p}, h) = \frac{2}{5} \sum_{h=2}^{-2} \varepsilon_{ab}(\vec{p}, h) \varepsilon_{a'b'}^+(\vec{p}, h) \gamma^b \Lambda_+(\vec{p}, \frac{1}{2}) \gamma^{b'} \\ \Lambda_{\pm\tau_\zeta\tau'_\zeta}(\vec{p}, \frac{1}{2}) = \frac{1}{2} \Lambda_{\pm ma\tau_\zeta a'\tau'_\zeta}(\vec{p}, \frac{3}{2}) \eta^{aa'}, \Lambda_{maa'}(\vec{p}, 1) = \frac{3}{4} (\frac{m}{E})^2 \Lambda_{\pm ma\tau_\zeta a'\tau'_\zeta}(\vec{p}, \frac{3}{2}) \Lambda_{\pm}^{\tau'_\zeta\tau_\zeta} \end{cases}$$

定理1.3.5.

$$\begin{cases} \sum_{h=1}^{-1} \varepsilon_a(\vec{p}, h) \varepsilon_{a'}^+(\vec{p}, h) = \eta_{aa'} + \frac{p_a p_{a'}}{m^2} \\ \sum_{h=3/2}^{-3/2} \varepsilon_{a\tau_\zeta}(\vec{p}, h) \varepsilon_{a'\tau'_\zeta}^+(\vec{p}, h) = \{[(\eta_{aa'} + \frac{p_a p_{a'}}{m^2}) - \frac{1}{3}(\gamma_a - \frac{ip_a}{m})(\gamma_b \eta_{a'}^b + \frac{ip_{a'}}{m})] \frac{(m - i\gamma^c p_c) \gamma^4}{2m}\}_{\tau_\zeta\tau'_\zeta} \Delta(x - x') \\ \sum_{h=2}^{-2} \varepsilon_{ab}(\vec{p}, h) \varepsilon_{a'b'}^+(\vec{p}, h) = \frac{1}{4} \{[\eta_{\{a\{a'}} + \frac{p_{\{a} p_{\{a'}}}{m^2}] [\eta_{b\} b'} + \frac{p_{b\} p_{b'}}{m^2}] - \frac{1}{3} [\delta_{\{ab\}} + \frac{p_{\{a} p_{b\}}}{m^2}] [\delta_{\{a'b'\}} + \frac{p_{\{a'} p_{b'}}}{m^2}]\} \end{cases}$$

↓

### 1.4 再回到自旋- $\frac{3}{2}$ 粒子Bargmann-Wigner方程<sup>[17]</sup>的平面波解

定理1.4.1.  $(\gamma^a \partial_a + m)_{\kappa\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) = [im\gamma^a(\zeta)C - 2S^{ab}(e, \zeta)C\partial_b] \frac{A_{a\eta_\zeta}(\vec{r}, t)}{2}$

$$\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=3/2}^{-3/2} \sqrt{\frac{m^3}{E}} [a(\vec{p}, h) U_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r} - Et)} + b^+(\vec{p}, h) V_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r} - Et)}] d^3\vec{p}$$

$$U_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, h) = \frac{1}{2\sqrt{2m}} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(p) \varepsilon_{a\eta_\zeta}(\vec{p}, h), V_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, h) = \frac{1}{2\sqrt{2m}} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(-p) \tilde{\varepsilon}_{a\eta_\zeta}(\vec{p}, h)$$

$$\begin{cases} a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m^3}{E^5}} U^{+\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, h) \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r} - Et)} d^3\vec{r} \\ b^+(\vec{p}, s) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m^3}{E^5}} V^{+\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{p}, h) \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) e^{i(\vec{p}\cdot\vec{r} - Et)} d^3\vec{r} \end{cases}$$

### 1.5 自旋- $\frac{3}{2}$ 粒子场 $F_{ab\tau_\zeta}, \psi_{\alpha_\kappa\tau_\zeta}$ 的反对易关系

定理1.5.1.

$$\begin{cases} \{A_{a\tau_\zeta}(x), A_{a'\tau'_\zeta}^+(x')\} = \frac{i}{2} \{[(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) - \frac{1}{3}(\gamma_a - \frac{\partial_a}{m})(\gamma_b \eta_{a'}^b + \frac{\partial_{a'}}{m})] (m - \gamma^c \partial_c) \gamma^4\}_{\tau_\zeta\tau'_\zeta} \Delta(x - x') \\ F_{ab\tau_\zeta} = \partial_a A_{b\tau_\zeta} - \partial_b A_{a\tau_\zeta} \\ \Rightarrow \{F_{ab\tau_\zeta}(x), F_{a'b'\tau'_\zeta}^+(x')\} = -\frac{i}{2} [(\eta_{[a\langle a'} - \frac{1}{3} \gamma_{[a} \eta_{\langle a'}^d \gamma_d) \gamma^c \gamma^4]_{\tau_\zeta\tau'_\zeta} \partial_b] \partial_{b'}^+ \partial_c \Delta(x - x') \end{cases}$$

定理1.5.2.

$$\begin{cases} \{F_{ab\tau_\zeta}(x), F_{a'b'\tau'_\zeta}^+(x')\} = -\frac{i}{2} [(\eta_{[a\langle a'} - \frac{1}{3} \gamma_{[a} \eta_{\langle a'}^d \gamma_d) \gamma^c \gamma^4]_{\tau_\zeta\tau'_\zeta} \partial_b] \partial_{b'}^+ \partial_c \Delta(x - x') \\ \psi_{\alpha_\kappa\tau_\zeta} := -\frac{1}{2\sqrt{2}} \sigma_{\kappa\alpha_\kappa}^{ab} F_{ab\tau_\zeta} \\ \Rightarrow \{\psi_{\alpha_\kappa\tau_\zeta}(x), \psi_{\alpha'_\kappa\tau'_\zeta}^+(x')\} = \frac{i}{2} [(\sigma_{\alpha_\kappa\alpha'_\kappa}^{ab} + \frac{1}{6} \sigma_{\kappa\alpha_\kappa}^{a'a'} \gamma_a \gamma_{b'} \sigma_{-\kappa\alpha_\kappa}^{b'b}) \gamma^c \gamma^4]_{\tau_\zeta\tau'_\zeta} \partial_a \partial_b \partial_c \Delta(x - x') \end{cases}$$

### 1.6 有质量引力微子场能量动量算符的提取

定理1.6.1.  $P_u(\frac{3}{2}) = \int \psi^{+\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) d^3\vec{r}$

定理1.6.2.  $P_u(\frac{3}{2}) = \int [\frac{1}{2} F^{+ab\eta_\zeta}(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} F_{ab\eta_\zeta}(\vec{r}, t) + m^2 A^{+a\eta_\zeta}(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} A_{a\eta_\zeta}(\vec{r}, t)] d^3\vec{r}$

$$\begin{aligned} \text{证明: } P_u(\frac{3}{2}) &= \int \psi^{+\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(\vec{r}, t) d^3\vec{r} \\ &= \int \{ \bar{C}[-im\gamma^{a'}(\zeta) - 2S^{a'b'}(e, \zeta)\partial_{b'}^+] \}_{\lambda_\zeta \mu_\zeta} \frac{A_{a'\eta_\zeta}(\vec{r}, t)}{2} \frac{-i\partial_u}{m^2 - \nabla^2} [im\gamma^a(\zeta)C - 2S^{ab}(e, \zeta)C\partial_b]_{\lambda_\zeta \mu_\zeta} \frac{A_{a\eta_\zeta}(\vec{r}, t)}{2} d^3\vec{r} \\ &= \frac{1}{4} \int tr \{ \bar{C}[-im\gamma^{a'}(\zeta) - 2S^{a'b'}(e, \zeta)\partial_{b'}^+] A_{a'\eta_\zeta}^+(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} [im\gamma^a(\zeta) - 2S^{ab}(e, \zeta)\partial_b] C A_{a\eta_\zeta}(\vec{r}, t) \} d^3\vec{r} \\ &= \frac{1}{4} \int tr \{ [-im\gamma^{a'}(\zeta) - 2S^{a'b'}(e, \zeta)\partial_{b'}^+] A_{a'\eta_\zeta}^+(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} [im\gamma^a(\zeta) - 2S^{ab}(e, \zeta)\partial_b] A_{a\eta_\zeta}(\vec{r}, t) \} d^3\vec{r} \\ &= \frac{1}{4} \int tr \{ [-im\gamma^{a'}(\zeta) - 2S^{a'b'}(e, \zeta)\partial_{b'}^+] A_{a'\eta_\zeta}^+(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} [im\gamma^a(\zeta) - 2S^{ab}(e, \zeta)\partial_b] A_{a\eta_\zeta}(\vec{r}, t) \} d^3\vec{r} \\ &= \int m^2 A^{+a\eta_\zeta}(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} A_{a\eta_\zeta}(\vec{r}, t) d^3\vec{r} + \int S^{a'b'ab} \partial_{b'}^+ A_{a'\eta_\zeta}^+(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} \partial_b A_{a\eta_\zeta}(\vec{r}, t) d^3\vec{r} \\ &= \int m^2 A^{+a\eta_\zeta}(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} A_{a\eta_\zeta}(\vec{r}, t) d^3\vec{r} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \int S^{a'b'ab} [\partial_{a'}^+ A_{b'}^{+\eta_\zeta}(\vec{r}, t) - \partial_{b'}^+ A_{a'}^{+\eta_\zeta}(\vec{r}, t)] \frac{-i\partial_u}{m^2 - \nabla^2} [\partial_a A_{b\eta_\zeta}(\vec{r}, t) - \partial_b A_{a\eta_\zeta}(\vec{r}, t)] d^3\vec{r} \\
& = \int m^2 A^{+a\eta_\zeta}(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} A_{a\eta_\zeta}(\vec{r}, t) d^3\vec{r} + \frac{1}{4} \int (\delta^{a'a} \delta^{b'b} - \delta^{a'b} \delta^{b'a}) F_{a'b'}^{+\eta_\zeta}(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} F_{ab\eta_\zeta}(\vec{r}, t) d^3\vec{r} \\
& = \int [\frac{1}{2} F^{+ab\eta_\zeta}(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} F_{ab\eta_\zeta}(\vec{r}, t) + m^2 A^{+a\eta_\zeta}(\vec{r}, t) \frac{-i\partial_u}{m^2 - \nabla^2} A_{a\eta_\zeta}(\vec{r}, t)] d^3\vec{r}
\end{aligned}$$

□

# 第三十三章 有质量引力子的协变量子化方案

自我评述：对于Bargmann-Wigner方程描述的粒子，一般来说既可以描述带荷的复粒子，也可以描述不带荷的马约拉纳粒子。两种情形的主对易规则形式一致，但其余对易或反对易括号，对带荷的复粒子一般为零；对不带荷的马约拉纳粒子，其余对易或反对易括号由主对易规则和马约拉纳条件自然得到，一般不为零。在本章只讨论复粒子情形，一般也只给出主对易规则，不再专门讨论马约拉纳粒子情形，若要得到马约拉纳粒子情形的量子场论，只需在复粒子情形加上马约拉纳条件即可自然得到。

## 1 有质量引力子两种等价描述的相互转换

### 1.1 自旋-2粒子的B-W全对称方程和K-G矢量方程两种等价描述 [17, 19, 22]

定义1.1.1.  $\mathbb{X}_a(x) := [im\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)\partial^b]C$ ,  $\mathbb{X}_a(p) := i[m\gamma_a(\varsigma) - 2S_{ab}(e, \varsigma)p^b]C$ ,  $C = \gamma_2\gamma_4$

定理1.1.1.

$$\begin{cases} (\gamma^a \partial_a + m)_{\kappa\varsigma} \lambda_{\varsigma} \psi_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma} = 0, \psi_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma} = \frac{1}{4!} \psi_{\{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma\}} \\ A_{ab} = \frac{1}{(2im)^2} [\bar{C}\gamma_a(\varsigma)]^{\lambda\varsigma\mu\varsigma} [\bar{C}\gamma_b(\varsigma)]^{\eta\varsigma\xi\varsigma} \psi_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma} \\ C = \gamma_2(\varsigma)\gamma_4(\varsigma) \end{cases} \Leftrightarrow \begin{cases} \partial^c F_{c|ab} + m^2 A_{ab} = 0, F_{c|ab} = \partial_c A_{ab} - \partial_a A_{cb} \\ \psi_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma} = \frac{1}{4} \mathbb{X}_{\lambda\varsigma\mu\varsigma}^a(x) \mathbb{X}_{\eta\varsigma\xi\varsigma}^b(x) A_{ab} \\ A_{ab} = A_{ba}, \delta^{ab} A_{ab} = 0 \end{cases}$$

### 1.2 自旋-2粒子Bargmann-Wigner方程 [17]的平面波解

定理1.2.1.  $(\gamma^a \partial_a + m)_{\kappa\varsigma} \lambda_{\varsigma} \psi_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{r}, t) = 0$ ,  $\psi_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{r}, t) = \frac{1}{4!} \psi_{\{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma\}}(\vec{r}, t)$

$$\begin{aligned} \psi_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{r}, t) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \sqrt{\frac{m^4}{E^7}} [a(\vec{p}, h) U_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) V_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p} \\ \begin{cases} a(\vec{p}, h) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{r}=-\infty}^{+\infty} \sqrt{\frac{m^4}{E^7}} U^{+\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{p}, h) \psi_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \\ b^+(\vec{p}, s) &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{r}=-\infty}^{+\infty} \sqrt{\frac{m^4}{E^7}} V^{+\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{p}, h) \psi_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{r}, t) e^{i(\vec{p}\cdot\vec{r}-Et)} d^3\vec{r} \end{cases} \end{aligned}$$

定理1.2.2.  $[\psi_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(x), \psi_{\lambda'\varsigma'\mu'\varsigma'\eta'\xi'\varsigma'}(x')]$

$$\begin{aligned} &= \frac{i}{2^3} \frac{1}{(4!)^2} [(m - \gamma^a \partial_a) \gamma^4]_{\{\lambda\varsigma(\lambda'_\varsigma [(m - \gamma^b \partial_b) \gamma^4]_{\mu\varsigma\mu'_\varsigma} [(m - \gamma^a \partial_a) \gamma^4]_{\eta\varsigma\eta'_\varsigma} [(m - \gamma^b \partial_b) \gamma^4]_{\xi\varsigma\xi'_\varsigma} \}} \Delta(x - x') \\ &= \frac{i}{2^5} \frac{1}{(4!)^2} \mathbb{X}_{\{\lambda\varsigma\mu\varsigma}^a(x) \mathbb{X}_{\eta\varsigma\xi\varsigma}^b(x) \mathbb{X}_{\lambda'\varsigma'\mu'\varsigma'}^{+a'}(x') \mathbb{X}_{\eta'\xi'\varsigma'}^{+b'}(x') (\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) (\eta_{bb'} - \frac{\partial_b \partial_{b'}}{m^2}) \Delta(x - x') \end{aligned}$$

定义1.2.1.

$$\begin{cases} \Lambda_{+\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma\lambda'\varsigma'\mu'\varsigma'\eta'\xi'\varsigma'}(\vec{p}, 2) := \sum_{h=2}^{-2} U_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{p}, h) U_{\lambda'\varsigma'\mu'\varsigma'\eta'\xi'\varsigma'}^+(\vec{p}, h) \\ \Lambda_{-\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma\lambda'\varsigma'\mu'\varsigma'\eta'\xi'\varsigma'}(\vec{p}, 2) := \sum_{h=2}^{-2} V_{\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma}(\vec{p}, h) V_{\lambda'\varsigma'\mu'\varsigma'\eta'\xi'\varsigma'}^+(\vec{p}, h) \end{cases}$$

定理1.2.3.

$$\begin{cases} \Lambda_{+\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma\lambda'\varsigma'\mu'\varsigma'\eta'\xi'\varsigma'}(\vec{p}, 2) = \frac{1}{(4!)^2} \Lambda_{+\{\lambda\varsigma(\lambda'_\varsigma(\vec{p}, \frac{1}{2}) \Lambda_{+\mu\varsigma\mu'_\varsigma}(\vec{p}, \frac{1}{2}) \Lambda_{+\eta\varsigma\eta'_\varsigma}(\vec{p}, \frac{1}{2}) \Lambda_{+\xi\varsigma\xi'_\varsigma}(\vec{p}, \frac{1}{2})\}} \\ = \frac{1}{2^6 m^4} \frac{1}{(4!)^2} \mathbb{X}_{\{\lambda\varsigma\mu\varsigma}^a(p) \mathbb{X}_{\eta\varsigma\xi\varsigma}^b(p) \mathbb{X}_{\lambda'\varsigma'\mu'\varsigma'}^{+a'}(p) \mathbb{X}_{\eta'\xi'\varsigma'}^{+b'}(p) \Lambda_{maa'}(\vec{p}, 1) \Lambda_{m\eta\eta'}(\vec{p}, 1) \\ \Lambda_{-\lambda\varsigma\mu\varsigma\eta\varsigma\xi\varsigma\lambda'\varsigma'\mu'\varsigma'\eta'\xi'\varsigma'}(\vec{p}, 2) = \frac{1}{(4!)^2} \Lambda_{-\{\lambda\varsigma(\lambda'_\varsigma(\vec{p}, \frac{1}{2}) \Lambda_{-\mu\varsigma\mu'_\varsigma}(\vec{p}, \frac{1}{2}) \Lambda_{-\eta\varsigma\eta'_\varsigma}(\vec{p}, \frac{1}{2}) \Lambda_{-\xi\varsigma\xi'_\varsigma}(\vec{p}, \frac{1}{2})\}} \\ = \frac{1}{2^6 m^4} \frac{1}{(4!)^2} \mathbb{X}_{\{\lambda\varsigma\mu\varsigma}^a(p) \mathbb{X}_{\eta\varsigma\xi\varsigma}^b(p) \mathbb{X}_{\lambda'\varsigma'\mu'\varsigma'}^{+a'}(p) \mathbb{X}_{\eta'\xi'\varsigma'}^{+b'}(p) \Lambda_{maa'}(\vec{p}, 1) \Lambda_{m\eta\eta'}(\vec{p}, 1) \end{cases}$$

定理1.2.4.

$$\begin{cases}
\Lambda_{+\lambda_s\mu_s\eta_s\xi_s\lambda'_s\mu'_s\eta'_s\xi'_s}(\vec{p}, 2) \\
= \frac{1}{(2m)^4} \frac{1}{(4!)^2} [(m - i\gamma^a p_a)\gamma^4]_{\{\lambda_s(\lambda'_s[(m - i\gamma^b p_b)\gamma^4]_{\mu_s\mu'_s}[(m - i\gamma^a p_a)\gamma^4]_{\eta_s\eta'_s}[(m - i\gamma^b p_b)\gamma^4]_{\xi_s\xi'_s}\}} \\
= \frac{1}{(2\sqrt{2}m)^4} \frac{1}{(4!)^2} \mathbb{X}_{\{\lambda_s\mu_s\}}^a(p) \mathbb{X}_{\eta_s\xi_s}^b(p) \mathbb{X}_{\lambda'_s\mu'_s}^{+a'}(p) \mathbb{X}_{\eta'_s\xi'_s}^{+b'}(p) (\eta_{aa'} + \frac{p_a p_{a'}}{m^2}) (\eta_{bb'} + \frac{p_b p_{b'}}{m^2}) \\
\Lambda_{-\lambda_s\mu_s\eta_s\xi_s\lambda'_s\mu'_s\eta'_s\xi'_s}(\vec{p}, 2) \\
= \frac{1}{(2m)^4} \frac{1}{(4!)^2} [(-m - i\gamma^a p_a)\gamma^4]_{\{\lambda_s(\lambda'_s[(-m - i\gamma^b p_b)\gamma^4]_{\mu_s\mu'_s} [(-m - i\gamma^a p_a)\gamma^4]_{\eta_s\eta'_s} [(-m - i\gamma^b p_b)\gamma^4]_{\xi_s\xi'_s}\}} \\
= \frac{1}{(2\sqrt{2}m)^4} \frac{1}{(4!)^2} \mathbb{X}_{\{\lambda_s\mu_s\}}^a(-p) \mathbb{X}_{\eta_s\xi_s}^b(-p) \mathbb{X}_{\lambda'_s\mu'_s}^{+a'}(-p) \mathbb{X}_{\eta'_s\xi'_s}^{+b'}(-p) (\eta_{aa'} + \frac{p_a p_{a'}}{m^2}) (\eta_{bb'} + \frac{p_b p_{b'}}{m^2})
\end{cases}$$

↓

### 1.3 推导到自旋-2粒子Klein-Gordon方程<sup>[17]</sup>的平面波解

定理1.3.1.

$$\begin{cases}
\partial^c F_{c|ab} + m^2 A_{ab} = 0, F_{c|ab} = \partial_c A_{ab} - \partial_a A_{cb}, \delta^{ab} A_{ab} = 0, A_{ab} = A_{ba} \\
A_{ab} = (\frac{1}{2im})^2 (\bar{C}\gamma_a)^{\lambda_s\mu_s} (\bar{C}\gamma_b)^{\eta_s\xi_s} \psi_{\lambda_s\mu_s\eta_s\xi_s}
\end{cases}$$

$$A_{ab}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \frac{1}{\sqrt{2^2 E}} \varepsilon_{ab}(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\varepsilon_{ab}(\vec{p}, h) = \frac{1}{(i\sqrt{2})^2} (\bar{C}\gamma_a)^{\lambda_s\mu_s} (\bar{C}\gamma_b)^{\eta_s\xi_s} U_{\lambda_s\mu_s\eta_s\xi_s}(\vec{p}, h) = \frac{1}{(i\sqrt{2})^2} (\bar{C}\gamma_a)^{\lambda_s\mu_s} (\bar{C}\gamma_b)^{\eta_s\xi_s} V_{\lambda_s\mu_s\eta_s\xi_s}(\vec{p}, h)$$

推论1.3.1.

$$\begin{cases}
\varepsilon_{ab}(\vec{p}, 2) = \varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1) \\
\varepsilon_{ab}(\vec{p}, 1) = \frac{1}{\sqrt{2}} [\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, 0) + \varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, 1)] \\
\varepsilon_{ab}(\vec{p}, 0) = \frac{1}{\sqrt{6}} [\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, -1) + \varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, 1) + 2\varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, 0)] \\
\varepsilon_{ab}(\vec{p}, -1) = \frac{1}{\sqrt{2}} [\varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, 0) + \varepsilon_a(\vec{p}, 0)\varepsilon_b(\vec{p}, -1)] \\
\varepsilon_{ab}(\vec{p}, -2) = \varepsilon_a(\vec{p}, -1)\varepsilon_b(\vec{p}, -1)
\end{cases}$$

性质1.3.1.  $\varepsilon_{ab}(\vec{p}, h) = \varepsilon_{ba}(\vec{p}, h), \delta^{ab}\varepsilon_{ab}(\vec{p}, h) = 0$

$$\text{定理1.3.2. } \sum_{h=2}^{-2} \varepsilon_{ab}(\vec{p}, h)\varepsilon_{a'b'}^+(\vec{p}, h) = \frac{1}{4} \{ [\eta_{\{a(a'} + \frac{p_a p_{a'}}{m^2})}][\eta_{b\}b') + \frac{p_b p_{b'}}{m^2}] - \frac{1}{3} [\delta_{\{ab\} + \frac{p_{\{a}p_{b\}}}{m^2}}][\delta_{(a'b')} + \frac{p_{(a'}p_{b')}}{m^2}] \}$$

$$\text{定理1.3.3. } [A_{ab}(x), A_{a'b'}^+(x')] = \frac{i}{8} \{ [\eta_{\{a(a'} - \frac{\partial_{\{a}\partial_{a'}}}{m^2})}][\eta_{b\}b') - \frac{\partial_b\partial_{b'}}{m^2}] - \frac{1}{3} [\delta_{\{ab\} - \frac{\partial_{\{a}\partial_{b\}}}{m^2}}][\delta_{(a'b')} - \frac{\partial_{(a'}\partial_{b')}}{m^2}] \} \Delta(x - x')$$

$$\text{定理1.3.4. } \{A_{a\tau_s}(x), \bar{A}_{b\tau'_s}(x')\} = \frac{i}{2} \{ [(\delta_{ab} - \frac{\partial_a\partial_b}{m^2}) - \frac{1}{3}(\gamma_a - \frac{\partial_a}{m})(\gamma_b + \frac{\partial_b}{m})](m - \gamma^c\partial_c)\gamma^4\}_{\tau_s\tau'_s} \Delta(x - x')$$

$$\text{定理1.3.5. } [A_a(x), \bar{A}_b(x')] = i(\delta_{ab} - \frac{\partial_a\partial_b}{m^2})\Delta(x - x')$$

$$\text{引理1.3.1. } \eta^{bb'} = \delta^{bb'} - 2\delta^{b4}\delta^{b'4}$$

$$\text{定理1.3.6. } \Lambda_{\pm\tau_s\tau'_s}(\vec{p}, \frac{1}{2}) = \frac{1}{5}\Lambda_{maba'b'}(\vec{p}, 2)\eta^{bb'}\gamma^a\Lambda_{\mp}(\vec{p}, \frac{1}{2})\gamma^{a'}$$

$$\text{定理1.3.7. } \Lambda_{\pm ma\tau_s a'\tau'_s}(\vec{p}, \frac{3}{2}) = \frac{2}{5}\Lambda_{maba'b'}(\vec{p}, 2)\gamma^b\Lambda_{\mp}(\vec{p}, \frac{1}{2})\gamma^{b'}$$

$$\text{定理1.3.8. } \Lambda_{maa'}(\vec{p}, 1) = \frac{3}{5}\Lambda_{maba'b'}(\vec{p}, 2)\eta^{bb'}$$

### 1.4 再回到自旋-2粒子Bargmann-Wigner方程<sup>[17]</sup>的平面波解

$$\text{定理1.4.1. } (\gamma^a\partial_a + m)_{\kappa_s} \lambda_s \psi_{\lambda_s\mu_s\eta_s\xi_s}(\vec{r}, t) = 0, \psi_{\lambda_s\mu_s\eta_s\xi_s}(\vec{r}, t) = \frac{1}{4}\mathbb{X}_{\lambda_s\mu_s}^a(x)\mathbb{X}_{\eta_s\xi_s}^b(x)A_{ab}(\vec{r}, t)$$

$$\psi_{\lambda_s\mu_s\eta_s\xi_s}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \sqrt{\frac{m^4}{E}} [a(\vec{p}, h)U_{\lambda_s\mu_s\eta_s\xi_s}(\vec{p}, h)e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)V_{\lambda_s\mu_s\eta_s\xi_s}(\vec{p}, h)e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$U_{\lambda_s\mu_s\eta_s\xi_s}(\vec{p}, h) = \frac{1}{8m^2}\mathbb{X}_{\lambda_s\mu_s}^a(p)\mathbb{X}_{\eta_s\xi_s}^b(p)\varepsilon_{ab}(\vec{p}, h), V_{\lambda_s\mu_s\eta_s\xi_s}(\vec{p}, h) = \frac{1}{8m^2}\mathbb{X}_{\lambda_s\mu_s}^a(-p)\mathbb{X}_{\eta_s\xi_s}^b(-p)\varepsilon_{ab}(\vec{p}, h)$$

## 2 有质量引力子方程的第三种等价描述

### 2.1 有质量引力子自旋方程等价描述

定理2.1.1.  $(\partial_a + iS_{ab}\partial^b)_{\beta\zeta} \alpha_\zeta \psi_{\alpha_\zeta c} = \frac{i}{\sqrt{2}} im^2 \sigma_{\beta\zeta}^{ab} A_{bc}, \psi_{\alpha_\zeta c} := \frac{i}{\sqrt{2}} \frac{i}{2} \sigma_{\alpha\zeta}^{ab} F_{a|bc}, S_{ab} := i\sigma_{\zeta ab} \gamma_{\alpha\zeta}$

$$A_{bc}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \frac{1}{\sqrt{2^2 E}} \varepsilon_{bc}(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$F_{a|bc}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \frac{1}{\sqrt{2^2 E}} [ip_a \varepsilon_{bc}(\vec{p}, h) - ip_b \varepsilon_{ac}(\vec{p}, h)] [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

$$\psi_{\alpha_\zeta c}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \frac{1}{\sqrt{2^2 E}} \frac{-i}{\sqrt{2}} \sigma_{\alpha\zeta}^{ab} p_a \varepsilon_{bc}(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

### 2.2 有质量引力子场 $F_{ab}$ 的平面波解和投影算子

定义2.2.1.  $\lambda_{abc}(\vec{p}, h) := [ip_a \varepsilon_{bc}(\vec{p}, h) - ip_b \varepsilon_{ac}(\vec{p}, h)]$

$$\text{推论2.2.1. } F_{abc}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \frac{1}{\sqrt{2^2 E}} [ip_a \varepsilon_{bc}(\vec{p}, h) - ip_b \varepsilon_{ac}(\vec{p}, h)] [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} - b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

定理2.2.1.  $\sum_{h=2}^{-2} \lambda_{abc}(\vec{p}, h) \lambda_{a'b'c'}^+(\vec{p}, h) = \frac{1}{2} p_{[a} p_{[a'}^+ \eta_{b]c'} \eta_{b']c} + \frac{1}{2} p_{[a} p_{[a'}^+ \eta_{b]b'}] [\eta_{cc'} + \frac{p_c p_{c'}^+}{m^2}] - \frac{1}{3} p_{[a} \delta_{b]c} p_{[a'}^+ \delta_{b']c'}$

证明:  $\sum_{h=2}^{-2} \lambda_{abc}(\vec{p}, h) \lambda_{a'b'c'}^+(\vec{p}, h)$

$$= \sum_{h=2}^{-2} [ip_a \varepsilon_{bc}(\vec{p}, h) - ip_b \varepsilon_{ac}(\vec{p}, h)] [ip_{a'} \varepsilon_{b'c'}(\vec{p}, h) - ip_{b'} \varepsilon_{a'c'}(\vec{p}, h)]^+$$

$$= p_a p_{a'}^+ \sum_{h=2}^{-2} \varepsilon_{bc}(\vec{p}, h) \varepsilon_{b'c'}^+(\vec{p}, h) + p_b p_{b'}^+ \sum_{h=2}^{-2} \varepsilon_{ac}(\vec{p}, h) \varepsilon_{a'c'}^+(\vec{p}, h)$$

$$- p_a p_{b'}^+ \sum_{h=2}^{-2} \varepsilon_{bc}(\vec{p}, h) \varepsilon_{a'c'}^+(\vec{p}, h) - p_b p_{a'}^+ \sum_{h=2}^{-2} \varepsilon_{ac}(\vec{p}, h) \varepsilon_{b'c'}^+(\vec{p}, h)$$

$$= p_a p_{a'}^+ \frac{1}{4} \{ [\eta_{\{b(b'} + \frac{p_{\{b} p_{\{b'}^+}{m^2}}] [\eta_{c\}c'} + \frac{p_{c\} p_{c'}^+}{m^2}] - \frac{1}{3} [\delta_{\{bc\} + \frac{p_{\{b} p_{c\}}}{m^2}}] [\delta_{(b'c')} + \frac{p_{(b'}^+ p_{c'}^+}{m^2}] \}$$

$$+ p_b p_{b'}^+ \frac{1}{4} \{ [\eta_{\{a(a'} + \frac{p_{\{a} p_{\{a'}^+}{m^2}}] [\eta_{c\}c'} + \frac{p_{c\} p_{c'}^+}{m^2}] - \frac{1}{3} [\delta_{\{ac\} + \frac{p_{\{a} p_{c\}}}{m^2}}] [\delta_{(a'c')} + \frac{p_{(a'}^+ p_{c'}^+}{m^2}] \}$$

$$- p_a p_{b'}^+ \frac{1}{4} \{ [\eta_{\{b(a'} + \frac{p_{\{b} p_{\{a'}^+}{m^2}}] [\eta_{c\}c'} + \frac{p_{c\} p_{c'}^+}{m^2}] - \frac{1}{3} [\delta_{\{bc\} + \frac{p_{\{b} p_{c\}}}{m^2}}] [\delta_{(a'c')} + \frac{p_{(a'}^+ p_{c'}^+}{m^2}] \}$$

$$- p_b p_{a'}^+ \frac{1}{4} \{ [\eta_{\{a(b'} + \frac{p_{\{a} p_{\{b'}^+}{m^2}}] [\eta_{c\}c'} + \frac{p_{c\} p_{c'}^+}{m^2}] - \frac{1}{3} [\delta_{\{ac\} + \frac{p_{\{a} p_{c\}}}{m^2}}] [\delta_{(b'c')} + \frac{p_{(b'}^+ p_{c'}^+}{m^2}] \}$$

$$= + p_a p_{a'}^+ \{ \frac{1}{4} [\eta_{\{b(b'} + \frac{p_{\{b} p_{\{b'}^+}{m^2}}] [\eta_{c\}c'} + \frac{p_{c\} p_{c'}^+}{m^2}] - \frac{1}{3} [\delta_{bc} + \frac{p_b p_c}{m^2}] [\delta_{b'c'} + \frac{p_{b'}^+ p_{c'}^+}{m^2}] \}$$

$$+ p_b p_{b'}^+ \{ \frac{1}{4} [\eta_{\{a(a'} + \frac{p_{\{a} p_{\{a'}^+}{m^2}}] [\eta_{c\}c'} + \frac{p_{c\} p_{c'}^+}{m^2}] - \frac{1}{3} [\delta_{ac} + \frac{p_a p_c}{m^2}] [\delta_{a'c'} + \frac{p_{a'}^+ p_{c'}^+}{m^2}] \}$$

$$- p_a p_{b'}^+ \{ \frac{1}{4} [\eta_{\{b(a'} + \frac{p_{\{b} p_{\{a'}^+}{m^2}}] [\eta_{c\}c'} + \frac{p_{c\} p_{c'}^+}{m^2}] - \frac{1}{3} [\delta_{bc} + \frac{p_b p_c}{m^2}] [\delta_{a'c'} + \frac{p_{a'}^+ p_{c'}^+}{m^2}] \}$$

$$- p_b p_{a'}^+ \{ \frac{1}{4} [\eta_{\{a(b'} + \frac{p_{\{a} p_{\{b'}^+}{m^2}}] [\eta_{c\}c'} + \frac{p_{c\} p_{c'}^+}{m^2}] - \frac{1}{3} [\delta_{ac} + \frac{p_a p_c}{m^2}] [\delta_{b'c'} + \frac{p_{b'}^+ p_{c'}^+}{m^2}] \}$$

$$= \frac{1}{2} (p_a p_{a'}^+ \eta_{bc} \eta_{cb'} + p_b p_{b'}^+ \eta_{ac} \eta_{a'c} - p_a p_{b'}^+ \eta_{ca} \eta_{bc'} - p_b p_{a'}^+ \eta_{ac'} \eta_{cb'})$$

$$+ \frac{1}{2} (p_a p_{a'}^+ \eta_{bb'} + p_b p_{b'}^+ \eta_{aa'} - p_a p_{b'}^+ \eta_{ba'} - p_b p_{a'}^+ \eta_{ab'}) [\eta_{cc'} + \frac{p_c p_{c'}^+}{m^2}]$$

$$- \frac{1}{3} (p_a \delta_{bc} - p_b \delta_{ac}) (p_{a'}^+ \delta_{b'c'} - p_{b'}^+ \delta_{a'c'})$$

$$= \frac{1}{2} p_{[a} p_{[a'}^+ \eta_{b]c'} \eta_{b']c} + \frac{1}{2} p_{[a} p_{[a'}^+ \eta_{b]b'}] [\eta_{cc'} + \frac{p_c p_{c'}^+}{m^2}] - \frac{1}{3} p_{[a} \delta_{b]c} p_{[a'}^+ \delta_{b']c'}$$

□

定理2.2.2.  $[F_{abc}(x), F_{a'b'c'}^+(x')] = -i \{ \frac{1}{2} \partial_{[a} \partial_{[a'}^+ \eta_{b]c'} \eta_{b']c} + \frac{1}{2} \partial_{[a} \partial_{[a'}^+ \eta_{b]b'}] [\eta_{cc'} - \frac{\partial_c \partial_{c'}^+}{m^2}] - \frac{1}{3} \partial_{[a} \delta_{b]c} \partial_{[a'}^+ \delta_{b']c'} \} \Delta(x - x')$

### 2.3 有质量引力子场 $\Psi_{\alpha_\zeta}$ 的平面波解和投影算子

定义2.3.1.  $\lambda_{\alpha_\zeta c}(\vec{p}, h) := \frac{-i}{\sqrt{2}} \sigma_{\alpha\zeta}^{ab} p_a \varepsilon_{bc}(\vec{p}, h)$

$$\text{推论2.3.1. } \psi_{\alpha_\zeta c}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \lambda_{\alpha_\zeta c}(\vec{p}, h) [a(\vec{p}, h) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^3\vec{p}$$

定理2.3.1.  $\sum_{h=1}^{-1} \lambda_{\alpha_\zeta c}(\vec{p}, h) \lambda_{\alpha'_\zeta c'}^+(\vec{p}, h) = -\frac{1}{8} \sigma_{\alpha\zeta}^{ab} \sigma_{\alpha'\zeta'}^{a'b'} p_a p_{a'}^+ [\eta_{\{b(b'} \eta_{c\}c')} - \frac{1}{3} \delta_{\{bc\}} \delta_{(b'c')}] - \frac{1}{2m^2} \sigma_{\alpha\zeta}^{aa'} p_a p_{a'} p_c p_{c'}^+$

证明: 
$$\begin{aligned} & \sum_{h=1}^{-1} \lambda_{\alpha_\zeta c}(\vec{p}, h) \lambda_{\alpha'_\zeta c'}^+(\vec{p}, h) \\ &= \sum_{h=1}^{-1} \frac{-i}{\sqrt{2}} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} p_a \varepsilon_{bc}(\vec{p}, h) \frac{-i}{\sqrt{2}} \sigma_{\alpha'_\zeta \alpha_\zeta}^{a'b'} p_{a'} \varepsilon_{b'c'}^+(\vec{p}, h) \\ &= -\frac{1}{2} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\alpha'_\zeta \alpha_\zeta}^{a'b'} p_a p_{a'} \sum_{h=1}^{-1} \varepsilon_{bc}(\vec{p}, h) \varepsilon_{b'c'}^+(\vec{p}, h) \\ &= -\frac{1}{2} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\alpha'_\zeta \alpha_\zeta}^{a'b'} p_a p_{a'} \frac{1}{4} \{ [\eta_{\{b(b' + \frac{p_{\{b} p_{\{b'}}^+}{m^2})\}}] [\eta_{c\}c'} + \frac{p_c\}p_{c'}^+] - \frac{1}{3} [\delta_{\{bc\}} + \frac{p_{\{b} p_{c\}}}{m^2}] [\delta_{(b'c')} + \frac{p_{(b'}^+ p_{c'}^+}{m^2}] \} \\ &= -\frac{1}{8} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\alpha'_\zeta \alpha_\zeta}^{a'b'} p_a p_{a'} \{ 2[\eta_{bb'} + \frac{p_b p_{b'}}{m^2}] [\eta_{cc'} + \frac{p_c p_{c'}}{m^2}] + 2[\eta_{cb'} + \frac{p_c p_{b'}}{m^2}] [\eta_{bc'} + \frac{p_b p_{c'}}{m^2}] - \frac{4}{3} [\delta_{bc} + \frac{p_b p_c}{m^2}] [\delta_{b'c'} + \frac{p_{b'}^+ p_{c'}^+}{m^2}] \} \\ &= -\frac{1}{4} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\alpha'_\zeta \alpha_\zeta}^{a'b'} p_a p_{a'} \{ \eta_{bb'} [\eta_{cc'} + \frac{p_c p_{c'}}{m^2}] + \eta_{cb'} \eta_{bc'} - \frac{2}{3} \delta_{bc} \delta_{b'c'} \} \\ &= -\frac{1}{8} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\alpha'_\zeta \alpha_\zeta}^{a'b'} p_a p_{a'} [\eta_{\{b(b' \eta_{c\}c')\}} - \frac{1}{3} \delta_{\{bc\}} \delta_{(b'c')}] - \frac{1}{2m^2} \sigma_{\alpha_\zeta \alpha'_\zeta}^{aa'} p_a p_{a'} p_c p_c^+ \end{aligned}$$

□

定理2.3.2.  $[\psi_{\alpha_\zeta c}(x), \psi_{\alpha'_\zeta c'}^+(x')] = i \{ \frac{1}{2m^2} \sigma_{\alpha_\zeta \alpha'_\zeta}^{aa'} \partial_a \partial_{a'} \partial_c \partial_{c'}^+ - \frac{1}{8} \sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\alpha'_\zeta \alpha_\zeta}^{a'b'} \partial_a \partial_{a'}^+ [\eta_{\{b(b' \eta_{c\}c')\}} - \frac{1}{3} \delta_{\{bc\}} \delta_{(b'c')}] \} \Delta(x - x')$

### 3 有质量引力子方程的第四种等价描述

#### 3.1 有质量引力子各种物理量定义

定义3.1.1. 
$$\begin{cases} \text{外尔复张量 } C^{\alpha_\zeta \beta_\zeta} := \frac{i}{2} \sigma_{\alpha_\zeta \beta_\zeta}^{ab} C^{ab\beta_\zeta} = \frac{i}{2} \sigma_{\alpha_\zeta \beta_\zeta}^{\alpha_\zeta} \frac{i}{2} \sigma_{\kappa cd}^{\beta_\zeta} C^{abcd} \\ \psi_{\alpha_\zeta \beta_\zeta} = (\frac{i}{\sqrt{2}})^2 C_{\alpha_\zeta \beta_\zeta}, \psi_{\alpha_\zeta \beta_\zeta}^+ = -\psi_{\alpha_{-\zeta} \beta_{-\zeta}} \end{cases}$$

定义3.1.2. 引力曲率旋量:

$$\psi^{A_\zeta B_\zeta C_\zeta D_\zeta} := \frac{i_\zeta}{\sqrt{2}} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \frac{i_\kappa}{\sqrt{2}} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \psi_{\alpha_\zeta \beta_\zeta} = \frac{i}{\sqrt{2}} \frac{i_\zeta}{\sqrt{2}} S_{ab}^{A_\zeta B_\zeta} \frac{i_\kappa}{\sqrt{2}} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} C^{ab\beta_\zeta} = \frac{i}{\sqrt{2}} \frac{i_\zeta}{\sqrt{2}} S_{ab}^{A_\zeta B_\zeta} \frac{i}{\sqrt{2}} \frac{i_\kappa}{\sqrt{2}} S_{cd}^{C_\zeta D_\zeta} C^{abcd}$$

推论3.1.1.  $\psi^{\alpha_\zeta \beta_\zeta} = \frac{i_\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \frac{i_\kappa}{\sqrt{2}} \sigma_{C_\zeta D_\zeta}^{\beta_\zeta} \psi^{A_\zeta B_\zeta C_\zeta D_\zeta}$

推论3.1.2.  $\psi_{\alpha_\zeta \beta_\zeta} = \frac{i_\zeta}{\sqrt{2}} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \frac{i_\zeta}{\sqrt{2}} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \psi_{A_\zeta B_\zeta C_\zeta D_\zeta} = -\frac{1}{2} \sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta} \psi_{A_\zeta B_\zeta C_\zeta D_\zeta}, [\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta}]^* = \sigma_{\alpha'_\zeta}^{A'_\zeta B'_\zeta}$

定义3.1.3.

$$\Gamma_{\alpha_\zeta \beta_\zeta \dots}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots} (n) := \Gamma_{\alpha_\zeta \beta_\zeta \dots}^{k_\zeta} (n) \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots} (n) = (\frac{i_\zeta}{\sqrt{2}})^n \frac{1}{(2n)!} \underbrace{\sigma_{\alpha_\zeta}^{(A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta \dots)}}_n$$

$$\Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}^{\alpha_\zeta \beta_\zeta \dots} (n) := \Gamma_{k_\zeta}^{\alpha_\zeta \beta_\zeta \dots} (n) \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta \dots}^{k_\zeta} (n) = (\frac{i_\zeta}{\sqrt{2}})^n \frac{1}{(2n)!} \underbrace{\sigma_{(A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta \dots}^{\beta_\zeta)}}_n$$

#### 3.2 有质量粒子一般对易规则两种等价描述

定理3.2.1.

$$\begin{cases} [\psi_{\alpha_\zeta \beta_\zeta \dots}^{A_\zeta B_\zeta C_\zeta D_\zeta \dots} (x), \psi_{\alpha'_\zeta \beta'_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta D'_\zeta \dots} (x')] = i \frac{(i_\zeta)^{2n}}{2^{2n-1}} \overbrace{(\sigma_{\alpha_\zeta}^a (i_\zeta)_{A_\zeta A'_\zeta} (\sigma_{\beta_\zeta}^b (i_\zeta)_{B_\zeta B'_\zeta} (\sigma_{\gamma_\zeta}^c (i_\zeta)_{C_\zeta C'_\zeta} (\sigma_{\delta_\zeta}^d (i_\zeta)_{D_\zeta D'_\zeta} \dots \partial_a \partial_b \partial_c \partial_d \dots))}^{2n} \Delta(x - x') \\ \psi_{\alpha_\zeta \beta_\zeta \dots} (x) = (\frac{i_\zeta}{\sqrt{2}})^n \underbrace{\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta} \sigma_{\beta_\zeta}^{C_\zeta D_\zeta \dots}}_n \psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots} (x) \\ \Leftrightarrow \begin{cases} [\psi_{\alpha_\zeta \beta_\zeta \dots} (x), \psi_{\alpha'_\zeta \beta'_\zeta \dots}^+ (x')] = \frac{i}{2^{n-1}} \overbrace{\sigma_{\alpha_\zeta \alpha'_\zeta}^{ab} \sigma_{\beta_\zeta \beta'_\zeta}^{cd} \dots \partial_a \partial_b \partial_c \partial_d \dots}^{2n} \Delta(x - x') \\ \psi_{A_\zeta B_\zeta C_\zeta D_\zeta \dots} (x) = (\frac{i_\zeta}{\sqrt{2}})^n \underbrace{\sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \sigma_{C_\zeta D_\zeta \dots}^{\beta_\zeta}}_n \psi_{\alpha_\zeta \beta_\zeta \dots} (x) \end{cases} \end{cases}$$

### 3.3 线性引力场 $\psi_{\alpha\zeta\beta\zeta}$ 的对易规则

定理3.3.1.

$$\begin{cases} [\psi_{\alpha\zeta\beta\zeta}(x), \psi_{\alpha'\zeta\beta'\zeta}^+(x')] = \frac{i}{2} \sigma_{\alpha\zeta\alpha'\zeta}^{ab} \sigma_{\beta\zeta\beta'\zeta}^{cd} \partial_a \partial_b \partial_c \partial_d \Delta(x-x') \\ [\psi_{\alpha\zeta\beta\zeta}(x), \psi_{\rho\zeta\sigma\zeta}(x')] = \frac{i}{32} m^4 \delta_{\{\alpha\zeta(\rho\zeta\delta\beta\zeta\}\sigma\zeta} \Delta(x-x') \\ [\psi_{\alpha'\zeta\beta'\zeta}(x), \psi_{\rho'\zeta\sigma'\zeta}(x')] = \frac{i}{32} m^4 \delta_{\{\alpha'\zeta(\rho'\zeta\delta\beta'\zeta\}\sigma'\zeta} \Delta(x-x') \end{cases}$$

证明:  $[\psi_{\alpha\zeta\beta\zeta}(x), \psi_{\alpha'\zeta\beta'\zeta}^+(x')]$

$$\begin{aligned} &= \frac{1}{4} \sigma_{\alpha\zeta}^{A_1 B_1 C_1 D_1} \sigma_{\beta\zeta}^{C_2 D_2} \sigma_{\alpha'\zeta}^{A_2 B_2 C_2 D_2} \sigma_{\beta'\zeta}^{C_3 D_3} [\psi_{A_1 B_1 C_1 D_1}, \psi_{A_2 B_2 C_2 D_2}^+] \\ &= \frac{i}{2^5} \sigma_{\alpha\zeta}^{A_1 B_1 C_1 D_1} \sigma_{\beta\zeta}^{C_2 D_2} \sigma_{\alpha'\zeta}^{A_2 B_2 C_2 D_2} \sigma_{\beta'\zeta}^{C_3 D_3} (\sigma, i\zeta)_{A_1 A_2}^a (\sigma, i\zeta)_{B_2 B_1}^b (\sigma, i\zeta)_{C_2 C_1}^c (\sigma, i\zeta)_{D_2 D_1}^d \Delta(x-x') \\ &= \frac{i}{2} \Gamma_{\alpha\zeta\alpha'\zeta\beta\zeta\beta'\zeta}^{abcd} (2) \partial_a \partial_b \partial_c \partial_d \Delta(x-x') \\ &= \frac{i}{2} \Gamma_{\alpha\zeta\alpha'\zeta}^{ab} (1) \Gamma_{\beta\zeta\beta'\zeta}^{cd} (1) \partial_a \partial_b \partial_c \partial_d \Delta(x-x') \\ &= \frac{i}{2} \sigma_{\alpha\zeta\alpha'\zeta}^{ab} \sigma_{\beta\zeta\beta'\zeta}^{cd} \partial_a \partial_b \partial_c \partial_d \Delta(x-x') \end{aligned} \quad \square$$

### 3.4 线性引力场 $C_{abcd}$ 的对易规则

推论3.4.1.  $\sigma_{\alpha\zeta\alpha'\zeta}^{ab} = -\frac{1}{2} \sigma_{\zeta\alpha\zeta}^{ac} \delta_{cd} \sigma_{-\zeta\alpha'\zeta}^{db}, \sigma_{ab}^{\alpha'\zeta\alpha\zeta} = -\frac{1}{2} \sigma_{\zeta\alpha\zeta}^{ac} \delta^{cd} \sigma_{-\zeta\alpha\zeta}^{db}$

引理3.4.1.

$$\begin{aligned} 2\sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\zeta a'b'}^{\alpha'\zeta} \sigma_{\alpha\zeta\alpha'\zeta}^{cc'} \partial_c \partial_{c'} &= \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\zeta\alpha\zeta cd} \sigma_{\zeta a'b'}^{\alpha'\zeta} \sigma_{\zeta\alpha'\zeta c'd'} \eta^{cc'} \partial^d \partial^{d'} = (S_{abcd} - \zeta \varepsilon_{abcd}) (S_{a'b'c'd'} - \zeta \varepsilon_{a'b'c'd'}) \eta^{cc'} \partial^d \partial^{d'} \\ 2\sigma_{-\zeta ab}^{\alpha\zeta} \sigma_{-\zeta a'b'}^{\alpha'\zeta} \sigma_{\alpha\zeta\alpha'\zeta}^{cc'} \partial_c \partial_{c'} &= \sigma_{-\zeta ab}^{\alpha\zeta} \sigma_{-\zeta\alpha\zeta cd} \sigma_{-\zeta a'b'}^{\alpha'\zeta} \sigma_{-\zeta\alpha'\zeta c'd'} \eta^{cc'} \partial^d \partial^{d'} = (S_{abcd} + \zeta \varepsilon_{abcd}) (S_{a'b'c'd'} + \zeta \varepsilon_{a'b'c'd'}) \eta^{cc'} \partial^d \partial^{d'} \end{aligned}$$

推论3.4.2.  $(S_{ab\bar{c}\bar{d}} \varepsilon_{a'b'c'd'} + \varepsilon_{ab\bar{c}\bar{d}} S_{a'b'c'd'}) \eta^{\bar{c}\bar{c}'} \partial^{\bar{d}} \partial^{\bar{d}'} \Delta(x-x')$   
 $= [(\delta_{a\bar{c}} \delta_{b\bar{d}} - \delta_{a\bar{d}} \delta_{b\bar{c}}) \varepsilon_{a'b'c'd'} + \varepsilon_{ab\bar{c}\bar{d}} (\delta_{a'\bar{c}'} \delta_{b'\bar{d}'} - \delta_{a'\bar{d}'} \delta_{b'\bar{c}'})] \eta^{\bar{c}\bar{c}'} \partial^{\bar{d}} \partial^{\bar{d}'} \Delta(x-x')$   
 $= [(\eta_{a'}^{\bar{c}'} \partial_b \partial^{+\bar{d}'} - \eta_{b'}^{\bar{c}'} \partial_a \partial^{+\bar{d}'})] \varepsilon_{a'b'c'd'} + \varepsilon_{ab\bar{c}\bar{d}} (\eta_{a'}^{\bar{c}'} \partial_{b'}^+ \partial^{\bar{d}} - \eta_{b'}^{\bar{c}'} \partial_{a'}^+ \partial^{\bar{d}})] \Delta(x-x')$

推论3.4.3.  $(S_{\bar{a}\bar{b}\bar{c}\bar{d}} \varepsilon_{\bar{a}'\bar{b}'c'd'} + \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} S_{\bar{a}'\bar{b}'c'd'}) \eta^{\bar{a}\bar{a}'} \partial^{\bar{b}} \partial^{\bar{b}'} \Delta(x-x')$   
 $= [(\eta_{c'}^{\bar{a}'} \partial_d \partial^{+\bar{b}'} - \eta_{d'}^{\bar{a}'} \partial_c \partial^{+\bar{b}'})] \varepsilon_{\bar{a}'\bar{b}'c'd'} + \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} (\eta_{c'}^{\bar{a}'} \partial_{d'}^+ \partial^{\bar{b}} - \eta_{d'}^{\bar{a}'} \partial_{c'}^+ \partial^{\bar{b}})] \Delta(x-x')$

推论3.4.4.

$$\begin{aligned} &[(\eta_{a'}^{\bar{c}'} \partial_b \partial^{+\bar{d}'} - \eta_{b'}^{\bar{c}'} \partial_a \partial^{+\bar{d}'})] \varepsilon_{a'b'c'd'} + \varepsilon_{ab\bar{c}\bar{d}} (\eta_{a'}^{\bar{c}'} \partial_{b'}^+ \partial^{\bar{d}} - \eta_{b'}^{\bar{c}'} \partial_{a'}^+ \partial^{\bar{d}})] \\ &[(\eta_{c'}^{\bar{a}'} \partial_d \partial^{+\bar{b}'} - \eta_{d'}^{\bar{a}'} \partial_c \partial^{+\bar{b}'})] \varepsilon_{\bar{a}'\bar{b}'c'd'} + \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} (\eta_{c'}^{\bar{a}'} \partial_{d'}^+ \partial^{\bar{b}} - \eta_{d'}^{\bar{a}'} \partial_{c'}^+ \partial^{\bar{b}})] \Delta(x-x') \\ &= (\eta_{a'}^{\bar{c}'} \partial_{b'} \partial^{+\bar{d}'} \varepsilon_{a'b'c'd'} + \varepsilon_{ab\bar{c}\bar{d}} \eta_{c'a'}^{\bar{c}'} \partial_{b'}^+ \partial^{\bar{d}}) (\eta_{c'}^{\bar{a}'} \partial_{d'} \partial^{+\bar{b}'} \varepsilon_{\bar{a}'\bar{b}'c'd'} + \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} \eta_{c'd'}^{\bar{a}'} \partial_{d'}^+ \partial^{\bar{b}}) \Delta(x-x') \end{aligned}$$

推论3.4.5.  $C_{abcd} = \frac{1}{2} (\sigma_{-\zeta ab}^{\alpha'\zeta} \sigma_{-\zeta cd}^{\beta'\zeta} \psi_{\alpha'\zeta\beta'\zeta}^+ + \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\zeta cd}^{\beta\zeta} \psi_{\alpha\zeta\beta\zeta})$

定理3.4.1.  $\begin{cases} [C_{abcd}(x), C_{a'b'c'd'}^+(x')] = \frac{i}{4} \{ \eta_{[a\langle a'} \partial_{b]} \partial_{b'}^+ \eta_{[c\langle c'} \partial_{d]} \partial_{d'}^+ + \eta_{[c\langle c'} \partial_{d]} \partial_{b'}^+ \eta_{[a\langle a'} \partial_{b]} \partial_{d'}^+ \} \Delta(x-x') \\ [C_{abcd}(x), C_{a'b'c'd'}(x')] = \frac{i}{4} \{ \delta_{[a\langle a'} \partial_{b]} \partial_{b'} > \eta_{[c\langle c'} \partial_{d]} \partial_{d'} > + \delta_{[c\langle c'} \partial_{d]} \partial_{b'} > \eta_{[a\langle a'} \partial_{b]} \partial_{d'} > \} \Delta(x-x') \\ [C_{abcd}^+(x), C_{a'b'c'd'}^+(x')] = \frac{i}{4} \{ \delta_{[a\langle a'} \partial_{b]} \partial_{b'}^+ \eta_{[c\langle c'} \partial_{d]} \partial_{d'}^+ + \delta_{[c\langle c'} \partial_{d]} \partial_{b'}^+ \eta_{[a\langle a'} \partial_{b]} \partial_{d'}^+ \} \Delta(x-x') \\ [C_{ab}^{\alpha\zeta}(x), C_{a'b'}^{\alpha'\zeta}(x')] = -\frac{i}{2} \{ \eta_{[a\langle a'} \partial_{b]} \partial_{b'}^+ \sigma_{\bar{c}\bar{d}}^{\alpha'\zeta\alpha\zeta} \partial^c \partial^d + \frac{1}{2} (\sigma_{-\zeta c[a}^{\alpha'\zeta} \partial_{b]} \partial^c) (\sigma_{-\zeta c'\langle a'}^{\alpha\zeta} \partial_{b']} \partial^{c'}) \} \Delta(x-x') \end{cases}$

证明:  $[C_{abcd}(x), C_{a'b'c'd'}^+(x')]$

$$\begin{aligned} &= \frac{1}{4} [ [\sigma_{-\zeta ab}^{\alpha'\zeta} \sigma_{-\zeta cd}^{\beta'\zeta} \psi_{\alpha'\zeta\beta'\zeta}^+(x) + \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\zeta cd}^{\beta\zeta} \psi_{\alpha\zeta\beta\zeta}(x)], [\sigma_{-\zeta a'b'}^{\alpha'\zeta} \sigma_{-\zeta c'd'}^{\beta'\zeta} \psi_{\alpha'\zeta\beta'\zeta}^+(x') + \sigma_{\zeta a'b'}^{\alpha\zeta} \sigma_{\zeta c'd'}^{\beta\zeta} \psi_{\alpha\zeta\beta\zeta}^+(x')] ] \\ &= \frac{1}{4} \{ \sigma_{-\zeta ab}^{\alpha'\zeta} \sigma_{-\zeta cd}^{\beta'\zeta} \sigma_{-\zeta a'b'}^{\alpha'\zeta} \sigma_{-\zeta c'd'}^{\beta'\zeta} [\psi_{\alpha'\zeta\beta'\zeta}^+(x), \psi_{\alpha\zeta\beta\zeta}(x')] + \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\zeta cd}^{\beta\zeta} \sigma_{\zeta a'b'}^{\alpha'\zeta} \sigma_{\zeta c'd'}^{\beta'\zeta} [\psi_{\alpha\zeta\beta\zeta}(x), \psi_{\alpha'\zeta\beta'\zeta}^+(x')] \} \\ &= \frac{i}{8} \{ \sigma_{-\zeta ab}^{\alpha'\zeta} \sigma_{-\zeta cd}^{\beta'\zeta} \sigma_{-\zeta a'b'}^{\alpha'\zeta} \sigma_{-\zeta c'd'}^{\beta'\zeta} + \sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\zeta cd}^{\beta\zeta} \sigma_{\zeta a'b'}^{\alpha'\zeta} \sigma_{\zeta c'd'}^{\beta'\zeta} \} \sigma_{\alpha\zeta\alpha'\zeta}^{\bar{a}\bar{b}} \sigma_{\beta\zeta\beta'\zeta}^{\bar{c}\bar{d}} \partial_{\bar{a}} \partial_{\bar{b}} \partial_{\bar{c}} \partial_{\bar{d}} \Delta(x-x') \\ &= \frac{i}{32} \{ 2\sigma_{-\zeta ab}^{\alpha'\zeta} \sigma_{-\zeta a'b'}^{\alpha'\zeta} 2\sigma_{-\zeta cd}^{\beta'\zeta} \sigma_{-\zeta c'd'}^{\beta'\zeta} + 2\sigma_{\zeta ab}^{\alpha\zeta} \sigma_{\zeta a'b'}^{\alpha'\zeta} 2\sigma_{\zeta cd}^{\beta\zeta} \sigma_{\zeta c'd'}^{\beta'\zeta} \} \sigma_{\alpha\zeta\alpha'\zeta}^{\bar{c}\bar{d}} \partial_{\bar{c}} \partial_{\bar{d}} \sigma_{\beta\zeta\beta'\zeta}^{\bar{a}\bar{b}} \partial_{\bar{a}} \partial_{\bar{b}} \Delta(x-x') \\ &= \frac{i}{32} \{ (S_{ab\bar{c}\bar{d}} - \zeta \varepsilon_{ab\bar{c}\bar{d}}) (S_{a'b'c'd'} - \zeta \varepsilon_{a'b'c'd'}) (S_{cd\bar{a}\bar{b}} - \zeta \varepsilon_{cd\bar{a}\bar{b}}) (S_{c'd'a'b'} - \zeta \varepsilon_{c'd'a'b'}) \\ &+ (S_{ab\bar{c}\bar{d}} + \zeta \varepsilon_{ab\bar{c}\bar{d}}) (S_{a'b'c'd'} + \zeta \varepsilon_{a'b'c'd'}) (S_{cd\bar{a}\bar{b}} + \zeta \varepsilon_{cd\bar{a}\bar{b}}) (S_{c'd'a'b'} + \zeta \varepsilon_{c'd'a'b'}) \} \eta^{\bar{c}\bar{c}'} \partial^{\bar{d}} \partial^{\bar{d}'} \eta^{\bar{a}\bar{a}'} \partial^{\bar{b}} \partial^{\bar{b}'} \Delta(x-x') \\ &= \frac{i}{32} \{ (S_{ab\bar{c}\bar{d}} - \zeta \varepsilon_{ab\bar{c}\bar{d}}) (S_{a'b'c'd'} - \zeta \varepsilon_{a'b'c'd'}) (S_{\bar{a}\bar{b}\bar{c}\bar{d}} - \zeta \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}}) (S_{\bar{a}'\bar{b}'c'd'} - \zeta \varepsilon_{\bar{a}'\bar{b}'c'd'}) \end{aligned}$$



$$\begin{aligned}
& + (S_{abc\bar{d}} + \varsigma \varepsilon_{abc\bar{d}})(S_{a'b'c'd'} + \varsigma \varepsilon_{a'b'c'd'}) (S_{\bar{a}b\bar{c}d} + \varsigma \varepsilon_{\bar{a}b\bar{c}d})(S_{\bar{a}'b'c'd'} + \varsigma \varepsilon_{\bar{a}'b'c'd'}) \eta^{\bar{a}\bar{a}'} \eta^{\bar{c}\bar{c}'} \partial^{\bar{b}} \partial^{+\bar{b}'} \partial^{\bar{d}} \partial^{+\bar{d}'} \Delta(x - x') \\
& = \frac{i}{16} \{ (S_{abc\bar{d}} S_{a'b'c'd'} S_{\bar{a}b\bar{c}d} S_{\bar{a}'b'c'd'} + \varepsilon_{abc\bar{d}} \varepsilon_{a'b'c'd'} \varepsilon_{\bar{a}b\bar{c}d} \varepsilon_{\bar{a}'b'c'd'}) \\
& + S_{abc\bar{d}} S_{a'b'c'd'} \varepsilon_{\bar{a}b\bar{c}d} \varepsilon_{\bar{a}'b'c'd'} + S_{abc\bar{d}} \varepsilon_{a'b'c'd'} S_{\bar{a}b\bar{c}d} \varepsilon_{\bar{a}'b'c'd'} + S_{abc\bar{d}} \varepsilon_{a'b'c'd'} \varepsilon_{\bar{a}b\bar{c}d} S_{\bar{a}'b'c'd'} \\
& + \varepsilon_{abc\bar{d}} \varepsilon_{a'b'c'd'} S_{\bar{a}b\bar{c}d} S_{\bar{a}'b'c'd'} + \varepsilon_{abc\bar{d}} S_{a'b'c'd'} \varepsilon_{\bar{a}b\bar{c}d} S_{\bar{a}'b'c'd'} + \varepsilon_{abc\bar{d}} S_{a'b'c'd'} S_{\bar{a}b\bar{c}d} \varepsilon_{\bar{a}'b'c'd'} \} \\
& \eta^{\bar{a}\bar{a}'} \eta^{\bar{c}\bar{c}'} \partial^{\bar{b}} \partial^{+\bar{b}'} \partial^{\bar{d}} \partial^{+\bar{d}'} \Delta(x - x') \\
& = \frac{i}{16} \{ (S_{abc\bar{d}} S_{a'b'c'd'} + \varepsilon_{abc\bar{d}} \varepsilon_{a'b'c'd'}) (S_{\bar{a}b\bar{c}d} S_{\bar{a}'b'c'd'} + \varepsilon_{\bar{a}b\bar{c}d} \varepsilon_{\bar{a}'b'c'd'}) \\
& + (S_{abc\bar{d}} \varepsilon_{a'b'c'd'} + \varepsilon_{abc\bar{d}} S_{a'b'c'd'}) (S_{\bar{a}b\bar{c}d} \varepsilon_{\bar{a}'b'c'd'} + \varepsilon_{\bar{a}b\bar{c}d} S_{\bar{a}'b'c'd'}) \} \\
& \eta^{\bar{a}\bar{a}'} \eta^{\bar{c}\bar{c}'} \partial^{\bar{b}} \partial^{+\bar{b}'} \partial^{\bar{d}} \partial^{+\bar{d}'} \Delta(x - x') \\
& = \frac{i}{16} \{ 4\eta_{[a<a'} \partial_{b]} \partial_{b'}^+ \eta_{[c<c'} \partial_{d]} \partial_{d'}^+ \Delta(x - x') \\
& + (S_{abc\bar{d}} \varepsilon_{a'b'c'd'} + \varepsilon_{abc\bar{d}} S_{a'b'c'd'}) (S_{\bar{a}b\bar{c}d} \varepsilon_{\bar{a}'b'c'd'} + \varepsilon_{\bar{a}b\bar{c}d} S_{\bar{a}'b'c'd'}) \eta^{\bar{a}\bar{a}'} \eta^{\bar{c}\bar{c}'} \partial^{\bar{b}} \partial^{+\bar{b}'} \partial^{\bar{d}} \partial^{+\bar{d}'} \Delta(x - x') \} \\
& = \frac{i}{16} \{ 4\eta_{[a<a'} \partial_{b]} \partial_{b'}^+ \eta_{[c<c'} \partial_{d]} \partial_{d'}^+ + 4\eta_{[c<c'} \partial_{d]} \partial_{d'}^+ \eta_{[a<a'} \partial_{b]} \partial_{b'}^+ \} \Delta(x - x') \\
& = \frac{i}{4} \{ \eta_{[a<a'} \partial_{b]} \partial_{b'}^+ \eta_{[c<c'} \partial_{d]} \partial_{d'}^+ + \eta_{[c<c'} \partial_{d]} \partial_{d'}^+ \eta_{[a<a'} \partial_{b]} \partial_{b'}^+ \} \Delta(x - x') \quad \square
\end{aligned}$$

## 4 有质量引力子势 $A_{ab}$ 的协变对易规则复杂的直接计算

### 4.1 数学准备

#### 4.1.1 反向推论

推论4.1.1.  $[A_{ab}(x), A_{a'b'}^+(x')]$

$$\begin{aligned}
& = \frac{1}{m^4} \frac{i}{29(4!)^2} \{ 64m^2 (\delta_{ab} \delta_{cd} - 2\delta_{c\{a} \delta_{b\}d}) + 64 (\delta_{ab} S_{cedf} + \delta_{\{a[c} S_{e]b\}df}) \partial^e \partial^f \} \{ 64m^2 (\delta_{a'b'} \delta_{c'd'} - 2\delta_{c'(a'} \delta_{b')d'}) + 64 (\delta_{a'b'} S_{c'e'd'f'} + \\
& \delta_{\{a'[c'} S_{e']b'}\}d'f') \partial^{e'} \partial^{f'} \} [\eta_{cc'} - \frac{\partial_c \partial_{c'}}{m^2}] [\eta_{dd'} - \frac{\partial_d \partial_{d'}}{m^2}] \Delta(x - x')
\end{aligned}$$

引理4.1.1.

$$\begin{aligned}
tr[\gamma_a(\varsigma) \gamma_b(\varsigma) \gamma_c(\varsigma) \gamma_d(\varsigma)] & = 4[\delta_{ab} \delta_{cd} - \delta_{a[c} \delta_{d]b}] \\
tr[\gamma_a(\varsigma) S_{bc}(e, \varsigma) \gamma_d(\varsigma) S_{ef}(e, \varsigma)] & = \delta_{ad} S_{bcef} + \delta_{a[b} S_{c]def} + \delta_{d[b} S_{c]aef} \\
tr[\gamma_5(\varsigma) \gamma_a(\varsigma) S_{bc}(e, \varsigma) \gamma_d(\varsigma) S_{ef}(e, \varsigma)] & = -\{ \delta_{ad} \varepsilon_{bcef} + \delta_{a[b} \varepsilon_{c]def} + \delta_{d[b} \varepsilon_{c]aef} \}
\end{aligned}$$

#### 4.1.2 系列引理

引理4.1.2.  $tr[\bar{C}\gamma_a(\varsigma) \mathbb{X}_b(x)] = 4im\delta_{ab}$

证明:  $tr[\bar{C}\gamma_a(\varsigma) \mathbb{X}_b(x)]$

$$\begin{aligned}
& = tr\{ \bar{C}\gamma_a(\varsigma) [im\gamma_b(\varsigma) C - 2S_{bc}(e, \varsigma) C \partial^c] \} \\
& = tr\{ \gamma_a(\varsigma) [im\gamma_b(\varsigma) - 2S_{bc}(e, \varsigma) \partial^c] \} \\
& = imtr\{ \gamma_a(\varsigma) \gamma_b(\varsigma) \} \\
& = 4im\delta_{ab} \quad \square
\end{aligned}$$

引理4.1.3.  $tr[\bar{C}\gamma_a(\varsigma) \mathbb{X}_b(x) \bar{C}\gamma_c(\varsigma) \mathbb{X}_d(x)] = -4m^2 (\delta_{ab} \delta_{cd} - \delta_{a[c} \delta_{d]b}) + 4(\delta_{ac} S_{bedf} + \delta_{\{a[b} S_{e]c\}df}) \partial^e \partial^f$

证明:  $tr[\bar{C}\gamma_a(\varsigma) \mathbb{X}_b(x) \bar{C}\gamma_c(\varsigma) \mathbb{X}_d(x)]$

$$\begin{aligned}
& = tr\{ \bar{C}\gamma_a(\varsigma) [im\gamma_b(\varsigma) C - 2S_{bc}(e, \varsigma) C \partial^e] \bar{C}\gamma_c(\varsigma) [im\gamma_d(\varsigma) C - 2S_{df}(e, \varsigma) C \partial^f] \} \\
& = tr\{ \gamma_a(\varsigma) [im\gamma_b(\varsigma) - 2S_{bc}(e, \varsigma) \partial^e] \gamma_c(\varsigma) [im\gamma_d(\varsigma) - 2S_{df}(e, \varsigma) \partial^f] \} \\
& = (im)^2 tr[\gamma_a(\varsigma) \gamma_b(\varsigma) \gamma_c(\varsigma) \gamma_d(\varsigma)] + 4tr[\gamma_a(\varsigma) S_{bc}(e, \varsigma) \gamma_c(\varsigma) S_{df}(e, \varsigma)] \partial^e \partial^f \\
& = -4m^2 (\delta_{ab} \delta_{cd} - \delta_{a[c} \delta_{d]b}) + 4(\delta_{ac} S_{bedf} + \delta_{\{a[b} S_{e]c\}df}) \partial^e \partial^f \quad \square
\end{aligned}$$

引理4.1.4.  $[\bar{C}\gamma_a(\varsigma)]^{\lambda\varsigma\mu\varsigma} [\bar{C}\gamma_b(\varsigma)]^{\eta\varsigma\xi\varsigma} \mathbb{X}_{\{\lambda\varsigma\mu\varsigma\}}^c(x) \mathbb{X}_{\{\eta\varsigma\xi\varsigma\}}^d(x)$

$$= 64m^2 (\delta_{ab} \delta_{cd} - 2\delta_{c\{a} \delta_{b\}d}) + 64 (\delta_{ab} S_{cedf} + \delta_{\{a[c} S_{e]b\}df}) \partial^e \partial^f$$

证明:  $[\bar{C}\gamma_a(\varsigma)]^{\lambda\varsigma\mu\varsigma} [\bar{C}\gamma_b(\varsigma)]^{\eta\varsigma\xi\varsigma} \mathbb{X}_{\{\lambda\varsigma\mu\varsigma\}}^c(x) \mathbb{X}_{\{\eta\varsigma\xi\varsigma\}}^d(x)$

$$= 4tr[\bar{C}\gamma_a(\varsigma) \mathbb{X}^c(x)] tr[\bar{C}\gamma_b(\varsigma) \mathbb{X}^d(x)] + 4tr[\bar{C}\gamma_a(\varsigma) \mathbb{X}^d(x)] tr[\bar{C}\gamma_b(\varsigma) \mathbb{X}^c(x)] + 8tr[\bar{C}\gamma_a(\varsigma) \mathbb{X}^c(x) \bar{C}\gamma_b(\varsigma) \mathbb{X}^d(x)] + 8tr[\bar{C}\gamma_a(\varsigma) \mathbb{X}^d(x) \bar{C}\gamma_b(\varsigma) \mathbb{X}^c(x)]$$

$$\begin{aligned}
&= 4tr[\bar{C}\gamma_a(\zeta)\mathbb{X}_c(x)]tr[\bar{C}\gamma_b(\zeta)\mathbb{X}_d(x)] + 4tr[\bar{C}\gamma_a(\zeta)\mathbb{X}_d(x)]tr[\bar{C}\gamma_b(\zeta)\mathbb{X}_c(x)] + 16tr[\bar{C}\gamma_a(\zeta)\mathbb{X}_c(x)\bar{C}\gamma_b(\zeta)\mathbb{X}_d(x)] \\
&= -64m^2\delta_{ac}\delta_{bd} - 64m^2\delta_{ad}\delta_{bc} - 64m^2\{\delta_{ac}\delta_{bd} - \delta_{a[b}\delta_{d]c}\} + 64(\delta_{ab}S_{cedf} + \delta_{\{a[c}S_{e]b\}df})\partial^e\partial^f \\
&= 64m^2(\delta_{ab}\delta_{cd} - 2\delta_{c\{a}\delta_{b\}d}) + 64(\delta_{ab}S_{cedf} + \delta_{\{a[c}S_{e]b\}df})\partial^e\partial^f
\end{aligned}$$

□

$$\begin{aligned}
&\text{引理4.1.5. } [\gamma_{a'}(\zeta)C]^{\lambda'_\zeta\mu'_\zeta}[\gamma_{b'}(\zeta)C]^{\eta'_\zeta\xi'_\zeta}\mathbb{X}_{\{\lambda'_\zeta\mu'_\zeta\}}^{+c'}(x)\mathbb{X}_{\{\eta'_\zeta\xi'_\zeta\}}^{+d'}(x') \\
&= 64m^2(\delta_{a'b'}\delta_{c'd'} - 2\delta_{c'\{a'}\delta_{b'}d'}) + 64(\delta_{a'b'}S_{c'e'd'f'} + \delta_{\{a'[c'S_{e']b'}\}d'f'})\partial'^{+e'}\partial'^{+f'}
\end{aligned}$$

### 4.1.3 系列计算I

$$\begin{aligned}
&\text{引理4.1.6. } \delta_{\{a[c}S_{e]b\}df}\bar{\delta}_{\{a'[c'S_{e']b'}\}d'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= [m^2\eta_{\{a(a'} - \partial_{\{a'}\partial_{a'}^+\})}[m^2\eta_{b\}b') - \partial_b\partial_b^+] + 4(m^2\delta_{ab} + \partial_a\partial_b)(m^2\delta_{a'b'} + \partial_a^+\partial_b^+) - 4m^4\delta_{ab}\delta_{a'b'}
\end{aligned}$$

$$\begin{aligned}
&\text{证明: } \delta_{ac}S_{ebdf}\delta_{a'e'}S_{e'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= \delta_{ac}\delta_{a'e'}(\delta_{ed}\delta_{fb} - \delta_{ef}\delta_{db})(\delta_{e'd'}\delta_{f'b'} - \delta_{e'f'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= \delta_{ac}\delta_{a'e'}(\partial_d\partial_b - m^2\delta_{db})(\partial_{d'}^+\partial_{b'}^+ - m^2\delta_{d'b'})\eta^{cc'}\eta^{dd'} \\
&= m^4(\eta_{aa'}\eta_{bb'} - \eta_{aa'}\frac{\partial_b\partial_b^+}{m^2})
\end{aligned}$$

□

$$\begin{aligned}
&\text{证明: } \delta_{ae}S_{cbdf}\delta_{a'e'}S_{c'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= \delta_{ae}\delta_{a'e'}(\delta_{cd}\delta_{fb} - \delta_{cf}\delta_{db})(\delta_{c'd'}\delta_{f'b'} - \delta_{c'f'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= \partial_a\partial_a^+(\delta_{cd}\partial_b - \partial_c\delta_{db})(\delta_{c'd'}\partial_{b'}^+ - \partial_{c'}^+\delta_{d'b'})\eta^{cc'}\eta^{dd'} \\
&= m^2\partial_a\partial_a^+(\eta_{bb'} + \frac{2\partial_b\partial_b^+}{m^2})
\end{aligned}$$

□

$$\begin{aligned}
&\text{证明: } -\delta_{ac}S_{ebdf}\delta_{a'e'}S_{c'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= -\delta_{ac}\delta_{a'e'}(\delta_{ed}\delta_{fb} - \delta_{ef}\delta_{db})(\delta_{c'd'}\delta_{f'b'} - \delta_{c'f'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= -\delta_{ac}\partial_a^+(\partial_d\partial_b - m^2\delta_{db})(\delta_{c'd'}\partial_{b'}^+ - \partial_{c'}^+\delta_{d'b'})\eta^{cc'}\eta^{dd'} \\
&= m^2(\delta_{ab}\partial_a^+\partial_{b'}^+ - \eta_{bb'}\partial_a\partial_a^+)
\end{aligned}$$

□

$$\text{证明: } -\delta_{ae}S_{cbdf}\delta_{a'e'}S_{c'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = m^2(\delta_{a'b'}\partial_a\partial_b - \eta_{bb'}\partial_a\partial_a^+)$$

□

$$\begin{aligned}
&\text{证明: } \delta_{a[c}S_{e]bdf}\delta_{a'[c'S_{e']b'}\}d'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= m^4(\eta_{aa'}\eta_{bb'} - \eta_{aa'}\frac{\partial_b\partial_b^+}{m^2}) + m^2\partial_a\partial_a^+(\eta_{bb'} + \frac{2\partial_b\partial_b^+}{m^2}) + m^2(\delta_{ab}\partial_a^+\partial_{b'}^+ - \eta_{bb'}\partial_a\partial_a^+) + m^2(\delta_{a'b'}\partial_a\partial_b - \eta_{bb'}\partial_a\partial_a^+) \\
&= (m^4\eta_{aa'}\eta_{bb'} + 2\partial_a\partial_a^+\partial_b\partial_b^+) + m^2(\delta_{ab}\partial_a^+\partial_{b'}^+ - \eta_{bb'}\partial_a\partial_a^+) + m^2(\delta_{a'b'}\partial_a\partial_b - \eta_{aa'}\partial_b\partial_b^+) \\
&= (m^2\eta_{aa'} - \partial_a\partial_a^+)(m^2\eta_{bb'} - \partial_b\partial_b^+) + (m^2\delta_{ab} + \partial_a\partial_b)(m^2\delta_{a'b'} + \partial_a^+\partial_{b'}^+) - m^4\delta_{ab}\delta_{a'b'}
\end{aligned}$$

□

### 4.1.4 系列计算II

$$\text{引理4.1.7. } \delta_{ab}S_{cedf}\delta_{a'b'}S_{c'e'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = 3m^4\delta_{ab}\delta_{a'b'}$$

$$\begin{aligned}
&\text{证明: } \delta_{ab}S_{cedf}\delta_{a'b'}S_{c'e'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= \delta_{ab}\delta_{a'b'}(\delta_{cd}\delta_{fe} - \delta_{cf}\delta_{de})(\delta_{c'd'}\delta_{f'e'} - \delta_{c'f'}\delta_{d'e'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= \delta_{ab}\delta_{a'b'}(\delta_{cd}m^2 - \partial_c\partial_d)(\delta_{c'd'}m^2 - \partial_{c'}^+\partial_{d'}^+)\eta^{cc'}\eta^{dd'} \\
&= 3m^4\delta_{ab}\delta_{a'b'}
\end{aligned}$$

□

### 4.1.5 系列计算III

引理4.1.8.

$$\begin{aligned}
&\delta_{ab}S_{cedf}\delta_{\{a'[c'S_{e']b'}\}d'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = -2m^4(\delta_{ab}\delta_{a'b'} + 2\delta_{ab}\frac{\partial_a^+\partial_b^+}{m^2}) \\
&\delta_{a'b'}S_{c'e'd'f'}\delta_{\{a[c}S_{e]b\}df}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = -2m^4(\delta_{ab}\delta_{a'b'} + 2\delta_{a'b'}\frac{\partial_a\partial_b}{m^2})
\end{aligned}$$

$$\begin{aligned}
&\text{证明: } \delta_{ab}S_{cedf}\delta_{a'e'}S_{e'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= \delta_{ab}\delta_{a'e'}(\delta_{cd}\delta_{fe} - \delta_{cf}\delta_{de})(\delta_{e'd'}\delta_{f'b'} - \delta_{e'f'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} \\
&= \delta_{ab}\delta_{a'e'}(\delta_{cd}m^2 - \partial_c\partial_d)(\partial_{d'}^+\partial_{b'}^+ - m^2\delta_{d'b'})\eta^{cc'}\eta^{dd'} \\
&= -m^4(\delta_{ab}\delta_{a'b'} - \delta_{ab}\frac{\partial_a^+\partial_b^+}{m^2})
\end{aligned}$$

□

证明:  $-\delta_{ab}S_{cedf}\delta_{a'e'}S_{c'b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$

$$= -\delta_{ab}\delta_{a'e'}(\delta_{cd}\delta_{fe} - \delta_{cf}\delta_{de})(\delta_{c'd'}\delta_{f'b'} - \delta_{c'f'}\delta_{d'b'})\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$$

$$= -\delta_{ab}(\delta_{cd}m^2 - \partial_c\partial_d)(\delta_{c'd'}\partial_a^+\partial_b^+ - \partial_a^+\partial_{c'}^+\delta_{d'b'})\eta^{cc'}\eta^{dd'}$$

$$= -3m^2\delta_{ab}\partial_a^+\partial_b^+ \quad \square$$

$$\text{证明: } \delta_{ab}S_{cedf}\delta_{a'[c'S_{e'}]b'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = -m^4(\delta_{ab}\delta_{a'b'} + 2\delta_{ab}\frac{\partial_a^+\partial_b^+}{m^2}) \quad \square$$

$$\text{证明: } \delta_{ab}S_{cedf}\delta_{b'[c'S_{e'}]a'd'f'}\eta^{cc'}\eta^{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'} = -m^4(\delta_{ab}\delta_{a'b'} + 2\delta_{ab}\frac{\partial_a^+\partial_b^+}{m^2}) \quad \square$$

#### 4.1.6 系列计算IV

引理4.1.9.  $(\delta_{ab}S_{cedf} + \delta_{\{a[cS_{e}]b\}df})(\delta_{a'b'}S_{c'e'd'f'} + \delta_{\{a'[c'S_{e'}]b'\}d'f'})\eta_{cc'}\eta_{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$

$$= -4m^4(\frac{1}{4}\delta_{ab}\delta_{a'b'} - \frac{\partial_a\partial_b\partial_a^+\partial_b^+}{m^4}) + m^2[\eta_{\{a(a'} - \frac{\partial_{\{a}\partial_{(a'}}}{m^2}})[\eta_{b\}b')] - \frac{\partial_b\partial_b^+}{m^2}]$$

证明:  $(\delta_{ab}S_{cedf} + \delta_{\{a[cS_{e}]b\}df})(\delta_{a'b'}S_{c'e'd'f'} + \delta_{\{a'[c'S_{e'}]b'\}d'f'})\eta_{cc'}\eta_{dd'}\partial^e\partial^f\partial^{+e'}\partial^{+f'}$

$$= 3m^4\delta_{ab}\delta_{a'b'} - 2m^2(m^2\delta_{ab}\delta_{a'b'} + 2\delta_{ab}\partial_a^+\partial_b^+) - 2m^2(m^2\delta_{ab}\delta_{a'b'} + 2\delta_{a'b'}\partial_a\partial_b) + [m^2\eta_{\{a(a'} - \partial_{\{a}\partial_{(a'}}^+)[m^2\eta_{b\}b')] - \partial_b\partial_b^+] +$$

$$4(m^2\delta_{ab} + \partial_a\partial_b)(m^2\delta_{a'b'} + \partial_a^+\partial_b^+) - 4m^4\delta_{ab}\delta_{a'b'}$$

$$= -4m^4(\frac{1}{4}\delta_{ab}\delta_{a'b'} - \frac{\partial_a\partial_b\partial_a^+\partial_b^+}{m^4}) + m^2(\eta_{\{a(a'} - \frac{\partial_{\{a}\partial_{(a'}}^+}{m^2}})(\eta_{b\}b') - \frac{\partial_b\partial_b^+}{m^2}) \quad \square$$

#### 4.1.7 $A_{ab}$ 对易关系的证明

定理4.1.1.

$$\begin{cases} [A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a\partial_{a'}^+}{m^2})\Delta(x-x'), \bar{A}_{a'b'} := \eta_{a'}^c\eta_b^d A_{c'd}^+ \\ [A_{ab}(x), A_{a'b'}^+(x')] = \frac{i}{8}\{[\eta_{\{a(a'} - \frac{\partial_{\{a}\partial_{(a'}}^+}{m^2}})[\eta_{b\}b')] - \frac{\partial_b\partial_b^+}{m^2}\} - \frac{1}{3}[\delta_{\{ab\}} - \frac{\partial_{\{a}\partial_b\}}{m^2}][\delta_{(a'b')} - \frac{\partial_{(a'}\partial_{b')}}{m^2}]\}\Delta(x-x') \\ [A_{ab}(x), \bar{A}_{a'b'}(x')] = \frac{i}{8}\{[\delta_{\{a(a'} - \frac{\partial_{\{a}\partial_{(a'}}^+}{m^2}})[\delta_{b\}b')] - \frac{\partial_b\partial_b^+}{m^2}\} - \frac{1}{3}[\delta_{\{ab\}} - \frac{\partial_{\{a}\partial_b\}}{m^2}][\delta_{(a'b')} - \frac{\partial_{(a'}\partial_{b')}}{m^2}]\}\Delta(x-x') \end{cases}$$

证明:  $[A_{ab}(x), A_{a'b'}^+(x')]$

$$= \frac{1}{(i2m)^4}[\bar{C}\gamma_a(\varsigma)]^{\lambda\mu\varsigma}[\bar{C}\gamma_b(\varsigma)]^{\eta\kappa\xi\varsigma}[\gamma_{a'}(\varsigma)C]^{\lambda'\mu'\varsigma}[\gamma_{b'}(\varsigma)C]^{\eta'\kappa'\xi'\varsigma}[\psi_{\lambda\mu\varsigma\eta\kappa\xi\varsigma}(x), \psi_{\lambda'\mu'\eta'\kappa'\xi'\varsigma}^+(x')]$$

$$= \frac{1}{(i2m)^4} \frac{i}{2^5(4!)^2}[\bar{C}\gamma_a(\varsigma)]^{\lambda\mu\varsigma}[\bar{C}\gamma_b(\varsigma)]^{\eta\kappa\xi\varsigma}[\gamma_{a'}(\varsigma)C]^{\lambda'\mu'\varsigma}[\gamma_{b'}(\varsigma)C]^{\eta'\kappa'\xi'\varsigma}\mathbb{X}_{\{\lambda\mu\varsigma}^c}^d(x)\mathbb{X}_{\eta\kappa\xi\varsigma}^+(x')\mathbb{X}_{\lambda'\mu'}^+(x')\mathbb{X}_{\eta'\kappa'\xi'\varsigma}^+(x')$$

$$(\eta_{cc'} - \frac{\partial_c\partial_{c'}^+}{m^2})(\eta_{dd'} - \frac{\partial_d\partial_{d'}^+}{m^2})\Delta(x-x')$$

$$= \frac{1}{m^4} \frac{i}{2^{13}(3!)^2}[\bar{C}\gamma_a(\varsigma)]^{\lambda\mu\varsigma}[\bar{C}\gamma_b(\varsigma)]^{\eta\kappa\xi\varsigma}\mathbb{X}_{\{\lambda\mu\varsigma}^c}^d(x)\mathbb{X}_{\eta\kappa\xi\varsigma}^d(x)[\gamma_{a'}(\varsigma)C]^{\lambda'\mu'\varsigma}[\gamma_{b'}(\varsigma)C]^{\eta'\kappa'\xi'\varsigma}\mathbb{X}_{\lambda'\mu'}^+(x')\mathbb{X}_{\eta'\kappa'\xi'\varsigma}^+(x')$$

$$(\eta_{cc'} - \frac{\partial_c\partial_{c'}^+}{m^2})(\eta_{dd'} - \frac{\partial_d\partial_{d'}^+}{m^2})\Delta(x-x')$$

$$= \frac{1}{m^4} \frac{i}{2^{13}(3!)^2}\{64m^2(\delta_{ab}\delta_{cd} - 2\delta_{c\{a\}d\}b) + 64(\delta_{ab}S_{cedf} + \delta_{\{a[cS_{e}]b\}df})\partial^e\partial^f\}$$

$$\{64m^2(\delta_{a'b'}\delta_{c'd'} - 2\delta_{c'(a'}\delta_{b')d'}) + 64(\delta_{a'b'}S_{c'e'd'f'} + \delta_{\{a'[c'S_{e'}]b'\}d'f'})\partial^{+e'}\partial^{+f'}\}$$

$$(\eta_{cc'} - \frac{\partial_c\partial_{c'}^+}{m^2})(\eta_{dd'} - \frac{\partial_d\partial_{d'}^+}{m^2})\Delta(x-x')$$

$$= \frac{i}{72m^4}\{m^2(\delta_{ab}\delta_{cd} - 2\delta_{c\{a\}d\}b) + (\delta_{ab}S_{cedf} + \delta_{\{a[cS_{e}]b\}df})\partial^e\partial^f\}$$

$$\{m^2(\delta_{a'b'}\delta_{c'd'} - 2\delta_{c'(a'}\delta_{b')d'}) + (\delta_{a'b'}S_{c'e'd'f'} + \delta_{\{a'[c'S_{e'}]b'\}d'f'})\partial^{+e'}\partial^{+f'}\}$$

$$(\eta_{cc'} - \frac{\partial_c\partial_{c'}^+}{m^2})(\eta_{dd'} - \frac{\partial_d\partial_{d'}^+}{m^2})\Delta(x-x')$$

$$= \frac{1}{36}\{-2i(\frac{1}{4}\delta_{ab}\delta_{a'b'} - \frac{\partial_a\partial_b\partial_a^+\partial_b^+}{m^4}) - 2i(\delta_{ab} - \frac{\partial_a\partial_b}{m^2})(\delta_{a'b'} - \frac{\partial_{a'}\partial_{b'}}{m^2}) + 2i(\eta_{\{a(a'} - \frac{\partial_{\{a}\partial_{(a'}}^+}{m^2}})(\eta_{b\}b') - \frac{\partial_b\partial_b^+}{m^2})\}$$

$$+ 4i(\frac{1}{4}\delta_{ab}\delta_{a'b'} - \frac{\partial_a\partial_b\partial_a^+\partial_b^+}{m^4}) - 4i(\delta_{ab} - \frac{\partial_a\partial_b}{m^2})(\delta_{a'b'} - \frac{\partial_{a'}\partial_{b'}}{m^2}) + 2i(\eta_{\{a(a'} - \frac{\partial_{\{a}\partial_{(a'}}^+}{m^2}})(\eta_{b\}b') - \frac{\partial_b\partial_b^+}{m^2})$$

$$- 2i(\frac{1}{4}\delta_{ab}\delta_{a'b'} - \frac{\partial_a\partial_b\partial_a^+\partial_b^+}{m^4}) + \frac{i}{2}(\eta_{\{a(a'} - \frac{\partial_{\{a}\partial_{(a'}}^+}{m^2}})(\eta_{b\}b') - \frac{\partial_b\partial_b^+}{m^2})\Delta(x-x')$$

$$= \frac{i}{8}\{[\eta_{\{a(a'} - \frac{\partial_{\{a}\partial_{(a'}}^+}{m^2}})[\eta_{b\}b')] - \frac{\partial_b\partial_b^+}{m^2}\} - \frac{1}{3}[\delta_{\{ab\}} - \frac{\partial_{\{a}\partial_b\}}{m^2}][\delta_{(a'b')} - \frac{\partial_{(a'}\partial_{b')}}{m^2}]\}\Delta(x-x') \quad \square$$

#### 4.2 反过来验证场对易关系

引理4.2.1.  $\psi_{\lambda\mu\varsigma\eta\kappa\xi\varsigma} = \frac{1}{4}\mathbb{X}_{\lambda\mu\varsigma}^a\mathbb{X}_{\eta\kappa\xi\varsigma}^b A_{ab}, \psi_{\lambda'\mu'\eta'\kappa'\xi'\varsigma}^+ = \frac{1}{4}\mathbb{X}_{\lambda'\mu'}^{a'}\mathbb{X}_{\eta'\kappa'\xi'\varsigma}^{b'} A_{a'b'}$

引理4.2.2.  $\mathbb{X}_{\{\lambda\mu\varsigma}^a}^b\mathbb{X}_{\eta\kappa\xi\varsigma}^c[\delta_{ab} - \frac{\partial_a\partial_b}{m^2}] =$

$$\begin{bmatrix} \frac{1}{2}(\sigma, i\varsigma)_{[a}(\sigma, -i\varsigma)_{c]}\sigma_y\partial^c & im(\sigma, i\varsigma)_{a\sigma_y} \\ im(\sigma, -i\varsigma)_{a\sigma_y} & -\frac{1}{2}(\sigma, -i\varsigma)_{[a}(\sigma, i\varsigma)_{c]}\sigma_y\partial^c \end{bmatrix}_{\{\lambda\mu\varsigma} \begin{bmatrix} \frac{1}{2}(\sigma, i\varsigma)_{[b}(\sigma, -i\varsigma)_{d]}\sigma_y\partial^d & im(\sigma, i\varsigma)_{b\sigma_y} \\ im(\sigma, -i\varsigma)_{b\sigma_y} & -\frac{1}{2}(\sigma, -i\varsigma)_{[b}(\sigma, i\varsigma)_{d]}\sigma_y\partial^d \end{bmatrix}_{\eta\kappa\xi\varsigma} [\delta^{ab} - \frac{\partial^a\partial^b}{m^2}]$$

$$= 0$$

$$\text{定理4.2.1. } [\psi_{\lambda_c \mu_c \eta_c \xi_c}, \psi_{\lambda'_c \mu'_c \eta'_c \xi'_c}] = \frac{i}{2^5(4!)^2} \mathbb{X}_{\{\lambda_c \mu_c\}}^a \mathbb{X}_{\{\eta_c \xi_c\}}^b \mathbb{X}_{\{\lambda'_c \mu'_c\}}^{+a'} \mathbb{X}_{\{\eta'_c \xi'_c\}}^{+b'} [\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] [\eta_{bb'} - \frac{\partial_b \partial_{b'}}{m^2}] \Delta(x - x')$$

$$\begin{aligned} \text{证明: } & [\psi_{\lambda_c \mu_c \eta_c \xi_c}, \psi_{\lambda'_c \mu'_c \eta'_c \xi'_c}] \\ &= \frac{1}{2^4} \mathbb{X}_{\lambda_c \mu_c}^a \mathbb{X}_{\eta_c \xi_c}^b \mathbb{X}_{\lambda'_c \mu'_c}^{+a'} \mathbb{X}_{\eta'_c \xi'_c}^{+b'} [A_{ab}, A_{a'b'}^+] \\ &= \frac{1}{2^4(4!)^2} \mathbb{X}_{\{\lambda_c \mu_c\}}^a \mathbb{X}_{\{\eta_c \xi_c\}}^b \mathbb{X}_{\{\lambda'_c \mu'_c\}}^{+a'} \mathbb{X}_{\{\eta'_c \xi'_c\}}^{+b'} [A_{ab}, A_{a'b'}^+] \\ &= \frac{i}{2^5(4!)^2} \mathbb{X}_{\{\lambda_c \mu_c\}}^a \mathbb{X}_{\{\eta_c \xi_c\}}^b \mathbb{X}_{\{\lambda'_c \mu'_c\}}^{+a'} \mathbb{X}_{\{\eta'_c \xi'_c\}}^{+b'} \{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] [\eta_{bb'} - \frac{\partial_b \partial_{b'}}{m^2}] - \frac{1}{3} [\delta_{ab} - \frac{\partial_a \partial_b}{m^2}] [\delta_{a'b'} - \frac{\partial_{a'} \partial_{b'}}{m^2}]\} \Delta(x - x') \\ &= \frac{i}{2^5(4!)^2} \mathbb{X}_{\{\lambda_c \mu_c\}}^a \mathbb{X}_{\{\eta_c \xi_c\}}^b \mathbb{X}_{\{\lambda'_c \mu'_c\}}^{+a'} \mathbb{X}_{\{\eta'_c \xi'_c\}}^{+b'} [\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}] [\eta_{bb'} - \frac{\partial_b \partial_{b'}}{m^2}] \Delta(x - x') \quad \square \end{aligned}$$

### 4.3 有质量引力子场能量动量算符的提取

$$\text{定理4.3.1. } P_u(2) = \int \psi^{+\lambda_c \mu_c \eta_c \xi_c}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} \psi_{\lambda_c \mu_c \eta_c \xi_c}(\vec{r}, t) d^3 \vec{r}$$

$$\text{定理4.3.2. } P_u(2) = \int m^4 A^{+ab}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} A_{ab}(\vec{r}, t) d^3 \vec{r} + \int m^2 F^{+abc}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} F_{abc}(\vec{r}, t) d^3 \vec{r} + \frac{1}{4} \int F^{+abcd}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} F_{abcd}(\vec{r}, t) d^3 \vec{r}$$

$$\begin{aligned} \text{证明: } & P_u(2) = \int \psi^{+\lambda_c \mu_c \eta_c \xi_c}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} \psi_{\lambda_c \mu_c \eta_c \xi_c}(\vec{r}, t) d^3 \vec{r} \\ &= \int \{ \bar{C}[-im\gamma^{a'}(\zeta) - 2S^{a'c'}(e, \zeta)\partial_{c'}^+] \}_{\lambda_c \mu_c} \{ \bar{C}[-im\gamma^{b'}(\zeta) - 2S^{b'd'}(e, \zeta)\partial_{d'}^+] \}_{\eta_c \xi_c} \frac{A_{a'b'}^+(\vec{r}, t)}{4} \\ &\quad \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} [im\gamma^a(\zeta)C - 2S^{ac}(e, \zeta)C\partial_c]_{\lambda_c \mu_c} [im\gamma^b(\zeta)C - 2S^{bd}(e, \zeta)C\partial_d]_{\eta_c \xi_c} \frac{A_{ab}(\vec{r}, t)}{4} d^3 \vec{r} \\ &= \frac{1}{16} \int \text{tr} \{ \bar{C}[-im\gamma^{a'}(\zeta) - 2S^{a'c'}(e, \zeta)\partial_{c'}^+] \{ \bar{C}[-im\gamma^{b'}(\zeta) - 2S^{b'd'}(e, \zeta)\partial_{d'}^+] \}_{\eta_c \xi_c} A_{a'b'}^+(\vec{r}, t) \\ &\quad \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} [im\gamma^a(\zeta)C - 2S^{ac}(e, \zeta)C\partial_c] [im\gamma^b(\zeta)C - 2S^{bd}(e, \zeta)C\partial_d]_{\eta_c \xi_c} A_{ab}(\vec{r}, t) \} d^3 \vec{r} \\ &= \frac{1}{16} \int \text{tr} \{ \bar{C}[-im\gamma^{a'}(\zeta) - 2S^{a'c'}(e, \zeta)\partial_{c'}^+] \{ \bar{C}[-im\gamma^{b'}(\zeta) - 2S^{b'd'}(e, \zeta)\partial_{d'}^+] \}_{\eta_c \xi_c} A_{a'b'}^+(\vec{r}, t) \\ &\quad \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} [im\gamma^a(\zeta)C - 2S^{ac}(e, \zeta)C\partial_c] [im\gamma^b(\zeta)C - 2S^{bd}(e, \zeta)C\partial_d]_{\eta_c \xi_c} A_{ab}(\vec{r}, t) \} d^3 \vec{r} \\ &= \frac{1}{4} \int \{ \bar{C}[-im\gamma^{b'}(\zeta) - 2S^{b'd'}(e, \zeta)\partial_{d'}^+] \}_{\eta_c \xi_c} A_{a'b'}^+(\vec{r}, t) m^2 \delta^{a'a} \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} [im\gamma^b(\zeta)C - 2S^{bd}(e, \zeta)C\partial_d]_{\eta_c \xi_c} A_{ab}(\vec{r}, t) \} d^3 \vec{r} \\ &\quad + \frac{1}{4} \int S^{a'c'ac} \partial_{c'}^+ \{ \bar{C}[-im\gamma^{b'}(\zeta) - 2S^{b'd'}(e, \zeta)\partial_{d'}^+] \}_{\eta_c \xi_c} A_{a'b'}^+(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} \partial_c [im\gamma^b(\zeta)C - 2S^{bd}(e, \zeta)C\partial_d]_{\eta_c \xi_c} A_{ab}(\vec{r}, t) \} d^3 \vec{r} \\ &= \frac{1}{4} \int \text{tr} \{ \bar{C}[-im\gamma^{b'}(\zeta) - 2S^{b'd'}(e, \zeta)\partial_{d'}^+] A_{a'b'}^+(\vec{r}, t) m^2 \delta^{a'a} \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} [im\gamma^b(\zeta)C - 2S^{bd}(e, \zeta)C\partial_d] A_{ab}(\vec{r}, t) \} d^3 \vec{r} \\ &\quad + \frac{1}{4} \int S^{a'c'ac} \partial_{c'}^+ \text{tr} \{ \bar{C}[-im\gamma^{b'}(\zeta) - 2S^{b'd'}(e, \zeta)\partial_{d'}^+] A_{a'b'}^+(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} \partial_c [im\gamma^b(\zeta)C - 2S^{bd}(e, \zeta)C\partial_d] A_{ab}(\vec{r}, t) \} d^3 \vec{r} \\ &= \int m^4 \delta^{a'a} \delta^{b'b} A_{a'b'}^+(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} A_{ab}(\vec{r}, t) d^3 \vec{r} \\ &\quad + \int m^2 S^{b'd'bd} \delta^{a'a} \partial_{d'}^+ A_{a'b'}^+(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} \partial_d A_{ab}(\vec{r}, t) d^3 \vec{r} \\ &\quad + \int m^2 S^{a'c'ac} \delta^{b'b} \partial_{c'}^+ A_{a'b'}^+(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} \partial_c A_{ab}(\vec{r}, t) d^3 \vec{r} \\ &\quad + \int S^{a'c'ac} S^{b'd'bd} \partial_{c'}^+ \partial_{d'}^+ A_{a'b'}^+(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} \partial_c \partial_d A_{ab}(\vec{r}, t) d^3 \vec{r} \\ &= \int m^4 A^{+ab}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} A_{ab}(\vec{r}, t) d^3 \vec{r} \\ &\quad + \int \frac{1}{2} m^2 F^{+bca}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} F_{bca}(\vec{r}, t) d^3 \vec{r} + \int \frac{1}{2} m^2 F^{+acb}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} F_{acb}(\vec{r}, t) d^3 \vec{r} \\ &\quad + \int S^{b'd'bd} \partial_{d'}^+ F_{b'}^{+ac}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} \partial_d F_{acb}(\vec{r}, t) d^3 \vec{r} \\ &= \int m^4 A^{+ab}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} A_{ab}(\vec{r}, t) d^3 \vec{r} \\ &\quad + \int \frac{1}{2} m^2 F^{+bca}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} F_{bca}(\vec{r}, t) d^3 \vec{r} + \int \frac{1}{2} m^2 F^{+acb}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} F_{acb}(\vec{r}, t) d^3 \vec{r} \\ &\quad + \int \frac{1}{4} F^{+acbd}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} F_{acbd}(\vec{r}, t) d^3 \vec{r} \\ &= \int m^4 A^{+ab}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} A_{ab}(\vec{r}, t) d^3 \vec{r} + \int m^2 F^{+abc}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} F_{abc}(\vec{r}, t) d^3 \vec{r} \\ &\quad + \frac{1}{4} \int F^{+abcd}(\vec{r}, t) \frac{(-i\partial_u)(i\partial_t)}{(m^2 - \nabla^2)^2} F_{abcd}(\vec{r}, t) d^3 \vec{r} \quad \square \end{aligned}$$

$$\text{性质4.3.1. } F_{ab|cd} = \partial_a F_{b|cd} - \partial_b F_{a|cd} = \partial_c F_{ab|d} - \partial_d F_{ab|c} = \partial_a \partial_c A_{bd} + \partial_b \partial_d A_{ac} - \partial_a \partial_d A_{bc} - \partial_b \partial_c A_{ad}$$

$$\begin{aligned} \text{证明: } & F_{ab|cd} = \partial_a F_{b|cd} - \partial_b F_{a|cd} \\ &= \partial_a (\partial_c A_{bd} - \partial_d A_{bc}) - \partial_b (\partial_c A_{ad} - \partial_d A_{ac}) \\ &= \partial_a \partial_c A_{bd} + \partial_b \partial_d A_{ac} - \partial_a \partial_d A_{bc} - \partial_b \partial_c A_{ad} \quad \square \end{aligned}$$

$$\begin{aligned} \text{证明: } & F_{ab|cd} = \partial_c F_{ab|d} - \partial_d F_{ab|c} \\ &= \partial_c (\partial_a A_{bd} - \partial_b A_{ad}) - \partial_d (\partial_a A_{bc} - \partial_b A_{ac}) \\ &= \partial_a \partial_c A_{bd} + \partial_b \partial_d A_{ac} - \partial_a \partial_d A_{bc} - \partial_b \partial_c A_{ad} \quad \square \end{aligned}$$

性质4.3.2.  $F_{ab|cd} = F_{cd|ab}, F_{ab|cd} = -F_{ba|cd}, F_{ab|cd} = -F_{ab|dc}, F_{ab|cd} = F_{ba|dc}$

## 5 四维时空中有质量自旋 $s = 2$ 的势对易关系求解新方法(试探还有错)

### 5.1 四维时空中有质量自旋 $s = 2$ 的Bargmann-Wigner方程势对易关系

引理5.1.1.  $K := (m - \gamma_a \partial^a) \gamma_0, \tilde{K} := CK^T \bar{C} = -\gamma_0 (m + \gamma_a \partial^a), Q := (m - \gamma_a \partial^a), \tilde{Q} := (m + \gamma_a \partial^a)$

引理5.1.2. 
$$\begin{cases} tr[\gamma_a Q \gamma_{a'} \tilde{Q}] = -tr[\gamma_5 \gamma_a Q \gamma_5 \gamma_{a'} Q] = -tr[\gamma_a \gamma_5 \tilde{Q} \gamma_{a'} \gamma_5 \tilde{Q}] \\ tr[\gamma_a Q \gamma_{a'} \tilde{Q} \gamma_b Q \gamma_{b'} \tilde{Q}] = tr[\gamma_5 \gamma_a Q \gamma_5 \gamma_{a'} Q \gamma_5 \gamma_b Q \gamma_5 \gamma_{b'} Q] = tr[\gamma_a \gamma_5 \tilde{Q} \gamma_{a'} \gamma_5 \tilde{Q} \gamma_b \gamma_5 \tilde{Q} \gamma_{b'} \gamma_5 \tilde{Q}] \end{cases}$$

引理5.1.3. 
$$\begin{aligned} tr[\gamma_a Q \gamma_{a'} \tilde{Q}] &= tr[\gamma_a (m - \gamma_{a_1} \partial^{a_1}) \gamma_{a'} (m + \gamma_{a_2} \partial^{a_2})] \\ &= m^2 tr(\gamma_a \gamma_{a'}) - tr(\gamma_a \gamma_{a_1} \gamma_{a'} \gamma_{a_2}) \partial^{a_1} \partial^{a_2} \\ &= 4m^2 \delta_{aa'} - 4(\delta_{aa_1} \delta_{a'a_2} - \delta_{aa'} \delta_{a_1 a_2} + \delta_{aa_2} \delta_{a_1 a'}) \partial^{a_1} \partial^{a_2} \\ &= 4m^2 \delta_{aa'} - 4(2\partial_a \partial_{a'} - \delta_{aa'} m^2) \\ &= 8(m^2 \delta_{aa'} - \partial_a \partial_{a'}) \end{aligned}$$

证明: 
$$\begin{aligned} tr[\gamma_a Q \gamma_{a'} \tilde{Q} \gamma_b Q \gamma_{b'} \tilde{Q}] &= tr[\gamma_a (m - \gamma_{a_1} \partial^{a_1}) \gamma_{a'} (m + \gamma_{a_2} \partial^{a_2}) \gamma_b (m - \gamma_{a_3} \partial^{a_3}) \gamma_{b'} (m + \gamma_{a_4} \partial^{a_4})] \\ &= tr[\gamma_a \gamma_5 (m + \gamma_{a_1} \partial^{a_1}) \gamma_{a'} \gamma_5 (m + \gamma_{a_2} \partial^{a_2}) \gamma_b \gamma_5 (m + \gamma_{a_3} \partial^{a_3}) \gamma_{b'} \gamma_5 (m + \gamma_{a_4} \partial^{a_4})] \\ &= tr\{ \\ &[(\gamma_a \gamma_5 m \gamma_{a'} \gamma_5 m + \gamma_a \gamma_5 \gamma_{a_1} \partial^{a_1} \gamma_{a'} \gamma_5 \gamma_{a_2} \partial^{a_2}) + (\gamma_a \gamma_5 \gamma_{a_1} \partial^{a_1} \gamma_{a'} \gamma_5 m + \gamma_a \gamma_5 m \gamma_{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})] \\ &[(\gamma_b \gamma_5 m \gamma_{b'} \gamma_5 m + \gamma_b \gamma_5 \gamma_{a_3} \partial^{a_3} \gamma_{b'} \gamma_5 \gamma_{a_4} \partial^{a_4}) + (\gamma_b \gamma_5 \gamma_{a_3} \partial^{a_3} \gamma_{b'} \gamma_5 m + \gamma_b \gamma_5 m \gamma_{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] \} \end{aligned}$$

$$\begin{aligned} &= tr[(\gamma_a \gamma_5 m \gamma_{a'} \gamma_5 m + \gamma_a \gamma_5 \gamma_{a_1} \partial^{a_1} \gamma_{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_b \gamma_5 m \gamma_{b'} \gamma_5 m + \gamma_b \gamma_5 \gamma_{a_3} \partial^{a_3} \gamma_{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] \\ &+ tr[(\gamma_a \gamma_5 \gamma_{a_1} \partial^{a_1} \gamma_{a'} \gamma_5 m + \gamma_a \gamma_5 m \gamma_{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_b \gamma_5 \gamma_{a_3} \partial^{a_3} \gamma_{b'} \gamma_5 m + \gamma_b \gamma_5 m \gamma_{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] \end{aligned}$$

$$\begin{aligned} &= tr[(\gamma_a \gamma_5 m)(\gamma_{a'} \gamma_5 m)(\gamma_b \gamma_5 m)(\gamma_{b'} \gamma_5 m)] + tr[(\gamma_a \gamma_5 \gamma_{a_1} \partial^{a_1})(\gamma_{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_b \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] \\ &+ tr[(\gamma_a \gamma_5 m)(\gamma_{a'} \gamma_5 m)(\gamma_b \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] + tr[(\gamma_a \gamma_5 \gamma_{a_1} \partial^{a_1})(\gamma_{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_b \gamma_5 m)(\gamma_{b'} \gamma_5 m)] \\ &+ tr[(\gamma_a \gamma_5 \gamma_{a_1} \partial^{a_1})(\gamma_{a'} \gamma_5 m)(\gamma_b \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{b'} \gamma_5 m)] + tr[(\gamma_a \gamma_5 m)(\gamma_{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_b \gamma_5 m)(\gamma_{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] \\ &+ tr[(\gamma_a \gamma_5 \gamma_{a_1} \partial^{a_1})(\gamma_{a'} \gamma_5 m)(\gamma_b \gamma_5 m)(\gamma_{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] + tr[(\gamma_a \gamma_5 m)(\gamma_{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_b \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{b'} \gamma_5 m)] \end{aligned}$$

□

证明: 
$$tr[\gamma_{\{a} Q \gamma_{\{a'} \tilde{Q} \gamma_{\{b} Q \gamma_{\{b'} \tilde{Q}}$$

$$\begin{aligned} &= tr[(\gamma_{\{a} \gamma_5 m)(\gamma_{\{a'} \gamma_5 m)(\gamma_{\{b} \gamma_5 m)(\gamma_{\{b'} \gamma_5 m)] + tr[(\gamma_{\{a} \gamma_5 \gamma_{a_1} \partial^{a_1})(\gamma_{\{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_{\{b} \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{\{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] \\ &+ tr[(\gamma_{\{a} \gamma_5 m)(\gamma_{\{a'} \gamma_5 m)(\gamma_{\{b} \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{\{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] + tr[(\gamma_{\{a} \gamma_5 \gamma_{a_1} \partial^{a_1})(\gamma_{\{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_{\{b} \gamma_5 m)(\gamma_{\{b'} \gamma_5 m)] \\ &+ tr[(\gamma_{\{a} \gamma_5 \gamma_{a_1} \partial^{a_1})(\gamma_{\{a'} \gamma_5 m)(\gamma_{\{b} \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{\{b'} \gamma_5 m)] + tr[(\gamma_{\{a} \gamma_5 m)(\gamma_{\{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_{\{b} \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{\{b'} \gamma_5 m)] \\ &= tr[(\gamma_{\{a} \gamma_5 m)(\gamma_{\{a'} \gamma_5 m)(\gamma_{\{b} \gamma_5 m)(\gamma_{\{b'} \gamma_5 m)] + tr[(\gamma_{\{a} \gamma_5 \gamma_{a_1} \partial^{a_1})(\gamma_{\{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_{\{b} \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{\{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] \\ &+ 2tr[(\gamma_{\{a} \gamma_5 m)(\gamma_{\{a'} \gamma_5 m)(\gamma_{\{b} \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{\{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] \\ &+ tr[(\gamma_{\{a} \gamma_5 \gamma_{a_1} \partial^{a_1})(\gamma_{\{a'} \gamma_5 m)(\gamma_{\{b} \gamma_5 \gamma_{a_3} \partial^{a_3})(\gamma_{\{b'} \gamma_5 m)] + tr[(\gamma_{\{a} \gamma_5 m)(\gamma_{\{a'} \gamma_5 \gamma_{a_2} \partial^{a_2})(\gamma_{\{b} \gamma_5 m)(\gamma_{\{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] \\ &+ 2tr[(\gamma_{\{a} \gamma_5 \gamma_{a_1} \partial^{a_1})(\gamma_{\{a'} \gamma_5 m)(\gamma_{\{b} \gamma_5 m)(\gamma_{\{b'} \gamma_5 \gamma_{a_4} \partial^{a_4})] \\ &= m^4 tr[\gamma_{\{a} \gamma_{\{a'} \gamma_{\{b} \gamma_{\{b'}}] + tr[\gamma_{\{a} \gamma_{a_1} \gamma_{\{a'} \gamma_{a_2} \gamma_{\{b} \gamma_{a_3} \gamma_{\{b'} \gamma_{a_4}}] \partial^{a_1} \partial^{a_2} \partial^{a_3} \partial^{a_4} \\ &- 2m^2 tr[\gamma_{\{a} \gamma_{\{a'} \gamma_{\{b} \gamma_{\{b'}} \gamma_{a_3} \gamma_{a_4}}] \partial^{a_3} \partial^{a_4} \\ &+ m^2 tr[\gamma_{\{a} \gamma_{a_1} \gamma_{\{a'} \gamma_{a_2} \gamma_{\{b} \gamma_{a_3} \gamma_{a_4}}] \partial^{a_1} \partial^{a_2} + m^2 tr[\gamma_{\{a} \gamma_{\{a'} \gamma_{a_2} \gamma_{\{b} \gamma_{a_3}}] \partial^{a_2} \partial^{a_4} \\ &- 2m^2 tr[\gamma_{\{a} \gamma_{a_1} \gamma_{\{a'} \gamma_{a_2} \gamma_{\{b} \gamma_{a_4}}] \partial^{a_1} \partial^{a_4} \\ &= m^4 tr[\gamma_{\{a} \gamma_{\{a'} \gamma_{\{b} \gamma_{\{b'}}] + tr[\gamma_{\{a} \gamma_{a_1} \gamma_{\{a'} \gamma_{a_2} \gamma_{\{b} \gamma_{a_3} \gamma_{a_4}}] \partial^{a_1} \partial^{a_2} \partial^{a_3} \partial^{a_4} \\ &- 2m^2 tr[\gamma_{\{a} \gamma_{\{a'} \gamma_{\{b} \gamma_{\{b'}} \gamma_{a_1} \gamma_{a_2}}] \partial^{a_1} \partial^{a_2} \\ &+ m^2 tr[\gamma_{\{a} \gamma_{a_1} \gamma_{\{a'} \gamma_{a_2} \gamma_{\{b} \gamma_{a_3}}] \partial^{a_1} \partial^{a_2} + m^2 tr[\gamma_{\{a} \gamma_{\{a'} \gamma_{a_1} \gamma_{\{b} \gamma_{a_2}}] \partial^{a_1} \partial^{a_2} \\ &- 2m^2 tr[\gamma_{\{a} \gamma_{a_1} \gamma_{\{a'} \gamma_{a_2} \gamma_{\{b} \gamma_{a_2}}] \partial^{a_1} \partial^{a_2} \end{aligned}$$

□

$$\begin{aligned}
& \text{引理5.1.4. } \text{tr}[\gamma_{\{a}\gamma_{a_1}\gamma(a'\gamma_{a_2}\gamma_b)\gamma_{a_3}\gamma_{b'}}\gamma_{a_4}] \partial^{a_1} \partial^{a_2} \partial^{a_3} \partial^{a_4} \\
& = [\gamma_{\{a_1}\gamma_{a_2}\gamma(a'_3\gamma_{a_4}\gamma_{a_5})\gamma_{a_6}\gamma_{a_7}\}\gamma_{a_8}] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
& = \delta_{a_1 a_2} [\delta_{a_3 a_4} (\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8} + \delta_{a_5 a_8} \delta_{a_6 a_7}) - \delta_{a_3 a_5} (\delta_{a_4 a_6} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_6 a_8} + \delta_{a_4 a_8} \delta_{a_6 a_7}) \\
& + \delta_{a_3 a_6} (\delta_{a_4 a_5} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_5 a_8} + \delta_{a_4 a_8} \delta_{a_5 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
& - \delta_{a_1 a_3} [\delta_{a_2 a_4} (\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8} + \delta_{a_5 a_8} \delta_{a_6 a_7}) - \delta_{a_2 a_5} (\delta_{a_4 a_6} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_6 a_8} + \delta_{a_4 a_8} \delta_{a_6 a_7}) \\
& + \delta_{a_2 a_6} (\delta_{a_4 a_5} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_5 a_8} + \delta_{a_4 a_8} \delta_{a_5 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
& + \delta_{a_1 a_4} [\delta_{a_2 a_3} (\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8} + \delta_{a_5 a_8} \delta_{a_6 a_7}) - \delta_{a_2 a_5} (\delta_{a_3 a_6} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_6 a_8} + \delta_{a_3 a_8} \delta_{a_6 a_7}) \\
& + \delta_{a_2 a_6} (\delta_{a_3 a_5} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_5 a_8} + \delta_{a_3 a_8} \delta_{a_5 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
& - \delta_{a_1 a_5} [\delta_{a_2 a_3} (\delta_{a_4 a_6} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_6 a_8} + \delta_{a_4 a_8} \delta_{a_6 a_7}) - \delta_{a_2 a_4} (\delta_{a_3 a_6} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_6 a_8} + \delta_{a_3 a_8} \delta_{a_6 a_7}) \\
& + \delta_{a_2 a_6} (\delta_{a_3 a_4} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_4 a_8} + \delta_{a_3 a_8} \delta_{a_4 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
& + \delta_{a_1 a_6} [\delta_{a_2 a_3} (\delta_{a_4 a_5} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_5 a_8} + \delta_{a_4 a_8} \delta_{a_5 a_7}) - \delta_{a_2 a_4} (\delta_{a_3 a_5} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_5 a_8} + \delta_{a_3 a_8} \delta_{a_5 a_7}) \\
& + \delta_{a_2 a_5} (\delta_{a_3 a_4} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_4 a_8} + \delta_{a_3 a_8} \delta_{a_4 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
& - \delta_{a_1 a_7} [\delta_{a_2 a_3} (\delta_{a_4 a_5} \delta_{a_6 a_8} - \delta_{a_4 a_6} \delta_{a_5 a_8} + \delta_{a_4 a_8} \delta_{a_5 a_6}) - \delta_{a_2 a_4} (\delta_{a_3 a_5} \delta_{a_6 a_8} - \delta_{a_3 a_6} \delta_{a_5 a_8} + \delta_{a_3 a_8} \delta_{a_5 a_6}) \\
& + \delta_{a_2 a_6} (\delta_{a_3 a_4} \delta_{a_6 a_8} - \delta_{a_3 a_6} \delta_{a_4 a_8} + \delta_{a_3 a_8} \delta_{a_4 a_6})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
& + \delta_{a_1 a_8} [\delta_{a_2 a_3} (\delta_{a_4 a_5} \delta_{a_6 a_7} - \delta_{a_4 a_6} \delta_{a_5 a_7} + \delta_{a_4 a_7} \delta_{a_5 a_6}) - \delta_{a_2 a_4} (\delta_{a_3 a_5} \delta_{a_6 a_7} - \delta_{a_3 a_6} \delta_{a_5 a_7} + \delta_{a_3 a_7} \delta_{a_5 a_6}) \\
& + \delta_{a_2 a_5} (\delta_{a_3 a_4} \delta_{a_6 a_7} - \delta_{a_3 a_6} \delta_{a_4 a_7} + \delta_{a_3 a_7} \delta_{a_4 a_6})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8}
\end{aligned}$$

$$\begin{aligned}
& \text{定理5.1.1. } [A_{ab\dots}(x), A_{a'b'\dots}^+(x')] = \frac{1}{m^{2n}} \frac{i}{2^{4n-1} [(2n)!]^2} \overbrace{(\bar{C}\gamma_a)^{\lambda_c \mu_c} (\bar{C}\gamma_b)^{\eta_c \xi_c} \dots (\gamma_{a'} C)^{\lambda'_c \mu'_c} (\gamma_{b'} C)^{\eta'_c \xi'_c} \dots}^n \\
& \underbrace{[(m - \gamma^a \partial_a) \gamma^A]_{\{\lambda_c \lambda'_c [(m - \gamma^b \partial_b) \gamma^A]_{\mu_c \mu'_c} \dots\}}}_{2n} \Delta(x - x')
\end{aligned}$$

$$\begin{aligned}
& \text{定理5.1.2. } [A_{ab}(x), A_{a'b'}^+(x')] = \frac{1}{m^4} \frac{i}{2^7 (4!)^2} [(\bar{C}\gamma_a)^{\lambda_c \mu_c} (\bar{C}\gamma_b)^{\eta_c \xi_c}] [(\gamma_{a'} C)^{\lambda'_c \mu'_c} (\gamma_{b'} C)^{\eta'_c \xi'_c}] \\
& [(m - \gamma^{a_1} \partial_{a_1}) \gamma^A]_{\{\lambda_c \lambda'_c [(m - \gamma^{a_2} \partial_{a_2}) \gamma^A]_{\mu_c \mu'_c} [(m - \gamma^{a_3} \partial_{a_3}) \gamma^A]_{\eta_c \eta'_c} [(m - \gamma^{a_4} \partial_{a_4}) \gamma^A]_{\xi_c \xi'_c}\}} \Delta(x - x') \\
& = \frac{1}{m^4} \frac{i}{2^3 (4!)^2} \eta_{a''}^{a'} \eta_{b''}^{b'} \left\{ \frac{3!}{112!} \frac{2!}{210!} \text{tr}[\gamma_{\{a} Q \gamma_{(a''} \tilde{Q}) \text{tr}[\gamma_{\{b} Q \gamma_{b''} \tilde{Q})] + \frac{3!}{21!} \frac{2!}{1110!} \text{tr}[\gamma_{\{a} Q \gamma_{(a''} \tilde{Q}) \gamma_{\{b} Q \gamma_{b''} \tilde{Q})} \right\} \Delta(x - x')
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } [A_{ab}(x), A_{a'b'}^+(x')] = \frac{1}{m^4} \frac{i}{2^7 (4!)^2} [(\bar{C}\gamma_a)^{\lambda_c \mu_c} (\bar{C}\gamma_b)^{\eta_c \xi_c}] [(\gamma_{a'} C)^{\lambda'_c \mu'_c} (\gamma_{b'} C)^{\eta'_c \xi'_c}] \\
& [(m - \gamma^{a_1} \partial_{a_1}) \gamma^A]_{\{\lambda_c \lambda'_c [(m - \gamma^{a_2} \partial_{a_2}) \gamma^A]_{\mu_c \mu'_c} [(m - \gamma^{a_3} \partial_{a_3}) \gamma^A]_{\eta_c \eta'_c} [(m - \gamma^{a_4} \partial_{a_4}) \gamma^A]_{\xi_c \xi'_c}\}} \Delta(x - x') \\
& = \frac{1}{m^4} \frac{i}{2^7 (4!)^2} [(\bar{C}\gamma_a)^{\lambda_c \mu_c} (\bar{C}\gamma_b)^{\eta_c \xi_c}] [(\gamma_{a'} C)^{\lambda'_c \mu'_c} (\gamma_{b'} C)^{\eta'_c \xi'_c}] \\
& [(m - \gamma_{a_1} \partial^{a_1}) \gamma_0]_{\{\lambda_c \lambda'_c [(m - \gamma_{a_2} \partial^{a_2}) \gamma_0]_{\mu_c \mu'_c} [(m - \gamma_{a_3} \partial^{a_3}) \gamma_0]_{\eta_c \eta'_c} [(m - \gamma_{a_4} \partial^{a_4}) \gamma_0]_{\xi_c \xi'_c}\}} \Delta(x - x') \\
& = \frac{1}{m^4} \frac{i}{2^7 (4!)^2} [(\bar{C}\gamma_a)^{\lambda_c \mu_c} (\bar{C}\gamma_b)^{\eta_c \xi_c}] [(\gamma_{a'} C)^{\lambda'_c \mu'_c} (\gamma_{b'} C)^{\eta'_c \xi'_c}] \\
& [(m - \gamma_{a_1} \partial^{a_1}) \gamma_0]_{\lambda_c \lambda'_c} [(m - \gamma_{a_2} \partial^{a_2}) \gamma_0]_{\mu_c \mu'_c} [(m - \gamma_{a_3} \partial^{a_3}) \gamma_0]_{\eta_c \eta'_c} [(m - \gamma_{a_4} \partial^{a_4}) \gamma_0]_{\xi_c \xi'_c} \Delta(x - x') \\
& = \frac{1}{m^4} \frac{i}{2^3 (4!)^2} \\
& [(\bar{C}\gamma_a)^{\lambda_c \mu_c} (\bar{C}\gamma_b)^{\eta_c \xi_c} + (\bar{C}\gamma_a)^{\eta_c \xi_c} (\bar{C}\gamma_b)^{\lambda_c \mu_c} + (\bar{C}\gamma_a)^{\lambda_c \eta_c} (\bar{C}\gamma_b)^{\mu_c \xi_c} + (\bar{C}\gamma_a)^{\mu_c \xi_c} (\bar{C}\gamma_b)^{\lambda_c \eta_c} \\
& + (\bar{C}\gamma_a)^{\lambda_c \xi_c} (\bar{C}\gamma_b)^{\mu_c \eta_c} + (\bar{C}\gamma_a)^{\mu_c \eta_c} (\bar{C}\gamma_b)^{\lambda_c \xi_c}] \\
& [(\gamma_{a'} C)^{\lambda'_c \mu'_c} (\gamma_{b'} C)^{\eta'_c \xi'_c} + (\gamma_{a'} C)^{\eta'_c \xi'_c} (\gamma_{b'} C)^{\lambda'_c \mu'_c} + (\gamma_{a'} C)^{\lambda'_c \eta'_c} (\gamma_{b'} C)^{\mu'_c \xi'_c} + (\gamma_{a'} C)^{\mu'_c \xi'_c} (\gamma_{b'} C)^{\lambda'_c \eta'_c} \\
& + (\gamma_{a'} C)^{\lambda'_c \xi'_c} (\gamma_{b'} C)^{\mu'_c \eta'_c} + (\gamma_{a'} C)^{\mu'_c \eta'_c} (\gamma_{b'} C)^{\lambda'_c \xi'_c}] \\
& [(m - \gamma_{a_1} \partial^{a_1}) \gamma_0]_{\lambda_c \lambda'_c} [(m - \gamma_{a_2} \partial^{a_2}) \gamma_0]_{\mu_c \mu'_c} [(m - \gamma_{a_3} \partial^{a_3}) \gamma_0]_{\eta_c \eta'_c} [(m - \gamma_{a_4} \partial^{a_4}) \gamma_0]_{\xi_c \xi'_c} \Delta(x - x') \\
& = \frac{1}{m^4} \frac{i}{2^3 (4!)^2} \\
& [(\bar{C}\gamma_a)^{\lambda_c \mu_c} (\bar{C}\gamma_b)^{\eta_c \xi_c} + (\bar{C}\gamma_a)^{\eta_c \xi_c} (\bar{C}\gamma_b)^{\lambda_c \mu_c} + (\bar{C}\gamma_a)^{\lambda_c \eta_c} (\bar{C}\gamma_b)^{\mu_c \xi_c} + (\bar{C}\gamma_a)^{\mu_c \xi_c} (\bar{C}\gamma_b)^{\lambda_c \eta_c} \\
& + (\bar{C}\gamma_a)^{\lambda_c \xi_c} (\bar{C}\gamma_b)^{\mu_c \eta_c} + (\bar{C}\gamma_a)^{\mu_c \eta_c} (\bar{C}\gamma_b)^{\lambda_c \xi_c}] \\
& [(\gamma_{a'} C)^{\lambda'_c \mu'_c} (\gamma_{b'} C)^{\eta'_c \xi'_c} + (\gamma_{a'} C)^{\eta'_c \xi'_c} (\gamma_{b'} C)^{\lambda'_c \mu'_c} + (\gamma_{a'} C)^{\lambda'_c \eta'_c} (\gamma_{b'} C)^{\mu'_c \xi'_c} + (\gamma_{a'} C)^{\mu'_c \xi'_c} (\gamma_{b'} C)^{\lambda'_c \eta'_c} \\
& + (\gamma_{a'} C)^{\lambda'_c \xi'_c} (\gamma_{b'} C)^{\mu'_c \eta'_c} + (\gamma_{a'} C)^{\mu'_c \eta'_c} (\gamma_{b'} C)^{\lambda'_c \xi'_c}] K_{\lambda_c \lambda'_c} K_{\mu_c \mu'_c} K_{\eta_c \eta'_c} K_{\xi_c \xi'_c} \Delta(x - x') \\
& = \frac{1}{m^4} \frac{i}{2^3 (4!)^2} \\
& [(\bar{C}\gamma_{\{a} \}^{\lambda_c \mu_c} (\bar{C}\gamma_{\{b} \}^{\eta_c \xi_c} + (\bar{C}\gamma_{\{a} \}^{\lambda_c \eta_c} (\bar{C}\gamma_{\{b} \}^{\mu_c \xi_c} + (\bar{C}\gamma_{\{a} \}^{\lambda_c \xi_c} (\bar{C}\gamma_{\{b} \}^{\mu_c \eta_c} \\
& [(\gamma_{(a'} C)^{\lambda'_c \mu'_c} (\gamma_{(b'} C)^{\eta'_c \xi'_c} + (\gamma_{(a'} C)^{\lambda'_c \eta'_c} (\gamma_{(b'} C)^{\mu'_c \xi'_c} + (\gamma_{(a'} C)^{\lambda'_c \xi'_c} (\gamma_{(b'} C)^{\mu'_c \eta'_c} \\
& K_{\lambda_c \lambda'_c} K_{\mu_c \mu'_c} K_{\eta_c \eta'_c} K_{\xi_c \xi'_c} \Delta(x - x')
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{m^4} \frac{i}{2^3(4!)^2} \\
&\{6tr[(\bar{C}\gamma_a)K(\gamma_{a'}C)K^T]tr[(\bar{C}\gamma_b)K(\gamma_{b'}C)K^T] + 6tr[(\bar{C}\gamma_a)K(\gamma_{b'}C)K^T]tr[(\bar{C}\gamma_b)K(\gamma_{a'}C)K^T] \\
&+ 12tr[(\bar{C}\gamma_a)K(\gamma_{a'}C)K^T(\bar{C}\gamma_b)K(\gamma_{b'}C)K^T] + 12tr[(\bar{C}\gamma_a)K(\gamma_{b'}C)K^T(\bar{C}\gamma_b)K(\gamma_{a'}C)K^T]\}\Delta(x-x') \\
&= \frac{1}{m^4} \frac{i}{2^3(4!)^2} \{3tr[\bar{C}\gamma_{\{a}K\gamma_{(a'}CK^T]tr[\bar{C}\gamma_{b)}K\gamma_{b')}CK^T] + 6tr[\bar{C}\gamma_{\{a}K\gamma_{(a'}CK^T\bar{C}\gamma_{b)}K\gamma_{b')}CK^T]\}\Delta(x-x') \\
&= \frac{1}{m^4} \frac{i}{2^3(4!)^2} \{3tr[\gamma_{\{a}K\gamma_{(a'}\tilde{K}]tr[\gamma_{b)}K\gamma_{b')\tilde{K}}] + 6tr[\gamma_{\{a}K\gamma_{(a'}\tilde{K}\gamma_{b)}K\gamma_{b')\tilde{K}}]\}\Delta(x-x') \\
&= \frac{1}{m^4} \frac{i}{2^3(4!)^2} \eta_{a''}^{a'} \eta_{b''}^{b'} \{ \frac{3!}{1!2!} \frac{2!}{2!0!0!} tr[\gamma_{\{a}Q\gamma_{(a''}\tilde{Q}]tr[\gamma_{b)}Q\gamma_{b''}\tilde{Q}] + \frac{3!}{2!1!} \frac{2!}{1!1!0!} tr[\gamma_{\{a}Q\gamma_{(a''}\tilde{Q}\gamma_{b)}Q\gamma_{b''}\tilde{Q}]\}\Delta(x-x') \\
&= \frac{1}{m^4} \frac{i}{2^3(4!)^2} \eta_{a''}^{a'} \eta_{b''}^{b'} \\
&\{c_3^1 \frac{2!}{2!0!0!} tr[\gamma_{\{a}\gamma_5\tilde{Q}\gamma_{(a''}\gamma_5\tilde{Q}]tr[\gamma_{b)}\gamma_5\tilde{Q}\gamma_{b''}\gamma_5\tilde{Q}] + c_3^2 \frac{2!}{1!1!0!} tr[\gamma_{\{a}\gamma_5\tilde{Q}\gamma_{(a''}\gamma_5\tilde{Q}\gamma_{b)}\gamma_5\tilde{Q}\gamma_{b''}\gamma_5\tilde{Q}]\}\Delta(x-x') \\
&= \frac{1}{m^4} \frac{i}{2^3(4!)^2} \eta_{a''}^{a'} \eta_{b''}^{b'} \\
&\{c_3^1 \frac{2!}{2!0!0!} tr[\gamma_5\gamma_{\{a}Q\gamma_5\gamma_{(a''}Q]tr[\gamma_5\gamma_{b)}Q\gamma_5\gamma_{b''}Q] + c_3^2 \frac{2!}{1!1!0!} tr[\gamma_5\gamma_{\{a}Q\gamma_5\gamma_{(a''}Q\gamma_5\gamma_{b)}Q\gamma_5\gamma_{b''}Q]\}\Delta(x-x') \\
&= \frac{1}{m^4} \frac{i}{2^3(4!)^2} \eta_{a''}^{a'} \eta_{b''}^{b'} \\
&\{ \frac{3!}{1!2!} \frac{2!}{2!0!0!} tr[\gamma_5\gamma_{\{a}Q\gamma_5\gamma_{(a''}Q]tr[\gamma_5\gamma_{b)}Q\gamma_5\gamma_{b''}Q] + \frac{3!}{2!1!} \frac{2!}{1!1!0!} tr[\gamma_5\gamma_{\{a}Q\gamma_5\gamma_{(a''}Q\gamma_5\gamma_{b)}Q\gamma_5\gamma_{b''}Q]\}\Delta(x-x') \quad \square
\end{aligned}$$

$$\text{猜想5.1.1. } [A_{a_1 a_2 \dots a_n}(x), A_{a'_1 a'_2 \dots a'_n}^+(x')] = \frac{1}{m^{2n}} \frac{i}{2^{4n-1}[(2n)!]^2} \overbrace{(\bar{C}\gamma_{a_1})^{\lambda_1 \mu_1} (\bar{C}\gamma_{a_2})^{\eta_1 \xi_1} \dots}^n \overbrace{(\gamma_{a'_1} C)^{\lambda'_1 \mu'_1} (\gamma_{a'_2} C)^{\eta'_1 \xi'_1} \dots}^n$$

$$\{ (m - \gamma^a \partial_a) \gamma^A \}_{\lambda_1 \dots \lambda_n} \{ (m - \gamma^b \partial_b) \gamma^A \}_{\mu_1 \dots \mu_n} \dots \Delta(x-x')$$

$$\begin{aligned}
&= \frac{1}{m^{2n}} \frac{i}{2^{2n-1}[(2n)!]^2} \eta_{a'_1}^{a_1} \eta_{a'_2}^{a_2} \dots \eta_{a'_n}^{a_n} \sum_{j_1 \dots j_n} \frac{[3^{n-2}(2n-1)!]}{j_1! \dots j_n! [3^{n-2}(2n-1) - (j_1 + \dots + j_n)]!} \frac{n!}{(1!)^{j_1} \dots (n!)^{j_n}} \\
&tr^{j_1 \dots j_n} [\gamma_{\{a_1} Q \gamma_{(a'_1} \tilde{Q} | \gamma_{a_2} Q \gamma_{a'_2} \tilde{Q} | \gamma_{a_3} Q \gamma_{a'_3} \tilde{Q} | \dots | \gamma_{a_n} \} Q \gamma_{a'_n} \tilde{Q}] \Delta(x-x'), 1j_1 + 2j_2 + \dots + nj_n = n
\end{aligned}$$

## 5.2 逐项求解一

证明:

$$\begin{aligned}
&\delta_{a_1 a_2} [\delta_{a_3 a_4} (\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8} + \delta_{a_5 a_8} \delta_{a_6 a_7}) - \delta_{a_3 a_5} (\delta_{a_4 a_6} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_6 a_8} + \delta_{a_4 a_8} \delta_{a_6 a_7}) \\
&+ \delta_{a_3 a_6} (\delta_{a_4 a_5} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_5 a_8} + \delta_{a_4 a_8} \delta_{a_5 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
&= \delta_{a_1 a_2} [\delta_{a_3 a_4} (2\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8}) - \delta_{a_3 a_5} (\delta_{a_4 a_6} \delta_{a_7 a_8}) + \delta_{a_3 a_6} (\delta_{a_4 a_8} \delta_{a_5 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
&= \partial_{a_1} [\partial_{a_3} (2\partial_{a_5} \partial_{a_7} - \delta_{a_5 a_7} m^2) - \delta_{a_3 a_5} (m^2 \partial_{a_7}) + \partial_{a_3} (m^2 \delta_{a_5 a_7})] \\
&= 2\partial_{a_1} \partial_{a_3} \partial_{a_5} \partial_{a_7} - m^2 \partial_{a_1} \delta_{a_3 a_5} \partial_{a_7} \quad \square
\end{aligned}$$

证明:

$$\begin{aligned}
&-\delta_{a_1 a_3} [\delta_{a_2 a_4} (\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8} + \delta_{a_5 a_8} \delta_{a_6 a_7}) - \delta_{a_2 a_5} (\delta_{a_4 a_6} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_6 a_8} + \delta_{a_4 a_8} \delta_{a_6 a_7}) \\
&+ \delta_{a_2 a_6} (\delta_{a_4 a_5} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_5 a_8} + \delta_{a_4 a_8} \delta_{a_5 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
&= -\delta_{a_1 a_3} [\delta_{a_2 a_4} (2\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8}) - \delta_{a_2 a_5} (\delta_{a_4 a_6} \delta_{a_7 a_8}) + \delta_{a_2 a_6} (\delta_{a_4 a_8} \delta_{a_5 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
&= -\delta_{a_1 a_3} [m^2 (2\partial_{a_5} \partial_{a_7} - \delta_{a_5 a_7} m^2) - \partial_{a_5} m^2 \partial_{a_7} + m^4 \delta_{a_5 a_7}] \\
&= -m^2 \delta_{a_1 a_3} \partial_{a_5} \partial_{a_7} \quad \square
\end{aligned}$$

证明:

$$\begin{aligned}
&+ \delta_{a_1 a_4} [\delta_{a_2 a_3} (\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8} + \delta_{a_5 a_8} \delta_{a_6 a_7}) - \delta_{a_2 a_5} (\delta_{a_3 a_6} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_6 a_8} + \delta_{a_3 a_8} \delta_{a_6 a_7}) \\
&+ \delta_{a_2 a_6} (\delta_{a_3 a_5} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_5 a_8} + \delta_{a_3 a_8} \delta_{a_5 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
&= \delta_{a_1 a_4} [\delta_{a_2 a_3} (2\delta_{a_5 a_6} \delta_{a_7 a_8} - \delta_{a_5 a_7} \delta_{a_6 a_8}) - \delta_{a_2 a_5} (2\delta_{a_3 a_6} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_6 a_8}) \\
&+ \delta_{a_2 a_6} (\delta_{a_3 a_5} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_5 a_8} + \delta_{a_3 a_8} \delta_{a_5 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\
&= +\partial_{a_1} [\partial_{a_3} (2\partial_{a_5} \partial_{a_7} - \delta_{a_5 a_7} m^2) - \partial_{a_5} (2\partial_{a_3} \partial_{a_7} - \delta_{a_3 a_7} m^2) + m^2 (\delta_{a_3 a_5} \partial_{a_7} - \delta_{a_3 a_7} \partial_{a_5} + \partial_{a_3} \delta_{a_5 a_7})] \\
&= m^2 \partial_{a_1} \delta_{a_3 a_5} \partial_{a_7} \quad \square
\end{aligned}$$

证明:

$$\begin{aligned}
&-\delta_{a_1 a_5} [\delta_{a_2 a_3} (\delta_{a_4 a_6} \delta_{a_7 a_8} - \delta_{a_4 a_7} \delta_{a_6 a_8} + \delta_{a_4 a_8} \delta_{a_6 a_7}) - \delta_{a_2 a_4} (\delta_{a_3 a_6} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_6 a_8} + \delta_{a_3 a_8} \delta_{a_6 a_7}) \\
&+ \delta_{a_2 a_6} (\delta_{a_3 a_4} \delta_{a_7 a_8} - \delta_{a_3 a_7} \delta_{a_4 a_8} + \delta_{a_3 a_8} \delta_{a_4 a_7})] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8}
\end{aligned}$$





$$\begin{aligned} & \text{证明: } [\gamma_a \gamma_{a_2} \gamma_{a'} \gamma_{a_4} \gamma_b \gamma_{a_6} \gamma_{b'} \gamma_{a_8}] \partial^{a_2} \partial^{a_4} \partial^{a_6} \partial^{a_8} \\ & = 6\partial_a \partial_{a'} \partial_b \partial_{b'} - m^2 \delta_{aa'} \partial_b \partial_{b'} - m^2 \delta_{ab} \partial_{a'} \partial_{b'} - 2m^2 \delta_{a'b} \partial_a \partial_{b'} - 2m^2 \delta_{ab'} \partial_{a'} \partial_b + m^4 \delta_{ab'} \delta_{a'b} \end{aligned} \quad \square$$

$$\begin{aligned} & \text{证明: } \frac{1}{2^2} \text{tr}[\gamma_{\{a} \gamma_{a_1} \gamma_{(a'} \gamma_{a_2} \gamma_b \gamma_{a_3} \gamma_{b')} \gamma_{a_4}] \partial^{a_1} \partial^{a_2} \partial^{a_3} \partial^{a_4} \\ & = 24\partial_a \partial_{a'} \partial_b \partial_{b'} - m^2 \delta_{\{a(a'} \partial_b \partial_{b')} - 4m^2 \delta_{ab} \partial_{a'} \partial_{b'} - 2m^2 \delta_{\{a(b'} \partial_a \partial_{b')} - 2m^2 \delta_{\{a(b'} \partial_{a'} \partial_b\} + m^4 \delta_{\{a(b'} \delta_{a')b\}} \\ & = 24\partial_a \partial_{a'} \partial_b \partial_{b'} - m^2 \delta_{\{a(a'} \partial_b \partial_{b')} - 4m^2 \delta_{ab} \partial_{a'} \partial_{b'} - 2m^2 \delta_{\{a(a'} \partial_b \partial_{b')} - 2m^2 \delta_{\{a(a'} \partial_b \partial_{b')} + m^4 \delta_{\{a(a'} \delta_b \delta_{b')\}} \\ & = 24\partial_a \partial_{a'} \partial_b \partial_{b'} - 5m^2 \delta_{\{a(a'} \partial_b \partial_{b')} - 4m^2 \delta_{ab} \partial_{a'} \partial_{b'} + m^4 \delta_{\{a(a'} \delta_b \delta_{b')\}} \end{aligned} \quad \square$$

### 5.3 逐项求解二

$$\begin{aligned} & \text{证明: } \frac{1}{2^2} \text{tr}[\gamma_{\{a} Q \gamma_{(a'} \tilde{Q} \gamma_b \gamma_{b')} Q \tilde{Q}] \\ & = m^4 \text{tr}[\gamma_{\{a} \gamma_{(a'} \gamma_b \gamma_{b')}] + \text{tr}[\gamma_{\{a} \gamma_{a_1} \gamma_{(a'} \gamma_{a_2} \gamma_b \gamma_{a_3} \gamma_{b')} \gamma_{a_4}] \partial^{a_1} \partial^{a_2} \partial^{a_3} \partial^{a_4} \\ & + m^2 \text{tr}[\gamma_{\{a} \gamma_{a_1} \gamma_{(a'} \gamma_b \gamma_{a_2} \gamma_{b')} \partial^{a_1} \partial^{a_2} + m^2 \text{tr}[\gamma_{\{a} \gamma_{(a'} \gamma_{a_1} \gamma_b \gamma_{b')} \gamma_{a_2}] \partial^{a_1} \partial^{a_2} \\ & - 2m^2 \text{tr}[\gamma_{\{a} \gamma_{(a'} \gamma_b \gamma_{a_1} \gamma_{b')} \gamma_{a_2}] \partial^{a_1} \partial^{a_2} - 2m^2 \text{tr}[\gamma_{\{a} \gamma_{a_1} \gamma_{(a'} \gamma_b \gamma_{b')} \gamma_{a_2}] \partial^{a_1} \partial^{a_2} \end{aligned} \quad \square$$

$$\begin{aligned} & \text{证明: } \frac{1}{2^2} m^4 \text{tr}[\gamma_{\{a} \gamma_{(a'} \gamma_b \gamma_{b')}] \\ & = m^4 \text{tr}[\gamma_{\{a} \gamma_{(a'} \gamma_b \gamma_{b')}] \\ & = m^4 (\delta_{\{a(a'} \delta_b \delta_{b')\}} - 4\delta_{ab} \delta_{a'b'} + \delta_{\{a(b'} \delta_b \delta_{a')\}}) \\ & = 2m^4 \delta_{\{a(a'} \delta_b \delta_{b')\}} - 4m^4 \delta_{ab} \delta_{a'b'} \end{aligned} \quad \square$$

### 5.4 逐项求解三

引理5.4.1.  $\gamma_{a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5} \gamma_{a_6}$  的常数项

$$\begin{aligned} & = \delta_{a_1 a_2} (\delta_{a_3 a_4} \delta_{a_5 a_6} - \delta_{a_3 a_5} \delta_{a_4 a_6} + \delta_{a_3 a_6} \delta_{a_4 a_5}) - \delta_{a_1 a_3} (\delta_{a_2 a_4} \delta_{a_5 a_6} - \delta_{a_2 a_5} \delta_{a_4 a_6} + \delta_{a_2 a_6} \delta_{a_4 a_5}) \\ & + \delta_{a_1 a_4} (\delta_{a_2 a_3} \delta_{a_5 a_6} - \delta_{a_2 a_5} \delta_{a_3 a_6} + \delta_{a_2 a_6} \delta_{a_3 a_5}) - \delta_{a_1 a_5} (\delta_{a_2 a_3} \delta_{a_4 a_6} - \delta_{a_2 a_4} \delta_{a_3 a_6} + \delta_{a_2 a_6} \delta_{a_3 a_4}) \\ & + \delta_{a_1 a_6} (\delta_{a_2 a_3} \delta_{a_4 a_5} - \delta_{a_2 a_4} \delta_{a_3 a_5} + \delta_{a_2 a_5} \delta_{a_3 a_4}) \end{aligned}$$

$$\begin{aligned} & \text{证明: } \frac{1}{2^2} \text{tr}(\gamma_{a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5} \gamma_{a_6}) \partial^{a_2} \partial^{a_5} \\ & = \delta_{a_1 a_2} (\delta_{a_3 a_4} \delta_{a_5 a_6} - \delta_{a_3 a_5} \delta_{a_4 a_6} + \delta_{a_3 a_6} \delta_{a_4 a_5}) \partial^{a_2} \partial^{a_5} - \delta_{a_1 a_3} (\delta_{a_2 a_4} \delta_{a_5 a_6} - \delta_{a_2 a_5} \delta_{a_4 a_6} + \delta_{a_2 a_6} \delta_{a_4 a_5}) \partial^{a_2} \partial^{a_5} \\ & + \delta_{a_1 a_4} (\delta_{a_2 a_3} \delta_{a_5 a_6} - \delta_{a_2 a_5} \delta_{a_3 a_6} + \delta_{a_2 a_6} \delta_{a_3 a_5}) \partial^{a_2} \partial^{a_5} - \delta_{a_1 a_5} (\delta_{a_2 a_3} \delta_{a_4 a_6} - \delta_{a_2 a_4} \delta_{a_3 a_6} + \delta_{a_2 a_6} \delta_{a_3 a_4}) \partial^{a_2} \partial^{a_5} \\ & + \delta_{a_1 a_6} (\delta_{a_2 a_3} \delta_{a_4 a_5} - \delta_{a_2 a_4} \delta_{a_3 a_5} + \delta_{a_2 a_5} \delta_{a_3 a_4}) \partial^{a_2} \partial^{a_5} \\ & = \partial_{a_1} (\delta_{a_3 a_4} \partial_{a_6} - \partial_{a_3} \delta_{a_4 a_6} + \delta_{a_3 a_6} \partial_{a_4}) - \delta_{a_1 a_3} (2\partial_{a_4} \partial_{a_6} - m^2 \delta_{a_4 a_6}) \\ & + \delta_{a_1 a_4} (2\partial_{a_3} \partial_{a_6} - m^2 \delta_{a_3 a_6}) - \partial_{a_1} (\partial_{a_3} \delta_{a_4 a_6} - \partial_{a_4} \delta_{a_3 a_6} + \partial_{a_6} \delta_{a_3 a_4}) \\ & + \delta_{a_1 a_6} (m^2 \delta_{a_3 a_4}) \\ & = 2\partial_{a_1} (-\partial_{a_3} \delta_{a_4 a_6} + \delta_{a_3 a_6} \partial_{a_4}) - \delta_{a_1 a_3} (2\partial_{a_4} \partial_{a_6} - m^2 \delta_{a_4 a_6}) + \delta_{a_1 a_4} (2\partial_{a_3} \partial_{a_6} - m^2 \delta_{a_3 a_6}) + \delta_{a_1 a_6} (m^2 \delta_{a_3 a_4}) \\ & = -2\partial_{a_1} \partial_{a_3} \delta_{a_4 a_6} + 2\delta_{a_3 a_6} \partial_{a_1} \partial_{a_4} - 2\delta_{a_1 a_3} \partial_{a_4} \partial_{a_6} + 2\delta_{a_1 a_4} \partial_{a_3} \partial_{a_6} + m^2 \delta_{a_1 a_3} \delta_{a_4 a_6} - m^2 \delta_{a_1 a_4} \delta_{a_3 a_6} + m^2 \delta_{a_1 a_6} \delta_{a_3 a_4} \\ & \frac{1}{2^2} m^2 \text{tr}[\gamma_{\{a} \gamma_{a_1} \gamma_{(a'} \gamma_b \gamma_{a_2} \gamma_{b')} \partial^{a_1} \partial^{a_2} \\ & = m^2 | -2\partial_{\{a} \partial_{(a'} \delta_b \delta_{b')\}} + 8\delta_{a'b'} \partial_a \partial_b - 2\delta_{\{a(a'} \partial_b \partial_{b')\}} + 8\delta_{ab} \partial_{a'} \partial_{b'} + m^2 \delta_{\{a(a'} \delta_b \delta_{b')\}} - 4m^2 \delta_{ab} \delta_{a'b'} + m^2 \delta_{\{a(b'} \delta_b \delta_{a')\}} \\ & = 8m^2 \delta_{a'b'} \partial_a \partial_b - 4m^2 \delta_{\{a(a'} \partial_b \partial_{b')\}} + 8m^2 \delta_{ab} \partial_{a'} \partial_{b'} + 2m^4 \delta_{\{a(a'} \delta_b \delta_{b')\}} - 4m^4 \delta_{ab} \delta_{a'b'} \end{aligned} \quad \square$$

$$\begin{aligned} & \text{证明: } \frac{1}{2^2} \text{tr}(\gamma_{a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5} \gamma_{a_6}) \partial^{a_3} \partial^{a_6} \\ & = \delta_{a_1 a_2} (\delta_{a_3 a_4} \delta_{a_5 a_6} - \delta_{a_3 a_5} \delta_{a_4 a_6} + \delta_{a_3 a_6} \delta_{a_4 a_5}) \partial^{a_3} \partial^{a_6} - \delta_{a_1 a_3} (\delta_{a_2 a_4} \delta_{a_5 a_6} - \delta_{a_2 a_5} \delta_{a_4 a_6} + \delta_{a_2 a_6} \delta_{a_4 a_5}) \partial^{a_3} \partial^{a_6} \\ & + \delta_{a_1 a_4} (\delta_{a_2 a_3} \delta_{a_5 a_6} - \delta_{a_2 a_5} \delta_{a_3 a_6} + \delta_{a_2 a_6} \delta_{a_3 a_5}) \partial^{a_3} \partial^{a_6} - \delta_{a_1 a_5} (\delta_{a_2 a_3} \delta_{a_4 a_6} - \delta_{a_2 a_4} \delta_{a_3 a_6} + \delta_{a_2 a_6} \delta_{a_3 a_4}) \partial^{a_3} \partial^{a_6} \\ & + \delta_{a_1 a_6} (\delta_{a_2 a_3} \delta_{a_4 a_5} - \delta_{a_2 a_4} \delta_{a_3 a_5} + \delta_{a_2 a_5} \delta_{a_3 a_4}) \partial^{a_3} \partial^{a_6} \\ & = \delta_{a_1 a_2} (m^2 \delta_{a_4 a_5}) - 2\partial_{a_1} (\delta_{a_2 a_4} \partial_{a_5} - \delta_{a_2 a_5} \partial_{a_4}) \\ & + \delta_{a_1 a_4} (2\partial_{a_2} \partial_{a_5} - \delta_{a_2 a_5} m^2) - \delta_{a_1 a_5} (2\partial_{a_2} \partial_{a_4} - \delta_{a_2 a_4} m^2) \\ & = -2\partial_{a_1} \delta_{a_2 a_4} \partial_{a_5} + 2\partial_{a_1} \delta_{a_2 a_5} \partial_{a_4} + 2\delta_{a_1 a_4} \partial_{a_2} \partial_{a_5} - 2\delta_{a_1 a_5} \partial_{a_2} \partial_{a_4} + m^2 \delta_{a_1 a_2} \delta_{a_4 a_5} - m^2 \delta_{a_1 a_4} \delta_{a_2 a_5} + m^2 \delta_{a_1 a_5} \delta_{a_2 a_4} \\ & \frac{1}{2^2} m^2 \text{tr}[\gamma_{\{a} \gamma_{(a'} \gamma_{a_1} \gamma_b \gamma_{b')} \gamma_{a_2}] \partial^{a_1} \partial^{a_2} \end{aligned}$$

$$\begin{aligned}
&= m^2 | - 2\partial_{\{a}\delta_{\{a'b\}}\partial_{b'}} + 8\partial_a\delta_{a'b'}\partial_b + 8\delta_{ab}\partial_{a'}\partial_{b'} - 2\delta_{\{a(b'}\partial_{a')\}b\}} + m^2\delta_{\{a(a'}\delta_{b\}}b') - 4m^2\delta_{ab}\delta_{a'b'} + m^2\delta_{\{a(b'}\delta_{a')b\}} \\
&= 8m^2\delta_{a'b'}\partial_a\partial_b - 4m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b'}) + 8m^2\delta_{ab}\partial_{a'}\partial_{b'} + 2m^4\delta_{\{a(a'}\delta_{b\}}b') - 4m^4\delta_{ab}\delta_{a'b'} \quad \square
\end{aligned}$$

证明:  $\frac{1}{2^2}tr(\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5}\gamma_{a_6})\partial^{a_4}\partial^{a_6}$

$$\begin{aligned}
&= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5})\partial^{a_4}\partial^{a_6} - \delta_{a_1a_3}(\delta_{a_2a_4}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_4a_6} + \delta_{a_2a_6}\delta_{a_4a_5})\partial^{a_4}\partial^{a_6} \\
&+ \delta_{a_1a_4}(\delta_{a_2a_3}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_5})\partial^{a_4}\partial^{a_6} - \delta_{a_1a_5}(\delta_{a_2a_3}\delta_{a_4a_6} - \delta_{a_2a_4}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_4})\partial^{a_4}\partial^{a_6} \\
&+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\partial^{a_4}\partial^{a_6} \\
&= \delta_{a_1a_2}(2\partial_{a_3}\partial_{a_5} - \delta_{a_3a_5}m^2) - \delta_{a_1a_3}(2\partial_{a_2}\partial_{a_5} - \delta_{a_2a_5}m^2) \\
&+ \partial_{a_1}(\delta_{a_2a_3}\partial_{a_5} - \delta_{a_2a_5}\partial_{a_3} + \partial_{a_2}\delta_{a_3a_5}) - \delta_{a_1a_5}(\delta_{a_2a_3}m^2) \\
&+ \partial_{a_1}(\delta_{a_2a_3}\partial_{a_5} - \partial_{a_2}\delta_{a_3a_5} + \delta_{a_2a_5}\partial_{a_3}) \\
&= \delta_{a_1a_2}(2\partial_{a_3}\partial_{a_5} - \delta_{a_3a_5}m^2) - \delta_{a_1a_3}(2\partial_{a_2}\partial_{a_5} - \delta_{a_2a_5}m^2) + 2\delta_{a_2a_3}\partial_{a_1}\partial_{a_5} - \delta_{a_1a_5}(\delta_{a_2a_3}m^2) \\
&= 2\delta_{a_1a_2}\partial_{a_3}\partial_{a_5} - 2\delta_{a_1a_3}\partial_{a_2}\partial_{a_5} + 2\delta_{a_2a_3}\partial_{a_1}\partial_{a_5} - m^2\delta_{a_1a_5}\delta_{a_2a_3} + m^2\delta_{a_1a_3}\delta_{a_2a_5} - m^2\delta_{a_1a_2}\delta_{a_3a_5} \\
&- 2m^2\frac{1}{2^2}tr[\gamma_{\{a}\gamma_{\{a'}\gamma_{b\}}\gamma_{a_1}\gamma_{b'}}\gamma_{a_2}]\partial^{a_1}\partial^{a_2} \\
&= -2m^2|2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 8\delta_{ab}\partial_{a'}\partial_{b'} + 2\delta_{\{a'\{b\}a\}}\partial_{b'} - m^2\delta_{\{a(b'}\delta_{a')b\}} + 4m^2\delta_{ab}\delta_{a'b'} - m^2\delta_{\{a(a'}\delta_{b\}}b') \\
&= -2m^2|4\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 8\delta_{ab}\partial_{a'}\partial_{b'} + 4m^2\delta_{ab}\delta_{a'b'} - 2m^2\delta_{\{a(a'}\delta_{b\}}b') \\
&= -8m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 16m^2\delta_{ab}\partial_{a'}\partial_{b'} - 8m^4\delta_{ab}\delta_{a'b'} + 4m^4\delta_{\{a(a'}\delta_{b\}}b') \quad \square
\end{aligned}$$

证明:  $\frac{1}{2^2}tr(\gamma_{a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5}\gamma_{a_6})\partial^{a_2}\partial^{a_6}$

$$\begin{aligned}
&= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5})\partial^{a_2}\partial^{a_6} - \delta_{a_1a_3}(\delta_{a_2a_4}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_4a_6} + \delta_{a_2a_6}\delta_{a_4a_5})\partial^{a_2}\partial^{a_6} \\
&+ \delta_{a_1a_4}(\delta_{a_2a_3}\delta_{a_5a_6} - \delta_{a_2a_5}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_5})\partial^{a_2}\partial^{a_6} - \delta_{a_1a_5}(\delta_{a_2a_3}\delta_{a_4a_6} - \delta_{a_2a_4}\delta_{a_3a_6} + \delta_{a_2a_6}\delta_{a_3a_4})\partial^{a_2}\partial^{a_6} \\
&+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\partial^{a_2}\partial^{a_6} \\
&= \delta_{a_1a_2}(\delta_{a_3a_4}\delta_{a_5a_6} - \delta_{a_3a_5}\delta_{a_4a_6} + \delta_{a_3a_6}\delta_{a_4a_5})\partial^{a_2}\partial^{a_6} - \delta_{a_1a_3}(m^2\delta_{a_4a_5})\partial^{a_2}\partial^{a_6} \\
&+ \delta_{a_1a_4}(m^2\delta_{a_3a_5})\partial^{a_2}\partial^{a_6} - \delta_{a_1a_5}(m^2\delta_{a_3a_4})\partial^{a_2}\partial^{a_6} \\
&+ \delta_{a_1a_6}(\delta_{a_2a_3}\delta_{a_4a_5} - \delta_{a_2a_4}\delta_{a_3a_5} + \delta_{a_2a_5}\delta_{a_3a_4})\partial^{a_2}\partial^{a_6} \\
&= 2\partial_{a_1}(\delta_{a_3a_4}\partial_{a_5} - \delta_{a_3a_5}\partial_{a_4} + \partial_{a_3}\delta_{a_4a_5}) - \delta_{a_1a_3}(m^2\delta_{a_4a_5}) \\
&+ \delta_{a_1a_4}(m^2\delta_{a_3a_5}) - \delta_{a_1a_5}(m^2\delta_{a_3a_4}) \\
&= 2\delta_{a_3a_4}\partial_{a_1}\partial_{a_5} - 2\delta_{a_3a_5}\partial_{a_1}\partial_{a_4} + 2\partial_{a_1}\partial_{a_3}\delta_{a_4a_5} - m^2\delta_{a_1a_3}\delta_{a_4a_5} + m^2\delta_{a_1a_4}\delta_{a_3a_5} - m^2\delta_{a_1a_5}\delta_{a_3a_4} \\
&- 2m^2\frac{1}{2^2}tr[\gamma_{\{a}\gamma_{a_1}\gamma_{\{a'}\gamma_{b\}}\gamma_{b'}}\gamma_{a_2}]\partial^{a_1}\partial^{a_2} \\
&= -2m^2|2\delta_{\{a'\{b\}a\}}\partial_{b'} - 8\delta_{a'b'}\partial_a\partial_b + 2\partial_{\{a}\partial_{\{a'}\delta_{b\}}b'\} - m^2\delta_{\{a(a'}\delta_{b\}}b') + 4m^2\delta_{ab}\delta_{a'b'} - m^2\delta_{\{a(b'}\delta_{a')b\}} \\
&= -8m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 16m^2\delta_{a'b'}\partial_a\partial_b + 4m^4\delta_{\{a(a'}\delta_{b\}}b') - 8m^4\delta_{ab}\delta_{a'b'} \quad \square
\end{aligned}$$

证明:  $\frac{1}{2^2}|m^2tr[\gamma_{\{a}\gamma_{a_1}\gamma_{\{a'}\gamma_{b\}}\gamma_{b'}}]\partial^{a_1}\partial^{a_2} + m^2tr[\gamma_{\{a}\gamma_{\{a'}\gamma_{a_1}\gamma_{b\}}\gamma_{b'}}\gamma_{a_2}]\partial^{a_1}\partial^{a_2}$

$$\begin{aligned}
&- 2m^2tr[\gamma_{\{a}\gamma_{\{a'}\gamma_{b\}}\gamma_{a_1}\gamma_{b'}}\gamma_{a_2}]\partial^{a_1}\partial^{a_2} - 2m^2tr[\gamma_{\{a}\gamma_{a_1}\gamma_{\{a'}\gamma_{b\}}\gamma_{b'}}\gamma_{a_2}]\partial^{a_1}\partial^{a_2} \\
&= 8m^2\delta_{a'b'}\partial_a\partial_b - 4m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 8m^2\delta_{ab}\partial_{a'}\partial_{b'} + 2m^4\delta_{\{a(a'}\delta_{b\}}b') - 4m^4\delta_{ab}\delta_{a'b'} \\
&+ 8m^2\delta_{a'b'}\partial_a\partial_b - 4m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 8m^2\delta_{ab}\partial_{a'}\partial_{b'} + 2m^4\delta_{\{a(a'}\delta_{b\}}b') - 4m^4\delta_{ab}\delta_{a'b'} \\
&- 8m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 16m^2\delta_{ab}\partial_{a'}\partial_{b'} - 8m^4\delta_{ab}\delta_{a'b'} + 4m^4\delta_{\{a(a'}\delta_{b\}}b') \\
&- 8m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 16m^2\delta_{a'b'}\partial_a\partial_b + 4m^4\delta_{\{a(a'}\delta_{b\}}b') - 8m^4\delta_{ab}\delta_{a'b'} \\
&= 32m^2\delta_{a'b'}\partial_a\partial_b + 32m^2\delta_{ab}\partial_{a'}\partial_{b'} - 24m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 12m^4\delta_{\{a(a'}\delta_{b\}}b') - 24m^4\delta_{ab}\delta_{a'b'} \quad \square
\end{aligned}$$

定理5.4.1.  $[A_{ab}(x), A_{a'b'}^+(x')] = \frac{i}{8}\{[\eta_{\{a(a'} - \frac{\partial_{\{a}\partial_{a'}}^+}{m^2}][\eta_{b\}b'} - \frac{\partial_{b\}\partial_{b'}}^+]{] - \frac{1}{3}[\delta_{\{ab\}} - \frac{\partial_{\{a}\partial_{b\}}}{m^2}][\delta_{\{a'b'\}} - \frac{\partial_{\{a'}\partial_{b'}^+}{m^2}]]\Delta(x-x')$

证明:  $2m^4\delta_{\{a(a'}\delta_{b\}}b') - 4m^4\delta_{ab}\delta_{a'b'}$

$$\begin{aligned}
&+ 24\partial_a\partial_{a'}\partial_b\partial_{b'} - 5m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 4m^2\delta_{ab}\partial_{a'}\partial_{b'} + m^4\delta_{\{a(a'}\delta_{b\}}b') \\
&+ 32m^2\delta_{a'b'}\partial_a\partial_b + 32m^2\delta_{ab}\partial_{a'}\partial_{b'} - 24m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 12m^4\delta_{\{a(a'}\delta_{b\}}b') - 24m^4\delta_{ab}\delta_{a'b'} \\
&= +24\partial_a\partial_{a'}\partial_b\partial_{b'} - 5m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} - 4m^2\delta_{ab}\partial_{a'}\partial_{b'} \\
&+ 32m^2\delta_{a'b'}\partial_a\partial_b + 32m^2\delta_{ab}\partial_{a'}\partial_{b'} - 24m^2\delta_{\{a(a'}\partial_{b\}}\partial_{b')} + 15m^4\delta_{\{a(a'}\delta_{b\}}b') - 28m^4\delta_{ab}\delta_{a'b'} \quad \square
\end{aligned}$$

# 第三十四章 高维时空有质量粒子的协变量子化

自我评述：本章将有质量粒子协变量子化推广到了一般 $N+1$ 维时空中。在 $N+1$ 维时空中与四维时空类似，对于Bargmann-Wigner方程或Dirac方程描述的粒子，一般来说既可以描述带荷的复粒子，也可以描述不带荷的马约拉纳粒子。两种情形的主对易规则形式一致，但其余对易或反对易括号，对带荷的复粒子一般为零；对不带荷的马约拉纳粒子，其余对易或反对易括号由主对易规则和马约拉纳条件自然得到，一般不为零。在本章只讨论复粒子情形，一般也只给出主对易规则，不再专门讨论马约拉纳粒子情形，若要得到马约拉纳粒子情形的量子场论，只需在复粒子情形加上马约拉纳条件即可自然得到。

## 1 $N+1$ 维时空中的洛伦兹推动变换

### 1.1 $N+1$ 维时空中矢量的洛伦兹推动变换

定义1.1.1.  $\Omega(s) := \frac{1}{2}(\Gamma \otimes I_{l^{2s-1}} + I_l \otimes \Gamma \otimes I_{l^{2s-2}} + \cdots + I_{l^{2s-1}} \otimes \Gamma), l = 2^{\lfloor \frac{N-1}{2} \rfloor}$

定义1.1.2.  $L = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & \dots & 0 & 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & -i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & -i & 0 & 0 \end{bmatrix} \right\}$

推论1.1.1.  $L_{\vec{v}} = e^{-\ln[\gamma_v(1+v)]\hat{v}\cdot L} = 1 - \gamma_v(\vec{v}\cdot L) + \frac{\gamma_v-1}{v^2}(\vec{v}\cdot L)^2, L_{\vec{v}}L_{-\vec{v}} = L_{-\vec{v}}L_{\vec{v}} = I$

推论1.1.2.  $L_{\vec{v}} \begin{bmatrix} \vec{0} \\ i \end{bmatrix} = e^{-\ln[\gamma_v(1+v)]\hat{v}\cdot L} \begin{bmatrix} \vec{0} \\ i \end{bmatrix} = \begin{bmatrix} \gamma_v \vec{v} \\ i\gamma_v \end{bmatrix}, L_{\vec{v}} \begin{bmatrix} \vec{0} \\ im \end{bmatrix} = e^{-\ln[\gamma_v(1+v)]\hat{v}\cdot L} \begin{bmatrix} \vec{0} \\ im \end{bmatrix} = \begin{bmatrix} \vec{p} \\ iE_{\vec{p}} \end{bmatrix}$

### 1.2 $N+1$ 维时空中中微子旋量的洛伦兹推动变换

性质1.2.1.  $(\vec{v}\cdot\Gamma)^2 = v^2, (\vec{v}\cdot i\vec{\gamma}\gamma_0)^2 = v^2$

推论1.2.1.  $\Lambda_{\zeta\vec{v}} = e^{-\zeta\ln[\gamma_v(1+v)]\hat{v}\cdot\frac{1}{2}\Gamma} = \frac{1}{\sqrt{2(\gamma_v+1)}}(1 + \gamma_v - \zeta\gamma_v\vec{v}\cdot\Gamma), c = \frac{(1+\gamma_v)}{\sqrt{2(\gamma_v+1)}}, s = -\frac{\zeta\gamma_v}{\sqrt{2(\gamma_v+1)}}$

### 1.3 $N+1$ 维时空中电子旋量的洛伦兹推动变换

推论1.3.1.  $D_{\zeta\vec{v}} = e^{-\zeta\ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_0)} = \frac{1}{\sqrt{2(\gamma_v+1)}}[1 + \gamma_v - i\zeta\gamma_v\vec{v}\cdot\vec{\gamma}\gamma_0]$

推论1.3.2.  $D_{\vec{v}} = e^{-\ln\frac{E+|\vec{p}|}{m}\hat{p}\cdot(\frac{i}{2}\vec{\gamma}\gamma_0)} = \frac{m-i\gamma^\alpha p_\alpha\gamma_0}{\sqrt{2m(E+m)}}$

证明:  $D_{\vec{v}} = e^{-\ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_0)} = \frac{1+\gamma_v-i\gamma_v\vec{v}\cdot\vec{\gamma}\gamma_0}{\sqrt{2(\gamma_v+1)}} = \frac{E+m-i\vec{p}\cdot\vec{\gamma}\gamma_0}{\sqrt{2m(E+m)}} = \frac{m-i\gamma^\alpha p_\alpha\gamma_0}{\sqrt{2m(E+m)}}$  □

### 1.4 $N+1$ 维时空中光子旋量的洛伦兹推动变换的多项式表示

推论1.4.1.  $e^{-\zeta\ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(1)} = 1 - \zeta\gamma_v v[\hat{v}\cdot\Omega(1)] + (\gamma_v - 1)[\hat{v}\cdot\Omega(1)]^2$

### 1.5 $N+1$ 维时空中引力微子旋量的洛伦兹推动变换的多项式表示

推论1.5.1.  $e^{-\zeta\ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(\frac{3}{2})} = \frac{(\gamma_v+1)}{\sqrt{2(\gamma_v+1)}}(1 - \frac{\gamma_v-1}{4}) - \frac{2\zeta\gamma_v v}{\sqrt{2(\gamma_v+1)}}(1 - \frac{\gamma_v-1}{12})[\hat{v}\cdot\Omega(\frac{3}{2})]$   
 $+ \frac{\gamma_v^2-1}{\sqrt{2(\gamma_v+1)}}[\hat{v}\cdot\Omega(\frac{3}{2})]^2 - \frac{1}{3}\frac{2\zeta\gamma_v v(\gamma_v-1)}{\sqrt{2(\gamma_v+1)}}[\hat{v}\cdot\Omega(\frac{3}{2})]^3$

### 1.6 $N+1$ 维时空中引力子旋量的洛伦兹推动变换的多项式表示

推论1.6.1.  $e^{-\zeta\ln[\gamma_v(1+v)]\hat{v}\cdot\Omega(2)} = 1 - \zeta\gamma_v(1 - \frac{\gamma_v-1}{3})[\vec{v}\cdot\Omega(2)] + \frac{\gamma_v-1}{v^2}(1 - \frac{\gamma_v-1}{6})[\vec{v}\cdot\Omega(2)]^2$   
 $- \frac{1}{3}\frac{\zeta\gamma_v(\gamma_v-1)}{v^2}[\vec{v}\cdot\Omega(2)]^3 + \frac{1}{6}\frac{(\gamma_v-1)^2}{v^4}[\vec{v}\cdot\Omega(2)]^4$

## 2 N+1维时空中电子的协变量子化

### 2.1 N+1维时空中的电子方程<sup>[4]</sup>

偶数维时空中的电子方程:

$$\text{定义2.1.1. } (\gamma^a \partial_a + m)\psi = 0, \psi = \begin{bmatrix} \varphi \\ \eta \end{bmatrix}, \gamma^a = (\Gamma \otimes \sigma_y, \varsigma I_* \otimes \sigma_z) = (\vec{\gamma}, \varsigma I_* \otimes \sigma_z)$$

奇数维时空中的电子方程:

$$\text{定义2.1.2. } (\gamma^a \partial_a + m)\psi = 0, \psi = \begin{bmatrix} \varphi \\ \eta \end{bmatrix}, \gamma^a = (\Gamma \otimes \sigma_y, I_* \otimes \sigma_x, \varsigma I_* \otimes \sigma_z) = (\vec{\gamma}, \varsigma I_* \otimes \sigma_z)$$

### 2.2 N+1维时空中的电子静止解和运动解

N+1维时空中的电子方程统一写法:

$$\text{定义2.2.1. } (\gamma^a \partial_a + m)\psi = 0, \psi = \begin{bmatrix} \varphi \\ \eta \end{bmatrix}, \gamma^a = (\vec{\gamma}, \varsigma I_* \otimes \sigma_z)$$

$$\text{推论2.2.1. } \partial_{t_0} \psi(\vec{0}) = -im\gamma_0 \psi(\vec{0}) \Leftrightarrow \psi(\vec{0}) = e^{-i\gamma_0 m t_0} \psi_0, \forall \psi_0$$

$$\text{推论2.2.2. } \psi(\vec{p}) = \frac{m - i\gamma^a p_a \gamma_0}{\sqrt{2m(E+m)}} e^{i\gamma_0(\vec{p}\cdot\vec{r} - Et)} \psi_p = \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) e^{i\gamma_0(\vec{p}\cdot\vec{r} - Et)} \psi_p$$

$$\text{推论2.2.3. } \psi^{(+\varsigma)}(\vec{p}) = \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \begin{bmatrix} \varphi \\ 0 \end{bmatrix} e^{i\varsigma(\vec{p}\cdot\vec{r} - Et)}, \psi^{(-\varsigma)}(\vec{p}) = \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \begin{bmatrix} 0 \\ \eta \end{bmatrix} e^{-i\varsigma(\vec{p}\cdot\vec{r} - Et)}$$

### 2.3 N+1维时空中电子平面波解的性质

$$\text{推论2.3.1. } \begin{cases} \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right]^+ = \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \\ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] = -\frac{i\gamma^a p_a \gamma_0}{m} \\ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \gamma_0 \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] = \gamma_0 \\ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \left[\sqrt{\frac{E+m}{2m}} \left(1 + \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] = 1 \end{cases}$$

$$\text{性质2.3.1. } \begin{cases} \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] = \frac{(\varsigma m - i\gamma^a p_a) \gamma_0}{2m} \\ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] = \frac{(-\varsigma m - i\gamma^a p_a) \gamma_0}{2m} \end{cases}$$

$$\text{性质2.3.2. } \begin{bmatrix} \varphi \\ 0 \end{bmatrix}^+ \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m} \begin{bmatrix} \eta \\ 0 \end{bmatrix} \equiv 0, \begin{bmatrix} 0 \\ \varphi \end{bmatrix}^+ \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m} \begin{bmatrix} 0 \\ \eta \end{bmatrix} \equiv 0$$

$$\text{性质2.3.3. } \begin{cases} \begin{bmatrix} \varphi \\ 0 \end{bmatrix}^+ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \begin{bmatrix} \eta \\ 0 \end{bmatrix} = \frac{E}{m} \varphi^+ \eta \\ \begin{bmatrix} 0 \\ \varphi \end{bmatrix}^+ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \begin{bmatrix} 0 \\ \eta \end{bmatrix} = \frac{E}{m} \varphi^+ \eta \end{cases}$$

$$\text{性质2.3.4. } \begin{cases} \begin{bmatrix} \varphi \\ 0 \end{bmatrix}^+ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \left[\sqrt{\frac{E+m}{2m}} \left(1 + \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \begin{bmatrix} 0 \\ \eta \end{bmatrix} = 0 \\ \begin{bmatrix} 0 \\ \varphi \end{bmatrix}^+ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \left[\sqrt{\frac{E+m}{2m}} \left(1 + \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \begin{bmatrix} \eta \\ 0 \end{bmatrix} = 0 \end{cases}$$

$$\text{性质2.3.5. } \begin{cases} \begin{bmatrix} \varphi \\ 0 \end{bmatrix}^+ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \gamma_0 \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \begin{bmatrix} \eta \\ 0 \end{bmatrix} = \varphi^+ \eta \\ \begin{bmatrix} 0 \\ \varphi \end{bmatrix}^+ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \gamma_0 \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \begin{bmatrix} 0 \\ \eta \end{bmatrix} = -\varphi^+ \eta \end{cases}$$

$$\text{性质2.3.6. } \begin{cases} \begin{bmatrix} \varphi \\ 0 \end{bmatrix}^+ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \gamma_0 \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \begin{bmatrix} 0 \\ \eta \end{bmatrix} = 0 \\ \begin{bmatrix} 0 \\ \varphi \end{bmatrix}^+ \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \gamma_0 \left[\sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right)\right] \begin{bmatrix} \eta \\ 0 \end{bmatrix} = 0 \end{cases}$$

## 2.4 N+1维时空中的电子自旋基

$$\text{定义2.4.1. } u_\zeta(\vec{p}, h) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \begin{bmatrix} 1 \\ 0 \\ 0_{l-2} \\ 0_l \end{bmatrix}, \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \begin{bmatrix} 0 \\ 1 \\ 0_{l-2} \\ 0_l \end{bmatrix}, \dots, \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \begin{bmatrix} 0_{l-2} \\ 0 \\ 1 \\ 0_l \end{bmatrix}$$

$$\text{定义2.4.2. } v_\zeta(\vec{p}, h) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \begin{bmatrix} 0_l \\ 1 \\ 0 \\ 0_{l-2} \end{bmatrix}, \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \begin{bmatrix} 0_l \\ 0 \\ 1 \\ 0_{l-2} \end{bmatrix}, \dots, \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \begin{bmatrix} 0_l \\ 0_{l-2} \\ 0 \\ 1 \end{bmatrix}$$

## 2.5 N+1维时空中Dirac自旋基是自旋、螺旋度和电荷三个算符的共同本征态

性质2.5.1.

$$\begin{cases} \sigma^2(\frac{1}{2}) \otimes I_* u(\vec{p}, h) = \frac{1}{2}(\frac{1}{2} + 1)u(\vec{p}, h) \\ \sigma(\frac{1}{2}) \cdot \hat{p} \otimes I_* u(\vec{p}, h) = (-1)^{h+1} \frac{1}{2} u(\vec{p}, h) \\ \hat{Q}(\vec{p}) u(\vec{p}, h) = -u(\vec{p}, h), h = 1, 2, \dots, l \\ \frac{E+m}{2m} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \left[\left(\frac{l+1}{2}\right) - \sigma_z\left(\frac{l-1}{2}\right)\right] \left(1 + \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) u(\vec{p}, h) = hu(\vec{p}, h) \\ \text{描述电子: } (s, h; Q) = (\frac{1}{2}; h, -1) \end{cases}$$

$$\begin{cases} \sigma^2(\frac{1}{2}) \otimes I_* v(\vec{p}, h) = \frac{1}{2}(\frac{1}{2} + 1)v(\vec{p}, h) \\ \sigma(\frac{1}{2}) \cdot \hat{p} \otimes I_* v(\vec{p}, h) = (-1)^{h+1} \frac{1}{2} v(\vec{p}, h) \\ \hat{Q}(\vec{p}) v(\vec{p}, h) = v(\vec{p}, h), h = 1, 2, \dots, l \\ \frac{E+m}{2m} \left(1 - \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) \left[\left(\frac{l+1}{2}\right) - \sigma_z\left(\frac{l-1}{2}\right)\right] \left(1 + \frac{i\vec{p}\cdot\vec{\gamma}\gamma_0}{E+m}\right) v(\vec{p}, h) = hv(\vec{p}, h) \\ \text{描述正电子: } (s, h; Q) = (\frac{1}{2}; h, 1) \end{cases}$$

## 2.6 N+1维时空中的电子自旋空间

### 2.7 N+1维时空中的电子自旋基性质(在一般表象下也成立)

$$\text{推论2.7.1. } \bar{u}_\zeta(\vec{p}, h) u_\zeta(\vec{p}, h') = \zeta \delta_{hh'}, \bar{v}_\zeta(\vec{p}, h) v_\zeta(\vec{p}, h') = -\zeta \delta_{hh'}, \bar{u}_\zeta(\vec{p}, h) v_\zeta(\vec{p}, h') = 0, \bar{v}_\zeta(\vec{p}, h) u_\zeta(\vec{p}, h') = 0$$

推论2.7.2.

$$u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}, h') = \frac{E}{m} \delta_{hh'}, v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}, h') = \frac{E}{m} \delta_{hh'}, u_\zeta^+(\vec{p}, h) v_\zeta(-\vec{p}, h') = 0, v_\zeta^+(\vec{p}, h) u_\zeta(-\vec{p}, h') = 0$$

$$\text{推论2.7.3. } \sum_h u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h) = \frac{\zeta m - i\gamma^a p_a}{2m}, \sum_h v_\zeta(\vec{p}, h) \bar{v}_\zeta(\vec{p}, h) = \frac{-\zeta m - i\gamma^a p_a}{2m}$$

$$\text{推论2.7.4. } \sum_h u_\zeta(\vec{p}, h) u_\zeta^+(\vec{p}, h) = \frac{(\zeta m - i\gamma^a p_a) \gamma_0}{2m}, \sum_h v_\zeta(\vec{p}, h) v_\zeta^+(\vec{p}, h) = \frac{(-\zeta m - i\gamma^a p_a) \gamma_0}{2m}$$

$$\text{推论2.7.5. } \begin{cases} \sum_h [u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h) - v_\zeta(\vec{p}, h) \bar{v}_\zeta(\vec{p}, h)] = \zeta \\ \sum_h [u_\zeta(\vec{p}, h) \bar{u}_\zeta(\vec{p}, h) + v_\zeta(\vec{p}, h) \bar{v}_\zeta(\vec{p}, h)] = \frac{-i\gamma^a p_a}{m} \\ \sum_h [u_\zeta(\vec{p}, h) u_\zeta^+(\vec{p}, h) + v_\zeta(-\vec{p}, h) v_\zeta^+(-\vec{p}, h)] = \frac{E}{m} \end{cases}$$

## 2.8 N+1维时空中电子的平面波解

推论2.8.1.

$$\psi(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^l [a_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta(\vec{p}, h) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} + b_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} v_\zeta(\vec{p}, h) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)}] d^N \vec{p}$$

$$\begin{cases} a_\zeta(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{E}{m}} u_\zeta^+(\vec{p}, h) \psi(\vec{r}, t) e^{-i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3 \vec{r} \\ b_\zeta^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{E}{m}} v_\zeta^+(\vec{p}, h) \psi(\vec{r}, t) e^{i\zeta(\vec{p}\cdot\vec{r}-Et)} d^3 \vec{r} \end{cases}$$

## 2.9 N+1维时空中电子的协变量子化规则

$$\text{推论2.9.1. } \begin{cases} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}') \\ \{a_\zeta(\vec{p}, h), a_\zeta(\vec{p}', h')\} = 0 \\ \{a_\zeta^+(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} = 0 \end{cases} \Rightarrow \begin{cases} \{\psi_\alpha(\vec{r}, t), \psi_\beta^+(\vec{r}', t)\} = \delta_{\alpha\beta} \delta^3(\vec{r} - \vec{r}') \\ \{\psi_\alpha(\vec{r}, t), \psi_\beta(\vec{r}', t)\} = 0 \\ \{\psi_\alpha^+(\vec{r}, t), \psi_\beta^+(\vec{r}', t)\} = 0 \end{cases}$$

$$\text{定理2.9.1. } \{\psi(x), \psi^+(x')\} = i(m - \gamma^a \partial_a) \gamma^0 \Delta(x - x')$$

证明:

$$\begin{aligned} \{\psi(x), \psi^+(x')\} &= \frac{1}{(2\pi)^N} \int \sqrt{\frac{m}{E}} \sqrt{\frac{m}{E'}} \sum_{h, h'=1}^l u_\zeta(\vec{p}, h) u_\zeta^+(\vec{p}', h') e^{i\zeta(p \cdot x - p' \cdot x')} \{a_\zeta(\vec{p}, h), a_\zeta^+(\vec{p}', h')\} \\ &+ v_\zeta(\vec{p}, h) v_\zeta^+(\vec{p}', h') e^{-i\zeta(p \cdot x - p' \cdot x')} \{b_\zeta^+(\vec{p}, h), b_\zeta(\vec{p}', h')\} d^N \vec{p} d^N \vec{p}' \\ &= \frac{1}{(2\pi)^N} \int \sqrt{\frac{m}{E}} \sqrt{\frac{m}{E'}} \sum_{h, h'=1}^l \delta_{hh'} \delta^N(\vec{p} - \vec{p}') [u_\zeta(\vec{p}, h) u_\zeta^+(\vec{p}', h') e^{i\zeta(p \cdot x - p' \cdot x')} + v_\zeta(\vec{p}, h) v_\zeta^+(\vec{p}', h') e^{-i\zeta(p \cdot x - p' \cdot x')}] d^N \vec{p} d^N \vec{p}' \\ &= \frac{1}{(2\pi)^N} \int \frac{m}{E} \left[ \sum_{h=1}^l u_\zeta(\vec{p}, h) u_\zeta^+(\vec{p}, h) e^{i\zeta p \cdot (x-x')} + \sum_{h=1}^l v_\zeta(\vec{p}, h) v_\zeta^+(\vec{p}, h) e^{-i\zeta p \cdot (x-x')} \right] d^N \vec{p} \\ &= \frac{1}{(2\pi)^N} \int \frac{m}{E} \left[ \frac{(\zeta m - i\gamma^a p_a) \gamma^0}{2m} e^{i\zeta p \cdot (x-x')} + \frac{(-\zeta m - i\gamma^a p_a) \gamma^0}{2m} e^{-i\zeta p \cdot (x-x')} \right] d^N \vec{p} \\ &= \frac{1}{(2\pi)^N} \int \frac{1}{2E} \zeta (m - \gamma^a \partial_a) \gamma^0 [e^{i\zeta p \cdot (x-x')} - e^{-i\zeta p \cdot (x-x')}] d^N \vec{p} \\ &= i(m - \gamma^a \partial_a) \gamma^0 \frac{-i\zeta}{(2\pi)^N} \int \frac{1}{2E} [e^{i\zeta p \cdot (x-x')} - e^{-i\zeta p \cdot (x-x')}] d^N \vec{p} \\ &= i(m - \gamma^a \partial_a) \gamma^0 \Delta(x - x') \end{aligned} \quad \square$$

## 2.10 N+1维时空中Dirac方程的守恒荷

$$\text{推论2.10.1. } Q = \int \psi^+ \psi d^N r = \int \sum_h [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) + b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^N \vec{p}$$

$$\text{证明: } Q = \int \psi^+ \psi d^N r$$

$$\begin{aligned} &= \frac{1}{(2\pi)^N} \int \sum_{h, h'} [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h) e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} + b_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} v_\zeta^+(\vec{p}, h) e^{i\zeta(\vec{p} \cdot \vec{r} - Et)}] \\ &[a_\zeta(\vec{p}', h') \sqrt{\frac{m}{E'}} u_\zeta(\vec{p}', h') e^{i\zeta(\vec{p}' \cdot \vec{r} - E't)} + b_\zeta^+(\vec{p}', h') \sqrt{\frac{m}{E'}} v_\zeta(\vec{p}', h') e^{-i\zeta(\vec{p}' \cdot \vec{r} - E't)}] d^N \vec{p}' d^N \vec{p} d^N r \\ &= \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}', h') + b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}', h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}', h')] \delta^N(\vec{p} - \vec{p}') d^N \vec{p}' d^N \vec{p} \\ &= \int \sum_{h, h'} \frac{m}{E} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}, h') + b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}, h')] d^N \vec{p} \\ &= \int \sum_h [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) + b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^N \vec{p} \end{aligned} \quad \square$$

$$\text{推论2.10.2. } H = i \int \psi^+ \partial_t \psi d^N r = \zeta \int \sum_h E(\vec{p}) [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^N \vec{p}$$

$$\text{证明: } H = i \int \psi^+ \partial_t \psi d^N r$$

$$\begin{aligned} &= i \frac{1}{(2\pi)^N} \int \sum_{h, h'} [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h) e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} + b_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} v_\zeta^+(\vec{p}, h) e^{i\zeta(\vec{p} \cdot \vec{r} - Et)}] \\ &(-i\zeta E') [a_\zeta(\vec{p}', h') \sqrt{\frac{m}{E'}} u_\zeta(\vec{p}', h') e^{i\zeta(\vec{p}' \cdot \vec{r} - E't)} - b_\zeta^+(\vec{p}', h') \sqrt{\frac{m}{E'}} v_\zeta(\vec{p}', h') e^{-i\zeta(\vec{p}' \cdot \vec{r} - E't)}] d^N \vec{p}' d^N \vec{p} d^N r \\ &= -i \int \sum_{h, h'} \frac{m}{E} (-i\zeta E') [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}', h') - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}', h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}', h')] \delta^N(\vec{p} - \vec{p}') d^N \vec{p}' d^N \vec{p} \\ &= -i \int \sum_{h, h'} \frac{m}{E} (-i\zeta E') [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}, h') - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}, h')] d^N \vec{p} \\ &= \zeta \int \sum_h E(\vec{p}) [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^N \vec{p} \end{aligned} \quad \square$$

$$\text{推论2.10.3. } \vec{P} = -i \int \psi^+ \nabla \psi d^N r = \zeta \int \sum_h \vec{p} [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}, h) - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}, h)] d^N \vec{p}$$

$$\text{证明: } \vec{P} = -i \int \psi^+ \nabla \psi d^N r$$

$$\begin{aligned} &= -i \frac{1}{(2\pi)^N} \int \sum_{h, h'} [a_\zeta^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_\zeta^+(\vec{p}, h) e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} + b_\zeta(\vec{p}, h) \sqrt{\frac{m}{E}} v_\zeta^+(\vec{p}, h) e^{i\zeta(\vec{p} \cdot \vec{r} - Et)}] \\ &(i\zeta \vec{p}') [a_\zeta(\vec{p}', h') \sqrt{\frac{m}{E'}} u_\zeta(\vec{p}', h') e^{i\zeta(\vec{p}' \cdot \vec{r} - E't)} - b_\zeta^+(\vec{p}', h') \sqrt{\frac{m}{E'}} v_\zeta(\vec{p}', h') e^{-i\zeta(\vec{p}' \cdot \vec{r} - E't)}] d^N \vec{p}' d^N \vec{p} d^N r \\ &= -i \int \sum_{h, h'} \frac{m}{E} (i\zeta \vec{p}') [a_\zeta^+(\vec{p}, h) a_\zeta(\vec{p}', h') u_\zeta^+(\vec{p}, h) u_\zeta(\vec{p}', h') - b_\zeta(\vec{p}, h) b_\zeta^+(\vec{p}', h') v_\zeta^+(\vec{p}, h) v_\zeta(\vec{p}', h')] \delta^N(\vec{p} - \vec{p}') d^N \vec{p}' d^N \vec{p} \end{aligned}$$

$$\begin{aligned}
&= -i \int \sum_{h, h'} \frac{m}{E} (i\zeta \vec{p}) [a_{\zeta}^+(\vec{p}, h) a_{\zeta}(\vec{p}, h') u_{\zeta}^+(\vec{p}, h) u_{\zeta}(\vec{p}, h') - b_{\zeta}(\vec{p}, h) b_{\zeta}^+(\vec{p}, h') v_{\zeta}^+(\vec{p}, h) v_{\zeta}(\vec{p}, h')] d^N \vec{p} \\
&= \zeta \int \sum_h \vec{p} [a_{\zeta}^+(\vec{p}, h) a_{\zeta}(\vec{p}, h) - b_{\zeta}(\vec{p}, h) b_{\zeta}^+(\vec{p}, h)] d^N \vec{p} \quad \square
\end{aligned}$$

$$\text{推论2.10.4. } P_u = -i \int \psi^+ \partial_u \psi d^N r = \zeta \int \sum_h p_u [a_{\zeta}^+(\vec{p}, h) a_{\zeta}(\vec{p}, h) - b_{\zeta}(\vec{p}, h) b_{\zeta}^+(\vec{p}, h)] d^N \vec{p}$$

$$\begin{aligned}
&\text{证明: } P_u = -i \int \psi^+ \partial_u \psi d^N r \\
&= i \frac{1}{(2\pi)^N} \int \sum_{h, h'} [a_{\zeta}^+(\vec{p}, h) \sqrt{\frac{m}{E}} u_{\zeta}^+(\vec{p}, h) e^{-i\zeta(\vec{p} \cdot \vec{r} - Et)} + b_{\zeta}(\vec{p}, h) \sqrt{\frac{m}{E}} v_{\zeta}^+(\vec{p}, h) e^{i\zeta(\vec{p} \cdot \vec{r} - Et)}] \\
&\quad (i\zeta \vec{p}'_u) [a_{\zeta}(\vec{p}', h') \sqrt{\frac{m}{E'}} u_{\zeta}(\vec{p}', h') e^{i\zeta(\vec{p}' \cdot \vec{r} - E't)} - b_{\zeta}^+(\vec{p}', h') \sqrt{\frac{m}{E'}} v_{\zeta}(\vec{p}', h') e^{-i\zeta(\vec{p}' \cdot \vec{r} - E't)}] d^N \vec{p}' d^N \vec{p} d^N r \\
&= -i \int \sum_{h, h'} \frac{m}{E} (i\zeta \vec{p}'_u) [a_{\zeta}^+(\vec{p}, h) a_{\zeta}(\vec{p}', h') u_{\zeta}^+(\vec{p}, h) u_{\zeta}(\vec{p}', h') - b_{\zeta}(\vec{p}, h) b_{\zeta}^+(\vec{p}', h') v_{\zeta}^+(\vec{p}, h) v_{\zeta}(\vec{p}', h')] \delta^N(\vec{p} - \vec{p}') d^N \vec{p}' d^N \vec{p} \\
&= -i \int \sum_{h, h'} \frac{m}{E} (i\zeta \vec{p}'_u) [a_{\zeta}^+(\vec{p}, h) a_{\zeta}(\vec{p}, h') u_{\zeta}^+(\vec{p}, h) u_{\zeta}(\vec{p}, h') - b_{\zeta}(\vec{p}, h) b_{\zeta}^+(\vec{p}, h') v_{\zeta}^+(\vec{p}, h) v_{\zeta}(\vec{p}, h')] d^N \vec{p} \\
&= \zeta \int \sum_h p_u [a_{\zeta}^+(\vec{p}, h) a_{\zeta}(\vec{p}, h) - b_{\zeta}(\vec{p}, h) b_{\zeta}^+(\vec{p}, h)] d^N \vec{p} \quad \square
\end{aligned}$$

### 3 N+1维时空中Bargmann-Wigner方程的协变量子化

本节证明可以参照四维时空中的方法, 完全可以将四维时空中的证明方法一一对应平移到N+1维时空中来, 故一般不再详细给出。

#### 3.1 N+1维时空中的电子正常自旋基性质(在一般表象下也成立)

$$\text{定义3.1.1. } u(\vec{p}, h) := \begin{cases} u_+(\vec{p}, h), \zeta = 1 \\ v_-(\vec{p}, h), \zeta = -1 \end{cases}, v(\vec{p}, h) := \begin{cases} v_+(\vec{p}, h), \zeta = 1 \\ u_-(\vec{p}, h), \zeta = -1 \end{cases}$$

$$\text{推论3.1.1. } \bar{u}(\vec{p}, h) u(\vec{p}, h') = \delta_{hh'}, \bar{v}(\vec{p}, h) v(\vec{p}, h') = -\delta_{hh'}, \bar{u}(\vec{p}, h) v(\vec{p}, h') = 0, \bar{v}(\vec{p}, h) u(\vec{p}, h') = 0$$

$$\text{推论3.1.2. } u^+(\vec{p}, h) u(\vec{p}, h') = \frac{E}{m} \delta_{hh'}, v^+(\vec{p}, h) v(\vec{p}, h') = \frac{E}{m} \delta_{hh'}, u^+(\vec{p}, h) v(-\vec{p}, h') = 0, v^+(\vec{p}, h) u(-\vec{p}, h') = 0$$

$$\text{推论3.1.3. } \sum_h u(\vec{p}, h) \bar{u}(\vec{p}, h) = \frac{m - i\gamma^a p_a}{2m}, \sum_h v(\vec{p}, h) \bar{v}(\vec{p}, h) = \frac{-m - i\gamma^a p_a}{2m}$$

$$\text{推论3.1.4. } \sum_h u(\vec{p}, h) u^+(\vec{p}, h) = \frac{(m - i\gamma^a p_a) \gamma_0}{2m}, \sum_h v(\vec{p}, h) v^+(\vec{p}, h) = \frac{(-m - i\gamma^a p_a) \gamma_0}{2m}$$

$$\text{推论3.1.5. } \begin{cases} \sum_h [u(\vec{p}, h) \bar{u}(\vec{p}, h) - v(\vec{p}, h) \bar{v}(\vec{p}, h)] = 1 \\ \sum_h [u(\vec{p}, h) \bar{u}(\vec{p}, h) + v(\vec{p}, h) \bar{v}(\vec{p}, h)] = \frac{-i\gamma^a p_a}{m} \\ \sum_h [u(\vec{p}, h) u^+(\vec{p}, h) + v(-\vec{p}, h) v^+(-\vec{p}, h)] = \frac{E}{m} \end{cases}$$

#### 3.2 N+1维时空中Dirac方程自旋基的广义多项式定理

定理3.2.1.

$$\begin{aligned}
&\sum_{n_i=2s} \frac{(2s)!}{n_1! n_2! \cdots n_l!} \underbrace{u_{\{\lambda_{\zeta}(\vec{p}, 1) \cdots u_{\mu_{\zeta}(\vec{p}, 2) \cdots u_{\tau_{\zeta}(\vec{p}, l) \cdots}}}_{n_1} \underbrace{u_{\{\lambda'_{\zeta}(\vec{p}, 1) \cdots u_{\mu'_{\zeta}(\vec{p}, 2) \cdots u_{\tau'_{\zeta}(\vec{p}, l) \cdots}}}_{n_1} \underbrace{u_{\{\lambda_{\zeta}(\vec{p}, 1) \cdots u_{\mu_{\zeta}(\vec{p}, 2) \cdots u_{\tau_{\zeta}(\vec{p}, l) \cdots}}}_{n_2} \underbrace{u_{\{\lambda_{\zeta}(\vec{p}, 1) \cdots u_{\mu_{\zeta}(\vec{p}, 2) \cdots u_{\tau_{\zeta}(\vec{p}, l) \cdots}}}_{n_l} \cdots}}_{n_l} \cdots}}_{2s} \\
&= \sum_{h=1}^l u_{\{\lambda_{\zeta}(\vec{p}, h) u_{\{\lambda'_{\zeta}(\vec{p}, h) \cdots \sum_{h=1}^l u_{\mu_{\zeta}(\vec{p}, h) u_{\{\mu'_{\zeta}(\vec{p}, h) \cdots \sum_{h=1}^l u_{\tau_{\zeta}(\vec{p}, h) u_{\{\tau'_{\zeta}(\vec{p}, h) \cdots \}}}_{n_1} \underbrace{u_{\{\lambda_{\zeta}(\vec{p}, 1) \cdots u_{\mu_{\zeta}(\vec{p}, 2) \cdots u_{\tau_{\zeta}(\vec{p}, l) \cdots}}}_{n_2} \underbrace{u_{\{\lambda_{\zeta}(\vec{p}, 1) \cdots u_{\mu_{\zeta}(\vec{p}, 2) \cdots u_{\tau_{\zeta}(\vec{p}, l) \cdots}}}_{n_l} \cdots}}}_{n_l} \cdots}}_{n_l} \cdots}
\end{aligned}$$

推论3.2.1.

$$\begin{aligned}
&\sum_{n_1 \cdots n_l} \frac{(2s)!}{n_1! n_2! \cdots n_l!} \underbrace{u_{\{i_{\zeta}(\vec{p}, 1) \cdots u_{i_{\zeta}(\vec{p}, 2) \cdots u_{i_{\zeta}(\vec{p}, l) \cdots}}}_{n_1} \underbrace{u_{\{i'_{\zeta}(\vec{p}, 1) \cdots u_{i'_{\zeta}(\vec{p}, 2) \cdots u_{i'_{\zeta}(\vec{p}, l) \cdots}}}_{n_1} \underbrace{u_{\{i_{\zeta}(\vec{p}, 1) \cdots u_{i_{\zeta}(\vec{p}, 2) \cdots u_{i_{\zeta}(\vec{p}, l) \cdots}}}_{n_2} \underbrace{u_{\{i_{\zeta}(\vec{p}, 1) \cdots u_{i_{\zeta}(\vec{p}, 2) \cdots u_{i_{\zeta}(\vec{p}, l) \cdots}}}_{n_l} \cdots}}}_{n_l} \cdots}}_{2s} \\
&= \sum_{h=1}^l u_{\{i_{\zeta}(\vec{p}, h) u_{\{i'_{\zeta}(\vec{p}, h) \cdots \sum_{h=1}^l u_{i_{\zeta}(\vec{p}, h) u_{\{i'_{\zeta}(\vec{p}, h) \cdots \sum_{h=1}^l u_{i_{\zeta}(\vec{p}, h) u_{\{i'_{\zeta}(\vec{p}, h) \cdots \}}}_{n_1} \underbrace{u_{\{i_{\zeta}(\vec{p}, 1) \cdots u_{i_{\zeta}(\vec{p}, 2) \cdots u_{i_{\zeta}(\vec{p}, l) \cdots}}}_{n_2} \underbrace{u_{\{i_{\zeta}(\vec{p}, 1) \cdots u_{i_{\zeta}(\vec{p}, 2) \cdots u_{i_{\zeta}(\vec{p}, l) \cdots}}}_{n_l} \cdots}}}_{n_l} \cdots}
\end{aligned}$$

$$\Leftrightarrow \sum_{n_1 \cdots n_l} \frac{(2s)!}{n_1! n_2! \cdots n_l!} [u_{i_\zeta}(\vec{p}, 1) u_{i'_\zeta}^\dagger(\vec{p}, 1)]^{n_1} [u_{i_\zeta}(\vec{p}, 2) u_{i'_\zeta}^\dagger(\vec{p}, 2)]^{n_2} \cdots [u_{i_\zeta}(\vec{p}, l) u_{i'_\zeta}^\dagger(\vec{p}, l)]^{n_l} = \left[ \sum_{h=1}^l u_{i_\zeta}(\vec{p}, h) u_{i'_\zeta}^\dagger(\vec{p}, h) \right]^{2s}$$

以上推论正好就是多项式展开定理。

### 3.3 N+1维时空中Bargmann-Wigner方程的自旋基

定义3.3.1.

$$\begin{cases} U_{\underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l) := \frac{1}{\sqrt{(2s)! n_1! n_2! \cdots n_l!}} \underbrace{u_{\lambda_\zeta}(\vec{p}, 1)}_{n_1} \cdots \underbrace{u_{\mu_\zeta}(\vec{p}, 2)}_{n_2} \cdots \underbrace{u_{\tau_\zeta}(\vec{p}, l)}_{n_l}, n_1 + n_2 + \cdots + n_l = 2s \\ V_{\underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l) := \frac{1}{\sqrt{(2s)! n_1! n_2! \cdots n_l!}} \underbrace{v_{\lambda_\zeta}(\vec{p}, 1)}_{n_1} \cdots \underbrace{v_{\mu_\zeta}(\vec{p}, 2)}_{n_2} \cdots \underbrace{v_{\tau_\zeta}(\vec{p}, l)}_{n_l}, n_1 + n_2 + \cdots + n_l = 2s \end{cases}$$

### 3.4 N+1维时空中Bargmann-Wigner方程自旋基的正交性质(可以直接看出来)

推论3.4.1.

$$\begin{cases} \overline{U}_{\underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l) U_{\underbrace{\lambda'_\zeta \cdots \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}; n'_1, n'_2, \cdots, n'_l) = \delta_{n_1 n'_1} \delta_{n_2 n'_2} \cdots \delta_{n_l n'_l} \\ \overline{V}_{\underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l) V_{\underbrace{\lambda'_\zeta \cdots \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}; n'_1, n'_2, \cdots, n'_l) = \delta_{n_1 n'_1} \delta_{n_2 n'_2} \cdots \delta_{n_l n'_l} \\ \overline{U}_{\underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l) V_{\underbrace{\lambda'_\zeta \cdots \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}; n'_1, n'_2, \cdots, n'_l) = 0 \\ \overline{V}_{\underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l) U_{\underbrace{\lambda'_\zeta \cdots \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}; n'_1, n'_2, \cdots, n'_l) = 0 \end{cases}$$

推论3.4.2.

$$\begin{cases} U_{\underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l) U_{\underbrace{\lambda'_\zeta \cdots \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}; n'_1, n'_2, \cdots, n'_l) = \left(\frac{E}{m}\right)^{2s} \delta_{n_1 n'_1} \delta_{n_2 n'_2} \cdots \delta_{n_l n'_l} \\ V_{\underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l) V_{\underbrace{\lambda'_\zeta \cdots \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(\vec{p}; n'_1, n'_2, \cdots, n'_l) = \left(\frac{E}{m}\right)^{2s} \delta_{n_1 n'_1} \delta_{n_2 n'_2} \cdots \delta_{n_l n'_l} \\ U_{\underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l) V_{\underbrace{\lambda'_\zeta \cdots \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(-\vec{p}; n'_1, n'_2, \cdots, n'_l) = 0 \\ V_{\underbrace{\lambda_\zeta \cdots \mu_\zeta \cdots \tau_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l) U_{\underbrace{\lambda'_\zeta \cdots \mu'_\zeta \cdots \tau'_\zeta}_{2s}}(-\vec{p}; n'_1, n'_2, \cdots, n'_l) = 0 \end{cases}$$

### 3.5 N+1维时空中Bargmann-Wigner方程U-自旋基的分解

定理3.5.1.  $U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta \lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_l)$

$$= \sum_{n'_1 + \cdots + n'_l} \frac{\sqrt{C_{n'_1}^{n'_1} C_{n'_2}^{n'_2} \cdots C_{n'_l}^{n'_l}}}{\sqrt{C_{2s}^{2s'}}} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2(s-s')}}(\vec{p}; n_1 - n'_1, n_2 - n'_2, \cdots, n_l - n'_l) U_{\underbrace{\lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}_{2s'}}(\vec{p}; n'_1, n'_2, \cdots, n'_l)$$

证明:  $U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta \lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}_{2s}}(\vec{p}, h) = \frac{1}{\sqrt{(2s)! n_1! n_2! \cdots n_l!}}$

$$\sum_{n'_1 + \cdots + n'_l}^{=2s'} C_{n'_1}^{n'_1} C_{n'_2}^{n'_2} \cdots C_{n'_l}^{n'_l} \underbrace{u_{\lambda_\zeta}(\vec{p}, 1) u_{\mu_\zeta}(\vec{p}, 1) \cdots u_{\sigma_\zeta}(\vec{p}, l) u_{\tau_\zeta}(\vec{p}, l)}_{n_1 - n'_1} \underbrace{u_{\lambda'_\zeta}(\vec{p}, 1) u_{\mu'_\zeta}(\vec{p}, 1) \cdots u_{\sigma'_\zeta}(\vec{p}, l) u_{\tau'_\zeta}(\vec{p}, l)}_{n'_l}$$

$$= \frac{1}{\sqrt{(2s)! n_1! n_2! \cdots n_l!}} \sum_{n'_1 + \cdots + n'_l}^{=2s'} C_{n'_1}^{n'_1} C_{n'_2}^{n'_2} \cdots C_{n'_l}^{n'_l}$$

$$\sqrt{(2s - 2s')! (n_1 - n'_1)! (n_2 - n'_2)! \cdots (n_l - n'_l)!} U_{\underbrace{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}_{2(s-s')}}(\vec{p}; n_1 - n'_1, n_2 - n'_2, \cdots, n_l - n'_l)$$



$$\begin{aligned} & \sqrt{(2s')!n_1!n_2!\cdots n_l!} \underbrace{U_{\lambda'_s\mu'_s\cdots\sigma'_s\tau'_s}}_{2s'}(\vec{p}; n'_1, n'_2, \cdots, n'_l) \\ &= \sum_{n'_1+\cdots+n'_l=2s'}^{=2s'} \frac{\sqrt{C_{n'_1}^{n'_1}C_{n'_2}^{n'_2}\cdots C_{n'_l}^{n'_l}}}{\sqrt{C_{2s'}^{2s'}}} \underbrace{U_{\lambda_s\mu_s\cdots\sigma_s\tau_s}}_{2(s-s')}(\vec{p}; n_1 - n'_1, n_2 - n'_2, \cdots, n_l - n'_l) \underbrace{U_{\lambda'_s\mu'_s\cdots\sigma'_s\tau'_s}}_{2s'}(\vec{p}; n'_1, n'_2, \cdots, n'_l) \quad \square \end{aligned}$$

$$\begin{aligned} \text{推论3.5.1. } & \underbrace{U_{\lambda_s\mu_s\cdots\sigma_s\tau_s}}_{2s}(\vec{p}; n_1, n_2, \cdots, n_l) = \frac{\sqrt{n_1}}{\sqrt{2s}} \underbrace{U_{\lambda_s\mu_s\cdots\sigma_s}}_{2s-1}(\vec{p}; n_1 - 1, n_2, \cdots, n_l) U_{\tau_s}(\vec{p}; 1, 0, \cdots, 0) \\ & + \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{U_{\lambda_s\mu_s\cdots\sigma_s}}_{2s-1}(\vec{p}; n_1, n_2 - 1, \cdots, n_l) U_{\tau_s}(\vec{p}; 0, 1, \cdots, 0) + \cdots + \frac{\sqrt{n_l}}{\sqrt{2s}} \underbrace{U_{\lambda_s\mu_s\cdots\sigma_s}}_{2s-1}(\vec{p}; n_1, n_2, \cdots, n_l - 1) U_{\tau_s}(\vec{p}; 0, 0, \cdots, 1) \end{aligned}$$

$$\begin{aligned} \text{推论3.5.2. } & \underbrace{U_{\lambda_s\mu_s\cdots\sigma_s\tau_s}}_{2s}(\vec{p}; 0, n_2, \cdots, n_l) \\ &= \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{U_{\lambda_s\mu_s\cdots\sigma_s}}_{2s-1}(\vec{p}; 0, n_2 - 1, \cdots, n_l) U_{\tau_s}(\vec{p}; 0, 1, \cdots, 0) + \cdots + \frac{\sqrt{n_l}}{\sqrt{2s}} \underbrace{U_{\lambda_s\mu_s\cdots\sigma_s}}_{2s-1}(\vec{p}; 0, n_2, \cdots, n_l - 1) U_{\tau_s}(\vec{p}; 0, 0, \cdots, 1) \end{aligned}$$

### 3.6 N+1维时空中Bargmann-Wigner方程V-自旋基的分解

$$\begin{aligned} \text{定理3.6.1. } & \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s\tau_s\lambda'_s\mu'_s\cdots\sigma'_s\tau'_s}}_{2s}(\vec{p}; n_1, n_2, \cdots, n_l) \\ &= \sum_{n'_1+\cdots+n'_l=2s'}^{=2s'} \frac{\sqrt{C_{n'_1}^{n'_1}C_{n'_2}^{n'_2}\cdots C_{n'_l}^{n'_l}}}{\sqrt{C_{2s'}^{2s'}}} \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s\tau_s}}_{2(s-s')}(\vec{p}; n_1 - n'_1, n_2 - n'_2, \cdots, n_l - n'_l) \underbrace{V_{\lambda'_s\mu'_s\cdots\sigma'_s\tau'_s}}_{2s'}(\vec{p}; n'_1, n'_2, \cdots, n'_l) \end{aligned}$$

$$\begin{aligned} \text{证明: } & \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s\tau_s\lambda'_s\mu'_s\cdots\sigma'_s\tau'_s}}_{2s}(\vec{p}, h) = \frac{1}{\sqrt{(2s)!n_1!n_2!\cdots n_l!}} \\ & \sum_{n'_1+\cdots+n'_l=2s'}^{=2s'} C_{n'_1}^{n'_1} C_{n'_2}^{n'_2} \cdots C_{n'_l}^{n'_l} \underbrace{v_{\lambda_s}(\vec{p}, 1)v_{\mu_s}(\vec{p}, 1)\cdots v_{\sigma_s}(\vec{p}, l)v_{\tau_s}(\vec{p}, l)}}_{n_1-n'_1} \underbrace{v_{\lambda'_s}(\vec{p}, 1)v_{\mu'_s}(\vec{p}, 1)\cdots v_{\sigma'_s}(\vec{p}, l)v_{\tau'_s}(\vec{p}, l)}}_{n_l-n'_l} \\ &= \frac{1}{\sqrt{(2s)!n_1!n_2!\cdots n_l!}} \sum_{n'_1+\cdots+n'_l=2s'}^{=2s'} C_{n'_1}^{n'_1} C_{n'_2}^{n'_2} \cdots C_{n'_l}^{n'_l} \\ & \sqrt{(2s-2s')!(n_1-n'_1)!(n_2-n'_2)\cdots(n_l-n'_l)!} \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s\tau_s}}_{2(s-s')}(\vec{p}; n_1 - n'_1, n_2 - n'_2, \cdots, n_l - n'_l) \\ & \sqrt{(2s')!n_1!n_2!\cdots n_l!} \underbrace{V_{\lambda'_s\mu'_s\cdots\sigma'_s\tau'_s}}_{2s'}(\vec{p}; n'_1, n'_2, \cdots, n'_l) \\ &= \sum_{n'_1+\cdots+n'_l=2s'}^{=2s'} \frac{\sqrt{C_{n'_1}^{n'_1}C_{n'_2}^{n'_2}\cdots C_{n'_l}^{n'_l}}}{\sqrt{C_{2s'}^{2s'}}} \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s\tau_s}}_{2(s-s')}(\vec{p}; n_1 - n'_1, n_2 - n'_2, \cdots, n_l - n'_l) \underbrace{V_{\lambda'_s\mu'_s\cdots\sigma'_s\tau'_s}}_{2s'}(\vec{p}; n'_1, n'_2, \cdots, n'_l) \quad \square \end{aligned}$$

$$\begin{aligned} \text{推论3.6.1. } & \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s\tau_s}}_{2s}(\vec{p}; n_1, n_2, \cdots, n_l) = \frac{\sqrt{n_1}}{\sqrt{2s}} \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s}}_{2s-1}(\vec{p}; n_1 - 1, n_2, \cdots, n_l) V_{\tau_s}(\vec{p}; 1, 0, \cdots, 0) \\ & + \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s}}_{2s-1}(\vec{p}; n_1, n_2 - 1, \cdots, n_l) V_{\tau_s}(\vec{p}; 0, 1, \cdots, 0) + \cdots + \frac{\sqrt{n_l}}{\sqrt{2s}} \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s}}_{2s-1}(\vec{p}; n_1, n_2, \cdots, n_l - 1) V_{\tau_s}(\vec{p}; 0, 0, \cdots, 1) \end{aligned}$$

$$\begin{aligned} \text{推论3.6.2. } & \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s\tau_s}}_{2s}(\vec{p}; 0, n_2, \cdots, n_l) \\ &= \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s}}_{2s-1}(\vec{p}; 0, n_2 - 1, \cdots, n_l) V_{\tau_s}(\vec{p}; 0, 1, \cdots, 0) + \cdots + \frac{\sqrt{n_l}}{\sqrt{2s}} \underbrace{V_{\lambda_s\mu_s\cdots\sigma_s}}_{2s-1}(\vec{p}; 0, n_2, \cdots, n_l - 1) V_{\tau_s}(\vec{p}; 0, 0, \cdots, 1) \end{aligned}$$

### 3.7 N+1维时空中Bargmann-Wigner方程的准投影算子

定义3.7.1.

$$\begin{cases} \Lambda_{+ \underbrace{\lambda_s\mu_s\cdots\sigma_s}_{2s} \underbrace{\lambda'_s\mu'_s\cdots\sigma'_s}_{2s}}(\vec{p}, s) := \sum_{n_1\cdots n_l} U_{\lambda_s\mu_s\cdots\sigma_s}(\vec{p}; n_1, n_2, \cdots, n_l) U_{\lambda'_s\mu'_s\cdots\sigma'_s}^+(\vec{p}; n_1, n_2, \cdots, n_l) \\ \Lambda_{- \underbrace{\lambda_s\mu_s\cdots\sigma_s}_{2s} \underbrace{\lambda'_s\mu'_s\cdots\sigma'_s}_{2s}}(\vec{p}, s) := \sum_{n_1\cdots n_l} U_{\lambda_s\mu_s\cdots\sigma_s}(\vec{p}; n_1, n_2, \cdots, n_l) U_{\lambda'_s\mu'_s\cdots\sigma'_s}^+(\vec{p}; n_1, n_2, \cdots, n_l) \end{cases}$$

推论3.7.1.

$$\begin{cases} \Lambda_{+\lambda_s\mu_s\cdots\tau_s\lambda'_s\mu'_s\cdots\tau'_s}(\vec{p}, s) = \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{+\{\lambda_s(\lambda'_s(\vec{p}, \frac{1}{2})\Lambda_{+\mu_s\mu'_s}(\vec{p}, \frac{1}{2}) \cdot \Lambda_{+\tau_s\tau'_s})\}}(\vec{p}, \frac{1}{2})}_{2s} \\ \Lambda_{-\lambda_s\mu_s\cdots\tau_s\lambda'_s\mu'_s\cdots\tau'_s}(\vec{p}, s) = \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{-\{\lambda_s(\lambda'_s(\vec{p}, \frac{1}{2})\Lambda_{-\mu_s\mu'_s}(\vec{p}, \frac{1}{2}) \cdot \Lambda_{-\tau_s\tau'_s})\}}(\vec{p}, \frac{1}{2})}_{2s} \end{cases}$$

从对称指标广义多项式定理便可以直接得到以上推论。

### 3.8 N+1维时空中Bargmann-Wigner方程<sup>[17]</sup>平面波解猜想 (后面章节会给出证明)

定理3.8.1.  $(\gamma^a \partial_a + m)_{\kappa_s} \psi_{\lambda_s\mu_s\cdots\tau_s}(\vec{r}, t) = 0, \psi_{\lambda_s\mu_s\cdots\tau_s}(\vec{r}, t) = \frac{1}{(2s)!} \psi_{\lambda_s\mu_s\cdots\tau_s}(\vec{r}, t)$

$$\begin{aligned} & \psi_{\lambda_s\mu_s\cdots\tau_s}(\vec{r}, t) \\ &= \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{n_1+\cdots+n_l=2s}^{=2s} \frac{m^s}{\sqrt{E}} [a(\vec{p}; n_1, \cdots, n_l) U_{\lambda_s\mu_s\cdots\tau_s}(\vec{p}; n_1, \cdots, n_l) e^{ip \cdot x} + b^+(\vec{p}; n_1, \cdots, n_l) V_{\lambda_s\mu_s\cdots\tau_s}(\vec{p}; n_1, \cdots, n_l) e^{-ip \cdot x}] d^N \vec{p} \\ U_{\lambda_s\cdots\mu_s\cdots\tau_s}(\vec{p}; n_1, n_2, \cdots, n_l) &:= \frac{1}{\sqrt{(2s)!n_1!n_2!\cdots n_l!}} \underbrace{u_{\{\lambda_s(\vec{p}, 1) \cdots u_{\mu_s}(\vec{p}, 2) \cdots u_{\tau_s}(\vec{p}, l) \cdots\}}}_{n_1, n_2, \cdots, n_l} \\ V_{\lambda_s\cdots\mu_s\cdots\tau_s}(\vec{p}; n_1, n_2, \cdots, n_l) &:= \frac{1}{\sqrt{(2s)!n_1!n_2!\cdots n_l!}} \underbrace{v_{\{\lambda_s(\vec{p}, 1) \cdots v_{\mu_s}(\vec{p}, 2) \cdots v_{\tau_s}(\vec{p}, l) \cdots\}}}_{n_1, n_2, \cdots, n_l} \end{aligned}$$

### 3.9 N+1维时空中Bargmann-Wigner方程的协变量子化规则

定义3.9.1.  $\vec{h} := (n_1, \cdots, n_l), \delta_{\vec{h}\vec{h}'} := \delta_{n_1 n'_1} \cdots \delta_{n_l n'_l}$

定理3.9.1.

$$\begin{cases} [a(\vec{p}; \vec{h}), a^+(\vec{p}'; \vec{h}')]_{-2s+1} = \delta_{\vec{h}\vec{h}'} \delta^N(\vec{p} - \vec{p}') \\ [b(\vec{p}; \vec{h}), b^+(\vec{p}'; \vec{h}')]_{-2s+1} = \delta_{\vec{h}\vec{h}'} \delta^N(\vec{p} - \vec{p}') \\ [a(\vec{p}; \vec{h}), b^+(\vec{p}'; \vec{h}')]_{-2s+1} = 0 \\ [a^+(\vec{p}; \vec{h}), b(\vec{p}'; \vec{h}')]_{-2s+1} = 0 \\ [b(\vec{p}; \vec{h}), a^+(\vec{p}'; \vec{h}')]_{-2s+1} = 0 \\ [b^+(\vec{p}; \vec{h}), a(\vec{p}'; \vec{h}')]_{-2s+1} = 0 \end{cases}$$

$$\begin{cases} [\psi_{\lambda_s\mu_s\cdots}(x), \psi_{\lambda'_s\mu'_s\cdots}^+(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^0]_{\mu_s\mu'_s \cdots})\}}}_{2s} \Delta(x - x') \\ [\psi_{\lambda_s\mu_s\cdots}^{(+)}(x), \psi_{\lambda'_s\mu'_s\cdots}^{(++)}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^0]_{\mu_s\mu'_s \cdots})\}}}_{2s} \Delta^{(+)}(x - x') \\ [\psi_{\lambda_s\mu_s\cdots}^{(-)}(x), \psi_{\lambda'_s\mu'_s\cdots}^{(-+)}(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^0]_{\mu_s\mu'_s \cdots})\}}}_{2s} \Delta^{(-)}(x - x') \end{cases}$$

证明:  $[\psi_{\lambda_s\mu_s\cdots}(x), \psi_{\lambda'_s\mu'_s\cdots}^+(x')] = \frac{1}{(2\pi)^N} \int d^N \vec{p} d^N \vec{p}' \sum_{\vec{h}, \vec{h}'} (EE')^{s-\frac{1}{2}} \sqrt{\frac{m^2}{EE'}}^{2s}$

$$\begin{aligned} & [[a(\vec{p}, \vec{h}) U_{\lambda_s\mu_s\cdots}(\vec{p}, \vec{h}) e^{ip \cdot x} + b^+(\vec{p}, \vec{h}) V_{\lambda_s\mu_s\cdots}(\vec{p}, \vec{h}) e^{-ip \cdot x}, [a^+(\vec{p}', \vec{h}') U_{\lambda'_s\mu'_s\cdots}^+(\vec{p}', \vec{h}') e^{-ip' \cdot x'} + b(\vec{p}', \vec{h}') V_{\lambda'_s\mu'_s\cdots}^+(\vec{p}', \vec{h}') e^{ip' \cdot x'}] \\ &= \frac{1}{(2\pi)^N} \int d^N \vec{p} d^N \vec{p}' \sum_{\vec{h}, \vec{h}'} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} \\ & [U_{\lambda_s\mu_s\cdots}(\vec{p}, \vec{h}) U_{\lambda'_s\mu'_s\cdots}^+(\vec{p}', \vec{h}') [a(\vec{p}, \vec{h}), a^+(\vec{p}', \vec{h}')] e^{i(p \cdot x - p' \cdot x')} + V_{\lambda_s\mu_s\cdots}(\vec{p}, \vec{h}) V_{\lambda'_s\mu'_s\cdots}^+(\vec{p}', \vec{h}') [b^+(\vec{p}, \vec{h}), b(\vec{p}', \vec{h}')] e^{-i(p \cdot x - p' \cdot x')} \\ &= \frac{1}{(2\pi)^N} \int d^N \vec{p} d^N \vec{p}' \sum_{\vec{h}, \vec{h}'} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} \\ & [U_{\lambda_s\mu_s\cdots}(\vec{p}, \vec{h}) U_{\lambda'_s\mu'_s\cdots}^+(\vec{p}', \vec{h}') \delta_{\vec{h}\vec{h}'} \delta^N(\vec{p} - \vec{p}') e^{i(p \cdot x - p' \cdot x')} + (-1)^{2s+1} V_{\lambda_s\mu_s\cdots}(\vec{p}, \vec{h}) V_{\lambda'_s\mu'_s\cdots}^+(\vec{p}', \vec{h}') \delta_{\vec{h}\vec{h}'} \delta^N(\vec{p} - \vec{p}') e^{-i(p \cdot x - p' \cdot x')}] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} \left[ \sum_{\vec{h}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, \vec{h}) \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}, \vec{h}) e^{ip \cdot (x-x')} + (-1)^{2s+1} \sum_{\vec{h}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}, \vec{h}) e^{-ip \cdot (x-x')} \right] \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} \left[ \Lambda_{+\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(\vec{p}, s) e^{ip \cdot (x-x')} + (-1)^{2s+1} \Lambda_{-\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(\vec{p}, s) e^{-ip \cdot (x-x')} \right] \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} \left[ \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{+\{\lambda_\zeta(\lambda'_\zeta(\vec{p}, \frac{1}{2})\Lambda_{+\mu_\zeta \mu'_\zeta}(\vec{p}, \frac{1}{2}) \dots)}_{2s}}}_{2s} e^{ip \cdot (x-x')} \right. \\
&\quad \left. + (-1)^{2s+1} \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{-\{\lambda_\zeta(\lambda'_\zeta(\vec{p}, \frac{1}{2})\Lambda_{-\mu_\zeta \mu'_\zeta}(\vec{p}, \frac{1}{2}) \dots)}_{2s}}}_{2s} e^{-ip \cdot (x-x')} \right] \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta(\lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots)}^{2s} e^{ip \cdot (x-x')} \right. \\
&\quad \left. + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m + \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta(\lambda'_\zeta} [(-m + \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots)}^{2s} e^{-ip \cdot (x-x')} \right\} \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta(\lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots)}^{2s} \frac{-i}{(2\pi)^N} \int d^N \vec{p} \frac{1}{2E} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta(\lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots)}^{2s} \Delta(x-x') \\
&= \frac{i(2m)^{2s}}{2^{2s-1}} \Lambda_{+\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(-i\partial, s) \Delta(x-x')
\end{aligned}$$

□

证明:  $[\psi_{\lambda_\zeta \mu_\zeta \dots}^{(+)}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^{(+)}(x')]_{-2s+1}$

$$\begin{aligned}
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} d^N \vec{p}' \sum_{\vec{h}, \vec{h}'} (EE')^{s-\frac{1}{2}} \sqrt{\frac{m^2}{EE'}} [a(\vec{p}, \vec{h}) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, \vec{h}) e^{ip \cdot x}, a^+(\vec{p}', \vec{h}') \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}', \vec{h}') e^{-ip' \cdot x'}] \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} d^N \vec{p}' \sum_{\vec{h}, \vec{h}'} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} [U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, \vec{h}) \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}', \vec{h}') [a(\vec{p}, \vec{h}), a^+(\vec{p}', \vec{h}')] e^{i(p \cdot x - p' \cdot x')} \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} d^N \vec{p}' \sum_{\vec{h}, \vec{h}'} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p}, \vec{h}) \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}', \vec{h}') \delta_{\vec{h}\vec{h}'} \delta^N(\vec{p} - \vec{p}') e^{i(p \cdot x - p' \cdot x')} \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} \sum_{\vec{h}} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, \vec{h}) \underbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}, \vec{h}) e^{ip \cdot (x-x')} \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} \Lambda_{+\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(\vec{p}, s) e^{ip \cdot (x-x')} \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{+\{\lambda_\zeta(\lambda'_\zeta(\vec{p}, \frac{1}{2})\Lambda_{+\mu_\zeta \mu'_\zeta}(\vec{p}, \frac{1}{2}) \dots)}_{2s}}}_{2s} e^{ip \cdot (x-x')} \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta(\lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots)}^{2s} e^{ip \cdot (x-x')} \right. \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta(\lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots)}^{2s} \frac{-i}{(2\pi)^N} \int d^N \vec{p} \frac{1}{2E} e^{ip \cdot (x-x')} \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta(\lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots)}^{2s} \Delta^{(+)}(x-x') \\
&= \frac{i(2m)^{2s}}{2^{2s-1}} \Lambda_{+\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(-i\partial, s) \Delta^{(+)}(x-x')
\end{aligned}$$

□

证明:  $[\psi_{\lambda_\zeta \mu_\zeta \dots}^{(-)}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^{(-)}(x')]_{-2s+1}$

$$\begin{aligned}
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} d^N \vec{p}' \sum_{\vec{h}, \vec{h}'} (EE')^{s-\frac{1}{2}} \sqrt{\frac{m^2}{EE'}} [b^+(\vec{p}, \vec{h}) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, \vec{h}) e^{-ip \cdot x}, b(\vec{p}', \vec{h}') \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots}^+}_{2s}(\vec{p}', \vec{h}') e^{ip' \cdot x'}]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} d^N \vec{p}' \sum_{\vec{h}, \vec{h}'} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots}}_{2s}(\vec{p}', \vec{h}') [b^+(\vec{p}, \vec{h}), b(\vec{p}', \vec{h}')] e^{-i(p \cdot x - p' \cdot x')} \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} d^N \vec{p}' \sum_{\vec{h}, \vec{h}'} \sqrt{\frac{m^{2s}}{E}} \sqrt{\frac{m^{2s}}{E'}} (-1)^{2s+1} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots}}_{2s}(\vec{p}', \vec{h}') \delta_{\vec{h}\vec{h}'}^N \delta^N(\vec{p} - \vec{p}') e^{-i(p \cdot x - p' \cdot x')} \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} (-1)^{2s+1} \sum_{\vec{h}} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_\zeta \mu'_\zeta \dots}}_{2s}(\vec{p}, \vec{h}) e^{-ip \cdot (x - x')} \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} (-1)^{2s+1} \Lambda_{-\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(\vec{p}, s) e^{-ip \cdot (x - x')} \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} (-1)^{2s+1} \frac{1}{[(2s)!]^2} \underbrace{\Lambda_{-\{\lambda_\zeta(\lambda'_\zeta(\vec{p}, \frac{1}{2})\Lambda_{-\mu_\zeta \mu'_\zeta}(\vec{p}, \frac{1}{2})\dots}}_{2s}} \dots e^{-ip \cdot (x - x')} \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{p} \frac{m^{2s}}{E} (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m + \gamma^a \partial_a) \gamma^0]_{\{\lambda_\zeta(\lambda'_\zeta[(-m + \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})}}_{2s}} e^{-ip \cdot (x - x')} \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})}}_{2s}} \frac{i}{(2\pi)^N} \int d^N \vec{p} \frac{1}{2E} e^{-ip \cdot (x - x')} \\
&= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})}}_{2s}} \Delta^{(-)}(x - x') \\
&= \frac{i(2m)^{2s}}{2^{2s-1}} \Lambda_{+\lambda_\zeta \mu_\zeta \dots \lambda'_\zeta \mu'_\zeta \dots}(-i\partial, s) \Delta^{(-)}(x - x') \quad \square
\end{aligned}$$

### 3.10 N+1维时空中Bargmann-Wigner方程协变对易规则的反向推理

$$\begin{aligned}
\text{定理3.10.1. } & [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} = \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})}}_{2s}} \Delta(x - x') \\
& \Rightarrow [a(\vec{p}, \vec{h}), a^+(\vec{p}', \vec{h}')]_{-2s+1} = \delta_{\vec{h}\vec{h}'}^N \delta^N(\vec{p} - \vec{p}'), [b(\vec{p}, \vec{h}), b^+(\vec{p}', \vec{h}')]_{-2s+1} = \delta_{\vec{h}\vec{h}'}^N \delta^N(\vec{p} - \vec{p}'), [rest]_{-2s+1} = 0
\end{aligned}$$

下面给出几个主要对易括号的详细证明过程。

$$\begin{aligned}
\text{证明: } & [a(\vec{p}, \vec{h}), a^+(\vec{p}', \vec{h}')]_{-2s+1} \\
&= \frac{1}{(2\pi)^N} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U_{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}, \vec{h}) \overbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}}^{2s}(\vec{p}', \vec{h}') [\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]_{-2s+1} e^{-i(p \cdot x - p' \cdot x')} d^N \vec{r} d^N \vec{r}' \\
&= \frac{1}{(2\pi)^N} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U_{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}, \vec{h}) \overbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}}^{2s}(\vec{p}', \vec{h}') \\
&\quad \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})}}_{2s}} \Delta(x - x') e^{-i(p \cdot x - p' \cdot x')} d^N \vec{r} d^N \vec{r}' \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{r} d^N \vec{r}' \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U_{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}, \vec{h}) \overbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}}^{2s}(\vec{p}', \vec{h}') \\
&\quad \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_\zeta(\lambda'_\zeta[(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})}}_{2s}} \left\{ \frac{-i}{(2\pi)^N} \int \frac{1}{2E_0} [e^{ip_0 \cdot (x-x')} - e^{-ip_0 \cdot (x-x')}] d^N \vec{p}_0 \right\} e^{-i(p \cdot x - p' \cdot x')} \\
&= \left[ \frac{1}{(2\pi)^N} \right]^2 \int \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} U_{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}, \vec{h}) \overbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}}^{2s}(\vec{p}', \vec{h}') \\
&\quad \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_{0a}) \gamma^0]_{\{\lambda_\zeta(\lambda'_\zeta[(m - i\gamma^b p_{0b}) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})}}_{2s}} e^{ip_0 \cdot (x-x')} e^{-i(p \cdot x - p' \cdot x')} d^N \vec{r} d^N \vec{r}' \right. \\
&\quad \left. + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m - i\gamma^a p_{0a}) \gamma^0]_{\{\lambda_\zeta(\lambda'_\zeta[(-m - i\gamma^b p_{0b}) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})}}_{2s}} e^{-ip_0 \cdot (x-x')} e^{-i(p \cdot x - p' \cdot x')} \right\} d^N \vec{r} d^N \vec{r}' d^N \vec{p}_0 \\
&= \left[ \frac{1}{(2\pi)^N} \right]^2 \int d^N \vec{r} d^N \vec{r}' d^N \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
&\quad U_{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}, \vec{h}) \overbrace{U_{\lambda'_\zeta \mu'_\zeta \dots}}^{2s}(\vec{p}', \vec{h}') \left\{ \sum_{\vec{h}_0} U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}_0, \vec{h}_0) \overbrace{U_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}}^{2s}(\vec{p}_0, \vec{h}_0) e^{i(p_0 - p) \cdot x} e^{-i(p_0 - p') \cdot x'} \right.
\end{aligned}$$

$$\begin{aligned}
& + (-1)^{2s+1} \sum_{\vec{h}_0} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, \vec{h}_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(\vec{p}_0, \vec{h}_0) e^{-i(p_0+p) \cdot x} e^{i(p_0+p') \cdot x'} \} \\
& = \int d^N \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
& U^+_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', \vec{h}') \{ \sum_{\vec{h}_0} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, \vec{h}_0) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(\vec{p}_0, \vec{h}_0) \delta^N(\vec{p}_0 - \vec{p}) \delta^N(\vec{p}_0 - \vec{p}') \\
& + (-1)^{2s+1} \sum_{\vec{h}_0} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, \vec{h}_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(\vec{p}_0, \vec{h}_0) e^{2iE_0(t-t')} \delta^N(\vec{p}_0 + \vec{p}) \delta^N(\vec{p}_0 + \vec{p}') \} \\
& = \delta^N(\vec{p} - \vec{p}') \left(\frac{m}{E}\right)^{4s} U^+_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}, \vec{h}') \\
& \{ \sum_{\vec{h}_0} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}_0) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(\vec{p}, \vec{h}_0) + (-1)^{2s+1} \sum_{\vec{h}_0} V_{\lambda_s \mu_s \dots \tau_s}(-\vec{p}, \vec{h}_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(-\vec{p}, \vec{h}_0) e^{2iE(t-t')} \} \\
& = \delta^N(\vec{p} - \vec{p}') (\sum_{\vec{h}_0} \delta_{\vec{h}\vec{h}_0} \delta_{\vec{h}'\vec{h}_0} + 0) \\
& = \delta_{\vec{h}\vec{h}'} \delta^N(\vec{p} - \vec{p}')
\end{aligned}$$

□

证明:  $[b^+(\vec{p}, \vec{h}), b(\vec{p}', \vec{h}')]_{-2s+1}$ 

$$\begin{aligned}
& = \frac{1}{(2\pi)^N} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} V^+_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', \vec{h}') [\psi_{\lambda_s \mu_s \dots \tau_s}(x), \psi_{\lambda'_s \mu'_s \dots \tau'_s}^+(x')]_{-2s+1} e^{i(p \cdot x - p' \cdot x')} d^N \vec{r} d^N \vec{r}' \\
& = \frac{1}{(2\pi)^N} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} V^+_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', \vec{h}') \\
& \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^0]_{\mu_s \mu'_s \dots})\}}}^{2s} \Delta(x - x') e^{i(p \cdot x - p' \cdot x')} d^N \vec{r} d^N \vec{r}' \\
& = \frac{1}{(2\pi)^N} \int d^N \vec{r} d^N \vec{r}' \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} V^+_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', \vec{h}') \\
& \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_s(\lambda'_s [(m - \gamma^b \partial_b) \gamma^0]_{\mu_s \mu'_s \dots})\}}}^{2s} \left\{ \frac{-i}{2E_0} \int e^{ip_0 \cdot (x-x')} - e^{-ip_0 \cdot (x-x')} \right\} d^N \vec{p}_0 e^{i(p \cdot x - p' \cdot x')} \\
& = \left[\frac{1}{(2\pi)^N}\right]^2 \int \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} V^+_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', \vec{h}') \\
& \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_{0a}) \gamma^0]_{\{\lambda_s(\lambda'_s [(m - i\gamma^b p_{0b}) \gamma^0]_{\mu_s \mu'_s \dots})\}}}^{2s} e^{ip_0 \cdot (x-x')} e^{i(p \cdot x - p' \cdot x')} \right\} d^N \vec{r} d^N \vec{r}' \\
& + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m - i\gamma^a p_{0a}) \gamma^0]_{\{\lambda_s(\lambda'_s [(-m - i\gamma^b p_{0b}) \gamma^0]_{\mu_s \mu'_s \dots})\}}}^{2s} e^{-ip_0 \cdot (x-x')} e^{i(p \cdot x - p' \cdot x')} \} d^N \vec{r} d^N \vec{r}' d^N \vec{p}_0 \\
& = \left[\frac{1}{(2\pi)^N}\right]^2 \int d^N \vec{r} d^N \vec{r}' d^N \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
& V^+_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', \vec{h}') \{ \sum_{\vec{h}_0} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, \vec{h}_0) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(\vec{p}_0, \vec{h}_0) e^{i(p_0+p) \cdot x} e^{-i(p_0+p') \cdot x'} \\
& + (-1)^{2s+1} \sum_{\vec{h}_0} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, \vec{h}_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(\vec{p}_0, \vec{h}_0) e^{-i(p_0-p) \cdot x} e^{i(p_0-p') \cdot x'} \} \\
& = \int d^N \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
& V^+_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}', \vec{h}') \{ \sum_{\vec{h}_0} U_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, \vec{h}_0) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(\vec{p}_0, \vec{h}_0) e^{-2iE_0(t-t')} \delta^N(\vec{p}_0 + \vec{p}) \delta^N(\vec{p}_0 + \vec{p}') \\
& + (-1)^{2s+1} \sum_{\vec{h}_0} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}_0, \vec{h}_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(\vec{p}_0, \vec{h}_0) \delta^N(\vec{p}_0 - \vec{p}) \delta^N(\vec{p}_0 - \vec{p}') \} \\
& = \delta^N(\vec{p} - \vec{p}') \left(\frac{m}{E}\right)^{4s} V^+_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}}_{2s}(\vec{p}, \vec{h}') \\
& \{ \sum_{\vec{h}_0} U_{\lambda_s \mu_s \dots \tau_s}(-\vec{p}, \vec{h}_0) \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(-\vec{p}, \vec{h}_0) e^{-2iE(t-t')} + (-1)^{2s+1} \sum_{\vec{h}_0} V_{\lambda_s \mu_s \dots \tau_s}(\vec{p}, \vec{h}_0) \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}^+}_{2s}(\vec{p}, \vec{h}_0) \}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{2s+1} \delta^N(\vec{p} - \vec{p}') (0 + \sum_{\vec{h}_0} \delta_{\vec{h}\vec{h}_0} \delta_{\vec{h}'\vec{h}_0}) \\
&= (-1)^{2s+1} \delta_{\vec{h}\vec{h}'} \delta^N(\vec{p} - \vec{p}')
\end{aligned}$$

□

证明:  $[a(\vec{p}, \vec{h}), b(\vec{p}', \vec{h}')]_{-2s+1}$ 

$$\begin{aligned}
&= \frac{1}{(2\pi)^N} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{p}, \vec{h}) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \dots}^{2s}}(\vec{p}', \vec{h}') [\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}(x)}_{2s}, \underbrace{\psi^+_{\lambda'_\zeta \mu'_\zeta \dots}(x')}]_{-2s+1} e^{-i(p \cdot x + p' \cdot x')} d^N \vec{r} d^N \vec{r}' \\
&= \frac{1}{(2\pi)^N} \int \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{p}, \vec{h}) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \dots}^{2s}}(\vec{p}', \vec{h}') \\
&\quad \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2s} \Delta(x - x') e^{-i(p \cdot x + p' \cdot x')} d^N \vec{r} d^N \vec{r}' \\
&= \frac{1}{(2\pi)^N} \int d^N \vec{r} d^N \vec{r}' \sqrt{EE'} \left(\frac{m}{EE'}\right)^{2s} U^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{p}, \vec{h}) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \dots}^{2s}}(\vec{p}', \vec{h}') \\
&\quad \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2s} \left\{ \frac{-i}{(2\pi)^N} \int \frac{1}{2E_0} [e^{ip_0 \cdot (x-x')} - e^{-ip_0 \cdot (x-x')}] d^N \vec{p}_0 \right\} e^{-i(p \cdot x + p' \cdot x')} \\
&= \left[ \frac{1}{(2\pi)^N} \right]^2 \int \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} U^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{p}, \vec{h}) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \dots}^{2s}}(\vec{p}', \vec{h}') \\
&\quad \left\{ \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(m - i\gamma^a p_{0a}) \gamma^0]_{\{\lambda_\zeta (\lambda'_\zeta [(m - i\gamma^b p_{0b}) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2s} e^{ip_0 \cdot (x-x')} e^{-i(p \cdot x + p' \cdot x')} d^N \vec{r} d^N \vec{r}' \right. \\
&\quad \left. + (-1)^{2s+1} \frac{1}{(2m)^{2s}} \frac{1}{[(2s)!]^2} \overbrace{[(-m - i\gamma^a p_{0a}) \gamma^0]_{\{\lambda_\zeta (\lambda'_\zeta [(-m - i\gamma^b p_{0b}) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2s} e^{-ip_0 \cdot (x-x')} e^{-i(p \cdot x + p' \cdot x')} \right\} d^N \vec{r} d^N \vec{r}' d^N \vec{p}_0 \\
&= \left[ \frac{1}{(2\pi)^N} \right]^2 \int d^N \vec{r} d^N \vec{r}' d^N \vec{p}_0 \frac{\sqrt{EE'}}{E_0} \left(\frac{m^2}{EE'}\right)^{2s} \\
&\quad U^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{p}, \vec{h}) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \dots}^{2s}}(\vec{p}', \vec{h}') \left\{ \sum_{\vec{h}_0} U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}_0, \vec{h}_0) U^+_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}(\vec{p}_0, \vec{h}_0) e^{i(p_0 - p) \cdot x} e^{-i(p_0 + p') \cdot x'} \right. \\
&\quad \left. + (-1)^{2s+1} \sum_{\vec{h}_0} V_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}_0, \vec{h}_0) V^+_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}(\vec{p}_0, \vec{h}_0) e^{-i(p_0 + p) \cdot x} e^{i(p_0 - p') \cdot x'} \right\} \\
&= \int d^N \vec{p}_0 \frac{\sqrt{EE'}}{|\vec{p}_0|} \left(\frac{m^2}{EE'}\right)^{2s} \\
&\quad U^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{p}, \vec{h}) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \dots}^{2s}}(\vec{p}', \vec{h}') \left\{ \sum_{\vec{h}_0} U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}_0, \vec{h}_0) U^+_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}(\vec{p}_0, \vec{h}_0) e^{-iE_0 t'} \delta^N(\vec{p}_0 - \vec{p}) \delta^N(\vec{p}_0 + \vec{p}') \right. \\
&\quad \left. + (-1)^{2s+1} \sum_{\vec{h}_0} V_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}_0, \vec{h}_0) V^+_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}(\vec{p}_0, \vec{h}_0) e^{iE_0 t'} \delta^N(\vec{p}_0 + \vec{p}) \delta^N(\vec{p}_0 - \vec{p}') \right\} \\
&= \delta^N(\vec{p} + \vec{p}') \left(\frac{m}{E}\right)^{4s} U^{+\overbrace{\lambda_\zeta \mu_\zeta \dots}^{2s}}(\vec{p}, \vec{h}) V^{\overbrace{\lambda'_\zeta \mu'_\zeta \dots}^{2s}}(\vec{p}', \vec{h}') \\
&\quad \left\{ \sum_{\vec{h}_0} U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}, \vec{h}_0) U^+_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}(\vec{p}, \vec{h}_0) + (-1)^{2s+1} \sum_{\vec{h}_0} V_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}, \vec{h}_0) V^+_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}(\vec{p}, \vec{h}_0) e^{2iE(t-t')} \right\} \\
&= 0 + 0 = 0
\end{aligned}$$

□

### 3.11 N+1维时空中Bargmann-Wigner方程协变对易规则的小结

结合以上两节的证明, 便得到以下重要的定理。

定理3.11.1.

$$\begin{aligned}
[\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}(x)}_{2s}, \underbrace{\psi^+_{\lambda'_\zeta \mu'_\zeta \dots}(x')}]_{-2s+1} &= \frac{i}{2^{2s-1}} \frac{1}{[(2s)!]^2} \overbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda_\zeta (\lambda'_\zeta [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta \dots})\}}}_{2s} \Delta(x - x') \\
\Leftrightarrow [a(\vec{p}, \vec{h}), a^+(\vec{p}', \vec{h}')]_{-2s+1} &= \delta_{\vec{h}\vec{h}'} \delta^N(\vec{p} - \vec{p}'), [b(\vec{p}, \vec{h}), b^+(\vec{p}', \vec{h}')]_{-2s+1} = \delta_{\vec{h}\vec{h}'} \delta^N(\vec{p} - \vec{p}'), [rest]_{-2s+1} = 0
\end{aligned}$$

## 3.12 N+1维时空中Bargmann-Wigner方程的各种物理算符

定理3.12.1.

$$P_u(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) d^N \vec{r} = \int \sum_{\vec{h}} p_u [a^+(\vec{p}, \vec{h}) a(\vec{p}, \vec{h}) + (-1)^{2s} b(\vec{p}, \vec{h}) b^+(\vec{p}, \vec{h})] d^N \vec{p}$$

$$\begin{aligned} \text{证明: } P_u(s) &= \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) d^N \vec{r} \\ &= \int \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_{\vec{h}'} E'^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'}}^{2s} [a^+(\vec{p}', \vec{h}') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{-i(\vec{p}' \cdot \vec{r} - E't)} + b(\vec{p}', \vec{h}') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{i(\vec{p}' \cdot \vec{r} - E't)}] d^N \vec{p}' \\ &\quad \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{\vec{h}} E^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E}}^{2s} \frac{p_u E^{2s-1}}{(E^2)^{2s-1}} [a(\vec{p}, \vec{h}) U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{2s} b^+(\vec{p}, \vec{h}) V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p} d^N \vec{r} \\ &= \frac{1}{(2\pi)^N} \int \sum_{\vec{h}, \vec{h}'} \left(\frac{E'}{E}\right)^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'E}}^{2s} [a^+(\vec{p}', \vec{h}') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{-i(\vec{p}' \cdot \vec{r} - E't)} + b(\vec{p}', \vec{h}') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{i(\vec{p}' \cdot \vec{r} - E't)}] \\ &\quad p_u [a(\vec{p}, \vec{h}) U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{2s} b^+(\vec{p}, \vec{h}) V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}' d^N \vec{p} d^N \vec{r} \\ &= \int d^N \vec{p}' d^N \vec{p} \sum_{\vec{h}, \vec{h}'} \left(\frac{m}{E}\right)^{2s} p_u \\ &\quad \{ \delta^N(\vec{p} - \vec{p}') [a^+(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}') U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) + (-1)^{2s} b(\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}') V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h})] \\ &\quad + \delta^N(\vec{p} + \vec{p}') [(-1)^{2s} e^{2iEt} a^+(-\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) \\ &\quad + e^{-2iEt} b(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h})] \} \\ &= \int \sum_{\vec{h}} p_u [a^+(\vec{p}, \vec{h}) a(\vec{p}, \vec{h}) + (-1)^{2s} b(\vec{p}, \vec{h}) b^+(\vec{p}, \vec{h})] d^N \vec{p} \end{aligned} \quad \square$$

定理3.12.2.

$$Q(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) d^N \vec{r} = \int \sum_{\vec{h}} [a^+(\vec{p}, \vec{h}) a(\vec{p}, \vec{h}) + (-1)^{2s-1} b(\vec{p}, \vec{h}) b^+(\vec{p}, \vec{h})] d^N \vec{p}$$

$$\begin{aligned} \text{证明: } Q(s) &= \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) d^N \vec{r} \\ &= \int \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_{\vec{h}'} E'^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'}}^{2s} [a^+(\vec{p}', \vec{h}') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{-i(\vec{p}' \cdot \vec{r} - E't)} + b(\vec{p}', \vec{h}') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{i(\vec{p}' \cdot \vec{r} - E't)}] d^N \vec{p}' \\ &\quad \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{\vec{h}} E^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E}}^{2s} \frac{E^{2s-1}}{(E^2)^{2s-1}} [a(\vec{p}, \vec{h}) U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{2s-1} b^+(\vec{p}, \vec{h}) V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p} d^N \vec{r} \\ &= \frac{1}{(2\pi)^N} \int \sum_{\vec{h}, \vec{h}'} \left(\frac{E'}{E}\right)^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'E}}^{2s} [a^+(\vec{p}', \vec{h}') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{-i(\vec{p}' \cdot \vec{r} - E't)} + b(\vec{p}', \vec{h}') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{i(\vec{p}' \cdot \vec{r} - E't)}] \\ &\quad [a(\vec{p}, \vec{h}) U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{2s-1} b^+(\vec{p}, \vec{h}) V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}' d^N \vec{p} d^N \vec{r} \\ &= \int d^N \vec{p}' d^N \vec{p} \sum_{\vec{h}, \vec{h}'} \left(\frac{m}{E}\right)^{2s} \\ &\quad \{ \delta^N(\vec{p} - \vec{p}') [a^+(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}') U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) + (-1)^{2s-1} b(\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}') V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h})] \\ &\quad + \delta^N(\vec{p} + \vec{p}') [(-1)^{2s-1} e^{2iEt} a^+(-\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') V_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}) \\ &\quad + e^{-2iEt} b(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') U_{\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h})] \} \end{aligned}$$

$$+ e^{-2iEt} b(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) \}} \\ = \int \sum_{\vec{h}} [a^+(\vec{p}, \vec{h}) a(\vec{p}, \vec{h}) + (-1)^{2s-1} b(\vec{p}, \vec{h}) b^+(\vec{p}, \vec{h})] d^N \vec{p}$$

□

定理3.12.3.

$$N(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{r}, t) d^N \vec{r} = \int \sum_{\vec{h}} [a^+(\vec{p}, \vec{h}) a(\vec{p}, \vec{h}) + (-1)^{2s} b(\vec{p}, \vec{h}) b^+(\vec{p}, \vec{h})] d^N \vec{p}$$

$$\text{证明: } N(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{r}, t) d^N \vec{r} \\ = \int \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_{\vec{h}'} E'^{s-\frac{1}{2}} \sqrt{\frac{m}{E'}}^{2s} [a^+(\vec{p}', \vec{h}') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{-i(\vec{p}' \cdot \vec{r} - E't)} + b(\vec{p}', \vec{h}') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{i(\vec{p}' \cdot \vec{r} - E't)}] d^N \vec{p}' \\ \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{\vec{h}} E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} \frac{E^{2s}}{E^{4s-1}} [a(\vec{p}, \vec{h}) U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{2s} b^+(\vec{p}, \vec{h}) V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p} d^N \vec{r} \\ = \frac{1}{(2\pi)^N} \int \sum_{\vec{h}, \vec{h}'} \left(\frac{E'}{E}\right)^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'E}}^{2s} [a^+(\vec{p}', \vec{h}') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{-i(\vec{p}' \cdot \vec{r} - E't)} + b(\vec{p}', \vec{h}') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{i(\vec{p}' \cdot \vec{r} - E't)}] \\ [a(\vec{p}, \vec{h}) U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{2s} b^+(\vec{p}, \vec{h}) V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}' d^N \vec{p} d^N \vec{r} \\ = \int d^N \vec{p}' d^N \vec{p} \sum_{\vec{h}, \vec{h}'} \left(\frac{m}{E}\right)^{2s} \\ \{\delta^N(\vec{p} - \vec{p}') [a^+(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}') U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) + (-1)^{2s} b(\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}') V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h})] \\ + \delta^N(\vec{p} + \vec{p}') [(-1)^{2s} e^{2iEt} a^+(-\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) \\ + e^{-2iEt} b(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h})]\} \\ = \int \sum_{\vec{h}} [a^+(\vec{p}, \vec{h}) a(\vec{p}, \vec{h}) + (-1)^{2s} b(\vec{p}, \vec{h}) b^+(\vec{p}, \vec{h})] d^N \vec{p}$$

□

定理3.12.4.

$$\vec{S}(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla} (i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{r}, t) d^N \vec{r} = \int \sum_{\vec{h}} \hat{p} [a^+(\vec{p}, \vec{h}) a(\vec{p}, \vec{h}) + (-1)^{2s} b(\vec{p}, \vec{h}) b^+(\vec{p}, \vec{h})] d^N \vec{p}$$

$$\text{证明: } \vec{S}(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla} (i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{r}, t) d^N \vec{r} \\ = \int \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_{\vec{h}'} E'^{s-\frac{1}{2}} \sqrt{\frac{m}{E'}}^{2s} [a^+(\vec{p}', \vec{h}') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{-i(\vec{p}' \cdot \vec{r} - E't)} + b(\vec{p}', \vec{h}') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{i(\vec{p}' \cdot \vec{r} - E't)}] d^N \vec{p}' \\ \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{\vec{h}} E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} \frac{\hat{p} E^{2s-1}}{(E^2)^{2s-1}} [a(\vec{p}, \vec{h}) U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{2s} b^+(\vec{p}, \vec{h}) V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p} d^N \vec{r} \\ = \frac{1}{(2\pi)^N} \int \sum_{\vec{h}, \vec{h}'} \left(\frac{E'}{E}\right)^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'E}}^{2s} \hat{p} [a^+(\vec{p}', \vec{h}') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{-i(\vec{p}' \cdot \vec{r} - E't)} + b(\vec{p}', \vec{h}') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{i(\vec{p}' \cdot \vec{r} - E't)}] \\ [a(\vec{p}, \vec{h}) U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{2s} b^+(\vec{p}, \vec{h}) V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}' d^N \vec{p} d^N \vec{r} \\ = \int d^N \vec{p}' d^N \vec{p} \sum_{\vec{h}, \vec{h}'} \left(\frac{m}{E}\right)^{2s} \hat{p} \\ \{\delta^N(\vec{p} - \vec{p}') [a^+(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}') U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) + (-1)^{2s} b(\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}') V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h})] \\ + \delta^N(\vec{p} + \vec{p}') [(-1)^{2s} e^{2iEt} a^+(-\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) \\ + e^{-2iEt} b(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h})]\}$$



$$\begin{aligned}
& + \delta^N(\vec{p} + \vec{p}') [(-1)^{2s} e^{2iEt} a^+(-\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) \\
& + e^{-2iEt} b(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h})] \\
& = \int \sum_{\vec{h}} \hat{p} [a^+(\vec{p}, \vec{h}) a(\vec{p}, \vec{h}) + (-1)^{2s} b(\vec{p}, \vec{h}) b^+(\vec{p}, \vec{h})] d^N \vec{p} \quad \square
\end{aligned}$$

定理3.12.5.

$$\vec{M}(s) = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{r}, t) d^N \vec{r} = \int \sum_{\vec{h}} \hat{p} [a^+(\vec{p}, \vec{h}) a(\vec{p}, \vec{h}) + (-1)^{2s-1} b(\vec{p}, \vec{h}) b^+(\vec{p}, \vec{h})] d^N \vec{p}$$

$$\begin{aligned}
\text{证明: } \vec{M}(s) & = \int \psi^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{r}, t) d^N \vec{r} \\
& = \int \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} \sum_{\vec{h}'} E'^{s-\frac{1}{2}} \sqrt{\frac{m}{E'}}^{2s} [a^+(\vec{p}', \vec{h}') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{-i(\vec{p}' \cdot \vec{r} - E't)} + b(\vec{p}', \vec{h}') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{i(\vec{p}' \cdot \vec{r} - E't)}] d^N \vec{p}' \\
& \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{\vec{h}} E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} \frac{\hat{p} E^{2s}}{E^{4s-1}} [a(\vec{p}, \vec{h}) U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{2s-1} b^+(\vec{p}, \vec{h}) V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p} d^N \vec{r} \\
& = \frac{1}{(2\pi)^N} \int \sum_{\vec{h}, \vec{h}'} \left(\frac{E'}{E}\right)^{s-\frac{1}{2}} \sqrt{\frac{m^2}{E'E}}^{2s} \hat{p} [a^+(\vec{p}', \vec{h}') U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{-i(\vec{p}' \cdot \vec{r} - E't)} + b(\vec{p}', \vec{h}') V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}', \vec{h}') e^{i(\vec{p}' \cdot \vec{r} - E't)}] \\
& [a(\vec{p}, \vec{h}) U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{i(\vec{p} \cdot \vec{r} - Et)} + (-1)^{2s-1} b^+(\vec{p}, \vec{h}) V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}' d^N \vec{p} d^N \vec{r} \\
& = \int d^N \vec{p}' d^N \vec{p} \sum_{\vec{h}, \vec{h}'} \left(\frac{m}{E}\right)^{2s} \hat{p} \\
& \{ \delta^N(\vec{p} - \vec{p}') [a^+(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}') U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) + (-1)^{2s-1} b(\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(\vec{p}, \vec{h}') V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h})] \\
& + \delta^N(\vec{p} + \vec{p}') [(-1)^{2s-1} e^{2iEt} a^+(-\vec{p}, \vec{h}') b^+(\vec{p}, \vec{h}) U^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') V_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h}) \\
& + e^{-2iEt} b(\vec{p}, \vec{h}') a(\vec{p}, \vec{h}) V^{+\overbrace{\lambda_s \mu_s \dots}^{2s}}(-\vec{p}, \vec{h}') U_{\underbrace{\lambda_s \mu_s \dots}_{2s}}(\vec{p}, \vec{h})] \} \\
& = \int \sum_{\vec{h}} \hat{p} [a^+(\vec{p}, \vec{h}) a(\vec{p}, \vec{h}) + (-1)^{2s-1} b(\vec{p}, \vec{h}) b^+(\vec{p}, \vec{h})] d^N \vec{p} \quad \square
\end{aligned}$$

### 3.13 N+1维时空中的Bargmann-Wigner全对称方程势对易规则猜想(很可能错?)

猜想3.13.1.

$$\begin{cases} [A_a(x), A_{a'}^+(x')] = i(\eta_{aa'} - \frac{\partial_a \partial_{a'}^+}{m^2}) \Delta(x - x') \\ [A_{ab}(x), A_{a'b'}^+(x')] = \frac{i}{8} \{ -\frac{1}{N} [\delta_{\{ab\}} - \frac{\partial_{\{a} \partial_{b\}}}{m^2}] [\delta_{\{a'b'\}} - \frac{\partial_{\{a'} \partial_{b'}^+}{m^2}}] + [\eta_{\{a(a'} - \frac{\partial_{\{a} \partial_{a'}^+}{m^2}}] [\eta_{b\} b') - \frac{\partial_b \partial_{b'}^+}{m^2}] \} \Delta(x - x') \\ [A_{abc}(x), A_{a'b'c'}^+(x')] \\ ? = \frac{i}{144} \{ -\frac{1}{N} [\delta_{\{ab\}} - \frac{\partial_{\{a} \partial_{b\}}}{m^2}] [\delta_{\{a'b'\}} - \frac{\partial_{\{a'} \partial_{b'}^+}{m^2}}] + [\eta_{\{a(a'} - \frac{\partial_{\{a} \partial_{a'}^+}{m^2}}] [\eta_{bb'} - \frac{\partial_b \partial_{b'}^+}{m^2}] \} [\eta_{c\} c') - \frac{\partial_c \partial_{c'}^+}{m^2}] \Delta(x - x') \end{cases}$$

猜想3.13.2.  $[A_{\underbrace{abc \dots}_n}(x), A_{\underbrace{a'b'c' \dots}_n}^+(x')]$

$$? = \frac{i}{2^{n-1}(n!)^2} \{ -\frac{1}{N} [\delta_{\{ab\}} - \frac{\partial_{\{a} \partial_{b\}}}{m^2}] [\delta_{\{a'b'\}} - \frac{\partial_{\{a'} \partial_{b'}^+}{m^2}}] + [\eta_{\{a(a'} - \frac{\partial_{\{a} \partial_{a'}^+}{m^2}}] [\eta_{bb'} - \frac{\partial_b \partial_{b'}^+}{m^2}] \} \underbrace{[\eta_{cc'} - \frac{\partial_c \partial_{c'}^+}{m^2}] \dots}_{n-2} \Delta(x - x')$$

# 第三十五章 高维时空无质量粒子的协变量子化

自我评述：本章将无质量粒子协变量子化推广到了一般 $N+1$ 维时空中。在特殊的 $N+1$ 维时空中与四维时空不同的是对于全对称Penrose方程描述的粒子，一般来说既可以描述带荷的复粒子，也可以描述不带荷的马约拉纳粒子。两种情形的主对易规则形式一致，但其余对易或反对易括号，对带荷的复粒子一般为零；对不带荷的马约拉纳粒子，其余对易或反对易括号由主对易规则和马约拉纳条件自然得到，一般不为零。在本章只讨论复粒子情形，一般也只给出主对易规则，不再专门讨论马约拉纳粒子情形，若要得到马约拉纳粒子情形的量子场论，只需在复粒子情形加上马约拉纳条件即可自然得到。

## 1 $N+1$ 维时空中的螺旋度本征函数

### 1.1 $N+1=2n$ 偶数维时空中分离表象的电子方程<sup>[4]</sup>

定义1.1.1.  $(\gamma^a \partial_a + m)\psi = 0, \psi = \begin{bmatrix} \varphi \\ \eta \end{bmatrix}, \gamma^a = (\Gamma \otimes \sigma_y, \varsigma I_* \otimes \sigma_x) \Leftrightarrow \begin{cases} (\Gamma, -i\varsigma)^a \partial_a \varphi = im\eta \\ (\Gamma, i\varsigma)^a \partial_a \eta = -im\varphi \end{cases}$

### 1.2 $N+1=2n$ 偶数维时空中的中微子方程<sup>[5]</sup>

当质量 $m=0$ 时，则退化为两个Weyl中微子方程：

推论1.2.1.  $(\Gamma, -i\varsigma)^a \partial_a \varphi = 0, (\Gamma, i\varsigma)^a \partial_a \eta = 0$

### 1.3 $N+1=2n$ 偶数维时空中运动方向的螺旋度本征函数

定义1.3.1.

$$\begin{cases} (I_* \otimes \sigma_z) \lambda(\hat{p}, \frac{1}{2}) = \frac{1}{2} \lambda(\vec{p}, \frac{1}{2}), (I_* \otimes \sigma_z) \lambda(\hat{p}, -\frac{1}{2}) = -\frac{1}{2} \lambda(\vec{p}, -\frac{1}{2}), l = 2^{\lfloor \frac{N-1}{2} \rfloor} = 2^{n-1} \\ \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}; 1\right) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}; 2\right) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \dots, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}; \frac{l}{2} - 1\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}; \frac{l}{2}\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, -\frac{1}{2}; 1\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, -\frac{1}{2}; 2\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \dots, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, -\frac{1}{2}; \frac{l}{2} - 1\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, -\frac{1}{2}; \frac{l}{2}\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \end{cases}$$

### 1.4 $N+1=2n-1$ 奇数维时空中运动方向的螺旋度本征函数

定义1.4.1.

$$\begin{cases} (I_* \otimes \sigma_z) \lambda(\hat{p}, \frac{1}{2}) = \frac{1}{2} \lambda(\vec{p}, \frac{1}{2}), (I_* \otimes \sigma_z) \lambda(\hat{p}, -\frac{1}{2}) = -\frac{1}{2} \lambda(\vec{p}, -\frac{1}{2}), l = 2^{\lfloor \frac{N-1}{2} \rfloor} = 2^{n-2} \\ \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}; 1\right) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}; 2\right) = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \dots, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}; l-1\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}; l\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, -\frac{1}{2}; 1\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, -\frac{1}{2}; 2\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \dots, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, -\frac{1}{2}; l-1\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, -\frac{1}{2}; l\right) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \end{cases}$$

定义1.4.2.  $(\Gamma \cdot \hat{p}) \lambda(\hat{p}, \frac{1}{2}; h) = \lambda(\hat{p}, \frac{1}{2}; h), (\Gamma \cdot \hat{p}) \lambda(\hat{p}, -\frac{1}{2}; h) = -\lambda(\hat{p}, -\frac{1}{2}; h)$

### 1.5 $N+1$ 维时空中螺旋度 $\Gamma \cdot \hat{p}$ 的本征函数

定义1.5.1.  $r := \begin{cases} l/2, \text{偶数维时空} \\ l, \text{奇数维时空} \end{cases}, l = 2^{\lfloor \frac{N-1}{2} \rfloor}$

定义1.5.2.

$$\begin{cases} (\Gamma \cdot \hat{p}) \lambda(\hat{p}, -\frac{\varsigma}{2}; h) = -\varsigma \lambda(\hat{p}, -\frac{\varsigma}{2}; h), \lambda(\hat{p}, -\frac{\varsigma}{2}; h) := e^{\frac{1}{8} \vartheta^{ij}(\hat{p})[\Gamma_i, \Gamma_j]} \lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, -\frac{\varsigma}{2}; h\right) \\ e^{-\frac{1}{8} \vartheta^{ij}(\hat{p})[\Gamma_i, \Gamma_j]} (\Gamma \cdot \hat{p}) e^{\frac{1}{8} \vartheta^{ij}(\hat{p})[\Gamma_i, \Gamma_j]} = \Gamma_N, \Gamma \cdot \hat{p} = e^{\frac{1}{8} \vartheta^{ij}(\hat{p})[\Gamma_i, \Gamma_j]} \Gamma_N e^{-\frac{1}{8} \vartheta^{ij}(\hat{p})[\Gamma_i, \Gamma_j]} \end{cases}$$

定义1.5.3.  $(\Gamma \cdot \hat{p}) \lambda(\hat{p}, \frac{1}{2}; h) = \lambda(\hat{p}, \frac{1}{2}; h), (\Gamma \cdot \hat{p}) \lambda(\hat{p}, -\frac{1}{2}; h) = -\lambda(\hat{p}, -\frac{1}{2}; h)$

定义1.5.4.

$$\begin{cases} (\Gamma \cdot \hat{p})\lambda(\hat{p}, -\frac{\zeta}{2}; h) = -\zeta\lambda(\hat{p}, -\frac{\zeta}{2}; h), \lambda(\hat{p}, -\frac{\zeta}{2}; h) := e^{\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i, \Gamma_j]}\lambda\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, -\frac{\zeta}{2}; h\right) \\ e^{-\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i, \Gamma_j]}(\Gamma \cdot \hat{p})e^{\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i, \Gamma_j]} = \Gamma_N, \Gamma \cdot \hat{p} = e^{\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i, \Gamma_j]}\Gamma_N e^{-\frac{1}{8}\vartheta^{ij}(\hat{p})[\Gamma_i, \Gamma_j]} \end{cases}$$

## 1.6 N+1维时空中螺旋度 $\Gamma \cdot \hat{p}$ 本征函数的正交性与完备性

定义1.6.1.  $\lambda(\hat{p}, \frac{1}{2}; h) := \lambda(\hat{p}; h), \lambda(\hat{p}, -\frac{1}{2}; h) := \lambda(\hat{p}; -h), \lambda(\hat{p}, \frac{\zeta}{2}; h) := \lambda(\hat{p}; h\zeta), \lambda(\hat{p}, -\frac{\zeta}{2}; h) := \lambda(\hat{p}; -h\zeta)$

推论1.6.1.  $\lambda^+(\hat{p}; h)\lambda(\hat{p}; h') = \delta_{hh'}$

$$\sum_{h=1}^r [\lambda(\hat{p}; h)\lambda^+(\hat{p}; h) + \lambda(\hat{p}; -h)\lambda^+(\hat{p}; -h)] = 1, \sum_{h=1}^r [\lambda(\hat{p}; h)\lambda^+(\hat{p}; h) - \lambda(\hat{p}; -h)\lambda^+(\hat{p}; -h)] = \Gamma \cdot \hat{p}$$

推论1.6.2.  $\sum_{h=1}^r \lambda(\hat{p}; h)\lambda^+(\hat{p}; h) = \frac{1}{2}(\Gamma, -i)^a \hat{p}_a, \sum_{h=1}^r \lambda(\hat{p}; -h)\lambda^+(\hat{p}; -h) = -\frac{1}{2}(\Gamma, -i)^a \hat{p}_a$

推论1.6.3.  $\sum_{h=1}^r \lambda(\hat{p}; h\zeta)\lambda^+(\hat{p}; h\zeta) = -\frac{\zeta}{2}(\Gamma, i\zeta)^a \hat{p}_a$

## 1.7 N+1维时空中高自旋螺旋度的本征函数

定义1.7.1.

$$\begin{cases} \underbrace{\lambda_{A_\zeta \cdots B_\zeta \cdots C_\zeta}_{2s}(\vec{p}; n_1, n_2, \dots, n_{2r})}_{2s} := \frac{1}{\sqrt{(2s)!n_1!n_2! \cdots n_r!}} \underbrace{\lambda_{A_\zeta}(\vec{p}; -1\zeta)}_{n_1} \underbrace{\lambda_{B_\zeta}(\vec{p}; -2\zeta)}_{n_2} \cdots \underbrace{\lambda_{C_\zeta}(\vec{p}; -2r\zeta)}_{n_{2r}} \cdots \\ \lambda_{k_\zeta}(\vec{p}; n_1, n_2, \dots, n_{2r}) := \frac{\sqrt{(2s)!}}{\sqrt{n_1!n_2! \cdots n_{2r}!}} \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}(s; w) \underbrace{\lambda_{A_\zeta}(\vec{p}; -1\zeta)}_{n_1} \underbrace{\lambda_{B_\zeta}(\vec{p}; -2\zeta)}_{n_2} \cdots \underbrace{\lambda_{C_\zeta}(\vec{p}; -2r\zeta)}_{n_{2r}} \cdots \\ \lambda_{A_\zeta}(\vec{p}; -(r+k)\zeta) := \lambda_{A_\zeta}(\vec{p}; k\zeta), k = 1, 2, \dots, r, \sum_{k=1}^{2r} n_k = 2s \end{cases}$$

推论1.7.1.

$$\begin{cases} [\sigma(s, w) \cdot \hat{p}] \lambda_\zeta(\vec{p}; n_1, n_2, \dots, n_{2r}) = -\frac{\zeta}{2} \left[ \sum_{k=1}^r (n_k - n_{r+k}) \right] \lambda_\zeta(\vec{p}; n_1, n_2, \dots, n_{2r}) \\ \sigma(s; w) = s\bar{\Gamma}(s; w)(\Gamma \otimes I_*)\Gamma(s; w), \sum_{k=1}^{2r} n_k = 2s \end{cases}$$

推论1.7.2.

$$\begin{cases} [\sigma(s, w) \cdot \hat{p}] \lambda_\zeta(\vec{p}; n_1, n_2, \dots, n_r) = -\zeta s \lambda_\zeta(\vec{p}; n_1, n_2, \dots, n_r), \sum_{k=1}^r n_k = 2s \\ \lambda_{k_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) := \lambda_{k_\zeta}(\vec{p}; n_1, n_2, \dots, n_r; 0_1, 0_2, \dots, 0_r) \\ [\sigma(s, w) \cdot \hat{p}] \lambda_\zeta(\vec{p}; n_{r+1}, n_{r+2}, \dots, n_{2r}) = \zeta s \lambda_\zeta(\vec{p}; n_{r+1}, n_{r+2}, \dots, n_{2r}), \sum_{k=r+1}^{2r} n_k = 2s \\ \lambda_{k_\zeta}(\vec{p}; n_{r+1}, n_{r+2}, \dots, n_{2r}) := \lambda_{k_\zeta}(\vec{p}; 0_1, 0_2, \dots, 0_r; n_{r+1}, n_{r+2}, \dots, n_{2r}) \end{cases}$$

## 2 N+1维时空中Penrose全对称方程自旋基及其基本性质

### 2.1 N+1维时空中Penrose全对称方程自旋基的广义多项式定理

定理2.1.1.  $\sum_{n_1+\dots+n_r=2s} \frac{(2s)!}{n_1!n_2! \cdots n_r!}$

$$\underbrace{\lambda_{A_\zeta}(\vec{p}; -1\zeta)}_{n_1} \underbrace{\lambda_{B_\zeta}(\vec{p}; -2\zeta)}_{n_2} \cdots \underbrace{\lambda_{C_\zeta}(\vec{p}; -r\zeta)}_{n_r} \underbrace{\lambda_{A'_\zeta}(\vec{p}; -1\zeta)}_{n_1} \underbrace{\lambda_{B'_\zeta}(\vec{p}; -2\zeta)}_{n_2} \cdots \underbrace{\lambda_{C'_\zeta}(\vec{p}; -r\zeta)}_{n_r} \cdots$$

$$= \left[ \sum_{h=1}^r \lambda_{A_\zeta}(\vec{p}; -h\zeta) \lambda_{A'_\zeta}(\vec{p}; -h\zeta) \right] \cdot \left[ \sum_{h=1}^r \lambda_{B_\zeta}(\vec{p}; -h\zeta) \lambda_{B'_\zeta}(\vec{p}; -h\zeta) \right] \cdot \left[ \sum_{h=1}^r \lambda_{C_\zeta}(\vec{p}; -h\zeta) \lambda_{C'_\zeta}(\vec{p}; -h\zeta) \right] \cdots$$

推论2.1.1.  $\sum_{n_1+\dots+n_r=2s} \frac{(2s)!}{n_1!n_2! \cdots n_r!}$

$$\begin{aligned} & \left[ \sum_{h=1}^r \lambda_{A_\zeta}(\vec{p}; -1\zeta) \lambda_{A'_\zeta}^+(\vec{p}; -1\zeta) \right]^{n_1} \left[ \sum_{h=1}^r \lambda_{A_\zeta}(\vec{p}; -2\zeta) \lambda_{A'_\zeta}^+(\vec{p}; -2\zeta) \right]^{n_2} \cdots \left[ \sum_{h=1}^r \lambda_{A_\zeta}(\vec{p}; -r\zeta) \lambda_{A'_\zeta}^+(\vec{p}; -r\zeta) \right]^{n_r} \\ &= \left[ \sum_{h=1}^r \lambda_{A_\zeta}(\vec{p}; -h\zeta) \lambda_{A'_\zeta}^+(\vec{p}; -h\zeta) \right]^{2s} \end{aligned}$$

以上推论正好就是多项式展开定理。

## 2.2 N+1维时空中Penrose全对称方程的自旋基

定义2.2.1.

$$\begin{cases} [\sigma(\frac{1}{2}, w) \cdot \hat{p}] \lambda(\vec{p}; n_1, n_2, \dots, n_r) = -\frac{\zeta}{2} \lambda(\vec{p}; n_1, n_2, \dots, n_r), n_1 + n_2 + \dots + n_r = 2s \\ \lambda_{\underbrace{A_\zeta \cdots B_\zeta \cdots C_\zeta}_{2s}}(\vec{p}; n_1, n_2, \dots, n_r) := \frac{1}{\sqrt{(2s)! n_1! n_2! \cdots n_r!}} \underbrace{\lambda_{A_\zeta}(\vec{p}; -1\zeta)}_{n_1} \cdots \underbrace{\lambda_{B_\zeta}(\vec{p}; -2\zeta)}_{n_2} \cdots \underbrace{\lambda_{C_\zeta}(\vec{p}; -r\zeta)}_{n_r} \cdots \end{cases}$$

定义2.2.2.

$$\begin{cases} [\sigma(s, w) \cdot \hat{p}] \lambda(\vec{p}; -s\zeta; n_1, n_2, \dots, n_r) = -s\zeta \lambda(\vec{p}; -s\zeta; n_1, n_2, \dots, n_r), n_1 + n_2 + \dots + n_r = 2s \\ \lambda_{k_\zeta}(\vec{p}; -s\zeta; n_1, n_2, \dots, n_r) := \frac{\sqrt{(2s)!}}{\sqrt{n_1! n_2! \cdots n_r!}} \Gamma_{k_\zeta}^{\underbrace{A_\zeta B_\zeta C_\zeta}_{2s}}(s; w) \underbrace{\lambda_{A_\zeta}(\vec{p}; -1\zeta)}_{n_1} \cdots \underbrace{\lambda_{B_\zeta}(\vec{p}; -2\zeta)}_{n_2} \cdots \underbrace{\lambda_{C_\zeta}(\vec{p}; -r\zeta)}_{n_r} \cdots \end{cases}$$

## 2.3 N+1维时空中Penrose全对称方程自旋基的正交性质

$$\text{推论2.3.1. } \lambda_{\underbrace{A_\zeta \cdots B_\zeta \cdots C_\zeta}_{2s}}(\vec{p}, -s\zeta; n_1, n_2, \dots, n_r) \lambda_{\underbrace{A_\zeta \cdots B_\zeta \cdots C_\zeta}_{2s}}(\vec{p}, -s\zeta; n'_1, n'_2, \dots, n'_r) = \delta_{n_1 n'_1} \delta_{n_2 n'_2} \cdots \delta_{n_r n'_r}$$

$$\text{推论2.3.2. } \lambda_{\underbrace{A_\zeta \cdots B_\zeta \cdots C_\zeta}_{2s}}(-\vec{p}, -s\zeta; n_1, n_2, \dots, n_r) \lambda_{\underbrace{A_\zeta \cdots B_\zeta \cdots C_\zeta}_{2s}}(\vec{p}, -s\zeta; n'_1, n'_2, \dots, n'_r) = 0$$

$$\text{推论2.3.3. } \lambda_{\underbrace{A_\zeta B_\zeta}_{2s}}(\vec{p}, -s\zeta; \vec{h}) \lambda_{\underbrace{A_\zeta B_\zeta}_{2s}}(\vec{p}, -s\zeta; \vec{h}') = \delta_{\vec{h} \vec{h}'}, \vec{h} := (n_1, n_2, \dots, n_r)$$

$$\text{推论2.3.4. } \lambda_{\underbrace{A_\zeta B_\zeta}_{2s}}(-\vec{p}, -s\zeta; \vec{h}) \lambda_{\underbrace{A_\zeta B_\zeta}_{2s}}(\vec{p}, -s\zeta; \vec{h}') = 0$$

## 2.4 N+1维时空中Penrose全对称方程自旋基的分解

$$\begin{aligned} \text{定理2.4.1. } & \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta A'_\zeta B'_\zeta \cdots C'_\zeta D'_\zeta}_{2s}}(\vec{p}; n_1, n_2, \dots, n_r) \\ &= \sum_{n'_1 + \dots + n'_r = 2s'}^{=2s'} \frac{\sqrt{C_{n'_1}^{n'_1} C_{n'_2}^{n'_2} \cdots C_{n'_r}^{n'_r}}}{\sqrt{C_{2s'}^{2s'}}} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2(s-s')}}(\vec{p}; n_1 - n'_1, n_2 - n'_2, \dots, n_r - n'_r) \lambda_{\underbrace{A'_\zeta B'_\zeta \cdots C'_\zeta D'_\zeta}_{2s'}}(\vec{p}; n'_1, n'_2, \dots, n'_r) \end{aligned}$$

$$\begin{aligned} \text{证明: } & \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta A'_\zeta B'_\zeta \cdots C'_\zeta D'_\zeta}_{2s}}(\vec{p}; n_1, n_2, \dots, n_r) = \frac{1}{\sqrt{(2s)! n_1! n_2! \cdots n_r!}} \sum_{n'_1 + \dots + n'_r = 2s'}^{=2s'} C_{n'_1}^{n'_1} C_{n'_2}^{n'_2} \cdots C_{n'_r}^{n'_r} \\ & \underbrace{\lambda_{\{A_\zeta(\vec{p}; -1\zeta) \lambda_{B_\zeta}(\vec{p}; -2\zeta) \cdots \lambda_{C_\zeta}(\vec{p}; -r\zeta) \lambda_{D_\zeta}\}}_{n_1 - n'_1}(\vec{p}; -r\zeta)}_{n_1 - n'_1} \underbrace{\lambda_{\{A'_\zeta(\vec{p}; -1\zeta) \lambda_{B'_\zeta}(\vec{p}; -2\zeta) \cdots \lambda_{C'_\zeta}(\vec{p}; -r\zeta) \lambda_{D'_\zeta}\}}_{n'_1}(\vec{p}; -r\zeta)}_{n_r - n'_r} \\ &= \frac{1}{\sqrt{(2s)! n_1! n_2! \cdots n_r!}} \sum_{n'_1 + \dots + n'_r = 2s'}^{=2s'} C_{n'_1}^{n'_1} C_{n'_2}^{n'_2} \cdots C_{n'_r}^{n'_r} \\ & \sqrt{(2s - 2s')! (n_1 - n'_1)! (n_2 - n'_2)! \cdots (n_r - n'_r)!} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2(s-s')}}(\vec{p}; n_1 - n'_1, n_2 - n'_2, \dots, n_r - n'_r) \\ & \sqrt{(2s')! n'_1! n'_2! \cdots n'_r!} \lambda_{\underbrace{A'_\zeta B'_\zeta \cdots C'_\zeta D'_\zeta}_{2s'}}(\vec{p}; n'_1, n'_2, \dots, n'_r) \\ &= \sum_{n'_1 + \dots + n'_r = 2s'}^{=2s'} \frac{\sqrt{C_{n'_1}^{n'_1} C_{n'_2}^{n'_2} \cdots C_{n'_r}^{n'_r}}}{\sqrt{C_{2s'}^{2s'}}} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2(s-s')}}(\vec{p}; n_1 - n'_1, n_2 - n'_2, \dots, n_r - n'_r) \lambda_{\underbrace{A'_\zeta B'_\zeta \cdots C'_\zeta D'_\zeta}_{2s'}}(\vec{p}; n'_1, n'_2, \dots, n'_r) \end{aligned}$$

□

$$\begin{aligned} \text{推论2.4.1. } & \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(\vec{p}; n_1, n_2, \cdots, n_r) = \frac{\sqrt{n_1}}{\sqrt{2s}} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta}_{2s-1}}(\vec{p}; n_1 - 1, n_2, \cdots, n_r) \lambda_{D_\zeta}(\vec{p}; 1, 0, \cdots, 0) \\ & + \frac{\sqrt{n_2}}{\sqrt{2s}} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta}_{2s-1}}(\vec{p}; n_1, n_2 - 1, \cdots, n_r) \lambda_{D_\zeta}(\vec{p}; 0, 1, \cdots, 0) + \cdots + \frac{\sqrt{n_r}}{\sqrt{2s}} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta}_{2s-1}}(\vec{p}; n_1, n_2, \cdots, n_r - 1) \lambda_{D_\zeta}(\vec{p}; 0, 0, \cdots, 1) \end{aligned}$$

$$\begin{aligned} \text{推论2.4.2. } & \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(\vec{p}; 0, n_2, \cdots, n_r) \\ & = \frac{\sqrt{n_2}}{\sqrt{2s}} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta}_{2s-1}}(\vec{p}; 0, n_2 - 1, \cdots, n_r) \lambda_{D_\zeta}(\vec{p}; 0, 1, \cdots, 0) + \cdots + \frac{\sqrt{n_r}}{\sqrt{2s}} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta}_{2s-1}}(\vec{p}; 0, n_2, \cdots, n_r - 1) \lambda_{D_\zeta}(\vec{p}; 0, 0, \cdots, 1) \end{aligned}$$

## 2.5 N+1维时空中Penrose全对称方程的投影算子

$$\text{推论2.5.1. } \sum_{\vec{h}} \lambda_{\underbrace{A_\zeta B_\zeta \cdots}_{2s}}(\vec{p}, -s\zeta; \vec{h}) \lambda_{\underbrace{A'_\zeta B'_\zeta \cdots}_{2s}}^+(\vec{p}, -s\zeta; \vec{h}) = \left(-\frac{\zeta}{2}\right)^{2s} \frac{1}{[(2s)!]^2} \overbrace{(\Gamma, i\zeta)_{\{A_\zeta(A'_\zeta(\Gamma, i\zeta)_{B_\zeta B'_\zeta} \cdots\})}^a}^{2s} \overbrace{\hat{p}_a \hat{p}_b \cdots}^{2s}$$

$$\text{引理2.5.1. } \begin{cases} (\Gamma, i\zeta)_{A_\zeta A'_\zeta}^a p_a (\Gamma, i\zeta)_{B_\zeta B'_\zeta}^b p_b \neq (\Gamma, i\zeta)_{B_\zeta A'_\zeta}^a p_a (\Gamma, i\zeta)_{A_\zeta B'_\zeta}^b p_b, p^a p_a = 0 \\ (\Gamma, i\zeta)_{A_\zeta A'_\zeta}^a p_a (\Gamma, i\zeta)_{B_\zeta B'_\zeta}^b p_b \neq (\Gamma, i\zeta)_{A_\zeta B'_\zeta}^a p_a (\Gamma, i\zeta)_{B_\zeta A'_\zeta}^b p_b, p^a p_a = 0 \end{cases}$$

$$\text{引理2.5.2. } \begin{cases} (\Gamma, i\zeta)_{A_\zeta A'_\zeta}^a \partial_a (\Gamma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_b \neq (\Gamma, i\zeta)_{B_\zeta A'_\zeta}^a \partial_a (\Gamma, i\zeta)_{A_\zeta B'_\zeta}^b \partial_b, \partial^a \partial_a = 0 \\ (\Gamma, i\zeta)_{A_\zeta A'_\zeta}^a \partial_a (\Gamma, i\zeta)_{B_\zeta B'_\zeta}^b \partial_b \neq (\Gamma, i\zeta)_{A_\zeta B'_\zeta}^a \partial_a (\Gamma, i\zeta)_{B_\zeta A'_\zeta}^b \partial_b, \partial^a \partial_a = 0 \end{cases}$$

直接验证便可以证明以上两个引理。

$$\text{推论2.5.2. } \sum_{\vec{h}} \lambda_{\underbrace{A_\zeta B_\zeta \cdots}_{2s}}(\vec{p}, -s\zeta; \vec{h}) \lambda_{\underbrace{A'_\zeta B'_\zeta \cdots}_{2s}}^+(\vec{p}, -s\zeta; \vec{h}) \neq \left(-\frac{\zeta}{2}\right)^{2s} \overbrace{(\Gamma, i\zeta)_{A_\zeta A'_\zeta}^a}^{2s} \overbrace{(\Gamma, i\zeta)_{B_\zeta B'_\zeta}^b}^{2s} \hat{p}_a \hat{p}_b \cdots$$

## 3 N+1维时空中Penrose全对称方程的协变量子化

### 3.1 N+1维时空中Penrose全对称方程<sup>[1, 2]</sup>的平面波解及其对易规则猜想

$$\text{推论3.1.1. } (\Gamma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \psi_{\underbrace{A_\zeta B_\zeta \cdots}_{2s}}(x) = 0$$

$$\begin{cases} \psi_{\underbrace{A_\zeta B_\zeta \cdots}_{2s}}(x) = \frac{1}{(2\pi)^{N/2}} \int \sum_{\vec{p} \neq 0} \frac{1}{\vec{h}} |\vec{p}|^{(s-\frac{1}{2})} \lambda_{\underbrace{A_\zeta B_\zeta \cdots}_{2s}}(\vec{p}, -s\zeta; \vec{h}) [a_1(\vec{p}, -s\zeta; \vec{h}) e^{ip \cdot x} + a_2^+(\vec{p}, -s\zeta; \vec{h}) e^{-ip \cdot x}] d^N \vec{p} \\ |\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -s\zeta; \vec{h}) = \frac{1}{(2\pi)^{N/2}} \int \lambda^{+\underbrace{A_\zeta B_\zeta \cdots}_{2s}}(\vec{p}, -s\zeta; \vec{h}) \psi_{\underbrace{A_\zeta B_\zeta \cdots}_{2s}}(x) e^{-ip \cdot x} d^N \vec{r} \\ |\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}, -s\zeta; \vec{h}) = \frac{1}{(2\pi)^{N/2}} \int \lambda^{+\underbrace{A_\zeta B_\zeta \cdots}_{2s}}(\vec{p}, -s\zeta; \vec{h}) \psi_{\underbrace{A_\zeta B_\zeta \cdots}_{2s}}(x) e^{ip \cdot x} d^N \vec{r} \end{cases}$$

$$\text{推论3.1.2. } (\Gamma, -i\zeta)_a^{A'_\zeta A_\zeta} \partial^a \psi_{A_\zeta}(x) = 0$$

$$\begin{cases} \psi_{A_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int \sum_{\vec{p} \neq 0} \sum_{h=1}^{l/4} \lambda_{A_\zeta}(\hat{p}; -h\zeta) [a_1(\vec{p}, -s\zeta; \vec{h}) e^{ip \cdot x} + a_2^+(\vec{p}, -s\zeta; \vec{h}) e^{-ip \cdot x}] d^N \vec{p} \\ a_1(\vec{p}, -s\zeta; \vec{h}) = \frac{1}{(2\pi)^{N/2}} \int \lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}; \vec{h}) \psi_{A_\zeta}(x) e^{-ip \cdot x} d^N \vec{r} \\ a_2^+(\vec{p}, -s\zeta; \vec{h}) = \frac{1}{(2\pi)^{N/2}} \int \lambda^{+A_\zeta}(\hat{p}, -\frac{\zeta}{2}; \vec{h}) \psi_{A_\zeta}(x) e^{ip \cdot x} d^N \vec{r} \end{cases}$$

$$\text{猜想3.1.1. } [\psi_{\underbrace{A_\zeta B_\zeta \cdots}_{2s}}(x), \psi_{\underbrace{A'_\zeta B'_\zeta \cdots}_{2s}}^+(x')]_{-2s+1} = i \frac{(i\zeta)^{2s}}{2^{2s-1} [(2s)!]^2} \overbrace{(\Gamma, i\zeta)_{\{A_\zeta(A'_\zeta(\Gamma, i\zeta)_{B_\zeta B'_\zeta} \cdots\})}^a}^{2s} \overbrace{\partial_a \partial_b \cdots}^{2s} \Delta(x - x')$$

### 3.2 N+1维时空中Penrose全对称方程自旋基之间的递推关系

#### 3.2.1 关于对称性条件的Penrose自旋基引理

引理3.2.1.

$$\sum_{n_1+\dots+n_r}^{=2s} a_{E_\zeta}(\vec{p}; n_1, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r) = \sum_{n_1+\dots+n_r}^{=2s} a_{D_\zeta}(\vec{p}; n_1, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta E_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r)$$

$$\Leftrightarrow \begin{cases} \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1+1, n_2, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2+1, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r+1) \lambda_{D_\zeta]}(\vec{p}; r) = 0 \end{cases}$$

证明:

$$\sum_{n_1+\dots+n_r}^{=2s} a_{E_\zeta}(\vec{p}; n_1, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r) = \sum_{n_1+\dots+n_r}^{=2s} a_{D_\zeta}(\vec{p}; n_1, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta E_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r)$$

$$\Leftrightarrow \sum_{n_1+\dots+n_r}^{=2s} a_{E_\zeta}(\vec{p}; n_1, \dots, n_r) \left[ \frac{\sqrt{n_1}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; n_1-1, n_2, \dots, n_r) \lambda_{D_\zeta}(\vec{p}; 1) \right.$$

$$+ \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; n_1, n_2-1, \dots, n_r) \lambda_{D_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_r}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; n_1, n_2, \dots, n_r-1) \lambda_{D_\zeta}(\vec{p}; r) \left. \right]$$

$$= \sum_{n_1+\dots+n_r}^{=2s} a_{D_\zeta}(\vec{p}; n_1, \dots, n_r) \left[ \frac{\sqrt{n_1}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; n_1-1, n_2, \dots, n_r) \lambda_{E_\zeta}(\vec{p}; 1) \right.$$

$$+ \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; n_1, n_2-1, \dots, n_r) \lambda_{E_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_r}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; n_1, n_2, \dots, n_r-1) \lambda_{E_\zeta}(\vec{p}; r) \left. \right]$$

$$\Leftrightarrow \begin{cases} \frac{\sqrt{n_1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1-1, n_2+1, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1-1, n_2, \dots, n_r+1) \lambda_{D_\zeta]}(\vec{p}; r) = 0 \\ \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1+1, n_2-1, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2-1, \dots, n_r+1) \lambda_{D_\zeta]}(\vec{p}; r) = 0 \dots \dots \\ \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1+1, n_2, \dots, n_r-1) \lambda_{D_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2+1, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; r) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{\sqrt{n_1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1-1, n_2+1, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1-1, n_2, \dots, n_r+1) \lambda_{D_\zeta]}(\vec{p}; r) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1+1, n_2, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2+1, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r+1) \lambda_{D_\zeta]}(\vec{p}; r) = 0 \end{cases} \quad \square$$

引理3.2.2.

$$\begin{cases} a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) + \sum_{k=1}^r d(\vec{p}; n_1, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; -k) \\ \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1+1, n_2, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2+1, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r+1) \lambda_{D_\zeta]}(\vec{p}; r) = 0 \end{cases}$$

$$\Leftrightarrow a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k), d(\vec{p}; n_1, n_2, \dots, n_r; k) = 0$$

$$\begin{cases} a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2+1, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1-1, n_2, \dots, n_r+1; 1) \lambda_{E_\zeta}(\vec{p}; r), n_1 \geq 1 \\ a_{E_\zeta}(\vec{p}; 0, n_2, \dots, n_r) = c(\vec{p}; 0, n_2, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) \\ + \frac{\sqrt{n_2}}{\sqrt{n_2}} c(\vec{p}; 0, n_2, \dots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3+1, \dots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 3) + \dots \\ + \frac{\sqrt{n_r+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2-1, n_3, \dots, n_r+1; 2) \lambda_{E_\zeta}(\vec{p}; r), n_2 \geq 1 \end{cases}$$

$$\begin{cases} a_{E_\zeta}(\vec{p}; 0, 0, n_3, \dots, n_r) = c(\vec{p}; 0, 0, n_3, \dots, n_r; 1)\lambda_{E_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, 0, n_3, \dots, n_r; 2)\lambda_{E_\zeta}(\vec{p}; 2) \\ + \frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3, n_4, \dots, n_r; 3)\lambda_{E_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, n_4 + 1, \dots, n_r; 3)\lambda_{E_\zeta}(\vec{p}; 4) + \dots \\ + \frac{\sqrt{n_r+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, \dots, n_r + 1; 3)\lambda_{E_\zeta}(\vec{p}; r), n_3 \geq 1 \\ \dots\dots\dots \\ a_{E_\zeta}(\vec{p}; 0, \dots, 0, n_r) = c(\vec{p}; 0, \dots, 0, n_r; 1)\lambda_{E_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, \dots, 0, n_r; 2)\lambda_{E_\zeta}(\vec{p}; 2) \\ + \dots + c(\vec{p}; 0, \dots, 0, n_r; r-1)\lambda_{E_\zeta}(\vec{p}; r-1) + \frac{\sqrt{n_r}}{\sqrt{n_r}}c(\vec{p}; 0, \dots, 0, n_r; r)\lambda_{E_\zeta}(\vec{p}; r), n_r = 2s \geq 1 \end{cases}$$

证明:

$$\begin{cases} a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k)\lambda_{E_\zeta}(\vec{p}; k) + \sum_{k=1}^r d(\vec{p}; n_1, n_2, \dots, n_r; k)\lambda_{E_\zeta}(\vec{p}; -k) \\ \frac{\sqrt{n_1+1}}{\sqrt{2s}}a_{[E_\zeta]}(\vec{p}; n_1 + 1, n_2, \dots, n_r)\lambda_{D_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}}a_{[E_\zeta]}(\vec{p}; n_1, n_2 + 1, \dots, n_r)\lambda_{D_\zeta}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{2s}}a_{[E_\zeta]}(\vec{p}; n_1, n_2, \dots, n_r + 1)\lambda_{D_\zeta}(\vec{p}; r) = 0 \end{cases}$$

 $\Leftrightarrow$ 

$$a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k)\lambda_{E_\zeta}(\vec{p}; k), d(\vec{p}; n_1, n_2, \dots, n_r; k) = 0$$

$$\begin{cases} c(\vec{p}; n_1 + 1, n_2, \dots, n_r; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1+1}}c(\vec{p}; n_1, n_2 + 1, \dots, n_r; 1) \dots \\ c(\vec{p}; n_1 + 1, n_2, \dots, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_1+1}}c(\vec{p}; n_1, n_2, \dots, n_r + 1; 1) \\ c(\vec{p}; 0, n_2 + 1, \dots, n_r; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2+1}}c(\vec{p}; 0, n_2, n_3 + 1, \dots, n_r; 2) \dots \\ c(\vec{p}; 0, n_2 + 1, \dots, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_2+1}}c(\vec{p}; 0, n_2, \dots, n_r + 1; 2) \\ c(\vec{p}; 0, 0, n_3 + 1, \dots, n_r; 4) = \frac{\sqrt{n_4+1}}{\sqrt{n_3+1}}c(\vec{p}; 0, 0, n_3, n_4 + 1, \dots, n_r; 3) \dots \\ c(\vec{p}; 0, 0, n_3 + 1, \dots, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_3+1}}c(\vec{p}; 0, 0, n_3, n_4, \dots, n_r + 1; 3) \\ \dots \\ c(\vec{p}; 0, \dots, 0, n_{r-1} + 1, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_{r-1}+1}}c(\vec{p}; 0, \dots, 0, n_{r-1}, n_r + 1; r-1) \end{cases}$$

 $\Leftrightarrow$ 

$$a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k)\lambda_{E_\zeta}(\vec{p}; k), d(\vec{p}; n_1, n_2, \dots, n_r; k) = 0$$

$$\begin{cases} a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \frac{\sqrt{n_1}}{\sqrt{n_1}}c(\vec{p}; n_1, n_2, \dots, n_r; 1)\lambda_{E_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_r; 1)\lambda_{E_\zeta}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2, \dots, n_r + 1; 1)\lambda_{E_\zeta}(\vec{p}; r), n_1 \geq 1 \\ a_{E_\zeta}(\vec{p}; 0, n_2, \dots, n_r) = c(\vec{p}; 0, n_2, \dots, n_r; 1)\lambda_{E_\zeta}(\vec{p}; 1) \\ + \frac{\sqrt{n_2}}{\sqrt{n_2}}c(\vec{p}; 0, n_2, \dots, n_r; 2)\lambda_{E_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_r; 2)\lambda_{E_\zeta}(\vec{p}; 3) + \dots \\ + \frac{\sqrt{n_r+1}}{\sqrt{n_2}}c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_r + 1; 2)\lambda_{E_\zeta}(\vec{p}; r), n_2 \geq 1 \\ a_{E_\zeta}(\vec{p}; 0, 0, n_3, \dots, n_r) = c(\vec{p}; 0, 0, n_3, \dots, n_r; 1)\lambda_{E_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, 0, n_3, \dots, n_r; 2)\lambda_{E_\zeta}(\vec{p}; 2) \\ + \frac{\sqrt{n_3}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3, n_4, \dots, n_r; 3)\lambda_{E_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, n_4 + 1, \dots, n_r; 3)\lambda_{E_\zeta}(\vec{p}; 4) + \dots \\ + \frac{\sqrt{n_r+1}}{\sqrt{n_3}}c(\vec{p}; 0, 0, n_3 - 1, \dots, n_r + 1; 3)\lambda_{E_\zeta}(\vec{p}; r), n_3 \geq 1 \\ \dots\dots\dots \\ a_{E_\zeta}(\vec{p}; 0, \dots, 0, n_r) = c(\vec{p}; 0, \dots, 0, n_r; 1)\lambda_{E_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, \dots, 0, n_r; 2)\lambda_{E_\zeta}(\vec{p}; 2) \\ + \dots + c(\vec{p}; 0, \dots, 0, n_r; r-1)\lambda_{E_\zeta}(\vec{p}; r-1) + \frac{\sqrt{n_r}}{\sqrt{n_r}}c(\vec{p}; 0, \dots, 0, n_r; r)\lambda_{E_\zeta}(\vec{p}; r), n_r = 2s \geq 1 \end{cases} \quad \square$$

$$\text{推论3.2.1. } a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k)\lambda_{E_\zeta}(\vec{p}; k)$$

$$\begin{cases} c(\vec{p}; n_1, n_2, \dots, n_r; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_r; 1), n_1 \geq 1 \dots \\ c(\vec{p}; n_1, n_2, \dots, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_1}}c(\vec{p}; n_1 - 1, n_2, \dots, n_r + 1; 1), n_1 \geq 1 \end{cases}$$

$$\begin{cases}
c(\vec{p}; 0, n_2, \dots, n_r; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_r; 2), n_2 \geq 1 \cdots \\
c(\vec{p}; 0, n_2, \dots, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_r + 1; 2), n_2 \geq 1 \\
\cdots \\
c(\vec{p}; 0, \dots, 0, n_{r-1}, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_{r-1}}} c(\vec{p}; 0, \dots, 0, n_{r-1} - 1, n_r + 1; r), n_{r-1} \geq 1 \\
\Leftrightarrow \\
a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) \\
\begin{cases}
a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 2) \\
+ \cdots + \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_r + 1; 1) \lambda_{E_\zeta}(\vec{p}; r), n_1 \geq 1 \\
a_{E_\zeta}(\vec{p}; 0, n_2, \dots, n_r) = c(\vec{p}; 0, n_2, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) \\
+ \frac{\sqrt{n_2}}{\sqrt{n_2}} c(\vec{p}; 0, n_2, \dots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 3) + \cdots \\
+ \frac{\sqrt{n_r+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_r + 1; 2) \lambda_{E_\zeta}(\vec{p}; r), n_2 \geq 1 \\
a_{E_\zeta}(\vec{p}; 0, 0, n_3, \dots, n_r) = c(\vec{p}; 0, 0, n_3, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, 0, n_3, \dots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 2) \\
+ \frac{\sqrt{n_3}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3, n_4, \dots, n_r; 3) \lambda_{E_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, n_4 + 1, \dots, n_r; 3) \lambda_{E_\zeta}(\vec{p}; 4) + \cdots \\
+ \frac{\sqrt{n_r+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, \dots, n_r + 1; 3) \lambda_{E_\zeta}(\vec{p}; r), n_3 \geq 1 \\
\cdots \cdots \cdots \\
a_{E_\zeta}(\vec{p}; 0, \dots, 0, n_r) = c(\vec{p}; 0, \dots, 0, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, \dots, 0, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 2) \\
+ \cdots + c(\vec{p}; 0, \dots, 0, n_r; r - 1) \lambda_{E_\zeta}(\vec{p}; r - 1) + \frac{\sqrt{n_r}}{\sqrt{n_r}} c(\vec{p}; 0, \dots, 0, n_r; r) \lambda_{E_\zeta}(\vec{p}; r), n_r = 2s \geq 1
\end{cases}
\end{cases}$$

## 引理3.2.3.

$$\begin{cases}
c(\vec{p}; n_1, n_2, \dots, n_r; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_r; 1), n_1 \geq 1 \cdots \\
c(\vec{p}; n_1, n_2, \dots, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_r + 1; 1), n_1 \geq 1
\end{cases}$$

$\Leftrightarrow$

$$\begin{aligned}
& \sum_{n_1+\dots+n_r=2s} \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r) \\
&= \sum_{n_1+\dots+n_r=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \dots, n_r; 1) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1 + 1, n_2, \dots, n_r) \\
&+ \sum_{n_2+\dots+n_r=2s} \sum_{k=2}^r c(\vec{p}; 0, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; 0, n_2, \dots, n_r)
\end{aligned}$$

证明:

$$\begin{cases}
c(\vec{p}; n_1, n_2, \dots, n_r; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_r; 1), n_1 \geq 1 \cdots \\
c(\vec{p}; n_1, n_2, \dots, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_r + 1; 1), n_1 \geq 1
\end{cases}$$

$\Leftrightarrow$

$$\begin{aligned}
& \sum_{n_1+\dots+n_r=2s} \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r) \\
&= \sum_{n_1+\dots+n_r=2s}^{n_1 \neq 0} \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, n_2, \dots, n_r) \left[ \frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) \right. \\
&+ \left. \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 2) + \cdots + \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_r + 1; 1) \lambda_{E_\zeta}(\vec{p}; r) \right] \\
&+ \sum_{n_1+\dots+n_r=2s}^{n_1=0} \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; 0, n_2, \dots, n_r) [c(\vec{p}; 0, n_2, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) \\
&+ c(\vec{p}; 0, n_2, \dots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 2) + \cdots + c(\vec{p}; 0, n_2, \dots, n_r; r) \lambda_{E_\zeta}(\vec{p}; r)] \\
&= \sum_{n_1+\dots+n_r=2s}^{1 \leq n_1 \leq 2s} \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, n_2, \dots, n_r) c(\vec{p}; n_1, n_2, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1)
\end{aligned}$$



$$\begin{aligned}
& + \sum_{n_1 \cdots + n_r = 2s}^{0 \leq n_1 \leq 2s-1, 1 \leq n_2 \leq 2s} \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1 + 1, n_2 - 1, \cdots, n_r) \frac{\sqrt{n_2}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 2) + \cdots \\
& + \sum_{n_1 \cdots + n_r = 2s}^{0 \leq n_1 \leq 2s-1, 1 \leq n_r \leq 2s} \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1 + 1, n_2, \cdots, n_r - 1) \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \cdots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; r) \\
& + \sum_{n_1 \cdots + n_r = 2s}^{n_1=0} \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; 0, n_2, \cdots, n_r) [c(\vec{p}; 0, n_2, \cdots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) \\
& + c(\vec{p}; 0, n_2, \cdots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 2) + \cdots + c(\vec{p}; 0, n_2, \cdots, n_r; r) \lambda_{E_\zeta}(\vec{p}; r)] \\
& = \sum_{n_1 \cdots + n_r = 2s} \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, n_2, \cdots, n_r) c(\vec{p}; n_1, n_2, \cdots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) \\
& + \sum_{n_1 \cdots + n_r = 2s} \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1 + 1, n_2 - 1, \cdots, n_r) \frac{\sqrt{n_2}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 2) + \cdots \\
& + \sum_{n_1 \cdots + n_r = 2s} \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1 + 1, n_2, \cdots, n_r - 1) \frac{\sqrt{n_r}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; r) \\
& + \sum_{n_1 \cdots + n_r = 2s}^{n_1=0} \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; 0, n_2, \cdots, n_r) [c(\vec{p}; 0, n_2, \cdots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 2) + \cdots + c(\vec{p}; 0, n_2, \cdots, n_r; r) \lambda_{E_\zeta}(\vec{p}; r)] \\
& = \sum_{n_1 \cdots + n_r = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdots, n_r; 1) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1 + 1, n_2, \cdots, n_r) \\
& + \sum_{n_2 \cdots + n_r = 2s} \sum_{k=2}^r c(\vec{p}; 0, n_2, \cdots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; 0, n_2, \cdots, n_r)
\end{aligned}$$

□

**推论3.2.2.**

$$\begin{cases}
c(\vec{p}; n_1, n_2, \cdots, n_r; 2) = \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2 + 1, \cdots, n_r; 1), n_1 \geq 1 \cdots \\
c(\vec{p}; n_1, n_2, \cdots, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \cdots, n_r + 1; 1), n_1 \geq 1 \\
c(\vec{p}; 0, n_2, \cdots, n_r; 3) = \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3 + 1, \cdots, n_r; 2), n_2 \geq 1 \cdots \\
c(\vec{p}; 0, n_2, \cdots, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \cdots, n_r + 1; 2), n_2 \geq 1 \\
\cdots \\
c(\vec{p}; 0, \cdots, 0, n_{r-1}, n_r; r) = \frac{\sqrt{n_r+1}}{\sqrt{n_{r-1}}} c(\vec{p}; 0, \cdots, 0, n_{r-1} - 1, n_r + 1; r), n_{r-1} \geq 1
\end{cases}$$

$$\begin{aligned}
& \Leftrightarrow \sum_{n_1 \cdots + n_r = 2s} \sum_{k=1}^r c(\vec{p}; n_1, n_2, \cdots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, \cdots, n_r) \\
& = \sum_{n_1 + \cdots + n_r = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdots, n_r; 1) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1 + 1, n_2, \cdots, n_r) \\
& + \sum_{n_2 + \cdots + n_r = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} c(\vec{p}; 0, n_2, n_3, \cdots, n_r; 2) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, n_2 + 1, n_3, \cdots, n_r) \\
& + \sum_{n_3 + \cdots + n_r = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} c(\vec{p}; 0, 0, n_3, \cdots, n_r; 3) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3 + 1, \cdots, n_r) \\
& + \cdots + \sum_{n_r = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_r+1}} c(\vec{p}; 0, \cdots, 0, n_r; r) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, \cdots, 0, n_r + 1)
\end{aligned}$$

**引理3.2.4.**

$$\begin{aligned}
& \sum_{n_1 + \cdots + n_r = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \cdots, n_r; 1) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1 + 1, n_2, \cdots, n_r) \\
& + \sum_{n_2 + \cdots + n_r = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} c(\vec{p}; 0, n_2, n_3, \cdots, n_r; 2) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, n_2 + 1, n_3, \cdots, n_r) \\
& + \sum_{n_3 + \cdots + n_r = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} c(\vec{p}; 0, 0, n_3, \cdots, n_r; 3) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3 + 1, \cdots, n_r) \\
& + \cdots + \sum_{n_r = 2s} \frac{\sqrt{2s+1}}{\sqrt{n_r+1}} c(\vec{p}; 0, \cdots, 0, n_r; r) \underbrace{\lambda_{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, \cdots, 0, n_r + 1)
\end{aligned}$$

$$= \sum_{n_1+\dots+n_r=2s+1} a(\vec{p}; n_1, n_2, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_r)$$

$$\begin{cases} a(\vec{p}; n_1, n_2, \dots, n_r) := \frac{\sqrt{2s+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_r; 1), n_1 \neq 0 \\ a(\vec{p}; 0, n_2, \dots, n_r) := \frac{\sqrt{2s+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_r; 2), n_2 \neq 0 \\ a(\vec{p}; 0, 0, n_3, \dots, n_r) := \frac{\sqrt{2s+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, \dots, n_r; 3), n_3 \neq 0 \\ \dots \\ a(\vec{p}; 0, 0, \dots, 0, n_r) := \frac{\sqrt{2s+1}}{\sqrt{n_r}} c(\vec{p}; 0, \dots, 0, n_r - 1; r), n_r \neq 0 \end{cases}$$

**证明:**

$$\sum_{n_1+\dots+n_r=2s} \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r)$$

$$= \sum_{n_1+\dots+n_r=2s} \frac{\sqrt{2s+1}}{\sqrt{n_1+1}} c(\vec{p}; n_1, n_2, \dots, n_r; 1) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1 + 1, n_2, \dots, n_r)$$

$$+ \sum_{n_2+\dots+n_r=2s} \frac{\sqrt{2s+1}}{\sqrt{n_2+1}} c(\vec{p}; 0, n_2, n_3, \dots, n_r; 2) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, n_2 + 1, n_3, \dots, n_r)$$

$$+ \sum_{n_3+\dots+n_r=2s} \frac{\sqrt{2s+1}}{\sqrt{n_3+1}} c(\vec{p}; 0, 0, n_3, \dots, n_r; 3) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3 + 1, \dots, n_r)$$

$$+ \dots + \sum_{n_r=2s} \frac{\sqrt{2s+1}}{\sqrt{n_r+1}} c(\vec{p}; 0, \dots, 0, n_r; r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, \dots, 0, n_r + 1)$$

$$= \sum_{n_1+\dots+n_r=2s+1}^{n_1 \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_r; 1) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_r)$$

$$+ \sum_{n_2+\dots+n_r=2s+1}^{n_1=0, n_2 \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_r; 2) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, n_2, n_3, \dots, n_r)$$

$$+ \sum_{n_3+\dots+n_r=2s+1}^{n_1=0, n_2=0, n_3 \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, \dots, n_r; 3) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, 0, n_3, \dots, n_r)$$

$$+ \dots + \sum_{n_r=2s+1}^{n_1=0, \dots, n_{r-1}=0, n_r \neq 0} \frac{\sqrt{2s+1}}{\sqrt{n_r}} c(\vec{p}; 0, \dots, 0, n_r - 1; r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; 0, \dots, 0, n_r)$$

$$= \sum_{n_1+\dots+n_r=2s+1} a(\vec{p}; n_1, n_2, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_r)$$

$$\begin{cases} a(\vec{p}; n_1, n_2, \dots, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_r; 1), n_1 \neq 0 \\ a(\vec{p}; 0, n_2, \dots, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_r; 2), n_2 \neq 0 \\ a(\vec{p}; 0, 0, n_3, \dots, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, \dots, n_r; 3), n_3 \neq 0 \\ \dots \\ a(\vec{p}; 0, 0, \dots, 0, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_r}} c(\vec{p}; 0, \dots, 0, n_r - 1; r), n_r \neq 0 \end{cases}$$

□

### 3.2.2 几个推论

**推论3.2.3.**  $\underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, n_2, \dots, n_r) = \frac{\sqrt{n_1}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; n_1 - 1, n_2, \dots, n_r) \lambda_{D_\zeta}(\vec{p}; 1)$

$$+ \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; n_1, n_2 - 1, \dots, n_r) \lambda_{D_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_r}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; n_1, n_2, \dots, n_r - 1) \lambda_{D_\zeta}(\vec{p}; r)$$

**推论3.2.4.**  $\underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; 0, n_2, \dots, n_r) = \frac{\sqrt{n_2}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; 0, n_2 - 1, \dots, n_r) \lambda_{D_\zeta}(\vec{p}; 2)$

$$+ \frac{\sqrt{n_3}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; 0, n_2, n_3 - 1, \dots, n_r) \lambda_{D_\zeta}(\vec{p}; 0, 0, 1, \dots, 0) + \dots + \frac{\sqrt{n_r}}{\sqrt{2s}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta}}_{2s-1}(\vec{p}; 0, n_2, \dots, n_r - 1) \lambda_{D_\zeta}(\vec{p}; r)$$

**推论3.2.5.**  $\underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_r) = \frac{\sqrt{n_1}}{\sqrt{2s+1}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1 - 1, n_2, \dots, n_r) \lambda_{E_\zeta}(\vec{p}; 1)$

$$+ \frac{\sqrt{n_2}}{\sqrt{2s+1}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, n_2 - 1, \dots, n_r) \lambda_{E_\zeta}(\vec{p}; 2) + \dots + \frac{\sqrt{n_r}}{\sqrt{2s+1}} \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, n_2, \dots, n_r - 1) \lambda_{E_\zeta}(\vec{p}; r)$$

## 3.2.3 一个重要定理

定理3.2.1.

$$\begin{aligned}
& \sum_{n_1+\dots+n_r=2s} a_{E_\zeta}(\vec{p}; n_1, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r) = \sum_{n_1+\dots+n_r=2s} a_{D_\zeta}(\vec{p}; n_1, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta E_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r) \\
& \Leftrightarrow \\
& a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) \\
& \sum_{n_1+\dots+n_r=2s} a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r) = \sum_{n_1+\dots+n_r=2s+1} a(\vec{p}; n_1, n_2, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_r) \\
& \begin{cases} a(\vec{p}; n_1, n_2, \dots, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_r; 1), n_1 \neq 0 \\ a(\vec{p}; 0, n_2, \dots, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_r; 2), n_2 \neq 0 \\ a(\vec{p}; 0, 0, n_3, \dots, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, \dots, n_r; 3), n_3 \neq 0 \\ \dots \\ a(\vec{p}; 0, 0, \dots, 0, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_r}} c(\vec{p}; 0, \dots, 0, n_r - 1; r), n_r \neq 0 \end{cases}
\end{aligned}$$

证明:

$$\begin{aligned}
& \sum_{n_1+\dots+n_r=2s} a_{E_\zeta}(\vec{p}; n_1, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r) = \sum_{n_1+\dots+n_r=2s} a_{D_\zeta}(\vec{p}; n_1, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta E_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r) \\
& \Leftrightarrow \\
& \begin{cases} \frac{\sqrt{n_1+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1 + 1, n_2, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2 + 1, \dots, n_r) \lambda_{D_\zeta]}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{2s}} a_{[E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r + 1) \lambda_{D_\zeta]}(\vec{p}; r) = 0 \end{cases} \\
& \Leftrightarrow \\
& a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) \\
& \begin{cases} a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \frac{\sqrt{n_1}}{\sqrt{n_1}} c(\vec{p}; n_1, n_2, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) + \frac{\sqrt{n_2+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2 + 1, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 2) \\ + \dots + \frac{\sqrt{n_r+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_r + 1; 1) \lambda_{E_\zeta}(\vec{p}; r), n_1 \geq 1 \\ \begin{cases} a_{E_\zeta}(\vec{p}; 0, n_2, \dots, n_r) = c(\vec{p}; 0, n_2, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) \\ + \frac{\sqrt{n_2}}{\sqrt{n_2}} c(\vec{p}; 0, n_2, \dots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 2) + \frac{\sqrt{n_3+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3 + 1, \dots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 3) + \dots \\ + \frac{\sqrt{n_r+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_r + 1; 2) \lambda_{E_\zeta}(\vec{p}; r), n_2 \geq 1 \end{cases} \\ \begin{cases} a_{E_\zeta}(\vec{p}; 0, 0, n_3, \dots, n_r) = c(\vec{p}; 0, 0, n_3, \dots, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, 0, n_3, \dots, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 2) \\ + \frac{\sqrt{n_3}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3, n_4, \dots, n_r; 3) \lambda_{E_\zeta}(\vec{p}; 3) + \frac{\sqrt{n_4+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, n_4 + 1, \dots, n_r; 3) \lambda_{E_\zeta}(\vec{p}; 4) + \dots \\ + \frac{\sqrt{n_r+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, \dots, n_r + 1; 3) \lambda_{E_\zeta}(\vec{p}; r), n_3 \geq 1 \end{cases} \\ \dots \dots \dots \\ \begin{cases} a_{E_\zeta}(\vec{p}; 0, \dots, 0, n_r) = c(\vec{p}; 0, \dots, 0, n_r; 1) \lambda_{E_\zeta}(\vec{p}; 1) + c(\vec{p}; 0, \dots, 0, n_r; 2) \lambda_{E_\zeta}(\vec{p}; 2) \\ + \dots + c(\vec{p}; 0, \dots, 0, n_r; r - 1) \lambda_{E_\zeta}(\vec{p}; r - 1) + \frac{\sqrt{n_r}}{\sqrt{n_r}} c(\vec{p}; 0, \dots, 0, n_r; r) \lambda_{E_\zeta}(\vec{p}; r), n_r = 2s \geq 1 \end{cases} \end{cases} \\
& \Leftrightarrow \\
& a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) = \sum_{k=1}^r c(\vec{p}; n_1, n_2, \dots, n_r; k) \lambda_{E_\zeta}(\vec{p}; k) \\
& \sum_{n_1+\dots+n_r=2s} a_{E_\zeta}(\vec{p}; n_1, n_2, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta}}_{2s}(\vec{p}; n_1, \dots, n_r) = \sum_{n_1+\dots+n_r=2s+1} a(\vec{p}; n_1, n_2, \dots, n_r) \underbrace{\lambda_{A_\zeta B_\zeta \dots C_\zeta D_\zeta E_\zeta}}_{2s+1}(\vec{p}; n_1, n_2, \dots, n_r) \\
& \begin{cases} a(\vec{p}; n_1, n_2, \dots, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_1}} c(\vec{p}; n_1 - 1, n_2, \dots, n_r; 1), n_1 \neq 0 \\ a(\vec{p}; 0, n_2, \dots, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_2}} c(\vec{p}; 0, n_2 - 1, n_3, \dots, n_r; 2), n_2 \neq 0 \\ a(\vec{p}; 0, 0, n_3, \dots, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_3}} c(\vec{p}; 0, 0, n_3 - 1, \dots, n_r; 3), n_3 \neq 0 \\ \dots \\ a(\vec{p}; 0, 0, \dots, 0, n_r) = \frac{\sqrt{2s+1}}{\sqrt{n_r}} c(\vec{p}; 0, \dots, 0, n_r - 1; r), n_r \neq 0 \end{cases}
\end{aligned}$$

□

## 3.3 用数学归纳法严格求解N+1维时空中Penrose全对称方程的平面波解

定理3.3.1.  $(\Gamma, -i\zeta)_a^{A' A_\zeta} \partial^a \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(x) = 0, \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(x) = \frac{1}{(2s)!} \psi_{\{A_\zeta B_\zeta \cdots C_\zeta D_\zeta\}}(x)$

$$\Leftrightarrow \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(x)$$

$$= \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} d^N \vec{p} |\vec{p}|^{(s-\frac{1}{2})} \sum_{n_1+\cdots+n_r=2s} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(\vec{p}; n_1, \cdots, n_r) [a_1(\vec{p}; n_1, \cdots, n_r) e^{ip \cdot x} + a_2^+(\vec{p}; n_1, \cdots, n_r) e^{-ip \cdot x}]$$

$$\left\{ \begin{aligned} |\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}; n_1, \cdots, n_r) &= \frac{1}{(2\pi)^{N/2}} \int \lambda^{+\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(\vec{p}; n_1, \cdots, n_r) \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(x) e^{-ip \cdot x} d^N \vec{r} \\ |\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}; n_1, \cdots, n_r) &= \frac{1}{(2\pi)^{N/2}} \int \lambda^{+\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(\vec{p}; n_1, \cdots, n_r) \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(x) e^{ip \cdot x} d^N \vec{r} \end{aligned} \right.$$

证明: 采用数学归纳法证明此定理。

第一步:  $s' = 1/2$ 时成立:

$$(\Gamma, -i\zeta)_a^{A' A_\zeta} \partial^a \psi_{A_\zeta}(x) = 0, \psi_{A_\zeta}(x) = \psi_{A_\zeta}(x)$$

$\Leftrightarrow$

$$\psi_{A_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} d^N \vec{p} \sum_{n_1+\cdots+n_r=1} \lambda_{A_\zeta}(\vec{p}; n_1, \cdots, n_r) [a_1(\vec{p}; n_1, \cdots, n_r) e^{ip \cdot x} + a_2^+(\vec{p}; n_1, \cdots, n_r) e^{-ip \cdot x}]$$

第二步: 假设  $s' = s$ 时成立:

$$(\Gamma, -i\zeta)_a^{A' A_\zeta} \partial^a \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(x) = 0, \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(x) = \frac{1}{(2s)!} \psi_{\{A_\zeta B_\zeta \cdots C_\zeta D_\zeta\}}(x)$$

$\Leftrightarrow$

$$\psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(x) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} d^N \vec{p} |\vec{p}|^{(s-\frac{1}{2})}$$

$$\sum_{n_1+\cdots+n_r=2s} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(\vec{p}; n_1, \cdots, n_r) [a_1(\vec{p}; n_1, \cdots, n_r) e^{ip \cdot x} + a_2^+(\vec{p}; n_1, \cdots, n_r) e^{-ip \cdot x}]$$

第三步:  $s' = s + 1/2$ 时:

$$(\Gamma, -i\zeta)_a^{A' A_\zeta} \partial^a \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}_{2s+1}}(x) = 0, \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}_{2s+1}}(x) = \frac{1}{(2s+1)!} \psi_{\{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta\}}(x)$$

$\Leftrightarrow$

$$\left\{ \begin{aligned} \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}_{2s+1}}(x) &= \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} d^N \vec{p} |\vec{p}|^{(s-\frac{1}{2})} \\ &\sum_{n_1+\cdots+n_r=2s} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(\vec{p}; n_1, \cdots, n_r) [a_{1E_\zeta}(\vec{p}; n_1, \cdots, n_r) e^{ip \cdot x} + a_{2E_\zeta}^+(\vec{p}; n_1, \cdots, n_r) e^{-ip \cdot x}] \end{aligned} \right.$$

$$\psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}_{2s+1}}(x) = \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta E_\zeta D_\zeta}_{2s+1}}(x)$$

$\Leftrightarrow$

$$\left\{ \begin{aligned} \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}_{2s+1}}(x) &= \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} d^N \vec{p} |\vec{p}|^{(s-\frac{1}{2})} \\ &\sum_{n_1+\cdots+n_r=2s} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(\vec{p}; n_1, \cdots, n_r) [a_{1E_\zeta}(\vec{p}; n_1, \cdots, n_r) e^{ip \cdot x} + a_{2E_\zeta}^+(\vec{p}; n_1, \cdots, n_r) e^{-ip \cdot x}] \\ &\sum_{n_1+\cdots+n_r=2s} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(\vec{p}; n_1, \cdots, n_r) a_{1E_\zeta}(\vec{p}; n_1, \cdots, n_r) = \sum_{n_1+\cdots+n_r=2s} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta E_\zeta}_{2s}}(\vec{p}; n_1, \cdots, n_r) a_{1D_\zeta}(\vec{p}; n_1, \cdots, n_r) \\ &\sum_{n_1+\cdots+n_r=2s} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta}_{2s}}(\vec{p}; n_1, \cdots, n_r) a_{2E_\zeta}^+(\vec{p}; n_1, \cdots, n_r) = \sum_{n_1+\cdots+n_r=2s} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta E_\zeta}_{2s}}(\vec{p}; n_1, \cdots, n_r) a_{2D_\zeta}^+(\vec{p}; n_1, \cdots, n_r) \end{aligned} \right.$$

$$\Leftrightarrow \left\{ \begin{aligned} \psi_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}_{2s+1}}(x) \\ = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} d^N \vec{p} |\vec{p}|^s \sum_{n_1+\cdots+n_r=2s+1} \lambda_{\underbrace{A_\zeta B_\zeta \cdots C_\zeta D_\zeta E_\zeta}_{2s+1}}(\vec{p}; n_1, \cdots, n_r) [a_1(\vec{p}; n_1, \cdots, n_r) e^{ip \cdot x} + a_2^+(\vec{p}; n_1, \cdots, n_r) e^{-ip \cdot x}] \end{aligned} \right.$$

此步证明了  $s' = s + 1/2$  时命题成立。

第四步：根据以上归纳法推理，命题成立，定理得证。  $\square$

### 3.4 N+1维时空中Penrose全对称方程的协变对易规则

定理3.4.1.

$$\begin{cases} [a_\sigma(\vec{p}, -s; \vec{h}), a_{\sigma'}^+(\vec{p}', -s; \vec{h}')]_{-2s+1} = \delta_{\sigma\sigma'} \delta_{\vec{h}\vec{h}'} \delta^3(\vec{p} - \vec{p}') \\ [a_\sigma(\vec{p}, -s; \vec{h}), a_{\sigma'}^+(\vec{p}', -s; \vec{h}')]_{-2s+1} = 0, [a_\sigma^+(\vec{p}, -s; \vec{h}), a_{\sigma'}^+(\vec{p}', -s; \vec{h}')]_{-2s+1} = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} [\psi_{A_\zeta B_\zeta \dots}^{2s}(x), \psi_{A'_\zeta B'_\zeta \dots}^{2s}(x')]_{-2s+1} = i \frac{(i\zeta)^{2s}}{2^{2s-1} [(2s)!]^2} \overbrace{(\Gamma, i\zeta)_{\{A_\zeta(A'_\zeta(\Gamma, i\zeta)_{B_\zeta B'_\zeta \dots}^b \cdot \cdot \cdot)\}}^{2s}}^{2s} \overbrace{\partial_a \partial_b \dots}^{2s} \Delta(x - x') \\ [\psi_{A_\zeta B_\zeta \dots}^{2s}(x), \psi_{E_\zeta F_\zeta \dots}^{2s}(x')]_{-2s+1} = 0, [\psi_{A'_\zeta B'_\zeta \dots}^{2s}(x), \psi_{E'_\zeta F'_\zeta \dots}^{2s}(x')]_{-2s+1} = 0, s \geq 0 \end{cases}$$

证明：  $[\psi_{A_\zeta B_\zeta \dots}^{2s}(x), \psi_{A'_\zeta B'_\zeta \dots}^{2s}(x')]_{-2s+1}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int \sum_{\vec{h}, \vec{h}'} d^3 \vec{p} d^3 \vec{p}' (|\vec{p}| |\vec{p}'|)^{(s-\frac{1}{2})} \lambda_{A_\zeta B_\zeta \dots}^{2s}(\vec{p}, -s; \vec{h}) \lambda_{A'_\zeta B'_\zeta \dots}^{2s}(\vec{p}', -s; \vec{h}') \\ &[[a_1(\vec{p}, -s; \vec{h}) e^{ip \cdot x} + a_2^+(\vec{p}, -s; \vec{h}) e^{-ip \cdot x}], [a_1^+(\vec{p}', -s; \vec{h}') e^{-ip' \cdot x'} + a_2(\vec{p}', -s; \vec{h}') e^{ip' \cdot x'}]]_{-2s+1} \\ &= \frac{1}{(2\pi)^3} \int \sum_{\vec{h}, \vec{h}'} d^3 \vec{p} d^3 \vec{p}' (|\vec{p}| |\vec{p}'|)^{(s-\frac{1}{2})} \lambda_{A_\zeta B_\zeta \dots}^{2s}(\vec{p}, -s; \vec{h}) \lambda_{A'_\zeta B'_\zeta \dots}^{2s}(\vec{p}', -s; \vec{h}') \\ &\{[a_1(\vec{p}, -s; \vec{h}), a_1^+(\vec{p}', -s; \vec{h}')]_{-2s+1} e^{i(p \cdot x - p' \cdot x')} + [a_2^+(\vec{p}, -s; \vec{h}), a_2(\vec{p}', -s; \vec{h}')]_{-2s+1} e^{-i(p \cdot x - p' \cdot x')}\} \\ &= \frac{1}{(2\pi)^3} \int \sum_{\vec{h}, \vec{h}'} d^3 \vec{p} d^3 \vec{p}' (|\vec{p}| |\vec{p}'|)^{(s-\frac{1}{2})} \lambda_{A_\zeta B_\zeta \dots}^{2s}(\vec{p}, -s; \vec{h}) \lambda_{A'_\zeta B'_\zeta \dots}^{2s}(\vec{p}', -s; \vec{h}') \delta^3(\vec{p} - \vec{p}') \delta_{\vec{h}\vec{h}'} \\ &[e^{i(p \cdot x - p' \cdot x')} + (-1)^{2s+1} e^{-i(p \cdot x - p' \cdot x')}] \\ &= \frac{1}{(2\pi)^3} \int \sum_{\vec{h}} d^3 \vec{p} |\vec{p}|^{2s-1} \lambda_{A_\zeta B_\zeta \dots}^{2s}(\vec{p}, -s; \vec{h}) \lambda_{A'_\zeta B'_\zeta \dots}^{2s}(\vec{p}, -s; \vec{h}) [e^{i(p \cdot x - p \cdot x')} + (-1)^{2s+1} e^{-i(p \cdot x - p \cdot x')}] \\ &= i \frac{(-\zeta)^{2s}}{2^{2s-1} [(2s)!]^2} \frac{1}{(2\pi)^3} \int \frac{-i}{|\vec{p}|} \overbrace{(\Gamma, i\zeta)_{\{A_\zeta(A'_\zeta(\Gamma, i\zeta)_{B_\zeta B'_\zeta \dots}^b \cdot \cdot \cdot)\}}^{2s}}^{2s} \overbrace{p_a p_b \dots}^{2s} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3 \vec{p} \\ &= i \frac{(i\zeta)^{2s}}{2^{2s-1} [(2s)!]^2} \overbrace{(\Gamma, i\zeta)_{\{A_\zeta(A'_\zeta(\Gamma, i\zeta)_{B_\zeta B'_\zeta \dots}^b \cdot \cdot \cdot)\}}^{2s}}^{2s} \overbrace{\partial_a \partial_b \dots}^{2s} \frac{1}{(2\pi)^3} \int \frac{-i}{|\vec{p}|} [e^{ip \cdot (x-x')} - e^{-ip \cdot (x-x')}] d^3 \vec{p} \\ &= i \frac{(i\zeta)^{2s}}{2^{2s-1} [(2s)!]^2} \overbrace{(\Gamma, i\zeta)_{\{A_\zeta(A'_\zeta(\Gamma, i\zeta)_{B_\zeta B'_\zeta \dots}^b \cdot \cdot \cdot)\}}^{2s}}^{2s} \overbrace{\partial_a \partial_b \dots}^{2s} \Delta(x - x') \end{aligned} \quad \square$$

证明：  $[a_1(\vec{p}, -s; \vec{h}), a_1^+(\vec{p}', -s; \vec{h}')]_{-2s+1}$

$$\begin{aligned} &= \frac{1}{(2\pi)^3} \int |\vec{p}|^{-(2s-1)} \lambda^{+A_\zeta B_\zeta \dots}(\vec{p}, -s; \vec{h}) \lambda_{A'_\zeta B'_\zeta \dots}^{2s}(\vec{p}', -s; \vec{h}') [\psi_{A_\zeta B_\zeta \dots}^{2s}(x), \psi_{A'_\zeta B'_\zeta \dots}^{2s}(x')] e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int |\vec{p}|^{-(2s-1)} \lambda^{+A_\zeta B_\zeta \dots}(\vec{p}, -s; \vec{h}) \lambda_{A'_\zeta B'_\zeta \dots}^{2s}(\vec{p}, -s; \vec{h}') \\ &i \frac{(i\zeta)^{2s}}{2^{2s-1} [(2s)!]^2} \overbrace{(\Gamma, i\zeta)_{\{A_\zeta(A'_\zeta(\Gamma, i\zeta)_{B_\zeta B'_\zeta \dots}^b \cdot \cdot \cdot)\}}^{2s}}^{2s} \overbrace{\partial_a \partial_b \dots}^{2s} \Delta(x - x') e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\ &= \frac{1}{(2\pi)^3} \int |\vec{p}|^{-(2s-1)} \lambda^{+A_\zeta B_\zeta \dots}(\vec{p}, -s; \vec{h}) \lambda_{A'_\zeta B'_\zeta \dots}^{2s}(\vec{p}, -s; \vec{h}') \\ &i \frac{(i\zeta)^{2s}}{2^{2s} [(2s)!]^2} \overbrace{(\Gamma, i\zeta)_{\{A_\zeta(A'_\zeta(\Gamma, i\zeta)_{B_\zeta B'_\zeta \dots}^b \cdot \cdot \cdot)\}}^{2s-1}}^{2s-1} \overbrace{\partial_a \partial_b \dots}^{2s} \frac{-i}{(2\pi)^3} \int \frac{1}{2|\vec{p}_0|} [e^{ip_0 \cdot (x-x')} - e^{-ip_0 \cdot (x-x')}] d^3 \vec{p}_0 e^{-i(p \cdot x - p' \cdot x')} d^3 \vec{r} d^3 \vec{r}' \\ &= [\frac{1}{(2\pi)^3}]^2 \int d^3 \vec{p}_0 d^3 \vec{r} d^3 \vec{r}' |\vec{p}|^{-(2s-1)} \lambda^{+A_\zeta B_\zeta \dots}(\vec{p}, -s; \vec{h}) \lambda_{A'_\zeta B'_\zeta \dots}^{2s}(\vec{p}, -s; \vec{h}') (-\frac{\zeta}{2})^{2s} \frac{1}{[(2s)!]^2} |\vec{p}_0|^{(2s-1)} \end{aligned}$$

$$\begin{aligned}
& \overbrace{(\Gamma, i\zeta)_{\{A_\zeta(A'_\zeta(\Gamma, i\zeta)_{B_\zeta(B'_\zeta \cdots)}\}}^{2s}}^a \overbrace{\hat{p}_{0a}\hat{p}_{0b} \cdots}^{2s} [e^{i(p_0-p)\cdot x} e^{-i(p_0-p')\cdot x'} + (-1)^{2s+1} e^{-i(p_0+p)\cdot x} e^{i(p_0+p')\cdot x'}] \\
&= [\frac{1}{(2\pi)^3}]^2 \int d^3\vec{p}_0 d^3\vec{r} d^3\vec{r}' |\vec{p}'|^{-(2s-1)} |\vec{p}_0|^{(2s-1)} \lambda^{+A_\zeta B_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}) \lambda^{A'_\zeta B'_\zeta \cdots}(\vec{p}', -s\zeta; \vec{h}') \\
& \sum_{\vec{h}_0} \lambda_{A_\zeta B_\zeta \cdots}(\vec{p}_0, -s\zeta; \vec{h}_0) \lambda_{A'_\zeta B'_\zeta \cdots}^+(\vec{p}_0, -s\zeta; \vec{h}_0) [e^{i(p_0-p)\cdot x} e^{-i(p_0-p')\cdot x'} + (-1)^{2s+1} e^{-i(p_0+p)\cdot x} e^{i(p_0+p')\cdot x'}] \\
&= \int \lambda^{+A_\zeta B_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}) \lambda^{A'_\zeta B'_\zeta \cdots}(\vec{p}', -s\zeta; \vec{h}') \sum_{\vec{h}_0} \lambda_{A_\zeta B_\zeta \cdots}(\vec{p}_0, -s\zeta; \vec{h}_0) \lambda_{A'_\zeta B'_\zeta \cdots}^+(\vec{p}_0, -s\zeta; \vec{h}_0) \\
& [\delta^3(\vec{p}_0 - \vec{p}) \delta^3(\vec{p}_0 - \vec{p}') + (-1)^{2s+1} e^{2iE_0(t-t')} \delta^3(\vec{p}_0 + \vec{p}) \delta^3(\vec{p}_0 + \vec{p}')] d^3\vec{p}_0 \\
&= \int \sum_{\vec{h}_0} \lambda^{+A_\zeta B_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}) \lambda^{A'_\zeta B'_\zeta \cdots}(\vec{p}', -s\zeta; \vec{h}') \lambda_{A_\zeta B_\zeta \cdots}(\vec{p}_0, -s\zeta; \vec{h}_0) \lambda_{A'_\zeta B'_\zeta \cdots}^+(\vec{p}_0, -s\zeta; \vec{h}_0) \delta^3(\vec{p}_0 - \vec{p}) \delta^3(\vec{p}_0 - \vec{p}') d^3\vec{p}_0 \\
&= \sum_{\vec{h}_0} \lambda^{+A_\zeta B_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}) \lambda_{A_\zeta B_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}_0) \lambda_{A'_\zeta B'_\zeta \cdots}^+(\vec{p}, -s\zeta; \vec{h}_0) \lambda^{A'_\zeta B'_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}') \delta^3(\vec{p} - \vec{p}') \\
&= \sum_{\vec{h}_0} \lambda^{+A_\zeta B_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}) \lambda_{A_\zeta B_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}_0) \lambda_{A'_\zeta B'_\zeta \cdots}^+(\vec{p}, -s\zeta; \vec{h}_0) \lambda^{A'_\zeta B'_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}') \delta^3(\vec{p} - \vec{p}') \\
&= \sum_{\vec{h}_0} \delta_{\vec{h}\vec{h}_0} \delta_{\vec{h}'\vec{h}_0} \delta^3(\vec{p} - \vec{p}') = \delta_{\vec{h}\vec{h}'} \delta^3(\vec{p} - \vec{p}') \quad \square
\end{aligned}$$

自我评述: 以上的证法不再基于等时对易规则, 直接基于协变对易规则, 看似更难了, 其实是更简单了, 因为不需要求出复杂的等时对易规则, 即使求出也比较难于使用, 而协变对易规则本身已知且很有规律, 也可以分解为自旋基的乘积, 整个证明过程基本上只依赖于自旋基的性质, 没有复杂的计算。其他的几个对易括号也可按同样的方法求出, 不再列出。

定理3.4.2.

$$\begin{cases}
[\psi_{A_\zeta B_\zeta \cdots}(x), \psi_{A'_\zeta B'_\zeta \cdots}^+(x')]_{-2s+1} = i \frac{(i\zeta)^{2s}}{2^{2s-1} [(2s)!]^2} \overbrace{(\Gamma, i\zeta)_{\{A_\zeta(A'_\zeta(\Gamma, i\zeta)_{B_\zeta(B'_\zeta \cdots)}\}}^{2s}}^a \overbrace{\partial_a \partial_b \cdots}^{2s} \Delta(x-x') \\
[\psi_{A_\zeta B_\zeta \cdots}(x), \psi_{E_\zeta F_\zeta \cdots}(x')]_{-2s+1} = 0, [\psi_{A'_\zeta B'_\zeta \cdots}^+(x), \psi_{E'_\zeta F'_\zeta \cdots}^+(x')]_{-2s+1} = 0, s \geq 0 \\
\Leftrightarrow \begin{cases}
[\psi_{k_\zeta}(x), \psi_{k'_\zeta}^+(x')]_{-2s+1} = i \frac{(-1)^{2s}}{2^{2s-1}} \Gamma_{k_\zeta k'_\zeta}^{abc \cdots}(s, w) \overbrace{\partial_a \partial_b \partial_c \cdots}^{2s} \Delta(x-x'), \Gamma(0) := 1 \\
[\psi_{k_\zeta}(x), \psi_{l_\zeta}(x')]_{-2s+1} = 0, [\psi_{k'_\zeta}^+(x), \psi_{l'_\zeta}^+(x')]_{-2s+1} = 0, s \geq 0
\end{cases}
\end{cases}$$

### 3.5 N+1维时空中Penrose全对称方程的各种物理算符

$$\begin{aligned}
\text{定理3.5.1. } P_u(s) &= \int \psi^{+A_\zeta B_\zeta \cdots}(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi_{A_\zeta B_\zeta \cdots}(\vec{r}, t) d^3\vec{r} \\
&= \int \sum_{\vec{h}} p_u [a_1^+(\vec{p}, -s\zeta; \vec{h}) a_1(\vec{p}, -s\zeta; \vec{h}) + (-1)^{2s} a_2(\vec{p}, -s\zeta; \vec{h}) a_2^+(\vec{p}, -s\zeta; \vec{h})] d^3\vec{p}
\end{aligned}$$

$$\begin{aligned}
\text{证明: } P_u(s) &= \int \psi^{+A_\zeta B_\zeta \cdots}(\vec{r}, t) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \psi_{A_\zeta B_\zeta \cdots}(\vec{r}, t) d^3\vec{r} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r}' \sum_{\vec{h}, \vec{h}'} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^{+A_\zeta B_\zeta \cdots}(\vec{p}', -s\zeta; \vec{h}') \lambda_{A_\zeta B_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}) \frac{p_u}{|\vec{p}|^{2s-1}} \\
& [a_1^+(\vec{p}', -s\zeta; \vec{h}') e^{-i(\vec{p}'\cdot\vec{r}' - |\vec{p}'|t)} + a_2(\vec{p}', -s\zeta; \vec{h}') e^{i(\vec{p}'\cdot\vec{r}' - |\vec{p}'|t)}] [a_1(\vec{p}, -s\zeta; \vec{h}) e^{i(\vec{p}\cdot\vec{r} - |\vec{p}|t)} + (-1)^{2s} a_2^+(\vec{p}, -s\zeta; \vec{h}) e^{-i(\vec{p}\cdot\vec{r} - |\vec{p}|t)}] \\
&= \int \sum_{\vec{h}, \vec{h}'} \vec{p}'^{2s-1} \lambda^{+A_\zeta B_\zeta \cdots}(\vec{p}', -s\zeta; \vec{h}') \lambda_{A_\zeta B_\zeta \cdots}(\vec{p}, -s\zeta; \vec{h}) \frac{p_u}{|\vec{p}|^{2s-1}}
\end{aligned}$$

$$\begin{aligned}
& \{[a_1^+(\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})]\delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s}a_1^+(-\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h})e^{2i|\vec{p}|t}]\delta^3(\vec{p}' + \vec{p})\}d^3\vec{p}'d^3\vec{p} \\
& = \int \sum_{\vec{h}, \vec{h}'} \lambda^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}', -s; \vec{h}') \underbrace{\lambda_{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}, -s; \vec{h}) p_u [a_1^+(\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})]d^3\vec{p} \\
& = \int \sum_{\vec{h}} p_u [a_1^+(\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})]d^3\vec{p} \quad \square
\end{aligned}$$

$$\begin{aligned}
\text{定理3.5.2. } Q(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \underbrace{\psi_{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \int \sum_{\vec{h}} [a_1^+(\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h}) + (-1)^{2s-1}a_2(\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})]d^3\vec{p}
\end{aligned}$$

$$\begin{aligned}
\text{证明: } Q(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \underbrace{\psi_{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} \sum_{\vec{h}, \vec{h}'} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}', -s; \vec{h}') \underbrace{\lambda_{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}, -s; \vec{h}) \frac{1}{|\vec{p}|^{2s-1}} \\
& [a_1^+(\vec{p}', -s; \vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)} + a_2(\vec{p}', -s; \vec{h}')e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}] [a_1(\vec{p}, -s; \vec{h})e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)} + (-1)^{2s-1}a_2^+(\vec{p}, -s; \vec{h})e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}] \\
& = \int \sum_{\vec{h}, \vec{h}'} \vec{p}'^{2s-1} \lambda^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}', -s; \vec{h}') \underbrace{\lambda_{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}, -s; \vec{h}) \frac{1}{|\vec{p}|^{2s-1}} \\
& \{[a_1^+(\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h}) + (-1)^{2s-1}a_2(\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})]\delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s-1}a_1^+(-\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h})e^{2i|\vec{p}|t}]\delta^3(\vec{p}' + \vec{p})\}d^3\vec{p}'d^3\vec{p} \\
& = \int \sum_{\vec{h}} [a_1^+(\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h}) + (-1)^{2s-1}a_2(\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})]d^3\vec{p} \quad \square
\end{aligned}$$

$$\begin{aligned}
\text{定理3.5.3. } N(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \underbrace{\psi_{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \int \sum_{\vec{h}} [a_1^+(\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})]d^3\vec{p}
\end{aligned}$$

$$\begin{aligned}
\text{证明: } N(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) \frac{(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \underbrace{\psi_{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} \sum_{\vec{h}, \vec{h}'} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}', -s; \vec{h}') \underbrace{\lambda_{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}, -s; \vec{h}) \frac{1}{|\vec{p}|^{2s-1}} \\
& [a_1^+(\vec{p}', -s; \vec{h}')e^{-i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)} + a_2(\vec{p}', -s; \vec{h}')e^{i(\vec{p}'\cdot\vec{r}-|\vec{p}'|t)}] [a_1(\vec{p}, -s; \vec{h})e^{i(\vec{p}\cdot\vec{r}-|\vec{p}|t)} + (-1)^{2s}a_2^+(\vec{p}, -s; \vec{h})e^{-i(\vec{p}\cdot\vec{r}-|\vec{p}|t)}] \\
& = \int \sum_{\vec{h}, \vec{h}'} \vec{p}'^{2s-1} \lambda^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}', -s; \vec{h}') \underbrace{\lambda_{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}, -s; \vec{h}) \frac{1}{|\vec{p}|^{2s-1}} \\
& \{[a_1^+(\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})]\delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s}a_1^+(-\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h})e^{2i|\vec{p}|t}]\delta^3(\vec{p}' + \vec{p})\}d^3\vec{p}'d^3\vec{p} \\
& = \int \sum_{\vec{h}} [a_1^+(\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})]d^3\vec{p} \quad \square
\end{aligned}$$

$$\begin{aligned}
\text{定理3.5.4. } \vec{S}(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \underbrace{\psi_{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \int \sum_{\vec{h}} \hat{p} [a_1^+(\vec{p}, -s; \vec{h})a_1(\vec{p}, -s; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s; \vec{h})a_2^+(\vec{p}, -s; \vec{h})]d^3\vec{p}
\end{aligned}$$

$$\begin{aligned}
\text{证明: } \vec{S}(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(-\nabla^2)^{2s-1}} \underbrace{\psi_{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r} \sum_{\vec{h}, \vec{h}'} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}', -s; \vec{h}') \underbrace{\lambda_{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}, -s; \vec{h}) \frac{\hat{p}}{|\vec{p}|^{2s-1}}
\end{aligned}$$

$$\begin{aligned}
& [a_1^+(\vec{p}', -s_\zeta; \vec{h}')e^{-i(\vec{p}' \cdot \vec{r}' - |\vec{p}'|t)} + a_2(\vec{p}', -s_\zeta; \vec{h}')e^{i(\vec{p}' \cdot \vec{r}' - |\vec{p}'|t)}][a_1(\vec{p}, -s_\zeta; \vec{h})e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + (-1)^{2s}a_2^+(\vec{p}, -s_\zeta; \vec{h})e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] \\
&= \int \sum_{\vec{h}, \vec{h}'} |\vec{p}|^{2s-1} \lambda^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}', -s_\zeta; \vec{h}') \lambda_{\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}, -s_\zeta; \vec{h}) \frac{\hat{p}}{|\vec{p}|^{2s-1}} \\
& \{ [a_1^+(\vec{p}, -s_\zeta; \vec{h})a_1(\vec{p}, -s_\zeta; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s_\zeta; \vec{h})a_2^+(\vec{p}, -s_\zeta; \vec{h})] \delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s}a_1^+(\vec{p}, -s_\zeta; \vec{h})a_2^+(\vec{p}, -s_\zeta; \vec{h})e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s_\zeta; \vec{h})a_1(\vec{p}, -s_\zeta; \vec{h})e^{2i|\vec{p}|t}] \delta^3(\vec{p}' + \vec{p}) \} d^3\vec{p}' d^3\vec{p} \\
&= \int \sum_{\vec{h}} \hat{p} [a_1^+(\vec{p}, -s_\zeta; \vec{h})a_1(\vec{p}, -s_\zeta; \vec{h}) + (-1)^{2s}a_2(\vec{p}, -s_\zeta; \vec{h})a_2^+(\vec{p}, -s_\zeta; \vec{h})] d^3\vec{p} \quad \square
\end{aligned}$$

$$\begin{aligned}
\text{定理3.5.5. } \vec{M}(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \psi_{\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \int \sum_{\vec{h}} \hat{p} [a_1^+(\vec{p}, -s_\zeta; \vec{h})a_1(\vec{p}, -s_\zeta; \vec{h}) + (-1)^{2s-1}a_2(\vec{p}, -s_\zeta; \vec{h})a_2^+(\vec{p}, -s_\zeta; \vec{h})] d^3\vec{p}
\end{aligned}$$

$$\begin{aligned}
\text{证明: } \vec{M}(s) &= \int \psi^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{-\nabla^2})^{4s-1}} \psi_{\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{r}, t) d^3\vec{r} \\
&= \frac{1}{(2\pi)^3} \int d^3\vec{p}' d^3\vec{p} d^3\vec{r}' \sum_{\vec{h}, \vec{h}'} |\vec{p}'|^{s-\frac{1}{2}} |\vec{p}|^{s-\frac{1}{2}} \lambda^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}', -s_\zeta; \vec{h}') \lambda_{\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}, -s_\zeta; \vec{h}) \frac{\hat{p}}{|\vec{p}|^{2s-1}} \\
& [a_1^+(\vec{p}', -s_\zeta; \vec{h}')e^{-i(\vec{p}' \cdot \vec{r}' - |\vec{p}'|t)} + a_2(\vec{p}', -s_\zeta; \vec{h}')e^{i(\vec{p}' \cdot \vec{r}' - |\vec{p}'|t)}][a_1(\vec{p}, -s_\zeta; \vec{h})e^{i(\vec{p} \cdot \vec{r} - |\vec{p}|t)} + (-1)^{2s-1}a_2^+(\vec{p}, -s_\zeta; \vec{h})e^{-i(\vec{p} \cdot \vec{r} - |\vec{p}|t)}] \\
&= \int \sum_{\vec{h}, \vec{h}'} |\vec{p}|^{2s-1} \lambda^{+\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}', -s_\zeta; \vec{h}') \lambda_{\overbrace{A_\zeta B_\zeta \dots}^{2s}}(\vec{p}, -s_\zeta; \vec{h}) \frac{\hat{p}}{|\vec{p}|^{2s-1}} \\
& \{ [a_1^+(\vec{p}, -s_\zeta; \vec{h})a_1(\vec{p}, -s_\zeta; \vec{h}) + (-1)^{2s-1}a_2(\vec{p}, -s_\zeta; \vec{h})a_2^+(\vec{p}, -s_\zeta; \vec{h})] \delta^3(\vec{p}' - \vec{p}) \\
& + [(-1)^{2s-1}a_1^+(\vec{p}, -s_\zeta; \vec{h})a_2^+(\vec{p}, -s_\zeta; \vec{h})e^{-2i|\vec{p}|t} + a_2(-\vec{p}, -s_\zeta; \vec{h})a_1(\vec{p}, -s_\zeta; \vec{h})e^{2i|\vec{p}|t}] \delta^3(\vec{p}' + \vec{p}) \} d^3\vec{p}' d^3\vec{p} \\
&= \int \sum_{\vec{h}} \hat{p} [a_1^+(\vec{p}, -s_\zeta; \vec{h})a_1(\vec{p}, -s_\zeta; \vec{h}) + (-1)^{2s-1}a_2(\vec{p}, -s_\zeta; \vec{h})a_2^+(\vec{p}, -s_\zeta; \vec{h})] d^3\vec{p} \quad \square
\end{aligned}$$

### 3.6 Bargmann-Wigner方程的作用量

$$\text{定理3.6.1. } S^? = \int \psi_{A'_\zeta B'_\zeta \dots}^+(x) \gamma_0^{A'_\zeta Z_\zeta} \gamma_0^{B'_\zeta B_\zeta} \dots (\gamma^a_{Z_\zeta} A_\zeta \partial_a + m \delta_{Z_\zeta}^{A_\zeta}) \psi_{A_\zeta B_\zeta \dots}(x) d^4x$$



# 第三十六章 低维时空粒子的协变量子化

自我评述：对于Bargmann-Wigner方程或Dirac方程描述的粒子，一般来说既可以描述带荷的复粒子，也可以描述不带荷的马约拉纳粒子。两种情形的主对易规则形式一致，但其余对易或反对易括号，对带荷的复粒子一般为零；对不带荷的马约拉纳粒子，其余对易或反对易括号由主对易规则和马约拉纳条件自然得到，一般不为零。在本章只讨论复粒子情形，一般也只给出主对易规则，不再专门讨论马约拉纳粒子情形，若要得到马约拉纳粒子情形的量子场论，只需在复粒子情形加上马约拉纳条件即可自然得到。本章描述的二、三维时空粒子可以认为是四维时空粒子在y,z轴或z轴受限后的结果，因而有现实的意义，可以应用于凝聚态物理。另外二维时空粒子也可以认为是三维时空粒子在y轴进一步受限后的结果。三维时空粒子对应量子面；二维时空粒子对应量子线；一维时空粒子对应量子点。

## 1 三维时空中有质量粒子的协变量子化

### 1.1 三维时空中有质量粒子的Bargmann-Wigner方程

#### 1.1.1 三维时空中的Dirac方程自旋基及其平面波解

$$\text{定义1.1.1. } u(\vec{p}) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{\zeta\sigma\cdot\vec{p}}{E+m}\right) \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2}\sigma_x\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v(\vec{p}) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{\zeta\sigma\cdot\vec{p}}{E+m}\right) \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2}\sigma_x\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{推论1.1.1. } u(\vec{p}) = \sigma_x v^*(\vec{p}), v(\vec{p}) = \sigma_x u^*(\vec{p})$$

$$\text{定理1.1.1. } (\gamma^a \partial_a + m)\psi = 0, \gamma^a = (-\sigma_y, \sigma_x, \zeta\sigma_z)$$

$$\psi(\vec{r}, t) = \frac{1}{(2\pi)^{1/2}} \int_{\vec{p}=-\infty}^{+\infty} [a(\vec{p})\sqrt{\frac{m}{E}}u(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p})\sqrt{\frac{m}{E}}v(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d\vec{p}$$

$$a(\vec{p}) = \frac{1}{(2\pi)^{1/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{E}{m}}u^+(\vec{p})\psi(\vec{r}, t)e^{-i(\vec{p}\cdot\vec{r}-Et)}d\vec{r}, b^+(\vec{p}) = \frac{1}{(2\pi)^{1/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{E}{m}}v^+(\vec{p})\psi(\vec{r}, t)e^{i(\vec{p}\cdot\vec{r}-Et)}d\vec{r}$$

#### 1.1.2 三维时空中Dirac自旋基的性质

$$\text{推论1.1.2. } \begin{cases} \bar{u}(\vec{p})u(\vec{p}) = 1, \bar{v}(\vec{p})v(\vec{p}) = -1, \bar{u}(\vec{p})v(\vec{p}) = 0, \bar{v}(\vec{p})u(\vec{p}) = 0 \\ u^+(\vec{p})u(\vec{p}) = \frac{E}{m}, v^+(\vec{p})v(\vec{p}) = \frac{E}{m}, u^+(\vec{p})v(-\vec{p}) = 0, v^+(\vec{p})u(-\vec{p}) = 0 \end{cases}$$

$$\text{推论1.1.3. } \begin{cases} u(\vec{p})\bar{u}(\vec{p}) = \frac{m-i\gamma^a p_a}{2m} & \begin{cases} u(\vec{p})u^+(\vec{p}) = \frac{(m-i\gamma^a p_a)\gamma^0}{2m} = \frac{m\sigma_z - (\sigma, i\zeta)^a p_a}{\zeta 2m} \\ v(\vec{p})\bar{v}(\vec{p}) = \frac{-m-i\gamma^a p_a}{2m} & \begin{cases} v(\vec{p})v^+(\vec{p}) = \frac{(-m-i\gamma^a p_a)\gamma^0}{2m} = \frac{-m\sigma_z - (\sigma, i\zeta)^a p_a}{\zeta 2m} \end{cases} \end{cases} \end{cases}$$

$$\text{推论1.1.4. } u(\vec{p})\bar{u}(\vec{p}) - v(\vec{p}, h)\bar{v}(\vec{p}) = 1, u(\vec{p})\bar{u}(\vec{p}) + v(\vec{p}, h)\bar{v}(\vec{p}) = \frac{-i\gamma^a p_a}{m}, u(\vec{p})u^+(\vec{p}) + v(-\vec{p}, h)v^+(-\vec{p}) = \frac{E}{m}$$

#### 1.1.3 三维时空中Dirac方程的协变量子化规则

$$\text{推论1.1.5. } \begin{cases} \{a(\vec{p}), a^+(\vec{p}')\} = \delta(\vec{p} - \vec{p}') \\ \{a(\vec{p}), a(\vec{p}')\} = 0, \{a^+(\vec{p}), a^+(\vec{p}')\} = 0 \end{cases} \Rightarrow \{\psi_{\lambda_\zeta}(x), \psi_{\lambda'_\zeta}^+(x')\} = i[(m - \gamma^a \partial_a)\gamma^0]_{\lambda_\zeta \lambda'_\zeta} \Delta(x - x')$$

## 1.2 三维时空中的Bargmann-Wigner方程

### 1.2.1 三维时空中Bargmann-Wigner方程<sup>[17]</sup>的自旋基及其平面波解

$$\text{定义1.2.1. } \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p})}_{2s} := \underbrace{u_{\lambda_\zeta}(\vec{p})u_{\mu_\zeta}(\vec{p}) \dots}_{2s}, \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}(\vec{p})}_{2s} := \underbrace{v_{\lambda_\zeta}(\vec{p})v_{\mu_\zeta}(\vec{p}) \dots}_{2s}$$

$$\text{推论1.2.1. } \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p})}_{2s} = \underbrace{\sigma_x \otimes \sigma_x \dots}_{2s} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}^+(\vec{p})}_{2s}, \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}(\vec{p})}_{2s} = \underbrace{\sigma_x \otimes \sigma_x \dots}_{2s} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}^+(\vec{p})}_{2s}$$

定理1.2.1.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \psi_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{r}, t) = 0, \psi_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{r}, t) = \frac{1}{(2s)!} \psi_{\{\lambda_\zeta \mu_\zeta \dots\}}^{\lambda_\zeta}(\vec{r}, t)$

$$\psi_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} [a(\vec{p}) U_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) e^{i(\vec{p} \cdot \vec{r} - Et)} + b^+(\vec{p}) V_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}$$

$$\begin{cases} a(\vec{p}) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m}{E}}^{2s} U^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}') \psi_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{r}, t) e^{-i(\vec{p}' \cdot \vec{r} - Et)} d^N \vec{r}' \\ b^+(\vec{p}) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}'=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m}{E}}^{2s} V^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}') \psi_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{r}, t) e^{i(\vec{p}' \cdot \vec{r} - Et)} d^N \vec{r}' \end{cases}$$

### 1.2.2 三维时空中Bargmann-Wigner方程自旋基的正交性质

推论1.2.2.

$$\begin{cases} \bar{U}^{\lambda_\zeta \mu_\zeta \dots}(\vec{p}) U_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) = 1, \bar{V}^{\lambda_\zeta \mu_\zeta \dots}(\vec{p}) V_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) = 1 \\ \bar{U}^{\lambda_\zeta \mu_\zeta \dots}(\vec{p}) V_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) = 0, \bar{V}^{\lambda_\zeta \mu_\zeta \dots}(\vec{p}) U_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) = 0 \\ U^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}) U_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) = \left(\frac{E}{m}\right)^{2s}, V^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}) V_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) = \left(\frac{E}{m}\right)^{2s} \\ U^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}) V_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(-\vec{p}) = 0, V^{+\lambda_\zeta \mu_\zeta \dots}(\vec{p}) U_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(-\vec{p}) = 0 \end{cases}$$

### 1.2.3 三维时空中Bargmann-Wigner方程的准投影算子

推论1.2.3.

$$\begin{cases} U_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) U_{\lambda'_\zeta \mu'_\zeta \dots}^{+\lambda'_\zeta}(\vec{p}) = \frac{1}{(2m)^{2s}} \underbrace{[(m - i\gamma^b p_b) \gamma^0]_{\lambda_\zeta \lambda'_\zeta} [(m - i\gamma^c p_c) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots}_{2s} \\ V_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) V_{\lambda'_\zeta \mu'_\zeta \dots}^{+\lambda'_\zeta}(\vec{p}) = \frac{1}{(2m)^{2s}} \underbrace{[(-m - i\gamma^b p_b) \gamma^0]_{\lambda_\zeta \lambda'_\zeta} [(-m - i\gamma^c p_c) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots}_{2s} \end{cases}$$

推论1.2.4.

$$\begin{cases} U_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) U_{\lambda'_\zeta \mu'_\zeta \dots}^{+\lambda'_\zeta}(\vec{p}) = \frac{1}{(\zeta 2m)^{2s}} \underbrace{[m\sigma_z - (\sigma, i\zeta)^a p_a]_{\lambda_\zeta \lambda'_\zeta} [m\sigma_z - (\sigma, i\zeta)^b p_b]_{\mu_\zeta \mu'_\zeta} \dots}_{2s} \\ V_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) V_{\lambda'_\zeta \mu'_\zeta \dots}^{+\lambda'_\zeta}(\vec{p}) = \frac{1}{(\zeta 2m)^{2s}} \underbrace{[-m\sigma_z - (\sigma, i\zeta)^a p_a]_{\lambda_\zeta \lambda'_\zeta} [-m\sigma_z - (\sigma, i\zeta)^b p_b]_{\mu_\zeta \mu'_\zeta} \dots}_{2s} \end{cases}$$

推论1.2.5.  $U_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(\vec{p}) U_{\lambda'_\zeta \mu'_\zeta \dots}^{+\lambda'_\zeta}(\vec{p}) = (-1)^{2s} V_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(-\vec{p}) V_{\lambda'_\zeta \mu'_\zeta \dots}^{+\lambda'_\zeta}(-\vec{p})$

### 1.2.4 三维时空中Bargmann-Wigner方程的协变对易规则

定理1.2.2.  $[\psi_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^{+\lambda'_\zeta}(x')] = \frac{i}{2^{2s-1}} \underbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta \lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots}_{2s} \Delta(x - x')$

$\Leftrightarrow$

定理1.2.3.  $[\psi_{\lambda_\zeta \mu_\zeta \dots}^{\lambda_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^{+\lambda'_\zeta}(x')] = i \frac{(i\zeta)^{2s}}{2^{2s-1}} \underbrace{[-im\sigma_z + (\sigma, i\zeta)^a \partial_a]_{\lambda_\zeta \lambda'_\zeta} [-im\sigma_z + (\sigma, i\zeta)^b \partial_b]_{\mu_\zeta \mu'_\zeta} \dots}_{2s} \Delta(x - x')$

## 1.3 三维时空中有质量粒子势方程的具体表述

自我评述: 此节类比四维时空中的情形, 探究三维时空中是否存在与B-W方程等价的C-K或R-S方程?

## 1.3.1 三维时空中有质量自旋-1的B-W方程等价于类C-K方程 [17, 19]

定理1.3.1.  $(\gamma^a \partial_a + m)_{\kappa\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta} = \psi_{\mu_\zeta \lambda_\zeta}, A_a = \frac{1}{\sqrt{2im}} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} \psi_{\lambda_\zeta \mu_\zeta}, \gamma^a = (-\sigma_y, \sigma_x, \varsigma \sigma_z)$   
 $\Leftrightarrow \partial_a A_b - \partial_b A_a = i\varsigma m \varepsilon_{ab}^c A_c, \psi = im \gamma^a \varepsilon A_a \Rightarrow (\partial^b \partial_b - m^2) A_a = 0, \partial^a A_a = 0$

定理1.3.2.  $(\gamma^a \partial_a + m) \psi(x) = 0, \psi^T(x) = \psi(x), A_a = \frac{1}{\sqrt{2im}} \text{tr}(\bar{\varepsilon} \gamma_a \psi), \gamma^a = (-\sigma_y, \sigma_x, \varsigma \sigma_z)$   
 $\Leftrightarrow \partial_a A_b - \partial_b A_a = i\varsigma m \varepsilon_{ab}^c A_c, \psi = \frac{im}{\sqrt{2}} \gamma^a \varepsilon A_a \Rightarrow (\partial^b \partial_b - m^2) A_a = 0, \partial^a A_a = 0$

证明:  $(\gamma^a \partial_a + m) \psi(x) = 0, \psi = \frac{im}{\sqrt{2}} \gamma^a \varepsilon A_a$

$$\Leftrightarrow (\gamma^a \partial_a + m) \frac{im}{\sqrt{2}} \gamma^b \varepsilon A_b = 0$$

$$\Leftrightarrow (\gamma^a \partial_a + m) \gamma^b A_b = 0$$

$$\Leftrightarrow \delta^{ab} \partial_a A_b + i\varsigma \varepsilon^{abc} \partial_a A_b \gamma_c + m \gamma_c A^c = 0$$

$$\Leftrightarrow \partial^a A_a + (i\varsigma \varepsilon^{ab} \partial_a A_b + m A_c) \gamma^c = 0$$

$$\Leftrightarrow \partial^a A_a = 0, i\varsigma \varepsilon^{ab} \partial_a A_b + m A_c = 0$$

$$\Leftrightarrow \varepsilon^{ab} \partial_a A_b = i\varsigma m A_c \Leftrightarrow \nabla \times \vec{A} = i\varsigma m \vec{A}$$

$$\Leftrightarrow \varepsilon^{a'b'c} \varepsilon^{ab} \partial_a A_b = i\varsigma m \varepsilon_{a'b'}^c A_c$$

$$\Leftrightarrow (\delta^{a'a} \delta^{b'b} - \delta^{a'b} \delta^{b'a}) \partial_a A_b = i\varsigma m \varepsilon_{a'b'}^c A_c$$

$$\Leftrightarrow \partial_a A_b - \partial_b A_a = i\varsigma m \varepsilon_{ab}^c A_c$$

$$\Rightarrow \partial^a \partial_a A_b - \partial_b \partial^a A_a = i\varsigma m \varepsilon_{ab}^c \partial^a A_c$$

$$\Leftrightarrow (\partial^a \partial_a - m^2) A_b = 0$$

□

定理1.3.3.  $\gamma^a \partial_a \psi(x) = 0, \psi^T(x) = \psi(x), A_a = \frac{1}{\sqrt{2im}} \text{tr}(\bar{\varepsilon} \gamma_a \psi), \gamma^a = (-\sigma_y, \sigma_x, \varsigma \sigma_z)$

$$\Leftrightarrow \partial_a A_b - \partial_b A_a = 0, \partial^a A_a = 0, \psi = im \gamma^a \varepsilon A_a \Rightarrow \partial^b \partial_b A_a = 0, \partial^a A_a = 0$$

证明:  $\gamma^a \partial_a \psi(x) = 0, \psi = \frac{im}{\sqrt{2}} \gamma^a \varepsilon A_a$

$$\Leftrightarrow \gamma^a \partial_a \frac{im}{\sqrt{2}} \gamma^b \varepsilon A_b = 0$$

$$\Leftrightarrow \gamma^a \partial_a \gamma^b A_b = 0$$

$$\Leftrightarrow \delta^{ab} \partial_a A_b + i\varsigma \varepsilon^{abc} \partial_a A_b \gamma_c = 0$$

$$\Leftrightarrow \partial^a A_a + i\varsigma \varepsilon^{ab} \partial_a A_b = 0$$

$$\Leftrightarrow \partial^a A_a = 0, i\varsigma \varepsilon^{ab} \partial_a A_b = 0$$

$$\Leftrightarrow \partial^a A_a = 0, \varepsilon^{ab} \partial_a A_b = 0 \Leftrightarrow \partial^a A_a = 0, \nabla \times \vec{A} = 0$$

$$\Leftrightarrow \partial^a A_a = 0, \varepsilon^{a'b'c} \varepsilon^{ab} \partial_a A_b = 0$$

$$\Leftrightarrow \partial^a A_a = 0, (\delta^{a'a} \delta^{b'b} - \delta^{a'b} \delta^{b'a}) \partial_a A_b = 0$$

$$\Leftrightarrow \partial^a A_a = 0, \partial_a A_b - \partial_b A_a = 0$$

$$\Rightarrow \partial^a A_a = 0, \partial^a \partial_a A_b - \partial_b \partial^a A_a = 0$$

$$\Leftrightarrow \partial^a \partial_a A_b = 0, \partial^a A_a = 0$$

□

1.3.2 三维时空中有质量自旋- $\frac{3}{2}$ 的B-W方程等价于类R-S方程 [17, 19]

定理1.3.4.  $(\gamma^a \partial_a + m)_{\kappa\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{1}{3!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta\}}, A_{a\eta_\zeta} = \frac{1}{\sqrt{2im}} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}$

$$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\zeta} - \partial_b A_{a\eta_\zeta} = i\varsigma m \varepsilon_{ab}^c A_{c\eta_\zeta} \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{im}{\sqrt{2}} (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} A_{a\eta_\zeta}, \gamma^a A_{a[\eta_\zeta]} = 0 \end{cases} \Rightarrow (\gamma^b \partial_b + m) A_{a[\eta_\zeta]} = 0, \gamma^a A_{a[\eta_\zeta]} = 0$$

证明:  $\begin{cases} (\gamma^a \partial_a + m)_{\kappa\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{1}{3!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta\}} \\ A_{a\eta_\zeta} = \frac{1}{\sqrt{2im}} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}, \gamma^a = (\sigma_x, \sigma_y, \varsigma \sigma_z) \end{cases}$

$$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\zeta} - \partial_b A_{a\eta_\zeta} = i\varsigma m \varepsilon_{ab}^c A_{c\eta_\zeta} \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{im}{\sqrt{2}} (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} A_{a\eta_\zeta}, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \psi_{\lambda_\zeta \eta_\zeta \mu_\zeta} \end{cases}$$

$$\begin{aligned}
&\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\zeta} - \partial_b A_{a\eta_\zeta} = i\zeta m \varepsilon_{ab}{}^c A_{c\eta_\zeta} \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{im}{\sqrt{2}} (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} A_{a\eta_\zeta}, \varepsilon^{\mu_\zeta \eta_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = 0 \end{cases} \\
&\Leftrightarrow \partial_a A_{b\eta_\zeta} - \partial_b A_{a\eta_\zeta} = i\zeta m \varepsilon_{ab}{}^c A_{c\eta_\zeta}, \gamma^a A_{a[\eta_\zeta]} = 0 \\
&\Rightarrow \gamma^a \partial_a A_{b[\eta_\zeta]} - \partial_b \gamma^a A_{a[\eta_\zeta]} = i\zeta m \varepsilon_{ab}{}^c \gamma^a A_{c[\eta_\zeta]}, \gamma^a A_{a\eta_\zeta} = 0 \\
&\Leftrightarrow \gamma^a \partial_a A_{b[\eta_\zeta]} + \frac{1}{2} m [\gamma_c, \gamma_b] A_{[\eta_\zeta]}^c = 0, \gamma^a A_{a\eta_\zeta} = 0 \\
&\Leftrightarrow \gamma^a \partial_a A_{b\eta_\zeta} + \frac{1}{2} m \{\gamma_c, \gamma_b\} A_{[\eta_\zeta]}^c = 0, \gamma^a A_{a\eta_\zeta} = 0 \\
&\Leftrightarrow (\gamma^b \partial_b + m) A_{a[\eta_\zeta]} = 0, \gamma^a A_{a[\eta_\zeta]} = 0 \\
&\Leftrightarrow (\gamma^b \partial_b + m) A_{a[\eta_\zeta]} = 0, \gamma^a A_{a[\eta_\zeta]} = 0, \partial^a A_{a\eta_\zeta} = 0
\end{aligned}$$

□

### 1.3.3 三维时空中有质量自旋-2的B-W方程等价于类C-K方程 [17, 19]

定理1.3.5.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \frac{1}{4!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta\}}, A_{ab} = (\frac{1}{\sqrt{2im}})^2 (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}$

$$\begin{aligned}
&\Leftrightarrow \begin{cases} \partial_a A_{bd} - \partial_b A_{ad} = i\zeta m \varepsilon_{ab}{}^c A_{cd}, A_{ab} = A_{ba} \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = (\frac{im}{\sqrt{2}})^2 (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma^b \varepsilon)_{\eta_\zeta \xi_\zeta} A_{ab}, \delta^{ab} A_{ab} = 0 \end{cases} \Rightarrow \begin{cases} (\partial^c \partial_c - m^2) A_{ab} = 0, A_{ab} = A_{ba} \\ \delta^{ab} A_{ab} = 0, \partial^a A_{ab} = 0 \end{cases}
\end{aligned}$$

证明:

$$\begin{aligned}
&\begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \frac{1}{4!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta\}} \\ A_{ab} := (\frac{1}{\sqrt{2im}})^2 (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}, \gamma^a = (-\sigma_y, \sigma_x, \varsigma \sigma_z) \end{cases} \\
&\Leftrightarrow \begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \frac{1}{4!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta\}} \\ A_{a\eta_\zeta \xi_\zeta} := \frac{1}{\sqrt{2im}} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} \end{cases} \\
&\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\zeta \xi_\zeta} - \partial_b A_{a\eta_\zeta \xi_\zeta} = i\zeta m \varepsilon_{ab}{}^c A_{c\eta_\zeta \xi_\zeta}, A_{a\eta_\zeta \xi_\zeta} = A_{a\xi_\zeta \eta_\zeta} \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \frac{im}{\sqrt{2}} (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} A_{a\eta_\zeta \xi_\zeta}, \gamma^a A_{a[\eta_\zeta] \xi_\zeta} = 0 \end{cases} \\
&\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\zeta \xi_\zeta} - \partial_b A_{a\eta_\zeta \xi_\zeta} = i\zeta m \varepsilon_{ab}{}^c A_{c\eta_\zeta \xi_\zeta}, A_{a\eta_\zeta \xi_\zeta} = A_{a\xi_\zeta \eta_\zeta} \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \frac{im}{\sqrt{2}} (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} A_{a\eta_\zeta \xi_\zeta}, \gamma^a A_{a[\eta_\zeta] \xi_\zeta} = 0 \end{cases} \\
&\Leftrightarrow \begin{cases} \partial_a A_{bd} - \partial_b A_{ad} = i\zeta m \varepsilon_{ab}{}^c A_{cd}, A_{ab} = A_{ba}, \delta^{ab} A_{ab} = 0 \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = (\frac{im}{\sqrt{2}})^2 (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma^b \varepsilon)_{\eta_\zeta \xi_\zeta} A_{ab} \end{cases} \\
&\Rightarrow (\partial^c \partial_c - m^2) A_{ab} = 0, A_{ab} = A_{ba}, \delta^{ab} A_{ab} = 0, \partial^a A_{ab} = 0
\end{aligned}$$

□

## 1.4 三维时空中有质量玻色子势方程的一般表述

### 1.4.1 数学准备

性质1.4.1.  $(\gamma^a \varepsilon)_{\lambda'_\zeta \mu'_\zeta} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} = \delta_{\lambda'_\zeta}^{\lambda_\zeta} \delta_{\mu'_\zeta}^{\mu_\zeta}$

性质1.4.2.  $(\gamma^a \varepsilon)_{\lambda'_\zeta \mu'_\zeta} \eta_{aa'} (\bar{\varepsilon} \gamma^{a'})^{\lambda_\zeta \mu_\zeta} = \delta_{\lambda'_\zeta}^{\lambda_\zeta} \delta_{\mu'_\zeta}^{\mu_\zeta} - 2|\varepsilon_{\lambda'_\zeta \mu'_\zeta}| |\varepsilon^{\lambda_\zeta \mu_\zeta}|$

### 1.4.2 三维时空中有质量自旋-n的B-W方程等价于类C-K方程 [17, 19]

定理1.4.1.

$$\begin{aligned}
&\begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}_{2n}(x) = 0 \\ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}_{2n} = \frac{1}{(2n)!} \underbrace{\psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots\}}}_{2n} \\ \underbrace{A_{ab \dots}}_n = (\frac{1}{\sqrt{2im}})^n (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}_{2n} \end{cases} \Leftrightarrow \begin{cases} \partial_a \underbrace{A_{bd \dots}}_n(x) - \partial_b \underbrace{A_{ad \dots}}_n(x) = i\zeta m \varepsilon_{ab}{}^c \underbrace{A_{cd \dots}}_n(x) \\ \underbrace{A_{ab \dots}}_n = \frac{1}{n!} \underbrace{A_{\{ab \dots\}}}_n, \delta^{ab} \underbrace{A_{ab \dots}}_n = 0 \\ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}}_{2n} = (\frac{im}{\sqrt{2}})^n (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma^b \varepsilon)_{\eta_\zeta \xi_\zeta} \dots \underbrace{A_{ab \dots}}_n \end{cases} \\
&\Rightarrow \begin{cases} (\partial^c \partial_c - m^2) \underbrace{A_{ab \dots}}_n = 0 \\ \underbrace{A_{ab \dots}}_n = \frac{1}{n!} \underbrace{A_{\{ab \dots\}}}_n, \delta^{ab} \underbrace{A_{ab \dots}}_n = 0, \partial^a \underbrace{A_{ab \dots}}_n = 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}\psi_{\lambda_s \mu_s \dots}(\vec{r}, t) &= \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{n-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2n} [a(\vec{p}) U_{\lambda_s \mu_s \dots}(\vec{p}) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}) V_{\lambda_s \mu_s \dots}(\vec{p}) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^N \vec{p} \\ A_{ab \dots}(\vec{r}, t) &= \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2^n E}} [a(\vec{p}) \varepsilon_{ab \dots}(\vec{p}) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}) \tilde{\varepsilon}_{ab \dots}(\vec{p}) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^N \vec{p} \\ \varepsilon_{ab \dots}(\vec{p}) &= \frac{1}{i^n} \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_s \mu_s} (\bar{\varepsilon}\gamma_b)^{\eta_s \xi_s} \dots}^n U_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}), \tilde{\varepsilon}_{ab \dots}(\vec{p}) = \frac{1}{i^n} \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_s \mu_s} (\bar{\varepsilon}\gamma_b)^{\eta_s \xi_s} \dots}^n V_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p})\end{aligned}$$

### 1.4.3 三维时空中有质量自旋- $n$ 的B-W方程自旋基和类C-K方程自旋基之间关系

推论1.4.1.

$$\begin{cases} (i\gamma^a p_a + m) U_{[\lambda_s] \mu_s \eta_s \xi_s \dots}(\vec{p}) = 0 \\ U_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}) \text{ 全对称} \\ \varepsilon_{ab \dots}(\vec{p}) = \frac{1}{i^n} \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_s \mu_s} (\bar{\varepsilon}\gamma_b)^{\eta_s \xi_s} \dots}^n U_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}) \end{cases} \Leftrightarrow \begin{cases} (p^c p_c + m^2) \varepsilon_{ab \dots}(\vec{p}) = 0 \\ p_a \varepsilon_{bd \dots}(\vec{p}) - p_b \varepsilon_{ad \dots}(\vec{p}) = \zeta m \varepsilon_{ab}^c \varepsilon_{cd \dots} \\ \delta^{ab} \varepsilon_{ab \dots}(\vec{p}) = 0, p^a \varepsilon_{ab \dots}(\vec{p}) = 0, \varepsilon_{ab \dots}(\vec{p}) \text{ 全对称} \\ U_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}) = \left(\frac{i}{2}\right)^n \overbrace{(\gamma_a \varepsilon)_{\lambda_s \mu_s} (\gamma_b \varepsilon)_{\eta_s \xi_s} \dots}^n \varepsilon_{ab \dots}(\vec{p}) \end{cases}$$

推论1.4.2.

$$\begin{cases} (-i\gamma^a p_a + m) V_{[\lambda_s] \mu_s \eta_s \xi_s \dots}(\vec{p}) = 0 \\ V_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}) \text{ 全对称} \\ \tilde{\varepsilon}_{ab \dots}(\vec{p}) = \frac{1}{i^n} \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_s \mu_s} (\bar{\varepsilon}\gamma_b)^{\eta_s \xi_s} \dots}^n V_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}) \end{cases} \Leftrightarrow \begin{cases} (p^c p_c + m^2) \tilde{\varepsilon}_{ab \dots}(\vec{p}) = 0 \\ p_a \tilde{\varepsilon}_{bd \dots}(\vec{p}) - p_b \tilde{\varepsilon}_{ad \dots}(\vec{p}) = -\zeta m \varepsilon_{ab}^c \tilde{\varepsilon}_{cd \dots} \\ \delta^{ab} \tilde{\varepsilon}_{ab \dots}(\vec{p}) = 0, p^a \tilde{\varepsilon}_{ab \dots}(\vec{p}) = 0, \tilde{\varepsilon}_{ab \dots}(\vec{p}) \text{ 全对称} \\ V_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}) = \left(\frac{i}{2}\right)^n \overbrace{(\gamma_a \varepsilon)_{\lambda_s \mu_s} (\gamma_b \varepsilon)_{\eta_s \xi_s} \dots}^n \tilde{\varepsilon}_{ab \dots}(\vec{p}) \end{cases}$$

推论1.4.3.

$$\begin{cases} U_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}) = \left(\frac{i}{2}\right)^n \overbrace{(\gamma_a \varepsilon)_{\lambda_s \mu_s} (\gamma_b \varepsilon)_{\eta_s \xi_s} \dots}^n \varepsilon_{ab \dots}(\vec{p}) [\Leftrightarrow] \varepsilon_{ab \dots}(\vec{p}) = \frac{1}{i^n} \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_s \mu_s} (\bar{\varepsilon}\gamma_b)^{\eta_s \xi_s} \dots}^n U_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}) \\ V_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}) = \left(\frac{i}{2}\right)^n \overbrace{(\gamma_a \varepsilon)_{\lambda_s \mu_s} (\gamma_b \varepsilon)_{\eta_s \xi_s} \dots}^n \tilde{\varepsilon}_{ab \dots}(\vec{p}) [\Leftrightarrow] \tilde{\varepsilon}_{ab \dots}(\vec{p}) = \frac{1}{i^n} \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_s \mu_s} (\bar{\varepsilon}\gamma_b)^{\eta_s \xi_s} \dots}^n V_{\lambda_s \mu_s \eta_s \xi_s \dots}(\vec{p}) \end{cases}$$

$$\text{推论1.4.4. } \varepsilon_{ab \dots}(\vec{p}) = \varepsilon_a(\vec{p}) \varepsilon_b(\vec{p}) \dots, \tilde{\varepsilon}_{ab \dots}(\vec{p}) = \tilde{\varepsilon}_a(\vec{p}) \tilde{\varepsilon}_b(\vec{p}) \dots$$

### 1.4.4 三维时空中的类Klein-Gordon方程自旋基 $\varepsilon_a(\vec{p})$ 及其性质

$$\text{推论1.4.5. } u(\vec{p}) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{\zeta \sigma \cdot \vec{p}}{E+m}\right) \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{定理1.4.2. } \varepsilon_a(\vec{p}) = \left(i\zeta + \frac{i\zeta p_x (p_x + i\zeta p_y)}{m(E+m)}\right), -1 + \frac{i\zeta p_y (p_x + i\zeta p_y)}{m(E+m)}, -\zeta \frac{p_x + i\zeta p_y}{m}$$

证明:  $u^T(\vec{p})u(\vec{p})$

$$\begin{aligned}&= \frac{E+m}{2m} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x\right) \left(1 - \frac{\zeta \sigma \cdot \vec{p}}{E+m}\right) \left(1 - \frac{\zeta \sigma \cdot \vec{p}}{E+m}\right) \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\&= \frac{E+m}{2m} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x\right) \left(1 + \frac{\sigma \cdot \vec{p}}{E+m} \frac{\sigma \cdot \vec{p}}{E+m}\right) \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\&= \frac{E+m}{2m} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x\right) \left(1 + \frac{p_x^2 - p_y^2 + 2i\zeta p_x p_y \sigma_z}{(E+m)^2}\right) \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\&= \frac{1}{2m(E+m)} [(E+m)^2 + p_x^2 - p_y^2 + 2i\zeta p_x p_y] \\&= 1 + \frac{p_x (p_x + i\zeta p_y)}{m(E+m)}\end{aligned}$$

□

$$\begin{aligned}
& \text{证明: } u^T(\vec{p})\sigma_z u(\vec{p}) \\
&= \frac{E+m}{2m} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 - \frac{\zeta \sigma^* \cdot \vec{p}}{E+m} \right) \sigma_z \left( 1 - \frac{\zeta \sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{E+m}{2m} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 - \frac{\sigma^* \cdot \vec{p}}{E+m} \frac{\sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \zeta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{E+m}{2m} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 - \frac{p_x^2 - p_y^2 + 2ip_x p_y \sigma_z}{(E+m)^2} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \zeta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{\zeta}{2m(E+m)} [(E+m)^2 - p_x^2 + p_y^2 - 2i\zeta p_x p_y] \\
&= \zeta - \frac{ip_y(p_x + i\zeta p_y)}{m(E+m)} \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } u^T(\vec{p})\sigma_x u(\vec{p}) \\
&= \frac{E+m}{2m} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 - \frac{\zeta \sigma^* \cdot \vec{p}}{E+m} \right) \sigma_x \left( 1 - \frac{\zeta \sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{E+m}{2m} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \sigma_x \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 - \frac{\zeta \sigma \cdot \vec{p}}{E+m} \right) \left( 1 - \frac{\zeta \sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{E+m}{2m} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \sigma_x \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \frac{-2\zeta \sigma \cdot \vec{p}}{E+m} \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= -\zeta \frac{(p_x + i\zeta p_y)}{m} \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } \varepsilon_a(\vec{p}) = -i(\bar{\varepsilon}\gamma_a)^{\lambda\mu\zeta} U_{\lambda\mu\zeta}(\vec{p}) \\
&= -iu^T(\vec{p})(\bar{\varepsilon}\gamma_a)u(\vec{p}) \\
&= u^T(\vec{p})(1, i\sigma_z, -i\zeta\sigma_x)u(\vec{p}) \\
&= \left( 1 + \frac{p_x(p_x + i\zeta p_y)}{m(E+m)}, i\zeta + \frac{p_y(p_x + i\zeta p_y)}{m(E+m)}, i\frac{p_x + i\zeta p_y}{m} \right) \\
&= \left( 1 + \frac{p_x(p_x - i\zeta p_y)}{m(E+m)}, -i\zeta + \frac{p_y(p_x - i\zeta p_y)}{m(E+m)}, i\frac{(p_x - i\zeta p_y)}{m} \right) \quad \square
\end{aligned}$$

$$\text{推论1.4.6. } \varepsilon_a(\vec{p})\varepsilon_{a'}^+(\vec{p}) = \begin{bmatrix} 1 + \frac{p_x^2}{m^2} & \frac{p_x p_y}{m^2} - \frac{\zeta p_x}{m} & -\frac{p_x p_x}{m^2} - \frac{\zeta p_y}{m} \\ \frac{p_x p_y}{m^2} + \frac{\zeta p_x}{m} & 1 + \frac{p_y^2}{m^2} & -\frac{p_y p_x}{m^2} + \frac{\zeta p_x}{m} \\ \frac{p_x p_x}{m^2} - \frac{\zeta p_y}{m} & \frac{p_y p_x}{m^2} + \frac{\zeta p_x}{m} & -1 - \frac{p_y^2}{m^2} \end{bmatrix} = \frac{1}{m} \begin{bmatrix} m & -\zeta p_x & -\zeta p_y \\ \zeta p_x & m & \zeta p_x \\ -\zeta p_y & \zeta p_x & -m \end{bmatrix} + \frac{1}{m^2} \begin{bmatrix} p_x p_x^+ & p_x p_y^+ & p_x p_\pi^+ \\ p_y p_x^+ & p_y p_y^+ & p_y p_\pi^+ \\ p_\pi p_x^+ & p_\pi p_y^+ & p_\pi p_\pi^+ \end{bmatrix}$$

$$\text{推论1.4.7. } \varepsilon_a(\vec{p})\varepsilon_{a'}^+(\vec{p}) = \eta_{aa'} + \frac{p_a p_{a'}}{m^2} - \frac{\zeta \varepsilon_{acd} \eta_{a'}^c p^d}{m}$$

$$\text{推论1.4.8. } \varepsilon_a(\vec{p})\delta^{ab}\varepsilon_b(\vec{p}) = 0, \varepsilon_a(\vec{p})p^a = 0, \varepsilon_a^+(\vec{p})\eta^{aa'}\varepsilon_{a'}(\vec{p}) = 2, \varepsilon_a^+(\vec{p})\delta^{aa'}\varepsilon_{a'}(\vec{p}) = 2\left(\frac{E}{m}\right)^2$$

$$\text{推论1.4.9. } \underbrace{\varepsilon_{ab\dots}(\vec{p})}_{n} \underbrace{\varepsilon_{a'b'\dots}^+(\vec{p})}_{n} = \underbrace{\left( \eta_{aa'} + \frac{p_a p_{a'}}{m^2} - \frac{\zeta \varepsilon_{acd} \eta_{a'}^c p^d}{m} \right)}_n \underbrace{\left( \eta_{bb'} + \frac{p_b p_{b'}}{m^2} - \frac{\zeta \varepsilon_{bcd} \eta_{b'}^c p^d}{m} \right)}_n \dots$$

### 1.4.5 三维时空中的类Klein-Gordon方程自旋基 $\tilde{\varepsilon}_a(\vec{p})$ 及其性质

$$\text{推论1.4.10. } v(\vec{p}) := \sqrt{\frac{E+m}{2m}} \left( 1 - \frac{\zeta \sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{定理1.4.3. } \varepsilon_a(\vec{p}) = \left( i\zeta + \frac{i\zeta p_x(p_x + i\zeta p_y)}{m(E+m)}, -1 + \frac{i\zeta p_y(p_x + i\zeta p_y)}{m(E+m)}, -\zeta \frac{p_x + i\zeta p_y}{m} \right)$$

$$\begin{aligned}
& \text{证明: } v^T(\vec{p})v(\vec{p}) \\
&= \frac{E+m}{2m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 - \frac{\zeta \sigma^* \cdot \vec{p}}{E+m} \right) \left( 1 - \frac{\zeta \sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{E+m}{2m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 + \frac{\sigma^* \cdot \vec{p}}{E+m} \frac{\sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{E+m}{2m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 + \frac{p_x^2 - p_y^2 + 2ip_x p_y \sigma_z}{(E+m)^2} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{2m(E+m)} [(E+m)^2 + p_x^2 - p_y^2 - 2i\zeta p_x p_y] \\
&= 1 + \frac{p_x(p_x - i\zeta p_y)}{m(E+m)} \quad \square
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } v^T(\vec{p})\sigma_z v(\vec{p}) \\
&= \frac{E+m}{2m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 - \frac{\zeta \sigma^* \cdot \vec{p}}{E+m} \right) \sigma_z \left( 1 - \frac{\zeta \sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= -\frac{E+m}{2m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 - \frac{\sigma^* \cdot \vec{p}}{E+m} \frac{\sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \zeta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= -\frac{E+m}{2m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \left( 1 - \frac{p_x^2 - p_y^2 + 2ip_x p_y \sigma_z}{(E+m)^2} \right) \left( \frac{1+\zeta}{2} + \frac{1-\zeta}{2} \sigma_x \right) \zeta \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}$$

$$= \frac{-\varsigma}{2m(E+m)} [(E+m)^2 - p_x^2 + p_y^2 + 2i\varsigma p_x p_y]$$

$$= -\varsigma - \frac{ip_y(p_x - i\varsigma p_y)}{m(E+m)} \quad \square$$

证明:  $v^T(\vec{p})\sigma_x v(\vec{p})$

$$= \frac{E+m}{2m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \left( \frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x \right) \left( 1 - \frac{\varsigma\sigma^* \cdot \vec{p}}{E+m} \right) \sigma_x \left( 1 - \frac{\varsigma\sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{E+m}{2m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \sigma_x \left( \frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x \right) \left( 1 - \frac{\varsigma\sigma \cdot \vec{p}}{E+m} \right) \left( 1 - \frac{\varsigma\sigma \cdot \vec{p}}{E+m} \right) \left( \frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{E+m}{2m} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \sigma_x \left( \frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x \right) \frac{-2\varsigma\sigma \cdot \vec{p}}{E+m} \left( \frac{1+\varsigma}{2} + \frac{1-\varsigma}{2}\sigma_x \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -\varsigma \frac{(p_x - i\varsigma p_y)}{m} \quad \square$$

证明:  $\tilde{\varepsilon}_a(\vec{p}) = -i(\bar{\varepsilon}\gamma_a)^{\lambda\mu\varsigma} V_{\lambda\mu\varsigma}(\vec{p})$

$$= -iv^T(\vec{p})(\bar{\varepsilon}\gamma_a)v(\vec{p})$$

$$= v^T(\vec{p})(1, i\sigma_z, -i\sigma_x)v(\vec{p})$$

$$= \left( 1 + \frac{p_x(p_x - i\varsigma p_y)}{m(E+m)}, -i\varsigma + \frac{p_y(p_x - i\varsigma p_y)}{m(E+m)}, i \frac{(p_x - i\varsigma p_y)}{m} \right) \quad \square$$

推论1.4.11.  $\tilde{\varepsilon}_a(\vec{p}) = \varepsilon_{a'}^+(\vec{p})\eta_{a'}^{\prime\prime}, \tilde{\varepsilon}_{ab\dots}(\vec{p}) = \varepsilon_{a'b'\dots}^+(\vec{p})\eta_{a'b'\dots}^{\prime\prime}$

推论1.4.12.  $\tilde{\varepsilon}_a(\vec{p})\tilde{\varepsilon}_{a'}^+(\vec{p}) = \eta_{aa'} + \frac{p_a p_{a'}}{m^2} + \frac{\varsigma\varepsilon_{acd}\eta_{a'}^c p^d}{m}$

推论1.4.13.  $\tilde{\varepsilon}_a(\vec{p})\delta^{ab}\tilde{\varepsilon}_b(\vec{p}) = 0, \tilde{\varepsilon}_a(\vec{p})p^a = 0, \tilde{\varepsilon}_a^+(\vec{p})\eta^{aa'}\tilde{\varepsilon}_{a'}(\vec{p}) = 2, \tilde{\varepsilon}_a^+(\vec{p})\delta^{aa'}\tilde{\varepsilon}_{a'}(\vec{p}) = 2\left(\frac{E}{m}\right)^2$

推论1.4.14.  $\tilde{\varepsilon}_{ab\dots}(\vec{p})\tilde{\varepsilon}_{a'b'\dots}^+(\vec{p}) = \underbrace{\left( \eta_{aa'} + \frac{p_a p_{a'}}{m^2} + \frac{\varsigma\varepsilon_{acd}\eta_{a'}^c p^d}{m} \right)}_n \underbrace{\left( \eta_{bb'} + \frac{p_b p_{b'}}{m^2} + \frac{\varsigma\varepsilon_{bcd}\eta_{b'}^c p^d}{m} \right)}_n \dots$

### 1.4.6 三维时空中有质量玻色子各种准投影算子之间关系

推论1.4.15. 
$$\left\{ \begin{aligned} \varepsilon_{ab\dots}(\vec{p})\varepsilon_{a'b'\dots}^+(\vec{p}) &= \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda\mu\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta\varsigma\xi}}^n \dots \overbrace{(\gamma_{a'}\varepsilon)^{\lambda'\mu'\varsigma}(\gamma_{b'}\varepsilon)^{\eta'\xi'\varsigma}}^n \dots \underbrace{U_{\lambda\mu\varsigma\eta\varsigma\xi}}_{2n}(\vec{p}) \underbrace{U_{\lambda'\mu'\eta'\xi'\varsigma}}_{2n}^+(\vec{p}) \\ \tilde{\varepsilon}_{ab\dots}(\vec{p})\tilde{\varepsilon}_{a'b'\dots}^+(\vec{p}) &= \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda\mu\varsigma}(\bar{\varepsilon}\gamma_b)^{\eta\varsigma\xi}}^n \dots \overbrace{(\gamma_{a'}\varepsilon)^{\lambda'\mu'\varsigma}(\gamma_{b'}\varepsilon)^{\eta'\xi'\varsigma}}^n \dots \underbrace{V_{\lambda\mu\varsigma\eta\varsigma\xi}}_{2n}(\vec{p}) \underbrace{V_{\lambda'\mu'\eta'\xi'\varsigma}}_{2n}^+(\vec{p}) \end{aligned} \right.$$

推论1.4.16. 
$$\left\{ \begin{aligned} U_{\lambda\mu\varsigma\dots}(\vec{p})U_{\lambda'\mu'\varsigma'}^+(\vec{p}) &= \frac{1}{2^{2n}} \overbrace{(\gamma_a\varepsilon)^{\lambda\mu\varsigma}}^n \dots \overbrace{(\bar{\varepsilon}\gamma_{a'})^{\lambda'\mu'\varsigma'}}^n \dots \varepsilon_{ab\dots}(\vec{p})\varepsilon_{a'b'\dots}^+(\vec{p}) \\ V_{\lambda\mu\varsigma\dots}(\vec{p})V_{\lambda'\mu'\varsigma'}^+(\vec{p}) &= \frac{1}{2^{2n}} \overbrace{(\gamma_a\varepsilon)^{\lambda\mu\varsigma}}^n \dots \overbrace{(\bar{\varepsilon}\gamma_{a'})^{\lambda'\mu'\varsigma'}}^n \dots \tilde{\varepsilon}_{ab\dots}(\vec{p})\tilde{\varepsilon}_{a'b'\dots}^+(\vec{p}) \end{aligned} \right.$$

推论1.4.17. 
$$\left\{ \begin{aligned} [A_{ab\dots}(x), A_{a'b'\dots}^+(x')] &= \frac{1}{m^{2n}2^n} \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda\mu\varsigma}}^n \dots \overbrace{(\gamma_{a'}\varepsilon)^{\lambda'\mu'\varsigma}}^n \dots [\psi_{\lambda\mu\varsigma\dots}(x), \psi_{\lambda'\mu'\varsigma'}^+(x')] \\ [\psi_{\lambda\mu\varsigma\dots}(x), \psi_{\lambda'\mu'\varsigma'}^+(x')] &= \frac{m^{2n}}{2^n} \overbrace{(\gamma_a\varepsilon)^{\lambda\mu\varsigma}}^n \dots \overbrace{(\bar{\varepsilon}\gamma_{a'})^{\lambda'\mu'\varsigma}}^n \dots [A_{ab\dots}(x), A_{a'b'\dots}^+(x')] \end{aligned} \right.$$

### 1.4.7 三维时空中有质量玻色子准投影算子的等价表述

引理1.4.1.

$$\left\{ \begin{aligned} u(\vec{p})u^+(\vec{p}) &= \frac{(m-i\gamma^a p_a)\gamma^0}{2m}, u_{\lambda\varsigma}(\vec{p})u_{\lambda'\varsigma'}^+(\vec{p})u_{\mu\varsigma}(\vec{p})u_{\mu'\varsigma'}^+(\vec{p}) = u_{\lambda\varsigma}(\vec{p})u_{\mu'\varsigma'}^+(\vec{p})u_{\mu\varsigma}(\vec{p})u_{\lambda'\varsigma'}^+(\vec{p}) \\ \varepsilon_a(\vec{p})\varepsilon_{a'}^+(\vec{p}) &= \eta_{aa'} + \frac{p_a p_{a'}}{m^2} - \frac{\varsigma\varepsilon_{acd}\eta_{a'}^c p^d}{m}, \varepsilon_a(\vec{p})\varepsilon_{a'}^+(\vec{p})\varepsilon_b(\vec{p})\varepsilon_{b'}^+(\vec{p}) = \varepsilon_a(\vec{p})\varepsilon_{b'}^+(\vec{p})\varepsilon_b(\vec{p})\varepsilon_{a'}^+(\vec{p}) \end{aligned} \right.$$

$$\Leftrightarrow \begin{cases} [(m - i\gamma^a p_a)\gamma^0]_{\lambda_s \lambda'_s} [(m - i\gamma^b p_b)\gamma^0]_{\mu_s \mu'_s} = [(m - i\gamma^a p_a)\gamma^0]_{\mu_s \mu'_s} [(m - i\gamma^b p_b)\gamma^0]_{\lambda_s \lambda'_s} \\ (\eta_{aa'} + \frac{p_a p_{a'}}{m^2} - \frac{\varepsilon \varepsilon_{acd} \eta_{a'}^c p^d}{m}) (\eta_{bb'} + \frac{p_b p_{b'}}{m^2} - \frac{\varepsilon \varepsilon_{bcd} \eta_{b'}^c p^d}{m}) = (\eta_{ab'} + \frac{p_a p_{b'}}{m^2} - \frac{\varepsilon \varepsilon_{acd} \eta_{b'}^c p^d}{m}) (\eta_{ba'} + \frac{p_b p_{a'}}{m^2} - \frac{\varepsilon \varepsilon_{bcd} \eta_{a'}^c p^d}{m}) \\ [(m - i\gamma^b p_b)\gamma^0]_{\lambda_s \lambda'_s} [(m - i\gamma^c p_c)\gamma^0]_{\mu_s \mu'_s} = m^2 (\gamma^a \varepsilon)_{\lambda_s \mu_s} (\bar{\varepsilon} \gamma^{a'})_{\lambda'_s \mu'_s} (\eta_{aa'} + \frac{p_a p_{a'}}{m^2} - \frac{\varepsilon \varepsilon_{acd} \eta_{a'}^c p^d}{m}) \end{cases}$$

推论1.4.18.

$$\begin{cases} U_{\lambda_s \mu_s \dots}^{2n}(\vec{p}) U_{\lambda'_s \mu'_s \dots}^{2n}(\vec{p}) = \frac{1}{2^{2n}} \underbrace{[(\gamma^a \varepsilon)_{\lambda_s \mu_s} (\bar{\varepsilon} \gamma^{a'})_{\lambda'_s \mu'_s} (\eta_{aa'} + \frac{p_a p_{a'}}{m^2} - \frac{\varepsilon \varepsilon_{acd} \eta_{a'}^c p^d}{m})]}_n \dots \\ V_{\lambda_s \mu_s \dots}^{2n}(\vec{p}) V_{\lambda'_s \mu'_s \dots}^{2n}(\vec{p}) = \frac{1}{2^{2n}} \underbrace{[(\gamma^a \varepsilon)_{\lambda_s \mu_s} (\bar{\varepsilon} \gamma^{a'})_{\lambda'_s \mu'_s} (\eta_{aa'} + \frac{p_a p_{a'}}{m^2} + \frac{\varepsilon \varepsilon_{acd} \eta_{a'}^c p^d}{m})]}_n \dots \end{cases}$$

### 1.4.8 三维时空中有质量玻色子协变对易规则

$$\text{定理1.4.4. } [\psi_{\lambda_s \mu_s \dots}^{2n}(x), \psi_{\lambda'_s \mu'_s \dots}^{2n}(x')] = \frac{i}{2^{2n-1}} \underbrace{[(m - \gamma^a \partial_a)\gamma^0]_{\lambda_s \lambda'_s} [(m - \gamma^b \partial_b)\gamma^0]_{\mu_s \mu'_s}}_{2n} \dots \Delta(x - x')$$

$$\Leftrightarrow$$

$$\text{定理1.4.5. } [\psi_{\lambda_s \mu_s \dots}^{2n}(x), \psi_{\lambda'_s \mu'_s \dots}^{2n}(x')] = i \frac{i^{2n}}{2^{2n-1}} \underbrace{[-im\sigma_z + (\sigma, i\varepsilon)^a \partial_a]_{\lambda_s \lambda'_s} [-im\sigma_z + (\sigma, i\varepsilon)^b \partial_b]_{\mu_s \mu'_s}}_{2n} \dots \Delta(x - x')$$

$$\Leftrightarrow$$

$$\text{定理1.4.6. } [\psi_{\lambda_s \mu_s \dots}^{2n}(x), \psi_{\lambda'_s \mu'_s \dots}^{2n}(x')] = \frac{i}{2^{2n-1}} \underbrace{\mathbb{X}_{\lambda_s \mu_s}^a(x)}_n \dots \underbrace{\mathbb{X}_{\lambda'_s \mu'_s}^{a'}(x')}_{n'} \dots \underbrace{(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2} + \frac{i\varepsilon \varepsilon_{acd} \eta_{a'}^c \partial^d}{m})}_{n'} \dots \Delta(x - x')$$

$$\Leftrightarrow$$

$$\text{定理1.4.7. } [A_{ab \dots}^n(x), A_{a'b' \dots}^n(x')] = \frac{i}{2^{n-1}} \underbrace{(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2} + \frac{i\varepsilon \varepsilon_{acd} \eta_{a'}^c \partial^d}{m})}_{n} \dots \Delta(x - x')$$

## 1.5 三维时空中费米子势方程的一般表述

### 1.5.1 三维时空中有质量自旋- $n + \frac{1}{2}$ 的B-W方程等价于类R-S方程 [17, 19]

定理1.5.1.

$$\begin{cases} (\gamma^a \partial_a + m)_{\kappa_s} \psi_{\lambda_s \mu_s \eta_s \xi_s \dots \tau_s}^{2n+1}(x) = 0 \\ \psi_{\lambda_s \mu_s \eta_s \xi_s \dots \tau_s}^{2n+1} = \frac{1}{(2n+1)!} \psi_{\{\lambda_s \mu_s \eta_s \xi_s \dots \tau_s\}}^{2n+1} \\ A_{ab \dots \tau_s}^n \\ = (\frac{1}{\sqrt{2im}})^n \underbrace{(\bar{\varepsilon} \gamma_a)_{\lambda_s \mu_s} (\bar{\varepsilon} \gamma_b)_{\eta_s \xi_s} \dots}_{n} \psi_{\lambda_s \mu_s \eta_s \xi_s \dots \tau_s}^{2n+1} \end{cases} \Leftrightarrow \begin{cases} \partial_a A_{bd \dots \tau_s}^n - \partial_b A_{ad \dots \tau_s}^n = i\varepsilon m \varepsilon_{ab}^c A_{cd \dots \tau_s}^n \\ A_{ab \dots \tau_s}^n = \frac{1}{n!} A_{\{ab \dots\} \tau_s}^n, \delta^{ab} A_{ab \dots \tau_s}^n = 0, \gamma^a A_{ab \dots [\tau_s]}^n = 0 \\ \psi_{\lambda_s \mu_s \eta_s \xi_s \dots \tau_s}^{2n+1} = (\frac{im}{\sqrt{2}})^n \underbrace{(\gamma^a \varepsilon)_{\lambda_s \mu_s} (\gamma^b \varepsilon)_{\eta_s \xi_s} \dots}_{n} A_{ab \dots \tau_s}^n \end{cases}$$

$$\Rightarrow \begin{cases} (\gamma^c \partial_c + m) A_{ab \dots [\tau_s]}^n = 0, \gamma^a A_{ab \dots [\tau_s]}^n = 0 \\ A_{ab \dots \tau_s}^n = \frac{1}{n!} A_{\{ab \dots\} \tau_s}^n, \delta^{ab} A_{ab \dots \tau_s}^n = 0, \partial^a A_{ab \dots \tau_s}^n = 0 \end{cases}$$

$$\psi_{\lambda_s \mu_s \dots \tau_s}^{2n+1}(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^n \sqrt{\frac{m}{E}}^{2n+1} [a(\vec{p}) U_{\lambda_s \mu_s \dots \tau_s}^{2n+1}(\vec{p}) e^{i(\vec{p} \cdot \vec{r} - Et)} + b^+(\vec{p}) V_{\lambda_s \mu_s \dots \tau_s}^{2n+1}(\vec{p}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}$$

$$A_{ab \dots \tau_s}^n(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{\sqrt{m}}{\sqrt{2^n E}} [a(\vec{p}) \varepsilon_{ab \dots \tau_s}^n(\vec{p}) e^{i(\vec{p} \cdot \vec{r} - Et)} + b^+(\vec{p}) \tilde{\varepsilon}_{ab \dots \tau_s}^n(\vec{p}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}$$

$$\varepsilon_{ab \dots \tau_s}^n(\vec{p}) = \frac{1}{i^n} \underbrace{(\bar{\varepsilon} \gamma_a)_{\lambda_s \mu_s} (\bar{\varepsilon} \gamma_b)_{\eta_s \xi_s} \dots}_{2n+1} U_{\lambda_s \mu_s \eta_s \xi_s \dots \tau_s}^{2n+1}(\vec{p}), \tilde{\varepsilon}_{ab \dots \tau_s}^n(\vec{p}) = \frac{1}{i^n} \underbrace{(\bar{\varepsilon} \gamma_a)_{\lambda_s \mu_s} (\bar{\varepsilon} \gamma_b)_{\eta_s \xi_s} \dots}_{2n+1} V_{\lambda_s \mu_s \eta_s \xi_s \dots \tau_s}^{2n+1}(\vec{p})$$

$$\text{推论1.5.1. } \varepsilon_{ab \dots \tau_s}^n(\vec{p}) = \underbrace{\varepsilon_a(\vec{p}) \varepsilon_b(\vec{p})}_{n} \dots u_{\tau_s}(\vec{p}), \tilde{\varepsilon}_{ab \dots \tau_s}^n(\vec{p}) = \underbrace{\tilde{\varepsilon}_a(\vec{p}) \tilde{\varepsilon}_b(\vec{p})}_{n} \dots v_{\tau_s}(\vec{p})$$



1.5.2 三维时空中有质量自旋- $n + \frac{1}{2}$ 的B-W方程自旋基和类R-S方程自旋基之间关系

推论1.5.2.

$$\left\{ \begin{array}{l} (i\gamma^a p_a + m) U_{[\lambda_\zeta] \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) = 0 \\ U_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n}(\vec{p}) \text{ 全对称} \\ \varepsilon_{ab \dots \tau_\zeta}(\vec{p}) \\ = \frac{1}{i^n} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots U_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (p^c p_c + m^2) \varepsilon_{ab \dots \tau_\zeta}(\vec{p}) = 0, \varepsilon_{ab \dots \tau_\zeta}(\vec{p}) \text{ 全对称} \\ p_a \varepsilon_{bd \dots \tau_\zeta}(x) - p_b \varepsilon_{ad \dots \tau_\zeta} = \zeta m \varepsilon_{ab}{}^c \varepsilon_{cd \dots \tau_\zeta} \\ \delta^{ab} \varepsilon_{ab \dots \tau_\zeta}(\vec{p}) = 0, p^a \varepsilon_{ab \dots \tau_\zeta}(\vec{p}) = 0, \gamma^a \varepsilon_{ab \dots [\tau_\zeta]} = 0 \\ U_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) = \left(\frac{i}{2}\right)^n (\gamma_a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma_b \varepsilon)_{\eta_\zeta \xi_\zeta} \dots \varepsilon_{ab \dots \tau_\zeta}(\vec{p}) \end{array} \right.$$

推论1.5.3.

$$\left\{ \begin{array}{l} (-i\gamma^a p_a + m) V_{[\lambda_\zeta] \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) = 0 \\ V_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) \text{ 全对称} \\ \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}) \\ = \frac{1}{i^n} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots V_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (p^c p_c + m^2) \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}) = 0, \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}) \text{ 全对称} \\ p_a \tilde{\varepsilon}_{bd \dots \tau_\zeta}(x) - p_b \tilde{\varepsilon}_{ad \dots \tau_\zeta} = -\zeta m \varepsilon_{ab}{}^c \tilde{\varepsilon}_{cd \dots \tau_\zeta} \\ \delta^{ab} \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}) = 0, p^a \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}) = 0, \gamma^a \tilde{\varepsilon}_{ab \dots [\tau_\zeta]} = 0 \\ V_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) = \left(\frac{i}{2}\right)^n (\gamma_a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma_b \varepsilon)_{\eta_\zeta \xi_\zeta} \dots \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}) \end{array} \right.$$

推论1.5.4.

$$\left\{ \begin{array}{l} U_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) = \left(\frac{i}{2}\right)^n (\gamma_a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma_b \varepsilon)_{\eta_\zeta \xi_\zeta} \dots \varepsilon_{ab \dots \tau_\zeta}(\vec{p}) [\Leftrightarrow] \varepsilon_{ab \dots \tau_\zeta}(\vec{p}) = \frac{1}{i^n} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots U_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) \\ V_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) = \left(\frac{i}{2}\right)^n (\gamma_a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma_b \varepsilon)_{\eta_\zeta \xi_\zeta} \dots \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}) [\Leftrightarrow] \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}) = \frac{1}{i^n} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots V_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) \end{array} \right.$$

$$\text{推论1.5.5. } \tilde{\varepsilon}_{a[\tau_\zeta]}(\vec{p}) = \sigma_x \varepsilon_{a'[\tau'_\zeta]}^+(\vec{p}) \eta_{a'}^{\prime}, \tilde{\varepsilon}_{ab \dots [\tau_\zeta]}(\vec{p}) = \sigma_x \varepsilon_{a' b' \dots [\tau'_\zeta]}^+(\vec{p}) \eta_{a' b'}^{\prime} \dots$$

## 1.5.3 三维时空中有质量费米子各种准投影算子之间关系

推论1.5.6.

$$\left\{ \begin{array}{l} \varepsilon_{ab \dots \tau_\zeta}(\vec{p}) \varepsilon_{a' b' \dots \tau'_\zeta}^+(\vec{p}) = (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots (\gamma_{a'} \varepsilon)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} \varepsilon)^{\eta'_\zeta \xi'_\zeta} \dots U_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) U_{\lambda'_\zeta \mu'_\zeta \xi'_\zeta \dots \tau'_\zeta}^{2n+1}(\vec{p}) \\ \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}) \tilde{\varepsilon}_{a' b' \dots \tau'_\zeta}^+(\vec{p}) = (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \dots (\gamma_{a'} \varepsilon)^{\lambda'_\zeta \mu'_\zeta} (\gamma_{b'} \varepsilon)^{\eta'_\zeta \xi'_\zeta} \dots V_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) V_{\lambda'_\zeta \mu'_\zeta \xi'_\zeta \dots \tau'_\zeta}^{2n+1}(\vec{p}) \end{array} \right.$$

推论1.5.7.

$$\left\{ \begin{array}{l} U_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) U_{\lambda'_\zeta \mu'_\zeta \xi'_\zeta \dots \tau'_\zeta}^{2n+1}(\vec{p}) = \frac{1}{2^{2n}} (\gamma_a \varepsilon)^{\lambda_\zeta \mu_\zeta} \dots (\bar{\varepsilon} \gamma_{a'})^{\lambda'_\zeta \mu'_\zeta} \dots \varepsilon_{ab \dots \tau_\zeta}(\vec{p}) \varepsilon_{a' b' \dots \tau'_\zeta}^+(\vec{p}) \\ V_{\lambda_\zeta \mu_\zeta \xi_\zeta \dots \tau_\zeta}^{2n+1}(\vec{p}) V_{\lambda'_\zeta \mu'_\zeta \xi'_\zeta \dots \tau'_\zeta}^{2n+1}(\vec{p}) = \frac{1}{2^{2n}} (\gamma_a \varepsilon)^{\lambda_\zeta \mu_\zeta} \dots (\bar{\varepsilon} \gamma_{a'})^{\lambda'_\zeta \mu'_\zeta} \dots \tilde{\varepsilon}_{ab \dots \tau_\zeta}(\vec{p}) \tilde{\varepsilon}_{a' b' \dots \tau'_\zeta}^+(\vec{p}) \end{array} \right.$$

推论1.5.8.

$$\left\{ \begin{aligned} \{A_{ab \dots \tau_\zeta}(x), A_{a'b' \dots \tau'_\zeta}^+(x')\} &= \frac{1}{m^{2n} 2^n} \overbrace{(\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} \dots}^n \overbrace{(\gamma_{a'} \varepsilon)^{\lambda'_\zeta \mu'_\zeta} \dots}^n \{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(x')\} \\ \{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(x')\} &= \frac{m^{2n}}{2^n} \overbrace{(\gamma_a \varepsilon)^{\lambda_\zeta \mu_\zeta} \dots}^n \overbrace{(\bar{\varepsilon} \gamma_{a'})^{\lambda'_\zeta \mu'_\zeta} \dots}^n \{A_{ab \dots \tau_\zeta}(x), A_{a'b' \dots \tau'_\zeta}^+(x')\} \end{aligned} \right.$$

### 1.5.4 三维时空中有质量费米子准投影算子的等价表述

推论1.5.9.

$$\left\{ \begin{aligned} U_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}) U_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(\vec{p}) &= \frac{1}{2^{2n+1} m} \underbrace{[(\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma^{a'})_{\lambda'_\zeta \mu'_\zeta} (\eta_{aa'} + \frac{p_a p_{a'}}{m^2} - \frac{\varepsilon \varepsilon_{acd} \eta_{a'}^c p^d}{m})]}_n \dots [(m - i\gamma^c p_c) \gamma^0]_{\tau_\zeta \tau'_\zeta} \\ V_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(\vec{p}) V_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(\vec{p}) &= \frac{1}{2^{2n+1} m} \underbrace{[(\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma^{a'})_{\lambda'_\zeta \mu'_\zeta} (\eta_{aa'} + \frac{p_a p_{a'}}{m^2} + \frac{\varepsilon \varepsilon_{acd} \eta_{a'}^c p^d}{m})]}_n \dots [(-m - i\gamma^c p_c) \gamma^0]_{\tau_\zeta \tau'_\zeta} \end{aligned} \right.$$

### 1.5.5 三维时空中有质量费米子协变对易规则

$$\text{定理1.5.2. } \{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(x')\} = \frac{i}{2^{2n}} \underbrace{[(m - \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta \lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots \Delta(x - x')}_n$$

⟨⟩

$$\text{定理1.5.3. } \{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(x')\} = i \frac{(i\zeta)^{2n+1}}{2^{2n}} \underbrace{[-im\sigma_z + (\sigma, i\zeta)^a \partial_a]_{\lambda_\zeta \lambda'_\zeta} [-im\sigma_z + (\sigma, i\zeta)^b \partial_b]_{\mu_\zeta \mu'_\zeta} \dots \Delta(x - x')}_n$$

⟨⟩

$$\begin{aligned} \text{定理1.5.4. } & \{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(x')\} \\ &= \frac{i}{2^{2n} m} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x) \dots \mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x')}_n \underbrace{(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2} + \frac{i\zeta \varepsilon_{acd} \eta_{a'}^c \partial^d}{m})}_n \cdot [(m - \gamma^c \partial_c) \gamma^0]_{\tau_\zeta \tau'_\zeta} \Delta(x - x') \end{aligned}$$

⟨⟩

$$\text{定理1.5.5. } \{A_{ab \dots \tau_\zeta}(x), A_{a'b' \dots \tau'_\zeta}^+(x')\} = \frac{i}{2^n} \underbrace{(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2} + \frac{i\zeta \varepsilon_{acd} \eta_{a'}^c \partial^d}{m})}_n \cdot [(m - \gamma^c \partial_c) \gamma^0]_{\tau_\zeta \tau'_\zeta} \Delta(x - x')$$

自我评述: 三维时空中确实存在与B-W方程等价的类C-K或R-S方程, 且形式上比四维的更简单明晰。

## 1.6 三维时空中s-自旋方程

$$\text{定理1.6.1. } [\gamma^a \partial_a + m] \psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x) \text{ 全对称}, \gamma_a := [-\sigma_y, \sigma_x, \zeta \sigma_z]$$

$$\Leftrightarrow [s \partial_a + m \gamma_a(s) + i S_{ab}(s, \zeta) \partial^b] \psi(s) = 0, S_{ab}(s, \zeta) = -i[\gamma_a(s), \gamma_b(s)], \gamma_a(s) := [-\sigma_y(s), \sigma_x(s), \zeta \sigma_z(s)]$$

证明:  $[\gamma^a \partial_a + m] \psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x)$  全对称

$$\Leftrightarrow [\gamma^a \partial_a + m] \hat{\psi}(s) = 0$$

$$\Leftrightarrow (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a D^a \hat{\psi}(s) = im\sigma_z \otimes I_{2^{2s-1}} \hat{\psi}(s), D^a = (\partial^x, \partial^y, 0, \partial^\pi)$$

$$\Leftrightarrow (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a D^a [I \otimes \Gamma(s)] N(s) \psi(s) = im\sigma_z \otimes I_{2^{2s-1}} \hat{\psi}(s)$$

$$\Leftrightarrow [I \otimes \Gamma(s)] (\sigma \otimes I_{2s}, -i\zeta)_a D^a N(s) \psi(s) = im\sigma_z \otimes I_{2^{2s-1}} \hat{\psi}(s)$$

$$\Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)_a D^a N(s) \psi(s) = im[I \otimes \bar{\Gamma}(s)] (\sigma_z \otimes I_{2^{2s-1}}) \hat{\psi}(s)$$

$$\Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)_a D^a N(s) \psi(s) = im(\sigma_z \otimes I_{2s}) [I \otimes \bar{\Gamma}(s)] \hat{\psi}(s)$$

$$\Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)_a D^a N(s) \psi(s) = im(\sigma_z \otimes I_{2s}) N(s) \psi(s)$$

$$\Leftrightarrow Z_b D^b \psi(s) = im \frac{i\zeta}{\sqrt{2}} (\sigma_z \otimes I_{2s}) N(s) \psi(s)$$

$$\Leftrightarrow \bar{Z}_a Z_b D^b \psi(s) = im \frac{i\zeta}{\sqrt{2}} \bar{Z}_a (\sigma_z \otimes I_{2s}) N(s) \psi(s)$$

$$\Leftrightarrow \bar{Z}_a Z_b D^b \psi(s) = im \frac{i\varsigma}{\sqrt{2}} \frac{-i\varsigma}{\sqrt{2}} \bar{N}(s) (\sigma \otimes I_{2s}, i\varsigma)_a (\sigma_z \otimes I_{2s}) N(s) \psi(s)$$

$$\Leftrightarrow \bar{Z}_a Z_b D^b \psi(s) = -\frac{m}{2s} \bar{N}(s) [(-\sigma_y, \sigma_x, -i) \otimes I_{2s}, \varsigma \sigma_z \otimes I_{2s}]_a N(s) \psi(s)$$

$$\Leftrightarrow [sD_a + iS_{ab}(s, \varsigma; 4)D^b] \psi(s) = -m [(-\sigma_y(s), \sigma_x(s), -is), \varsigma \sigma_z(s)]_a \psi(s)$$

$$S_{ab}(s, \varsigma; 4) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -\varsigma \sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -\varsigma \sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -\varsigma \sigma_z(s) \\ \varsigma \sigma_x(s) & \varsigma \sigma_y(s) & \varsigma \sigma_z(s) & 0 \end{bmatrix}$$

$$\Leftrightarrow [s\partial_a + iS_{ab}(s, \varsigma)\partial^b] \psi(s) = -m\gamma_a(s)\psi(s), S_{ab}(s, \varsigma) = -i[\gamma_a(s), \gamma_b(s)] \succ \begin{bmatrix} 0 & \sigma_z(s) & -\varsigma \sigma_x(s) \\ -\sigma_z(s) & 0 & -\varsigma \sigma_y(s) \\ \varsigma \sigma_x(s) & \varsigma \sigma_y(s) & 0 \end{bmatrix}$$

$$\Leftrightarrow [s\partial_a + m\gamma_a(s) + iS_{ab}(s, \varsigma)\partial^b] \psi(s) = 0$$

□

$$\text{引理1.6.1. } \gamma_a(s) = [e^\theta]_a^b e^{\frac{1}{2}\theta^{ab}[\gamma_a(s), \gamma_b(s)]} \gamma_b(s) e^{-\frac{1}{2}\theta^{ab}[\gamma_a(s), \gamma_b(s)]} = [e^{i\omega R_z + \epsilon \cdot L}]_a^b e^{i\omega \sigma_z(s) + \varsigma \epsilon \cdot \sigma(s)} \gamma_b(s) e^{-i\omega \sigma_z(s) - \varsigma \epsilon \cdot \sigma(s)}$$

定理1.6.2.

$$\begin{cases} [\gamma^a \partial_a + m] \psi_{\underbrace{[\lambda_\varsigma] \mu_\varsigma \eta_\varsigma \xi_\varsigma}_{2s} \dots}(x) = 0, \psi_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma}_{2s} \dots}(x) \text{ 全对称} \\ \psi_{k_\varsigma}(x) := \Gamma_{k_\varsigma}^{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \dots} \psi_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma}_{2s} \dots}(x) \end{cases} \Rightarrow \begin{cases} [\gamma^a(s) \partial_a + sm] \psi_{[k_\varsigma]}(x) = 0 \\ \psi_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma}_{2s} \dots}(x) = \Gamma_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \dots}^{k_\varsigma} \psi_{k_\varsigma}(x) \end{cases}$$

### 1.7 三维时空中的Bargmann-Wigner方程<sup>[17]</sup>等价于Penrose方程<sup>[1, 2]</sup>

定理1.7.1.

$$\begin{cases} [\gamma^a \partial_a + m] \psi_{\underbrace{[\lambda_\varsigma] \mu_\varsigma \eta_\varsigma \xi_\varsigma}_{2s} \dots}(x) = 0, \psi_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma}_{2s} \dots}(x) \text{ 全对称} \\ \psi_{k_\varsigma}(x) := \Gamma_{k_\varsigma}^{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \dots} \psi_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma}_{2s} \dots}(x) \end{cases} \Leftrightarrow \begin{cases} [s\partial_a + iS_{ab}(s, \varsigma)\partial^b] \psi_{[k_\varsigma]}(x) = -m\gamma_a(s) \psi_{[k_\varsigma]}(x) \\ \psi_{\underbrace{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma}_{2s} \dots}(x) = \Gamma_{\lambda_\varsigma \mu_\varsigma \eta_\varsigma \xi_\varsigma \dots}^{k_\varsigma} \psi_{k_\varsigma}(x) \end{cases}$$

$$\psi_{\underbrace{\lambda_\varsigma \mu_\varsigma}_{2s} \dots}(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{n-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} [a(\vec{p}, h) U_{\underbrace{\lambda_\varsigma \mu_\varsigma}_{2s} \dots}(\vec{p}) e^{i(\vec{p} \cdot \vec{r} - Et)} + b^+(\vec{p}, h) V_{\underbrace{\lambda_\varsigma \mu_\varsigma}_{2s} \dots}(\vec{p}) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}$$

$$\psi_{k_\varsigma}(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{n-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} [a(\vec{p}, h) U_{k_\varsigma}(\vec{p}; s) e^{i(\vec{p} \cdot \vec{r} - Et)} + b^+(\vec{p}, h) V_{k_\varsigma}(\vec{p}; s) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}$$

$$\begin{cases} U_{k_\varsigma}(\vec{p}; s) := \Gamma_{k_\varsigma}^{\lambda_\varsigma \mu_\varsigma \dots} U_{\underbrace{\lambda_\varsigma \mu_\varsigma}_{2s} \dots}(\vec{p}) \Leftrightarrow U_{\underbrace{\lambda_\varsigma \mu_\varsigma}_{2s} \dots}(\vec{p}) = \Gamma_{\lambda_\varsigma \mu_\varsigma \dots}^{k_\varsigma} U_{k_\varsigma}(\vec{p}; s) \\ V_{k_\varsigma}(\vec{p}; s) := \Gamma_{k_\varsigma}^{\lambda_\varsigma \mu_\varsigma \dots} V_{\underbrace{\lambda_\varsigma \mu_\varsigma}_{2s} \dots}(\vec{p}) \Leftrightarrow V_{\underbrace{\lambda_\varsigma \mu_\varsigma}_{2s} \dots}(\vec{p}) = \Gamma_{\lambda_\varsigma \mu_\varsigma \dots}^{k_\varsigma} V_{k_\varsigma}(\vec{p}; s) \end{cases}$$

$$\text{定理1.7.2. } (\gamma^a \partial_a + m)\psi(x) = 0, \gamma^a = (-\sigma_y, \sigma_x, \varsigma \sigma_z) \Leftrightarrow (\sigma, -i\varsigma)^a \partial_a \psi(x) = im\sigma_z \psi(x), \sigma = (\sigma_x, \sigma_y)$$

$$\text{定理1.7.3. } (\gamma^a \partial_a + m)\psi(x) = M\sigma_x \psi^*(x), \gamma^a = (-\sigma_y, \sigma_x, \varsigma \sigma_z) \Leftrightarrow (\sigma, -i\varsigma)^a \partial_a \psi(x) = im\sigma_z \psi(x) + M\sigma_y \psi^*(x)$$

### 1.8 四维时空中z轴受限的Majorana方程等价于三维时空中的Penrose方程<sup>[1, 2]</sup>

推论1.8.1.

$$\begin{cases} (\sigma, -i\varsigma)_a \partial^a \nu(x) - me^{-2i\theta} \sigma_y \nu^*(x) = 0 \\ \psi(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} \nu(x) - ie^{-2i\theta} \sigma_y \nu^*(x) \\ -\nu(x) - ie^{-2i\theta} \sigma_y \nu^*(x) \end{bmatrix} \end{cases} \Leftrightarrow \begin{cases} (\gamma^a \partial_a + m)\psi(x) = 0, \gamma^a = (\sigma \otimes \sigma_y, \varsigma I \otimes \sigma_z) \\ \psi^*(x) = -e^{2i\theta} \sigma_y \otimes \sigma_y \psi(x) \\ \nu(x) = \frac{1}{\sqrt{2}} [\psi_1(x) + ie^{-2i\theta} \sigma_y \psi_1^*(x)] \end{cases}$$

$$\begin{cases} \nu(x) = \frac{1}{(2\pi)^{N/2}} \int \frac{E+m-\varsigma \vec{p} \cdot \sigma}{\sqrt{2m(E+m)}} \frac{1}{\sqrt{2}} (\xi_0 e^{i\varsigma p \cdot x} + ie^{-2i\theta} \sigma_y \xi_0^* e^{-i\varsigma p \cdot x}) d^N \vec{p} \\ \psi(x) = \frac{1}{(2\pi)^{N/2}} \int \frac{E+m+\varsigma \vec{p} \cdot \sigma \otimes \sigma_x}{\sqrt{2m(E+m)}} \begin{bmatrix} \xi_0 e^{i\varsigma p \cdot x} \\ -ie^{-2i\theta} \sigma_y \xi_0^* e^{-i\varsigma p \cdot x} \end{bmatrix} d^3 \vec{p} = \frac{1}{(2\pi)^{N/2}} \int \begin{bmatrix} \frac{(E+m)\xi_0 e^{i\varsigma p \cdot x} - \varsigma \vec{p} \cdot \sigma (ie^{-2i\theta} \sigma_y \xi_0^*) e^{-i\varsigma p \cdot x}}{\sqrt{2m(E+m)}} \\ \frac{-(E+m)(ie^{-2i\theta} \sigma_y \xi_0^*) e^{-i\varsigma p \cdot x} + \varsigma \vec{p} \cdot \sigma \xi_0 e^{i\varsigma p \cdot x}}{\sqrt{2m(E+m)}} \end{bmatrix} d^N \vec{p} \end{cases}$$

$$\xi_0 = a(\vec{p}, \frac{1}{2}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a(\vec{p}, -\frac{1}{2}) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ 或 } \xi_0 = a(\vec{p}, -\frac{\varsigma}{2}) \lambda(\hat{p}, -\frac{\varsigma}{2}) + a(\vec{p}, \frac{\varsigma}{2}) \lambda(\hat{p}, \frac{\varsigma}{2})$$

## 2 三维时空中广义Bargmann-Wigner方程

### 2.1 三维时空中广义的Bargmann-Wigner方程

推论2.1.1.

$$(\sigma, -i\varsigma)_a \partial^a \nu(x) - im\sigma_z \nu(x) - Me^{-2i\theta} \sigma_y \nu^*(x) = 0 \Leftrightarrow (\gamma^a \partial_a + m)\nu(x) = M\sigma_x e^{-2i\theta} \nu^*(x), \gamma^a = (-\sigma_y, \sigma_x, \varsigma\sigma_z)$$

$$\begin{cases} [i\varsigma\gamma^a(\vec{p}, iE_+)^a + (m+M)]\xi_+(\vec{p}) = 0, [-i\varsigma\gamma^a(\vec{p}, iE_+)^a + (m+M)]\eta_+(\vec{p}) = 0 \\ [i\varsigma\gamma^a(\vec{p}, iE_-)^a + (m-M)]\xi_-(\vec{p}) = 0, [-i\varsigma\gamma^a(\vec{p}, iE_-)^a + (m-M)]\eta_-(\vec{p}) = 0 \\ \nu(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \{ \xi_+(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} - e^{2i\theta} \sigma_x \xi_+^*(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} + \xi_-(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} + e^{2i\theta} \sigma_x \xi_-^*(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} \} \end{cases}$$

证明:

$$(\sigma, -i\varsigma)_a \partial^a \nu(x) - im\sigma_z \nu(x) - Me^{-2i\theta} \sigma_y \nu^*(x) = 0 \Leftrightarrow (\gamma^a \partial_a + m)\nu(x) = M\sigma_x e^{-2i\theta} \nu^*(x), \gamma^a = (-\sigma_y, \sigma_x, \varsigma\sigma_z)$$

$$\Rightarrow [\partial^a \partial_a - (m+M)^2][\partial^a \partial_a - (m-M)^2]\nu(x) = 0, E_+ = \sqrt{\vec{p}^2 + (m+M)^2}, E_- = \sqrt{\vec{p}^2 + (m-M)^2}$$

$$\nu(x) = \frac{1}{(2\pi)^{3/2}} \int d^3 \vec{p} \{ \xi_+(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} + \eta_+(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} + \xi_-(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} + \eta_-(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} \}$$

$\Leftrightarrow$

$$\begin{aligned} & i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a \{ \xi_+(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} - \eta_+(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} \} \\ & + i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a \{ \xi_-(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} - \eta_-(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} \} \\ & - im\sigma_z \end{aligned}$$

$$\begin{aligned} & \{ \xi_+(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} + \eta_+(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} \\ & + \xi_-(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} + \eta_-(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} \} \\ & - Me^{-2i\theta} \sigma_y \end{aligned}$$

$$\begin{aligned} & \{ \xi_+^*(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} + \eta_+^*(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} \\ & + \xi_-^*(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} + \eta_-^*(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} \} = 0 \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} & \{ [i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a - im\sigma_z] \xi_+(\vec{p}) - Me^{-2i\theta} \sigma_y \eta_+^*(\vec{p}) \} e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} \\ & + \{ [-i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a - im\sigma_z] \eta_+(\vec{p}) - Me^{-2i\theta} \sigma_y \xi_+^*(\vec{p}) \} e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} \\ & + \{ [i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a - im\sigma_z] \xi_-(\vec{p}) - Me^{-2i\theta} \sigma_y \eta_-^*(\vec{p}) \} e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} \\ & + \{ [-i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a - im\sigma_z] \eta_-(\vec{p}) - Me^{-2i\theta} \sigma_y \xi_-^*(\vec{p}) \} e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} = 0 \end{aligned}$$

$$\Leftrightarrow (M \neq 0, m \neq 0)$$

$$\begin{cases} [i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a - im\sigma_z] \xi_+(\vec{p}) - Me^{-2i\theta} \sigma_y \eta_+^*(\vec{p}) = 0 \\ [-i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a - im\sigma_z] \eta_+(\vec{p}) - Me^{-2i\theta} \sigma_y \xi_+^*(\vec{p}) = 0 \\ [i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a - im\sigma_z] \xi_-(\vec{p}) - Me^{-2i\theta} \sigma_y \eta_-^*(\vec{p}) = 0 \\ [-i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a - im\sigma_z] \eta_-(\vec{p}) - Me^{-2i\theta} \sigma_y \xi_-^*(\vec{p}) = 0 \end{cases}$$

$$\Leftrightarrow (M \neq 0, m \neq 0)$$

$$\begin{cases} M\eta_+(\vec{p}) = -e^{-2i\theta} \sigma_y [-i\varsigma(\sigma^*, -i\varsigma)_a(\vec{p}, iE_+)^a + im\sigma_z] \xi_+^*(\vec{p}) \\ [-i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a - im\sigma_z] \eta_+(\vec{p}) - Me^{-2i\theta} \sigma_y \xi_+^*(\vec{p}) = 0 \\ Me^{-2i\theta} \sigma_y \eta_-^*(\vec{p}) = -e^{-2i\theta} \sigma_y [-i\varsigma(\sigma^*, -i\varsigma)_a(\vec{p}, iE_-)^a + im\sigma_z] \xi_-^*(\vec{p}) \\ [-i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a - im\sigma_z] \eta_-(\vec{p}) - Me^{-2i\theta} \sigma_y \xi_-^*(\vec{p}) = 0 \end{cases}$$

$$\Leftrightarrow (M \neq 0, m \neq 0)$$

$$\begin{cases}
(\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a \xi_+(\vec{p}) = \varsigma\sigma_z(m+M)\xi_+(\vec{p}), (\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a \eta_+(\vec{p}) = -\varsigma\sigma_z(m+M)\eta_+(\vec{p}) \\
M\eta_+^*(\vec{p}) = e^{2i\theta}\sigma_y[i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a - im\sigma_z]\xi_+(\vec{p}) \\
(\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a \xi_-(\vec{p}) = \varsigma\sigma_z(m-M)\xi_-(\vec{p}), (\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a \eta_-(\vec{p}) = -\varsigma\sigma_z(m-M)\eta_-(\vec{p}) \\
M\eta_-^*(\vec{p}) = e^{2i\theta}\sigma_y[i\varsigma(\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a - im\sigma_z]\xi_-(\vec{p}) \\
\Leftrightarrow (M \neq 0, m \neq 0) \\
(\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a \xi_+(\vec{p}) = \varsigma\sigma_z(m+M)\xi_+(\vec{p}), (\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a \eta_+(\vec{p}) = -\varsigma\sigma_z(m+M)\eta_+(\vec{p}) \\
\eta_+(\vec{p}) = -e^{-2i\theta}\sigma_x\xi_+^*(\vec{p}) \\
(\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a \xi_-(\vec{p}) = \varsigma\sigma_z(m-M)\xi_-(\vec{p}), (\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a \eta_-(\vec{p}) = -\varsigma\sigma_z(m-M)\eta_-(\vec{p}) \\
\eta_-(\vec{p}) = e^{-2i\theta}\sigma_x\xi_-^*(\vec{p}) \\
\Leftrightarrow \\
(\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a \xi_+(\vec{p}) = \varsigma\sigma_z(m+M)\xi_+(\vec{p}), (\sigma, -i\varsigma)_a(\vec{p}, iE_+)^a \eta_+(\vec{p}) = -\varsigma\sigma_z(m+M)\eta_+(\vec{p}) \\
(\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a \xi_-(\vec{p}) = \varsigma\sigma_z(m-M)\xi_-(\vec{p}), (\sigma, -i\varsigma)_a(\vec{p}, iE_-)^a \eta_-(\vec{p}) = -\varsigma\sigma_z(m-M)\eta_-(\vec{p}) \\
\nu(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \{ \xi_+(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} - e^{-2i\theta}\sigma_x\xi_+^*(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} \\
+ \xi_-(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} + e^{-2i\theta}\sigma_x\xi_-^*(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} \} \\
\Leftrightarrow \\
[i\varsigma\gamma^a(\vec{p}, iE_+)^a + (m+M)]\xi_+(\vec{p}) = 0, [-i\varsigma\gamma^a(\vec{p}, iE_+)^a + (m+M)]\eta_+(\vec{p}) = 0 \\
[i\varsigma\gamma^a(\vec{p}, iE_-)^a + (m-M)]\xi_-(\vec{p}) = 0, [-i\varsigma\gamma^a(\vec{p}, iE_-)^a + (m-M)]\eta_-(\vec{p}) = 0 \\
\nu(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \{ \xi_+(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} - e^{-2i\theta}\sigma_x\xi_+^*(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_+t]} \\
+ \xi_-(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} + e^{-2i\theta}\sigma_x\xi_-^*(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_-t]} \}
\end{cases}$$

□

## 2.2 三维时空中的广义Majorana B-W方程

推论2.2.1.

$$(\sigma, -i\varsigma)_a \partial^a \nu(x) - i(m \pm M)\sigma_z \nu(x) = 0 \Leftrightarrow (\gamma^a \partial_a + m \pm M)\nu(x) = 0, \gamma^a = (-\sigma_y, \sigma_x, \varsigma\sigma_z)$$

$$\nu(x) = i\sigma_x \nu^*(x), e^{-2i\theta} = -\pm i$$

$$\begin{cases}
[i\varsigma\gamma^a(\vec{p}, iE_\pm)^a + (m \pm M)]\xi_\pm(\vec{p}) = 0, [-i\varsigma\gamma^a(\vec{p}, iE_\pm)^a + (m \pm M)]\sigma_x \xi_\pm^*(\vec{p}) = 0 \\
\nu(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \{ \xi_\pm(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-E_\pm t]} + i\sigma_x \xi_\pm^*(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-E_\pm t]} \}
\end{cases}$$

推论2.2.2.

$$(\sigma, -i\varsigma)_a \partial^a \nu(x) - im\sigma_z \nu(x) = 0 \Leftrightarrow (\gamma^a \partial_a + m)\nu(x) = 0, \gamma^a = (-\sigma_y, \sigma_x, \varsigma\sigma_z)$$

$$\nu(x) = i\sigma_x \nu^*(x), e^{-2i\theta} = -i$$

$$\begin{cases}
[i\varsigma\gamma^a(\vec{p}, iE)^a + m]\xi(\vec{p}) = 0, [-i\varsigma\gamma^a(\vec{p}, iE)^a + m]\sigma_x \xi^*(\vec{p}) = 0 \\
\nu(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \{ \xi(\vec{p}) e^{i\varsigma[\vec{p}\cdot\vec{r}-Et]} + i\sigma_x \xi^*(\vec{p}) e^{-i\varsigma[\vec{p}\cdot\vec{r}-Et]} \}
\end{cases}$$

## 2.3 三维时空中实表象的广义Bargmann-Wigner方程

推论2.3.1.

$$S_{xy} S_c(\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}, [S_{xy} S_c(\frac{1}{2})]^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}$$

$$S_{xy} S_c(\frac{1}{2}) \sigma_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, [S_{xy} S_c(\frac{1}{2}) \sigma_x]^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

$$\sigma_x [S_{xy} S_c(\frac{1}{2})]^+ (-\sigma_y, \sigma_x, \varsigma\sigma_z) S_{xy} S_c(\frac{1}{2}) \sigma_x^+ = \sigma_x (\sigma_z, \sigma_x, \varsigma\sigma_y) \sigma_x^+ = (-\sigma_z, \sigma_x, -\varsigma\sigma_y)$$

推论2.3.2.

$$(\sigma, -i\varsigma)_a \partial^a \nu(x) + im\sigma_y \nu(x) - Me^{-2i\theta} \sigma_y \nu^*(x) = 0 \Leftrightarrow (\gamma^a \partial_a + m)\nu(x) = -iMe^{-2i\theta} \nu^*(x), \gamma^a = (-\sigma_z, \sigma_x, -\varsigma\sigma_y)_{xz\pi}$$

$$\begin{cases} [i\zeta\gamma^a(\vec{p}, iE_+)^a + (m+M)]\xi_+(\vec{p}) = 0, [-i\zeta\gamma^a(\vec{p}, iE_+)^a + (m+M)]\eta_+(\vec{p}) = 0 \\ [i\zeta\gamma^a(\vec{p}, iE_-)^a + (m-M)]\xi_-(\vec{p}) = 0, [-i\zeta\gamma^a(\vec{p}, iE_-)^a + (m-M)]\eta_-(\vec{p}) = 0 \\ \nu(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \{ \xi_+(\vec{p}) e^{i\zeta[\vec{p}\cdot\vec{r}-E_+t]} + i e^{-2i\theta} \xi_+^*(\vec{p}) e^{-i\zeta[\vec{p}\cdot\vec{r}-E_+t]} + \xi_-(\vec{p}) e^{i\zeta[\vec{p}\cdot\vec{r}-E_-t]} - i e^{-2i\theta} \xi_-^*(\vec{p}) e^{-i\zeta[\vec{p}\cdot\vec{r}-E_-t]} \} \end{cases}$$

## 2.4 三维时空中实表象的广义Majorana B-W方程

推论2.4.1.

$$\begin{aligned} (\sigma, -i\zeta)_a \partial^a \nu(x) + i(m \pm M) \sigma_y \nu(x) = 0 &\Leftrightarrow (\gamma^a \partial_a + m \pm M) \nu(x) = 0, \gamma^a = (-\sigma_z, \sigma_x, -\zeta \sigma_y), \nu^*(x) = \nu(x) \\ \begin{cases} [i\zeta\gamma^a(\vec{p}, iE_\pm)^a + (m \pm M)]\xi_\pm(\vec{p}) = 0, [-i\zeta\gamma^a(\vec{p}, iE_\pm)^a + (m \pm M)]\xi_\pm^*(\vec{p}) = 0 \\ \nu(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \{ \xi_\pm(\vec{p}) e^{i\zeta[\vec{p}\cdot\vec{r}-E_\pm t]} + \xi_\pm^*(\vec{p}) e^{-i\zeta[\vec{p}\cdot\vec{r}-E_\pm t]} \} \end{cases} \end{aligned}$$

推论2.4.2.

$$\begin{aligned} (\sigma, -i\zeta)_a \partial^a \nu(x) + im \sigma_y \nu(x) = 0 &\Leftrightarrow (\gamma^a \partial_a + m) \nu(x) = 0, \gamma^a = (-\sigma_z, \sigma_x, -\zeta \sigma_y), \nu^*(x) = \nu(x) \\ \begin{cases} [i\zeta\gamma^a(\vec{p}, iE)^a + m]\xi(\vec{p}) = 0, [-i\zeta\gamma^a(\vec{p}, iE)^a + m]\xi^*(\vec{p}) = 0 \\ \nu(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \{ \xi(\vec{p}) e^{i\zeta[\vec{p}\cdot\vec{r}-Et]} + \xi^*(\vec{p}) e^{-i\zeta[\vec{p}\cdot\vec{r}-Et]} \} \end{cases} \end{aligned}$$

## 3 三维时空中直观表象下的Bargmann-Wigner方程

### 3.1 三维时空中直观表象下的Dirac方程

$$\text{证明: } D_{\vec{v}} = e^{-ln[\gamma_v(1+v)]\hat{v}\cdot(\frac{i}{2}\vec{\gamma}\gamma_0)} = \frac{1+\gamma_v-i\gamma_v\vec{v}\cdot\vec{\gamma}\gamma_0}{\sqrt{2(\gamma_v+1)}} = \frac{E+m-i\vec{p}\cdot\vec{\gamma}\gamma_0}{\sqrt{2m(E+m)}} = \frac{m-i\gamma^a p_a \gamma_0}{\sqrt{2m(E+m)}} \quad \square$$

$$\text{定义3.1.1. } (\gamma^a \partial_a + m)\psi = 0, \gamma^a = (1 \otimes \sigma_x, 1 \otimes \sigma_y, \zeta 1 \otimes \sigma_z)$$

$$\text{定义3.1.2. } u(\vec{p}) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\zeta\sigma\cdot\vec{p}\sigma_z}{E+m}\right) \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2}\sigma_x\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v(\vec{p}) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\zeta\sigma\cdot\vec{p}\sigma_z}{E+m}\right) \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2}\sigma_x\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{推论3.1.1. } u(\vec{p}) = \sigma_x v^*(\vec{p}), v(\vec{p}) = \sigma_x u^*(\vec{p})$$

$$\text{推论3.1.2. } S_{xy}(\sigma_x, \sigma_y, \sigma_z) S_{xy}^+ = (-\sigma_y, \sigma_x, \sigma_z), S_{xy} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}, S_{xy}^+ = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

### 3.2 三维时空中直观表象下的类Klein-Gordon方程自旋基 $\varepsilon_a(\vec{p})$ , $\tilde{\varepsilon}_a(\vec{p})$ 及其性质

$$\text{定理3.2.1. } u(\vec{p}) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\zeta\sigma\cdot\vec{p}\sigma_z}{E+m}\right) \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2}\sigma_x\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \gamma^a = (\sigma_x, \sigma_y, \zeta\sigma_z)$$

$$\Rightarrow \varepsilon_a(\vec{p}) = -i(\tilde{\varepsilon}\gamma_a)^{\lambda\zeta\mu\zeta} U_{\lambda\zeta\mu\zeta}(\vec{p}) = -iu^T(\vec{p})(\tilde{\varepsilon}\gamma_a)u(\vec{p})$$

$$= (i\zeta + \frac{i\zeta p_x(p_x+i\zeta p_y)}{m(E+m)}, -1 + \frac{i\zeta p_y(p_x+i\zeta p_y)}{m(E+m)}, -\zeta \frac{p_x+i\zeta p_y}{m}) = i\zeta \left(1 + \frac{p_x(p_x+i\zeta p_y)}{m(E+m)}\right), i\zeta + \frac{p_y(p_x+i\zeta p_y)}{m(E+m)}, i \frac{p_x+i\zeta p_y}{m}$$

$$\text{定理3.2.2. } v(\vec{p}) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\zeta\sigma\cdot\vec{p}\sigma_z}{E+m}\right) \left(\frac{1+\zeta}{2} + \frac{1-\zeta}{2}\sigma_x\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \gamma^a = (\sigma_x, \sigma_y, \zeta\sigma_z)$$

$$\Rightarrow \tilde{\varepsilon}_a(\vec{p}) = -i(\tilde{\varepsilon}\gamma_a)^{\lambda\zeta\mu\zeta} V_{\lambda\zeta\mu\zeta}(\vec{p})$$

$$= (-i\zeta - \frac{i\zeta p_x(p_x-i\zeta p_y)}{m(E+m)}, -1 - \frac{i\zeta p_y(p_x-i\zeta p_y)}{m(E+m)}, \zeta \frac{p_x-i\zeta p_y}{m}) = -i\zeta \left(1 + \frac{p_x(p_x-i\zeta p_y)}{m(E+m)}\right), -i\zeta + \frac{p_y(p_x-i\zeta p_y)}{m(E+m)}, i \frac{p_x-i\zeta p_y}{m}$$

$$\text{推论3.2.1. } \tilde{\varepsilon}_a(\vec{p}) = \varepsilon_{a'}^+(\vec{p}) \eta_{a'}^a, \tilde{\varepsilon}_{ab}(\vec{p}) = \varepsilon_{a'b'}^+(\vec{p}) \eta_{a'}^a \eta_b^{b'}$$

### 3.3 三维时空中直观表象下s-自旋方程

$$\text{定理3.3.1. } [\gamma^a \partial_a + m] \psi_{\underbrace{[\lambda_\zeta] \mu_\zeta}_{2s}} \dots(x) = 0, \psi_{\underbrace{[\lambda_\zeta] \mu_\zeta}_{2s}} \dots(x) \text{ 全对称}, \gamma_a := [\sigma_x, \sigma_y, \zeta\sigma_z]$$

$$\Leftrightarrow [s\partial_a + m\gamma_a(s) + iS_{ab}(s, \zeta)\partial^b] \psi(s) = \frac{i\zeta m}{s\sqrt{2}} \gamma_a(s) \psi(s), S_{ab}(s, \zeta) = -i[\gamma_a(s), \gamma_b(s)], \gamma_a(s) := [\sigma_x(s), \sigma_y(s), \zeta\sigma_z(s)]$$

$$\text{证明: } [\gamma^a \partial_a + m] \psi_{\underbrace{[\lambda_\zeta] \mu_\zeta}_{2s}} \dots(x) = 0, \psi_{\underbrace{[\lambda_\zeta] \mu_\zeta}_{2s}} \dots(x) \text{ 全对称}$$

$$\Leftrightarrow [\gamma^a \partial_a + m] \hat{\psi}(s) = 0$$

$$\begin{aligned}
&\Leftrightarrow (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a D^a \hat{\psi}(s) = -m\hat{\psi}(s), D^a = (\partial^x, \partial^y, \zeta \partial^\pi, 0) \\
&\Leftrightarrow (\sigma \otimes I_{2^{2s-1}}, -i\zeta)_a D^a [I \otimes \Gamma(s)] N(s) \psi(s) = -m\hat{\psi}(s) \\
&\Leftrightarrow [I \otimes \Gamma(s)] (\sigma \otimes I_{2s}, -i\zeta)_a D^a N(s) \psi(s) = -m\hat{\psi}(s) \\
&\Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)_a D^a N(s) \psi(s) = -m[I \otimes \bar{\Gamma}(s)] \hat{\psi}(s) \\
&\Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)_a D^a N(s) \psi(s) = -mN(s) \psi(s) \\
&\Leftrightarrow Z_b D^b \psi(s) = -m \frac{i\zeta}{\sqrt{2}} N(s) \psi(s) \\
&\Leftrightarrow \bar{Z}_a Z_b D^b \psi(s) = -m \frac{-i\zeta}{\sqrt{2}} \bar{Z}_a N(s) \psi(s) \\
&\Leftrightarrow \bar{Z}_a Z_b D^b \psi(s) = -\frac{m}{2} \bar{N}(s) (\sigma \otimes I_{2s}, i\zeta)_a N(s) \psi(s) \\
&\Leftrightarrow \bar{Z}_a Z_b D^b \psi(s) = -\frac{m}{2s} [\sigma(s), i\zeta]_a \psi(s) \\
&\Leftrightarrow [sD_a + iS_{ab}(s, \zeta; 4) D^b] \psi(s) = -m[\sigma(s), i\zeta]_a \psi(s), S_{ab}(s, \zeta; 4) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -\zeta \sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -\zeta \sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -\zeta \sigma_z(s) \\ \zeta \sigma_x(s) & \zeta \sigma_y(s) & \zeta \sigma_z(s) & 0 \end{bmatrix} \\
&\Leftrightarrow [s\partial_a + iS_{ab}(s, \zeta) \partial^b] \psi(s) = -m\gamma_a(s) \psi(s), S_{ab}(s, \zeta) = -i[\gamma_a(s), \gamma_b(s)] \succ \begin{bmatrix} 0 & \sigma_z(s) & -\zeta \sigma_y(s) \\ -\sigma_z(s) & 0 & \zeta \sigma_x(s) \\ \zeta \sigma_y(s) & -\zeta \sigma_x(s) & 0 \end{bmatrix} \\
&\Leftrightarrow [s\partial_a + m\gamma_a(s) + iS_{ab}(s, \zeta) \partial^b] \psi(s) = 0
\end{aligned}$$

□

## 4 三维时空中无质量粒子的Bargmann-Wigner方程(m只是参数)

### 4.1 三维时空中无质量粒子的Penrose方程

#### 4.1.1 三维时空中无质量粒子的螺旋度函数

定义4.1.1.  $\sigma(\frac{1}{2}) \cdot \hat{p} \lambda(\hat{p}, h) = h \lambda(\hat{p}, h), h = -\frac{1}{2}, \frac{1}{2}$

#### 4.1.2 三维时空中无质量粒子的Penrose方程<sup>[1, 2]</sup>及其螺旋度本征函数

定义4.1.2.  $\gamma^a \partial_a \psi(x) = 0, \gamma^a = (-\sigma_y, \sigma_x, \zeta \sigma_z) = S_{xy}(\sigma_x, \sigma_y, \zeta \sigma_z) S_{xy}^+ \Leftrightarrow (\sigma, -i\zeta)^a \partial_a \psi(x) = 0, \sigma = (\sigma_x, \sigma_y)$

证明:  $\gamma^a \partial_a \psi(x) = 0, \gamma^a = (-\sigma_y, \sigma_x, \zeta \sigma_z)$

$$\Leftrightarrow (-\sigma_y \partial_x + \sigma_x \partial_y) \psi(x) = -\zeta \sigma_z \partial_\pi \psi(x)$$

$$\Leftrightarrow (\sigma_x \partial_x + \sigma_y \partial_y) \psi(x) = i\zeta \partial_\pi \psi(x)$$

$$\Leftrightarrow (\sigma_y \partial_y + \sigma_x \partial_x) \psi(x) = i\zeta \partial_\pi \psi(x)$$

$$\Leftrightarrow (\sigma_y \partial_y + \sigma_x \partial_x) \psi(\vec{p}) e^{ip \cdot x} = i\zeta \partial_\pi \psi(\vec{p}) e^{ip \cdot x}$$

$$\Leftrightarrow (\sigma_x p_x + \sigma_y p_y) \psi(\vec{p}) e^{ip \cdot x} = i\zeta p_\pi \psi(\vec{p}) e^{ip \cdot x}$$

$$\Leftrightarrow (\sigma_x \hat{p}_x + \sigma_y \hat{p}_y) \lambda(\hat{p}, -\frac{\zeta}{2}) = -\zeta \lambda(\hat{p}, -\frac{\zeta}{2})$$

$$\Leftrightarrow (\sigma_x \hat{p}_x + \sigma_y \hat{p}_y) \lambda\left(\begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ 0 \end{bmatrix}, -\frac{\zeta}{2}\right) = -\zeta \lambda\left(\begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ 0 \end{bmatrix}, -\frac{\zeta}{2}\right)$$

□

$$\text{推论4.1.1. } \lambda(\hat{p}, \frac{1}{2}) = \lambda\left(\begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ 0 \end{bmatrix}, \frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \hat{p}_x + i\hat{p}_y \end{bmatrix}, \lambda(\hat{p}, -\frac{1}{2}) = \lambda\left(\begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ 0 \end{bmatrix}, -\frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{bmatrix} -(\hat{p}_x - i\hat{p}_y) \\ 1 \end{bmatrix}$$

$$\text{推论4.1.2. } \lambda(\hat{p}, \frac{1}{2}) \lambda^+(\hat{p}, \frac{1}{2}) = \begin{cases} -\frac{i}{2} \gamma^0 \gamma^a \hat{p}_a, \zeta = 1 \\ -\frac{i}{2} \gamma^a \hat{p}_a \gamma^0, \zeta = -1 \end{cases} \quad \lambda(\hat{p}, -\frac{1}{2}) \lambda^+(\hat{p}, -\frac{1}{2}) = \begin{cases} -\frac{i}{2} \gamma^a \hat{p}_a \gamma^0, \zeta = 1 \\ -\frac{i}{2} \gamma^0 \gamma^a \hat{p}_a, \zeta = -1 \end{cases}$$

$$\text{推论4.1.3. } \begin{cases} \lambda(\hat{p}, -\frac{\zeta}{2}) \lambda^+(\hat{p}, -\frac{\zeta}{2}) = -\frac{i}{2} \gamma^a \hat{p}_a \gamma^0 = -\frac{\zeta}{2} (\sigma, i\zeta)^a \hat{p}_a \\ \lambda(\hat{p}, \frac{\zeta}{2}) \lambda^+(\hat{p}, \frac{\zeta}{2}) = -\frac{i}{2} \gamma^0 \gamma^a \hat{p}_a = \frac{\zeta}{2} (\sigma, -i\zeta)^a \hat{p}_a \end{cases}$$

$$\text{推论4.1.4. } \sigma(\frac{1}{2}) \cdot \hat{p} \lambda(\hat{p}, h) = h \lambda(\hat{p}, h), h = -\frac{1}{2}, \frac{1}{2}$$

$$\text{推论4.1.5. } \lambda^+(\hat{p}, h) \lambda(\hat{p}, h') = \delta_{hh'}, \sum_{h=1/2}^{-1/2} \lambda(\hat{p}, h) \lambda^+(\hat{p}, h) = 1, \sum_{h=1/2}^{-1/2} h \lambda(\hat{p}, h) \lambda^+(\hat{p}, h) = \sigma(\frac{1}{2}) \cdot \hat{p}$$

### 4.1.3 三维时空中无质量粒子<sup>[1, 2]</sup>的平面波解及其自旋基

定理4.1.1.  $\gamma^a \partial_a \psi(x) = 0 \Leftrightarrow (\sigma, -i\zeta)_a \partial^a \psi(x) = 0$

$$\text{推论4.1.6.} \begin{cases} \psi(\vec{r}, t) := \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} \lambda(\hat{p}, -\frac{\zeta}{2}) [a_1(\vec{p}, -\frac{\zeta}{2}) e^{ip \cdot x} + a_2^\dagger(\vec{p}, -\frac{\zeta}{2}) e^{-ip \cdot x}] d^N \vec{p} \\ a_1(\vec{p}, -\frac{\zeta}{2}) = \frac{1}{(2\pi)^{N/2}} \int \lambda^+(\hat{p}, -\frac{\zeta}{2}) \psi(\vec{r}, t) e^{-ip \cdot x} d^N \vec{r} \\ a_2^\dagger(\vec{p}, -\frac{\zeta}{2}) = \frac{1}{(2\pi)^{N/2}} \int \lambda^+(\hat{p}, -\frac{\zeta}{2}) \psi(\vec{r}, t) e^{ip \cdot x} d^N \vec{r} \end{cases}$$

定理4.1.2.  $\gamma^a z_{\zeta}^{A_{\zeta}} \partial_a \psi_{\underbrace{A_{\zeta} B_{\zeta} \dots}_{2s}}(x) = 0 \Leftrightarrow (\sigma, -i\zeta)_a^{A_{\zeta} A_{\zeta}} \partial^a \psi_{\underbrace{A_{\zeta} B_{\zeta} \dots}_{2s}}(x) = 0$

推论4.1.7.

$$\begin{cases} \psi_{\underbrace{A_{\zeta} B_{\zeta} \dots}_{2s}}(x) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \underbrace{\lambda_{A_{\zeta}}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_{\zeta}}(\hat{p}, -\frac{\zeta}{2}) \dots}_{2s} \cdot [a_1(\vec{p}, -s\zeta) e^{ip \cdot x} + a_2^\dagger(\vec{p}, -s\zeta) e^{-ip \cdot x}] d^3 \vec{p} \\ |\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -s\zeta) = \frac{1}{(2\pi)^{3/2}} \int \underbrace{\lambda^{+A_{\zeta}}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_{\zeta}}(\hat{p}, -\frac{\zeta}{2}) \dots}_{2s} \psi_{\underbrace{A_{\zeta} B_{\zeta} \dots}_{2s}}(x) e^{-ip \cdot x} d^3 \vec{r} \\ |\vec{p}|^{(s-\frac{1}{2})} a_2^\dagger(\vec{p}, -s\zeta) = \frac{1}{(2\pi)^{3/2}} \int \underbrace{\lambda^{+A_{\zeta}}(\hat{p}, -\frac{\zeta}{2}) \lambda^{+B_{\zeta}}(\hat{p}, -\frac{\zeta}{2}) \dots}_{2s} \psi_{\underbrace{A_{\zeta} B_{\zeta} \dots}_{2s}}(x) e^{ip \cdot x} d^3 \vec{r} \end{cases}$$

定义4.1.3.  $\lambda_{\underbrace{A_{\zeta} B_{\zeta} \dots}_{2s}}(\hat{p}, -s\zeta) := \underbrace{\lambda_{A_{\zeta}}(\hat{p}, -\frac{\zeta}{2}) \lambda_{B_{\zeta}}(\hat{p}, -\frac{\zeta}{2}) \dots}_{2s}$

推论4.1.8.  $\lambda^{+\underbrace{A_{\zeta} B_{\zeta} \dots}_{2s}}(\hat{p}, -s\zeta) \lambda_{\underbrace{A_{\zeta} B_{\zeta} \dots}_{2s}}(\hat{p}, -s\zeta) = 1$

推论4.1.9.  $\lambda_{\underbrace{A_{\zeta} B_{\zeta} \dots}_{2s}}(\hat{p}, -s\zeta) \lambda_{\underbrace{A'_{\zeta} B'_{\zeta} \dots}_{2s}}^+(\hat{p}, -s\zeta) = \frac{1}{(2|\vec{p}|)^{2s}} \underbrace{[(-i\gamma^a p_a) \gamma^0]_{A_{\zeta} A'_{\zeta}} [(-i\gamma^b p_b) \gamma^0]_{B_{\zeta} B'_{\zeta}}}_{2s}$

推论4.1.10.  $\lambda_{\underbrace{A_{\zeta} B_{\zeta} \dots}_{2s}}(\hat{p}, -s\zeta) \lambda_{\underbrace{A'_{\zeta} B'_{\zeta} \dots}_{2s}}^+(\hat{p}, -s\zeta) = \frac{1}{(-s2|\vec{p}|)^{2s}} \underbrace{(\sigma, i\zeta)_{A_{\zeta} A'_{\zeta}}^a (\sigma, i\zeta)_{B_{\zeta} B'_{\zeta}}^b}_{2s} \dots \underbrace{p_a p_b}_{2s}$

## 4.2 三维时空中无质量粒子势方程的具体表述

### 4.2.1 三维时空中无质量自旋-1的B-W方程等价于类C-K方程<sup>[17, 19]</sup>

定理4.2.1.  $\gamma^a \partial_a \psi(x) = 0, \psi^T(x) = \psi(x), A_a = \frac{1}{\sqrt{2im}} \text{tr}(\bar{\varepsilon} \gamma_a \psi), \gamma^a = (-\sigma_y, \sigma_x, \zeta \sigma_z)$   
 $\Leftrightarrow \partial_a A_b - \partial_b A_a = 0, \partial^a A_a = 0, \psi = \frac{im}{\sqrt{2}} \gamma^a \varepsilon A_a \Rightarrow \partial^b \partial_b A_a = 0, \partial^a A_a = 0$

证明:  $\gamma^a \partial_a \psi(x) = 0, \psi = \frac{im}{\sqrt{2}} \gamma^a \varepsilon A_a$

$$\Leftrightarrow \gamma^a \partial_a \frac{im}{\sqrt{2}} \gamma^b \varepsilon A_b = 0$$

$$\Leftrightarrow \gamma^a \partial_a \gamma^b A_b = 0$$

$$\Leftrightarrow \delta^{ab} \partial_a A_b + i\zeta \varepsilon^{abc} \partial_a A_b \gamma_c = 0$$

$$\Leftrightarrow \partial^a A_a + i\zeta \varepsilon^{ab} \partial_a A_b \gamma^c = 0$$

$$\Leftrightarrow \partial^a A_a = 0, i\zeta \varepsilon^{ab} \partial_a A_b = 0$$

$$\Leftrightarrow \partial^a A_a = 0, \varepsilon^{ab} \partial_a A_b = 0 \Leftrightarrow \partial^a A_a = 0, \nabla \times \vec{A} = 0$$

$$\Leftrightarrow \partial^a A_a = 0, \varepsilon^{a'b'c} \varepsilon^{ab} \partial_a A_b = 0$$

$$\Leftrightarrow \partial^a A_a = 0, (\delta^{a'a} \delta^{b'b} - \delta^{a'b} \delta^{b'a}) \partial_a A_b = 0$$

$$\Leftrightarrow \partial^a A_a = 0, \partial_a A_b - \partial_b A_a = 0$$

$$\Rightarrow \partial^a A_a = 0, \partial^a \partial_a A_b - \partial_b \partial^a A_a = 0$$

$$\Leftrightarrow \partial^a \partial_a A_b = 0, \partial^a A_a = 0$$

□



## 4.2.2 三维时空中无质量自旋-\$\frac{3}{2}\$的B-W方程等价于类R-S方程 [17, 19]

定理4.2.2.  $(\gamma^a \partial_a)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{1}{3!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta\}}, A_{a\eta_\zeta} = \frac{1}{\sqrt{2im}} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}$

$$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\zeta} - \partial_b A_{a\eta_\zeta} = 0, \partial^a A_{a\eta_\zeta} = 0 \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{im}{\sqrt{2}} (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} A_{a\eta_\zeta}, \gamma^a A_{a[\eta_\zeta]} = 0 \end{cases} \Rightarrow \gamma^b \partial_b A_{a[\eta_\zeta]} = 0, \gamma^a A_{a[\eta_\zeta]} = 0, \partial^a A_{a\eta_\zeta} = 0$$

证明:  $\begin{cases} (\gamma^a \partial_a)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{1}{3!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta\}} \\ A_{a\eta_\zeta} = \frac{1}{\sqrt{2im}} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta}, \gamma^a = (-\sigma_y, \sigma_x, \varsigma \sigma_z) \end{cases}$

$$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\zeta} - \partial_b A_{a\eta_\zeta} = 0, \partial^a A_{a\eta_\zeta} = 0 \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{im}{\sqrt{2}} (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} A_{a\eta_\zeta}, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \psi_{\lambda_\zeta \eta_\zeta \mu_\zeta} \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\zeta} - \partial_b A_{a\eta_\zeta} = 0, \partial^a A_{a\eta_\zeta} = 0 \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = \frac{im}{\sqrt{2}} (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} A_{a\eta_\zeta}, \varepsilon^{\mu_\zeta \eta_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta} = 0 \end{cases}$$

$$\Leftrightarrow \partial_a A_{b\eta_\zeta} - \partial_b A_{a\eta_\zeta} = 0, \gamma^a A_{a[\eta_\zeta]} = 0, \partial^a A_{a\eta_\zeta} = 0$$

$$\Rightarrow \gamma^a \partial_a A_{b[\eta_\zeta]} - \partial_b \gamma^a A_{a[\eta_\zeta]} = 0, \gamma^a A_{a\eta_\zeta} = 0, \partial^a A_{a\eta_\zeta} = 0$$

$$\Leftrightarrow \gamma^a \partial_a A_{b[\eta_\zeta]} = 0, \gamma^a A_{a\eta_\zeta} = 0, \partial^a A_{a\eta_\zeta} = 0$$

$$\Leftrightarrow \gamma^a \partial_a A_{b\eta_\zeta} = 0, \gamma^a A_{a\eta_\zeta} = 0, \partial^a A_{a\eta_\zeta} = 0$$

$$\Leftrightarrow \gamma^b \partial_b A_{a[\eta_\zeta]} = 0, \gamma^a A_{a[\eta_\zeta]} = 0, \partial^a A_{a\eta_\zeta} = 0$$

□

## 4.2.3 三维时空中无质量自旋-2的B-W方程等价于类C-K方程 [17, 19]

定理4.2.3.  $(\gamma^a \partial_a)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \frac{1}{4!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta\}}, A_{ab} = \left(\frac{1}{\sqrt{2im}}\right)^2 (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}$

$$\Leftrightarrow \begin{cases} \partial_a A_{bd} - \partial_b A_{ad} = 0, \partial^a A_{ab} = 0, A_{ab} = A_{ba}, \delta^{ab} A_{ab} = 0 \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \left(\frac{im}{\sqrt{2}}\right)^2 (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma^b \varepsilon)_{\eta_\zeta \xi_\zeta} A_{ab} \end{cases} \Rightarrow \begin{cases} \partial^c \partial_c A_{ab} = 0, \partial^a A_{ab} = 0 \\ A_{ab} = A_{ba}, \delta^{ab} A_{ab} = 0 \end{cases}$$

证明:

$\begin{cases} (\gamma^a \partial_a)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \frac{1}{4!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta\}} \\ A_{ab} := \left(\frac{1}{\sqrt{2im}}\right)^2 (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)^{\eta_\zeta \xi_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}, \gamma^a = (-\sigma_y, \sigma_x, \varsigma \sigma_z) \end{cases}$

$$\Leftrightarrow \begin{cases} (\gamma^a \partial_a)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \frac{1}{4!} \psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta\}} \\ A_{a\eta_\zeta \xi_\zeta} := \frac{1}{\sqrt{2im}} (\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\zeta \xi_\zeta} - \partial_b A_{a\eta_\zeta \xi_\zeta} = 0, A_{a\eta_\zeta \xi_\zeta} = A_{a\xi_\zeta \eta_\zeta} \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \frac{im}{\sqrt{2}} (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} A_{a\eta_\zeta \xi_\zeta}, \gamma^a A_{a[\eta_\zeta] \xi_\zeta} = 0, \partial^a A_{a\eta_\zeta \xi_\zeta} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial_a A_{b\eta_\zeta \xi_\zeta} - \partial_b A_{a\eta_\zeta \xi_\zeta} = 0, A_{a\eta_\zeta \xi_\zeta} = A_{a\xi_\zeta \eta_\zeta} \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \frac{im}{\sqrt{2}} (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} A_{a\eta_\zeta \xi_\zeta}, \gamma^a A_{a[\eta_\zeta] \xi_\zeta} = 0, \partial^a A_{a\eta_\zeta \xi_\zeta} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial_a A_{bd} - \partial_b A_{ad} = 0, \partial^a A_{ab} = 0, A_{ab} = A_{ba}, \delta^{ab} A_{ab} = 0 \\ \psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta} = \left(\frac{im}{\sqrt{2}}\right)^2 (\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma^b \varepsilon)_{\eta_\zeta \xi_\zeta} A_{ab} \end{cases}$$

$$\Rightarrow \partial^c \partial_c A_{ab} = 0, \partial^a A_{ab} = 0, A_{ab} = A_{ba}, \delta^{ab} A_{ab} = 0$$

□

### 4.3 三维时空中无质量粒子势方程的一般表述

#### 4.3.1 三维时空中无质量自旋- $n$ 的B-W方程等价于类C-K方程 [17, 19]

定理4.3.1.

$$\begin{cases} (\gamma^a \partial_a)_{\kappa\zeta} \lambda_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n}(x) = 0 \\ \psi_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n} = \frac{1}{(2n)!} \psi_{\{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots\}}^{2n} \\ A_{ab\cdots}^n = \left(\frac{1}{\sqrt{2im}}\right)^n (\bar{\varepsilon}\gamma_a)^{\lambda\zeta\mu\zeta} (\bar{\varepsilon}\gamma_b)^{\eta\zeta\xi\zeta} \cdots \psi_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n} \end{cases} \Leftrightarrow \begin{cases} \partial_a A_{bd\cdots}^n - \partial_b A_{ad\cdots}^n = 0, \partial^a A_{ab\cdots}^n = 0 \\ A_{ab\cdots}^n = \frac{1}{n!} A_{\{ab\cdots\}}^n, \delta^{ab} A_{ab\cdots}^n = 0 \\ \psi_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n} = \left(\frac{im}{\sqrt{2}}\right)^n (\gamma^a \varepsilon)_{\lambda\zeta\mu\zeta} (\gamma^b \varepsilon)_{\eta\zeta\xi\zeta} \cdots A_{ab\cdots}^n \end{cases}$$

$$\Rightarrow \begin{cases} \partial^c \partial_c A_{ab\cdots}^n = 0, \partial^a A_{ab\cdots}^n = 0 \\ A_{ab\cdots}^n = \frac{1}{n!} A_{\{ab\cdots\}}^n, \delta^{ab} A_{ab\cdots}^n = 0 \end{cases}$$

推论4.3.1.

$$\psi_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n}(x) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(n-\frac{1}{2})} \lambda_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n}(\vec{p}, -\frac{n\zeta}{2}) [a_1(\vec{p}, -n\zeta) e^{ip \cdot x} + a_2^\dagger(\vec{p}, -n\zeta) e^{-ip \cdot x}] d^N \vec{p}$$

$$A_{ab\cdots}^n(x) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} \frac{1}{\sqrt{2^n E}} \lambda_{ab\cdots}^n(\vec{p}) [a_1(\vec{p}, -s\zeta) e^{ip \cdot x} + a_2^\dagger(\vec{p}, -s\zeta) e^{-ip \cdot x}] d^N \vec{p}$$

$$\lambda_{ab\cdots}^n(\vec{p}) = \frac{1}{i^n} (\bar{\varepsilon}\gamma_a)^{\lambda\zeta\mu\zeta} (\bar{\varepsilon}\gamma_b)^{\eta\zeta\xi\zeta} \cdots \lambda_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n}(\vec{p}, -\frac{n\zeta}{2})$$

#### 4.3.2 三维时空中有质量自旋- $n$ 的B-W方程自旋基和类C-K方程自旋基之间关系

推论4.3.2.

$$\begin{cases} (i\gamma^a p_a + m) \lambda_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n}(\vec{p}, -\frac{n\zeta}{2}) = 0 \\ \lambda_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n}(\vec{p}, -\frac{n\zeta}{2}) \text{ 全对称} \\ \lambda_{ab\cdots}^n(\vec{p}) \\ = \frac{1}{i^n} (\bar{\varepsilon}\gamma_a)^{\lambda\zeta\mu\zeta} (\bar{\varepsilon}\gamma_b)^{\eta\zeta\xi\zeta} \cdots \lambda_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n}(\vec{p}, -\frac{n\zeta}{2}) \end{cases} \Leftrightarrow \begin{cases} (p^c p_c + m^2) \lambda_{ab\cdots}^n(\vec{p}) = 0 \\ p_a \lambda_{bd\cdots}^n(x) - p_b \lambda_{ad\cdots}^n(x) = \zeta m \lambda_{ab}^c \lambda_{cd}^{\cdots} \\ \delta^{ab} \lambda_{ab\cdots}^n(\vec{p}) = 0, p^a \lambda_{ab\cdots}^n(\vec{p}) = 0, \lambda_{ab\cdots}^n(\vec{p}) \text{ 全对称} \\ \lambda_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n}(\vec{p}, -\frac{n\zeta}{2}) = \left(\frac{i}{2}\right)^n (\gamma_a \varepsilon)_{\lambda\zeta\mu\zeta} (\gamma_b \varepsilon)_{\eta\zeta\xi\zeta} \cdots \lambda_{ab\cdots}^n(\vec{p}) \end{cases}$$

推论4.3.3.

$$\lambda_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n}(\vec{p}, -\frac{n\zeta}{2}) = \left(\frac{i}{2}\right)^n (\gamma_a \varepsilon)_{\lambda\zeta\mu\zeta} (\gamma_b \varepsilon)_{\eta\zeta\xi\zeta} \cdots \lambda_{ab\cdots}^n(\vec{p}) \Leftrightarrow \lambda_{ab\cdots}^n(\vec{p}) = \frac{1}{i^n} (\bar{\varepsilon}\gamma_a)^{\lambda\zeta\mu\zeta} (\bar{\varepsilon}\gamma_b)^{\eta\zeta\xi\zeta} \cdots \lambda_{\lambda\zeta\mu\zeta\eta\zeta\xi\zeta\cdots}^{2n}(\vec{p}, -\frac{n\zeta}{2})$$

$$\text{推论4.3.4. } \lambda_{\lambda\zeta\mu\zeta}^{2n}(\vec{p}, -s\zeta) = \lambda_{\lambda\zeta}(\vec{p}, -\frac{\zeta}{2}) \lambda_{\mu\zeta}(\vec{p}, -\frac{\zeta}{2}) \cdots, \lambda_{ab\cdots}^n(\vec{p}) = \lambda_a(\vec{p}) \lambda_b(\vec{p}) \cdots$$

推论4.3.5.  $\lambda_a(\vec{p}, -\zeta)$

$$\begin{aligned} &= \frac{1}{i} (\bar{\varepsilon}\gamma_a)^{\lambda\zeta\mu\zeta} \lambda_{\lambda\zeta\mu\zeta}(\vec{p}, -\zeta) \\ &= \frac{1}{i} \lambda_{\lambda\zeta}(\vec{p}, -\frac{\zeta}{2}) (\bar{\varepsilon}\gamma_a)^{\lambda\zeta\mu\zeta} \lambda_{\mu\zeta}(\vec{p}, -\frac{\zeta}{2}) \\ &= (\hat{p}_x - i\zeta \hat{p}_y) (\hat{p}_x, \hat{p}_y, i)_a \\ &= (\hat{p}_x - i\zeta \hat{p}_y) \hat{p}_a \end{aligned}$$

$$\lambda^{+a}(\vec{p}, -\zeta) \lambda_a(\vec{p}, -\zeta) = 2, \lambda_a(\vec{p}, -\zeta) \lambda_a^+(\vec{p}, -\zeta) = \hat{p}_a \hat{p}_a^+$$

$$\text{推论4.3.6. } \lambda(\hat{p}, \frac{1}{2}) = \lambda\left(\begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ 0 \end{bmatrix}, \frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \hat{p}_x + i\hat{p}_y \end{bmatrix}, \lambda(\hat{p}, -\frac{1}{2}) = \lambda\left(\begin{bmatrix} \hat{p}_x \\ \hat{p}_y \\ 0 \end{bmatrix}, -\frac{1}{2}\right) = \frac{1}{\sqrt{2}} \begin{bmatrix} -(\hat{p}_x - i\hat{p}_y) \\ 1 \end{bmatrix}$$

4.3.3 三维时空中无质量自旋 $-n + \frac{1}{2}$ 的B-W方程等价于类R-S方程 [17, 19]

定理4.3.2.

$$\left\{ \begin{array}{l} (\gamma^a \partial_a)_{\kappa_\zeta} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1}(x) = 0 \\ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1} = \frac{1}{(2n+1)!} \underbrace{\psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta\}}}_{2n+1} \\ \underbrace{A_{ab \dots \tau_\zeta}}_n \\ = \left( \frac{1}{\sqrt{2im}} \right)^n \underbrace{(\bar{\varepsilon} \gamma_a)_{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)_{\eta_\zeta \xi_\zeta} \dots}_{n} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \partial_a \underbrace{A_{bd \dots \tau_\zeta}}_n - \partial_b \underbrace{A_{ad \dots \tau_\zeta}}_n = 0, \partial^a \underbrace{A_{ab \dots \tau_\zeta}}_n = 0 \\ \underbrace{A_{ab \dots \tau_\zeta}}_n = \frac{1}{n!} \underbrace{A_{\{ab \dots\}}}_{n, \tau_\zeta}, \delta^{ab} \underbrace{A_{ab \dots \tau_\zeta}}_n = 0, \gamma^a \underbrace{A_{ab \dots \tau_\zeta}}_n = 0 \\ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1} = \left( \frac{im}{\sqrt{2}} \right)^n \underbrace{(\gamma^a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma^b \varepsilon)_{\eta_\zeta \xi_\zeta} \dots}_{n} \underbrace{A_{ab \dots \tau_\zeta}}_n \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} \gamma^c \partial_c \underbrace{A_{ab \dots \tau_\zeta}}_n = 0, \partial^a \underbrace{A_{ab \dots \tau_\zeta}}_n = 0 \\ \underbrace{A_{ab \dots \tau_\zeta}}_n = \frac{1}{n!} \underbrace{A_{\{ab \dots\}}}_{n, \tau_\zeta}, \delta^{ab} \underbrace{A_{ab \dots \tau_\zeta}}_n = 0, \gamma^a \underbrace{A_{ab \dots \tau_\zeta}}_n = 0 \end{array} \right.$$

推论4.3.7.

$$\underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1}(x) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} |\vec{p}|^n \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1}(\hat{p}, -\frac{s\zeta}{2}) [a_1(\vec{p}, -s\zeta) e^{ip \cdot x} + a_2^\dagger(\vec{p}, -s\zeta) e^{-ip \cdot x}] d^N \vec{p}$$

$$\underbrace{A_{ab \dots \tau_\zeta}}_n(x) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} \frac{1}{\sqrt{2^n E}} \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) [a_1(\vec{p}, -s\zeta) e^{ip \cdot x} + a_2^\dagger(\vec{p}, -s\zeta) e^{-ip \cdot x}] d^N \vec{p}$$

$$\underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) = \frac{1}{i^n} \underbrace{(\bar{\varepsilon} \gamma_a)_{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)_{\eta_\zeta \xi_\zeta} \dots}_n \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1}(\vec{p}, -\frac{s\zeta}{2})$$

4.3.4 三维时空中有质量自旋 $-n + \frac{1}{2}$ 的B-W方程自旋基和类R-S方程自旋基之间关系

推论4.3.8.

$$\left\{ \begin{array}{l} (i\gamma^a p_a + m) \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1}(\hat{p}, -\frac{n\zeta}{2}) = 0 \\ \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1}(\hat{p}, -\frac{n\zeta}{2}) \text{全对称} \\ \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) = \frac{1}{i^n} \underbrace{(\bar{\varepsilon} \gamma_a)_{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)_{\eta_\zeta \xi_\zeta} \dots}_n \\ \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1}(\vec{p}, -\frac{n\zeta}{2}) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (p^c p_c + m^2) \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) = 0, p^a \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) = 0 \\ p_a \underbrace{\lambda_{bd \dots \tau_\zeta}}_n(x) - p_b \underbrace{\lambda_{ad \dots \tau_\zeta}}_n(x) = \zeta m \lambda_{ab}^c \underbrace{\lambda_{cd \dots \tau_\zeta}}_n \\ \delta^{ab} \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) = 0, \gamma^a \underbrace{\lambda_{ab \dots \tau_\zeta}}_n = 0, \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) \text{全对称} \\ \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1}(\vec{p}, -\frac{n\zeta}{2}) = \left( \frac{i}{2} \right)^n \underbrace{(\gamma_a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma_b \varepsilon)_{\eta_\zeta \xi_\zeta} \dots}_n \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) \end{array} \right.$$

$$\text{推论4.3.9. } \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1}(\vec{p}, -\frac{n\zeta}{2}) = \left( \frac{i}{2} \right)^n \underbrace{(\gamma_a \varepsilon)_{\lambda_\zeta \mu_\zeta} (\gamma_b \varepsilon)_{\eta_\zeta \xi_\zeta} \dots}_n \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p})$$

$$[\Leftrightarrow] \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) = \frac{1}{i^n} \underbrace{(\bar{\varepsilon} \gamma_a)_{\lambda_\zeta \mu_\zeta} (\bar{\varepsilon} \gamma_b)_{\eta_\zeta \xi_\zeta} \dots}_n \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \tau_\zeta}}_{2n+1}(\vec{p}, -\frac{n\zeta}{2})$$

推论4.3.10.

$$\underbrace{\lambda_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}_{2n+1}(\hat{p}, -s\zeta) = \underbrace{\lambda_{\lambda_\zeta}(\hat{p}, -\frac{\zeta}{2}) \lambda_{\mu_\zeta}(\hat{p}, -\frac{\zeta}{2}) \dots \lambda_{\tau_\zeta}(\hat{p}, -\frac{\zeta}{2})}_{2n+1}, \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) = \underbrace{\lambda_a(\vec{p}) \lambda_b(\vec{p}) \dots \lambda_{\tau_\zeta}(\vec{p})}_n = \underbrace{\lambda_{ab \dots \tau_\zeta}}_n(\vec{p}) \lambda_{\tau_\zeta}(\vec{p}, -\frac{\zeta}{2})$$

## 4.4 三维时空中无质量s-自旋方程

定理4.4.1.

$$\left\{ \begin{array}{l} \gamma^a \partial_a \underbrace{\psi_{[\lambda_\zeta] \mu_\zeta \dots}}_{2s}(x) = 0, \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(x) \text{全对称} \\ \psi_{k_\zeta}(s, \zeta) := \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}}_{2s}, \gamma^a = (-\sigma_y, \sigma_x, \zeta \sigma_z) \end{array} \right.$$

$$\Leftrightarrow \begin{cases} [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(s, \varsigma) = 0, S_{ab}(s, \varsigma) = -i[\gamma_a(s), \gamma_b(s)] \\ \psi_{\lambda_{\varsigma}\mu_{\varsigma}\dots} = \Gamma_{\lambda_{\varsigma}\mu_{\varsigma}\dots}^{k_{\varsigma}} \psi_{k_{\varsigma}}(s, \varsigma), S_{ab}(s, \varsigma) = \begin{bmatrix} 0 & \sigma_z(s) & -\varsigma\sigma_x(s) \\ -\sigma_z(s) & 0 & -\varsigma\sigma_y(s) \\ \varsigma\sigma_x(s) & \varsigma\sigma_y(s) & 0 \end{bmatrix} \end{cases}$$

$$\psi_{\lambda_{\varsigma}\mu_{\varsigma}\dots}(x) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda_{\lambda_{\varsigma}\mu_{\varsigma}\dots}(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma)e^{ip \cdot x} + a_2^+(\vec{p}, -s\varsigma)e^{-ip \cdot x}] d^N \vec{p}$$

$$\psi_{k_{\varsigma}}(x) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda_{k_{\varsigma}}(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma)e^{ip \cdot x} + a_2^+(\vec{p}, -s\varsigma)e^{-ip \cdot x}] d^N \vec{p}$$

$$\lambda_{k_{\varsigma}}(\vec{p}, -s\varsigma) := \Gamma_{k_{\varsigma}}^{\lambda_{\varsigma}\mu_{\varsigma}\dots} \lambda_{\lambda_{\varsigma}\mu_{\varsigma}\dots}(\vec{p}, -s\varsigma) \Leftrightarrow \lambda_{\lambda_{\varsigma}\mu_{\varsigma}\dots}(\vec{p}, -s\varsigma) = \Gamma_{\lambda_{\varsigma}\mu_{\varsigma}\dots}^{k_{\varsigma}} \lambda_{k_{\varsigma}}(\vec{p}, -s\varsigma)$$

#### 4.4.1 三维时空中无质量粒子的协变对易规则

$$\text{定理4.4.2. } [\psi_{\lambda_{\varsigma}\mu_{\varsigma}\dots}(x), \psi_{\lambda'_{\varsigma}\mu'_{\varsigma}\dots}^+(x')] = \frac{i}{2^{2s-1}} \underbrace{[(-\gamma^a \partial_a) \gamma^0]_{\lambda_{\varsigma}\lambda'_{\varsigma}} [(-\gamma^b \partial_b) \gamma^0]_{\mu_{\varsigma}\mu'_{\varsigma}}}_{2s} \cdot \Delta(x - x')$$

$$[\Downarrow]$$

$$\text{定理4.4.3. } [\psi_{\lambda_{\varsigma}\mu_{\varsigma}\dots}(x), \psi_{\lambda'_{\varsigma}\mu'_{\varsigma}\dots}^+(x')] = i \frac{(i\varsigma)^{2s}}{2^{2s-1}} \underbrace{[(\sigma, i\varsigma)^a \partial_a]_{\lambda_{\varsigma}\lambda'_{\varsigma}} [(\sigma, i\varsigma)^b \partial_b]_{\mu_{\varsigma}\mu'_{\varsigma}}}_{2s} \cdot \Delta(x - x')$$

$$[\Downarrow]$$

$$\text{定理4.4.4. } [\psi_{k_{\varsigma}}(x), \psi_{k'_{\varsigma}}^+(x')] = i \frac{(-1)^{2s}}{2^{2s-1}} \Gamma_{k_{\varsigma}k'_{\varsigma}}^{abc\dots} \underbrace{\partial_a \partial_b \partial_c}_{2s} \cdot \Delta(x - x')$$

$$[\Downarrow]$$

$$\text{定理4.4.5. } [A_{\underline{ab}\dots}(x), A_{\underline{a'b'}\dots}^+(x')] = \frac{i}{2^{n-1}} \underbrace{\frac{\partial_a \partial_{a'}^+}{\nabla^2} \frac{\partial_b \partial_{b'}^+}{\nabla^2}}_n \cdot \Delta(x - x')$$

$$[\Downarrow]$$

$$\text{定理4.4.6. } \{A_{\underline{ab}\dots\tau_{\varsigma}}(x), A_{\underline{a'b'}\dots\tau'_{\varsigma}}^+(x')\} = \frac{i}{2^n} \underbrace{\frac{\partial_a \partial_{a'}^+}{\nabla^2} \frac{\partial_b \partial_{b'}^+}{\nabla^2}}_n \cdot [(-\gamma^c \partial_c) \gamma^0]_{\tau_{\varsigma}\tau'_{\varsigma}} \Delta(x - x')$$

$$[\Downarrow]$$

$$\text{定理4.4.7. } \{A_{\underline{ab}\dots\tau_{\varsigma}}(x), A_{\underline{a'b'}\dots\tau'_{\varsigma}}^+(x')\} = i\varsigma \frac{i}{2^n} \underbrace{\frac{\partial_a \partial_{a'}^+}{\nabla^2} \frac{\partial_b \partial_{b'}^+}{\nabla^2}}_n \cdot [(\sigma, i\varsigma)^c \partial_c]_{\tau_{\varsigma}\tau'_{\varsigma}} \Delta(x - x')$$

## 5 二维时空中有质量粒子的协变量子化

### 5.1 二维时空中的Dirac方程

#### 5.1.1 二维时空中的Dirac自旋基

$$\text{定义5.1.1. } u(\vec{p}) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{sp_x \sigma_x}{E+m}\right) \left(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2} \sigma_x\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v(\vec{p}) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{sp_x \sigma_x}{E+m}\right) \left(\frac{1+\varsigma}{2} + \frac{1-\varsigma}{2} \sigma_x\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{推论5.1.1. } u(\vec{p}) = \sigma_x v(\vec{p}), v(\vec{p}) = \sigma_x u(\vec{p}), u^*(\vec{p}) = u(\vec{p}), v^*(\vec{p}) = v(\vec{p})$$

#### 5.1.2 二维时空中Dirac自旋基的性质

$$\text{性质5.1.1. } \begin{cases} \bar{u}(\vec{p})u(\vec{p}) = 1, \bar{v}(\vec{p})v(\vec{p}) = -1, \bar{u}(\vec{p})v(\vec{p}) = 0, \bar{v}(\vec{p})u(\vec{p}) = 0 \\ u^+(\vec{p})u(\vec{p}) = \frac{E}{m}, v^+(\vec{p})v(\vec{p}) = \frac{E}{m}, u^+(\vec{p})v(-\vec{p}) = 0, v^+(\vec{p})u(-\vec{p}) = 0 \end{cases}$$

$$\text{性质5.1.2. } \begin{cases} u(\vec{p})\bar{u}(\vec{p}) = \frac{m-i\gamma^a p_a}{2m} & \begin{cases} u(\vec{p})u^+(\vec{p}) = \frac{(m-i\gamma^a p_a)\gamma^0}{2m} = \frac{m\sigma_z - (\sigma, i\varsigma)^a p_a}{\varsigma 2m} \\ v(\vec{p})\bar{v}(\vec{p}) = \frac{-m-i\gamma^a p_a}{2m} & \begin{cases} v(\vec{p})v^+(\vec{p}) = \frac{(-m-i\gamma^a p_a)\gamma^0}{2m} = \frac{-m\sigma_z - (\sigma, i\varsigma)^a p_a}{\varsigma 2m} \end{cases} \end{cases} \end{cases}$$

$$\text{性质5.1.3. } u(\vec{p})\bar{u}(\vec{p}) - v(\vec{p}, h)\bar{v}(\vec{p}) = 1, u(\vec{p})\bar{u}(\vec{p}) + v(\vec{p}, h)\bar{v}(\vec{p}) = \frac{-i\gamma^a p_a}{m}, u(\vec{p})u^+(\vec{p}) + v(-\vec{p}, h)v^+(-\vec{p}) = \frac{E}{m}$$

### 5.1.3 二维时空中的Dirac方程<sup>[4]</sup>及其平面波解

定理5.1.1.  $(\gamma^a \partial_a + m)\psi = 0, \gamma^a = (-\sigma_y, \varsigma \sigma_z)$

$$\psi(\vec{r}, t) = \frac{1}{(2\pi)^{1/2}} \int_{\vec{p}=-\infty}^{+\infty} [a(\vec{p}) \sqrt{\frac{m}{E}} u(\vec{p}) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}) \sqrt{\frac{m}{E}} v(\vec{p}) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d\vec{p}$$

$$a(\vec{p}) = \frac{1}{(2\pi)^{1/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{E}{m}} u^+(\vec{p}) \psi(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r}-Et)} d\vec{r}, b^+(\vec{p}) = \frac{1}{(2\pi)^{1/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{E}{m}} v^+(\vec{p}) \psi(\vec{r}, t) e^{i(\vec{p}\cdot\vec{r}-Et)} d\vec{r}$$

### 5.1.4 二维时空中Dirac方程的协变量子化规则

$$\text{推论5.1.2. } \begin{cases} \{a(\vec{p}), a^+(\vec{p}')\} = \delta(\vec{p} - \vec{p}') \\ \{a(\vec{p}), a(\vec{p}')\} = 0, \{a^+(\vec{p}), a^+(\vec{p}')\} = 0 \end{cases} \Rightarrow \{\psi(x), \psi^+(x')\} = i(m - \gamma^a \partial_a) \gamma^0 \Delta(x - x')$$

## 5.2 二维时空中的Bargmann-Wigner方程

### 5.2.1 二维时空中Bargmann-Wigner方程<sup>[17]</sup>的自旋基及其平面波解

$$\text{定义5.2.1. } U_{\lambda_s \mu_s \dots}(\vec{p}) := \underbrace{u_{\lambda_s}(\vec{p})}_{2s} \underbrace{u_{\mu_s}(\vec{p})}_{2s} \dots, V_{\lambda_s \mu_s \dots}(\vec{p}) := \underbrace{v_{\lambda_s}(\vec{p})}_{2s} \underbrace{v_{\mu_s}(\vec{p})}_{2s} \dots$$

$$\text{推论5.2.1. } U_{\lambda_s \mu_s \dots}(\vec{p}) = \underbrace{\sigma_x}_{2s} \otimes \underbrace{\sigma_x}_{2s} \dots \underbrace{V_{\lambda_s \mu_s \dots}(\vec{p})}_{2s}, V_{\lambda_s \mu_s \dots}(\vec{p}) = \underbrace{\sigma_x}_{2s} \otimes \underbrace{\sigma_x}_{2s} \dots \underbrace{U_{\lambda_s \mu_s \dots}(\vec{p})}_{2s}$$

定理5.2.1.  $(\gamma^a \partial_a + m)_{\kappa_s} \psi_{\lambda_s \mu_s \dots}(\vec{r}, t) = 0, \psi_{\lambda_s \mu_s \dots}(\vec{r}, t) = \frac{1}{(2s)!} \psi_{\{\lambda_s \mu_s \dots\}}(\vec{r}, t)$

$$\psi_{\lambda_s \mu_s \dots}(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} [a(\vec{p}) U_{\lambda_s \mu_s \dots}(\vec{p}) e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}) V_{\lambda_s \mu_s \dots}(\vec{p}) e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^N \vec{p}$$

$$\begin{cases} a(\vec{p}) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m}{E}}^{2s} U^{+\lambda_s \mu_s \dots}(\vec{p}) \psi_{\lambda_s \mu_s \dots}(\vec{r}, t) e^{-i(\vec{p}\cdot\vec{r}-Et)} d^N \vec{r} \\ b^+(\vec{p}) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m}{E}}^{2s} V^{+\lambda_s \mu_s \dots}(\vec{p}) \psi_{\lambda_s \mu_s \dots}(\vec{r}, t) e^{i(\vec{p}\cdot\vec{r}-Et)} d^N \vec{r} \end{cases}$$

### 5.2.2 二维时空中Bargmann-Wigner方程自旋基的正交性质

$$\text{推论5.2.2. } \begin{cases} \bar{U}^{+\lambda_s \mu_s \dots}(\vec{p}) U_{\lambda_s \mu_s \dots}(\vec{p}) = 1, \bar{V}^{+\lambda_s \mu_s \dots}(\vec{p}) V_{\lambda_s \mu_s \dots}(\vec{p}) = 1 \\ \bar{U}^{+\lambda_s \mu_s \dots}(\vec{p}) V_{\lambda_s \mu_s \dots}(\vec{p}) = 0, \bar{V}^{+\lambda_s \mu_s \dots}(\vec{p}) U_{\lambda_s \mu_s \dots}(\vec{p}) = 0 \end{cases}$$

$$\text{推论5.2.3. } \begin{cases} U^{+\lambda_s \mu_s \dots}(\vec{p}) U_{\lambda_s \mu_s \dots}(\vec{p}) = (\frac{E}{m})^{2s}, U^{+\lambda_s \mu_s \dots}(\vec{p}) V_{\lambda_s \mu_s \dots}(-\vec{p}) = 0 \\ V^{+\lambda_s \mu_s \dots}(\vec{p}) V_{\lambda_s \mu_s \dots}(\vec{p}) = (\frac{E}{m})^{2s}, V^{+\lambda_s \mu_s \dots}(\vec{p}) U_{\lambda_s \mu_s \dots}(-\vec{p}) = 0 \end{cases}$$

### 5.2.3 二维时空中Bargmann-Wigner方程的准投影算子

$$\text{推论5.2.4. } \begin{cases} U_{\lambda_s \mu_s \dots}(\vec{p}) U_{\lambda'_s \mu'_s \dots}^+(\vec{p}) = \frac{1}{(2m)^{2s}} \underbrace{[(m - i\gamma^b p_b) \gamma^0]_{\lambda_s \lambda'_s} [(m - i\gamma^c p_c) \gamma^0]_{\mu_s \mu'_s} \dots}_{2s} \\ V_{\lambda_s \mu_s \dots}(\vec{p}) V_{\lambda'_s \mu'_s \dots}^+(\vec{p}) = \frac{1}{(2m)^{2s}} \underbrace{[(-m - i\gamma^b p_b) \gamma^0]_{\lambda_s \lambda'_s} [(-m - i\gamma^c p_c) \gamma^0]_{\mu_s \mu'_s} \dots}_{2s} \end{cases}$$

$$\text{推论5.2.5. } \begin{cases} U_{\lambda_s \mu_s \dots}(\vec{p}) U_{\lambda'_s \mu'_s \dots}^+(\vec{p}) = \frac{1}{(\varsigma 2m)^{2s}} \underbrace{[m\sigma_z - (\sigma, i\varsigma)^a p_a]_{\lambda_s \lambda'_s} [m\sigma_z - (\sigma, i\varsigma)^b p_b]_{\mu_s \mu'_s} \dots}_{2s} \\ V_{\lambda_s \mu_s \dots}(\vec{p}) V_{\lambda'_s \mu'_s \dots}^+(\vec{p}) = \frac{1}{(\varsigma 2m)^{2s}} \underbrace{[-m\sigma_z - (\sigma, i\varsigma)^a p_a]_{\lambda_s \lambda'_s} [-m\sigma_z - (\sigma, i\varsigma)^b p_b]_{\mu_s \mu'_s} \dots}_{2s} \end{cases}$$

$$\text{推论5.2.6. } \underbrace{U_{\lambda_s \mu_s \dots}(p)}_{2s} \underbrace{U_{\lambda'_s \mu'_s \dots}^+(-p)}_{2s} = (-1)^{2s} \underbrace{V_{\lambda_s \mu_s \dots}(-p)}_{2s} \underbrace{V_{\lambda'_s \mu'_s \dots}^+(-p)}_{2s}$$

### 5.2.4 二维时空中Bargmann-Wigner方程的协变对易规则

$$\text{定理5.2.2. } \underbrace{[\psi_{\lambda_s \mu_s \dots}(x), \psi_{\lambda'_s \mu'_s \dots}^+(x')]}_{2s} = \underbrace{\frac{i}{2^{2s-1}} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda_s \lambda'_s} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_s \mu'_s}}_{2s} \Delta(x - x')$$

$$\Leftrightarrow$$

$$\text{定理5.2.3. } \underbrace{[\psi_{\lambda_s \mu_s \dots}(x), \psi_{\lambda'_s \mu'_s \dots}^+(x')]}_{2s} = \underbrace{i \frac{(i\sigma)^{2s}}{2^{2s-1}} [-im\sigma_z + (\sigma, i\sigma)^a \partial_a]_{\lambda_s \lambda'_s} [-im\sigma_z + (\sigma, i\sigma)^b \partial_b]_{\mu_s \mu'_s}}_{2s} \Delta(x - x')$$

### 5.2.5 二维时空中 $X_{\lambda_s \mu_s}^a(p)$ 的性质

$$\text{定义5.2.2. } X_{\lambda_s \mu_s}^a(x) := [(im\gamma^a + \varsigma \sigma_x \varepsilon^{ab} \partial_b) \varepsilon]_{\lambda_s \mu_s}, X_{\lambda_s \mu_s}^a(p) := [(im\gamma^a + i\varsigma \sigma_x \varepsilon^{ab} p_b) \varepsilon]_{\lambda_s \mu_s}$$

$$\text{性质5.2.1. } (\gamma^a \varepsilon)_{\lambda'_s \mu'_s} (\bar{\varepsilon} \gamma_a)^{\lambda_s \mu_s} = -\delta_{\lambda'_s \lambda_s}^{\{\lambda_s \mu_s\}} - \sigma_x \lambda'_s \mu'_s \sigma_x^{\lambda_s \mu_s}$$

$$\text{性质5.2.2. } (\bar{\varepsilon} \gamma^{a'})^{\lambda_s \mu_s} X_{\lambda_s \mu_s}^a(p) = i2m\delta^{a'a}$$

$$\begin{aligned} \text{证明: } & (\bar{\varepsilon} \gamma^{a'})^{\lambda_s \mu_s} X_{\lambda_s \mu_s}^a(p) \\ &= (\bar{\varepsilon} \gamma^{a'})^{\lambda_s \mu_s} [(im\gamma^a + i\varsigma \sigma_x \varepsilon^{ab} p_b) \varepsilon]_{\lambda_s \mu_s} \\ &= \text{tr}[\bar{\varepsilon} \gamma^{a'} (im\gamma^a + i\varsigma \sigma_x \varepsilon^{ab} p_b) \varepsilon] \\ &= \text{tr}[\gamma^{a'} (im\gamma^a + i\varsigma \sigma_x \varepsilon^{ab} p_b)] \\ &= im \text{tr}(\gamma^{a'} \gamma^a) \\ &= i2m\delta^{a'a} \end{aligned}$$

□

$$\text{性质5.2.3. } [(im\gamma^a + \varsigma \sigma_x \varepsilon^{ab} \partial_b) \varepsilon]_{\lambda'_s \mu'_s} (\bar{\varepsilon} \gamma_a)^{\lambda_s \mu_s} = -im\delta_{\lambda'_s \lambda_s}^{\{\lambda_s \mu_s\}} - im\sigma_x \lambda'_s \mu'_s \sigma_x^{\lambda_s \mu_s} + \dots$$

### 5.2.6 二维时空中Bargmann-Wigner方程准投影算子的等价表述

引理5.2.1.

$$\begin{cases} u(\vec{p})u^+(\vec{p}) = \frac{(m - i\gamma^a p_a) \gamma^0}{2m}, u_{\lambda_s}(\vec{p})u_{\lambda'_s}^+(\vec{p})u_{\mu_s}(\vec{p})u_{\mu'_s}^+(\vec{p}) = u_{\lambda_s}(\vec{p})u_{\mu_s}(\vec{p})u_{\lambda'_s}^+(\vec{p})u_{\mu'_s}^+(\vec{p}) \\ \varepsilon_a(\vec{p})\varepsilon_{a'}^+(\vec{p}) = \eta_{aa'} + \frac{p_a p_{a'}}{m^2}, \varepsilon_a(\vec{p})\varepsilon_{a'}^+(\vec{p})\varepsilon_b(\vec{p})\varepsilon_{b'}^+(\vec{p}) = \varepsilon_a(\vec{p})\varepsilon_{b'}^+(\vec{p})\varepsilon_b(\vec{p})\varepsilon_{a'}^+(\vec{p}) \end{cases}$$

$$\Leftrightarrow \begin{cases} [(m - i\gamma^b p_b) \gamma^0]_{\lambda_s \lambda'_s} [(m - i\gamma^c p_c) \gamma^0]_{\mu_s \mu'_s} = [(m - i\gamma^b p_b) \gamma^0]_{\mu_s \lambda'_s} [(m - i\gamma^c p_c) \gamma^0]_{\lambda_s \mu'_s} \\ (\eta_{aa'} + \frac{p_a p_{a'}}{m^2})(\eta_{bb'} + \frac{p_b p_{b'}}{m^2}) = (\eta_{ba'} + \frac{p_b p_{a'}}{m^2})(\eta_{ab'} + \frac{p_a p_{b'}}{m^2}) \\ [(m - i\gamma^b p_b) \gamma^0]_{\lambda_s \lambda'_s} [(m - i\gamma^c p_c) \gamma^0]_{\mu_s \mu'_s} = X_{\lambda_s \mu_s}^a(p) X_{\lambda'_s \mu'_s}^{+a'}(-p) (\eta_{aa'} + \frac{p_a p_{a'}}{m^2}) \end{cases}$$

$$\text{推论5.2.7. } \begin{cases} \underbrace{U_{\lambda_s \mu_s \dots}(\vec{p})}_{2n} \underbrace{U_{\lambda'_s \mu'_s \dots}^+(\vec{p})}_{2n} = \frac{1}{(2m)^{2n}} \underbrace{[X_{\lambda_s \mu_s}^a(p) X_{\lambda'_s \mu'_s}^{+a'}(-p) (\eta_{aa'} + \frac{p_a p_{a'}}{m^2})]}_n \dots \\ \underbrace{V_{\lambda_s \mu_s \dots}(\vec{p})}_{2n} \underbrace{V_{\lambda'_s \mu'_s \dots}^+(\vec{p})}_{2n} = \frac{1}{(2m)^{2n}} \underbrace{[X_{\lambda_s \mu_s}^a(-p) X_{\lambda'_s \mu'_s}^{+a'}(p) (\eta_{aa'} + \frac{p_a p_{a'}}{m^2})]}_n \dots \end{cases}$$

$$\text{推论5.2.8. } \begin{cases} \underbrace{U_{\lambda_s \mu_s \dots \tau_s}(\vec{p})}_{2n+1} \underbrace{U_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p})}_{2n+1} = \frac{1}{(2m)^{2n+1}} \underbrace{[X_{\lambda_s \mu_s}^a(p) X_{\lambda'_s \mu'_s}^{+a'}(-p) (\eta_{aa'} + \frac{p_a p_{a'}}{m^2})]}_n \dots [(m - i\gamma^c p_c) \gamma^0]_{\tau_s \tau'_s} \\ \underbrace{V_{\lambda_s \mu_s \dots \tau_s}(\vec{p})}_{2n+1} \underbrace{V_{\lambda'_s \mu'_s \dots \tau'_s}^+(\vec{p})}_{2n+1} = \frac{1}{(2m)^{2n+1}} \underbrace{[X_{\lambda_s \mu_s}^a(-p) X_{\lambda'_s \mu'_s}^{+a'}(p) (\eta_{aa'} + \frac{p_a p_{a'}}{m^2})]}_n \dots [(-m - i\gamma^c p_c) \gamma^0]_{\tau_s \tau'_s} \end{cases}$$

### 5.2.7 二维时空中Bargmann-Wigner方程的协变对易规则等价表述

$$\text{定理5.2.4. } \underbrace{[\psi_{\lambda_s \mu_s \dots}(x), \psi_{\lambda'_s \mu'_s \dots}^+(x')]}_{2n} = \underbrace{\frac{i}{2^{2n-1}} \mathbb{X}_{\lambda_s \mu_s}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_s \mu'_s}^{+a'}(x')}_n \cdot \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \Delta(x - x')$$

## 定理5.2.5.

$$[\psi_{\lambda_s \mu_s \dots \tau_s}(x), \psi_{\lambda'_s \mu'_s \dots \tau'_s}(x')] = \frac{i}{2^{2n}} \underbrace{\mathbb{X}_{\lambda_s \mu_s}^a(x)}_{2n+1} \cdot \underbrace{\mathbb{X}_{\lambda'_s \mu'_s}^{+a'}(x')}_{2n+1} \cdot \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdot [(m - \gamma^c \partial_c) \gamma^0]_{\tau_s \tau'_s} \Delta(x - x')$$

[⇕]

## 定理5.2.6.

$$[\psi_{\lambda_s \mu_s \dots \tau_s}(x), \psi_{\lambda'_s \mu'_s \dots \tau'_s}(x')] = i \frac{i \zeta_s}{2^{2n}} \underbrace{\mathbb{X}_{\lambda_s \mu_s}^a(x)}_{2n+1} \cdot \underbrace{\mathbb{X}_{\lambda'_s \mu'_s}^{+a'}(x')}_{2n+1} \cdot \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdot [-im\sigma_z + (\sigma, i\zeta)^b \partial_b]_{\tau_s \tau'_s} \Delta(x - x')$$

## 5.3 二维时空中的势方程

自我评述: 此节类比四维时空中的情形, 探究二维时空中是否存在与B-W方程等价的C-K或R-S方程?

5.3.1 二维时空中自旋-1的B-W方程等价于K-G方程<sup>[17, 19]</sup>

定义5.3.1.  $(\gamma^a \partial_a + m)^{\kappa_s} \psi^{\lambda_s \mu_s \dots \zeta_s} = \overbrace{J^{\kappa_s \mu_s \dots \zeta_s}}^{2s}, \psi^{\lambda_s \mu_s \dots \zeta_s}$  全对称,  $\overbrace{J^{\kappa_s \mu_s \dots \zeta_s}}^{2s}$  除  $\kappa_s$  外全对称,  $\gamma^a = (-\sigma_y, \sigma_z)$

定理5.3.1.  $(\gamma^a \partial_a + m)\psi(x) = 0, \psi^T(x) = \psi(x), A_a = \frac{1}{\sqrt{2im}} \text{tr}(\bar{\varepsilon} \gamma_a \psi)$   
 $\Leftrightarrow (\partial^b \partial_b - m^2)A_a = 0, \partial^a A_a = 0; F_{ab} := \partial_a A_b - \partial_b A_a, \psi = (im\gamma^a + \zeta \varepsilon^{ab} \sigma_x \partial_b) \varepsilon \frac{A_a}{\sqrt{2}}, S_{ab}(e) = \frac{1}{2} \zeta \varepsilon^{ab} \sigma_x$

证明:  $(\gamma^a \partial_a + m)\psi(x) = 0, \psi^T(x) = \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \zeta \sigma_x \varepsilon F_{xy})$   
 $\Leftrightarrow (\gamma^a \partial_a + m)(\gamma^b \varepsilon im A_b - \zeta \sigma_x \varepsilon F_{xy}) = 0, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \zeta \sigma_x \varepsilon F_{xy})$   
 $\Leftrightarrow (\gamma^a \partial_a + m)(\gamma^b im A_b - \zeta \sigma_x F_{xy}) = 0, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \zeta \sigma_x \varepsilon F_{xy})$   
 $\Leftrightarrow (\delta^{ab} - i\zeta \varepsilon^{ab} \sigma_x) im \partial_a A_b + m(im \gamma_b A^b - \zeta \sigma_x F_{xy}) - i\varepsilon^{ab} \gamma_b \partial_a F_{xy} = 0, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \zeta \sigma_x \varepsilon F_{xy})$   
 $\Leftrightarrow im \partial^a A_a - m(F_{xy} - \varepsilon^{ab} \partial_a A_b) \zeta \sigma_x + i(m^2 A^b - \varepsilon^{ab} \partial_a F_{xy}) \gamma_b = 0, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \zeta \sigma_x \varepsilon F_{xy})$   
 $\Leftrightarrow im \partial^a A_a = 0, m(F_{xy} - \varepsilon^{ab} \partial_a A_b) = 0, i(m^2 A^b - \varepsilon^{ab} \partial_a F_{xy}) \gamma_b = 0, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \zeta \sigma_x \varepsilon F_{xy})$   
 $\Leftrightarrow \partial^a A_a = 0, F_{xy} = \varepsilon^{ab} \partial_a A_b, \varepsilon^{ab} \partial_a F_{xy} - m^2 A^b = 0, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a + \zeta \sigma_x \varepsilon F_{xy})$   
 $\Leftrightarrow \partial^a A_a = 0, F_{xy} = \partial_x A_y - \partial_y A_x, \partial_a F^{ab} - m^2 A^b = 0$   
 $F_{ab} := \partial_a A_b - \partial_b A_a = \varepsilon_{ab} F_{xy}, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \zeta \sigma_x \varepsilon F_{xy})$   
 $\Leftrightarrow \partial^a F_{ab} - m^2 A_b = 0; F_{ab} := \partial_a A_b - \partial_b A_a = \varepsilon_{ab} F_{xy}, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \zeta \sigma_x \varepsilon F_{xy})$   
 $\Leftrightarrow (\partial^b \partial_b - m^2)A_a = 0, \partial^a A_a = 0, \psi = (im\gamma^a + \zeta \varepsilon^{ab} \sigma_x \partial_b) \varepsilon \frac{A_a}{\sqrt{2}}$  □

证明:  $A_a = \frac{1}{\sqrt{2im}} (\bar{\varepsilon} \gamma_a)^{\lambda_s \mu_s} \psi_{\lambda_s \mu_s}$   
 $\Rightarrow [A_a(x), A_{a'}^{+}(x')] = \frac{1}{(\sqrt{2m})^2} (\bar{\varepsilon} \gamma_a)^{\lambda_s \mu_s} (\gamma_{a'} \varepsilon)^{\lambda'_s \mu'_s} [\psi_{\lambda_s \mu_s}, \psi_{\lambda'_s \mu'_s}^+]$   
 $= \frac{1}{(\sqrt{2m})^2} (\bar{\varepsilon} \gamma_a)^{\lambda_s \mu_s} (\gamma_{a'} \varepsilon)^{\lambda'_s \mu'_s} \frac{i}{8} [(m - \gamma^b \partial_b) \gamma^0]_{\{\lambda_s \lambda'_s\} \{\mu_s \mu'_s\}} [(m - \gamma^c \partial_c) \gamma^0]_{\mu_s \mu'_s} \Delta(x - x')$   
 $= \frac{i}{2} \frac{1}{(\sqrt{2m})^2} (\bar{\varepsilon} \gamma_a)^{\lambda_s \mu_s} (\gamma_{a'} \varepsilon)^{\lambda'_s \mu'_s} [(m - \gamma^b \partial_b) \gamma^0]_{\lambda_s \lambda'_s} [(m - \gamma^c \partial_c) \gamma^0]_{\mu_s \mu'_s} \Delta(x - x')$   
 $= \frac{i}{2} \frac{1}{(2m)^2} [(\bar{\varepsilon} \gamma_a)(m - \gamma^b \partial_b) \gamma^0 (\gamma_{a'} \varepsilon)]^{\mu_s \mu'_s} [(m - \gamma^c \partial_c) \gamma^0]_{\mu_s \mu'_s} \Delta(x - x')$   
 $= \frac{i}{2} \frac{1}{(\sqrt{2m})^2} \text{tr}\{[(\bar{\varepsilon} \gamma_a)(m - \gamma^b \partial_b) \gamma^0 (\gamma_{a'} \varepsilon)] [-\gamma^0 (m - \gamma^{*c} \partial_c)]\} \Delta(x - x')$   
 $= \frac{i}{2} \frac{1}{(\sqrt{2m})^2} \text{tr}\{[(\gamma^0 \gamma_a)(m - \gamma^b \partial_b) \gamma^0 (\gamma_{a'} \gamma^0)] [(-m - \gamma^c \partial_c) \gamma^0]\} \Delta(x - x')$   
 $= \frac{i}{2} \frac{1}{(2m)^2} \text{tr}\{\gamma_a (m - \gamma^b \partial_b) \gamma_{a'}^* (m + \gamma^c \partial_c)\} \Delta(x - x')$   
 $= \frac{i}{2} \frac{1}{(\sqrt{2m})^2} \text{tr}\{\gamma_a (m - \gamma_b \partial^b) \gamma_{b'} (m + \gamma_c \partial^c)\} \eta_{a'}^{b'} \Delta(x - x')$   
 $= \frac{i}{2} \frac{1}{(2m)^2} \text{tr}\{\gamma_a [\gamma_{b'} (m + \gamma_b \partial^b) - 2\delta_{bb'} \partial^b] (m + \gamma_c \partial^c)\} \eta_{a'}^{b'} \Delta(x - x')$   
 $= \frac{i}{(2m)^2} \text{tr}\{\gamma_a [m \gamma_{b'} - \partial_{b'}] (m + \gamma_c \partial^c)\} \eta_{a'}^{b'} \Delta(x - x')$   
 $= \frac{i}{(2m)^2} \text{tr}\{\gamma_a (m^2 \gamma_{b'} - \gamma_c \partial_{b'} \partial^c)\} \eta_{a'}^{b'} \Delta(x - x')$   
 $= \frac{i}{(\sqrt{2m})^2} \text{tr}\{(m^2 \delta_{ab'} - \delta_{ac} \partial_{b'} \partial^c)\} \eta_{a'}^{b'} \Delta(x - x')$   
 $= i(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x - x')$  □

定理5.3.2.  $[(m - \gamma^b \partial_b) \gamma^0]_{\{\lambda_s \lambda'_s\} \{\mu_s \mu'_s\}} \Delta(x - x') = X_{\{\lambda_s \mu_s\}}(x) X_{(\lambda'_s \mu'_s)}^+(x') (\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \Delta(x - x')$

证明:  $\psi = (im\gamma^a + \varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon\frac{A_a}{\sqrt{2}}$

$$\Rightarrow A_a = \frac{1}{\sqrt{2im}}tr(\bar{\varepsilon}\gamma_a\psi)$$

$$\Rightarrow \psi = [(im\gamma^a + \varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]\frac{1}{2im}tr(\bar{\varepsilon}\gamma_a\psi)$$

$$\Rightarrow \psi_{\lambda_s\mu_s} = [(im\gamma^a + \varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]_{\lambda_s\mu_s}[\frac{1}{2im}(\bar{\varepsilon}\gamma_a)^{\tilde{\lambda}_s\tilde{\mu}_s}\psi_{\tilde{\lambda}_s\tilde{\mu}_s}]$$

$$\Rightarrow [\psi_{\lambda_s\mu_s}(x), \psi_{\lambda'_s\mu'_s}(x')]$$

$$= [(im\gamma^a + \varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]_{\lambda_s\mu_s}[(im\gamma^{a'} + \varepsilon^{a'b'}\varsigma\sigma_x\partial_{b'})\varepsilon]_{\lambda'_s\mu'_s}[\frac{1}{2im}(\bar{\varepsilon}\gamma_a)^{\tilde{\lambda}_s\tilde{\mu}_s}][\frac{1}{-2im}(\gamma_a\varepsilon)^{\tilde{\lambda}'_s\tilde{\mu}'_s}][\psi_{\tilde{\lambda}'_s\tilde{\mu}'_s}(x'), \psi_{\tilde{\lambda}_s\tilde{\mu}_s}(x)]$$

$$\Rightarrow \frac{i}{8}[(m - \gamma^b\partial_b)\gamma^0]_{\{\lambda_s(\lambda'_s)\mu_s\}\mu'_s}\Delta(x - x')$$

$$= [(im\gamma^a + \varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]_{\lambda_s\mu_s}[(im\gamma^{a'} + \varepsilon^{a'b'}\varsigma\sigma_x\partial_{b'})\varepsilon]_{\lambda'_s\mu'_s}[\frac{1}{2im}(\bar{\varepsilon}\gamma_a)^{\tilde{\lambda}_s\tilde{\mu}_s}][\frac{1}{-2im}(\gamma_a\varepsilon)^{\tilde{\lambda}'_s\tilde{\mu}'_s}][\frac{i}{8}[(m - \gamma^b\partial_b)\gamma^0]_{\{\tilde{\lambda}_s(\tilde{\lambda}'_s)\tilde{\mu}_s\}\tilde{\mu}'_s}\Delta(x - x')]$$

$$= \frac{i}{2}[(im\gamma^a + \varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]_{\lambda_s\mu_s}[\bar{\varepsilon}(-im\gamma^{a'} + \varepsilon^{a'b'}\varsigma\sigma_x\partial_{b'}^+)]_{\lambda'_s\mu'_s}(\eta_{aa'} - \frac{\partial_a\partial_{a'}}{m^2})\Delta(x - x')$$

$$= \frac{i}{2}X_{\lambda_s\mu_s}^a(x)X_{\lambda'_s\mu'_s}^{+a'}(x')(\eta_{aa'} - \frac{\partial_a\partial_{a'}}{m^2})\Delta(x - x'), X_{\lambda_s\mu_s}^a(x) := [(im\gamma^a + \varsigma\varepsilon^{ab}\sigma_x\partial_b)\varepsilon]_{\lambda_s\mu_s}$$

$$\Rightarrow [(m - \gamma^b\partial_b)\gamma^0]_{\{\lambda_s(\lambda'_s)\mu_s\}\mu'_s}\Delta(x - x') = X_{\{\lambda_s\mu_s\}}^a(x)X_{(\lambda'_s\mu'_s)}^{+a'}(x')(\eta_{aa'} - \frac{\partial_a\partial_{a'}}{m^2})\Delta(x - x') \quad \square$$

定理5.3.3.  $[F_{ab}(x), F_{a'b'}^+(x')] = -i\eta_{[a<a'}\partial_b]\partial_{b'}^+\Delta(x - x')$

定理5.3.4.  $[F_{xy}(x), F_{xy}^+(x')] = im^2\Delta(x - x')$

### 5.3.2 二维时空中自旋-n的类Klein-Gordon方程 [17, 19]

定理5.3.5.

$$\begin{cases} [\gamma^a(\varsigma)\partial_a + m]\psi_{\underbrace{[\lambda_s\mu_s\eta_s\xi_s \dots]_{2n}}_n}(x) = 0, \psi_{\underbrace{\lambda_s\mu_s\eta_s\xi_s \dots]_{2n}}_n}(x) \text{全对称} \\ A_{\underbrace{ab \dots]_n}(x) := (\frac{1}{\sqrt{2im}})^n (\bar{\varepsilon}\gamma_a)^{\lambda_s\mu_s} (\bar{\varepsilon}\gamma_b)^{\eta_s\xi_s} \dots \psi_{\lambda_s\mu_s\eta_s\xi_s \dots]_{2n}}(x) \end{cases} \Rightarrow \begin{cases} (-\partial^c\partial_c + m^2)A_{\underbrace{ab \dots]_n}(x) = 0 \\ \partial^a A_{\underbrace{ab \dots]_n}(x) = 0, A_{\underbrace{ab \dots]_n}(x) \text{全对称} \end{cases}$$

$$\psi_{\underbrace{\lambda_s\mu_s \dots]_{2n}}_n}(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{n-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2n} [a(\vec{p}, h)U_{\lambda_s\mu_s \dots]_{2n}}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)V_{\lambda_s\mu_s \dots]_{2n}}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$A_{\underbrace{ab \dots]_n}(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2^n E}} [a(\vec{p})\varepsilon_{\underbrace{ab \dots]_n}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p})\tilde{\varepsilon}_{\underbrace{ab \dots]_n}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}]d^N\vec{p}$$

$$\varepsilon_{\underbrace{ab \dots]_n}(\vec{p}) = \frac{1}{i^n} (\bar{\varepsilon}\gamma_a)^{\lambda_s\mu_s} (\bar{\varepsilon}\gamma_b)^{\eta_s\xi_s} \dots U_{\lambda_s\mu_s\eta_s\xi_s \dots]_{2n}}(\vec{p}), \tilde{\varepsilon}_{\underbrace{ab \dots]_n}(\vec{p}) = \frac{1}{i^n} (\bar{\varepsilon}\gamma_a)^{\lambda_s\mu_s} (\bar{\varepsilon}\gamma_b)^{\eta_s\xi_s} \dots V_{\lambda_s\mu_s\eta_s\xi_s \dots]_{2n}}(\vec{p})$$

### 5.3.3 二维时空中类Klein-Gordon方程自旋基的性质

推论5.3.1.  $\varepsilon_{\underbrace{ab \dots]_n}(\vec{p}) = \varepsilon_a(\vec{p})\varepsilon_b(\vec{p}) \dots, \tilde{\varepsilon}_{\underbrace{ab \dots]_n}(\vec{p}) = \tilde{\varepsilon}_a(\vec{p})\tilde{\varepsilon}_b(\vec{p}) \dots$

推论5.3.2.  $\varepsilon_a(\vec{p}) = \tilde{\varepsilon}_a(\vec{p}) = \frac{1}{m}(E, ip_x), \varepsilon_{\underbrace{ab \dots]_n}(\vec{p}) = \tilde{\varepsilon}_{\underbrace{ab \dots]_n}(\vec{p})$

证明:  $\varepsilon_a(\vec{p}) = -i(\bar{\varepsilon}\gamma_a)^{\lambda_s\mu_s}U_{\lambda_s\mu_s}(\vec{p})$

$$= -iu^T(\vec{p})(\bar{\varepsilon}\gamma_a)u(\vec{p})$$

$$= -iu^+(\vec{p})(i, \varsigma\sigma_x)_a u(\vec{p})$$

$$= (\frac{E}{m}, -i\varsigma u^+(\vec{p})v(\vec{p}))_a$$

$$= (\frac{E}{m}, i\frac{p_x}{m})_a \quad \square$$

推论5.3.3.  $\varepsilon_a(\vec{p})\delta^{ab}\varepsilon_b(\vec{p}) = 1$

证明:  $X_a^{\lambda_s\mu_s}(\vec{p})\varepsilon^a(\vec{p})$

$$= X_a^{\lambda_s\mu_s}(-\frac{E}{m}, -i\frac{p_x}{m})^a$$

$$\neq U^{\lambda_s\mu_s}(\vec{p}) \quad \square$$



$$\text{推论5.3.4. } \varepsilon_{ab\dots}(\vec{p})\varepsilon_{a'b'\dots}^+(\vec{p}) = \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{\varepsilon}\gamma_b)^{\eta_\zeta\xi_\zeta}\dots}^n \overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\zeta\mu'_\zeta}(\gamma_{b'}\varepsilon)^{\eta'_\zeta\xi'_\zeta}\dots}^n U_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\dots}(x)U_{\lambda'_\zeta\mu'_\zeta\eta'_\zeta\xi'_\zeta\dots}^+(x)$$

$$\text{推论5.3.5. } \varepsilon_{ab\dots}(\vec{p})\varepsilon_{a'b'\dots}^+(\vec{p}) = \underbrace{(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})(\eta_{bb'} + \frac{p_b p_{b'}}{m^2})\dots}_n$$

$$\text{推论5.3.6. } \varepsilon_a(\vec{p})\varepsilon_a^+(\vec{p}) = \eta_{aa'} + \frac{p_a p_{a'}}{m^2}, \varepsilon_a(\vec{p})\varepsilon_a^+(\vec{p})\eta_b^{a'} = \varepsilon_a(\vec{p})\varepsilon_b(\vec{p}) = \delta_{ab} + \frac{p_a p_b}{m^2}$$

$$\text{推论5.3.7. } \begin{cases} U_{\lambda_\zeta\mu_\zeta\dots}(\vec{p})U_{\lambda'_\zeta\mu'_\zeta\dots}^+(\vec{p}) = \frac{1}{(2m)^{2n}} \underbrace{[X_{\lambda_\zeta\mu_\zeta}^a(p)X_{\lambda'_\zeta\mu'_\zeta}^{+a'}(-p)]\dots}_n \varepsilon_{ab\dots}(\vec{p})\varepsilon_{a'b'\dots}^+(\vec{p}) \\ V_{\lambda_\zeta\mu_\zeta\dots}(\vec{p})V_{\lambda'_\zeta\mu'_\zeta\dots}^+(\vec{p}) = \frac{1}{(2m)^{2n}} \underbrace{[X_{\lambda_\zeta\mu_\zeta}^a(-p)X_{\lambda'_\zeta\mu'_\zeta}^{+a'}(p)]\dots}_n \tilde{\varepsilon}_{ab\dots}(\vec{p})\tilde{\varepsilon}_{a'b'\dots}^+(\vec{p}) \end{cases}$$

$$\text{推论5.3.8. } \begin{cases} [A_{ab\dots}(x), A_{a'b'\dots}^+(x')] = \frac{1}{m^{2n-1}} \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\zeta\mu_\zeta}\dots}^n \overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\zeta\mu'_\zeta}\dots}^n [\psi_{\lambda_\zeta\mu_\zeta\dots}(x), \psi_{\lambda'_\zeta\mu'_\zeta\dots}^+(x')] \\ [\psi_{\lambda_\zeta\mu_\zeta\dots}(x), \psi_{\lambda'_\zeta\mu'_\zeta\dots}^+(x')] = \frac{1}{2^{2n-1}} \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a(x)\dots}_n \underbrace{\mathbb{X}_{\lambda'_\zeta\mu'_\zeta}^{+a'}(x')\dots}_n [A_{ab\dots}(x), A_{a'b'\dots}^+(x')] \end{cases}$$

### 5.3.4 二维时空中自旋 $-n + \frac{1}{2}$ 的类Rarita-Schwinger方程 [17, 19]

定理5.3.6.

$$\begin{cases} [\gamma^a(\zeta)\partial_a + m]\psi_{[\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\dots\tau_\zeta]}(x) = 0, \psi_{[\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\dots\tau_\zeta]}(x) \text{全对称} \\ A_{ab\dots\tau_\zeta}(x) := (\frac{1}{\sqrt{2im}})^n \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{\varepsilon}\gamma_b)^{\eta_\zeta\xi_\zeta}\dots}^n \psi_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\dots\tau_\zeta}(x) \end{cases} \Rightarrow \begin{cases} (-\partial^c\partial_c + m^2)A_{ab\dots\tau_\zeta}(x) = 0 \\ \partial^a A_{ab\dots\tau_\zeta}(x) = 0, A_{ab\dots\tau_\zeta}(x) \text{全对称} \end{cases}$$

$$\psi_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^n \sqrt{\frac{m}{E}}^{2n+1} [a(\vec{p}, h)U_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p}, h)V_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^N \vec{p}$$

$$A_{ab\dots\tau_\zeta}(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{\sqrt{m}}{\sqrt{2^n E}} [a(\vec{p})\varepsilon_{ab\dots\tau_\zeta}(\vec{p})e^{i(\vec{p}\cdot\vec{r}-Et)} + b^+(\vec{p})\tilde{\varepsilon}_{ab\dots\tau_\zeta}(\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)}] d^N \vec{p}$$

$$\varepsilon_{ab\dots\tau_\zeta}(\vec{p}) = \frac{1}{i^n} \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{\varepsilon}\gamma_b)^{\eta_\zeta\xi_\zeta}\dots}^n U_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\dots\tau_\zeta}(\vec{p}), \tilde{\varepsilon}_{ab\dots\tau_\zeta}(\vec{p}) = \frac{1}{i^n} \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{\varepsilon}\gamma_b)^{\eta_\zeta\xi_\zeta}\dots}^n V_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\dots\tau_\zeta}(\vec{p})$$

### 5.3.5 二维时空中类Rarita-Schwinger方程自旋基的性质

$$\text{推论5.3.9. } \varepsilon_{ab\dots\tau_\zeta}(\vec{p}) = \underbrace{I \otimes I \otimes \dots \otimes I}_{2n} \otimes \sigma_x \tilde{\varepsilon}_{ab\dots\tau_\zeta}(\vec{p})$$

$$\text{推论5.3.10. } \varepsilon_{ab\dots\tau_\zeta}(\vec{p})\varepsilon_{a'b'\dots\tau'_\zeta}^+(\vec{p}) = \overbrace{(\bar{\varepsilon}\gamma_a)^{\lambda_\zeta\mu_\zeta}(\bar{\varepsilon}\gamma_b)^{\eta_\zeta\xi_\zeta}\dots}^n \overbrace{(\gamma_{a'}\varepsilon)^{\lambda'_\zeta\mu'_\zeta}(\gamma_{b'}\varepsilon)^{\eta'_\zeta\xi'_\zeta}\dots}^n U_{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\dots\tau_\zeta}(x)U_{\lambda'_\zeta\mu'_\zeta\eta'_\zeta\xi'_\zeta\dots\tau'_\zeta}^+(x)$$

$$\text{推论5.3.11. } \varepsilon_{ab\dots\tau_\zeta}(\vec{p})\varepsilon_{a'b'\dots\tau'_\zeta}^+(\vec{p}) = \frac{1}{2m} \underbrace{(\eta_{aa'} + \frac{p_a p_{a'}}{m^2})(\eta_{bb'} + \frac{p_b p_{b'}}{m^2})\dots}_n [(m - i\gamma^c p_c)\gamma^0]_{\tau_\zeta\tau'_\zeta}$$

$$\text{推论5.3.12. } \begin{cases} U_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p})U_{\lambda'_\zeta\mu'_\zeta\dots\tau'_\zeta}^+(\vec{p}) = \frac{1}{(2m)^{2n+1}} \underbrace{[X_{\lambda_\zeta\mu_\zeta}^a(p)X_{\lambda'_\zeta\mu'_\zeta}^{+a'}(-p)]\dots}_n \varepsilon_{ab\dots\tau_\zeta}(\vec{p})\varepsilon_{a'b'\dots\tau'_\zeta}^+(\vec{p}) \\ V_{\lambda_\zeta\mu_\zeta\dots\tau_\zeta}(\vec{p})V_{\lambda'_\zeta\mu'_\zeta\dots\tau'_\zeta}^+(\vec{p}) = \frac{1}{(2m)^{2n+1}} \underbrace{[X_{\lambda_\zeta\mu_\zeta}^a(-p)X_{\lambda'_\zeta\mu'_\zeta}^{+a'}(p)]\dots}_n \tilde{\varepsilon}_{ab\dots\tau_\zeta}(\vec{p})\tilde{\varepsilon}_{a'b'\dots\tau'_\zeta}^+(\vec{p}) \end{cases}$$

$$\text{推论5.3.13.} \quad \left\{ \begin{aligned} [A_{ab \dots \tau_\zeta}^{\dots}(x), A_{a'b' \dots \tau'_\zeta}^{\dots}(x')] &= \frac{1}{m^{2n} 2^n} \overbrace{(\bar{\varepsilon} \gamma_a)^{\lambda_\zeta \mu_\zeta} \dots}^n \overbrace{(\gamma_{a'} \varepsilon)^{\lambda'_\zeta \mu'_\zeta} \dots}^n \cdot [\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(x')] \\ [\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}^+(x')] &= \frac{1}{2^{2n-1}} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x')}_n \cdot [A_{ab \dots \tau_\zeta}^{\dots}(x), A_{a'b' \dots \tau'_\zeta}^{\dots}(x')] \end{aligned} \right.$$

自我评述：二维时空中只有少部分B-W方程存在等价的类C-K方程，但仍存在类似的对易规则。

## 5.4 二维时空中自旋 $-\frac{3}{2}$ 的Bargmann-Wigner方程<sup>[17]</sup>

证明： $(\gamma^a \partial_a + m)^{\kappa_\zeta} \psi^{\lambda_\zeta \mu_\zeta \eta_\zeta} = 0$ ,  $\psi^{\lambda_\zeta \mu_\zeta \eta_\zeta}$  全对称,  $\gamma^a = (-\sigma_y, \varsigma \sigma_z)$

$$\Leftrightarrow (\partial^b \partial_b - m^2) A_{a\eta_\zeta} = 0, \partial^a A_{a\eta_\zeta} = 0, \psi^{\lambda_\zeta \mu_\zeta \eta_\zeta} = [(im\gamma^a + \varsigma \varepsilon^{ab} \sigma_x \partial_b) \varepsilon]^{\lambda_\zeta \mu_\zeta} A_a^{\eta_\zeta}, \bar{\varepsilon}_{\mu_\zeta \eta_\zeta} \psi^{\lambda_\zeta \mu_\zeta \eta_\zeta} = 0 \quad \square$$

证明： $\psi^{\lambda_\zeta \mu_\zeta \eta_\zeta} = [(im\gamma^a + \varsigma \varepsilon^{ab} \sigma_x \partial_b) \varepsilon]^{\lambda_\zeta \mu_\zeta} A_a^{\eta_\zeta}$ ,  $\psi^{\lambda_\zeta \mu_\zeta \eta_\zeta} = 0$

$$\Rightarrow [(im\gamma^a + \varsigma \varepsilon^{ab} \sigma_x \partial_b) \varepsilon]^{\lambda_\zeta \mu_\zeta} \bar{\varepsilon}_{\mu_\zeta \eta_\zeta} A_a^{\eta_\zeta} = 0$$

$$\Leftrightarrow [(im\gamma^a + \varsigma \varepsilon^{ab} \sigma_x \partial_b)]^{\lambda_\zeta} A_a^{\eta_\zeta} = 0 \quad \square$$

## 5.5 二维时空中的自旋方程

### 5.5.1 二维时空中自旋-s粒子的自旋方程

定理5.5.1.  $[\gamma^a \partial_a + m] \psi_{\underbrace{[\lambda_\zeta] \mu_\zeta}_{2s}} \dots(x) = 0$ ,  $\psi_{\underbrace{\lambda_\zeta \mu_\zeta}_{2s}} \dots(x)$  全对称,  $\gamma_a := [-\sigma_y, \varsigma \sigma_z]$

$$\Leftrightarrow \begin{cases} [s\partial_a + m\gamma_a(s) + iS_{ab}(s, \varsigma) \partial^b] \psi(s) = 0 \\ [\gamma_x(s) \partial_\pi - \gamma_\pi(s) \partial_x] \psi(s) = i\varsigma m \gamma_y(s) \psi(s) \end{cases}, S_{ab}(s, \varsigma) = -i[\gamma_a(s), \gamma_b(s)], \gamma_a(s) := [-\sigma_y(s), \varsigma \sigma_z(s)]$$

证明： $[\gamma^a \partial_a + m] \psi_{\underbrace{[\lambda_\zeta] \mu_\zeta}_{2s}} \dots(x) = 0$ ,  $\psi_{\underbrace{\lambda_\zeta \mu_\zeta}_{2s}} \dots(x)$  全对称

$$\Leftrightarrow [\gamma^a \partial_a + m] \hat{\psi}(s) = 0$$

$$\Leftrightarrow (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a D^a \hat{\psi}(s) = im\sigma_z \otimes I_{2^{2s-1}} \hat{\psi}(s), D^a = (\partial^x, 0, 0, \partial^\pi)$$

$$\Leftrightarrow (\sigma \otimes I_{2^{2s-1}}, -i\varsigma)_a D^a [I \otimes \Gamma(s)] N(s) \psi(s) = im\sigma_z \otimes I_{2^{2s-1}} \hat{\psi}(s)$$

$$\Leftrightarrow [I \otimes \Gamma(s)] (\sigma \otimes I_{2s}, -i\varsigma)_a D^a N(s) \psi(s) = im\sigma_z \otimes I_{2^{2s-1}} \hat{\psi}(s)$$

$$\Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_a D^a N(s) \psi(s) = im[I \otimes \bar{\Gamma}(s)] (\sigma_z \otimes I_{2^{2s-1}}) \hat{\psi}(s)$$

$$\Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_a D^a N(s) \psi(s) = im(\sigma_z \otimes I_{2s}) [I \otimes \bar{\Gamma}(s)] \hat{\psi}(s)$$

$$\Leftrightarrow (\sigma \otimes I_{2s}, -i\varsigma)_a D^a N(s) \psi(s) = im(\sigma_z \otimes I_{2s}) N(s) \psi(s)$$

$$\Leftrightarrow Z_b D^b \psi(s) = im \frac{i\varsigma}{\sqrt{2}} (\sigma_z \otimes I_{2s}) N(s) \psi(s)$$

$$\Leftrightarrow \bar{Z}_a Z_b D^b \psi(s) = im \frac{i\varsigma}{\sqrt{2}} \bar{Z}_a (\sigma_z \otimes I_{2s}) N(s) \psi(s)$$

$$\Leftrightarrow \bar{Z}_a Z_b D^b \psi(s) = im \frac{i\varsigma}{\sqrt{2}} \frac{-i\varsigma}{\sqrt{2}} \bar{N}(s) (\sigma \otimes I_{2s}, i\varsigma)_a (\sigma_z \otimes I_{2s}) N(s) \psi(s)$$

$$\Leftrightarrow \bar{Z}_a Z_b D^b \psi(s) = -\frac{m}{2s} \bar{N}(s) [(-\sigma_y, \sigma_x, -i) \otimes I_{2s}, \varsigma \sigma_z \otimes I_{2s}]_a N(s) \psi(s)$$

$$\Leftrightarrow [sD_a + iS_{ab}(s, \varsigma; 4) D^b] \psi(s) = -m [(-\sigma_y(s), \sigma_x(s), -i), \varsigma \sigma_z(s)]_a \psi(s)$$

$$S_{ab}(s, \varsigma; 4) \succ \begin{bmatrix} 0 & \sigma_z(s) & -\sigma_y(s) & -\varsigma \sigma_x(s) \\ -\sigma_z(s) & 0 & \sigma_x(s) & -\varsigma \sigma_y(s) \\ \sigma_y(s) & -\sigma_x(s) & 0 & -\varsigma \sigma_z(s) \\ \varsigma \sigma_x(s) & \varsigma \sigma_y(s) & \varsigma \sigma_z(s) & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} [s\partial_a + iS_{ab}(s, \varsigma) \partial^b] \psi(s) = -m\gamma_a(s) \psi(s) \\ [\gamma_x(s) \partial_\pi - \gamma_\pi(s) \partial_x] \psi(s) = i\varsigma m \gamma_y(s) \psi(s) \end{cases}, S_{ab}(s, \varsigma) = -i[\gamma_a(s), \gamma_b(s)] \succ \begin{bmatrix} 0 & -\varsigma \sigma_x(s) \\ \varsigma \sigma_x(s) & 0 \end{bmatrix}$$

$$\Leftrightarrow [s\partial_a + m\gamma_a(s) + iS_{ab}(s, \varsigma) \partial^b] \psi(s) = 0$$

$$\Leftrightarrow [\gamma_x(s) \partial_\pi - \gamma_\pi(s) \partial_x] \psi(s) = i\varsigma m \gamma_y(s) \psi(s) \quad \square$$

## 5.5.2 二维时空中自旋方程的平面波解及其自旋基

定理5.5.2.

$$\left\{ \begin{array}{l} [\gamma^a \partial_a + m] \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}(x)}_{2s} = 0, \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}(x)}_{2s} \text{全对称} \\ \psi_{k_\zeta}(x) := \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots} \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}(x)}_{2s} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} [s \partial_a + m \gamma_a(s) + i S_{ab}(s, \zeta) \partial^b] \psi(s) = 0 \\ [\gamma_x(s) \partial_\pi - \gamma_\pi(s) \partial_x] \psi(s) = i \zeta m \gamma_y(s) \psi(s) \\ \underbrace{\psi_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}(x)}_{2s} = \Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots}^{k_\zeta} \underbrace{\psi_{k_\zeta}(x)}_{2s} \end{array} \right.$$

$$\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots}(\vec{r}, t)}_{2s} = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{n-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} [a(\vec{p}, h) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p})}_{2s} e^{i(\vec{p} \cdot \vec{r} - Et)} + b^+(\vec{p}, h) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}(\vec{p})}_{2s} e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}$$

$$\psi_{k_\zeta}(\vec{r}, t) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{n-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} [a(\vec{p}, h) U_{k_\zeta}(\vec{p}; s) e^{i(\vec{p} \cdot \vec{r} - Et)} + b^+(\vec{p}, h) V_{k_\zeta}(\vec{p}; s) e^{-i(\vec{p} \cdot \vec{r} - Et)}] d^N \vec{p}$$

定理5.5.3.

$$\left\{ \begin{array}{l} U_{k_\zeta}(\vec{p}; s) := \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots} \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p})}_{2s} \Leftrightarrow \underbrace{U_{\lambda_\zeta \mu_\zeta \dots}(\vec{p})}_{2s} = \Gamma_{\lambda_\zeta \mu_\zeta \dots}^{k_\zeta} \underbrace{U_{k_\zeta}(\vec{p}; s)}_{2s} \\ V_{k_\zeta}(\vec{p}; s) := \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots} \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}(\vec{p})}_{2s} \Leftrightarrow \underbrace{V_{\lambda_\zeta \mu_\zeta \dots}(\vec{p})}_{2s} = \Gamma_{\lambda_\zeta \mu_\zeta \dots}^{k_\zeta} \underbrace{V_{k_\zeta}(\vec{p}; s)}_{2s} \end{array} \right.$$

## 5.5.3 二维时空中自旋方程的自旋基正交性

定理5.5.4.  $U^{+k_\zeta}(\vec{p}; s) U_{k_\zeta}(\vec{p}; s) = (\frac{E}{m})^{2s}, V^{+k_\zeta}(\vec{p}; s) V_{k_\zeta}(\vec{p}; s) = (\frac{E}{m})^{2s}$ 

$$\text{定理5.5.5.} \left\{ \begin{array}{l} U_{k_\zeta}(\vec{p}; s) U_{k'_\zeta}^+(\vec{p}; s) = \frac{1}{(\zeta 2m)^{2s}} \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots} \Gamma_{k'_\zeta}^{\lambda'_\zeta \mu'_\zeta \dots} \underbrace{[m\sigma_z - (\sigma, i\zeta)^a p_a]_{\lambda_\zeta \lambda'_\zeta} [m\sigma_z - (\sigma, i\zeta)^b p_b]_{\mu_\zeta \mu'_\zeta} \dots}_{2s} \\ V_{k_\zeta}(\vec{p}; s) V_{k'_\zeta}^+(\vec{p}; s) = \frac{1}{(\zeta 2m)^{2s}} \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots} \Gamma_{k'_\zeta}^{\lambda'_\zeta \mu'_\zeta \dots} \underbrace{[-m\sigma_z - (\sigma, i\zeta)^a p_a]_{\lambda_\zeta \lambda'_\zeta} [-m\sigma_z - (\sigma, i\zeta)^b p_b]_{\mu_\zeta \mu'_\zeta} \dots}_{2s} \end{array} \right.$$

## 5.6 二维时空中有质量粒子协变对易规则

## 5.6.1 二维时空中有质量玻色子协变对易规则梳理

$$\text{定理5.6.1.} \underbrace{[\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]}_{2n} = \underbrace{\frac{i}{2^{2n-1}} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta \lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots}_{2n} \Delta(x - x')$$

$$\Leftrightarrow$$

$$\text{定理5.6.2.} \underbrace{[\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]}_{2n} = i \underbrace{\frac{i^{2n}}{2^{2n-1}} [-im\sigma_z + (\sigma, i\zeta)^a \partial_a]_{\lambda_\zeta \lambda'_\zeta} [-im\sigma_z + (\sigma, i\zeta)^b \partial_b]_{\mu_\zeta \mu'_\zeta} \dots}_{2n} \Delta(x - x')$$

$$\Leftrightarrow$$

$$\text{定理5.6.3.} \underbrace{[\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]}_{2n} = \frac{i}{2^{2n-1}} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x')}_n \cdot \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \Delta(x - x')$$

## 5.6.2 二维时空中有质量费米子协变对易规则梳理

$$\text{定理5.6.4.} \underbrace{[\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]}_{2n+1} = \underbrace{\frac{i}{2^{2n}} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda_\zeta \lambda'_\zeta} [(m - \gamma^b \partial_b) \gamma^0]_{\mu_\zeta \mu'_\zeta} \dots}_{2n+1} \Delta(x - x')$$

$$\Leftrightarrow$$

$$\text{定理5.6.5.} \underbrace{[\psi_{\lambda_\zeta \mu_\zeta \dots}(x), \psi_{\lambda'_\zeta \mu'_\zeta \dots}^+(x')]}_{2n+1} = i \underbrace{\frac{i^{2n+1}}{2^{2n}} [-im\sigma_z + (\sigma, i\zeta)^a \partial_a]_{\lambda_\zeta \lambda'_\zeta} [-im\sigma_z + (\sigma, i\zeta)^b \partial_b]_{\mu_\zeta \mu'_\zeta} \dots}_{2n+1} \Delta(x - x')$$

$$\Leftrightarrow$$

定理5.6.6.

$$[\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}_{2n+1}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}}_{2n+1}(x')] = \frac{i}{2^{2n}} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x')}_n \cdot \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdot [(m - \gamma^c \partial_c) \gamma^0]_{\tau_\zeta \tau'_\zeta} \Delta(x - x')$$

$$[\Downarrow]$$

定理5.6.7.

$$[\underbrace{\psi_{\lambda_\zeta \mu_\zeta \dots \tau_\zeta}}_{2n+1}(x), \underbrace{\psi_{\lambda'_\zeta \mu'_\zeta \dots \tau'_\zeta}}_{2n+1}(x')] = i \frac{i\zeta}{2^{2n}} \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x)}_n \cdot \underbrace{\mathbb{X}_{\lambda'_\zeta \mu'_\zeta}^{+a'}(x')}_n \cdot \underbrace{[\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}]}_n \cdot [-im\sigma_z + (\sigma, i\zeta)^b \partial_b]_{\tau_\zeta \tau'_\zeta} \Delta(x - x')$$

## 6 二维时空中无质量粒子的Bargmann-Wigner方程

### 6.1 二维时空中的无质量Dirac方程

证明:  $\gamma^a \partial_a \psi(x) = 0, \gamma^a = (-\sigma_y, \varsigma \sigma_z)$

$$\Leftrightarrow (\sigma_x, -i\zeta)^a \partial_a \psi(x) = 0$$

$$\Leftrightarrow (\sigma_x, -i\zeta)^a p_a \lambda(\hat{p}, -\frac{\zeta}{2}) = 0$$

$$\Leftrightarrow \sigma_x p_x \lambda(\hat{p}, -\frac{\zeta}{2}) = -\zeta |p_x| \lambda(\hat{p}, -\frac{\zeta}{2})$$

$$\Leftrightarrow \sigma_x \hat{p}_x \lambda(\hat{p}, -\frac{\zeta}{2}) = -\zeta \lambda(\hat{p}, -\frac{\zeta}{2})$$

$$\Leftrightarrow \lambda(\hat{p}, -\frac{\zeta}{2}) = \frac{1}{2\sqrt{2}} \begin{bmatrix} (1-\zeta) - (1+\zeta)\hat{p}_x \\ (1+\zeta) + (1-\zeta)\hat{p}_x \end{bmatrix}$$

□

推论6.1.1.

$$\begin{cases} \psi(\vec{r}, t) := \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} \lambda(\hat{p}, -\frac{\zeta}{2}) [a_1(\vec{p}, -\frac{\zeta}{2}) e^{ip \cdot x} + a_2^+(\vec{p}, -\frac{\zeta}{2}) e^{-ip \cdot x}] d^N \vec{p} \\ a_1(\vec{p}, -\frac{\zeta}{2}) = \frac{1}{(2\pi)^{N/2}} \int \lambda^+(\hat{p}, -\frac{\zeta}{2}) \psi(\vec{r}, t) e^{-ip \cdot x} d^N \vec{r} \\ a_2^+(\vec{p}, -\frac{\zeta}{2}) = \frac{1}{(2\pi)^{N/2}} \int \lambda^+(\hat{p}, -\frac{\zeta}{2}) \psi(\vec{r}, t) e^{ip \cdot x} d^N \vec{r} \end{cases}$$

$$\lambda^+(\hat{p}, -\frac{\zeta}{2}) \lambda(\hat{p}, -\frac{\zeta}{2}) = 1, \lambda(\hat{p}, -\frac{\zeta}{2}) \lambda^+(\hat{p}, -\frac{\zeta}{2}) = \frac{1}{2} (1 - \varsigma \sigma_x \hat{p}_x) = -\frac{\zeta}{2} (\sigma_x, i\zeta)^a \hat{p}_a = -\frac{\zeta}{2} (\sigma, i\zeta)^a \hat{p}_a$$

### 6.2 二维时空中无质量Bargmann-Wigner方程<sup>[17]</sup>的平面波解

定理6.2.1.  $\gamma^a \kappa_\zeta \lambda_\zeta \partial_a \psi_{\lambda_\zeta \mu_\zeta \dots} (x) = 0, \gamma^a = (-\sigma_y, \varsigma \sigma_z)$

$$\begin{cases} \psi(\vec{r}, t) := \frac{1}{(2\pi)^{N/2}} \int_{\vec{p} \neq 0} \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\hat{p}, -s\zeta) [a_1(\vec{p}, -\frac{\zeta}{2}) e^{ip \cdot x} + a_2^+(\vec{p}, -\frac{\zeta}{2}) e^{-ip \cdot x}] d^N \vec{p} \\ a_1(\vec{p}, -s\zeta) = \frac{1}{(2\pi)^{N/2}} \int \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \dots}^+}_{2s}(\hat{p}, -s\zeta) \psi(\vec{r}, t) e^{-ip \cdot x} d^N \vec{r} \\ a_2^+(\vec{p}, -\frac{\zeta}{2}) = \frac{1}{(2\pi)^{N/2}} \int \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \dots}^+}_{2s}(\hat{p}, -s\zeta) \psi(\vec{r}, t) e^{ip \cdot x} d^N \vec{r} \end{cases}$$

$$\underbrace{\lambda_{\lambda_\zeta \mu_\zeta \dots}^+}_{2s}(\hat{p}, -s\zeta) \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\hat{p}, -s\zeta) = 1, \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \dots}}_{2s}(\hat{p}, -s\zeta) \underbrace{\lambda_{\lambda_\zeta \mu_\zeta \dots}^+}_{2s}(\hat{p}, -s\zeta) = (-\frac{\zeta}{2})^{2s} \overbrace{(\sigma, i\zeta)^a}^{2s} \overbrace{(\sigma, i\zeta)^b}^{2s} \cdot \hat{p}_a \hat{p}_b \dots$$

### 6.3 二维时空中无质量Bargmann-Wigner方程的势描述<sup>[17, 19]</sup>

定理6.3.1.  $\gamma^a \partial_a \psi(x) = 0, \psi^T(x) = \psi(x), A_a = \frac{1}{\sqrt{2im}} \text{tr}(\bar{\varepsilon} \gamma_a \psi), \gamma^a = (-\sigma_y, \varsigma \sigma_z)$

$$\Leftrightarrow (\partial^b \partial_b - m^2) A_a = 0, \partial^a A_a = 0; F_{ab} := \partial_a A_b - \partial_b A_a, \psi = (im\gamma^a + \varsigma \varepsilon^{ab} \sigma_x \partial_b) \varepsilon \frac{A_a}{\sqrt{2}}, S_{ab}(e) = \frac{1}{2} \varsigma \varepsilon^{ab} \sigma_x$$

证明:  $\gamma^a \partial_a \psi(x) = 0, \psi^T(x) = \psi(x) = \frac{1}{\sqrt{2}} (\gamma^a \varepsilon im A_a - \varsigma \sigma_x \varepsilon F_{xy})$

$$\Leftrightarrow \gamma^a \partial_a (\gamma^b \varepsilon im A_b - \varsigma \sigma_x \varepsilon F_{xy}) = 0, \psi(x) = \frac{1}{\sqrt{2}} (\gamma^a \varepsilon im A_a - \varsigma \sigma_x \varepsilon F_{xy})$$

$$\Leftrightarrow \gamma^a \partial_a (\gamma^b im A_b - \varsigma \sigma_x F_{xy}) = 0, \psi(x) = \frac{1}{\sqrt{2}} (\gamma^a \varepsilon im A_a - \varsigma \sigma_x \varepsilon F_{xy})$$

$$\Leftrightarrow (\delta^{ab} - i\zeta \varepsilon^{ab} \sigma_x) im \partial_a A_b - i\varepsilon^{ab} \gamma_b \partial_a F_{xy} = 0, \psi(x) = \frac{1}{\sqrt{2}} (\gamma^a \varepsilon im A_a - \varsigma \sigma_x \varepsilon F_{xy})$$

$$\Leftrightarrow im \partial^a A_a + m \varepsilon^{ab} \partial_a A_b \varsigma \sigma_x - i\varepsilon^{ab} \partial_a F_{xy} \gamma_b = 0, \psi(x) = \frac{1}{\sqrt{2}} (\gamma^a \varepsilon im A_a - \varsigma \sigma_x \varepsilon F_{xy})$$

$$\begin{aligned} &\Leftrightarrow im\partial^a A_a = 0, m\varepsilon^{ab}\partial_a A_b = 0, -i\varepsilon^{ab}\partial_a F_{xy}\gamma_b = 0, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \varsigma \sigma_x \varepsilon F_{xy}) \\ &\Leftrightarrow \partial^a A_a = 0, \varepsilon^{ab}\partial_a A_b = 0, \varepsilon^{ab}\partial_a F_{xy} = 0, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \varsigma \sigma_x \varepsilon F_{xy}) \\ &\Leftrightarrow \partial^a A_a = 0, \partial_a A_b - \partial_b A_a = 0, \partial_a F_{xy} = 0, \psi(x) = \frac{1}{\sqrt{2}}(\gamma^a \varepsilon im A_a - \varsigma \sigma_x \varepsilon F_{xy}) \quad \square \end{aligned}$$

## 6.4 二维时空中无质量s-自旋方程

定理6.4.1.  $\gamma^a \partial_a \psi_{\underbrace{[\lambda_\varsigma] \mu_\varsigma}_{2s}} \dots(x) = 0, \psi_{\underbrace{[\lambda_\varsigma] \mu_\varsigma}_{2s}} \dots(x)$  全对称,  $\gamma_a := [-\sigma_y, \varsigma \sigma_z]$

$$\Leftrightarrow \begin{cases} [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(s) = 0 \\ [\gamma_x(s)\partial_\pi - \gamma_\pi(s)\partial_x]\psi(s) = 0 \end{cases}, S_{ab}(s, \varsigma) = -i[\gamma_a(s), \gamma_b(s)], \gamma_a(s) := [-\sigma_y(s), \varsigma \sigma_z(s)]$$

## 6.5 二维时空中无质量Bargmann-Wigner方程的协变对易规则

定理6.5.1.  $[\psi_{\underbrace{[\lambda_\varsigma] \mu_\varsigma}_{2s}} \dots(x), \psi_{\underbrace{[\lambda'_\varsigma] \mu'_\varsigma}_{2s}} \dots(x')] = \frac{i}{2^{2s-1}} \underbrace{[(-\gamma^a \partial_a) \gamma^0]_{\lambda_\varsigma \lambda'_\varsigma} [(-\gamma^b \partial_b) \gamma^0]_{\mu_\varsigma \mu'_\varsigma}} \cdot \Delta(x - x')$

[⇔]

定理6.5.2.  $[\psi_{\underbrace{[\lambda_\varsigma] \mu_\varsigma}_{2s}} \dots(x), \psi_{\underbrace{[\lambda'_\varsigma] \mu'_\varsigma}_{2s}} \dots(x')] = i \frac{(i\varsigma)^{2s}}{2^{2s-1}} \underbrace{[(\sigma, i\varsigma)^a \partial_a]_{\lambda_\varsigma \lambda'_\varsigma} [(\sigma, i\varsigma)^b \partial_b]_{\mu_\varsigma \mu'_\varsigma}} \cdot \Delta(x - x')$

[⇔]

定理6.5.3.  $[\psi_{k_\varsigma}(x), \psi_{k'_\varsigma}^+(x')] = i \frac{(-1)^{2s}}{2^{2s-1}} \Gamma_{\underbrace{k_\varsigma k'_\varsigma}_{2s}}^{abc \dots}(s) \underbrace{\partial_a \partial_b \partial_c \dots}_{2s} \Delta(x - x')$

## 7 二维时空中无质量粒子的Penrose方程

### 7.1 二维时空中分离表象的Dirac方程 [4, 5]

定义7.1.1.  $(\gamma^a \partial_a + m)\psi = 0, \psi = \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix}, \gamma^a = (1 \otimes \sigma_y, \varsigma 1 \otimes \sigma_x) \Leftrightarrow \begin{cases} (1, -i\varsigma)^a \partial_a \varphi = im\bar{\varphi} \\ (1, i\varsigma)^a \partial_a \bar{\varphi} = -im\varphi \end{cases}$

定义7.1.2.  $\vartheta = \begin{bmatrix} 0 & i\varepsilon \\ -i\varepsilon & 0 \end{bmatrix}, S_{ab} = \begin{bmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{bmatrix}, S_{ab}(e, \varsigma) = -\frac{i}{4}[\sigma_a, \sigma_b] = \begin{bmatrix} 0 & \frac{-\varsigma}{2}\sigma_z \\ \frac{\varsigma}{2}\sigma_z & 0 \end{bmatrix}, \psi := \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix}$

推论7.1.1.  $\Lambda\left(\begin{bmatrix} \sigma \\ i\tau \end{bmatrix}\right) = e^{\frac{i}{2}\vartheta^{ab} S_{ab}} = e^{-\varepsilon \sigma_y}, \Lambda\left(\begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix}\right) = e^{\frac{i}{2}\vartheta^{ab} S_{ab}(e, \varsigma)} = e^{\frac{\varepsilon}{2}\varepsilon \sigma_z}$

当质量 $m=0$ 时, 则退化为两个Weyl中微子方程:

推论7.1.2.  $(1, -i\varsigma)^a \partial_a \varphi = 0, (1, i\varsigma)^a \partial_a \bar{\varphi} = 0, \Lambda(\varphi) = e^{\frac{\varepsilon}{2}\varepsilon}, \Lambda(\bar{\varphi}) = e^{-\frac{\varepsilon}{2}\varepsilon}$

### 7.2 二维时空中分离表象的无质量Dirac方程的螺旋度本征函数

定义7.2.1.  $\gamma^a \partial_a \psi(x) = 0, \psi = \begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix}, \gamma^a = (1 \otimes \sigma_y, \varsigma 1 \otimes \sigma_x) \Leftrightarrow \begin{cases} (1, -i\varsigma)^a \partial_a \varphi = 0 \\ (1, i\varsigma)^a \partial_a \bar{\varphi} = 0 \end{cases}$

证明:  $(\sigma_z, -i\varsigma)^a \partial_a \psi(x) = 0$

$\Leftrightarrow (\sigma_z, -i\varsigma)^a p_a \lambda(\hat{p}, -\frac{\varsigma}{2}) = 0$

$\Leftrightarrow \sigma_z p_z \lambda(\hat{p}, -\frac{\varsigma}{2}) = -\varsigma |p_z| \lambda(\hat{p}, -\frac{\varsigma}{2})$

$\Leftrightarrow \sigma_z \hat{p}_z \lambda(\hat{p}, -\frac{\varsigma}{2}) = -\varsigma \lambda(\hat{p}, -\frac{\varsigma}{2})$

$\Leftrightarrow \lambda(\hat{p}, -\varsigma) = \frac{1}{2} \begin{bmatrix} -1 + \varsigma \hat{p}_z \\ 1 + \varsigma \hat{p}_z \end{bmatrix}$

推论7.2.1.  $\lambda^+(\hat{p}, -\frac{\varsigma}{2})\lambda(\hat{p}, -\frac{\varsigma}{2}) = 1, \lambda(\hat{p}, -\frac{\varsigma}{2})\lambda^+(\hat{p}, -\frac{\varsigma}{2}) = -\frac{\varsigma}{2}(\sigma_z, i\varsigma)^a \hat{p}_a$

### 7.3 二维中的矢量和旋量 [44]

#### 7.3.1 二维中的光锥坐标和导数

定义7.3.1.  $z \equiv \tau + \sigma, \tilde{z} \equiv \tau - \sigma, \tau = \frac{1}{2}(z + \tilde{z}), \sigma = \frac{1}{2}(z - \tilde{z}), z_\zeta := \tau + \zeta\sigma, \bar{z}_\zeta := \tau - \zeta\sigma$

定义7.3.2.  $\begin{bmatrix} z \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} \begin{bmatrix} \sigma \\ i\tau \end{bmatrix}, \begin{bmatrix} \sigma \\ i\tau \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \begin{bmatrix} z \\ \tilde{z} \end{bmatrix}$

推论7.3.1.  $\begin{cases} dz = d\tau + d\sigma, d\tilde{z} = d\tau - d\sigma \\ \partial_z = \frac{1}{2}(\partial_\tau + \partial_\sigma), \partial_{\tilde{z}} = \frac{1}{2}(\partial_\tau - \partial_\sigma) \end{cases} \quad \begin{cases} d\tau = \frac{1}{2}(dz + d\tilde{z}), d\sigma = \frac{1}{2}(dz - d\tilde{z}) \\ \partial_\tau = \partial_z + \partial_{\tilde{z}}, \partial_\sigma = \partial_z - \partial_{\tilde{z}} \end{cases}$

推论7.3.2.  $dz \wedge d\tilde{z} = 2d\sigma \wedge d\tau$

定义7.3.3.  $P_z \equiv -i\partial_z, P_{\tilde{z}} \equiv -i\partial_{\tilde{z}}, P_\tau \equiv i\partial_\tau, P_\sigma \equiv -i\partial_\sigma$

推论7.3.3.  $P_z = -\frac{1}{2}(P_\tau - P_\sigma), P_{\tilde{z}} = -\frac{1}{2}(P_\tau + P_\sigma), -P_\tau = P_z + P_{\tilde{z}}, P_\sigma = P_z - P_{\tilde{z}}$

推论7.3.4.  $e^{i(P_\sigma\sigma - P_\tau\tau)} = e^{i(P_z z + P_{\tilde{z}}\tilde{z})}$

#### 7.3.2 二维中的矢量和旋量的洛伦兹变换规律

推论7.3.5. 矢量:  $\Lambda\left(\begin{bmatrix} \sigma \\ i\tau \end{bmatrix}\right) = e^{-\varepsilon\sigma_y}, \sigma, \tau \in R \Leftrightarrow$  光锥矢量:  $\Lambda\left(\begin{bmatrix} z \\ \tilde{z} \end{bmatrix}\right) = e^{-\varepsilon\sigma_z}, z, \tilde{z} \in R$

推论7.3.6. Dirac旋量:  $\Lambda(\psi) = e^{\frac{\varepsilon}{2}\sigma_z}, \psi \in C,$  Weyl旋量:  $\Lambda(\varphi) = e^{\frac{\varepsilon}{2}\varepsilon}, \Lambda(\bar{\varphi}) = e^{-\frac{\varepsilon}{2}\varepsilon}, \varphi, \bar{\varphi} \in C$

推论7.3.7. Majorana旋量:  $\Lambda(\psi) = e^{\frac{\varepsilon}{2}\sigma_z}, \psi \in R,$  Majorana-Weyl旋量:  $\Lambda(\varphi) = e^{\frac{\varepsilon}{2}\varepsilon}, \Lambda(\bar{\varphi}) = e^{-\frac{\varepsilon}{2}\varepsilon}, \varphi, \bar{\varphi} \in R$

命题7.3.1. 常数张量:  $(1, -i\zeta)^a = e^{-\varepsilon\sigma_y}|^a_b e^{-\frac{\varepsilon}{2}\varepsilon}(1, -i)^b e^{-\frac{\varepsilon}{2}\varepsilon}, (1, -i\zeta)^{a'} \partial_{a'} = e^{-\varepsilon\sigma_y}(1, -i\zeta)^a \partial_a$

命题7.3.2. 常数张量:  $(1, i\zeta)^a = e^{-\varepsilon\sigma_y}|^a_b e^{\frac{\varepsilon}{2}\varepsilon}(1, i\zeta)^b e^{\frac{\varepsilon}{2}\varepsilon}, (1, i\zeta)^{a'} \partial_{a'} = e^{\varepsilon\sigma_y}(1, i\zeta)^a \partial_a$

推论7.3.8.  $\partial_{z_\zeta} = \frac{1}{2}(1, i\zeta)^a \partial_a, \partial_{\bar{z}_\zeta} = -\frac{1}{2}(1, -i\zeta)^a \partial_a; \partial_z = \frac{1}{2}(1, i)^a \partial_a, \partial_{\tilde{z}} = -\frac{1}{2}(1, -i)^a \partial_a$

推论7.3.9.  $\begin{cases} \partial_{z'_\zeta} = e^{\varepsilon\sigma_y} \partial_{z_\zeta}, dz'_\zeta = e^{-\varepsilon\sigma_y} dz_\zeta, z'_\zeta = e^{-\varepsilon\sigma_y} z_\zeta \\ \partial_{\bar{z}'_\zeta} = e^{-\varepsilon\sigma_y} \partial_{\bar{z}_\zeta}, d\bar{z}'_\zeta = e^{\varepsilon\sigma_y} d\bar{z}_\zeta, \bar{z}'_\zeta = e^{\varepsilon\sigma_y} \bar{z}_\zeta \end{cases} \quad \begin{cases} \partial_{z'} = e^\varepsilon \partial_z, dz' = e^{-\varepsilon} dz, z' = e^{-\varepsilon} z \\ \partial_{\bar{z}'} = e^{-\varepsilon} \partial_{\bar{z}}, d\bar{z}' = e^\varepsilon d\bar{z}, \bar{z}' = e^\varepsilon \bar{z} \end{cases}$

推论7.3.10. 不变量:  $dz_\zeta \partial_{z_\zeta}, d\bar{z}_\zeta \partial_{\bar{z}_\zeta}, dz_\zeta d\bar{z}_\zeta, \partial_{z_\zeta} \partial_{\bar{z}_\zeta}; dz \partial_z, d\tilde{z} \partial_{\tilde{z}}, dz d\tilde{z}, \partial_z \partial_{\tilde{z}}$

推论7.3.11. 光锥矢量:  $\Lambda\left(\begin{bmatrix} z_\zeta \\ \bar{z}_\zeta \end{bmatrix}\right) = e^{-\varepsilon\sigma_z}, \Lambda\left(\begin{bmatrix} \partial_{z_\zeta} \\ \partial_{\bar{z}_\zeta} \end{bmatrix}\right) = e^{\varepsilon\sigma_z},$  旋量:  $\Lambda\left(\begin{bmatrix} \varphi \\ \bar{\varphi} \end{bmatrix}\right) = e^{\frac{\varepsilon}{2}\varepsilon\sigma_z}$

在二维中光锥矢量与旋量十分相似, 在二维中光锥矢量就是矢量的旋量表象。

#### 7.3.3 Wick转动(不采用)

定义7.3.4.  $z \equiv \sigma + i\tau, \bar{z} \equiv z^* = \sigma - i\tau, \sigma = \frac{1}{2}(z + \bar{z}), i\tau = \frac{1}{2}(z - \bar{z})$

推论7.3.12.  $\begin{cases} dz = d\sigma + id\tau, d\bar{z} = d\sigma - id\tau, d\sigma = \frac{1}{2}(dz + d\bar{z}), id\tau = \frac{1}{2}(dz - d\bar{z}) \\ \partial_z = \frac{1}{2}(\partial_\sigma + \partial_{i\tau}), \partial_{\bar{z}} = \frac{1}{2}(\partial_\sigma - \partial_{i\tau}), \partial_\sigma = \partial_z + \partial_{\bar{z}}, \partial_{i\tau} = \partial_z - \partial_{\bar{z}} \end{cases} \quad \begin{cases} dz = d^* \bar{z} \\ \partial_z = \partial_{\bar{z}}^* \end{cases}$

### 7.4 二维时空中Penrose方程 [1,2]的平面波解

定理7.4.1.  $(1, -i\zeta)_a \partial^a \varphi(x) = 0$

$\Rightarrow \varphi(x) = \frac{1}{\sqrt{\pi}} \int a(p, \zeta) e^{ip(\tau + \zeta\sigma)} dp = \begin{cases} \frac{1}{\sqrt{\pi}} \int_0^{+\infty} [a(p) e^{i(p\sigma - |p|\tau)} + b^+(p) e^{-i(p\sigma - |p|\tau)}] dp, \zeta = - \\ \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 [a(p) e^{i(p\sigma - |p|\tau)} + b^+(p) e^{-i(p\sigma - |p|\tau)}] dp, \zeta = + \end{cases}$

证明:  $(1, -i\varsigma)_a \partial^a \varphi(x) = 0$

$$\Leftrightarrow \partial^a \partial_a \varphi(x) = 0, (1, -i\varsigma)_a \partial^a \varphi(x) = 0$$

$$\Rightarrow \varphi(x) = \frac{1}{\sqrt{\pi}} \int [a(p)e^{i(p\sigma - |p|\tau)} + b^+(p)e^{-i(p\sigma - |p|\tau)}] dp$$

$$\Rightarrow (\partial_\sigma - \varsigma \partial_\tau) \varphi(x) = 0$$

$$\Leftrightarrow \frac{1}{\sqrt{\pi}} \int [i(p + \varsigma|p|)a(p)e^{i(p\sigma - |p|\tau)} - i(p + \varsigma|p|)b^+(p)e^{-i(p\sigma - |p|\tau)}] dp$$

$$\Leftrightarrow (p + \varsigma|p|)a(p) = 0, (p + \varsigma|p|)b^+(p) = 0$$

$$\Leftrightarrow a(\varsigma p > 0) = 0, b^+(\varsigma p > 0) = 0$$

$$\Leftrightarrow \begin{cases} \varphi(x) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} [a(p)e^{i(p\sigma - |p|\tau)} + b^+(p)e^{-i(p\sigma - |p|\tau)}] dp, \varsigma = - \\ \varphi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 [a(p)e^{i(p\sigma - |p|\tau)} + b^+(p)e^{-i(p\sigma - |p|\tau)}] dp, \varsigma = + \end{cases}$$

$$\Leftrightarrow \begin{cases} \varphi(x) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} [a(p)e^{ip(\sigma - \tau)} + b^+(p)e^{-ip(\sigma - \tau)}] dp, \varsigma = - \\ \varphi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 [a(p)e^{ip(\sigma + \tau)} + b^+(p)e^{-ip(\sigma + \tau)}] dp, \varsigma = + \end{cases}$$

$$\Leftrightarrow \begin{cases} \varphi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a(p)e^{ip(\sigma - \tau)}; a(p) := b^+(-p), p < 0; \varsigma = - \\ \varphi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a(p)e^{ip(\sigma + \tau)} dp; a(p) := b^+(-p), p > 0; \varsigma = + \end{cases}$$

$$\Leftrightarrow \begin{cases} \varphi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a'(p)e^{ip(\tau - \sigma)} dp, \varsigma = - \\ \varphi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a(p)e^{ip(\tau + \sigma)} dp, \varsigma = + \end{cases}$$

$$\Leftrightarrow \varphi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a(p, \varsigma) e^{ip(\tau + \varsigma \sigma)} dp$$

□

## 7.5 二维时空中的因果函数

$$\text{定义7.5.1. } \Delta(z_\varsigma) := \frac{i}{\pi} \int_0^{+\infty} \frac{1}{2p} (e^{ipz_\varsigma} - e^{-ipz_\varsigma}) dp = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{2p} (e^{ipz_\varsigma} - e^{-ipz_\varsigma}) dp = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{p} e^{ipz_\varsigma} dp$$

$$\text{定义7.5.2. } \begin{cases} \Delta^{(+)}(z_\varsigma) := \frac{i}{\pi} \int_{p=0}^{+\infty} \frac{1}{2p} e^{ipz_\varsigma} d\vec{p}, \Delta^{(-)}(z_\varsigma) := -\frac{i}{\pi} \int_{p=0}^{+\infty} \frac{1}{2p} e^{-ipz_\varsigma} d\vec{p}, \Delta^{(-)}(z_\varsigma) = -\Delta^{(+)}(-z_\varsigma) \\ \Delta(z_\varsigma) := \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{1}{2p} [e^{ipz_\varsigma} - e^{-ipz_\varsigma}] d\vec{p}, \Delta(z_\varsigma) = \Delta^{(+)}(z_\varsigma) + \Delta^{(-)}(z_\varsigma) \end{cases}$$

$$\text{定义7.5.3. } \begin{cases} \frac{1}{\sqrt{-\nabla^2}} \Delta(z_\varsigma) := \frac{i}{\pi} \int_0^{+\infty} \frac{1}{2p^2} (e^{ipz_\varsigma} - e^{-ipz_\varsigma}) dp = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{2p|p|} (e^{ipz_\varsigma} - e^{-ipz_\varsigma}) dp = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{p|p|} e^{ipz_\varsigma} dp \\ \sqrt{-\nabla^2} \Delta(z_\varsigma) := \frac{i}{\pi} \int_0^{+\infty} \frac{1}{2} (e^{ipz_\varsigma} - e^{-ipz_\varsigma}) dp = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{|p|}{2p} (e^{ipz_\varsigma} - e^{-ipz_\varsigma}) dp = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{|p|}{p} e^{ipz_\varsigma} dp \end{cases}$$

$$\text{性质7.5.1. } \Delta^*(z_\varsigma) = \Delta(z_\varsigma), \Delta(-z_\varsigma) = -\Delta(z_\varsigma), (\nabla^2 - \partial_\tau^2) \Delta(z_\varsigma) = 0, \partial_{z_\varsigma} \Delta(z_\varsigma)|_{\tau=0} = -\delta(\sigma)$$

$$\text{性质7.5.2. } \Delta(z_\varsigma - z'_\varsigma) := \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{2p} [e^{ip(z_\varsigma - z'_\varsigma)} - e^{-ip(z_\varsigma - z'_\varsigma)}] d\vec{p}$$

$$\begin{cases} \partial_u \Delta(z_\varsigma - z'_\varsigma) = -\partial'_u \Delta(z_\varsigma - z'_\varsigma) & \begin{cases} (\sqrt{-\nabla^2})^n \Delta(z_\varsigma - z'_\varsigma) = (\sqrt{-\nabla'^2})^n \Delta(z_\varsigma - z'_\varsigma) \\ \frac{1}{(\sqrt{-\nabla^2})^n} \Delta(z_\varsigma - z'_\varsigma) = \frac{1}{(\sqrt{-\nabla'^2})^n} \Delta(z_\varsigma - z'_\varsigma) \end{cases} \\ \nabla \Delta(z_\varsigma - z'_\varsigma) = -\nabla' \Delta(z_\varsigma - z'_\varsigma) \\ \partial_\pi \Delta(z_\varsigma - z'_\varsigma) = -\partial'_\pi \Delta(z_\varsigma - z'_\varsigma) & \partial_\pi^{2n} \Delta(z_\varsigma - z'_\varsigma) = \partial_\pi'^{2n} \Delta(z_\varsigma - z'_\varsigma) \end{cases}$$

## 7.6 二维时空中s-自旋方程的对易规则

$$\text{推论7.6.1. } [s\partial_a + iS_{ab}(s, \varsigma)\partial^b] \varphi(x) = 0, \varphi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a(p, \varsigma) e^{ip(\tau + \varsigma \sigma)} dp$$

定理7.6.1.  $[a(p, \varsigma), a^+(p', \varsigma)]_{-2s+1} = \delta(p - p') \Rightarrow [\varphi(z_\varsigma), \varphi^+(z'_\varsigma)]_{-2s+1} = i \frac{(i\varsigma)^{2s}}{2^{2s-1}} \overbrace{(1, i\varsigma)^a \partial_a (1, i\varsigma)^b \partial_b \cdots \Delta(z_\varsigma - z'_\varsigma)}^{2s}$

证明:  $[a(p, \varsigma), a^+(p', \varsigma)]_{-2s+1} = \delta(p - p')$

$\Rightarrow [a(p), a^+(p')]_{-2s+1} = [b(p), b^+(p')]_{-2s+1} = \delta(p - p')$

$\Rightarrow [\varphi(z_\varsigma), \varphi^+(z'_\varsigma)]_{-2s+1}$

$$= \begin{cases} \frac{1}{\pi} \int_0^{+\infty} \{e^{i(p\sigma - |p|\tau)} e^{-i(p'\sigma' - |p'|\tau')} [a(p), a^+(p')]_{-2s+1} + e^{-i(p\sigma - |p|\tau)} e^{i(p'\sigma' - |p'|\tau')} [b^+(p), b(p')]_{-2s+1}\} dp dp', \varsigma = - \\ \frac{1}{\pi} \int_0^{+\infty} \{e^{i(p\sigma - |p|\tau)} e^{-i(p'\sigma' - |p'|\tau')} [a(p), a^+(p')]_{-2s+1} + e^{-i(p\sigma - |p|\tau)} e^{i(p'\sigma' - |p'|\tau')} [b^+(p), b(p')]_{-2s+1}\} dp dp', \varsigma = + \end{cases}$$

$$= \begin{cases} \frac{1}{\pi} \int_0^{+\infty} \{e^{i(p\sigma - |p|\tau)} e^{-i(p'\sigma' - |p'|\tau')} + (-1)^{2s-1} e^{-i(p\sigma - |p|\tau)} e^{i(p'\sigma' - |p'|\tau')}\} [a(p), a^+(p')]_{-2s+1} dp dp', \varsigma = - \\ \frac{1}{\pi} \int_0^{+\infty} \{e^{i(p\sigma - |p|\tau)} e^{-i(p'\sigma' - |p'|\tau')} + (-1)^{2s-1} e^{-i(p\sigma - |p|\tau)} e^{i(p'\sigma' - |p'|\tau')}\} [a(p), a^+(p')]_{-2s+1} dp dp', \varsigma = + \end{cases}$$

$$= \begin{cases} \frac{1}{\pi} \int_0^{+\infty} \{e^{i(p\sigma - |p|\tau)} e^{-i(p'\sigma' - |p'|\tau')} + (-1)^{2s-1} e^{-i(p\sigma - |p|\tau)} e^{i(p'\sigma' - |p'|\tau')}\} |p|^{2s-1} \delta(p - p') dp dp', \varsigma = - \\ \frac{1}{\pi} \int_0^{+\infty} \{e^{i(p\sigma - |p|\tau)} e^{-i(p'\sigma' - |p'|\tau')} + (-1)^{2s-1} e^{-i(p\sigma - |p|\tau)} e^{i(p'\sigma' - |p'|\tau')}\} |p|^{2s-1} \delta(p - p') dp dp', \varsigma = + \end{cases}$$

$$= \begin{cases} \frac{1}{\pi} \int_0^{+\infty} \{e^{i(p\sigma - |p|\tau)} e^{-i(p\sigma' - |p|\tau')} + (-1)^{2s-1} e^{-i(p\sigma - |p|\tau)} e^{i(p\sigma' - |p|\tau')}\} |p|^{2s-1} dp, \varsigma = - \\ \frac{1}{\pi} \int_0^{+\infty} \{e^{i(p\sigma - |p|\tau)} e^{-i(p\sigma' - |p|\tau')} + (-1)^{2s-1} e^{-i(p\sigma - |p|\tau)} e^{i(p\sigma' - |p|\tau')}\} |p|^{2s-1} dp, \varsigma = + \end{cases}$$

$$= \begin{cases} \frac{1}{\pi} \int_0^{+\infty} \{e^{i(p\sigma - |p|\tau)} e^{-i(p\sigma' - |p|\tau')} + (-1)^{2s-1} e^{-i(p\sigma - |p|\tau)} e^{i(p\sigma' - |p|\tau')}\} |p|^{2s-1} dp, \varsigma = - \\ \frac{1}{\pi} \int_0^{+\infty} \{e^{i(-p\sigma - |p|\tau)} e^{-i(p\sigma' - |p|\tau')} + (-1)^{2s-1} e^{-i(-p\sigma - |p|\tau)} e^{i(p\sigma' - |p|\tau')}\} |p|^{2s-1} dp, \varsigma = + \end{cases}$$

$$= \begin{cases} \frac{1}{\pi} \int_0^{+\infty} \{e^{ip(\sigma - \tau)} e^{-ip(\sigma' - \tau')} + (-1)^{2s-1} e^{-ip(\sigma - \tau)} e^{ip(\sigma' - \tau')}\} p^{2s-1} dp, \varsigma = - \\ \frac{1}{\pi} \int_0^{+\infty} \{(-1)^{2s-1} e^{ip(\sigma + \tau)} e^{-ip(\sigma' + \tau')} + e^{-ip(\sigma + \tau)} e^{ip(\sigma' + \tau')}\} p^{2s-1} dp, \varsigma = + \end{cases}$$

$$= \begin{cases} \left[ \frac{-i}{2} (1, -i)^a \partial_a \right]^{2s} \frac{1}{\pi} \int_0^{+\infty} \{e^{ip(\sigma - \tau)} e^{-ip(\sigma' - \tau')} - e^{-ip(\sigma - \tau)} e^{ip(\sigma' - \tau')}\} p^{-1} dp, \varsigma = - \\ \left[ \frac{-i}{2} (1, i)^a \partial_a \right]^{2s} \frac{1}{\pi} \int_0^{+\infty} \{(-1)^{2s-1} e^{ip(\sigma + \tau)} e^{-ip(\sigma' + \tau')} + (-1)^{2s} e^{-ip(\sigma + \tau)} e^{ip(\sigma' + \tau')}\} p^{-1} dp, \varsigma = + \end{cases}$$

$$= \begin{cases} \left[ \frac{-i}{2} (1, -i)^a \partial_a \right]^{2s} \frac{1}{\pi} \int_0^{+\infty} \{e^{ip(\sigma - \tau)} e^{-ip(\sigma' - \tau')} - e^{-ip(\sigma - \tau)} e^{ip(\sigma' - \tau')}\} p^{-1} dp, \varsigma = - \\ \left[ \frac{i}{2} (1, i)^a \partial_a \right]^{2s} \frac{1}{\pi} \int_0^{+\infty} \{-e^{ip(\sigma + \tau)} e^{-ip(\sigma' + \tau')} + e^{-ip(\sigma + \tau)} e^{ip(\sigma' + \tau')}\} p^{-1} dp, \varsigma = + \end{cases}$$

$$= \begin{cases} i \left[ \frac{-i}{2} (1, -i)^a \partial_a \right]^{2s} \frac{i}{\pi} \int_0^{+\infty} \{e^{ip(\tau - \sigma)} e^{-ip(\tau' - \sigma')} - e^{-ip(\tau - \sigma)} e^{ip(\tau' - \sigma')}\} p^{-1} dp, \varsigma = - \\ + i \left[ \frac{i}{2} (1, i)^a \partial_a \right]^{2s} \frac{i}{\pi} \int_0^{+\infty} \{e^{ip(\tau + \sigma)} e^{-ip(\tau' + \sigma')} - e^{-ip(\tau + \sigma)} e^{ip(\tau' + \sigma')}\} p^{-1} dp, \varsigma = + \end{cases}$$

$$= 2i \left[ \frac{i\varsigma}{2} (1, i\varsigma)^a \partial_a \right]^{2s} \frac{i}{2\pi} \int_0^{+\infty} \{e^{ip(\tau + \varsigma\sigma)} e^{-ip(\tau' + \varsigma\sigma')} - e^{-ip(\tau + \varsigma\sigma)} e^{ip(\tau' + \varsigma\sigma')}\} p^{-1} dp$$

$$= i \frac{(i\varsigma)^{2s}}{2^{2s-1}} \overbrace{(1, i\varsigma)^a \partial_a (1, i\varsigma)^b \partial_b \cdots \Delta(z_\varsigma - z'_\varsigma)}^{2s}$$

$$= 2i^{2s+1} \partial_{z_\varsigma}^{2s} \Delta(z_\varsigma - z'_\varsigma)$$

$$= 2i^{2s-1} \partial_{z_\varsigma}^{2s-1} \delta(z_\varsigma - z'_\varsigma)$$

□

## 7.7 二维时空中Penrose方程 [1,2] 杂烩

定理7.7.1.  $(1, -i\varsigma)_a^{A_\varsigma A_\varsigma} \partial^a \varphi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma}_{2s}} \cdots(x) = 0, \varphi_{\underbrace{A_\varsigma B_\varsigma C_\varsigma}_{2s}} \cdots(x) = k(\tau + \varsigma\sigma) + \frac{1}{\sqrt{\pi}} \int a(p, \varsigma) e^{ip(\tau + \varsigma\sigma)} dp$



$$\text{定理7.7.2. } (1, -i\zeta)_a \underbrace{\partial^a \varphi_{A_\zeta B_\zeta C_\zeta \dots}}_{2s}(x) = 0, \partial^a \underbrace{\partial_a \varphi_{A_\zeta B_\zeta C_\zeta \dots}}_{2s}(x) = 0$$

$$\text{定理7.7.3. } (1, -i\zeta)_a \underbrace{\partial^a \varphi_{A_\zeta B_\zeta C_\zeta \dots}}_{2s}(x) = 0 \Leftrightarrow [s\partial_a + iS_{ab}(s)\partial^b] \underbrace{\varphi_{A_\zeta B_\zeta C_\zeta \dots}}_{2s}(x) = 0, iS_{ab}(s) = \begin{bmatrix} 0 & is \\ -is & 0 \end{bmatrix}$$

$$\text{命题7.7.1. } [s\partial_a + iS_{ab}(s)\partial^b]\varphi(s) = 0, \varphi'(s) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s)}\varphi(s) = e^{-s\varepsilon}\varphi(s), \vartheta^{ab} \succ \begin{bmatrix} 0 & i\varepsilon \\ -i\varepsilon & 0 \end{bmatrix}$$

## 7.8 二维时空中 $\frac{1}{3}$ -自旋方程的对易规则?

$$\text{推论7.8.1. } [\frac{1}{3}\partial_a + iS_{ab}(\frac{1}{3}, \varsigma)\partial^b]\varphi(x) = 0, \varphi(x) = \frac{1}{\sqrt{\pi}} \int a(p, \varsigma) e^{ip(\tau+\varsigma\sigma)} dp$$

$$\text{定理7.8.1. } [a(p, \varsigma), a^+(p', \varsigma)]_\varphi = \delta(p - p') \Rightarrow [\varphi(z_\varsigma), \varphi^+(z'_\varsigma)]_{-2s+1} = i \frac{(i\varsigma)^{2s}}{2^{2s-1}} \overbrace{(1, i\varsigma)^a \partial_a (1, i\varsigma)^b \partial_b \dots}^{2s} \Delta(z_\varsigma - z'_\varsigma)$$

$$\text{猜想7.8.1. } [a(p, \varsigma), a^+(p', \varsigma)]_{-\frac{5}{3}} = \delta(p - p') \Rightarrow [\varphi(z_\varsigma), \varphi^+(z'_\varsigma)]_{-\frac{5}{3}} = i \frac{(i\varsigma)^{\frac{2}{3}}}{2^{-\frac{1}{3}}} [(1, i\varsigma)^a \partial_a]^{\frac{2}{3}} \Delta(z_\varsigma - z'_\varsigma)$$

# 第三十七章 N+1维时空B-W方程的势分析

自我评述：本章仿照四维情形，对N+1维时空中的Bargmann-Wigner方程进行了势分解和详细的数学分析。在自旋-1情形下自然而然出现了反对称张量场，并基于Bargmann-Wigner方程的对易规则推导出了反对称张量场的对易规则。与四维情形对比，N+1维时空中的B-W方程的势分析复杂好多，而且全对称的B-W方程不再描述单个自旋态，而是描述多个基本场。正是这点导致了复杂性，从而使这种推广失去了部分美感，也使这样的描述变得丑陋，这似乎暗示着这种推广变得意义不大了。比如在自旋-1情形直接研究基本的反对称张量场即可，因为它是单个基本场，更简单、更基本。本章没有对自旋 $-\frac{3}{2}, 2$ 等更高自旋情形进行详细讨论，只是给出了高自旋情形的两个猜想，严格证明等以后有时间再说。

通过本章的研究，我发现高于四维时空的全对称Bargmann-Wigner方程不再是描述物理场的好方法了，此时B-W方程不像四维时空中那样描述的是基本场，而是描述混合场，所以此时应该直接采用反对称张量场的描述方法更合适。

## 1 N+1维时空Dirac矩阵

### 1.1 N+1维时空Dirac矩阵的常规表象

定义1.1.1.

$$\begin{cases} \gamma_a(1) = (1) \\ \gamma_1(1) = 1 \end{cases}$$

定义1.1.2.

$$\begin{cases} \gamma_a(2) := (\gamma_a(1) \otimes \sigma_x, 1 \otimes \sigma_y) = (\sigma_x, \sigma_y), \Gamma^a(2) := [\gamma_a(1), i\zeta] = (1, i\zeta) \\ C(2) := \gamma_2(2) = \sigma_y, \bar{C}(2) = C^+(2) = C(2), \gamma_1(2)\gamma_2(2) = i\sigma_z = i\gamma_0(2) \\ C^T(2) = -C(2), \gamma_a(2)C(2) = [\gamma_a(2)C(2)]^T, \gamma_{[a}(2)\gamma_{b]}(2)C(2) = \{\gamma_{[a}(2)\gamma_{b]}(2)C(2)\}^T \end{cases}$$

定义1.1.3.

$$\begin{cases} \gamma_a(3) = [\gamma_a(2), 1 \otimes \sigma_z] = (\sigma_x, \sigma_y, \sigma_z) \\ C(3) := \gamma_2(3) = \sigma_y, \bar{C}(3) = C^+(3) = C(3), \gamma_1(3) \cdots \gamma_3(3) = i = i\gamma_0(3) \\ C^T(3) = -C(3), [\gamma_a(3)C(3)]^T = \gamma_a(3)C(3) \end{cases}$$

定义1.1.4.

$$\begin{cases} \gamma_a(4) = [\gamma_a(3) \otimes \sigma_y, I \otimes \sigma_x] = (\sigma \otimes \sigma_y, I \otimes \sigma_x), \Gamma^a(4) = [\gamma_a(3), i\zeta] \\ C(4) := \gamma_2(4)\gamma_4(4) = -i\sigma_y \otimes \sigma_z, \bar{C}(4) = C^+(4) = -C(4), \gamma_1(4) \cdots \gamma_4(4) = I \otimes \sigma_z = \gamma_0(4) \\ [\gamma_a(4)C(4)]^T = \gamma_a(4)C(4), \{\gamma_{[a}(4)\gamma_{b]}(4)C(4)\}^T = \gamma_{[a}(4)\gamma_{b]}(4)C(4) \\ C^T(4) = -C(4), \{\gamma_{[a}(4)\gamma_{b}(4)\gamma_{c]}(4)C(4)\}^T = -\gamma_{[a}(4)\gamma_{b}(4)\gamma_{c]}(4)C(4) \\ \{\gamma_{[a}(4)\gamma_{b}(4)\gamma_{c}(4)\gamma_{d]}(4)C(4)\}^T = -\gamma_{[a}(4)\gamma_{b}(4)\gamma_{c}(4)\gamma_{d]}(4)C(4) \end{cases}$$

定义1.1.5.

$$\begin{cases} \gamma_a(5) = [\gamma_a(4), I \otimes \sigma_z] = (\sigma \otimes \sigma_y, I \otimes \sigma_x, I \otimes \sigma_z) \\ C(5) := \gamma_2(5)\gamma_4(5)\gamma_5(5) = -i\sigma_y \otimes I, \bar{C}(5) = C^+(5) = -C(5), \gamma_1(5) \cdots \gamma_5(5) = 1 = \gamma_0(5) \\ C^T(5) = -C(5), [\gamma_a(5)C(5)]^T = -\gamma_a(5)C(5), \{\gamma_{[a}(5)\gamma_{b]}(5)C(5)\}^T = \gamma_{[a}(5)\gamma_{b]}(5)C(5) \end{cases}$$

定义1.1.6.

$$\gamma_a(10) = [(((\sigma_x, \sigma_y, \sigma_z) \otimes \sigma_y, I \otimes \sigma_x, I \otimes \sigma_z) \otimes \sigma_y, I_4 \otimes \sigma_x, I_4 \otimes \sigma_z) \otimes \sigma_y, I_8 \otimes \sigma_x, I_8 \otimes \sigma_z] \otimes \sigma_y, I_{16} \otimes \sigma_x]$$

定义1.1.7.

$$\left\{ \begin{array}{l} \gamma_a(6) = [\gamma_a(5) \otimes \sigma_y, I_4 \otimes \sigma_x], \Gamma^a(6) = [\gamma_a(5), i\zeta] \\ C(6) := \gamma_2(6)\gamma_4(6)\gamma_5(6) = -i\sigma_y \otimes I \otimes \sigma_y, \bar{C}(6) = C^+(6) = -C(6), \gamma_1(6) \cdots \gamma_6(6) = -iI_4 \otimes \sigma_z = -i\gamma_0(6) \\ [\gamma_a(6)C(6)]^T = -\gamma_a(6)C(6), [\gamma_{[a}(6)\gamma_{b]}(6)C(6)]^T = -\gamma_{[a}(6)\gamma_{b]}(6)C(6) \\ [\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_d(6)\gamma_{e]}(6)C(6)]^T = -\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_d(6)\gamma_{e]}(6)C(6) \\ [\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_d(6)\gamma_e(6)\gamma_{f]}(6)C(6)]^T = -\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_d(6)\gamma_e(6)\gamma_{f]}(6)C(6) \\ C^T(6) = C(6), [\gamma_{[a}(6)\gamma_b(6)\gamma_{c]}(6)C(6)]^T = \gamma_{[a}(6)\gamma_b(6)\gamma_{c]}(6)C(6) \\ [\gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_{d]}(6)C(6)]^T = \gamma_{[a}(6)\gamma_b(6)\gamma_c(6)\gamma_{d]}(6)C(6) \end{array} \right.$$

定义1.1.8.

$$\left\{ \begin{array}{l} \gamma_a(7) = [\gamma_a(6), I_4 \otimes \sigma_z] \\ C(7) := \gamma_2(7)\gamma_4(7)\gamma_5(7) = -i\sigma_y \otimes I \otimes \sigma_y, C(7) = C(6), \bar{C}(7) = C^+(7) = -C(7), \gamma_1(7) \cdots \gamma_7(7) = -i = -i\gamma_0(7) \\ [\gamma_a(7)C(7)]^T = -\gamma_a(7)C(7), [\gamma_{[a}(7)\gamma_{b]}(7)C(7)]^T = -\gamma_{[a}(7)\gamma_{b]}(7)C(7) \\ C^T(7) = C(7), [\gamma_{[a}(7)\gamma_b(7)\gamma_{c]}(7)C(7)]^T = \gamma_{[a}(7)\gamma_b(7)\gamma_{c]}(7)C(7) \end{array} \right.$$

定义1.1.9.

$$\left\{ \begin{array}{l} \gamma_a(8) = [\gamma_a(7) \otimes \sigma_y, I_8 \otimes \sigma_x], \Gamma^a(8) = [\gamma_a(7), i\zeta] \\ C(8) := \gamma_2(8)\gamma_4(8)\gamma_5(8)\gamma_8(8) = -\sigma_y \otimes I \otimes \sigma_y \otimes \sigma_z, \bar{C}(8) = C^+(8) = C(8), \gamma_1(8) \cdots \gamma_8(8) = -I_8 \otimes \sigma_z = -\gamma_0(8) \\ [\gamma_a(8)C(8)]^T = -\gamma_a(8)C(8), [\gamma_{[a}(8)\gamma_{b]}(8)C(8)]^T = -\gamma_{[a}(8)\gamma_{b]}(8)C(8) \\ [\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_{e]}(8)C(8)]^T = -\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_{e]}(8)C(8) \\ [\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_{f]}(8)C(8)]^T = -\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_{f]}(8)C(8) \\ C^T(8) = C(8), [\gamma_{[a}(8)\gamma_b(8)\gamma_{c]}(8)C(8)]^T = \gamma_{[a}(8)\gamma_b(8)\gamma_{c]}(8)C(8) \\ [\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_{d]}(8)C(8)]^T = \gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_{d]}(8)C(8) \\ [\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_f(8)\gamma_{g]}(8)C(8)]^T = \gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_f(8)\gamma_{g]}(8)C(8) \\ [\gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_f(8)\gamma_g(8)\gamma_{h]}(8)C(8)]^T = \gamma_{[a}(8)\gamma_b(8)\gamma_c(8)\gamma_d(8)\gamma_e(8)\gamma_f(8)\gamma_g(8)\gamma_{h]}(8)C(8) \end{array} \right.$$

定义1.1.10.

$$\left\{ \begin{array}{l} \gamma_a(9) = [\gamma_a(8), I_8 \otimes \sigma_z] = [\gamma_a(7) \otimes \sigma_y, I_8 \otimes \sigma_x, I_8 \otimes \sigma_z] \\ C(9) := \gamma_2(9)\gamma_4(9)\gamma_5(9)\gamma_8(9)\gamma_9(9) = -\sigma_y \otimes I \otimes \sigma_y \otimes I, \bar{C}(9) = C^+(9) = C(9), \gamma_1(9) \cdots \gamma_9(9) = -1 = -\gamma_0(9) \\ [\gamma_{[a}(9)\gamma_{b]}(9)C(9)]^T = -\gamma_{[a}(9)\gamma_{b]}(9)C(9), [\gamma_{[a}(9)\gamma_b(9)\gamma_{c]}(9)C(9)]^T = -\gamma_{[a}(9)\gamma_b(9)\gamma_{c]}(9)C(9) \\ C^T(9) = C(9), [\gamma_a(9)C(9)]^T = \gamma_a(9)C(9), [\gamma_{[a}(9)\gamma_b(9)\gamma_c(9)\gamma_{d]}(9)C(9)]^T = \gamma_{[a}(9)\gamma_b(9)\gamma_c(9)\gamma_{d]}(9)C(9) \end{array} \right.$$

定义1.1.11.

$$\left\{ \begin{array}{l} \gamma_a(10) = [\gamma_a(9) \otimes \sigma_y, I_{16} \otimes \sigma_x], \Gamma^a(10) = [\gamma_a(9), i\zeta], \gamma_1(10) \cdots \gamma_{10}(10) = iI_{16} \otimes \sigma_z = i\gamma_0(10) \\ C(10) := \gamma_2(10)\gamma_4(10)\gamma_5(10)\gamma_8(10)\gamma_9(10) = -\sigma_y \otimes I \otimes \sigma_y \otimes I \otimes \sigma_y, \bar{C}(10) = C^+(10) = C(10) \end{array} \right.$$

$$\left\{ \begin{array}{l} C^T(10) = -C(10), [\gamma_a(10)C(10)]^T = \gamma_a(10)C(10) \\ [\gamma_{[a}(10)\gamma_{b]}(10)C(10)]^T = \gamma_{[a}(10)\gamma_{b]}(10)C(10) \\ [\gamma_{[a}(10)\gamma_b(10)\gamma_{c]}(10)C(10)]^T = -\gamma_{[a}(10)\gamma_b(10)\gamma_{c]}(10)C(10) \\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_{d]}(10)C(10)]^T = -\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_{d]}(10)C(10) \\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_{e]}(10)C(10)]^T = \gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_{e]}(10)C(10) \\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_{f]}(10)C(10)]^T = \gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_{f]}(10)C(10) \\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_{g]}(10)C(10)]^T = -\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_{g]}(10)C(10) \\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_{h]}(10)C(10)]^T = -\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_{h]}(10)C(10) \\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_h(10)\gamma_{i]}(10)C(10)]^T = \gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_h(10)\gamma_{i]}(10)C(10) \\ [\gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_h(10)\gamma_i(10)\gamma_{j]}(10)C(10)]^T = \gamma_{[a}(10)\gamma_b(10)\gamma_c(10)\gamma_d(10)\gamma_e(10)\gamma_f(10)\gamma_g(10)\gamma_h(10)\gamma_i(10)\gamma_{j]}(10)C(10) \end{array} \right.$$

定义1.1.12.  $\gamma_a(10) = [\sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y, \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y, I \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y, I \otimes \sigma_z \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y, I_4 \otimes \sigma_x \otimes \sigma_y \otimes \sigma_y, I_4 \otimes \sigma_z \otimes \sigma_y \otimes \sigma_y, I_8 \otimes \sigma_x \otimes \sigma_y, I_8 \otimes \sigma_z \otimes \sigma_y, I_{16} \otimes \sigma_x]$

定义1.1.13.

$$\left\{ \begin{array}{l} \gamma_a(11) = [\gamma_a(10), I_{16} \otimes \sigma_z], \gamma_1(11) \cdots \gamma_{11}(11) = i = i\gamma_0(11) \\ C(11) := \gamma_2(11)\gamma_4(11)\gamma_5(11)\gamma_8(11)\gamma_9(11) = -\sigma_y \otimes I \otimes \sigma_y \otimes I \otimes \sigma_y, C(11) = C(10), \bar{C}(11) = C^+(11) = C(11) \\ C^T(11) = -C(11), [\gamma_a(11)C(11)]^T = \gamma_a(11)C(11) \\ [\gamma_{[a}(11)\gamma_{b]}(11)C(11)]^T = \gamma_{[a}(11)\gamma_{b]}(11)C(11) \\ [\gamma_{[a}(11)\gamma_b(11)\gamma_{c]}(11)C(11)]^T = -\gamma_{[a}(11)\gamma_b(11)\gamma_{c]}(11)C(11) \\ [\gamma_{[a}(11)\gamma_b(11)\gamma_c(11)\gamma_{d]}(11)C(11)]^T = -\gamma_{[a}(11)\gamma_b(11)\gamma_c(11)\gamma_{d]}(11)C(11) \\ [\gamma_{[a}(11)\gamma_b(11)\gamma_c(11)\gamma_d(11)\gamma_{e]}(11)C(11)]^T = \gamma_{[a}(11)\gamma_b(11)\gamma_c(11)\gamma_d(11)\gamma_{e]}(11)C(11) \end{array} \right.$$

猜想1.1.1.

$$\left\{ \begin{array}{l} \bar{C}(n) = C^+(n), C^+(n) = (-1)^{[\frac{n}{4}]}C(n), C^T(n) = (-1)^{[\frac{n+2}{4}]}C(n) \\ [\gamma_a(n)C(n)]^T = (-1)^{[\frac{n-1}{4}]}[\gamma_a(n)C(n)], [C^+(n)\gamma_a(n)]^T = (-1)^{[\frac{n-1}{4}]}[C^+(n)\gamma_a(n)] \end{array} \right.$$

自我评述：以上N+1维时空中的Dirac矩阵的选取不是唯一的，原则上有无穷种，做个表象变换即可，此时C矩阵也会发生改变，不再是原来的形式。

## 2 N+1=n维时空中二阶矩阵的反对称张量场展开 [45]

### 2.1 N+1=n偶数维时空中一般矩阵的反对称张量场展开

定义2.1.1.

$$\begin{aligned} X &= \frac{1}{(0!)^2}F + \frac{1}{(1!)^2}F^{a_1}\gamma_{a_1} + \frac{1}{(2!)^2}F^{a_1a_2}\gamma_{[a_1}\gamma_{a_2]} + \frac{1}{(3!)^2}F^{a_1a_2a_3}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]} + \cdots + \frac{1}{(n!)^2}F^{a_1a_2a_3\cdots a_n}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\cdots\gamma_{a_n]} \\ \left\{ \begin{array}{l} F = 2^{-[\frac{n}{2}]}tr(X), F_{a_1} = 2^{-[\frac{n}{2}]}tr(\gamma_{a_1}X) \\ F_{a_1a_2} = -2^{-[\frac{n}{2}]}tr(\frac{1}{2!}\gamma_{[a_1}\gamma_{a_2]}X), F_{a_1a_2a_3} = -2^{-[\frac{n}{2}]}tr(\frac{1}{3!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3]}X) \\ F_{a_1a_2a_3a_4} = 2^{-[\frac{n}{2}]}tr(\frac{1}{4!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4]}X), F_{a_1a_2a_3a_4a_5} = 2^{-[\frac{n}{2}]}tr(\frac{1}{5!}\gamma_{[a_1}\gamma_{a_2}\gamma_{a_3}\gamma_{a_4}\gamma_{a_5]}X) \\ \cdots F_{a_1a_2\cdots a_n} = (-1)^{[(n\%4)/2]}2^{-[\frac{n}{2}]}tr(\frac{1}{n!}\gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_n]}X) \end{array} \right. \end{aligned}$$

定义2.1.2.

$$\left\{ \begin{array}{l} X_{\lambda_s\mu_s\eta_s}(x) = \sum_{i=0}^n \frac{1}{(i!)^2}F^{a_1\cdots a_i}|_{\eta_s}(x)(\gamma_{[a_1}\cdots\gamma_{a_i]})_{\lambda_s\mu_s} \\ F_{a_1\cdots a_i}|_{\eta_s}(x) = (-1)^{[(i\%4)/2]} \frac{2^{-[\frac{n}{2}]}i!}{i!}(\gamma_{[a_1}\cdots\gamma_{a_i]})^{\mu_s\lambda_s} X_{\lambda_s\mu_s\eta_s}(x) \end{array} \right.$$

定义2.1.3.

$$\begin{cases} \sum_{i \in \text{even}} \frac{1}{(i!)^2} F^{a_1 \cdots a_i} |_{\eta_\zeta}(x) (\gamma_{[a_1 \cdots a_i]} C)_{\lambda_\zeta \mu_\zeta} (C^+ \gamma_{[c_1 \cdots c_k]}) |_{\text{odd}}^{\eta_\zeta \mu_\zeta} = 0 \\ \sum_{i \in \text{even}} \frac{1}{(i!)^2} (\gamma_{[a_1 \cdots a_i]} C) (C^+ \gamma_{[c_1 \cdots c_k]}) |_{\text{odd}} F^{a_1 \cdots a_i}(x) = 0 \end{cases}$$

定义2.1.4.

$$\begin{cases} X_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}(x) = \sum_{i,j=0}^n \frac{1}{(i!j!)^2} F^{a_1 \cdots a_i | b_1 \cdots b_j}(x) (\gamma_{[a_1 \cdots a_i]})_{\lambda_\zeta \mu_\zeta} (\gamma_{[b_1 \cdots b_j]})_{\eta_\zeta \xi_\zeta} \\ F_{a_1 \cdots a_i | b_1 \cdots b_j}(x) = (-1)^{[(i\%4)/2] + [(j\%4)/2]} \frac{4^{-[\frac{n}{2}]}}{i!j!} (\gamma_{[a_1 \cdots a_i]})^{\mu_\zeta \lambda_\zeta} (\gamma_{[b_1 \cdots b_j]})^{\xi_\zeta \eta_\zeta} X_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}(x) \end{cases}$$

定义2.1.5.

$$\begin{cases} \sum_{i,j \in \text{even}} \frac{1}{(i!j!)^2} F^{a_1 \cdots a_i | b_1 \cdots b_j}(x) (\gamma_{[a_1 \cdots a_i]} C)_{\lambda_\zeta \mu_\zeta} (C^+ \gamma_{[c_1 \cdots c_k]}) |_{\text{odd}}^{\eta_\zeta \mu_\zeta} (\gamma_{[b_1 \cdots b_j]} C)_{\eta_\zeta \xi_\zeta} = 0 \\ \sum_{i,j \in \text{even}} \frac{1}{(i!j!)^2} F^{a_1 \cdots a_i | b_1 \cdots b_j}(x) (\gamma_{[a_1 \cdots a_i]} C) (C^+ \gamma_{[c_1 \cdots c_k]}) |_{\text{odd}} (\gamma_{[b_1 \cdots b_j]} C) = 0 \end{cases}$$

## 2.2 N+1=n奇数维时空中一般矩阵的反对称张量场展开

定义2.2.1.

$$\begin{cases} X = \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(3!)^2} F^{a_1 a_2 a_3} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} + \cdots + \frac{1}{\{[n/2]!\}^2} F^{a_1 a_2 \cdots a_{[n/2]}} \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_{[n/2]}} \\ \begin{cases} F = 2^{-[\frac{n}{2}]} \text{tr}(X), F_{a_1} = 2^{-[\frac{n}{2}]} \text{tr}(\gamma_{a_1} X) \\ F_{a_1 a_2} = -2^{-[\frac{n}{2}]} \text{tr}(\frac{1}{2!} \gamma_{[a_1} \gamma_{a_2]} X), F_{a_1 a_2 a_3} = -2^{-[\frac{n}{2}]} \text{tr}(\frac{1}{3!} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} X) \\ F_{a_1 a_2 a_3 a_4} = 2^{-[\frac{n}{2}]} \text{tr}(\frac{1}{4!} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} X), F_{a_1 a_2 a_3 a_4 a_5} = 2^{-[\frac{n}{2}]} \text{tr}(\frac{1}{5!} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} X) \\ \cdots F_{a_1 a_2 \cdots a_{[n/2]}} = (-1)^{[(\frac{n}{2}\%4)/2]} 2^{-[\frac{n}{2}]} \text{tr}(\frac{1}{n!} \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_{[n/2]}} X) \end{cases} \end{cases}$$

定义2.2.2.

$$\begin{cases} X_{\lambda_\zeta \mu_\zeta \eta_\zeta}(x) = \sum_{i=0}^{[n/2]} \frac{1}{(i!)^2} F^{a_1 \cdots a_i} |_{\eta_\zeta}(x) (\gamma_{[a_1 \cdots a_i]})_{\lambda_\zeta \mu_\zeta} \\ F_{a_1 \cdots a_i | \eta_\zeta}(x) = (-1)^{[(i\%4)/2]} \frac{2^{-[\frac{n}{2}]}}{i!} (\gamma_{[a_1 \cdots a_i]})^{\mu_\zeta \lambda_\zeta} X_{\lambda_\zeta \mu_\zeta \eta_\zeta}(x) \end{cases}$$

定义2.2.3.

$$\begin{cases} \sum_{i \in \text{even}} \frac{1}{(i!)^2} F^{a_1 \cdots a_i} |_{\eta_\zeta}(x) (\gamma_{[a_1 \cdots a_i]} C)_{\lambda_\zeta \mu_\zeta} (C^+ \gamma_{[c_1 \cdots c_k]}) |_{\text{odd}}^{\eta_\zeta \mu_\zeta} = 0 \\ \sum_{i \in \text{even}} \frac{1}{(i!)^2} (\gamma_{[a_1 \cdots a_i]} C) (C^+ \gamma_{[c_1 \cdots c_k]}) |_{\text{odd}} F^{a_1 \cdots a_i}(x) = 0 \end{cases}$$

定义2.2.4.

$$\begin{cases} X_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}(x) = \sum_{i,j=0}^{[n/2]} \frac{1}{(i!j!)^2} F^{a_1 \cdots a_i | b_1 \cdots b_j}(x) (\gamma_{[a_1 \cdots a_i]})_{\lambda_\zeta \mu_\zeta} (\gamma_{[b_1 \cdots b_j]})_{\eta_\zeta \xi_\zeta} \\ F_{a_1 \cdots a_i | b_1 \cdots b_j}(x) = (-1)^{[(i\%4)/2] + [(j\%4)/2]} \frac{4^{-[\frac{n}{2}]}}{i!j!} (\gamma_{[a_1 \cdots a_i]})^{\mu_\zeta \lambda_\zeta} (\gamma_{[b_1 \cdots b_j]})^{\xi_\zeta \eta_\zeta} X_{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta}(x) \end{cases}$$

定义2.2.5.

$$\begin{cases} \sum_{i,j \in \text{even}} \frac{1}{(i!j!)^2} F^{a_1 \cdots a_i | b_1 \cdots b_j}(x) (\gamma_{[a_1 \cdots a_i]} C)_{\lambda_\zeta \mu_\zeta} (C^+ \gamma_{[c_1 \cdots c_k]}) |_{\text{odd}}^{\eta_\zeta \mu_\zeta} (\gamma_{[b_1 \cdots b_j]} C)_{\eta_\zeta \xi_\zeta} = 0 \\ \sum_{i,j \in \text{even}} \frac{1}{(i!j!)^2} F^{a_1 \cdots a_i | b_1 \cdots b_j}(x) (\gamma_{[a_1 \cdots a_i]} C) (C^+ \gamma_{[c_1 \cdots c_k]}) |_{\text{odd}} (\gamma_{[b_1 \cdots b_j]} C) = 0 \end{cases}$$

自我评述: 二阶Dirac张量(自旋-1)可以自然分解为反对称张量集合, 所以具体展示了自旋-1的理论必定是规范理论。

## 2.3 N+1=n偶数维时空中对称矩阵的展开

$$\text{性质2.3.1. } X(2) = \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} \right\} C$$

$$\text{性质2.3.2. } X(4) = \left\{ \frac{1}{1!} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} \right\} C$$

$$\text{性质2.3.3. } X(6) = \left\{ \frac{1}{0!} F + \frac{1}{(3!)^2} F^{a_1 a_2 a_3} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 a_2 a_3 a_4} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} \right\} C$$

性质2.3.4.

$$X(8) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{3!} F^{a_1 a_2 a_3} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \dots a_4} \gamma_{[a_1} \dots \gamma_{a_4]} + \frac{1}{(7!)^2} F^{a_1 \dots a_7} \gamma_{[a_1} \dots \gamma_{a_7]} + \frac{1}{(8!)^2} F^{a_1 \dots a_8} \gamma_{[a_1} \dots \gamma_{a_8]} \right\} C$$

性质2.3.5.

$$X(10) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{2!} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \dots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \dots a_6} \gamma_{[a_1} \dots \gamma_{a_6]} + \frac{1}{(9!)^2} F^{a_1 \dots a_9} \gamma_{[a_1} \dots \gamma_{a_9]} + \frac{1}{(10!)^2} F^{a_1 \dots a_{10}} \gamma_{[a_1} \dots \gamma_{a_{10}]} \right\} C$$

## 2.4 N+1=n偶数维时空中反对称矩阵的展开

性质2.4.1.  $X(2) = \frac{1}{(0!)^2} FC$

性质2.4.2.  $X(4) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \dots \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \dots a_4} \gamma_{[a_1} \dots \gamma_{a_4]} \right\} C$

性质2.4.3.  $X(6) = \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \dots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \dots a_6} \gamma_{[a_1} \dots \gamma_{a_6]} \right\} C$

性质2.4.4.  $X(8) = \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \dots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \dots a_6} \gamma_{[a_1} \dots \gamma_{a_6]} \right\} C$

性质2.4.5.  $X(10) =$

$$\left\{ \frac{1}{1!} F^{a_1} \gamma_{a_1} + \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \dots \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \dots a_4} \gamma_{[a_1} \dots \gamma_{a_4]} + \frac{1}{(7!)^2} F^{a_1 \dots a_7} \gamma_{[a_1} \dots \gamma_{a_7]} + \frac{1}{(8!)^2} F^{a_1 \dots a_8} \gamma_{[a_1} \dots \gamma_{a_8]} \right\} C$$

## 2.5 N+1=n奇数维时空中对称矩阵的展开

性质2.5.1.  $X(3) = \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} C$

性质2.5.2.  $X(5) = \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} C$

性质2.5.3.  $X(7) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 a_2 a_3} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} \right\} C$

性质2.5.4.  $X(9) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{1!} F^{a_1} \gamma_{a_1} + \frac{1}{(4!)^2} F^{a_1 a_2 a_3 a_4} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} \right\} C$

性质2.5.5.  $X(11) = \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \dots \gamma_{a_5]} \right\} C$

## 2.6 N+1=n奇数维时空中反对称矩阵的展开

性质2.6.1.  $X(3) = \frac{1}{(0!)^2} FC$

性质2.6.2.  $X(5) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} \right\} C$

性质2.6.3.  $X(7) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \right\} C$

性质2.6.4.  $X(9) = \left\{ \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \dots \gamma_{a_3]} \right\} C$

性质2.6.5.  $X(11) = \left\{ \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \dots \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \dots a_4} \gamma_{[a_1} \dots \gamma_{a_4]} \right\} C$

# 3 N+1维时空中基本反对称张量场的共同性质

## 3.1 N+1维时空中的无质量反对称张量场

定义3.1.1.  $\partial^{[a_0} A^{a_1 \dots a_i]} = 0, \partial_{a_1} A^{a_1 \dots a_i} = 0; \partial^{[a_0} F^{a_1 \dots a_i]} = 0, \partial_{a_1} F^{a_1 \dots a_i} = 0$

## 3.2 N+1维时空中的有质量反对称张量场

定义3.2.1.  $\frac{1}{l!} \partial^{[a_0} A^{a_1 \dots a_i]} + m F^{a_0 \dots a_i} = 0, \partial_{a_0} F^{a_0 \dots a_i} + m A^{a_1 \dots a_i} = 0$

$$\Leftrightarrow \partial_{a_0} \partial^{a_0} A^{a_1 \dots a_i} - m^2 A^{a_1 \dots a_i} = 0, \partial_{a_1} A^{a_1 \dots a_i} = 0, F^{a_0 \dots a_i} = -\frac{1}{(l+1)!m} \partial^{[a_0} A^{a_1 \dots a_i]}$$

## 3.3 N+1维时空中反对称张量场对偶基之间的关系

定理3.3.1.  $\frac{1}{l!} \gamma_{[a_1} \dots \gamma_{a_l]} = -(-1)^{(n-l-1)(n-l)/2} i^{-[n/2]} \varepsilon_{a_1 \dots a_n} \frac{1}{[(n-l)!]^2} \Gamma_0 \gamma^{[a_{l+1} \dots \gamma_{a_n}]}, \Gamma_0 := -i^{[n/2]} \gamma_1 \dots \gamma_n$

推论3.3.1.  $\begin{cases} \frac{1}{l!} \gamma_{[a_1} \dots \gamma_{a_l]} = -i^{[n/2]+l+l(-1)^n} \varepsilon_{a_1 \dots a_n} \frac{1}{[(n-l)!]^2} \Gamma_0 \gamma^{[a_{l+1} \dots \gamma_{a_n}]}, \Gamma_0 := -i^{[n/2]} \gamma_1 \dots \gamma_n, \Gamma_0|_{\text{odd}} = 1 \\ \frac{1}{l!} \gamma_{[a_1} \dots \gamma_{a_l]} = -i^{[n/2]+l(l-1)} \varepsilon_{a_1 \dots a_n} \frac{1}{[(n-l)!]^2} \gamma^{[a_{l+1} \dots \gamma_{a_n}]}, \Gamma_0 := -i^{[n/2]} \gamma_1 \dots \gamma_n, \Gamma_0|_{\text{odd}} = 1 \end{cases}$

## 3.4 N+1维时空中反对称张量场的等价对偶表示

引理3.4.1.  $*A^{a_1 \cdots a_l} = \frac{1}{(n-l)!} \varepsilon^{a_1 a_2 \cdots a_n} A_{a_{l+1} \cdots a_n} \Leftrightarrow A_{a_{l+1} \cdots a_n} = \frac{1}{l!} \varepsilon_{a_1 a_2 \cdots a_n} *A^{a_1 \cdots a_l} = (-1)^{Nl} **A_{a_{l+1} \cdots a_n}$

证明:  $*A^{a_1 \cdots a_l} = \frac{1}{(n-l)!} \varepsilon^{a_1 a_2 \cdots a_n} A_{a_{l+1} \cdots a_n}$

$$\Rightarrow \varepsilon_{a_1 \cdots a_l b_{l+1} \cdots b_n} *A^{a_1 \cdots a_l}$$

$$= \varepsilon_{a_1 \cdots a_l b_{l+1} \cdots b_n} \frac{1}{(n-l)!} \varepsilon^{a_1 \cdots a_l a_{l+1} \cdots a_n} A_{a_{l+1} \cdots a_n}$$

$$= \frac{l!}{(n-l)!} \delta_{b_{l+1}}^{[a_{l+1}} \cdots \delta_{b_n}^{a_n]} A_{a_{l+1} \cdots a_n}$$

$$= \frac{l!}{(n-l)!} (n-l)! \delta_{a_{l+1}}^{b_{l+1}} \cdots \delta_{a_n}^{b_n} A_{a_{l+1} \cdots a_n}$$

$$= l! A_{b_{l+1} \cdots b_n}$$

$$\Rightarrow A_{a_{l+1} \cdots a_n} = \frac{1}{l!} \varepsilon_{a_1 a_2 \cdots a_n} *A^{a_1 \cdots a_l} = (-1)^{Nl} **A_{a_{l+1} \cdots a_n} \quad \square$$

证明:  $A_{a_{l+1} \cdots a_n} = (-1)^{Nl} **A_{a_{l+1} \cdots a_n} = \frac{1}{l!} \varepsilon_{a_1 a_2 \cdots a_n} *A^{a_1 \cdots a_l}$

$$\Rightarrow \varepsilon^{b_1 \cdots b_l a_{l+1} \cdots a_n} A_{a_{l+1} \cdots a_n}$$

$$= \varepsilon^{b_1 \cdots b_l a_{l+1} \cdots a_n} \frac{1}{l!} \varepsilon_{a_1 a_2 \cdots a_n} *A^{a_1 \cdots a_l}$$

$$= \frac{(n-l)!}{l!} \delta_{[a_1}^{b_1} \cdots \delta_{a_l]}^{b_l} *A^{a_1 \cdots a_l}$$

$$= \frac{(n-l)!}{l!} l! \delta_{a_1}^{b_1} \cdots \delta_{a_l}^{b_l} *A^{a_1 \cdots a_l}$$

$$= (n-l)! *A^{b_1 \cdots b_l}$$

$$\Rightarrow *A^{a_1 \cdots a_l} = \frac{1}{(n-l)!} \varepsilon^{a_1 a_2 \cdots a_n} A_{a_{l+1} \cdots a_n} \quad \square$$

定理3.4.1.

$$\begin{cases} \frac{1}{l!} \partial^{[a_0} A^{a_1 \cdots a_l]} + m F^{a_0 \cdots a_l} = 0 \\ *A_{a_{l+1} \cdots a_n} = \frac{(-1)^{Nl}}{l!} \varepsilon_{a_1 \cdots a_n} A^{a_1 \cdots a_l} \\ *F_{a_{l+1} \cdots a_{n-1}} = \frac{(-1)^{N(l+1)}}{(l+1)!} \varepsilon_{a_0 \cdots a_{n-1}} F^{a_0 \cdots a_l} \end{cases} \Leftrightarrow \begin{cases} \partial^{a_0} *A_{a_0 a_1 \cdots a_{n-l-1}} + (-1)^{n-l-1} m *F_{a_1 \cdots a_{n-l-1}} = 0 \\ A^{a_1 \cdots a_l} = \frac{(-1)^{Nl}}{(n-l)!} \varepsilon^{a_1 \cdots a_n} *A_{a_{l+1} \cdots a_n} \\ F^{a_0 \cdots a_l} = \frac{(-1)^{N(l+1)}}{(n-l-1)!} \varepsilon^{a_0 \cdots a_{n-1}} *F_{a_{l+1} \cdots a_{n-1}} \end{cases}$$

证明:  $\frac{1}{l!} \partial^{[a_0} A^{a_1 \cdots a_l]} + m F^{a_0 \cdots a_l} = 0 \Leftrightarrow \frac{1}{(l+1)!} \varepsilon_{a_0 a_1 a_2 \cdots a_{n-1}} \{ \frac{1}{l!} \partial^{[a_0} A^{a_1 \cdots a_l]} + m F^{a_0 \cdots a_l} \} = 0$

$$\Leftrightarrow \frac{1}{l!} \varepsilon_{a_0 a_1 a_2 \cdots a_{n-1}} \partial^{a_0} A^{a_1 \cdots a_l} + \frac{1}{(l+1)!} \varepsilon_{a_0 a_1 a_2 \cdots a_{n-1}} m F^{a_0 \cdots a_l} = 0$$

$$\Leftrightarrow (-1)^{Nl-l} \partial^{a_0} *A_{a_0 a_{l+1} \cdots a_{n-1}} + m (-1)^{N(l+1)} *F_{a_{l+1} \cdots a_{n-1}} = 0$$

$$\Leftrightarrow \partial^{a_0} *A_{a_0 a_{l+1} \cdots a_{n-1}} - (-1)^{n-l} m *F_{a_{l+1} \cdots a_{n-1}} = 0$$

$$\Leftrightarrow \partial^{a_0} *A_{a_0 a_1 \cdots a_{n-l-1}} + (-1)^{n-l-1} m *F_{a_1 \cdots a_{n-l-1}} = 0 \quad \square$$

定理3.4.2.

$$\begin{cases} \partial_{a_0} F^{a_0 \cdots a_l} + m A^{a_1 \cdots a_l} = 0 \\ *A_{a_{l+1} \cdots a_n} = \frac{(-1)^{Nl}}{l!} \varepsilon_{a_1 \cdots a_n} A^{a_1 \cdots a_l} \\ *F_{a_{l+1} \cdots a_{n-1}} = \frac{(-1)^{N(l+1)}}{(l+1)!} \varepsilon_{a_0 \cdots a_{n-1}} F^{a_0 \cdots a_l} \end{cases} \Leftrightarrow \begin{cases} \frac{1}{(n-l-1)!} \partial_{[a_0} *F_{a_1 \cdots a_{n-l-1}]} + (-1)^{n-l-1} m *A_{a_0 a_1 \cdots a_{n-l-1}} = 0 \\ A^{a_1 \cdots a_l} = \frac{(-1)^{Nl}}{(n-l)!} \varepsilon^{a_1 \cdots a_n} *A_{a_{l+1} \cdots a_n} \\ F^{a_0 \cdots a_l} = \frac{(-1)^{N(l+1)}}{(n-l-1)!} \varepsilon^{a_0 \cdots a_{n-1}} *F_{a_{l+1} \cdots a_{n-1}} \end{cases}$$

证明:  $\partial_{a_0} F^{a_0 \cdots a_l} + m A^{a_1 \cdots a_l} = 0 \Leftrightarrow \frac{1}{l!} \varepsilon_{a_1 a_2 \cdots a_n} \{ \partial_{a_0} F^{a_0 a_1 \cdots a_l} + m A^{a_1 \cdots a_l} \} = 0$

$$\Leftrightarrow \frac{1}{l!} \varepsilon_{a_1 a_2 \cdots a_n} \{ \partial_{a_0} \frac{(-1)^{N(l+1)}}{(n-l-1)!} \varepsilon^{a_0 a_1 \cdots b_{l+1} b_{n-1}} *F_{b_{l+1} \cdots b_{n-1}} + m A^{a_1 \cdots a_l} \} = 0$$

$$\Leftrightarrow \frac{1}{l!} \varepsilon_{a_1 a_2 \cdots a_n} \partial_{a_0} \frac{(-1)^{Nl}}{(n-l-1)!} \varepsilon^{a_1 \cdots b_{l+1} b_{n-1} a_0} *F_{b_{l+1} \cdots b_{n-1}} + (-1)^{Nl} m *A^{a_1 \cdots a_n} = 0$$

$$\Leftrightarrow \frac{1}{(n-l-1)!} \delta_{[a_{l+1}}^{b_{l+1}} \cdots \delta_{a_{n-1}}^{b_{n-1}]} \partial_{a_0} *F_{b_{l+1} \cdots b_{n-1}} + m *A_{a_{l+1} \cdots a_n} = 0$$

$$\Leftrightarrow \frac{1}{(n-l-1)!} \partial_{[a_n} *F_{a_{l+1} \cdots a_{n-1}]} + m *A_{a_{l+1} \cdots a_n} = 0$$

$$\Leftrightarrow \frac{1}{(n-l-1)!} \partial_{[a_0} *F_{a_1 \cdots a_{n-l-1}]} + (-1)^{n-l-1} m *A_{a_0 a_1 \cdots a_{n-l-1}} = 0 \quad \square$$

推论3.4.1.

$$\begin{cases} \frac{1}{l!} \partial^{[a_0} A^{a_1 \cdots a_l]} + m F^{a_0 \cdots a_l} = 0 \\ \partial_{a_0} F^{a_0 \cdots a_l} + m A^{a_1 \cdots a_l} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} *A^{a_0 a_1 \cdots a_{n-l-1}} + (-1)^{n-l-1} m *F^{a_1 \cdots a_{n-l-1}} = 0 \\ \frac{1}{(n-l-1)!} \partial^{[a_0} *F^{a_1 \cdots a_{n-l-1}]} + (-1)^{n-l-1} m *A^{a_0 a_1 \cdots a_{n-l-1}} = 0 \end{cases}$$

推论3.4.2.

$$\begin{cases} \frac{1}{l!} \partial^{[a_0} A^{a_1 \cdots a_l]} + m F^{a_0 \cdots a_l} = 0 \\ \partial_{a_0} F^{a_0 \cdots a_l} + m A^{a_1 \cdots a_l} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} * A^{a_0 a_1 \cdots a_{n-l-1}} + (-1)^{n-l-1} m * F^{a_1 \cdots a_{n-l-1}} = 0 \\ \partial_{a_0} F^{a_0 \cdots a_l} + m A^{a_1 \cdots a_l} = 0 \end{cases}$$

推论3.4.3.

$$\begin{cases} \frac{1}{l!} \partial^{[a_0} A^{a_1 \cdots a_l]} + m F^{a_0 \cdots a_l} = 0 \\ \partial_{a_0} F^{a_0 \cdots a_l} + m A^{a_1 \cdots a_l} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{l!} \partial^{[a_0} A^{a_1 \cdots a_l]} + m F^{a_0 \cdots a_l} = 0 \\ \frac{1}{(n-l-1)!} \partial^{[a_0} * F^{a_1 \cdots a_{n-l-1}]} + (-1)^{n-l-1} m * A^{a_0 a_1 \cdots a_{n-l-1}} = 0 \end{cases}$$

推论3.4.4.

$$\begin{cases} \partial^{[a_0} F^{a_1 \cdots a_l]} = 0 \\ \partial_{a_1} F^{a_1 \cdots a_l} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_1} * F^{a_1 \cdots a_{n-l}} = 0 \\ \partial^{[a_0} * F^{a_1 \cdots a_{n-l}]} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_1} * F^{a_1 \cdots a_{n-l}} = 0 \\ \partial_{a_1} F^{a_1 \cdots a_l} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1 \cdots a_l]} = 0 \\ \partial^{[a_0} * F^{a_1 \cdots a_{n-l}]} = 0 \end{cases}$$

### 3.5 N+1=n偶维时空中B-W方程导出基本的反对称张量场

引理3.5.1.  $(\gamma^a \partial_a + m) \left\{ \frac{1}{(l!)^2} F^{a_1 \cdots a_l} \gamma_{[a_1 \cdots a_l]} + \frac{1}{[(l+1)!]^2} F^{a_1 \cdots a_{l+1}} \gamma_{[a_1 \cdots a_{l+1}]} \right\} C = 0$

$$\Leftrightarrow \begin{cases} \frac{1}{l!} \partial^{[a_0} F^{a_1 \cdots a_l]} + m F^{a_0 a_1 \cdots a_l} = 0, \partial_{a_1} F^{a_1 \cdots a_l} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_{l+1}]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_l} + m F^{a_1 \cdots a_l} = 0 \end{cases}; 1 \leq l \leq n-1$$

证明:  $(\gamma^a \partial_a + m) \left\{ \frac{1}{(l!)^2} F^{a_1 \cdots a_l} \gamma_{[a_1 \cdots a_l]} + \frac{1}{[(l+1)!]^2} F^{a_1 \cdots a_{l+1}} \gamma_{[a_1 \cdots a_{l+1}]} \right\} = 0; 1 \leq l \leq n-1$

$$\begin{aligned} &\Leftrightarrow (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{l!} F^{a_1 \cdots a_l} \gamma_{a_1 \cdots a_l} + \frac{1}{(l+1)!} F^{a_1 \cdots a_{l+1}} \gamma_{a_1 \cdots a_{l+1}} \right\} = 0 \\ &\Leftrightarrow \left\{ \frac{1}{(l+1)!} \gamma_{[a_0} \gamma_{a_1 \cdots a_l]} + \frac{1}{(l-1)!} (\delta_{a_0 a_1} \gamma_{[a_2 \cdots a_l]} + \cdots) + \cdots \right\} \partial^{a_0} \frac{1}{l!} F^{a_1 \cdots a_l} + m \frac{1}{(l+1)!} F^{a_1 \cdots a_{l+1}} \gamma_{a_1 \cdots a_{l+1}} \\ &+ \left\{ \frac{1}{(l+2)!} \gamma_{[a_0} \gamma_{a_1 \cdots a_{l+1}]} + \frac{1}{l!} (\delta_{a_0 a_1} \gamma_{[a_2 \cdots a_{l+1}]} + \cdots) + \cdots \right\} \partial^{a_0} \frac{1}{(l+1)!} F^{a_1 \cdots a_{l+1}} + m \frac{1}{l!} F^{a_1 \cdots a_l} \gamma_{a_1 \cdots a_l} = 0 \\ &\Leftrightarrow \left\{ \frac{1}{(l+1)!} \gamma_{[a_0} \gamma_{a_1 \cdots a_l]} + \frac{l}{(l-1)!} \delta_{a_0 a_1} \gamma_{[a_2 \cdots a_l]} \right\} \partial^{a_0} \frac{1}{l!} F^{a_1 \cdots a_l} + \frac{1}{(l+1)!} m \frac{1}{(l+1)!} F^{a_1 \cdots a_{l+1}} \gamma_{[a_1 \cdots a_{l+1}]} \\ &+ \left\{ \frac{1}{(l+2)!} \gamma_{[a_0} \gamma_{a_1 \cdots a_{l+1}]} + \frac{l+1}{l!} \delta_{a_0 a_1} \gamma_{[a_2 \cdots a_{l+1}]} \right\} \partial^{a_0} \frac{1}{(l+1)!} F^{a_1 \cdots a_{l+1}} + \frac{1}{l!} m \frac{1}{l!} F^{a_1 \cdots a_l} \gamma_{[a_1 \cdots a_l]} = 0 \\ &\Leftrightarrow \begin{cases} \frac{1}{(l+1)!} \partial^{[a_0} \frac{1}{l!} F^{a_1 \cdots a_l]} + m \frac{1}{(l+1)!} F^{a_0 a_1 \cdots a_l} = 0, \partial_{a_1} \frac{1}{l!} F^{a_1 \cdots a_l} = 0 \\ \partial^{[a_0} \frac{1}{(l+1)!} F^{a_1 \cdots a_{l+1}]} = 0, \partial_{a_0} \frac{1}{(l+1)!} F^{a_0 a_1 \cdots a_l} + \frac{1}{l+1} m \frac{1}{l!} F^{a_1 \cdots a_l} = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} \frac{1}{l!} \partial^{[a_0} F^{a_1 \cdots a_l]} + m F^{a_0 a_1 \cdots a_l} = 0, \partial_{a_1} F^{a_1 \cdots a_l} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_{l+1}]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_l} + m F^{a_1 \cdots a_l} = 0 \end{cases}; 1 \leq l \leq n-1 \quad \square \end{aligned}$$

引理3.5.2.  $(\gamma^a \partial_a + m) \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} \right\} C = 0 \Leftrightarrow \begin{cases} \frac{1}{0!} \partial^{a_0} F + m F^{a_0} = 0 \\ \partial^{[a_0} F^{a_1]} = 0, \partial_{a_0} F^{a_0} + m F = 0 \end{cases}; l = 0$

证明:  $(\gamma^a \partial_a + m) \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} \right\} C$

$$\begin{aligned} &\Leftrightarrow (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{0!} F + \frac{1}{1!} F^{a_1} \gamma_{a_1} \right\} = 0 \\ &\Leftrightarrow \frac{1}{1!} \gamma_{a_0} \partial^{a_0} \frac{1}{0!} F + m \frac{1}{1!} F^{a_1} \gamma_{a_1} + \left\{ \frac{1}{2!} \gamma_{[a_0} \gamma_{a_1]} + \frac{1}{0!} \delta_{a_0 a_1} \right\} \partial^{a_0} \frac{1}{1!} F^{a_1} + m \frac{1}{0!} F = 0 \\ &\Leftrightarrow \begin{cases} \frac{1}{1!} \partial^{a_0} \frac{1}{0!} F + m \frac{1}{1!} F^{a_0} = 0 \\ \partial^{[a_0} \frac{1}{1!} F^{a_1]} = 0, \partial_{a_0} \frac{1}{1!} F^{a_0} + \frac{1}{1} m \frac{1}{0!} F = 0 \end{cases} \\ &\Leftrightarrow \begin{cases} \frac{1}{0!} \partial^{a_0} F + m F^{a_0} = 0 \\ \partial^{[a_0} F^{a_1]} = 0, \partial_{a_0} F^{a_0} + m F = 0 \end{cases}; l = 0 \quad \square \end{aligned}$$

引理3.5.3.  $(\gamma^a \partial_a + m) \left\{ \frac{1}{(n!)^2} F^{a_1 \cdots a_n} \gamma_{[a_1 \cdots a_n]} \right\} C = 0 \Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1 \cdots a_n]} = 0, \partial_{a_1} F^{a_1 \cdots a_n} = 0 \\ m F^{a_1 \cdots a_n} = 0 \end{cases}; l = n$

证明:  $(\gamma^a \partial_a + m) \left\{ \frac{1}{(n!)^2} F^{a_1 \cdots a_n} \gamma_{[a_1 \cdots a_n]} \right\} C = 0; l = n$

$$\begin{aligned} &\Leftrightarrow (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{n!} F^{a_1 \cdots a_n} \gamma_{a_1 \cdots a_n} \right\} = 0 \\ &\Leftrightarrow \left\{ \frac{1}{(n+1)!} \gamma_{[a_0} \gamma_{a_1 \cdots a_n]} + \frac{1}{(n-1)!} (\delta_{a_0 a_1} \gamma_{[a_2 \cdots a_n]} + \cdots) + \cdots \right\} \partial^{a_0} \frac{1}{n!} F^{a_1 \cdots a_n} + m \frac{1}{n!} F^{a_1 \cdots a_n} \gamma_{a_1 \cdots a_n} = 0 \end{aligned}$$



$$\begin{aligned} & \Leftrightarrow \left\{ \frac{1}{(n+1)!} \gamma_{[a_0} \gamma_{a_1} \cdots \gamma_{a_n]} + \frac{n}{(n-1)!} \delta_{a_0 a_1} \gamma_{[a_2} \cdots \gamma_{a_n]} \right\} \partial^{a_0} \frac{1}{n!} F^{a_1 \cdots a_n} + \frac{1}{n!} m \frac{1}{n!} F^{a_1 \cdots a_n} \gamma_{[a_1} \cdots \gamma_{a_n]} = 0 \\ & \Leftrightarrow \begin{cases} \frac{1}{(n+1)!} \partial^{[a_0} \frac{1}{n!} F^{a_1 \cdots a_n]} = 0, \partial_{a_1} \frac{1}{n!} F^{a_1 \cdots a_n} = 0 \\ \frac{1}{n+1} m \frac{1}{n!} F^{a_1 \cdots a_n} = 0 \end{cases} \\ & \Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1 \cdots a_n]} = 0, \partial_{a_1} F^{a_1 \cdots a_n} = 0 \\ m F^{a_1 \cdots a_n} = 0 \end{cases} \quad ; l = n \end{aligned} \quad \square$$

### 3.6 N+1=n偶维时空中基本反对称张量场的性质

推论3.6.1.  $\frac{1}{l!} \gamma_{[a_1} \cdots \gamma_{a_l]} = -i^{[n/2]+l(l+1)} \varepsilon_{a_1 \cdots a_n} \frac{1}{[(n-l)!]^2} \Gamma_0 \gamma^{[a_{l+1} \cdots a_n]}$

定理3.6.1.  $(\gamma^a \partial_a + m) \left\{ \frac{1}{(l!)^2} F^{a_1 \cdots a_l} \gamma_{[a_1} \cdots \gamma_{a_l]} + \frac{1}{[(l+1)!]^2} F^{a_1 \cdots a_{l+1}} \gamma_{[a_1} \cdots \gamma_{a_{l+1}]} \right\} C = 0$

$$\Leftrightarrow \begin{cases} \frac{1}{l!} \partial^{[a_0} F^{a_1 \cdots a_l]} + m F^{a_0 a_1 \cdots a_l} = 0, \partial_{a_1} F^{a_1 \cdots a_l} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_{l+1}]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_l} + m F^{a_1 \cdots a_l} = 0 \end{cases}$$

推论3.6.2.  $\gamma^a \partial_a \left\{ \frac{1}{(l!)^2} F^{a_1 \cdots a_l} \gamma_{[a_1} \cdots \gamma_{a_l]} + \frac{1}{[(l+1)!]^2} F^{a_1 \cdots a_{l+1}} \gamma_{[a_1} \cdots \gamma_{a_{l+1}]} \right\} C = 0$

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1 \cdots a_l]} = 0, \partial_{a_1} F^{a_1 \cdots a_l} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_{l+1}]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_l} = 0 \end{cases}$$

推论3.6.3.  $(\gamma^a \partial_a + m) \left\{ \frac{1}{(l!)^2} F^{a_1 \cdots a_l} \gamma_{[a_1} \cdots \gamma_{a_l]} + \frac{1}{[(l+1)!]^2} F^{a_1 \cdots a_{l+1}} \gamma_{[a_1} \cdots \gamma_{a_{l+1}]} \right\} C = 0, m \neq 0$

$$\Leftrightarrow \begin{cases} \frac{1}{l!} \partial^{[a_0} F^{a_1 \cdots a_l]} + m F^{a_0 a_1 \cdots a_l} = 0 \\ \partial_{a_0} F^{a_0 a_1 \cdots a_l} + m F^{a_1 \cdots a_l} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_l} - m^2 F^{a_1 \cdots a_l} = 0, \partial_{a_1} F^{a_1 \cdots a_l} = 0 \\ F^{a_0 a_1 \cdots a_l} = -\frac{1}{l!m} \partial^{[a_0} F^{a_1 \cdots a_l]} \end{cases}$$

推论3.6.4.

$$\begin{cases} \frac{1}{l!} \partial^{[a_0} F^{a_1 \cdots a_l]} + m F^{a_0 a_1 \cdots a_l} = 0 \\ \partial_{a_0} F^{a_0 a_1 \cdots a_l} + m F^{a_1 \cdots a_l} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{(n-l-1)!} \partial^{[a_0} * F^{a_1 \cdots a_{n-l-1}]} + (-1)^{n-l-1} m * F^{a_0 a_1 \cdots a_{n-l-1}} = 0 \\ \partial_{a_0} * F^{a_0 a_1 \cdots a_{n-l-1}} + (-1)^{n-l-1} m * F^{a_1 \cdots a_{n-l-1}} = 0 \end{cases}$$

### 3.7 N+1=n奇数维时空中基本反对称张量场的性质

推论3.7.1.  $\frac{1}{l!} \gamma_{[a_1} \cdots \gamma_{a_l]} = -i^{[n/2]+l(l-1)} \varepsilon_{a_1 \cdots a_n} \frac{1}{[(n-l)!]^2} \gamma^{[a_{l+1} \cdots a_n]}$

推论3.7.2.  $\frac{1}{([\frac{n}{2}]+1)!} \gamma_{[a_1} \cdots \gamma_{a_{[n/2]+1}]} = -(-i)^{[\frac{n}{2}] \% 2} \varepsilon_{a_1 \cdots a_n} \frac{1}{([\frac{n}{2}!]^2)} \gamma^{[a_{[n/2]+2} \cdots a_n]}$

定理3.7.1.  $(\gamma^a \partial_a + m) \left\{ \frac{1}{(l!)^2} F^{a_1 \cdots a_l} \gamma_{[a_1} \cdots \gamma_{a_l]} + \frac{1}{[(l+1)!]^2} F^{a_1 \cdots a_{l+1}} \gamma_{[a_1} \cdots \gamma_{a_{l+1}]} \right\} C = 0$

$$\Leftrightarrow \begin{cases} \frac{1}{l!} \partial^{[a_0} F^{a_1 \cdots a_l]} + m F^{a_0 a_1 \cdots a_l} = 0, \partial_{a_1} F^{a_1 \cdots a_l} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_{l+1}]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_l} + m F^{a_1 \cdots a_l} = 0 \end{cases} \quad ; l \leq [\frac{n}{2}] - 2$$

引理3.7.1.  $(\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{([\frac{n}{2}!]^2)} F^{a_1 \cdots a_{[n/2]}} \gamma_{[a_1} \cdots \gamma_{a_{[n/2]}}] \right\} C = 0$

$$\Leftrightarrow \frac{1}{([\frac{n}{2}]!)^2} \varepsilon^{a_1 \cdots a_n} \partial_{a_{[n/2]+1}} F_{a_{[n/2]+2} \cdots a_n} - i^{[\frac{n}{2}] \% 2} m F^{a_1 \cdots a_{[n/2]}} = 0, \partial_{a_1} F^{a_1 \cdots a_{[n/2]}} = 0$$

证明:  $(\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{([\frac{n}{2}]!)^2} F^{a_1 \cdots a_{[n/2]}} \gamma_{[a_1} \cdots \gamma_{a_{[n/2]}}] \right\} C = 0$

$$\Leftrightarrow (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{([\frac{n}{2}]!)^2} F^{a_1 \cdots a_{[n/2]}} \gamma_{[a_1} \cdots \gamma_{a_{[n/2]}}] \right\} = 0$$

$$\Leftrightarrow (\gamma_{a_0} \partial^{a_0} + m) \left\{ F^{a_1 \cdots a_{[n/2]}} \gamma_{a_1} \cdots \gamma_{a_{[n/2]}} \right\} = 0$$

$$\Leftrightarrow \gamma_{a_0} \gamma_{a_1} \cdots \gamma_{a_{[n/2]}} \partial^{a_0} F^{a_1 \cdots a_{[n/2]}} + m F^{a_1 \cdots a_{[n/2]}} \gamma_{a_1} \cdots \gamma_{a_{[n/2]}} = 0$$

$$\Leftrightarrow \left\{ \frac{1}{([\frac{n}{2}]+1)!} \gamma_{[a_0} \gamma_{a_1} \cdots \gamma_{a_{[n/2]}}] + \frac{1}{([\frac{n}{2}-1)!} (\delta_{a_0 a_1} \gamma_{[a_2} \cdots \gamma_{a_{[n/2]}}] + \cdots) + \cdots \right\} \partial^{a_0} F^{a_1 \cdots a_{[n/2]}} + m F^{a_1 \cdots a_{[n/2]}} \gamma_{a_1} \cdots \gamma_{a_{[n/2]}} = 0$$

$$\Leftrightarrow \frac{1}{([\frac{n}{2}]+1)!} \gamma_{[a_0} \gamma_{a_1} \cdots \gamma_{a_{[n/2]}}] \partial^{a_0} F^{a_1 \cdots a_{[n/2]}} + \frac{[\frac{n}{2}]}{([\frac{n}{2}-1)!} \delta_{a_0 a_1} \gamma_{[a_2} \cdots \gamma_{a_{[n/2]}}] \partial^{a_0} F^{a_1 \cdots a_{[n/2]}} + m F^{a_1 \cdots a_{[n/2]}} \gamma_{a_1} \cdots \gamma_{a_{[n/2]}} = 0$$

$$\Leftrightarrow -(-i)^{[\frac{n}{2}] \% 2} \varepsilon^{a_0 \cdots a_{n-1}} \frac{1}{([\frac{n}{2}!]^2)} \gamma_{[a_{[n/2]+1} \cdots \gamma_{a_{n-1]}}] \partial_{a_0} F_{a_1 \cdots a_{[n/2]}}$$

$$+ \frac{[\frac{n}{2}]}{([\frac{n}{2}-1)!} \delta_{a_0 a_1} \gamma_{[a_2} \cdots \gamma_{a_{[n/2]}}] \partial^{a_0} F^{a_1 \cdots a_{[n/2]}} + \frac{1}{[\frac{n}{2}]!} m F^{a_{[n/2]+1} \cdots a_{n-1}} \gamma_{[a_{[n/2]+1} \cdots \gamma_{a_{n-1]}}] = 0$$

$$\Leftrightarrow \frac{1}{[\frac{n}{2}]!} \varepsilon^{a_0 \cdots a_{n-1}} \partial_{a_0} F_{a_1 \cdots a_{[n/2]}} - i^{[\frac{n}{2}] \% 2} m F^{a_{[n/2]+1} \cdots a_{n-1}} = 0, \partial_{a_1} F^{a_1 \cdots a_{[n/2]}} = 0$$

$$\Leftrightarrow \frac{1}{[\frac{n}{2}]!} \varepsilon^{a_1 \cdots a_n} \partial_{a_{[n/2]+1}} F_{a_{[n/2]+2} \cdots a_n} - i^{[\frac{n}{2}] \% 2} m F^{a_1 \cdots a_{[n/2]}} = 0, \partial_{a_1} F^{a_1 \cdots a_{[n/2]}} = 0 \quad \square$$

推论3.7.3.

$$\gamma_{a_0} \partial^{a_0} \left\{ \frac{1}{([\frac{n}{2}]!)^2} F^{a_1 \cdots a_{[n/2]}} \gamma_{[a_1 \cdots a_{[n/2]}]} \right\} C = 0 \Leftrightarrow \varepsilon^{a_1 \cdots a_n} \partial_{a_{[n/2]+1}} F_{a_{[n/2]+2} \cdots a_n} = 0, \partial_{a_1} F^{a_1 \cdots a_{[n/2]}} = 0$$

$$\text{推论3.7.4. } (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{([\frac{n}{2}]!)^2} F^{a_1 \cdots a_{[n/2]}} \gamma_{[a_1 \cdots a_{[n/2]}]} \right\} C = 0, m \neq 0$$

$$\Leftrightarrow \frac{1}{[\frac{n}{2}]!} \varepsilon^{a_1 \cdots a_n} \partial_{a_{[n/2]+1}} F_{a_{[n/2]+2} \cdots a_n} - i^{[\frac{n}{2}] \% 2} m F^{a_1 \cdots a_{[n/2]}} = 0$$

$$\text{推论3.7.5. } \frac{1}{[\frac{n}{2}]!} \varepsilon^{a_1 \cdots a_n} \partial_{a_{[n/2]+1}} F_{a_{[n/2]+2} \cdots a_n} - i^{[\frac{n}{2}] \% 2} m F^{a_1 \cdots a_{[n/2]}} = 0$$

$$\Leftrightarrow \partial_{a_0} * F^{a_0 \cdots a_{[n/2]}} - (-i)^{[\frac{n}{2}] \% 2} m F^{a_1 \cdots a_{[n/2]}} = 0 \Leftrightarrow \frac{1}{[\frac{n}{2}]!} \partial_{[a_0} F_{a_1 \cdots a_{[n/2]}}] - i^{[\frac{n}{2}] \% 2} m * F_{a_0 \cdots a_{[n/2]}} = 0$$

## 4 N+1维时空中基本反对称张量场的协变对易规则

### 4.1 从对称和反对称B-W方程导出基本反对称张量场的对易规则

猜想4.1.1.

$$\left\{ \begin{array}{l} \bar{C}(n) = C^+(n), C^+(n) = (-1)^{[\frac{n}{4}]} C(n), C^T(n) = (-1)^{[\frac{n+2}{4}]} C(n) \\ [\gamma_a(n) C(n)]^T = (-1)^{[\frac{n-1}{4}]} [\gamma_a(n) C(n)], [C^+(n) \gamma_a(n)]^T = (-1)^{[\frac{n-1}{4}]} [C^+(n) \gamma_a(n)] \\ \gamma_0^T(n) = \gamma_0(n), n \geq 3; \gamma_0^T(2) = -\gamma_0(2) \\ C(n) \gamma_0^T(n) C^+(n) = (-1)^{\xi(n)} \gamma_0^T(n), C(n) \gamma_0^T(n) \gamma_a^T(n) C^+(n) = (-1)^{\xi(n) + \eta(n)} \gamma_0^T(n) \gamma_a(n) \\ (-1)^{\xi(n)} = (-1)^{\eta(n)} := (-1)^{[\frac{n-1}{4}]} (-1)^{[\frac{n+2}{4}]}, n \geq 3 \\ (-1)^{\xi(n)+1} = (-1)^{\eta(n)} := (-1)^{[\frac{n-1}{4}]} (-1)^{[\frac{n+2}{4}]}, n \geq 3 \end{array} \right.$$

$$\text{定理4.1.1. } [F_{a_1 a_2 \cdots a_l}(x), F_{a'_1 a'_2 \cdots a'_l}(x')] = -i \frac{(-1)^{\xi_{2,n}}}{2^{[\frac{n}{2}]}} \left\{ \begin{array}{l} \frac{1}{(l+1)!} \eta_{[a'_1}^{a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_l}^{a_l}] \partial_a \partial^{a'} \Delta(x-x'), (-1)^{\eta(n)+l} = 1 \\ -\frac{1}{(l-1)!} \eta_{[a'_1}^{a_1} \cdots \eta_{a'_{l-1}}^{a_{l-1}} \partial^{a_l}] \partial_{a'_l} \Delta(x-x'), (-1)^{\eta(n)+l} = -1 \end{array} \right.$$

$$\text{证明: } [F_{a_1 a_2 \cdots a_l}(x), F_{a'_1 a'_2 \cdots a'_l}(x')]$$

$$= \frac{4^{-[\frac{n}{2}]}}{(l!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]})^{\mu\lambda} (C^+ \gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l]})^{*\mu'\lambda'} [\psi_{\lambda\mu}(x), \psi_{\lambda'\mu'}^+(x')]$$

$$= \frac{4^{-[\frac{n}{2}]}}{(l!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]})^{\mu\lambda} (C^+ \gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l]})^{+\mu'\lambda'} [\psi_{\lambda\mu}(x), \psi_{\lambda'\mu'}^+(x')]$$

$$= \frac{4^{-[\frac{n}{2}]}}{(l!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]})^{\mu\lambda} (\gamma_{[a'_1} \cdots \gamma_{a'_2} \gamma_{a'_1}] C)^{\lambda'\mu'} [\psi_{\lambda\mu}(x), \psi_{\lambda'\mu'}^+(x')]$$

$$= \frac{4^{-[\frac{n}{2}]}}{(l!)^2} i^{l(l-1)} (C^+ \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]})^{\mu\lambda} (\gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l}] C)^{\lambda'\mu'} [\psi_{\lambda\mu}(x), \psi_{\lambda'\mu'}^+(x')]$$

$$= i \frac{4^{-[\frac{n}{2}]}}{2(l!)^2} i^{l(l-1)} (C^+ \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]})^{\mu\lambda} (\gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l}] C)^{\lambda'\mu'} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda\lambda'} [(m - \gamma^b \partial_b) \gamma^0]_{\mu\mu'} \Delta(x-x')$$

$$= i \frac{4^{-[\frac{n}{2}]}}{2(l!)^2} i^{l(l-1)} (C^+ \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]})^{\mu\lambda} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda\lambda'} (\gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l}] C)^{\lambda'\mu'} [(m - \gamma^b \partial_b) \gamma^0]_{\mu'\mu}^T \Delta(x-x')$$

$$= i \frac{4^{-[\frac{n}{2}]}}{2(l!)^2} i^{l(l-1)} \text{tr} \{ C^+ \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]} [(m - \gamma^a \partial_a) \gamma^0] \gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l}] C [(m - \gamma^b \partial_b) \gamma^0]^T \} \Delta(x-x')$$

$$= i \frac{4^{-[\frac{n}{2}]}}{2(l!)^2} i^{l(l-1)} \text{tr} \{ \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda\lambda'} \gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l}] C [(m - \gamma^b \partial_b) \gamma^0]^T C^+ \} \Delta(x-x')$$

$$= i \frac{4^{-[\frac{n}{2}]}}{2(l!)^2} i^{l(l-1)} (-1)^{\xi(n)} \text{tr} \{ \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]} [(m - \gamma^a \partial_a) \gamma^0] \gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l}] [\gamma^0 (m - (-1)^{\eta(n)} \gamma^b \partial_b)] \} \Delta(x-x')$$

$$= i \frac{4^{-[\frac{n}{2}]}}{2(l!)^2} i^{l(l-1)} (-1)^{\xi(n)} \text{tr} \{ \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]} [(m - \gamma^a \partial_a) \gamma^0] \gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l}] [(m + (-1)^{\eta(n)} \gamma^b \partial_b^+) \gamma^0] \} \Delta(x-x')$$

$$= i \frac{4^{-[\frac{n}{2}]}}{2(l!)^2} i^{l(l-1)} (-1)^{\xi(n)}$$

$$\{ \text{tr} (m^2 \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]} \gamma^0 \gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l]} \gamma^0) - (-1)^{\eta(n)} \text{tr} (\gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_l]} \gamma_a \gamma_0 \gamma_{[a'_1} \gamma_{a'_2} \cdots \gamma_{a'_l]} \gamma_{a'} \gamma_0) \partial^a \partial^{a'} \} \Delta(x-x')$$

$$= i \frac{4^{-[\frac{n}{2}]}}{2(l!)^2} i^{l(l-1)} (-1)^{\xi(n)} \{ i^{l(l+1)} 2^{[\frac{n}{2}]} (l!)^2 m^2 \frac{1}{l!} \eta_{[a'_1}^{a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_l}^{a_l}]$$

$$- (-1)^{\eta(n)} i^{(l+1)(l+2)} 2^{[\frac{n}{2}]} (l!)^2 \{ \frac{1}{(l+1)!} \eta_{[a'_1}^{a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_l}^{a_l}] - \frac{1}{(l-1)!} \eta_{[a'_1}^{a_1} \cdots \eta_{a'_{l-1}}^{a_{l-1}} \delta^{a_l}]^a \delta_{a'_l a'} \} \partial_a \partial^{a'} \} \Delta(x-x')$$

$$\begin{aligned}
&= i2^{-[\frac{n}{2}]-1}i^{2l^2}(-1)^{\xi(n)} \\
&\left\{ \frac{1}{l!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_l}^{a_l]} m^2 + (-1)^{\eta(n)+l} \left\{ \frac{1}{(l+1)!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_l}^{a_l]} - \frac{1}{(l-1)!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_{l-1}}^{a_{l-1}} \delta^{a_l]} \delta_{a'_l]} \right\} \partial_a \partial^{+a'} \right\} \Delta(x-x') \\
&= i2^{-[\frac{n}{2}]}(-1)^{\xi(n)+l} \\
&\left\{ \frac{1+(-1)^{\eta(n)+l}}{2} \frac{1}{l!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_l}^{a_l]} m^2 - (-1)^{\eta(n)+l} \frac{1}{(l-1)!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_{l-1}}^{a_{l-1}} \partial^{a_l]} \partial_{a'_l]} \right\} \Delta(x-x') \\
&= i \frac{(-1)^{\xi(n)+l}}{2^{[\frac{n}{2}]}} \begin{cases} \frac{1}{(l+1)!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_l}^{a_l]} \partial_a \partial^{+a'} \Delta(x-x'), & (-1)^{\eta(n)+l} = 1 \\ \frac{1}{(l-1)!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_{l-1}}^{a_{l-1}} \partial^{a_l]} \partial_{a'_l]} \Delta(x-x'), & (-1)^{\eta(n)+l} = -1 \end{cases} \\
&= -i \frac{(-1)^{\delta_{2,n}}}{2^{[\frac{n}{2}]}} \begin{cases} \frac{1}{(l+1)!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_l}^{a_l]} \partial_a \partial^{+a'} \Delta(x-x'), & (-1)^{\eta(n)+l} = 1 \\ -\frac{1}{(l-1)!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_{l-1}}^{a_{l-1}} \partial^{a_l]} \partial_{a'_l]} \Delta(x-x'), & (-1)^{\eta(n)+l} = -1 \end{cases} \quad \square
\end{aligned}$$

## 4.2 N+1=n维时空中基本反对称张量场的通用对易规则猜想

定义4.2.1.  $\frac{1}{l!} \partial^{[a_0} A^{a_1 \cdots a_l]} + m F^{a_0 \cdots a_l} = 0, \partial_{a_0} F^{a_0 \cdots a_l} + m A^{a_1 \cdots a_l} = 0$

$$\Leftrightarrow \partial_{a_0} \partial^{a_0} A^{a_1 \cdots a_l} - m^2 A^{a_1 \cdots a_l} = 0, \partial_{a_1} A^{a_1 \cdots a_l} = 0, F^{a_0 \cdots a_l} = -\frac{1}{(l+1)! m} \partial^{[a_0} A^{a_1 \cdots a_l]}$$

推论4.2.1.

$$\begin{cases} \frac{1}{l!} \partial^{[a_0} A^{a_1 \cdots a_l]} + m F^{a_0 a_1 \cdots a_l} = 0 \\ \partial_{a_0} F^{a_0 a_1 \cdots a_l} + m A^{a_1 \cdots a_l} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{(n-l-1)!} \partial^{[a_0} * F^{a_1 \cdots a_{n-l-1}]} + (-1)^{n-l-1} m * A^{a_0 a_1 \cdots a_{n-l-1}} = 0 \\ \partial_{a_0} * A^{a_0 a_1 \cdots a_{n-l-1}} + (-1)^{n-l-1} m * F^{a_1 \cdots a_{n-l-1}} = 0 \end{cases}$$

引理4.2.1.  $\frac{1}{(l+1)!} \eta_{[a_1} \langle a'_1 \eta_{a_2} a'_2 \cdots \eta_{a_{l-1}} a'_{l-1} \eta_{a_l} a'_l \eta_{a_{l+1}} a'_{l+1} \rangle \partial^{a_{l+1}} \partial^{+a'_{l+1}} \Delta(x-x')$   
 $= \left\{ \frac{1}{l!} \eta_{[a_1} \langle a'_1 \eta_{a_2} a'_2 \cdots \eta_{a_{l-1}} a'_{l-1} \eta_{a_l} a'_l \rangle m^2 - \frac{1}{(l-1)!} \eta_{[a_1} \langle a'_1 \eta_{a_2} a'_2 \cdots \eta_{a_{l-1}} a'_{l-1} \partial_{a_l} \rangle \partial_{a'_l}^+ \right\} \Delta(x-x')$

猜想4.2.1.

$$\begin{cases} [A_{a_1 \cdots a_l}(x), A_{a'_1 \cdots a'_l}^+(x')] = i \frac{2^{-[\frac{n}{2}]}}{(l+1)!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_l}^{a_l]} \partial_a \partial^{+a'} \Delta(x-x') \\ [F_{a_0 a_1 \cdots a_l}(x), F_{a'_0 a'_1 \cdots a'_l}^+(x')] = -i \frac{2^{-[\frac{n}{2}]}}{l!} \eta_{[a'_0}^{[a_0} \cdots \eta_{a'_{l-1}}^{a_{l-1}} \partial^{a_l]} \partial_{a'_l]} \Delta(x-x') \end{cases} \\
\begin{cases} [*A_{a_0 \cdots a_{n-l-1}}(x), *A_{a'_0 \cdots a'_{n-l-1}}^+(x')] = -i \frac{2^{-[\frac{n}{2}]}}{(n-l-1)!} \eta_{[a'_0}^{[a_0} \cdots \eta_{a'_{n-l-2}}^{a_{n-l-2}} \partial^{a_{n-l-1}}] \partial_{a'_{n-l-1}}] \Delta(x-x') \\ [*F_{a_1 \cdots a_{n-l-1}}(x), *F_{a'_1 \cdots a'_{n-l-1}}^+(x')] = i \frac{2^{-[\frac{n}{2}]}}{(n-l)!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_{n-l-1}}^{a_{n-l-1}} \partial^{a'}] \Delta(x-x') \end{cases}$$

## 5 全耦合反对称张量场集

### 5.1 N+1=n偶维时空中的Bargmann-Wigner一般矢量场方程

定义5.1.1.

$$\begin{aligned}
X &= \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(3!)^2} F^{a_1 a_2 a_3} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} + \cdots + \frac{1}{(n!)^2} F^{a_1 a_2 a_3 \cdots a_n} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \cdots \gamma_{a_n]} \\
\begin{cases} F = 2^{-[\frac{n}{2}]} \text{tr}(X), F_{a_1} = 2^{-[\frac{n}{2}]} \text{tr}(\gamma_{a_1} X) \\ F_{a_1 a_2} = -2^{-[\frac{n}{2}]} \text{tr}(\frac{1}{2!} \gamma_{[a_1} \gamma_{a_2]} X), F_{a_1 a_2 a_3} = -2^{-[\frac{n}{2}]} \text{tr}(\frac{1}{3!} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} X) \\ F_{a_1 a_2 a_3 a_4} = 2^{-[\frac{n}{2}]} \text{tr}(\frac{1}{4!} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} X), F_{a_1 a_2 a_3 a_4 a_5} = 2^{-[\frac{n}{2}]} \text{tr}(\frac{1}{5!} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} X) \\ \cdots F_{a_1 a_2 \cdots a_n} = (-1)^{[(n\%4)/2]} 2^{-[\frac{n}{2}]} \text{tr}(\frac{1}{n!} \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_n]} X) \end{cases}
\end{aligned}$$

定理5.1.1.  $(\gamma_a \partial^a + m)\psi(x) = 0 \Leftrightarrow$

$$\begin{cases} mF + \partial_{a_0} F^{a_0} = 0, \frac{1}{0!} \partial^{a_1} F + mF^{a_1} + \partial_{a_0} F^{a_0 a_1} = 0, \frac{1}{1!} \partial^{[a_1} F^{a_2]} + mF^{a_1 a_2} + \partial_{a_0} F^{a_0 a_1 a_2} = 0 \\ \frac{1}{2!} \partial^{[a_1} F^{a_2 a_3]} + mF^{a_1 a_2 a_3} + \partial_{a_0} F^{a_0 a_1 a_2 a_3} = 0, \cdots, \frac{1}{(n-2)!} \partial^{[a_1} F^{a_2 \cdots a_{n-1}]} + mF^{a_1 \cdots a_{n-1}} + \partial_{a_0} F^{a_0 a_1 \cdots a_{n-1}} = 0 \\ \frac{1}{(n-1)!} \partial^{[a_1} F^{a_2 \cdots a_n]} + mF^{a_1 \cdots a_n} = 0, \frac{1}{n!} \partial^{[a_0} F^{a_1 \cdots a_n]} = 0 \end{cases}$$

证明:

$$(\gamma_a \partial^a + m)\psi(x) = 0 \Leftrightarrow \begin{cases} (\gamma_a \partial^a + m)\psi(x) = 0 \\ X = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \cdots + \frac{1}{(n!)^2} F^{a_1 \cdots a_n} \gamma_{[a_1} \cdots \gamma_{a_n]} \right\} C \end{cases}$$

⇔

$$(\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \cdots + \frac{1}{(n!)^2} F^{a_1 \cdots a_n} \gamma_{[a_1} \cdots \gamma_{a_n]} \right\} C = 0$$

⇔

$$(\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{0!} F + \frac{1}{1!} F^{a_1} \gamma_{a_1} + \frac{1}{2!} F^{a_1 a_2} \gamma_{a_1} \gamma_{a_2} + \cdots + \frac{1}{n!} F^{a_1 \cdots a_n} \gamma_{a_1} \cdots \gamma_{a_n} \right\} = 0$$

⇔

$$\partial^{a_0} \left\{ \frac{1}{0!} \gamma_{a_0} F + \frac{1}{1!} F^{a_1} \gamma_{a_0} \gamma_{a_1} + \frac{1}{2!} F^{a_1 a_2} \gamma_{a_0} \gamma_{a_1} \gamma_{a_2} + \cdots + \frac{1}{n!} F^{a_1 \cdots a_n} \gamma_{a_0} \gamma_{a_1} \cdots \gamma_{a_n} \right\} + m \left\{ \frac{1}{0!} F + \frac{1}{1!} F^{a_1} \gamma_{a_1} + \frac{1}{2!} F^{a_1 a_2} \gamma_{a_1} \gamma_{a_2} + \cdots + \frac{1}{n!} F^{a_1 \cdots a_n} \gamma_{a_1} \cdots \gamma_{a_n} \right\} = 0$$

⇔

$$\left\{ \frac{1}{0!} \gamma_{a_0} \partial^{a_0} F + \frac{1}{1!} \partial^{a_0} F^{a_1} \left( \frac{1}{2!} \gamma_{[a_0} \gamma_{a_1]} + \frac{1}{0!} \delta_{a_0 a_1} \right) + \frac{1}{2!} \partial^{a_0} F^{a_1 a_2} \left( \frac{1}{3!} \gamma_{[a_0} \gamma_{a_1} \gamma_{a_2]} + \frac{1}{1!} \delta_{a_0 [a_1} \gamma_{a_2]} \right) + \frac{1}{3!} \partial^{a_0} F^{a_1 \cdots a_3} \left( \frac{1}{4!} \gamma_{[a_0} \gamma_{a_1} \cdots \gamma_{a_3]} + \frac{1}{2!} \delta_{a_0 [a_1} \gamma_{a_2} \cdots \gamma_{a_3]} \right) + \frac{1}{4!} \partial^{a_0} F^{a_1 \cdots a_4} \left( \frac{1}{5!} \gamma_{[a_0} \gamma_{a_1} \cdots \gamma_{a_4]} + \frac{1}{3!} \delta_{a_0 [a_1} \gamma_{a_2} \cdots \gamma_{a_4]} \right) + \cdots + \frac{1}{n!} \partial^{a_0} F^{a_1 \cdots a_n} \left( \frac{1}{(n+1)!} \gamma_{[a_0} \gamma_{a_1} \cdots \gamma_{a_n]} + \frac{1}{(n-1)!} \delta_{a_0 [a_1} \gamma_{a_2} \cdots \gamma_{a_n]} \right) \right\} + m \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_0} \gamma_{a_0} + \frac{1}{(2!)^2} F^{a_0 a_1} \gamma_{[a_0} \gamma_{a_1]} + \cdots + \frac{1}{(n!)^2} F^{a_0 \cdots a_{n-1}} \gamma_{[a_0} \cdots \gamma_{a_{n-1}]} \right\} = 0$$

⇔

$$mF + \partial_{a_0} F^{a_0} = 0$$

$$\frac{1}{0!} \partial^{a_1} F + mF^{a_1} + \partial_{a_0} F^{a_0 a_1} = 0$$

$$\frac{1}{1!} \partial^{[a_1} F^{a_2]} + mF^{a_1 a_2} + \partial_{a_0} F^{a_0 a_1 a_2} = 0$$

$$\frac{1}{2!} \partial^{[a_1} F^{a_2 a_3]} + mF^{a_1 a_2 a_3} + \partial_{a_0} F^{a_0 a_1 a_2 a_3} = 0$$

$$\frac{1}{3!} \partial^{[a_1} F^{a_2 \cdots a_4]} + mF^{a_1 \cdots a_4} + \partial_{a_0} F^{a_0 a_1 \cdots a_4} = 0$$

...

$$\frac{1}{(n-3)!} \partial^{[a_1} F^{a_2 \cdots a_{n-2}]} + mF^{a_1 \cdots a_{n-2}} + \partial_{a_0} F^{a_0 a_1 \cdots a_{n-2}} = 0$$

$$\frac{1}{(n-2)!} \partial^{[a_1} F^{a_2 \cdots a_{n-1}]} + mF^{a_1 \cdots a_{n-1}} + \partial_{a_0} F^{a_0 a_1 \cdots a_{n-1}} = 0$$

$$\frac{1}{(n-1)!} \partial^{[a_1} F^{a_2 \cdots a_n]} + mF^{a_1 \cdots a_n} = 0$$

$$\frac{1}{n!} \partial^{[a_0} F^{a_1 \cdots a_n]} \equiv 0$$

□

$$\text{推论5.1.1. } \gamma_a \partial^a \psi(x) = 0 \Leftrightarrow \begin{cases} \partial_{a_0} F^{a_0} = 0, \frac{1}{0!} \partial^{a_1} F + \partial_{a_0} F^{a_0 a_1} = 0, \frac{1}{1!} \partial^{[a_1} F^{a_2]} + \partial_{a_0} F^{a_0 a_1 a_2} = 0 \\ \frac{1}{2!} \partial^{[a_1} F^{a_2 a_3]} + \partial_{a_0} F^{a_0 a_1 a_2 a_3} = 0, \dots, \frac{1}{(n-2)!} \partial^{[a_1} F^{a_2 \cdots a_{n-1}]} + \partial_{a_0} F^{a_0 a_1 \cdots a_{n-1}} = 0 \\ \frac{1}{(n-1)!} \partial^{[a_1} F^{a_2 \cdots a_n]} = 0, \frac{1}{n!} \partial^{[a_0} F^{a_1 \cdots a_n]} = 0 \end{cases}$$

## 5.2 全耦合反对称张量场的对易规则

引理5.2.1.  $\frac{1}{(l+1)!} \eta_{[a_1} \langle a'_1 \eta_{a_2} a'_2 \cdots \eta_{a_{l-1}} a'_{l-1} \eta_{a_l} a'_l \eta_{a_{l+1}} a'_{l+1} \rangle \partial^{a_{l+1}} \partial^{+a'_{l+1}} \Delta(x-x')$

$$= \left\{ \frac{1}{l!} \eta_{[a_1} \langle a'_1 \eta_{a_2} a'_2 \cdots \eta_{a_{l-1}} a'_{l-1} \eta_{a_l} a'_l \rangle m^2 - \frac{1}{(l-1)!} \eta_{[a_1} \langle a'_1 \eta_{a_2} a'_2 \cdots \eta_{a_{l-1}} a'_{l-1} \partial_{a_l} \partial_{a'_l}^+ \rangle \right\} \Delta(x-x')$$

猜想5.2.1.

$$\begin{cases} [F^{a_1 a_2 \cdots a_l}(x), F_{a'_1 a'_2 \cdots a'_l}^+(x')] = -i \frac{(-1)^{\delta_{2,n}}}{2^{\lfloor \frac{n}{2} \rfloor + 1}} \left\{ \frac{1}{(l+1)!} \eta_{[a'_1}^{a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_l}^{a_l} \eta_{a'_l}^{a_l}] \partial_a \partial^{+a'} - \frac{1}{(l-1)!} \eta_{[a'_1}^{a_1} \cdots \eta_{a'_{l-1}}^{a_{l-1}} \partial_{a_l} \partial_{a'_l}^+] \right\} \Delta(x-x') \\ [F^{a_1 a_2 \cdots a_l}(x), F_{a'_1 a'_2 \cdots a'_l}^+(x')] = -i \frac{(-1)^{\delta_{2,n}}}{2^{\lfloor \frac{n}{2} \rfloor + 1}} \left\{ \frac{1}{l!} \eta_{[a'_1}^{a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_l}^{a_l}] m^2 - \frac{2}{(l-1)!} \eta_{[a'_1}^{a_1} \cdots \eta_{a'_{l-1}}^{a_{l-1}} \partial_{a_l} \partial_{a'_l}^+] \right\} \Delta(x-x') \end{cases}$$

## 5.3 N+1=n偶维时空中反对称张量场集基和B-W集基之间的关系

定义5.3.1.

$$\begin{cases} u(\vec{p}, h) u^T(\vec{p}, h') = \sum_{l=0}^n \frac{1}{(l!)^2} U_{a_1 \cdots a_l}(\vec{p}; h, h') \gamma^{[a_1} \cdots \gamma^{a_l]} C, v(\vec{p}, h) v^T(\vec{p}, h') = \sum_{l=0}^n \frac{1}{(l!)^2} V_{a_1 \cdots a_l}(\vec{p}; h, h') \gamma^{[a_1} \cdots \gamma^{a_l]} C \\ u(\vec{p}, h) v^T(\vec{p}, h') = \sum_{l=0}^n \frac{1}{(l!)^2} X_{a_1 \cdots a_l}(\vec{p}; h, h') \gamma^{[a_1} \cdots \gamma^{a_l]} C, v(\vec{p}, h) u^T(\vec{p}, h') = \sum_{l=0}^n \frac{1}{(l!)^2} Y_{a_1 \cdots a_l}(\vec{p}; h, h') \gamma^{[a_1} \cdots \gamma^{a_l]} C \\ h, h' = 1, \dots, 2^{\lfloor \frac{N-1}{2} \rfloor} \end{cases}$$

$$\Leftrightarrow \begin{cases} u(\vec{p}, h)u^T(\vec{p}, h') = \sum_{l=0}^n \frac{1}{l!} U_{a_1 \dots a_l}(\vec{p}; h, h') \frac{1}{l!} \gamma^{[a_1 \dots a_l]}; h, h' = -2^{\lfloor \frac{N-1}{2} \rfloor}, \dots, -1, 1, \dots, 2^{\lfloor \frac{N-1}{2} \rfloor} \\ U_{a_1 a_2 \dots a_l}(\vec{p}; h, h') = (-1)^{\lfloor (l\%4)/2 \rfloor} 2^{-\lfloor \frac{n}{2} \rfloor} u^T(\vec{p}, h') \frac{1}{l!} C^+ \gamma_{[a_1 \dots a_l]} u(\vec{p}, h) \end{cases}$$

定义5.3.2.

$$\begin{cases} \frac{1}{l!} \gamma^{[a_1 \dots a_l]} = \frac{m^2}{E^2} \sum_{h, h'} W^{a_1 \dots a_l}(\vec{p}; h, h') u(\vec{p}, h) u^T(\vec{p}, h'); h, h' = -2^{\lfloor \frac{N-1}{2} \rfloor}, \dots, -1, 1, \dots, 2^{\lfloor \frac{N-1}{2} \rfloor} \\ W^{a_1 \dots a_l}(\vec{p}; h, h') = u^+(\vec{p}, h) \frac{1}{l!} \gamma^{[a_1 \dots a_l]} u^*(\vec{p}, h') \end{cases}$$

推论5.3.1.

$$\begin{cases} u(\vec{p}, h)u^T(\vec{p}, h') = \frac{m^2}{E^2} \sum_{l=0}^n \sum_{h'', h'''} \frac{1}{l!} U_{a_1 \dots a_l}(\vec{p}; h, h') W^{a_1 \dots a_l}(\vec{p}; h'', h''') u(\vec{p}, h'') u^T(\vec{p}, h''') \\ \sum_{l=0}^n \frac{1}{l!} U_{a_1 \dots a_l}(\vec{p}; h, h') W^{a_1 \dots a_l}(\vec{p}; h, h') = 1 \end{cases}$$

推论5.3.2.

$$\begin{cases} \frac{1}{l!} \gamma^{[a_1 \dots a_l]} = \frac{m^2}{E^2} \sum_{l=0}^n \sum_{h, h'} \frac{1}{l!} W^{a_1 \dots a_l}(\vec{p}; h, h') U_{a'_1 \dots a'_l}(\vec{p}; h, h') \frac{1}{l!} \gamma^{[a'_1 \dots a'_l]} \\ \frac{m^2}{E^2} \sum_{h, h'} W^{a_1 \dots a_l}(\vec{p}; h, h') U_{a_1 \dots a_l}(\vec{p}; h, h') = 1 \end{cases}$$

## 5.4 N+1维时空中无质量反对称张量场的对易规则猜想

猜想5.4.1.  $\partial_{[a_0} F_{a_1 \dots a_l]} = 0, \partial^{a_1} F_{a_1 \dots a_l} = 0, F_{a_1 \dots a_l} = \frac{1}{l!} F_{[a_1 \dots a_l]}$   
 $\Rightarrow [F^{a_1 a_2 \dots a_l}(x), F_{a'_1 a'_2 \dots a'_l}(x')] = ? - i \frac{1}{2^{\lfloor \frac{n}{2} \rfloor}} \frac{1}{(l-1)!} \eta_{[a'_1}^{[a_1} \dots \eta_{a'_l-1}^{a_{l-1}] \partial^{a_l]} \Delta(x-x')$

## 6 偶数维时空中的对称与反对称B-W矢量场方程

### 6.1 二维时空中的对称Bargmann-Wigner矢量场方程

引理6.1.1.  $\begin{cases} (\gamma^a \partial_a + m)X(2) = 0 \\ X(2) = \{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \} C \end{cases} \Leftrightarrow \begin{cases} \partial^{[a} F^{b]} + mF^{ab} = 0, \partial_a F^a = 0 \\ \partial^{[a} F^{bc]} = 0, \partial_a F^{ab} + mF^b = 0 \end{cases}$

证明:  $(\gamma^a \partial_a + m)X(2) = 0, X(2) = \{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \} C$   
 $\Leftrightarrow (\gamma_a \partial^a + m) \{ \frac{1}{(1!)^2} F^b \gamma_b + \frac{1}{(2!)^2} F^{bc} \gamma_{[b} \gamma_{c]} \} C = 0$   
 $\Leftrightarrow (\gamma_a \partial^a + m) \{ \frac{1}{1!} F^b \gamma_b + \frac{1}{2!} F^{bc} \gamma_b \gamma_c \} = 0$   
 $\Leftrightarrow \gamma_a \gamma_b \partial^a \frac{1}{1!} F^b + \gamma_a \gamma_b \gamma_c \partial^a \frac{1}{2!} F^{bc} + m \frac{1}{1!} F^b \gamma_b + m \frac{1}{2!} F^{bc} \gamma_b \gamma_c = 0$   
 $\Leftrightarrow \{ \frac{1}{2!} \gamma_{[a} \gamma_{b]} + \delta_{ab} \} \partial^a \frac{1}{1!} F^b + m \frac{1}{2!} F^{bc} \gamma_b \gamma_c + \{ \frac{1}{3!} \gamma_{[a} \gamma_b \gamma_{c]} + (\delta_{a[b} \gamma_{c]} + \gamma_a \delta_{bc}) \} \partial^a \frac{1}{2!} F^{bc} + m \frac{1}{1!} F^b \gamma_b = 0$   
 $\Leftrightarrow \frac{1}{2!} \gamma_{[a} \gamma_{b]} \partial^a \frac{1}{1!} F^b + \delta_{ab} \partial^a \frac{1}{1!} F^b + m \frac{1}{2!} F^{bc} \gamma_b \gamma_c + \frac{1}{3!} \gamma_{[a} \gamma_b \gamma_{c]} \partial^a \frac{1}{2!} F^{bc} + \frac{2}{1!} \delta_{ab} \gamma_c \partial^a \frac{1}{2!} F^{bc} + m \frac{1}{1!} F^b \gamma_b = 0$   
 $\Leftrightarrow \frac{1}{2!} \gamma_{[a} \gamma_{b]} \partial^a \frac{1}{1!} F^b + \partial_a \frac{1}{1!} F^a + \frac{1}{2!} m \frac{1}{2!} F^{ab} \gamma_{[a} \gamma_{b]} + \frac{1}{3!} \gamma_{[a} \gamma_b \gamma_{c]} \partial^a \frac{1}{2!} F^{bc} + \frac{2}{1!} \gamma_b \partial_a \frac{1}{2!} F^{ab} + m \frac{1}{1!} F^b \gamma_b = 0$   
 $\Leftrightarrow \begin{cases} \frac{1}{2!} \partial^{[a} \frac{1}{1!} F^{b]} + m \frac{1}{2!} F^{ab} = 0, \partial_a \frac{1}{1!} F^a = 0 \\ \partial^{[a} \frac{1}{2!} F^{bc]} = 0, \partial_a \frac{1}{2!} F^{ab} + \frac{1}{2} m \frac{1}{1!} F^b = 0 \end{cases}$   
 $\Leftrightarrow \begin{cases} \partial^{[a} F^{b]} + mF^{ab} = 0, \partial_a F^a = 0 \\ \partial^{[a} F^{bc]} = 0, \partial_a F^{ab} + mF^b = 0 \end{cases}$  □

推论6.1.1.  $\begin{cases} \gamma^a \partial_a X(2) = 0 \\ X(2) = \{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \} C \end{cases} \Leftrightarrow \begin{cases} \partial^{[a} F^{b]} = 0, \partial_a F^a = 0 \\ \partial^{[a} F^{bc]} = 0, \partial_a F^{ab} = 0 \end{cases}$

推论6.1.2.  $(\gamma^a \partial_a + m)X(2) = 0, X(2) = \{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \} C, m \neq 0$   
 $\Leftrightarrow \partial_a F^{ab} + mF^b = 0, \partial^{[a} F^{b]} + mF^{ab} = 0 \Leftrightarrow \partial_b \partial^b F^a - m^2 F^a = 0, \partial_a F^a = 0, F^{ab} = -\frac{1}{m} \partial^{[a} F^{b]}$

定理6.1.1.  $\begin{cases} (\gamma^a \partial_a + m)X(2) = 0 \\ X(2) = X^T(2) \end{cases} \Leftrightarrow \begin{cases} \partial_b \partial^b F^a - m^2 F^a = 0, \partial_a F^a = 0 \\ X(2) = \{ \frac{1}{1!} \gamma_a + \frac{1}{2!m} \gamma_{[a} \gamma_{b]} \partial^b \} C F^a \end{cases}$

$$\text{推论6.1.3. } \begin{cases} \partial^{[a} F^{b]} + mF^{ab} = 0 \\ \partial_a F^{ab} + mF^b = 0 \end{cases} \Leftrightarrow \begin{cases} \partial^a *F + m*F^a = 0 \\ \partial_a *F^a + m*F = 0 \end{cases}$$

## 6.2 二维时空中的反对称Bargmann-Wigner矢量场方程

$$\text{引理6.2.1. } (\gamma^a \partial_a + m)X(2) = 0, X(2) = \frac{1}{(0!)^2} FC \Leftrightarrow \partial^a F = 0, mF = 0$$

$$\text{证明: } (\gamma^a \partial_a + m)X(2) = 0, X(2) = \frac{1}{(0!)^2} FC$$

$$\Leftrightarrow (\gamma_a \partial^a + m) \frac{1}{(0!)^2} FC = 0$$

$$\Leftrightarrow (\gamma_a \partial^a + m) \frac{1}{0!} F = 0$$

$$\Leftrightarrow \partial^a F = 0, mF = 0 \quad \square$$

$$\text{推论6.2.1. } \gamma^a \partial_a X(2) = 0, X(2) = \frac{1}{(0!)^2} FC \Leftrightarrow \partial^a F = 0$$

$$\text{推论6.2.2. } (\gamma^a \partial_a + m)X(2) = 0, X(2) = \frac{1}{(0!)^2} FC, m \neq 0 \Leftrightarrow F = 0$$

$$\text{定理6.2.1. } (\gamma^a \partial_a + m)X(2) = 0, X(2) = -X^T(2) \Leftrightarrow F = 0$$

$$\text{推论6.2.3. } F = 0 \Leftrightarrow *F^{ab} = 0$$

## 6.3 四维时空中的对称Bargmann-Wigner矢量场方程

$$\text{引理6.3.1. } \begin{cases} (\gamma^a \partial_a + m)X(4) = 0 \\ X(4) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \right\} C \end{cases} \Leftrightarrow \begin{cases} \partial^{[a} F^{b]} + mF^{ab} = 0, \partial_a F^a = 0 \\ \partial^{[a} F^{bc]} = 0, \partial_a F^{ab} + mF^b = 0 \end{cases}$$

$$\text{证明: } (\gamma^a \partial_a + m)X(4) = 0, X(4) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \right\} C$$

$$\Leftrightarrow (\gamma_a \partial^a + m) \left\{ \frac{1}{(1!)^2} F^b \gamma_b + \frac{1}{(2!)^2} F^{bc} \gamma_{[b} \gamma_{c]} \right\} C = 0$$

$$\Leftrightarrow (\gamma_a \partial^a + m) \left\{ \frac{1}{1!} F^b \gamma_b + \frac{1}{2!} F^{bc} \gamma_b \gamma_c \right\} = 0$$

$$\Leftrightarrow \gamma_a \gamma_b \partial^a \frac{1}{1!} F^b + \gamma_a \gamma_b \gamma_c \partial^a \frac{1}{2!} F^{bc} + m \frac{1}{1!} F^b \gamma_b + m \frac{1}{2!} F^{bc} \gamma_b \gamma_c = 0$$

$$\Leftrightarrow \left\{ \frac{1}{2!} \gamma_{[a} \gamma_{b]} + \delta_{ab} \right\} \partial^a \frac{1}{1!} F^b + m \frac{1}{2!} F^{bc} \gamma_b \gamma_c + \left\{ \frac{1}{3!} \gamma_{[a} \gamma_b \gamma_{c]} + (\delta_{a[b} \gamma_{c]} + \gamma_a \delta_{bc}) \right\} \partial^a \frac{1}{2!} F^{bc} + m \frac{1}{1!} F^b \gamma_b = 0$$

$$\Leftrightarrow \frac{1}{2!} \gamma_{[a} \gamma_{b]} \partial^a \frac{1}{1!} F^b + \delta_{ab} \partial^a \frac{1}{1!} F^b + m \frac{1}{2!} F^{bc} \gamma_b \gamma_c + \frac{1}{3!} \gamma_{[a} \gamma_b \gamma_{c]} \partial^a \frac{1}{2!} F^{bc} + \frac{2}{1!} \delta_{ab} \gamma_c \partial^a \frac{1}{2!} F^{bc} + m \frac{1}{1!} F^b \gamma_b = 0$$

$$\Leftrightarrow \frac{1}{2!} \gamma_{[a} \gamma_{b]} \partial^a \frac{1}{1!} F^b + \partial_a \frac{1}{1!} F^a + \frac{1}{2!} m \frac{1}{2!} F^{ab} \gamma_{[a} \gamma_{b]} + \frac{1}{3!} \gamma_{[a} \gamma_b \gamma_{c]} \partial^a \frac{1}{2!} F^{bc} + \frac{2}{1!} \gamma_b \partial_a \frac{1}{2!} F^{ab} + m \frac{1}{1!} F^b \gamma_b = 0$$

$$\Leftrightarrow \begin{cases} \frac{1}{2!} \partial^{[a} \frac{1}{1!} F^{b]} + m \frac{1}{2!} F^{ab} = 0, \partial_a \frac{1}{1!} F^a = 0 \\ \partial^{[a} \frac{1}{2!} F^{bc]} = 0, \partial_a \frac{1}{2!} F^{ab} + \frac{1}{2} m \frac{1}{1!} F^b = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \partial^{[a} F^{b]} + mF^{ab} = 0, \partial_a F^a = 0 \\ \partial^{[a} F^{bc]} = 0, \partial_a F^{ab} + mF^b = 0 \end{cases} \quad \square$$

$$\text{推论6.3.1. } \begin{cases} \gamma^a \partial_a X(4) = 0 \\ X(4) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \right\} C \end{cases} \Leftrightarrow \begin{cases} \partial^{[a} F^{b]} = 0, \partial_a F^a = 0 \\ \partial^{[a} F^{bc]} = 0, \partial_a F^{ab} = 0 \end{cases}$$

$$\text{推论6.3.2. } (\gamma^a \partial_a + m)X(4) = 0, X(4) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \right\} C, m \neq 0$$

$$\Leftrightarrow \partial_a F^{ab} + mF^b = 0, \partial^{[a} F^{b]} + mF^{ab} = 0 \Leftrightarrow \partial_b \partial^b F^a - m^2 F^a = 0, \partial_a F^a = 0, F^{ab} = -\frac{1}{m} \partial^{[a} F^{b]} \quad \square$$

$$\text{定理6.3.1. } \begin{cases} (\gamma^a \partial_a + m)X(4) = 0 \\ X(4) = X^T(4) \end{cases} \Leftrightarrow \begin{cases} \partial_b \partial^b F^a - m^2 F^a = 0, \partial_a F^a = 0 \\ X(4) = \left\{ \frac{1}{1!} \gamma_a + \frac{1}{2!m} \gamma_{[a} \gamma_{b]} \partial^b \right\} C F^a \\ = \left\{ -\frac{1}{2!} \gamma^{[a_1} \gamma^{a_2]} + \frac{1}{3!m} \gamma^{[a_1} \gamma^{a_2} \gamma^{a_3]} \partial_{a_3} \right\} \Gamma_0 C \frac{1}{2!} *F_{a_1 a_2} \end{cases}$$

$$\text{推论6.3.3. } \begin{cases} \partial^{[a} F^{b]} + mF^{ab} = 0 \\ \partial_a F^{ab} + mF^b = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{2!} \partial^{[a} *F^{bc]} + m*F^{abc} = 0 \\ \partial_a *F^{abc} + m*F^{bc} = 0 \end{cases}$$

## 引理6.3.2.

$$\begin{aligned}\frac{1}{2!}\gamma_{[a_1}\gamma_{a_2]} &= -\varepsilon_{a_1 a_2 a_3 a_4} \frac{1}{(2!)^2} \Gamma_0 \gamma^{[a_3} \gamma^{a_4]} = -\varepsilon_{a_1 a_2 a_3 a_4} \frac{1}{(2!)^2} \gamma^{[a_3} \gamma^{a_4]} \Gamma_0 \\ \frac{1}{1!}\gamma_{a_1} &= -\varepsilon_{a_1 a_2 a_3 a_4} \frac{1}{(3!)^2} \Gamma_0 \gamma^{[a_2} \gamma^{a_3} \gamma^{a_4]} = \varepsilon_{a_1 a_2 a_3 a_4} \frac{1}{(3!)^2} \gamma^{[a_2} \gamma^{a_3} \gamma^{a_4]} \Gamma_0\end{aligned}$$

$$\text{推论6.3.4. } X(4) = \frac{1}{2!}\gamma^{[a_1}\gamma^{a_2]} C \frac{1}{2!} F_{a_1 a_2} + \frac{1}{3!m} \gamma^{[a_1} \gamma^{a_2} \gamma^{a_3]} \partial_{a_3} \Gamma_0 C \frac{1}{2!} *F_{a_1 a_2}$$

证明:

$$\begin{aligned}X(4) &= \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \right\} C \\ &= \left\{ \frac{1}{1!} F^{a_1} \varepsilon_{a_1 a_2 a_3 a_4} \frac{1}{(3!)^2} \gamma^{[a_2} \gamma^{a_3} \gamma^{a_4]} - \frac{1}{2!} F^{a_1 a_2} \varepsilon_{a_1 a_2 a_3 a_4} \frac{1}{(2!)^2} \gamma^{[a_3} \gamma^{a_4]} \right\} \Gamma_0 C \\ &= \left\{ -\frac{1}{(3!)^2} *F_{a_2 a_3 a_4} \gamma^{[a_2} \gamma^{a_3} \gamma^{a_4]} - \frac{1}{(2!)^2} *F_{a_3 a_4} \gamma^{[a_3} \gamma^{a_4]} \right\} \Gamma_0 C \\ &= \left\{ -\frac{1}{(2!)^2} *F_{a_1 a_2} \gamma^{[a_1} \gamma^{a_2]} - \frac{1}{(3!)^2} *F_{a_1 a_2 a_3} \gamma^{[a_1} \gamma^{a_2} \gamma^{a_3]} \right\} \Gamma_0 C \\ &= \left\{ -\frac{1}{2!} \gamma^{[a_1} \gamma^{a_2]} + \frac{1}{3!m} \gamma^{[a_1} \gamma^{a_2} \gamma^{a_3]} \partial_{a_3} \Gamma_0 C \frac{1}{2!} *F_{a_1 a_2} \right. \\ &= \frac{1}{2!} \gamma^{[a_1} \gamma^{a_2]} C \frac{1}{2!} F_{a_1 a_2} + \frac{1}{3!m} \gamma^{[a_1} \gamma^{a_2} \gamma^{a_3]} \partial_{a_3} \Gamma_0 C \frac{1}{2!} *F_{a_1 a_2} \\ &= \left. \left\{ \frac{1}{(2!)^2} \gamma^{[a_3} \gamma^{a_4]} + \frac{1}{3!m} \gamma^{[a_1} \gamma^{a_2} \gamma^{a_3]} \partial_{a_3} \Gamma_0 C \frac{1}{(2!)^2} \varepsilon_{a_1 a_2 a_3 a_4} \right\} F^{a_3 a_4} \right\} \Gamma_0 C\end{aligned}$$

□

## 定义6.3.1.

$$\begin{cases} \gamma_a(4) = [\gamma_a(3) \otimes \sigma_y, I \otimes \sigma_x] = (\sigma \otimes \sigma_y, I \otimes \sigma_x), \Gamma^a(4) = [\gamma_a(3), i\zeta] \\ C(4) := \gamma_2(4) \gamma_4(4) = -i\sigma_y \otimes \sigma_z, \gamma_1(4) \cdots \gamma_4(4) = I \otimes \sigma_z = \gamma_0(4) \\ [\gamma_a(4) C(4)]^T = \gamma_a(4) C(4), \{\gamma_{[a}(4) \gamma_{b]}(4) C(4)\}^T = \gamma_{[a}(4) \gamma_{b]}(4) C(4) \\ C^T(4) = -C(4), \{\gamma_{[a}(4) \gamma_b(4) \gamma_c(4) C(4)\}^T = -\gamma_{[a}(4) \gamma_b(4) \gamma_c(4) C(4) \\ \{\gamma_{[a}(4) \gamma_b(4) \gamma_c(4) \gamma_d(4) C(4)\}^T = -\gamma_{[a}(4) \gamma_b(4) \gamma_c(4) \gamma_d(4) C(4) \end{cases}$$

证明:  $[F_{a_1}(x), F_{a_1}^+(x')]$ 

$$\begin{aligned}&= \frac{2^{-4}}{(1!)^2} \bar{C}^{\lambda\eta} (\gamma_{a_1})_{\eta}^{\mu} (\gamma_{a_1}')^{\mu'} \bar{C}^{+\eta'\lambda'} [\psi_{\lambda\mu}(x), \psi_{\lambda'\mu'}^+(x')] \\ &= \frac{2^{-4}}{(1!)^2} (\bar{C} \gamma_{a_1})^{\lambda\mu} (\gamma_{a_1}' C)^{\lambda'\mu'} \frac{i}{2^3} [(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda(\lambda' [(m - \gamma^b \partial_b) \gamma^0]_{\mu\mu'}) \Delta(x - x') \\ &= i \frac{2^{-5}}{(1!)^2} (\bar{C} \gamma_{a_1})^{\lambda\mu} (\gamma_{a_1}' C)^{\lambda'\mu'} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda\lambda'} [(m - \gamma^b \partial_b) \gamma^0]_{\mu\mu'} \Delta(x - x') \\ &= i \frac{2^{-5}}{(1!)^2} (\bar{C} \gamma_{a_1})^{\mu\lambda} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda\lambda'} (\gamma_{a_1}' C)^{\lambda'\mu'} [(m - \gamma^b \partial_b) \gamma^0]_{\mu'\mu}^T \Delta(x - x') \\ &= i \frac{2^{-5}}{(1!)^2} \text{tr} \{ (\bar{C} \gamma_{a_1}) [(m - \gamma^a \partial_a) \gamma^0] (\gamma_{a_1}' C) [(m - \gamma^b \partial_b) \gamma^0]^T \} \Delta(x - x') \\ &= i \frac{2^{-5}}{(1!)^2} \text{tr} \{ \gamma_{a_1} [(m - \gamma^a \partial_a) \gamma^0] \gamma_{a_1}' C [(m - \gamma^b \partial_b) \gamma^0]^T \bar{C} \} \Delta(x - x') \\ &= -i \frac{2^{-5}}{(1!)^2} \text{tr} \{ \gamma_{a_1} [(m - \gamma^a \partial_a) \gamma^0] \gamma_{a_1}' [\gamma^0 (m + \gamma^b \partial_b)] \} \Delta(x - x') \\ &= -i \frac{2^{-5}}{(1!)^2} \text{tr} \{ \gamma_{a_1} [(m - \gamma^a \partial_a) \gamma^0] \gamma_{a_1}' [(m - \gamma^b \partial_b^+) \gamma^0] \} \Delta(x - x') \\ &= -i \frac{2^{-5}}{(1!)^2} \{ m^2 \text{tr} (\gamma_{a_1} \gamma^0 \gamma_{a_1}' \gamma^0) + \text{tr} (\gamma_{a_1} \gamma^a \partial_a \gamma^0 \gamma_{a_1}' \gamma^b \partial_b^+ \gamma^0) \} \Delta(x - x') \\ &= \frac{i}{4} (m^2 \eta_{a_1 a_1}' - \partial_{a_1} \partial_{a_1}^+) \Delta(x - x')\end{aligned}$$

□

## 6.4 四维时空中的反对称Bargmann-Wigner矢量场方程

引理6.4.1.

$$\begin{cases} (\gamma^a \partial_a + m) X(4) = 0 \\ X(4) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{abc} \gamma_{[a} \gamma_b \gamma_c] + \frac{1}{(4!)^2} F^{abcd} \gamma_{[a} \gamma_b \gamma_c \gamma_d] \right\} C \end{cases} \Leftrightarrow \begin{cases} F = 0 \\ \frac{1}{3!} \partial^{[a} F^{bcd]} + m F^{abcd} = 0, \partial_a F^{abc} = 0 \\ \partial^{[a} F^{bcde]} = 0, \partial_a F^{abcd} + m F^{abcd} = 0 \end{cases}$$

推论6.4.1.

$$\begin{cases} \gamma^a \partial_a X(4) = 0 \\ X(4) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{abc} \gamma_{[a} \gamma_b \gamma_c] + \frac{1}{(4!)^2} F^{abcd} \gamma_{[a} \gamma_b \gamma_c \gamma_d] \right\} C \end{cases} \Leftrightarrow \begin{cases} F = 0 \\ \partial^{[a} F^{bcd]} = 0, \partial_a F^{abc} = 0 \\ \partial^{[a} F^{bcde]} = 0, \partial_a F^{abcd} = 0 \end{cases}$$

推论6.4.2.  $(\gamma^a \partial_a + m)X(4) = 0, X(4) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{abc} \gamma_{[a} \gamma_b \gamma_c] + \frac{1}{(4!)^2} F^{abcd} \gamma_{[a} \gamma_b \gamma_c \gamma_d] \right\} C, m \neq 0$

$$\Leftrightarrow \begin{cases} F = 0 \\ \frac{1}{3!} \partial^{[a} F^{bcd]} + m F^{abcd} = 0 \\ \partial_a F^{abcd} + m F^{abcd} = 0 \end{cases} \Leftrightarrow \begin{cases} F = 0, \partial_a F^{abc} = 0 \\ \partial_a \partial^a F^{bcd} - m^2 F^{bcd} = 0 \\ F^{abcd} = -\frac{1}{3!m} \partial^{[a} F^{bcd]} \end{cases}$$

定理6.4.1.  $\begin{cases} (\gamma^a \partial_a + m)X(4) = 0 \\ X(4) = -X^T(4) \end{cases} \Leftrightarrow \begin{cases} \partial_d \partial^d F^{abc} - m^2 F^{abc} = 0, \partial_a F^{abc} = 0 \\ X(4) = \left\{ \frac{1}{3!} \gamma_{[a} \gamma_b \gamma_c] + \frac{1}{4!m} \gamma_{[a} \gamma_b \gamma_c \gamma_d] \partial^d \right\} C \frac{1}{3!} F^{abcd}$

推论6.4.3.  $\begin{cases} \frac{1}{3!} \partial^{[a} F^{bcd]} + m F^{abcd} = 0 \\ \partial_a F^{abcd} + m F^{abcd} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{0!} \partial^a *F + m *F^a = 0 \\ \partial_a *F^a + m *F = 0 \end{cases}$

## 6.5 六维时空中的对称Bargmann-Wigner矢量场方程

引理6.5.1.

$$\begin{cases} (\gamma^a \partial_a + m)X(6) = 0 \\ X(6) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{abc} \gamma_{[a} \gamma_b \gamma_c] + \frac{1}{(4!)^2} F^{abcd} \gamma_{[a} \gamma_b \gamma_c \gamma_d] \right\} C \end{cases} \Leftrightarrow \begin{cases} F = 0 \\ \frac{1}{3!} \partial^{[a} F^{bcd]} + m F^{abcd} = 0, \partial_a F^{abc} = 0 \\ \partial^{[a} F^{bcde]} = 0, \partial_a F^{abcd} + m F^{abcd} = 0 \end{cases}$$

证明:  $(\gamma^a \partial_a + m)X(6) = 0, X(6) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{abc} \gamma_{[a} \gamma_b \gamma_c] + \frac{1}{(4!)^2} F^{abcd} \gamma_{[a} \gamma_b \gamma_c \gamma_d] \right\} C$

$$\Leftrightarrow (\gamma_a \partial^a + m) \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{bcd} \gamma_{[b} \gamma_c \gamma_d] + \frac{1}{(4!)^2} F^{bcde} \gamma_{[b} \gamma_c \gamma_d \gamma_e] \right\} C = 0$$

$$\Leftrightarrow (\gamma_a \partial^a + m) \left\{ \frac{1}{3!} F^{bcd} \gamma_b \gamma_c \gamma_d + \frac{1}{4!} F^{bcde} \gamma_b \gamma_c \gamma_d \gamma_e \right\} = 0, F = 0$$

$$\Leftrightarrow \gamma_a \gamma_b \gamma_c \gamma_d \partial^a \frac{1}{3!} F^{bcd} + \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \partial^a \frac{1}{4!} F^{bcde} + m \frac{1}{3!} F^{bcd} \gamma_b \gamma_c \gamma_d + m \frac{1}{4!} F^{bcde} \gamma_b \gamma_c \gamma_d \gamma_e = 0$$

$$\Leftrightarrow \left\{ \frac{1}{4!} \gamma_{[a} \gamma_b \gamma_c \gamma_d] + \frac{1}{2!} (\delta_{ab} \gamma_{[c} \gamma_d] + \delta_{ac} \gamma_{[d} \gamma_b] + \delta_{ad} \gamma_{[b} \gamma_c] + \gamma_{[a} \gamma_b] \delta_{cd} + \gamma_{[c} \gamma_a] \delta_{bd} + \gamma_{[a} \gamma_d] \delta_{bc}) \right. \\ \left. + (\delta_{ab} \delta_{cd} - \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \right\} \partial^a \frac{1}{3!} F^{bcd} + \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \partial^a \frac{1}{4!} F^{bcde} + m \frac{1}{3!} F^{bcd} \gamma_b \gamma_c \gamma_d + m \frac{1}{4!} F^{bcde} \gamma_b \gamma_c \gamma_d \gamma_e = 0$$

$$\Leftrightarrow \left\{ \frac{1}{4!} \gamma_{[a} \gamma_b \gamma_c \gamma_d] + 3 \delta_{ab} \gamma_{[c} \gamma_d] \right\} \partial^a \frac{1}{3!} F^{bcd} + \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \partial^a \frac{1}{4!} F^{bcde} + m \frac{1}{3!} F^{bcd} \gamma_b \gamma_c \gamma_d + m \frac{1}{4!} F^{bcde} \gamma_b \gamma_c \gamma_d \gamma_e = 0$$

$$\Leftrightarrow \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \partial^a \frac{1}{4!} F^{bcde} + m \frac{1}{3!} F^{bcd} \gamma_b \gamma_c \gamma_d = 0, \partial_a \frac{1}{3!} F^{abc} = 0, \frac{1}{4!} \partial^{[a} \frac{1}{3!} F^{bcd]} + m \frac{1}{4!} F^{abcd} = 0$$

$$\Leftrightarrow \left\{ \frac{1}{5!} \gamma_{[a} \gamma_b \gamma_c \gamma_d \gamma_e] + \frac{1}{3!} (\delta_{ab} \gamma_{[c} \gamma_d \gamma_e] + \dots) + \frac{1}{1!} (\delta_{ab} \delta_{cd} \gamma_e + \dots) \right\} \partial^a \frac{1}{4!} F^{bcde} + m \frac{1}{3!} F^{bcd} \gamma_b \gamma_c \gamma_d = 0$$

$$\Leftrightarrow \left\{ \frac{1}{5!} \gamma_{[a} \gamma_b \gamma_c \gamma_d \gamma_e] + \frac{4}{3!} \delta_{ab} \gamma_{[c} \gamma_d \gamma_e] \right\} \partial^a \frac{1}{4!} F^{bcde} + m \frac{1}{3!} F^{bcd} \gamma_b \gamma_c \gamma_d = 0$$

$$\Leftrightarrow \begin{cases} \frac{1}{4!} \partial^{[a} \frac{1}{3!} F^{bcd]} + m \frac{1}{4!} F^{abcd} = 0, \partial_a \frac{1}{3!} F^{abc} = 0 \\ \partial^{[a} \frac{1}{4!} F^{bcde]} = 0, \partial_a \frac{1}{4!} F^{abcd} + \frac{1}{4} m \frac{1}{3!} F^{bcd} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{3!} \partial^{[a} F^{bcd]} + m F^{abcd} = 0, \partial_a F^{abc} = 0 \\ \partial^{[a} F^{bcde]} = 0, \partial_a F^{abcd} + m F^{abcd} = 0 \end{cases}$$

□

推论6.5.1.  $\begin{cases} \gamma^a \partial_a X(6) = 0 \\ X(6) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{abc} \gamma_{[a} \gamma_b \gamma_c] + \frac{1}{(4!)^2} F^{abcd} \gamma_{[a} \gamma_b \gamma_c \gamma_d] \right\} C \end{cases} \Leftrightarrow \begin{cases} F = 0 \\ \partial^{[a} F^{bcd]} = 0, \partial_a F^{abc} = 0 \\ \partial^{[a} F^{bcde]} = 0, \partial_a F^{abcd} = 0 \end{cases}$

推论6.5.2.  $(\gamma^a \partial_a + m)X(6) = 0, X(6) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{abc} \gamma_{[a} \gamma_b \gamma_c] + \frac{1}{(4!)^2} F^{abcd} \gamma_{[a} \gamma_b \gamma_c \gamma_d] \right\} C, m \neq 0$

$$\Leftrightarrow \begin{cases} F = 0 \\ \frac{1}{3!} \partial^{[a} F^{bcd]} + m F^{abcd} = 0 \\ \partial_a F^{abcd} + m F^{abcd} = 0 \end{cases} \Leftrightarrow \begin{cases} F = 0, \partial_a F^{abc} = 0 \\ \partial_a \partial^a F^{bcd} - m^2 F^{bcd} = 0 \\ F^{abcd} = -\frac{1}{3!m} \partial^{[a} F^{bcd]} \end{cases}$$

定理6.5.1.  $\begin{cases} (\gamma^a \partial_a + m)X(6) = 0 \\ X(6) = X^T(6) \end{cases} \Leftrightarrow \begin{cases} \partial_d \partial^d F^{abc} - m^2 F^{abc} = 0, \partial_a F^{abc} = 0 \\ X(6) = \left\{ \frac{1}{3!} \gamma_{[a} \gamma_b \gamma_c] + \frac{1}{4!m} \gamma_{[a} \gamma_b \gamma_c \gamma_d] \partial^d \right\} C \frac{1}{3!} F^{abcd}$



$$\text{推论6.5.3. } \begin{cases} \frac{1}{3!} \partial^{[a} F^{bcd]} + m F^{abcd} = 0 \\ \partial_a F^{abcd} + m F^{abcd} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{2!} \partial^{[a} * F^{bc]} + m * F^{abc} = 0 \\ \partial_a * F^{abc} + m * F^{bc} = 0 \end{cases}$$

## 6.6 六维时空中的反对称Bargmann-Wigner矢量场方程

引理6.6.1.  $(\gamma^a \partial_a + m)X(6) = 0$

$$X(6) = \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \dots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \dots a_6} \gamma_{[a_1} \dots \gamma_{a_6]} \right\} C$$

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1]} + m F^{a_0 a_1} = 0, \partial_{a_1} F^{a_1} = 0 \\ \partial^{[a_0} F^{a_1 a_2]} = 0, \partial_{a_0} F^{a_0 a_1} + m F^{a_1} = 0 \end{cases} \begin{cases} \frac{1}{5!} \partial^{[a_0} F^{a_1 \dots a_5]} + m F^{a_0 a_1 \dots a_5} = 0, \partial_{a_1} F^{a_1 \dots a_5} = 0 \\ \partial^{[a_0} F^{a_1 \dots a_6]} = 0, \partial_{a_0} F^{a_0 a_1 \dots a_5} + m F^{a_1 \dots a_5} = 0 \end{cases}$$

推论6.6.1.  $\gamma^a \partial_a X(6) = 0, X(6) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \dots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \dots a_6} \gamma_{[a_1} \dots \gamma_{a_6]} \right\} C$

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1]} = 0, \partial_{a_1} F^{a_1} = 0 \\ \partial^{[a_0} F^{a_1 a_2]} = 0, \partial_{a_0} F^{a_0 a_1} = 0 \end{cases} \begin{cases} \partial^{[a_0} F^{a_1 \dots a_5]} = 0, \partial_{a_1} F^{a_1 \dots a_5} = 0 \\ \partial^{[a_0} F^{a_1 \dots a_6]} = 0, \partial_{a_0} F^{a_0 a_1 \dots a_5} = 0 \end{cases}$$

推论6.6.2.  $(\gamma^a \partial_a + m)X(6) = 0, m \neq 0$

$$X(6) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \dots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \dots a_6} \gamma_{[a_1} \dots \gamma_{a_6]} \right\} C$$

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1]} + m F^{a_0 a_1} = 0 \\ \partial_{a_0} F^{a_0 a_1} + m F^{a_1} = 0 \\ \frac{1}{5!} \partial^{[a_0} F^{a_1 \dots a_5]} + m F^{a_0 a_1 \dots a_5} = 0 \\ \partial_{a_0} F^{a_0 a_1 \dots a_5} + m F^{a_1 \dots a_5} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1} - m^2 F^{a_1} = 0, \partial_{a_1} F^{a_1} = 0 \\ F^{a_0 a_1} = -\frac{1}{m} \partial^{[a_0} F^{a_1]} \\ \partial_{a_0} \partial^{a_0} F^{a_1 \dots a_5} - m^2 F^{a_1 \dots a_5} = 0, \partial_{a_1} F^{a_1 \dots a_5} = 0 \\ F^{a_0 a_1 \dots a_5} = -\frac{1}{5!m} \partial^{[a_0} F^{a_1 \dots a_5]} \end{cases}$$

定理6.6.1.  $(\gamma^a \partial_a + m)X(6) = 0, X(6) = -X^T(6)$

$$\Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1} - m^2 F^{a_1} = 0, \partial_{a_1} F^{a_1} = 0; \partial_{a_0} \partial^{a_0} F^{a_1 \dots a_5} - m^2 F^{a_1 \dots a_5} = 0, \partial_{a_1} F^{a_1 \dots a_5} = 0 \\ X(6) = \left\{ \frac{1}{1!} \gamma_{a_1} + \frac{1}{2!m} \gamma_{[a_1} \gamma_{a_2]} \partial^{a_2} \right\} C \frac{1}{1!} F^{a_1} + \left\{ \frac{1}{5!} \gamma_{[a_1} \dots \gamma_{a_5]} + \frac{1}{6!} \gamma_{[a_1} \dots \gamma_{a_6]} \partial^{a_6} \right\} C \frac{1}{5!} F^{a_1 \dots a_5} \end{cases}$$

$$\text{推论6.6.3. } \begin{cases} \frac{1}{1!} \partial^{[a_0} F^{a_1]} + m F^{a_0 a_1} = 0 \\ \partial_{a_0} F^{a_0 a_1} + m F^{a_1} = 0 \\ \frac{1}{5!} \partial^{[a_0} F^{a_1 \dots a_5]} + m F^{a_0 a_1 \dots a_5} = 0 \\ \partial_{a_0} F^{a_0 a_1 \dots a_5} + m F^{a_1 \dots a_5} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{4!} \partial^{[a_0} * F^{a_1 \dots a_4]} + m * F^{a_0 a_1 \dots a_4} = 0 \\ \partial_{a_0} * F^{a_0 a_1 \dots a_4} + m * F^{a_1 \dots a_4} = 0 \\ \frac{1}{0!} \partial^{a_0} * F + m * F^{a_0} = 0 \\ \partial_{a_0} * F^{a_0} + m * F = 0 \end{cases}$$

## 6.7 八维时空中的对称Bargmann-Wigner矢量场方程

引理6.7.1.  $(\gamma^a \partial_a + m)X(8) = 0$

$$X(8) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 a_2 a_3} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \dots a_4} \gamma_{[a_1} \dots \gamma_{a_4]} + \frac{1}{(7!)^2} F^{a_1 \dots a_7} \gamma_{[a_1} \dots \gamma_{a_7]} + \frac{1}{(8!)^2} F^{a_1 \dots a_8} \gamma_{[a_1} \dots \gamma_{a_8]} \right\} C$$

$$\Leftrightarrow \begin{cases} F = 0 \\ \frac{1}{3!} \partial^{[a_0} F^{a_1 \dots a_3]} + m F^{a_0 \dots a_3} = 0, \partial_{a_1} F^{a_1 \dots a_3} = 0; \partial^{[a_0} F^{a_1 \dots a_4]} = 0, \partial_{a_0} F^{a_0 a_1 \dots a_3} + m F^{a_1 \dots a_3} = 0 \\ \frac{1}{7!} \partial^{[a_0} F^{a_1 \dots a_7]} + m F^{a_0 a_1 \dots a_7} = 0, \partial_{a_1} F^{a_1 \dots a_7} = 0; \partial^{[a_0} F^{a_1 \dots a_8]} = 0, \partial_{a_0} F^{a_0 a_1 \dots a_7} + m F^{a_1 \dots a_7} = 0 \end{cases}$$

证明:  $(\gamma^a \partial_a + m)X(8) = 0$

$$X(8) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 a_2 a_3} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \dots a_4} \gamma_{[a_1} \dots \gamma_{a_4]} + \frac{1}{(7!)^2} F^{a_1 \dots a_7} \gamma_{[a_1} \dots \gamma_{a_7]} + \frac{1}{(8!)^2} F^{a_1 \dots a_8} \gamma_{[a_1} \dots \gamma_{a_8]} \right\} C$$

$$\Leftrightarrow \begin{cases} F = 0 \\ (\gamma^a \partial_a + m) \left\{ \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \dots \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \dots a_4} \gamma_{[a_1} \dots \gamma_{a_4]} \right\} = 0 \\ (\gamma^a \partial_a + m) \left\{ \frac{1}{(7!)^2} F^{a_1 \dots a_7} \gamma_{[a_1} \dots \gamma_{a_7]} + \frac{1}{(8!)^2} F^{a_1 \dots a_8} \gamma_{[a_1} \dots \gamma_{a_8]} \right\} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} F = 0 \\ \frac{1}{3!} \partial^{[a_0} F^{a_1 \cdots a_3]} + m F^{a_0 \cdots a_3} = 0, \partial_{a_1} F^{a_1 \cdots a_3} = 0; \partial^{[a_0} F^{a_1 \cdots a_4]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_3} + m F^{a_1 \cdots a_3} = 0 \\ \frac{1}{7!} \partial^{[a_0} F^{a_1 \cdots a_7]} + m F^{a_0 a_1 \cdots a_7} = 0, \partial_{a_1} F^{a_1 \cdots a_7} = 0; \partial^{[a_0} F^{a_1 \cdots a_8]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_7} + m F^{a_1 \cdots a_7} = 0 \end{cases} \quad \square$$

推论6.7.1.  $\gamma^a \partial_a X(8) = 0$

$$X(8) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 a_2 a_3} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \cdots a_4} \gamma_{[a_1} \cdots \gamma_{a_4]} + \frac{1}{(7!)^2} F^{a_1 \cdots a_7} \gamma_{[a_1} \cdots \gamma_{a_7]} + \frac{1}{(8!)^2} F^{a_1 \cdots a_8} \gamma_{[a_1} \cdots \gamma_{a_8]} \right\} C$$

$$\Leftrightarrow \begin{cases} F = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_3]} = 0, \partial_{a_1} F^{a_1 \cdots a_3} = 0; \partial^{[a_0} F^{a_1 \cdots a_4]} = 0, \partial_{a_1} F^{a_1 \cdots a_4} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_7]} = 0, \partial_{a_1} F^{a_1 \cdots a_7} = 0; \partial^{[a_0} F^{a_1 \cdots a_8]} = 0, \partial_{a_1} F^{a_1 \cdots a_8} = 0 \end{cases}$$

推论6.7.2.  $(\gamma^a \partial_a + m)X(8) = 0, m \neq 0$

$$X(8) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 a_2 a_3} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \cdots a_4} \gamma_{[a_1} \cdots \gamma_{a_4]} + \frac{1}{(7!)^2} F^{a_1 \cdots a_7} \gamma_{[a_1} \cdots \gamma_{a_7]} + \frac{1}{(8!)^2} F^{a_1 \cdots a_8} \gamma_{[a_1} \cdots \gamma_{a_8]} \right\} C$$

$$\Leftrightarrow \begin{cases} F = 0 \\ \frac{1}{3!} \partial^{[a_0} F^{a_1 \cdots a_3]} + m F^{a_0 \cdots a_3} = 0, \partial_{a_0} F^{a_0 \cdots a_3} + m F^{a_1 \cdots a_3} = 0 \\ \frac{1}{7!} \partial^{[a_0} F^{a_1 \cdots a_7]} + m F^{a_0 \cdots a_7} = 0, \partial_{a_0} F^{a_0 \cdots a_7} + m F^{a_1 \cdots a_7} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} F = 0 \\ \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_3} - m^2 F^{a_1 \cdots a_3} = 0, \partial_{a_1} F^{a_1 \cdots a_3} = 0, F^{a_0 \cdots a_3} = -\frac{1}{3!m} \partial^{[a_0} F^{a_1 \cdots a_3]} \\ \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_7} - m^2 F^{a_1 \cdots a_7} = 0, \partial_{a_1} F^{a_1 \cdots a_7} = 0, F^{a_0 \cdots a_7} = -\frac{1}{7!m} \partial^{[a_0} F^{a_1 \cdots a_7]} \end{cases}$$

定理6.7.1.

$$\begin{cases} (\gamma^a \partial_a + m)X(8) = 0 \\ X(8) = X^T(8) \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_3} - m^2 F^{a_1 \cdots a_3} = 0, \partial_{a_1} F^{a_1 \cdots a_3} = 0 \\ \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_7} - m^2 F^{a_1 \cdots a_7} = 0, \partial_{a_1} F^{a_1 \cdots a_7} = 0 \\ X(8) = \left\{ \frac{1}{3!} \gamma_{[a_1} \cdots \gamma_{a_3]} + \frac{1}{4!m} \gamma_{[a_1} \cdots \gamma_{a_4]} \partial^{a_4} \right\} C \frac{1}{3!} F^{a_1 \cdots a_3} \\ + \left\{ \frac{1}{7!} \gamma_{[a_1} \cdots \gamma_{a_7]} + \frac{1}{8!m} \gamma_{[a_1} \cdots \gamma_{a_8]} \partial^{a_8} \right\} C \frac{1}{7!} F^{a_1 \cdots a_7} \end{cases}$$

推论6.7.3.

$$\begin{cases} \frac{1}{3!} \partial^{[a_0} F^{a_1 \cdots a_3]} + m F^{a_0 \cdots a_3} = 0 \\ \partial_{a_0} F^{a_0 \cdots a_3} + m F^{a_1 \cdots a_3} = 0 \\ \frac{1}{7!} \partial^{[a_0} F^{a_1 \cdots a_7]} + m F^{a_0 \cdots a_7} = 0 \\ \partial_{a_0} F^{a_0 \cdots a_7} + m F^{a_1 \cdots a_7} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{4!} \partial^{[a_0} * F^{a_1 \cdots a_4]} + m * F^{a_0 \cdots a_4} = 0 \\ \partial_{a_0} * F^{a_0 \cdots a_4} + m * F^{a_1 \cdots a_4} = 0 \\ \partial^{a_0} * F + m * F^{a_0} = 0 \\ \partial_{a_0} * F^{a_0} + m * F = 0 \end{cases}$$

## 6.8 八维时空中的反对称Bargmann-Wigner矢量场方程

引理6.8.1.  $(\gamma^a \partial_a + m)X(8) = 0$

$$X(8) = \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \cdots a_5} \gamma_{[a_1} \cdots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \cdots a_6} \gamma_{[a_1} \cdots \gamma_{a_6]} \right\} C$$

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1]} + m F^{a_0 a_1} = 0, \partial_{a_1} F^{a_1} = 0 \\ \partial^{[a_0} F^{a_1 a_2]} = 0, \partial_{a_0} F^{a_0 a_1} + m F^{a_1} = 0 \end{cases} \begin{cases} \frac{1}{5!} \partial^{[a_0} F^{a_1 \cdots a_5]} + m F^{a_0 a_1 \cdots a_5} = 0, \partial_{a_1} F^{a_1 \cdots a_5} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_6]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_5} + m F^{a_1 \cdots a_5} = 0 \end{cases}$$

推论6.8.1.  $\gamma^a \partial_a X(8) = 0, X(8) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} + \frac{1}{(5!)^2} F^{a_1 \cdots a_5} \gamma_{[a_1} \cdots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \cdots a_6} \gamma_{[a_1} \cdots \gamma_{a_6]} \right\} C$

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1]} = 0, \partial_{a_1} F^{a_1} = 0 \\ \partial^{[a_0} F^{a_1 a_2]} = 0, \partial_{a_0} F^{a_0 a_1} = 0 \end{cases} \begin{cases} \partial^{[a_0} F^{a_1 \cdots a_5]} = 0, \partial_{a_1} F^{a_1 \cdots a_5} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_6]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_5} = 0 \end{cases}$$

推论6.8.2.  $(\gamma^a \partial_a + m)X(8) = 0, m \neq 0$

$$X(8) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} + \frac{1}{(5!)^2} F^{a_1 \cdots a_5} \gamma_{[a_1} \cdots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \cdots a_6} \gamma_{[a_1} \cdots \gamma_{a_6]} \right\} C$$

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1]} + m F^{a_0 a_1} = 0 \\ \partial_{a_0} F^{a_0 a_1} + m F^{a_1} = 0 \\ \frac{1}{5!} \partial^{[a_0} F^{a_1 \cdots a_5]} + m F^{a_0 a_1 \cdots a_5} = 0 \\ \partial_{a_0} F^{a_0 a_1 \cdots a_5} + m F^{a_1 \cdots a_5} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1} - m^2 F^{a_1} = 0, \partial_{a_1} F^{a_1} = 0 \\ F^{a_0 a_1} = -\frac{1}{m} \partial^{[a_0} F^{a_1]} \\ \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_5} - m^2 F^{a_1 \cdots a_5} = 0, \partial_{a_1} F^{a_1 \cdots a_5} = 0 \\ F^{a_0 a_1 \cdots a_5} = -\frac{1}{5!m} \partial^{[a_0} F^{a_1 \cdots a_5]} \end{cases}$$

定理6.8.1.  $(\gamma^a \partial_a + m)X(8) = 0, X(8) = -X^T(8)$

$$\Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1} - m^2 F^{a_1} = 0, \partial_{a_1} F^{a_1} = 0; \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_5} - m^2 F^{a_1 \cdots a_5} = 0, \partial_{a_1} F^{a_1 \cdots a_5} = 0 \\ X(6) = \left\{ \frac{1}{1!} \gamma_{a_1} + \frac{1}{2!m} \gamma_{[a_1} \gamma_{a_2]} \partial^{a_2} \right\} C F^{a_1} + \left\{ \frac{1}{5!} \gamma_{[a_1} \cdots \gamma_{a_5]} + \frac{1}{6!} \gamma_{[a_1} \cdots \gamma_{a_6]} \partial^{a_6} \right\} C \frac{1}{5!} F^{a_1 \cdots a_5} \end{cases}$$

$$\text{推论6.8.3.} \begin{cases} \frac{1}{1!} \partial^{[a_0} F^{a_1]} + m F^{a_0 a_1} = 0 \\ \partial_{a_0} F^{a_0 a_1} + m F^{a_1} = 0 \\ \frac{1}{5!} \partial^{[a_0} F^{a_1 \cdots a_5]} + m F^{a_0 a_1 \cdots a_5} = 0 \\ \partial_{a_0} F^{a_0 a_1 \cdots a_5} + m F^{a_1 \cdots a_5} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{6!} \partial^{[a_0 * F^{a_1 \cdots a_6]} + m * F^{a_0 a_1 \cdots a_6} = 0 \\ \partial_{a_0} * F^{a_0 a_1 \cdots a_6} + m * F^{a_1 \cdots a_6} = 0 \\ \frac{1}{2!} \partial^{[a_0 * F^{a_1 a_2]} + m * F^{a_0 a_1 a_2} = 0 \\ \partial_{a_0} * F^{a_0 a_1 a_2} + m * F^{a_1 a_2} = 0 \end{cases}$$

## 6.9 十维时空中的对称Bargmann-Wigner矢量场方程

引理6.9.1.  $(\gamma^a \partial_a + m)X(10) = 0, X(10) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{2!} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} \right.$

$$\left. + \frac{1}{(5!)^2} F^{a_1 \cdots a_5} \gamma_{[a_1} \cdots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \cdots a_6} \gamma_{[a_1} \cdots \gamma_{a_6]} + \frac{1}{(9!)^2} F^{a_1 \cdots a_9} \gamma_{[a_1} \cdots \gamma_{a_9]} + \frac{1}{(10!)^2} F^{a_1 \cdots a_{10}} \gamma_{[a_1} \cdots \gamma_{a_{10}]} \right\} C$$

$$\Leftrightarrow \begin{cases} F = 0, F^{a_1 a_2} = 0 \\ \frac{1}{5!} \partial^{[a_0} F^{a_1 \cdots a_5]} + m F^{a_0 a_1 \cdots a_5} = 0, \partial_{a_1} F^{a_1 \cdots a_5} = 0; \partial^{[a_0} F^{a_1 \cdots a_6]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_5} + m F^{a_1 \cdots a_5} = 0 \\ \frac{1}{9!} \partial^{[a_0} F^{a_1 \cdots a_9]} + m F^{a_0 a_1 \cdots a_9} = 0, \partial_{a_1} F^{a_1 \cdots a_9} = 0; \partial^{[a_0} F^{a_1 \cdots a_{10}]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_9} + m F^{a_1 \cdots a_9} = 0 \end{cases}$$

证明:  $(\gamma^a \partial_a + m)X(10) = 0, X(10) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{2!} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} \right.$

$$\left. + \frac{1}{(5!)^2} F^{a_1 \cdots a_5} \gamma_{[a_1} \cdots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \cdots a_6} \gamma_{[a_1} \cdots \gamma_{a_6]} + \frac{1}{(9!)^2} F^{a_1 \cdots a_9} \gamma_{[a_1} \cdots \gamma_{a_9]} + \frac{1}{(10!)^2} F^{a_1 \cdots a_{10}} \gamma_{[a_1} \cdots \gamma_{a_{10}]} \right\} C$$

$$\Leftrightarrow \begin{cases} F = 0, F^{a_1 a_2} = 0 \\ (\gamma^b \partial_b + m) \left\{ \frac{1}{(5!)^2} F^{a_1 \cdots a_5} \gamma_{[a_1} \cdots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \cdots a_6} \gamma_{[a_1} \cdots \gamma_{a_6]} \right\} = 0 \\ (\gamma^b \partial_b + m) \left\{ \frac{1}{(9!)^2} F^{a_1 \cdots a_9} \gamma_{[a_1} \cdots \gamma_{a_9]} + \frac{1}{(10!)^2} F^{a_1 \cdots a_{10}} \gamma_{[a_1} \cdots \gamma_{a_{10}]} \right\} = 0 \\ \frac{1}{5!} \partial^{[a_0} F^{a_1 \cdots a_5]} + m F^{a_0 a_1 \cdots a_5} = 0, \partial_{a_1} F^{a_1 \cdots a_5} = 0; \partial^{[a_0} F^{a_1 \cdots a_6]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_5} + m F^{a_1 \cdots a_5} = 0 \\ \frac{1}{9!} \partial^{[a_0} F^{a_1 \cdots a_9]} + m F^{a_0 a_1 \cdots a_9} = 0, \partial_{a_1} F^{a_1 \cdots a_9} = 0; \partial^{[a_0} F^{a_1 \cdots a_{10}]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_9} + m F^{a_1 \cdots a_9} = 0 \end{cases} \quad \square$$

推论6.9.1.  $\gamma^a \partial_a X(10) = 0, X(10) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{2!} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \cdots a_5} \gamma_{[a_1} \cdots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \cdots a_6} \gamma_{[a_1} \cdots \gamma_{a_6]} \right.$

$$\left. + \frac{1}{(9!)^2} F^{a_1 \cdots a_9} \gamma_{[a_1} \cdots \gamma_{a_9]} + \frac{1}{(10!)^2} F^{a_1 \cdots a_{10}} \gamma_{[a_1} \cdots \gamma_{a_{10}]} \right\} C$$

$$\Leftrightarrow \begin{cases} F = 0, F^{a_1 a_2} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_5]} = 0, \partial_{a_1} F^{a_1 \cdots a_5} = 0; \partial^{[a_0} F^{a_1 \cdots a_6]} = 0, \partial_{a_1} F^{a_1 \cdots a_6} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_9]} = 0, \partial_{a_1} F^{a_1 \cdots a_9} = 0; \partial^{[a_0} F^{a_1 \cdots a_{10}]} = 0, \partial_{a_1} F^{a_1 \cdots a_{10}} = 0 \end{cases}$$

推论6.9.2.

$$(\gamma^b \partial_b + m)X(10) = 0, X(10) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{2!} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \cdots a_5} \gamma_{[a_1} \cdots \gamma_{a_5]} + \frac{1}{(6!)^2} F^{a_1 \cdots a_6} \gamma_{[a_1} \cdots \gamma_{a_6]} \right.$$

$$\left. + \frac{1}{(9!)^2} F^{a_1 \cdots a_9} \gamma_{[a_1} \cdots \gamma_{a_9]} + \frac{1}{(10!)^2} F^{a_1 \cdots a_{10}} \gamma_{[a_1} \cdots \gamma_{a_{10}]} \right\} C, m \neq 0$$

$$\Leftrightarrow \begin{cases} F = 0, F^{a_1 a_2} = 0 \\ \frac{1}{5!} \partial^{[a_0} F^{a_1 \cdots a_5]} + m F^{a_0 \cdots a_5} = 0, \partial_{a_0} F^{a_0 \cdots a_5} + m F^{a_1 \cdots a_5} = 0 \\ \frac{1}{9!} \partial^{[a_0} F^{a_1 \cdots a_9]} + m F^{a_0 \cdots a_9} = 0, \partial_{a_0} F^{a_0 \cdots a_9} + m F^{a_1 \cdots a_9} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} F = 0, F^{a_1 a_2} = 0 \\ \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_5} - m^2 F^{a_1 \cdots a_5} = 0, \partial_{a_1} F^{a_1 \cdots a_5} = 0, F^{a_0 \cdots a_5} = -\frac{1}{5!m} \partial^{[a_0} F^{a_1 \cdots a_5]} \\ \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_9} - m^2 F^{a_1 \cdots a_9} = 0, \partial_{a_1} F^{a_1 \cdots a_9} = 0, F^{a_0 \cdots a_9} = -\frac{1}{9!m} \partial^{[a_0} F^{a_1 \cdots a_9]} \end{cases}$$

定理6.9.1.

$$\begin{cases} (\gamma^a \partial_a + m)X(10) = 0 \\ X(10) = X^T(10) \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_5} - m^2 F^{a_1 \cdots a_5} = 0, \partial_{a_1} F^{a_1 \cdots a_5} = 0 \\ \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_9} - m^2 F^{a_1 \cdots a_9} = 0, \partial_{a_1} F^{a_1 \cdots a_9} = 0 \\ X(10) = \left\{ \frac{1}{5!} \gamma_{[a_1} \cdots \gamma_{a_5]} + \frac{1}{6!m} \gamma_{[a_1} \cdots \gamma_{a_6]} \partial^{a_6} \right\} C \frac{1}{5!} F^{a_1 \cdots a_5} \\ + \left\{ \frac{1}{9!} \gamma_{[a_1} \cdots \gamma_{a_9]} + \frac{1}{10!m} \gamma_{[a_1} \cdots \gamma_{a_{10}}] \partial^{a_{10}} \right\} C \frac{1}{9!} F^{a_1 \cdots a_9} \end{cases}$$

推论6.9.3.

$$\Leftrightarrow \begin{cases} \frac{1}{5!} \partial^{[a_0} F^{a_1 \cdots a_5]} + m F^{a_0 \cdots a_5} = 0, \partial_{a_0} F^{a_0 \cdots a_5} + m F^{a_1 \cdots a_5} = 0 \\ \frac{1}{9!} \partial^{[a_0} F^{a_1 \cdots a_9]} + m F^{a_0 \cdots a_9} = 0, \partial_{a_0} F^{a_0 \cdots a_9} + m F^{a_1 \cdots a_9} = 0 \\ \frac{1}{4!} \partial^{[a_0} *F^{a_1 \cdots a_4]} + m *F^{a_0 \cdots a_4} = 0, \partial_{a_0} *F^{a_0 \cdots a_4} + m *F^{a_1 \cdots a_4} = 0 \\ \partial^{a_0} *F + m *F^{a_0} = 0, \partial_{a_0} *F^{a_0} + m *F = 0 \end{cases}$$

推论6.9.4.

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1 \cdots a_5]} = 0, \partial_{a_1} F^{a_1 \cdots a_5} = 0; \partial^{[a_0} F^{a_1 \cdots a_6]} = 0, \partial_{a_1} F^{a_1 \cdots a_6} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_9]} = 0, \partial_{a_1} F^{a_1 \cdots a_9} = 0; \partial^{[a_0} F^{a_1 \cdots a_{10}}] = 0, \partial_{a_1} F^{a_1 \cdots a_{10}} = 0 \\ \partial^{[a_0} *F^{a_1 \cdots a_5]} = 0, \partial_{a_1} *F^{a_1 \cdots a_5} = 0; \partial^{[a_0} *F^{a_1 \cdots a_4]} = 0, \partial_{a_1} *F^{a_1 \cdots a_4} = 0 \\ \partial^{[a_0} *F^{a_1]} = 0, \partial_{a_1} *F^{a_1} = 0; \partial^{a_0} *F = 0 \end{cases}$$

## 6.10 十维时空中的反对称Bargmann-Wigner矢量场方程

引理6.10.1.  $(\gamma^b \partial_b + m)X(10) = 0, X(10) =$

$$\left\{ \frac{1}{1!} F^{a_1} \gamma_{a_1} + \frac{1}{(3!)^2} F^{a_1 \cdots a_3} \gamma_{[a_1} \cdots \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \cdots a_4} \gamma_{[a_1} \cdots \gamma_{a_4]} + \frac{1}{(7!)^2} F^{a_1 \cdots a_7} \gamma_{[a_1} \cdots \gamma_{a_7]} + \frac{1}{(8!)^2} F^{a_1 \cdots a_8} \gamma_{[a_1} \cdots \gamma_{a_8]} \right\} C$$

$$\Leftrightarrow \begin{cases} F^{a_1} = 0 \\ \frac{1}{3!} \partial^{[a_0} F^{a_1 \cdots a_3]} + m F^{a_0 a_1 \cdots a_3} = 0, \partial_{a_1} F^{a_1 \cdots a_3} = 0; \partial^{[a_0} F^{a_1 \cdots a_4]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_3} + m F^{a_1 \cdots a_3} = 0 \\ \frac{1}{7!} \partial^{[a_0} F^{a_1 \cdots a_7]} + m F^{a_0 a_1 \cdots a_7} = 0, \partial_{a_1} F^{a_1 \cdots a_7} = 0; \partial^{[a_0} F^{a_1 \cdots a_8]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_7} + m F^{a_1 \cdots a_7} = 0 \end{cases}$$

推论6.10.1.  $\gamma^b \partial_b X(10) = 0, X(10) =$

$$\left\{ \frac{1}{1!} F^{a_1} \gamma_{a_1} + \frac{1}{(3!)^2} F^{a_1 \cdots a_3} \gamma_{[a_1} \cdots \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \cdots a_4} \gamma_{[a_1} \cdots \gamma_{a_4]} + \frac{1}{(7!)^2} F^{a_1 \cdots a_7} \gamma_{[a_1} \cdots \gamma_{a_7]} + \frac{1}{(8!)^2} F^{a_1 \cdots a_8} \gamma_{[a_1} \cdots \gamma_{a_8]} \right\} C$$

$$\Leftrightarrow \begin{cases} F^{a_1} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_3]} = 0, \partial_{a_1} F^{a_1 \cdots a_3} = 0; \partial^{[a_0} F^{a_1 \cdots a_4]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_3} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_7]} = 0, \partial_{a_1} F^{a_1 \cdots a_7} = 0; \partial^{[a_0} F^{a_1 \cdots a_8]} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_7} = 0 \end{cases}$$

推论6.10.2.

$(\gamma^b \partial_b + m)X(10) = 0, m \neq 0, X(10) =$

$$\left\{ \frac{1}{1!} F^{a_1} \gamma_{a_1} + \frac{1}{(3!)^2} F^{a_1 \cdots a_3} \gamma_{[a_1} \cdots \gamma_{a_3]} + \frac{1}{(4!)^2} F^{a_1 \cdots a_4} \gamma_{[a_1} \cdots \gamma_{a_4]} + \frac{1}{(7!)^2} F^{a_1 \cdots a_7} \gamma_{[a_1} \cdots \gamma_{a_7]} + \frac{1}{(8!)^2} F^{a_1 \cdots a_8} \gamma_{[a_1} \cdots \gamma_{a_8]} \right\} C$$

$$\Leftrightarrow \begin{cases} F^{a_1} = 0 \\ \frac{1}{3!} \partial^{[a_0} F^{a_1 \cdots a_3]} + m F^{a_0 a_1 \cdots a_3} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_3} + m F^{a_1 \cdots a_3} = 0 \\ \frac{1}{7!} \partial^{[a_0} F^{a_1 \cdots a_7]} + m F^{a_0 a_1 \cdots a_7} = 0, \partial_{a_0} F^{a_0 a_1 \cdots a_7} + m F^{a_1 \cdots a_7} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} F^{a_1} = 0 \\ \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_3} - m^2 F^{a_1 \cdots a_3} = 0, \partial_{a_1} F^{a_1 \cdots a_3} = 0, F^{a_0 a_1 \cdots a_3} = -\frac{1}{3!m} \partial^{[a_0} F^{a_1 \cdots a_3]} \\ \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_7} - m^2 F^{a_1 \cdots a_7} = 0, \partial_{a_1} F^{a_1 \cdots a_7} = 0, F^{a_0 a_1 \cdots a_7} = -\frac{1}{7!m} \partial^{[a_0} F^{a_1 \cdots a_7]} \end{cases}$$

定理6.10.1.  $(\gamma^a \partial_a + m)X(10) = 0, X(10) = -X^T(10)$

$$\Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_3} - m^2 F^{a_1 \cdots a_3} = 0, \partial_{a_1} F^{a_1 \cdots a_3} = 0; \partial_{a_0} \partial^{a_0} F^{a_1 \cdots a_7} - m^2 F^{a_1 \cdots a_7} = 0, \partial_{a_1} F^{a_1 \cdots a_7} = 0 \\ X(10) = \left\{ \frac{1}{3!} \gamma_{[a_1} \cdots \gamma_{a_3]} + \frac{1}{4!m} \gamma_{[a_1} \cdots \gamma_{a_4]} \partial^{a_4} \right\} C \frac{1}{3!} F^{a_1 \cdots a_3} + \left\{ \frac{1}{7!} \gamma_{[a_1} \cdots \gamma_{a_7]} + \frac{1}{8!m} \gamma_{[a_1} \cdots \gamma_{a_8]} \partial^{a_8} \right\} C \frac{1}{7!} F^{a_1 \cdots a_7} \end{cases}$$

推论6.10.3.

$$\begin{cases} \frac{1}{3!} \partial^{[a_0} F^{a_1 \cdots a_3]} + m F^{a_0 a_1 \cdots a_3} = 0 \\ \partial_{a_0} F^{a_0 a_1 \cdots a_3} + m F^{a_1 \cdots a_3} = 0 \\ \frac{1}{7!} \partial^{[a_0} F^{a_1 \cdots a_7]} + m F^{a_0 a_1 \cdots a_7} = 0 \\ \partial_{a_0} F^{a_0 a_1 \cdots a_7} + m F^{a_1 \cdots a_7} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{6!} \partial^{[a_0 * F^{a_1 \cdots a_6]} + m * F^{a_0 \cdots a_6} = 0 \\ \partial_{a_0} * F^{a_0 \cdots a_6} + m * F^{a_1 \cdots a_6} = 0 \\ \frac{1}{2!} \partial^{[a_0 * F^{a_1 a_2]} + m * F^{a_0 a_1 a_2} = 0 \\ \partial_{a_0} * F^{a_0 a_1 a_2} + m * F^{a_1 a_2} = 0 \end{cases}$$

## 6.11 十维时空中Bargmann-Wigner矢量场方程的对易规则

引理6.11.1.  $\frac{1}{(l+1)!} \eta_{[a_1} \langle a'_1 \eta_{a_2 a'_2} \cdots \eta_{a_{l-1} a'_{l-1}} \eta_{a_l a'_l} \eta_{a_{l+1} a'_{l+1}} \rangle \partial^{a_{l+1}} \partial^{+a'_{l+1}} \Delta(x-x')$   
 $= \left\{ \frac{1}{l!} \eta_{[a_1} \langle a'_1 \eta_{a_2 a'_2} \cdots \eta_{a_{l-1} a'_{l-1}} \eta_{a_l} \rangle a'_l m^2 - \frac{1}{(l-1)!} \eta_{[a_1} \langle a'_1 \eta_{a_2 a'_2} \cdots \eta_{a_{l-1} a'_{l-1}} \partial_{a_l} \rangle \partial_{a'_l}^+ \right\} \Delta(x-x')$

推论6.11.1.  $F_{a_1 a_2 a_3 a_4 a_5}(x) = 2^{-5} \text{tr} \left\{ \frac{1}{5!} \bar{C} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} X(x) \right\} = \frac{2^{-5}}{5!} \bar{C}^{\lambda \eta} \left\{ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} \right\} \eta^\mu X_{\lambda \mu}(x)$

性质6.11.1.  $\text{tr} \left\{ \frac{1}{5!} \gamma^{[b_1} \cdots \gamma^{b_5]} \frac{1}{5!} \gamma_{[a_1} \cdots \gamma_{a_5]} \right\} = 2^5 \delta_{[a_1}^{b_1} \cdots \delta_{a_5]}^{b_5}$

推论6.11.2.

$$\begin{cases} U_{a_1 a_2 a_3 a_4 a_5}(\vec{p}, h) := \frac{2^{-5}}{5!} \bar{C}^{\lambda \eta} \left\{ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} \right\} \eta^\mu U_{\lambda \mu}(\vec{p}, h) = 2^{-5} \text{tr} \left\{ \bar{C} \frac{1}{5!} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} U(\vec{p}, h) \right\} \\ V_{a_1 a_2 a_3 a_4 a_5}(\vec{p}, h) := \frac{2^{-5}}{5!} \bar{C}^{\lambda \eta} \left\{ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} \right\} \eta^\mu V_{\lambda \mu}(\vec{p}, h) = 2^{-5} \text{tr} \left\{ \bar{C} \frac{1}{5!} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} V(\vec{p}, h) \right\} \\ C(10) := \gamma_2(10) \gamma_4(10) \gamma_5(10) \gamma_8(10) \gamma_9(10) = -\sigma_y \otimes I \otimes \sigma_y \otimes I \otimes \sigma_y \\ \bar{C} = C^+, C^T = -C \end{cases}$$

猜想6.11.1.

$$\begin{cases} \bar{C}(n) = C^+(n), C^+(n) = (-1)^{\lfloor \frac{n}{4} \rfloor} C(n), C^T(n) = (-1)^{\lfloor \frac{n+2}{4} \rfloor} C(n) \\ [\gamma_a(n) C(n)]^T = (-1)^{\lfloor \frac{n-1}{4} \rfloor} [\gamma_a(n) C(n)], [C^+(n) \gamma_a(n)]^T = (-1)^{\lfloor \frac{n-1}{4} \rfloor} [C^+(n) \gamma_a(n)] \end{cases}$$

定理6.11.1.  $[F_{a_1 a_2 a_3 a_4 a_5}(x), F_{a'_1 a'_2 a'_3 a'_4 a'_5}^+(x')] = -\frac{i}{2^5} \frac{1}{6!} \eta_{[a_1} \langle a'_1 \eta_{a_2 a'_2} \cdots \eta_{a_5 a'_5} \eta_{a_6 a'_6} \rangle \partial^{a_6} \partial^{+a'_6} \Delta(x-x')$

证明:  $[F_{a_1 a_2 a_3 a_4 a_5}(x), F_{a'_1 a'_2 a'_3 a'_4 a'_5}^+(x')]$

$$\begin{aligned} &= \frac{2^{-10}}{(5!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]})^{\mu \lambda} (C^+ \gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5]})^{* \mu' \lambda'} [\psi_{\lambda \mu}(x), \psi_{\lambda' \mu'}^+(x')] \\ &= \frac{2^{-10}}{(5!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]})^{\lambda \mu} (C^+ \gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5]})^{+ \lambda' \mu'} [\psi_{\lambda \mu}(x), \psi_{\lambda' \mu'}^+(x')] \\ &= \frac{2^{-10}}{(5!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]})^{\lambda \mu} (\gamma_{[a'_5} \gamma_{a'_4} \gamma_{a'_3} \gamma_{a'_2} \gamma_{a'_1}])^{\lambda' \mu'} [\psi_{\lambda \mu}(x), \psi_{\lambda' \mu'}^+(x')] \\ &= \frac{2^{-10}}{(5!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]})^{\lambda \mu} (\gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5}])^{\lambda' \mu'} [\psi_{\lambda \mu}(x), \psi_{\lambda' \mu'}^+(x')] \\ &= \frac{2^{-10}}{(5!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]})^{\lambda \mu} (\gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5}])^{\lambda' \mu'} \frac{i}{2^3} [(m - \gamma^a \partial_a) \gamma^0]_{\{\lambda(\lambda' [(m - \gamma^b \partial_b) \gamma^0]_{\mu} \mu')\}} \Delta(x-x') \\ &= i \frac{2^{-11}}{(5!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]})^{\lambda \mu} (\gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5}])^{\lambda' \mu'} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda \lambda'} [(m - \gamma^b \partial_b) \gamma^0]_{\mu \mu'} \Delta(x-x') \\ &= i \frac{2^{-11}}{(5!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]})^{\lambda \mu} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda \lambda'} (\gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5}])^{\lambda' \mu'} [(m - \gamma^b \partial_b) \gamma^0]_{\mu \mu'}^T \Delta(x-x') \\ &= i \frac{2^{-11}}{(5!)^2} \text{tr} \left\{ C^+ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda \lambda'} \gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5]} C [(m - \gamma^b \partial_b) \gamma^0]^T \right\} \Delta(x-x') \\ &= i \frac{2^{-11}}{(5!)^2} \text{tr} \left\{ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} [(m - \gamma^a \partial_a) \gamma^0]_{\lambda \lambda'} \gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5]} C [(m - \gamma^b \partial_b) \gamma^0]^T C^+ \right\} \Delta(x-x') \end{aligned}$$

$$\begin{aligned}
&= -i \frac{2^{-11}}{(5!)^2} \text{tr} \{ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} [ (m - \gamma^a \partial_a) \gamma^0 ]_{\lambda\lambda'} \gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5]} [ \gamma^0 (m + \gamma^b \partial_b) ] \} \Delta(x - x') \\
&= -i \frac{2^{-11}}{(5!)^2} \text{tr} \{ \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]} [ (m - \gamma^a \partial_a) \gamma^0 ]_{\lambda\lambda'} \gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5]} [ (m - \gamma^b \partial_b^+) \gamma^0 ] \} \Delta(x - x') \\
&= -i \frac{2^{-11}}{(5!)^2} \text{tr} \{ m^2 (\gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]}) \gamma^0 (\gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5]}) \gamma^0 \} \Delta(x - x') \\
&\quad - i \frac{2^{-11}}{(5!)^2} \text{tr} \{ (\gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4} \gamma_{a_5]}) \gamma_a \gamma_0 (\gamma_{[a'_1} \gamma_{a'_2} \gamma_{a'_3} \gamma_{a'_4} \gamma_{a'_5]}) \gamma_b \gamma_0 \} \partial^a \partial^b \Delta(x - x') \\
&= -i \frac{2^{-11}}{(5!)^2} i^{5*6} 2^5 (5!)^2 m^2 \frac{1}{5!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_5]}^{a_5]} \Delta(x - x') \\
&\quad - i \frac{2^{-11}}{(5!)^2} i^{6*7} 2^5 (5!)^2 \{ \frac{1}{6!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_5]}^{a_5} \eta_b] - \frac{1}{4!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_4}^{a_4} \delta^{a_5]a} \delta_{a'_5]b} \} \partial_a \partial^{+b} \Delta(x - x') \\
&= \frac{i}{2^6} \{ \frac{1}{5!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_5]}^{a_5]} m^2 + (\frac{1}{6!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_5]}^{a_5} \eta_b] - \frac{1}{4!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_4}^{a_4} \delta^{a_5]a} \delta_{a'_5]b}) \partial_a \partial^{+b} \} \Delta(x - x') \\
&= \frac{i}{2^6} \{ \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_5]}^{a_5]} m^2 + (\eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_5]}^{a_5} \eta_b] - \frac{1}{4!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_4}^{a_4} \delta^{a_5]a} \delta_{a'_5]b}) \partial_a \partial^{+b} \} \Delta(x - x') \\
&= \frac{i}{2^6} \{ \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_5]}^{a_5]} m^2 + (\eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_5]}^{a_5} \eta_b] \partial_a \partial^{+b} - \frac{1}{4!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_4}^{a_4} \delta^{a_5]a} \delta_{a'_5]b}^+) \} \Delta(x - x') \\
&= \frac{i}{2^5} \{ \frac{1}{5!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_5]}^{a_5]} m^2 - \frac{1}{4!} \eta_{[a'_1}^{[a_1} \cdots \eta_{a'_4}^{a_4} \delta^{a_5]a} \delta_{a'_5]b}^+ \} \Delta(x - x') \\
&= \frac{i}{2^5} \frac{1}{6!} \eta_{[a'_1}^{[a_1} \eta_{a'_2}^{a_2} \cdots \eta_{a'_5]}^{a_5} \eta_{a'_6]} \partial_{a_6} \partial^{+a_6} \Delta(x - x') \\
&= \frac{i}{2^5} \frac{1}{6!} \eta_{[a_1} \langle a'_1 \eta_{a_2} a'_2 \cdots \eta_{a_5} a'_5 \eta_{a_6} \rangle a'_6 \} \partial^{a_6} \partial^{+a'_6} \Delta(x - x') \quad \square
\end{aligned}$$

$$\text{推论 6.11.3. } [F_{a_1 a_2 \cdots a_6}(x), F_{a'_1 a'_2 \cdots a'_6}^+(x')] = -i \frac{2^{-5}}{5!} \eta_{[a_1} \langle a'_1 \eta_{a_2} a'_2 \cdots \eta_{a_5} a'_5 \partial_{a_6} \rangle a'_6 \} \Delta(x - x')$$

## 7 奇数维时空中的对称和反对称B-W矢量场方程

### 7.1 三维时空中的对称Bargmann-Wigner矢量场方程

$$\text{引理 7.1.1. } (\gamma^a \partial_a + m)X(3) = 0, X(3) = \frac{1}{(1!)^2} F^a \gamma_a C \Leftrightarrow \varepsilon^{abc} \partial_b F_c - imF^a = 0, \partial_a F^a = 0$$

$$\text{证明: } (\gamma^a \partial_a + m)X(3) = 0, X(3) = \frac{1}{(1!)^2} F^a \gamma_a C$$

$$\Leftrightarrow (\gamma_a \partial^a + m) \frac{1}{(1!)^2} F^b \gamma_b C = 0$$

$$\Leftrightarrow (\gamma_a \partial^a + m) F^b \gamma_b = 0$$

$$\Leftrightarrow \gamma_a \gamma_b \partial^a F^b + m F^b \gamma_b = 0$$

$$\Leftrightarrow \{ \frac{1}{2!} \gamma_{[a} \gamma_b] + \delta_{ab} \} \partial^a F^b + \frac{1}{1!} m F^b \gamma_b = 0$$

$$\Leftrightarrow \frac{1}{2!} \gamma_{[a} \gamma_b] \partial^a F^b + \delta_{ab} \partial^a F^b + \frac{1}{1!} m F^b \gamma_b = 0$$

$$\Leftrightarrow i \varepsilon^{abc} \frac{1}{(1!)^2} \gamma_c \partial_a F_b + \delta_{ab} \partial^a F^b + \frac{1}{1!} m F^c \gamma_c = 0$$

$$\Leftrightarrow \varepsilon^{abc} \partial_a F_b - imF^c = 0, \partial_a F^a = 0$$

$$\Leftrightarrow \varepsilon^{abc} \partial_b F_c - imF^a = 0, \partial_a F^a = 0 \quad \square$$

$$\text{推论 7.1.1. } \gamma^a \partial_a X(3) = 0, X(3) = \frac{1}{(1!)^2} F^a \gamma_a C \Leftrightarrow \varepsilon^{abc} \partial_b F_c = 0, \partial_a F^a = 0 \Rightarrow \partial_b \partial^b F^a = 0$$

$$\text{推论 7.1.2. } (\gamma^a \partial_a + m)X(3) = 0, X(3) = \frac{1}{(1!)^2} F^a \gamma_a C, m \neq 0 \Leftrightarrow \varepsilon^{abc} \partial_b F_c - imF^a = 0 \Rightarrow \partial_b \partial^b F^a - m^2 F^a = 0$$

$$\text{定理 7.1.1. } (\gamma^a \partial_a + m)X(3) = 0, X(3) = X^T(3), m \neq 0 \Leftrightarrow \varepsilon^{abc} \partial_b F_c - imF^a = 0, X(3) = \frac{1}{(1!)^2} F^a \gamma_a C$$

$$\text{推论 7.1.3. } \varepsilon^{abc} \partial_b F_c - imF^a = 0 \Leftrightarrow \partial_a * F^{ab} + imF^b = 0 \Leftrightarrow \partial_{[a} F_{b]} - im * F_{ab} = 0$$

### 7.2 三维时空中的反对称Bargmann-Wigner矢量场方程

$$\text{引理 7.2.1. } (\gamma^a \partial_a + m)X(3) = 0, X(3) = \frac{1}{(0!)^2} FC \Leftrightarrow \partial^a F = 0, mF = 0$$

$$\text{证明: } (\gamma^a \partial_a + m)X(3) = 0, X(3) = \frac{1}{(0!)^2} FC$$

$$\Leftrightarrow (\gamma_a \partial^a + m) \frac{1}{(0!)^2} FC = 0$$

$$\Leftrightarrow (\gamma_a \partial^a + m)F = 0$$

$$\Leftrightarrow \partial^a F = 0, mF = 0 \quad \square$$

$$\text{推论 7.2.1. } \gamma^a \partial_a \frac{1}{(0!)^2} FC = 0 \Leftrightarrow \partial^a F = 0$$

$$\text{推论 7.2.2. } (\gamma^a \partial_a + m) \frac{1}{(0!)^2} FC = 0, m \neq 0 \Leftrightarrow F = 0$$

$$\text{定理 7.2.1. } (\gamma^a \partial_a + m)X(3) = 0, X(3) = -X^T(3), m \neq 0 \Leftrightarrow F = 0, X(3) = \frac{1}{(0!)^2} FC = 0$$

$$\text{推论 7.2.3. } F = 0 \Leftrightarrow *F_{ab} = 0$$

### 7.3 五维时空中的对称Bargmann-Wigner矢量场方程

引理7.3.1.  $(\gamma^a \partial_a + m)X(5) = 0, X(5) = \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} C \Leftrightarrow \varepsilon^{abcde} \partial_c F_{de} - mF^{ab} = 0, \partial_a F^{ab} = 0$

证明:  $(\gamma^a \partial_a + m)X(5) = 0, X(5) = \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} C$   
 $\Leftrightarrow (\gamma_a \partial^a + m) \frac{1}{(2!)^2} F^{bc} \gamma_{[b} \gamma_{c]} C = 0$   
 $\Leftrightarrow (\gamma_a \partial^a + m) F^{bc} \gamma_b \gamma_c = 0$   
 $\Leftrightarrow \gamma_a \gamma_b \gamma_c \partial^a F^{bc} + mF^{bc} \gamma_b \gamma_c = 0$   
 $\Leftrightarrow \left\{ \frac{1}{3!} \gamma_{[a} \gamma_b \gamma_{c]} + (\delta_{a[b} \gamma_{c]} + \gamma_a \delta_{bc}) \right\} \partial^a F^{bc} + \frac{1}{2!} mF^{bc} \gamma_{[b} \gamma_{c]} = 0$   
 $\Leftrightarrow \frac{1}{3!} \gamma_{[a} \gamma_b \gamma_{c]} \partial^a F^{bc} + 2\delta_{ab} \gamma_c \partial^a F^{bc} + \frac{1}{2!} mF^{bc} \gamma_{[b} \gamma_{c]} = 0$   
 $\Leftrightarrow -\varepsilon^{abcde} \frac{1}{(2!)^2} \gamma_{[d} \gamma_{e]} \partial_a F_{bc} + 2\delta_{ab} \gamma_c \partial^a F^{bc} + \frac{1}{2!} mF^{de} \gamma_{[d} \gamma_{e]} = 0$   
 $\Leftrightarrow \frac{1}{2!} \varepsilon^{abcde} \partial_a F_{bc} - mF^{de} = 0, \partial_a F^{ab} = 0$   
 $\Leftrightarrow \frac{1}{2!} \varepsilon^{abcde} \partial_c F_{de} - mF^{ab} = 0, \partial_a F^{ab} = 0$  □

推论7.3.1.  $\gamma^a \partial_a X(5) = 0, X(5) = \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} C \Leftrightarrow \varepsilon^{abcde} \partial_c F_{de} = 0, \partial_a F^{ab} = 0$

推论7.3.2.  $(\gamma^a \partial_a + m)X(5) = 0, X(5) = \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} C, m \neq 0 \Leftrightarrow \frac{1}{2!} \varepsilon^{abcde} \partial_c F_{de} - mF^{ab} = 0$

定理7.3.1.  $(\gamma^a \partial_a + m)X(5) = 0, X(5) = X^T(5), m \neq 0 \Leftrightarrow \frac{1}{2!} \varepsilon^{abcde} \partial_c F_{de} - mF^{ab} = 0, X(5) = \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} C$

推论7.3.3.  $\frac{1}{2!} \varepsilon^{abcde} \partial_c F_{de} - mF^{ab} = 0 \Leftrightarrow \partial_a *F^{abc} - mF^{bc} = 0 \Leftrightarrow \frac{1}{2!} \partial_{[a} F_{bc]} - m*F_{abc} = 0$

### 7.4 五维时空中的反对称Bargmann-Wigner矢量场方程

引理7.4.1. 
$$\begin{cases} (\gamma^a \partial_a + m)X(5) = 0 \\ X(5) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} \right\} C \end{cases} \Leftrightarrow \begin{cases} \frac{1}{0!} \partial^{a_0} F + mF^{a_0} = 0 \\ \partial^{[a_0} F^{a_1]} = 0, \partial_{a_0} F^{a_0} + mF = 0 \end{cases}$$

推论7.4.1. 
$$\begin{cases} \gamma^a \partial_a X(5) = 0 \\ X(5) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} \right\} C \end{cases} \Leftrightarrow \begin{cases} \partial^{a_0} F = 0 \\ \partial^{[a_0} F^{a_1]} = 0, \partial_{a_0} F^{a_0} = 0 \end{cases}$$

推论7.4.2. 
$$\begin{cases} (\gamma^a \partial_a + m)X(5) = 0, m \neq 0 \\ X(5) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} \right\} C \end{cases} \Leftrightarrow \begin{cases} \frac{1}{0!} \partial^{a_0} F + mF^{a_0} = 0 \\ \partial_{a_0} F^{a_0} + mF = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F - m^2 F = 0 \\ F^{a_0} = -\frac{1}{0!m} \partial^{a_0} F \end{cases}$$

定理7.4.1. 
$$\begin{cases} (\gamma^a \partial_a + m)X(5) = 0, m \neq 0 \\ X(5) = -X^T(5) \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F - m^2 F = 0 \\ X(5) = \left\{ \frac{1}{0!} F - \frac{1}{1!m} \gamma_{a_1} \partial^{a_1} \right\} C \end{cases}$$

推论7.4.3. 
$$\begin{cases} \frac{1}{0!} \partial^{a_0} F + mF^{a_0} = 0 \\ \partial_{a_0} F^{a_0} + mF = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{4!} \partial^{[a_0} *F^{a_1 \dots a_4]} + m*F^{a_0 a_1 \dots a_4} = 0 \\ \partial_{a_0} *F^{a_0 a_1 \dots a_4} + m*F^{a_1 \dots a_4} = 0 \end{cases}$$

### 7.5 七维时空中的对称Bargmann-Wigner矢量场方程

引理7.5.1.  $(\gamma^{a_0} \partial_{a_0} + m)X(7) = 0, X(7) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \dots \gamma_{a_3]} \right\} C$   
 $\Leftrightarrow \varepsilon^{a_1 \dots a_7} \partial_{a_4} F_{a_5 \dots a_7} - imF^{a_1 \dots a_3} = 0, \partial_{a_1} F^{a_1 \dots a_3} = 0$

证明:  $(\gamma^{a_0} \partial_{a_0} + m)X(7) = 0, X(7) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \dots \gamma_{a_3]} \right\} C$   
 $\Leftrightarrow (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \dots \gamma_{a_3]} \right\} C = 0$   
 $\Leftrightarrow F = 0, (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \dots \gamma_{a_3]} \right\} C = 0$   
 $\Leftrightarrow \frac{1}{3!} \varepsilon^{a_1 \dots a_7} \partial_{a_4} F_{a_5 \dots a_7} - imF^{a_1 \dots a_3} = 0, \partial_{a_1} F^{a_1 \dots a_3} = 0$  □

推论7.5.1.  $\gamma^{a_0} \partial_{a_0} = 0, X(7) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \dots \gamma_{a_3]} \right\} C$   
 $\Leftrightarrow \varepsilon^{a_1 \dots a_7} \partial_{a_4} F_{a_5 \dots a_7} = 0, \partial_{a_1} F^{a_1 \dots a_3} = 0$

$$\text{推论7.5.2. } (\gamma^{a_0} \partial_{a_0} + m)X(7) = 0, X(7) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(3!)^2} F^{a_1 \cdots a_3} \gamma_{[a_1} \cdots \gamma_{a_3]} \right\} C, m \neq 0$$

$$\Leftrightarrow \frac{1}{3!} \varepsilon^{a_1 \cdots a_7} \partial_{a_4} F_{a_5 \cdots a_7} - im F^{a_1 \cdots a_3} = 0$$

$$\text{定理7.5.1. } (\gamma^a \partial_a + m)X(7) = 0, X(7) = X^T(7), m \neq 0$$

$$\Leftrightarrow \frac{1}{3!} \varepsilon^{a_1 \cdots a_7} \partial_{a_4} F_{a_5 \cdots a_7} - im F^{a_1 \cdots a_3} = 0, X(7) = \frac{1}{(3!)^2} F^{a_1 \cdots a_3} \gamma_{[a_1} \cdots \gamma_{a_3]} C$$

推论7.5.3.

$$\frac{1}{3!} \varepsilon^{a_1 \cdots a_7} \partial_{a_4} F_{a_5 \cdots a_7} - im F^{a_1 \cdots a_3} = 0 \Leftrightarrow \partial_{a_0} * F^{a_0 \cdots a_3} + im F^{a_1 \cdots a_3} = 0 \Leftrightarrow \frac{1}{3!} \partial_{[a_0} F_{a_1 \cdots a_3]} - im * F_{a_0 \cdots a_3} = 0$$

## 7.6 七维时空中的反对称Bargmann-Wigner矢量场方程

$$\text{引理7.6.1. } \begin{cases} (\gamma^a \partial_a + m)X(7) = 0 \\ X(7) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \right\} C \end{cases} \Leftrightarrow \begin{cases} \partial^{[a} F^{b]} + m F^{ab} = 0, \partial_a F^a = 0 \\ \partial^{[a} F^{bc]} = 0, \partial_a F^{ab} + m F^b = 0 \end{cases}$$

$$\text{推论7.6.1. } \begin{cases} \gamma^a \partial_a X(7) = 0 \\ X(7) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \right\} C \end{cases} \Leftrightarrow \begin{cases} \partial^{[a} F^{b]} = 0, \partial_a F^a = 0 \\ \partial^{[a} F^{bc]} = 0, \partial_a F^{ab} = 0 \end{cases}$$

$$\text{推论7.6.2. } (\gamma^a \partial_a + m)X(7) = 0, X(7) = \left\{ \frac{1}{(1!)^2} F^a \gamma_a + \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} \right\} C, m \neq 0$$

$$\Leftrightarrow \partial_a F^{ab} + m F^b = 0, \partial^{[a} F^{b]} + m F^{ab} = 0 \Leftrightarrow \partial_b \partial^b F^a - m^2 F^a = 0, \partial_a F^a = 0, F^{ab} = -\frac{1}{m} \partial^{[a} F^{b]}$$

$$\text{定理7.6.1. } \begin{cases} (\gamma^a \partial_a + m)X(7) = 0 \\ X(7) = -X^T(7) \end{cases} \Leftrightarrow \begin{cases} \partial_b \partial^b F^a - m^2 F^a = 0, \partial_a F^a = 0 \\ X(7) = \left\{ \frac{1}{1!} \gamma_a + \frac{1}{2!m} \gamma_{[a} \gamma_{b]} \partial^b \right\} C F^a \end{cases}$$

$$\text{推论7.6.3. } \begin{cases} \frac{1}{1!} \partial^{[a_0} F^{a_1]} + m F^{a_0 a_1} = 0 \\ \partial_{a_0} F^{a_0 a_1} + m F^{a_1} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{5!} \partial^{[a_0} * F^{a_1 \cdots a_5]} - m * F^{a_0 a_1 \cdots a_5} = 0 \\ \partial_{a_0} * F^{a_0 a_1 \cdots a_5} - m * F^{a_1 \cdots a_5} = 0 \end{cases}$$

## 7.7 九维时空中的对称Bargmann-Wigner矢量场方程

$$\text{引理7.7.1. } (\gamma^{a_0} \partial_{a_0} + m)X(9) = 0, X(9) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(4!)^2} F^{a_1 a_2 a_3 a_4} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} \right\} C$$

$$\Leftrightarrow \begin{cases} \partial^{a_1} F + m F^{a_1} = 0, \partial^{[a_0} F^{a_1]} = 0, \partial_{a_1} F^{a_1} + m F = 0 \\ \frac{1}{4!} \varepsilon^{a_1 \cdots a_9} \partial_{a_5} F_{a_6 \cdots a_9} - m F^{a_1 \cdots a_4} = 0, \partial_{a_1} F^{a_1 \cdots a_4} = 0 \end{cases}$$

$$\text{证明: } (\gamma^{a_0} \partial_{a_0} + m)X(9) = 0, X(9) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(4!)^2} F^{a_1 a_2 a_3 a_4} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} \right\} C$$

$$\Leftrightarrow (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(4!)^2} F^{a_1 a_2 a_3 a_4} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} \right\} C = 0$$

$$\Leftrightarrow (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} \right\} C = 0, (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{(4!)^2} F^{a_1 a_2 a_3 a_4} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} \right\} C = 0$$

$$\Leftrightarrow \begin{cases} \partial^{a_1} F + m F^{a_1} = 0, \partial^{[a_0} F^{a_1]} = 0, \partial_{a_1} F^{a_1} + m F = 0 \\ \frac{1}{4!} \varepsilon^{a_1 \cdots a_9} \partial_{a_5} F_{a_6 \cdots a_9} - m F^{a_1 \cdots a_4} = 0, \partial_{a_1} F^{a_1 \cdots a_4} = 0 \end{cases} \quad \square$$

$$\text{推论7.7.1. } \gamma^{a_0} \partial_{a_0} X(9) = 0, X(9) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(4!)^2} F^{a_1 a_2 a_3 a_4} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} \right\} C$$

$$\Leftrightarrow \begin{cases} \partial^{a_1} F = 0, \partial^{[a_0} F^{a_1]} = 0, \partial_{a_1} F^{a_1} = 0 \\ \varepsilon^{a_1 \cdots a_9} \partial_{a_5} F_{a_6 \cdots a_9} = 0, \partial_{a_1} F^{a_1 \cdots a_4} = 0 \end{cases}$$

$$\text{推论7.7.2. } (\gamma^{a_0} \partial_{a_0} + m)X(9) = 0, X(9) = \left\{ \frac{1}{(0!)^2} F + \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(4!)^2} F^{a_1 a_2 a_3 a_4} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} \right\} C, m \neq 0$$

$$\Leftrightarrow \begin{cases} \partial^{a_1} F + m F^{a_1} = 0, \partial_{a_1} F^{a_1} + m F = 0 \Leftrightarrow \partial_{a_1} \partial^{a_1} F - m^2 F = 0, F^{a_1} = -\frac{1}{m} \partial^{a_1} F \\ \frac{1}{4!} \varepsilon^{a_1 \cdots a_9} \partial_{a_5} F_{a_6 \cdots a_9} - m F^{a_1 \cdots a_4} = 0 \end{cases}$$

$$\text{定理7.7.1. } \begin{cases} (\gamma^a \partial_a + m)X(9) = 0 \\ X(9) = X^T(9) \end{cases} \Leftrightarrow \begin{cases} \partial_{a_1} \partial^{a_1} F - m^2 F = 0, \frac{1}{4!} \varepsilon^{a_1 \cdots a_9} \partial_{a_5} F_{a_6 \cdots a_9} - m F^{a_1 \cdots a_4} = 0 \\ X(9) = \left\{ \left(1 - \frac{1}{m} \gamma_{a_1} \partial^{a_1}\right) F + \frac{1}{(4!)^2} F^{a_1 a_2 a_3 a_4} \gamma_{[a_1} \gamma_{a_2} \gamma_{a_3} \gamma_{a_4]} \right\} C \end{cases}$$

$$\text{推论7.7.3. } \frac{1}{4!} \varepsilon^{a_1 \cdots a_9} \partial_{a_5} F_{a_6 \cdots a_9} - m F^{a_1 \cdots a_4} = 0 \Leftrightarrow \partial_{a_0} * F^{a_0 \cdots a_4} - m F^{a_1 \cdots a_4} = 0 \Leftrightarrow \frac{1}{4!} \partial_{[a_0} F_{a_1 \cdots a_4]} - m * F_{a_0 \cdots a_4} = 0$$



## 7.8 九维时空中的反对称Bargmann-Wigner矢量场方程

引理7.8.1.

$$\begin{cases} (\gamma^a \partial_a + m)X(9) = 0 \\ X(9) = \left\{ \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \cdot \cdot \gamma_{a_3]} \right\} C \end{cases} \Leftrightarrow \begin{cases} \frac{1}{2!} \partial^{[a_0} F^{a_1 a_2]} + m F^{a_0 a_1 a_2} = 0, \partial_{a_1} F^{a_1 a_2} = 0 \\ \partial^{[a_0} F^{a_1 a_2 a_3]} = 0, \partial_{a_0} F^{a_0 a_1 a_2} + m F^{a_1 a_2} = 0 \end{cases}$$

推论7.8.1.

$$\begin{cases} \gamma^a \partial_a X(9) = 0 \\ X(9) = \left\{ \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \cdot \cdot \gamma_{a_3]} \right\} C \end{cases} \Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1 a_2]} = 0, \partial_{a_1} F^{a_1 a_2} = 0 \\ \partial^{[a_0} F^{a_1 a_2 a_3]} = 0, \partial_{a_0} F^{a_0 a_1 a_2} = 0 \end{cases}$$

推论7.8.2.  $(\gamma^a \partial_a + m)X(9) = 0, m \neq 0, X(9) = \left\{ \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(3!)^2} F^{a_1 \dots a_3} \gamma_{[a_1} \cdot \cdot \gamma_{a_3]} \right\} C$ 

$$\Leftrightarrow \begin{cases} \frac{1}{2!} \partial^{[a_0} F^{a_1 a_2]} + m F^{a_0 a_1 a_2} = 0 \\ \partial_{a_0} F^{a_0 a_1 a_2} + m F^{a_1 a_2} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1 a_2} - m^2 F^{a_1 a_2} = 0, \partial_{a_1} F^{a_1 a_2} = 0 \\ F^{a_0 a_1 a_2} = -\frac{1}{2!m} \partial^{[a_0} F^{a_1 a_2]} \end{cases}$$

定理7.8.1.

$$\begin{cases} (\gamma^a \partial_a + m)X(9) = 0, m \neq 0 \\ X(9) = -X^T(9) \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1 a_2} - m^2 F^{a_1 a_2} = 0, \partial_{a_1} F^{a_1 a_2} = 0 \\ X(9) = \left\{ \frac{1}{2!} \gamma_{[a_1} \gamma_{a_2]} - \frac{1}{3!m} F^{a_1 \dots a_3} \gamma_{[a_1} \cdot \cdot \gamma_{a_3]} \partial^{a_3} \right\} C \frac{1}{2!} F^{a_1 a_2} \end{cases}$$

推论7.8.3.

$$\begin{cases} \frac{1}{2!} \partial^{[a_0} F^{a_1 a_2]} + m F^{a_0 a_1 a_2} = 0 \\ \partial_{a_0} F^{a_0 a_1 a_2} + m F^{a_1 a_2} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{6!} \partial^{[a_0} * F^{a_1 \dots a_6]} + m * F^{a_0 a_1 \dots a_6} = 0 \\ \partial_{a_0} * F^{a_0 a_1 \dots a_6} + m * F^{a_1 \dots a_6} = 0 \end{cases}$$

## 7.9 十一维时空中的对称Bargmann-Wigner矢量场方程

引理7.9.1.  $(\gamma^{a_0} \partial_{a_0} + m)X(11) = 0, X(11) = \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \cdot \cdot \gamma_{a_5]} \right\} C$ 

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1]} + m F^{a_0 a_1} = 0, \partial_{a_1} F^{a_1} = 0; \partial^{[a_0} F^{a_1 a_2]} = 0, \partial_{a_0} F^{a_0 a_1} + m F^{a_1} = 0 \\ \frac{1}{5!} \varepsilon^{a_1 \dots a_{11}} \partial_{a_6} F_{a_7 \dots a_{11}} - m F^{a_1 \dots a_5} = 0, \partial_{a_1} F^{a_1 \dots a_5} = 0 \end{cases}$$

证明:  $(\gamma^{a_0} \partial_{a_0} + m)X(11) = 0, X(11) = \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \cdot \cdot \gamma_{a_5]} \right\} C$ 

$$\Leftrightarrow (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \cdot \cdot \gamma_{a_5]} \right\} C = 0$$

$$\Leftrightarrow (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} \right\} C = 0, (\gamma_{a_0} \partial^{a_0} + m) \left\{ \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \cdot \cdot \gamma_{a_5]} \right\} C = 0$$

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1]} + m F^{a_0 a_1} = 0, \partial_{a_1} F^{a_1} = 0; \partial^{[a_0} F^{a_1 a_2]} = 0, \partial_{a_0} F^{a_0 a_1} + m F^{a_1} = 0 \\ \frac{1}{5!} \varepsilon^{a_1 \dots a_{11}} \partial_{a_6} F_{a_7 \dots a_{11}} - m F^{a_1 \dots a_5} = 0, \partial_{a_1} F^{a_1 \dots a_5} = 0 \end{cases} \quad \square$$

推论7.9.1.  $\gamma^{a_0} \partial_{a_0} X(11) = 0, X(11) = \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \cdot \cdot \gamma_{a_5]} \right\} C$ 

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1]} = 0, \partial_{a_1} F^{a_1} = 0, \partial^{[a_0} F^{a_1 a_2]} = 0, \partial_{a_0} F^{a_0 a_1} = 0 \\ \varepsilon^{a_1 \dots a_{11}} \partial_{a_6} F_{a_7 \dots a_{11}} = 0, \partial_{a_1} F^{a_1 \dots a_5} = 0 \end{cases}$$

推论7.9.2.  $(\gamma^{a_0} \partial_{a_0} + m)X(11) = 0, X(11) = \left\{ \frac{1}{(1!)^2} F^{a_1} \gamma_{a_1} + \frac{1}{(2!)^2} F^{a_1 a_2} \gamma_{[a_1} \gamma_{a_2]} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \cdot \cdot \gamma_{a_5]} \right\} C, m \neq 0$ 

$$\Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1]} + m F^{a_0 a_1} = 0, \partial_{a_0} F^{a_0 a_1} + m F^{a_1} = 0 \\ \varepsilon^{a_1 \dots a_{11}} \partial_{a_6} F_{a_7 \dots a_{11}} - m F^{a_1 \dots a_5} = 0 \\ \partial_{a_0} \partial^{a_0} F^{a_1} - m^2 F^{a_1} = 0, \partial_{a_1} F^{a_1} = 0, F^{a_0 a_1} = -\frac{1}{m} \partial^{[a_0} F^{a_1]} \\ \frac{1}{5!} \varepsilon^{a_1 \dots a_{11}} \partial_{a_6} F_{a_7 \dots a_{11}} - m F^{a_1 \dots a_5} = 0 \end{cases}$$

定理7.9.1.  $\begin{cases} (\gamma^a \partial_a + m)X(11) = 0 \\ X(11) = X^T(11), m \neq 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1} - m^2 F^{a_1} = 0, \frac{1}{5!} \varepsilon^{a_1 \dots a_{11}} \partial_{a_6} F_{a_7 \dots a_{11}} - m F^{a_1 \dots a_5} = 0 \\ X(11) = \left\{ \left( \frac{1}{1!} \gamma_{a_1} + \frac{1}{2!m} \gamma_{[a_1} \gamma_{a_2]} \partial^{a_2} \right) F^{a_1} + \frac{1}{(5!)^2} F^{a_1 \dots a_5} \gamma_{[a_1} \cdot \cdot \gamma_{a_5]} \right\} C \end{cases}$ 推论7.9.3.  $\frac{1}{5!} \varepsilon^{a_1 \dots a_{11}} \partial_{a_6} F_{a_7 \dots a_{11}} - m F^{a_1 \dots a_5} = 0 \Leftrightarrow \partial_{a_0} * F^{a_0 \dots a_5} + m F^{a_1 \dots a_5} = 0 \Leftrightarrow \frac{1}{5!} \partial_{[a_0} F_{a_1 \dots a_5]} - m * F_{a_0 \dots a_5} = 0$

## 7.10 十一维时空中的反对称Bargmann-Wigner矢量场方程

引理7.10.1.

$$\begin{cases} (\gamma^a \partial_a + m)X(11) = 0, X(11) = \\ \left\{ \frac{1}{(3!)^2} F^{a_1 \cdots a_3} \gamma_{[a_1 \cdots a_3]} + \frac{1}{(4!)^2} F^{a_1 \cdots a_4} \gamma_{[a_1 \cdots a_4]} \right\} C \end{cases} \Leftrightarrow \begin{cases} \frac{1}{3!} \partial^{[a_0} F^{a_1 a_2 a_3]} + m F^{a_0 a_1 a_2 a_3} = 0, \partial_{a_1} F^{a_1 a_2 a_3} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_4]} = 0, \partial_{a_0} F^{a_0 a_1 a_2 a_3} + m F^{a_1 a_2 a_3} = 0 \end{cases}$$

推论7.10.1.

$$\begin{cases} \gamma^a \partial_a X(11) = 0 \\ X(11) = \left\{ \frac{1}{(3!)^2} F^{a_1 \cdots a_3} \gamma_{[a_1 \cdots a_3]} + \frac{1}{(4!)^2} F^{a_1 \cdots a_4} \gamma_{[a_1 \cdots a_4]} \right\} C \end{cases} \Leftrightarrow \begin{cases} \partial^{[a_0} F^{a_1 a_2 a_3]} = 0, \partial_{a_1} F^{a_1 a_2 a_3} = 0 \\ \partial^{[a_0} F^{a_1 \cdots a_4]} = 0, \partial_{a_0} F^{a_0 a_1 a_2 a_3} = 0 \end{cases}$$

推论7.10.2.  $(\gamma^a \partial_a + m)X(11) = 0, m \neq 0, X(11) = \left\{ \frac{1}{(3!)^2} F^{a_1 \cdots a_3} \gamma_{[a_1 \cdots a_3]} + \frac{1}{(4!)^2} F^{a_1 \cdots a_4} \gamma_{[a_1 \cdots a_4]} \right\} C$

$$\Leftrightarrow \begin{cases} \frac{1}{3!} \partial^{[a_0} F^{a_1 a_2 a_3]} + m F^{a_0 a_1 a_2 a_3} = 0 \\ \partial_{a_0} F^{a_0 a_1 a_2 a_3} + m F^{a_1 a_2 a_3} = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_{a_0} \partial^{a_0} F^{a_1 a_2 a_3} - m^2 F^{a_1 a_2 a_3} = 0, \partial_{a_1} F^{a_1 a_2 a_3} = 0 \\ F^{a_0 a_1 a_2 a_3} = -\frac{1}{m} \partial^{[a_0} F^{a_1 a_2 a_3]} \end{cases}$$

定理7.10.1.

$$\begin{cases} (\gamma^a \partial_a + m)X(11) = 0, m \neq 0 \\ X(11) = -X^T(11) \end{cases} \Leftrightarrow \begin{cases} \partial_{a_1} \partial^{a_1} F^{a_1 a_2 a_3} - m^2 F^{a_1 a_2 a_3} = 0, \partial_{a_1} F^{a_1 a_2 a_3} = 0 \\ X(11) = \left\{ \frac{1}{3!} \gamma_{[a_1 \cdots a_3]} + \frac{1}{4!} \gamma_{[a_1 \cdots a_4]} \partial^{a_4} \right\} C \frac{1}{3!} F^{a_1 a_2 a_3} \end{cases}$$

推论7.10.3.

$$\begin{cases} \frac{1}{3!} \partial^{[a_0} F^{a_1 a_2 a_3]} + m F^{a_0 a_1 a_2 a_3} = 0 \\ \partial_{a_0} F^{a_0 a_1 a_2 a_3} + m F^{a_1 a_2 a_3} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{1}{7!} \partial^{[a_0 * F^{a_1 \cdots a_7]} - m * F^{a_0 a_1 \cdots a_7} = 0 \\ \partial_{a_0} * F^{a_0 a_1 \cdots a_7} - m * F^{a_1 \cdots a_7} = 0 \end{cases}$$

## 8 N+1=n偶维时空中反对称B-W矢量场方程

### 8.1 关于反对称关系的引理

引理8.1.1.  $\sum_{h=1}^l a_{\{\mu_\zeta\}}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0 \Leftrightarrow [a(\vec{p}; h, h') + a(\vec{p}; h', h)] = 0, c(\vec{p}; h, h') = 0$

证明:  $\sum_{h=1}^l a_{\{\mu_\zeta\}}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0$

$$\Leftrightarrow \sum_{h=1}^l u_{\{\lambda_\zeta\}}(\vec{p}, h) a_{\mu_\zeta}(\vec{p}, h) = 0$$

$$\Leftrightarrow \sum_{h, h'=1}^l u_{\{\lambda_\zeta\}}(\vec{p}, h) [a(\vec{p}; h, h') u_{\mu_\zeta}(\vec{p}, h') + c(\vec{p}; h, h') v_{\mu_\zeta}(\vec{p}, h')] = 0$$

$$\Leftrightarrow \sum_{h, h'=1}^l [a(\vec{p}; h, h') u_{\{\lambda_\zeta\}}(\vec{p}, h) u_{\mu_\zeta}(\vec{p}, h') + c(\vec{p}; h, h') u_{\{\lambda_\zeta\}}(\vec{p}, h) v_{\mu_\zeta}(\vec{p}, h')] = 0$$

$$\Leftrightarrow \sum_{h, h'=1}^l [a(\vec{p}; h, h') + a(\vec{p}; h', h)] u_{\lambda_\zeta}(\vec{p}, h) u_{\mu_\zeta}(\vec{p}, h') + \sum_{h, h'=1}^l c(\vec{p}; h, h') u_{\{\lambda_\zeta\}}(\vec{p}, h) v_{\mu_\zeta}(\vec{p}, h')] = 0$$

$$\Leftrightarrow [a(\vec{p}; h, h') + a(\vec{p}; h', h)] = 0, c(\vec{p}; h, h') = 0 \quad \square$$

### 8.2 N+1=n偶维时空中反对称B-W矢量场方程的平面波解

定理8.2.1.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta}(x) = -\psi_{\mu_\zeta \lambda_\zeta}(x)$

$\Leftrightarrow$

$$\psi_{\lambda_\zeta \mu_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{+\infty} \frac{m}{\sqrt{E}} \sum_{h, h'=1}^l [a(\vec{p}; h, h') \frac{1}{\sqrt{2}} u_{[\lambda_\zeta}(\vec{p}, h) u_{\mu_\zeta]}(\vec{p}, h') e^{ip \cdot x} + b^+(\vec{p}; h, h') \frac{1}{\sqrt{2}} v_{[\lambda_\zeta}(\vec{p}, h) v_{\mu_\zeta]}(\vec{p}, h') e^{-ip \cdot x}] d^N \vec{p}$$

证明:

$$(\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \psi_{\lambda_\zeta \mu_\zeta}(x) = 0, \psi_{\lambda_\zeta \mu_\zeta}(x) = -\psi_{\mu_\zeta \lambda_\zeta}(x)$$

$\Leftrightarrow$

$$\psi_{\lambda_\zeta \mu_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{+\infty} \sqrt{\frac{m}{E}} \sum_{h=1}^l [a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) e^{ip \cdot x} + b_{\mu_\zeta}^+(\vec{p}, h) v_{\lambda_\zeta}(\vec{p}, h) e^{-ip \cdot x}] d^N \vec{p}, \psi_{\lambda_\zeta \mu_\zeta}(x) = -\psi_{\mu_\zeta \lambda_\zeta}(x)$$

$\Leftrightarrow$

$$\begin{cases}
\psi_{\lambda_\zeta, \mu_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{+\infty} \sqrt{\frac{m}{E}} \sum_{h=1}^l [a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) e^{ip \cdot x} + b_{\mu_\zeta}^+(\vec{p}, h) v_{\lambda_\zeta}(\vec{p}, h) e^{-ip \cdot x}] d^N \vec{p} \\
\int_{-\infty}^{+\infty} \sqrt{\frac{m}{E}} \sum_{h=1}^l [a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) e^{ip \cdot x} + b_{\mu_\zeta}^+(\vec{p}, h) v_{\lambda_\zeta}(\vec{p}, h) e^{-ip \cdot x}] d^N \vec{p} \\
= - \int_{-\infty}^{+\infty} \sqrt{\frac{m}{E}} \sum_{h=1}^l [a_{\lambda_\zeta}(\vec{p}, h) u_{\mu_\zeta}(\vec{p}, h) e^{ip \cdot x} + b_{\lambda_\zeta}^+(\vec{p}, h) v_{\mu_\zeta}(\vec{p}, h) e^{-ip \cdot x}] d^N \vec{p} \\
\Leftrightarrow \\
\psi_{\lambda_\zeta, \mu_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{+\infty} \sqrt{\frac{m}{E}} \sum_{h=1}^l [a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) e^{ip \cdot x} + b_{\mu_\zeta}^+(\vec{p}, h) v_{\lambda_\zeta}(\vec{p}, h) e^{-ip \cdot x}] d^N \vec{p} \\
\int_{-\infty}^{+\infty} \sqrt{\frac{m}{E}} \sum_{h=1}^l [a_{\{\mu_\zeta\}}(\vec{p}, h) u_{\{\lambda_\zeta\}}(\vec{p}, h) e^{ip \cdot x} + b_{\{\mu_\zeta\}}^+(\vec{p}, h) v_{\{\lambda_\zeta\}}(\vec{p}, h) e^{-ip \cdot x}] d^N \vec{p} = 0 \\
\Leftrightarrow \\
\psi_{\lambda_\zeta, \mu_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{+\infty} \sqrt{\frac{m}{E}} \sum_{h=1}^l [a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) e^{ip \cdot x} + b_{\mu_\zeta}^+(\vec{p}, h) v_{\lambda_\zeta}(\vec{p}, h) e^{-ip \cdot x}] d^N \vec{p} \\
\sum_{h=1}^l a_{\{\mu_\zeta\}}(\vec{p}, h) u_{\{\lambda_\zeta\}}(\vec{p}, h) = 0, \sum_{h=1}^l b_{\{\mu_\zeta\}}^+(\vec{p}, h) v_{\{\lambda_\zeta\}}(\vec{p}, h) = 0 \\
\Leftrightarrow \\
\psi_{\lambda_\zeta, \mu_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{+\infty} \sqrt{\frac{m}{E}} \sum_{h, h'=1}^l [a(\vec{p}; h, h') u_{\lambda_\zeta}(\vec{p}, h) u_{\mu_\zeta}(\vec{p}, h') e^{ip \cdot x} + b^+(\vec{p}; h, h') v_{\lambda_\zeta}(\vec{p}, h) v_{\mu_\zeta}(\vec{p}, h') e^{-ip \cdot x}] d^N \vec{p} \\
a(\vec{p}; h, h') + a(\vec{p}; h', h) = 0, b^+(\vec{p}; h, h') + b^+(\vec{p}; h', h) = 0 \\
\Leftrightarrow \\
\psi_{\lambda_\zeta, \mu_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{+\infty} \sqrt{\frac{m}{E}} \sum_{h, h'=1}^l [a(\vec{p}; h, h') \frac{1}{2!} u_{[\lambda_\zeta}(\vec{p}, h) u_{\mu_\zeta]}(\vec{p}, h') e^{ip \cdot x} + b^+(\vec{p}; h, h') \frac{1}{2!} v_{[\lambda_\zeta}(\vec{p}, h) v_{\mu_\zeta]}(\vec{p}, h') e^{-ip \cdot x}] d^N \vec{p} \\
a(\vec{p}; h, h') + a(\vec{p}; h', h) = 0, b^+(\vec{p}; h, h') + b^+(\vec{p}; h', h) = 0 \\
\Leftrightarrow \\
\psi_{\lambda_\zeta, \mu_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{+\infty} \frac{m}{\sqrt{E}} \sum_{h, h'=1}^l [a(\vec{p}; h, h') \frac{1}{\sqrt{2}} u_{[\lambda_\zeta}(\vec{p}, h) u_{\mu_\zeta]}(\vec{p}, h') e^{ip \cdot x} + b^+(\vec{p}; h, h') \frac{1}{\sqrt{2}} v_{[\lambda_\zeta}(\vec{p}, h) v_{\mu_\zeta]}(\vec{p}, h') e^{-ip \cdot x}] d^N \vec{p} \\
a(\vec{p}; h, h') = \frac{\sqrt{2}}{\sqrt{m}} a(\vec{p}; h, h'), b^+(\vec{p}; h, h') = \frac{\sqrt{2}}{\sqrt{m}} b^+(\vec{p}; h, h')
\end{cases}$$

□

## 9 N+1维时空中反对称张量场的直接求解

### 9.1 N+1维时空中的电子方程<sup>[4]</sup>

偶数维时空中的电子方程:

$$\text{定义9.1.1. } (\gamma^a \partial_a + m)\psi = 0, \psi = \begin{bmatrix} \varphi \\ \eta \end{bmatrix}, \gamma^a = (\Gamma \otimes \sigma_y, \varsigma I_* \otimes \sigma_z) = (\vec{\gamma}, \varsigma I_* \otimes \sigma_z)$$

奇数维时空中的电子方程:

$$\text{定义9.1.2. } (\gamma^a \partial_a + m)\psi = 0, \psi = \begin{bmatrix} \varphi \\ \eta \end{bmatrix}, \gamma^a = (\Gamma \otimes \sigma_y, I_* \otimes \sigma_x, \varsigma I_* \otimes \sigma_z) = (\vec{\gamma}, \varsigma I_* \otimes \sigma_z)$$

### 9.2 N+1维时空中的电子自旋基

$$\text{定义9.2.1. } u_\zeta(\vec{p}, h) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p} \cdot \vec{\gamma} \gamma_0}{E+m}\right) \begin{bmatrix} 1 \\ 0 \\ 0_{l-2} \\ 0_l \end{bmatrix}, \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p} \cdot \vec{\gamma} \gamma_0}{E+m}\right) \begin{bmatrix} 0 \\ 1 \\ 0_{l-2} \\ 0_l \end{bmatrix}, \dots, \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p} \cdot \vec{\gamma} \gamma_0}{E+m}\right) \begin{bmatrix} 0_{l-2} \\ 0 \\ 1 \\ 0_l \end{bmatrix}$$

$$\text{定义9.2.2. } v_\zeta(\vec{p}, h) := \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p} \cdot \vec{\gamma} \gamma_0}{E+m}\right) \begin{bmatrix} 0_l \\ 1 \\ 0 \\ 0_{l-2} \end{bmatrix}, \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p} \cdot \vec{\gamma} \gamma_0}{E+m}\right) \begin{bmatrix} 0_l \\ 0 \\ 1 \\ 0_{l-2} \end{bmatrix}, \dots, \sqrt{\frac{E+m}{2m}} \left(1 - \frac{i\vec{p} \cdot \vec{\gamma} \gamma_0}{E+m}\right) \begin{bmatrix} 0_l \\ 0_{l-2} \\ 0 \\ 1 \end{bmatrix}$$

### 9.3 N+1维时空中电子的平面波解

推论9.3.1.

$$\begin{cases}
\psi_{\lambda_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{m}{E}} \sum_{h=1}^l [a(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) v_{\lambda_\zeta}(\vec{p}, h) e^{-ip \cdot x}] d^N \vec{p} \\
a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{E}{m}} u^+(\vec{p}, h) \psi(x) e^{-ip \cdot x} d^3 \vec{r}, b^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sqrt{\frac{E}{m}} v^+(\vec{p}, h) \psi(x) e^{ip \cdot x} d^3 \vec{r}
\end{cases}$$

定理9.3.1.

$$\begin{cases} A_{a_1 \dots a_l}(x) = 2^{-[\frac{n}{2}]} \text{tr} \left\{ \frac{1}{l!} \bar{C} \gamma_{[a_1} \dots \gamma_{a_l]} \psi(x) \right\} = \frac{1}{(2\pi)^{N/2}} \int \sqrt{\frac{E}{m}} \sum_{h=1}^l [a(\vec{p}, h) U_{a_1 \dots a_l}(\vec{p}, h) e^{ip \cdot x} + b^+(\vec{p}, h) V_{a_1 \dots a_l}^+(\vec{p}, h) e^{-ip \cdot x}] d^N \vec{p} \\ U_{a_1 \dots a_l}(\vec{p}, h) = 2^{-[\frac{n}{2}]} \text{tr} \left\{ \frac{1}{l!} \bar{C} \gamma_{[a_1} \dots \gamma_{a_l]} u(\vec{p}, h) \right\}, V_{a_1 \dots a_l}(\vec{p}, h) = 2^{-[\frac{n}{2}]} \text{tr} \left\{ \frac{1}{l!} \bar{C} \gamma_{[a_1} \dots \gamma_{a_l]} v(\vec{p}, h) \right\} \end{cases}$$

## 9.4 N+1维时空中B-W矢量场的自旋基

定义9.4.1.

$$\begin{cases} U_{\lambda_s \mu_s}(\vec{p}; h, h) = \frac{1}{2} u_{\{\lambda_s}(\vec{p}, h) u_{\mu_s\}}(\vec{p}, h), U_{\lambda_s \mu_s}(\vec{p}; h < h') = \frac{1}{\sqrt{2}} u_{\{\lambda_s}(\vec{p}, h) u_{\mu_s\}}(\vec{p}, h'); h, h' = 1, 2, \dots, 2^{[\frac{N-1}{2}]} \\ V_{\lambda_s \mu_s}(\vec{p}; h, h) = \frac{1}{2} v_{\{\lambda_s}(\vec{p}, h) v_{\mu_s\}}(\vec{p}, h), V_{\lambda_s \mu_s}(\vec{p}; h < h') = \frac{1}{\sqrt{2}} v_{\{\lambda_s}(\vec{p}, h) v_{\mu_s\}}(\vec{p}, h'); h, h' = 1, 2, \dots, 2^{[\frac{N-1}{2}]} \\ X_{\lambda_s \mu_s}(\vec{p}; h < h') = \frac{1}{\sqrt{2}} u_{[\lambda_s}(\vec{p}, h) u_{\mu_s]}(\vec{p}, h'); h, h' = 1, 2, \dots, 2^{[\frac{N-1}{2}]} \\ Y_{\lambda_s \mu_s}(\vec{p}; h < h') = \frac{1}{\sqrt{2}} v_{[\lambda_s}(\vec{p}, h) v_{\mu_s]}(\vec{p}, h'); h, h' = 1, 2, \dots, 2^{[\frac{N-1}{2}]} \end{cases}$$

## 9.5 N+1维时空中反对称张量场的自旋基及其性质

$$\text{定理9.5.1.} \quad \begin{cases} U_{a_1 \dots a_l}(\vec{p}; h, h) = 2^{-[\frac{n}{2}]} \left\{ \left( \frac{1}{l!} C^+ \gamma_{[a_1} \dots \gamma_{a_l]} \right)^{\mu_s \lambda_s} u_{\lambda_s}(\vec{p}, h) u_{\mu_s}(\vec{p}, h) \right\} \\ U_{a_1 \dots a_l}(\vec{p}; h < h') = 2^{-[\frac{n}{2}]} \sqrt{2} \left\{ \left( \frac{1}{l!} C^+ \gamma_{[a_1} \dots \gamma_{a_l]} \right)^{\mu_s \lambda_s} u_{\lambda_s}(\vec{p}, h) u_{\mu_s}(\vec{p}, h') \right\} \\ U_{a_1 \dots a_l}(\vec{p}; h, h') = 2^{-[\frac{n}{2}]} \sqrt{2}^{1-\delta_{hh'}} \left\{ \left( \frac{1}{l!} C^+ \gamma_{[a_1} \dots \gamma_{a_l]} \right)^{\mu_s \lambda_s} u_{\lambda_s}(\vec{p}, h) u_{\mu_s}(\vec{p}, h') \right\} \end{cases}$$

$$\begin{aligned} \text{证明:} \quad & U_{a_1 \dots a_l}(\vec{p}; h, h') = 2^{-[\frac{n}{2}]} \sqrt{2}^{1-\delta_{hh'}} \left\{ \left( \frac{1}{l!} C^+ \gamma_{[a_1} \dots \gamma_{a_l]} \right)^{\mu_s \lambda_s} u_{\lambda_s}(\vec{p}, h) u_{\mu_s}(\vec{p}, h') \right\} \\ & = 2^{-[\frac{n}{2}]} \sqrt{2}^{1-\delta_{hh'}} \left\{ \frac{1}{l!} u_{\mu_s}(\vec{p}, h') (C^+ \gamma_{[a_1} \dots \gamma_{a_l]})^{\mu_s \lambda_s} u_{\lambda_s}(\vec{p}, h) \right\} \\ & = 2^{-[\frac{n}{2}]} \sqrt{2}^{1-\delta_{hh'}} \frac{1}{l!} u^T(\vec{p}, h') C^+ \gamma_{[a_1} \dots \gamma_{a_l]} u(\vec{p}, h) \end{aligned} \quad \square$$

$$\begin{aligned} \text{证明:} \quad & \sum_{h \leq h'} \sqrt{2}^{1-\delta_{hh'}} \left[ \frac{1}{l!} u^T(\vec{p}, h') C^+ \gamma_{[a_1} \dots \gamma_{a_l]} u(\vec{p}, h) \right] \left[ \frac{1}{l!} u^T(\vec{p}, h') C^+ \gamma_{[a'_1} \dots \gamma_{a'_l]} u(\vec{p}, h) \right]^+ \\ & = \sum_{h \leq h'} \sqrt{2}^{1-\delta_{hh'}} \left[ \frac{1}{l!} u^T(\vec{p}, h') (C^+ \gamma_{[a_1} \dots \gamma_{a_l]}) u(\vec{p}, h) \frac{1}{l!} u^+(\vec{p}, h) (C^+ \gamma_{[a'_1} \dots \gamma_{a'_l]})^+ u^*(\vec{p}, h') \right] \\ & = \frac{1}{(l!)^2} \sum_{h \leq h'} \sqrt{2}^{1-\delta_{hh'}} \left[ u^T(\vec{p}, h') (C^+ \gamma_{[a_1} \dots \gamma_{a_l]}) u(\vec{p}, h) u^+(\vec{p}, h) (\gamma_{[a'_1} \dots \gamma_{a'_l]} C) u^*(\vec{p}, h') \right] \\ & = \frac{i^{l(l-1)}}{(l!)^2} \sum_{h \leq h'} \sqrt{2}^{1-\delta_{hh'}} \left[ u^T(\vec{p}, h') (C^+ \gamma_{[a_1} \dots \gamma_{a_l]}) u(\vec{p}, h) u^+(\vec{p}, h) (\gamma_{[a'_1} \dots \gamma_{a'_l]} C) u^*(\vec{p}, h') \right] \\ & = \frac{i^{l(l-1)}}{(l!)^2} \sum_{h \leq h'} \sqrt{2}^{1-\delta_{hh'}} \left[ u_{\mu}(\vec{p}, h') (C^+ \gamma_{[a_1} \dots \gamma_{a_l]})^{\mu \lambda} u_{\lambda}(\vec{p}, h) u_{\lambda'}^+(\vec{p}, h) (\gamma_{[a'_1} \dots \gamma_{a'_l]} C)^{\lambda' \mu'} u_{\mu'}^+(\vec{p}, h') \right] \\ & = \frac{i^{l(l-1)}}{(l!)^2} \left[ (C^+ \gamma_{[a_1} \dots \gamma_{a_l]})^{\mu \lambda} \left[ \sum_h u_{\lambda}(\vec{p}, h) u_{\lambda'}^+(\vec{p}, h) \right] (\gamma_{[a'_1} \dots \gamma_{a'_l]} C)^{\lambda' \mu'} \left[ \sum_{h'} u_{\mu}(\vec{p}, h') u_{\mu'}^+(\vec{p}, h') \right] \right] \\ & = \frac{i^{l(l-1)}}{4m^2(l!)^2} \left[ (C^+ \gamma_{[a_1} \dots \gamma_{a_l]})^{\mu \lambda} [(m - i\gamma^a p_a) \gamma^0]_{\lambda \lambda'} (\gamma_{[a'_1} \dots \gamma_{a'_l]} C)^{\lambda' \mu'} [(m - i\gamma^b p_b) \gamma^0]_{\mu' \mu}^T \right] \\ & = \frac{i^{l(l-1)}}{4m^2(l!)^2} \text{tr} \left\{ \left[ (C^+ \gamma_{[a_1} \dots \gamma_{a_l]}) [(m - i\gamma^a p_a) \gamma^0] (\gamma_{[a'_1} \dots \gamma_{a'_l]} C) [(m - i\gamma^b p_b) \gamma^0]^T \right] \right\} \\ & = i \frac{4^{-[\frac{n}{2}]} (-1)^{\delta_{2,n}}}{2(l!)^2} \text{tr} \left\{ C^+ \gamma_{[a_1} \gamma_{a_2} \dots \gamma_{a_l]} [(m - i\gamma^a p_a) \gamma^0] \gamma_{[a'_1} \gamma_{a'_2} \dots \gamma_{a'_l]} C [(m - i\gamma^b p_b) \gamma^0]^T \right\} \Delta(x - x') \\ & = -\frac{2^{[\frac{n}{2}]} (-1)^{\delta_{2,n}+l}}{2m^2} \begin{cases} \frac{1}{(l+1)!} \eta_{[a'_1}^{a_1} \eta_{a'_2}^{a_2} \dots \eta_{a'_l}^{a_l} \eta_{a'_l}^{a_l}] p_a p^{+a'}, (-1)^{\eta(n)+l} = 1 \\ \frac{1}{(l-1)!} \eta_{[a'_1}^{a_1} \dots \eta_{a'_{l-1}}^{a_{l-1}} p^{a_l}] p_{a'_l}, (-1)^{\eta(n)+l} = -1 \end{cases} \\ & = \frac{2^{[\frac{n}{2}]} (-1)^{\delta_{2,n}}}{2m^2} \begin{cases} \frac{1}{(l+1)!} \eta_{[a'_1}^{a_1} \eta_{a'_2}^{a_2} \dots \eta_{a'_l}^{a_l} \eta_{a'_l}^{a_l}] p_a p^{+a'}, (-1)^{\eta(n)+l} = 1 \\ -\frac{1}{(l-1)!} \eta_{[a'_1}^{a_1} \dots \eta_{a'_{l-1}}^{a_{l-1}} p^{a_l}] p_{a'_l}, (-1)^{\eta(n)+l} = -1 \end{cases} \quad \square \end{aligned}$$

$$\text{推论9.5.1.} \quad \sum_{h \leq h'} U_{a_1 \dots a_l}(\vec{p}; h, h') U_{a'_1 \dots a'_l}^+(\vec{p}; h, h') = \frac{(-1)^{\delta_{2,n}}}{2m^2 2^{[\frac{n}{2}]}} \begin{cases} \frac{1}{(l+1)!} \eta_{[a'_1}^{a_1} \eta_{a'_2}^{a_2} \dots \eta_{a'_l}^{a_l} \eta_{a'_l}^{a_l}] p_a p^{+a'}, (-1)^{\eta(n)+l} = 1 \\ -\frac{1}{(l-1)!} \eta_{[a'_1}^{a_1} \dots \eta_{a'_{l-1}}^{a_{l-1}} p^{a_l}] p_{a'_l}, (-1)^{\eta(n)+l} = -1 \end{cases}$$

证明:  $U^{+a'_1 \dots a'_l}(\vec{p}; h, h') \eta_{a'_1}^{a_1} \dots \eta_{a'_l}^{a_l} U_{a_1 \dots a_l}(\vec{p}; h, h')$

$$\begin{aligned} & = 4^{-[\frac{n}{2}]} 2^{1-\delta_{hh'}} i^{l(l-1)} \left\{ \frac{1}{l!} u_{\mu_s}^+(\vec{p}, h') u_{\lambda_s}^+(\vec{p}, h) (\gamma_{[a'_1} \dots \gamma_{a'_l]} C)^{\lambda_s \mu_s} \eta_{a'_1}^{a_1} \dots \eta_{a'_l}^{a_l} \left\{ \left( \frac{1}{l!} C^+ \gamma_{[a_1} \dots \gamma_{a_l]} \right)^{\mu_s \lambda_s} u_{\lambda_s}(\vec{p}, h) u_{\mu_s}(\vec{p}, h') \right\} \right\} \\ & = 4^{-[\frac{n}{2}]} 2^{1-\delta_{hh'}} i^{l(l-1)} \left[ u_{\lambda_s}^+(\vec{p}, h) u_{\lambda_s}(\vec{p}, h) \right] \left[ u_{\mu_s}^+(\vec{p}, h') u_{\mu_s}(\vec{p}, h') \right] \left\{ \left( \frac{1}{l!} \gamma_{[a'_1} \dots \gamma_{a'_l]} C \right)^{\lambda_s \mu_s} \eta_{a'_1}^{a_1} \dots \eta_{a'_l}^{a_l} \left( \frac{1}{l!} C^+ \gamma_{[a_1} \dots \gamma_{a_l]} \right)^{\mu_s \lambda_s} \right\} \end{aligned}$$

$$\begin{aligned}
&= 4^{-[\frac{n}{2}]} 2^{1-\delta_{nh'}} i^{l(l+1)} [u_{\lambda'_\zeta}^+(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h)] [u_{\mu'_\zeta}^+(\vec{p}, h') u_{\mu_\zeta}(\vec{p}, h')] \{ (\frac{1}{l!} \gamma_0 \gamma^{[a_1 \dots \gamma_{a_l}] C} \gamma_0)^{\lambda'_\zeta \mu'_\zeta} (\frac{1}{l!} C^+ \gamma_{[a_1 \dots \gamma_{a_l}])^{\mu_\zeta \lambda_\zeta} \} \\
&= 4^{-[\frac{n}{2}]} 2^{1-\delta_{nh'}} i^{l(l+1)} [\bar{u}_{\lambda'_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h)] [\bar{u}_{\mu'_\zeta}(\vec{p}, h') u_{\mu_\zeta}(\vec{p}, h')] \{ (\frac{1}{l!} \gamma^{[a_1 \dots \gamma_{a_l}] C})^{\lambda'_\zeta \mu'_\zeta} (\frac{1}{l!} C^+ \gamma_{[a_1 \dots \gamma_{a_l}])^{\mu_\zeta \lambda_\zeta} \} \quad \square
\end{aligned}$$

## 9.6 N+1维时空中Dirac矩阵的正交性?

证明:  $(\frac{1}{l!} \gamma^{[a_1 \dots \gamma_{a_l}] C})^{\lambda'_\zeta \mu'_\zeta} (\frac{1}{l!} C^+ \gamma_{[a_1 \dots \gamma_{a_l}])^{\mu_\zeta \lambda_\zeta} = ??? \delta^{\lambda'_\zeta \lambda_\zeta} \delta^{\mu_\zeta \mu'_\zeta} + \dots$  □

## 10 四维时空中特殊的反对称张量场

### 10.1 相关性质汇总

推论10.1.1.  $v(\vec{p}, h) = -\gamma_5 u(\vec{p}, h), u(\vec{p}, h) = -\gamma_5 v(\vec{p}, h)$

$$\text{性质10.1.1.} \quad \begin{cases} u^T(\vec{p}, h) C^+ u(\vec{p}, h') = 0, u^T(\vec{p}, h) C^+ v(\vec{p}, h) = 0 \\ u^T(\vec{p}, \frac{1}{2}) C^+ v(\vec{p}, -\frac{1}{2}) = -\varsigma, u^T(\vec{p}, -\frac{1}{2}) C^+ v(\vec{p}, \frac{1}{2}) = \varsigma \end{cases}$$

$$\text{性质10.1.2.} \quad \begin{cases} u^T(\vec{p}, h) C^+ \gamma_5 u(\vec{p}, h) = 0, u^T(\vec{p}, h) C^+ \gamma_5 v(\vec{p}, h') = 0 \\ u^T(\vec{p}, -\frac{1}{2}) C^+ \gamma_5 u(\vec{p}, \frac{1}{2}) = -1, u^T(\vec{p}, \frac{1}{2}) C^+ \gamma_5 u(\vec{p}, -\frac{1}{2}) = 1 \end{cases}$$

$$\text{性质10.1.3.} \quad \begin{cases} u^T(\hat{p}, \frac{\kappa}{2}) \bar{C} \gamma_a u(\hat{p}, \frac{\kappa}{2}) = i\sqrt{2} \varepsilon_a(\vec{p}, \kappa), u^T(\hat{p}, \frac{\kappa}{2}) \bar{C} \gamma_a u(\hat{p}, -\frac{\kappa}{2}) = i\varepsilon_a(\vec{p}, 0) \\ u^T(\vec{p}, \frac{\kappa}{2}) C^+ \gamma_a v(\vec{p}, \frac{\kappa}{2}) = 0, u^T(\vec{p}, \frac{1}{2}) C^+ \gamma_a v(\vec{p}, -\frac{1}{2}) = \frac{-i s p_a}{m}, u^T(\vec{p}, -\frac{1}{2}) C^+ \gamma_a v(\vec{p}, \frac{1}{2}) = \frac{i s p_a}{m} \end{cases}$$

### 10.2 四维时空中特殊的反对称张量场平面波解

引理10.2.1.

$$\begin{cases} (\gamma^a \partial_a + m) X(4) = 0 \\ X(4) = \{ \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} + \frac{1}{(3!)^2} F^{abc} \gamma_{[a} \gamma_b \gamma_{c]} \} C \end{cases} \Leftrightarrow \begin{cases} \frac{1}{2!} \partial^{[a} F^{bc]} + m F^{abc} = 0, \partial_a F^{ab} = 0 \\ \partial^{[a} F^{bcd]} = 0, \partial_a F^{abc} + m F^{bc} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{l!} \partial^{[a} * F^{b]} + m * F^{ab} = 0 \\ \partial_a * F^{ab} + m * F^b = 0 \end{cases}$$

引理10.2.2.

$$\begin{cases} (\gamma^a \partial_a + m) \psi(x) = 0 \\ tr\{C^+ \psi(x)\} = 0, tr\{C^+ \gamma_a \psi(x)\} = 0, tr\{C^+ \gamma_{[a} \gamma_b \gamma_c \gamma_d] \psi(x)\} = 0 \\ \psi(x) = \{ \frac{1}{(2!)^2} F^{ab} \gamma_{[a} \gamma_{b]} + \frac{1}{(3!)^2} F^{abc} \gamma_{[a} \gamma_b \gamma_{c]} \} C \end{cases} \Leftrightarrow \begin{cases} \frac{1}{2!} \partial^{[a} F^{bc]} + m F^{abc} = 0, \partial_a F^{ab} = 0 \\ \partial^{[a} F^{bcd]} = 0, \partial_a F^{abc} + m F^{bc} = 0 \end{cases}$$

证明:

$$\begin{cases} (\gamma^a \partial_a + m) \psi(x) = 0 \\ tr\{C^+ \psi(x)\} = 0, tr\{C^+ \gamma_a \psi(x)\} = 0, tr\{C^+ \gamma_{[a} \gamma_b \gamma_c \gamma_d] \psi(x)\} = 0 \end{cases}$$

$$\begin{cases} \psi_{\lambda_\zeta \mu_\zeta}(x) = \frac{1}{(2\pi)^{N/2}} \int \sqrt{\frac{m}{E}} \sum_{h=1/2}^{-1/2} [a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) e^{ip \cdot x} + b_{\mu_\zeta}^+(\vec{p}, h) v_{\lambda_\zeta}(\vec{p}, h) e^{-ip \cdot x}] d^N \vec{p} \\ \sum_{h=1/2}^{-1/2} C^{+\mu_\zeta \lambda_\zeta} a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0, \sum_{h=1/2}^{-1/2} (C^+ \gamma_a)^{\mu_\zeta \lambda_\zeta} a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0 \\ \sum_{h=1/2}^{-1/2} (C^+ \gamma_{[a} \gamma_b \gamma_c \gamma_d])^{\mu_\zeta \lambda_\zeta} a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0 \end{cases} \quad \square$$

证明:

$$\begin{cases} \sum_{h=1/2}^{-1/2} C^{+\mu_\zeta \lambda_\zeta} a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0 \\ \sum_{h=1/2}^{-1/2} (C^+ \gamma_a)^{\mu_\zeta \lambda_\zeta} a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0 \\ \sum_{h=1/2}^{-1/2} (C^+ \gamma_{[a} \gamma_b \gamma_c \gamma_d])^{\mu_\zeta \lambda_\zeta} a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0 \end{cases} \Leftrightarrow \begin{cases} \sum_{h=1/2}^{-1/2} C^{+\mu_\zeta \lambda_\zeta} a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0 \\ \sum_{h=1/2}^{-1/2} (C^+ \gamma_a)^{\mu_\zeta \lambda_\zeta} a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0 \\ \sum_{h=1/2}^{-1/2} (C^+ \gamma_5)^{\mu_\zeta \lambda_\zeta} a_{\mu_\zeta}(\vec{p}, h) u_{\lambda_\zeta}(\vec{p}, h) = 0 \end{cases}$$

$$\begin{aligned}
& \Leftrightarrow \begin{cases} \sum_{h,h'=1/2}^{-1/2} C^{+\mu_c \lambda_c} [a(\vec{p}; h, h') u_{\mu_c}(\vec{p}, h') + c(\vec{p}; h, h') v_{\mu_c}(\vec{p}, h')] u_{\lambda_c}(\vec{p}, h) = 0 \\ \sum_{h,h'=1/2}^{-1/2} (C^+ \gamma_a)^{\mu_c \lambda_c} [a(\vec{p}; h, h') u_{\mu_c}(\vec{p}, h') + c(\vec{p}; h, h') v_{\mu_c}(\vec{p}, h')] u_{\lambda_c}(\vec{p}, h) = 0 \\ \sum_{h,h'=1/2}^{-1/2} (C^+ \gamma_5)^{\mu_c \lambda_c} [a(\vec{p}; h, h') u_{\mu_c}(\vec{p}, h') + c(\vec{p}; h, h') v_{\mu_c}(\vec{p}, h')] u_{\lambda_c}(\vec{p}, h) = 0 \end{cases} \\
& \Leftrightarrow \begin{cases} \sum_{h,h'=1/2}^{-1/2} C^{+\mu_c \lambda_c} [a(\vec{p}; h, h') u_{[\lambda_c}(\vec{p}, h) u_{\mu_c]}(\vec{p}, h') + c(\vec{p}; h, h') u_{[\lambda_c}(\vec{p}, h) v_{\mu_c]}(\vec{p}, h')] = 0 \\ \sum_{h,h'=1/2}^{-1/2} (C^+ \gamma_a)^{\mu_c \lambda_c} [a(\vec{p}; h, h') u_{\{\lambda_c}(\vec{p}, h) u_{\mu_c\}}(\vec{p}, h') + c(\vec{p}; h, h') u_{\{\lambda_c}(\vec{p}, h) v_{\mu_c\}}(\vec{p}, h')] = 0 \\ \sum_{h,h'=1/2}^{-1/2} (C^+ \gamma_5)^{\mu_c \lambda_c} [a(\vec{p}; h, h') u_{[\lambda_c}(\vec{p}, h) u_{\mu_c]}(\vec{p}, h') + c(\vec{p}; h, h') u_{[\lambda_c}(\vec{p}, h) v_{\mu_c]}(\vec{p}, h')] = 0 \end{cases} \\
& \Leftrightarrow \begin{cases} \sum_{h,h'=1/2}^{-1/2} C^{+\lambda_c \mu_c} [a(\vec{p}; h, h') u_{\lambda_c}(\vec{p}, h) u_{\mu_c}(\vec{p}, h') + c(\vec{p}; h, h') u_{\lambda_c}(\vec{p}, h) v_{\mu_c}(\vec{p}, h')] = 0 \\ \sum_{h,h'=1/2}^{-1/2} (C^+ \gamma_a)^{\lambda_c \mu_c} [a(\vec{p}; h, h') u_{\lambda_c}(\vec{p}, h) u_{\mu_c}(\vec{p}, h') + c(\vec{p}; h, h') u_{\lambda_c}(\vec{p}, h) v_{\mu_c}(\vec{p}, h')] = 0 \\ \sum_{h,h'=1/2}^{-1/2} (C^+ \gamma_5)^{\lambda_c \mu_c} [a(\vec{p}; h, h') u_{\lambda_c}(\vec{p}, h) u_{\mu_c}(\vec{p}, h') + c(\vec{p}; h, h') u_{\lambda_c}(\vec{p}, h) v_{\mu_c}(\vec{p}, h')] = 0 \end{cases} \\
& \Leftrightarrow \begin{cases} \sum_{h,h'=1/2}^{-1/2} [a(\vec{p}; h, h') u^T(\vec{p}, h) C^+ u(\vec{p}, h') + c(\vec{p}; h, h') u^T(\vec{p}, h) C^+ v(\vec{p}, h')] = 0 \\ \sum_{h,h'=1/2}^{-1/2} [a(\vec{p}; h, h') u^T(\vec{p}, h) C^+ \gamma_a u(\vec{p}, h') + c(\vec{p}; h, h') u^T(\vec{p}, h) C^+ \gamma_a v(\vec{p}, h')] = 0 \\ \sum_{h,h'=1/2}^{-1/2} [a(\vec{p}; h, h') u^T(\vec{p}, h) C^+ \gamma_5 u(\vec{p}, h') + c(\vec{p}; h, h') u^T(\vec{p}, h) C^+ \gamma_5 v(\vec{p}, h')] = 0 \end{cases} \\
& \Leftrightarrow \begin{cases} c(\vec{p}; \frac{1}{2}, -\frac{1}{2}) = c(\vec{p}; -\frac{1}{2}, \frac{1}{2}) \\ i\sqrt{2}\varepsilon_a(\vec{p}, 1)a(\vec{p}; \frac{1}{2}, \frac{1}{2}) + i\sqrt{2}\varepsilon_a(\vec{p}, -1)a(\vec{p}; -\frac{1}{2}, -\frac{1}{2}) + i\varepsilon_a(\vec{p}, 0)[a(\vec{p}; \frac{1}{2}, -\frac{1}{2}) + a(\vec{p}; -\frac{1}{2}, \frac{1}{2})] = 0 \\ a(\vec{p}; \frac{1}{2}, -\frac{1}{2}) = a(\vec{p}; -\frac{1}{2}, \frac{1}{2}) \end{cases} \\
& \Leftrightarrow \begin{cases} c(\vec{p}; \frac{1}{2}, -\frac{1}{2}) = c(\vec{p}; -\frac{1}{2}, \frac{1}{2}) \\ a(\vec{p}; h, h') = 0 \end{cases} \\
& \Rightarrow \psi_{\lambda_c \mu_c}(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \sqrt{\frac{m}{E}} \sum_{h,h'=1/2}^{-1/2} [a(\vec{p}; h, h') u_{\lambda_c}(\vec{p}, h) u_{\mu_c}(\vec{p}, h') + c(\vec{p}; h, h') u_{\lambda_c}(\vec{p}, h) v_{\mu_c}(\vec{p}, h')] e^{ip \cdot x} + \dots \\
& \Rightarrow \psi_{\lambda_c \mu_c}(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \sqrt{\frac{m}{E}} \sum_{h,h'=1/2}^{-1/2} c(\vec{p}; h, h') u_{\lambda_c}(\vec{p}, h) v_{\mu_c}(\vec{p}, h') e^{ip \cdot x} + \dots \\
& \Rightarrow \psi_{\lambda_c \mu_c}(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \sqrt{\frac{m}{E}} e^{ip \cdot x} \\
& \{c(\vec{p}; \frac{1}{2}, \frac{1}{2}) u_{\lambda_c}(\vec{p}, \frac{1}{2}) v_{\mu_c}(\vec{p}, \frac{1}{2}) + c(\vec{p}; -\frac{1}{2}, -\frac{1}{2}) u_{\lambda_c}(\vec{p}, -\frac{1}{2}) v_{\mu_c}(\vec{p}, -\frac{1}{2}) + c(\vec{p}; \frac{1}{2}, -\frac{1}{2}) [u_{\lambda_c}(\vec{p}, \frac{1}{2}) v_{\mu_c}(\vec{p}, -\frac{1}{2}) + u_{\lambda_c}(\vec{p}, -\frac{1}{2}) v_{\mu_c}(\vec{p}, \frac{1}{2})] \} + \dots \\
& \Rightarrow \psi_{\lambda_c \mu_c}(x) = \frac{1}{(2\pi)^{N/2}} \int d^N \vec{p} \sqrt{\frac{m}{E}} e^{ip \cdot x} \\
& \{c(\vec{p}; 1) u_{\lambda_c}(\vec{p}, \frac{1}{2}) v_{\mu_c}(\vec{p}, \frac{1}{2}) + c(\vec{p}; -1) u_{\lambda_c}(\vec{p}, -\frac{1}{2}) v_{\mu_c}(\vec{p}, -\frac{1}{2}) + c(\vec{p}; 0) \frac{1}{\sqrt{2}} [u_{\lambda_c}(\vec{p}, \frac{1}{2}) v_{\mu_c}(\vec{p}, -\frac{1}{2}) + u_{\lambda_c}(\vec{p}, -\frac{1}{2}) v_{\mu_c}(\vec{p}, \frac{1}{2})] \} + \dots \\
& \dots \quad \square
\end{aligned}$$

## 10.3 四维时空中特殊的反对称张量场的B-W准投影算子

证明:

$$\begin{aligned}
& u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})v_{\mu_\zeta}(\vec{p}, \frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2}) + u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})v_{\mu_\zeta}(\vec{p}, -\frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2}) \\
& + \frac{1}{\sqrt{2}}[u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})v_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) + u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})v_{\mu_\zeta}(\vec{p}, \frac{1}{2})] \frac{1}{\sqrt{2}}[u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2}) + u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2})] \\
& = [u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2})][v_{\mu_\zeta}(\vec{p}, \frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2})] + [u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})][v_{\mu_\zeta}(\vec{p}, -\frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
& + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2})][v_{\mu_\zeta}(\vec{p}, -\frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2})] + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})][v_{\mu_\zeta}(\vec{p}, \frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2})] \\
& + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})][v_{\mu_\zeta}(\vec{p}, -\frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2})] + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2})][v_{\mu_\zeta}(\vec{p}, \frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
& = \frac{1}{2} \sum_{h=1/2}^{-1/2} [u_{\lambda_\zeta}(\vec{p}, h)u_{\lambda'_\zeta}^+(\vec{p}, h)] \sum_{h'=1/2}^{-1/2} [v_{\mu_\zeta}(\vec{p}, h')v_{\mu'_\zeta}^+(\vec{p}, h')] \\
& + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2})][v_{\mu_\zeta}(\vec{p}, \frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2})] + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})][v_{\mu_\zeta}(\vec{p}, -\frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
& + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})][v_{\mu_\zeta}(\vec{p}, -\frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2})] + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2})][v_{\mu_\zeta}(\vec{p}, \frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
& = \frac{1}{2} \sum_{h=1/2}^{-1/2} [u_{\lambda_\zeta}(\vec{p}, h)u_{\lambda'_\zeta}^+(\vec{p}, h)] \sum_{h'=1/2}^{-1/2} [v_{\mu_\zeta}(\vec{p}, h')v_{\mu'_\zeta}^+(\vec{p}, h')] \\
& + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2})][v_{\mu_\zeta}(\vec{p}, \frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2})] + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2})][v_{\mu_\zeta}(\vec{p}, -\frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})] \\
& + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, \frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, \frac{1}{2})][v_{\mu_\zeta}(\vec{p}, -\frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, -\frac{1}{2})] + \frac{1}{2}[u_{\lambda_\zeta}(\vec{p}, -\frac{1}{2})v_{\mu'_\zeta}^+(\vec{p}, -\frac{1}{2})][v_{\mu_\zeta}(\vec{p}, \frac{1}{2})u_{\lambda'_\zeta}^+(\vec{p}, \frac{1}{2})] \\
& = \frac{1}{2} \sum_{h=1/2}^{-1/2} [u_{\lambda_\zeta}(\vec{p}, h)u_{\lambda'_\zeta}^+(\vec{p}, h)] \sum_{h'=1/2}^{-1/2} [v_{\mu_\zeta}(\vec{p}, h')v_{\mu'_\zeta}^+(\vec{p}, h')] + \frac{1}{2} \sum_{h=1/2}^{-1/2} [u_{\lambda_\zeta}(\vec{p}, h)v_{\mu'_\zeta}^+(\vec{p}, h)] \sum_{h'=1/2}^{-1/2} [v_{\mu_\zeta}(\vec{p}, h')u_{\lambda'_\zeta}^+(\vec{p}, h')] \\
& = \frac{1}{2} \sum_{h=1/2}^{-1/2} [u_{\lambda_\zeta}(\vec{p}, h)u_{\lambda'_\zeta}^+(\vec{p}, h)] \sum_{h'=1/2}^{-1/2} [v_{\mu_\zeta}(\vec{p}, h')v_{\mu'_\zeta}^+(\vec{p}, h')] \\
& + \frac{1}{2} \sum_{h=1/2}^{-1/2} [u_{\lambda_\zeta}(\vec{p}, h)[u^+(\vec{p}, h)\gamma_5]_{\mu'_\zeta}] \sum_{h'=1/2}^{-1/2} [v_{\mu_\zeta}(\vec{p}, h')[v^+(\vec{p}, h')\gamma_5]_{\lambda'_\zeta}] \\
& = \frac{1}{2} \frac{[(m-i\gamma^a p_a)\gamma_0]_{\lambda_\zeta \lambda'_\zeta} [(-m-i\gamma^a p_a)\gamma_0]_{\mu_\zeta \mu'_\zeta}}{2m} + \frac{1}{2} \frac{[(m-i\gamma^a p_a)\gamma_0 \gamma_5]_{\lambda_\zeta \mu'_\zeta} [(-m-i\gamma^a p_a)\gamma_0 \gamma_5]_{\mu_\zeta \lambda'_\zeta}}{2m} \quad \square
\end{aligned}$$

## 10.4 四维时空中特殊的反对称张量场的势对易规则

定理10.4.1.  $[F_{a_1 a_2}(x), F_{a'_1 a'_2}^+(x')] = \frac{i}{2^2} \{ \frac{1}{2!} \eta_{[a'_1}^{[a_1} \eta_{a'_2]}^{a_2]} m^2 - \frac{1}{1!} \eta_{[a'_1}^{[a_1} \partial^{a_2]} \partial_{a'_2}^+ \} \Delta(x-x') = \frac{i}{2^2} \frac{1}{3!} \eta_{[a'_1}^{[a_1} \eta_{a'_2]}^{a_2} \eta_b^a] \partial_a \partial^{+b} \Delta(x-x')$ 证明:  $[F_{a_1 a_2}(x), F_{a'_1 a'_2}^+(x')]$ 

$$\begin{aligned}
& = \frac{2^{-4}}{(2!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2]})^{\mu\lambda} (C^+ \gamma_{[a'_1} \gamma_{a'_2]})^{*\mu'\lambda'} [\psi_{\lambda\mu}(x), \psi_{\lambda'\mu'}^+(x')] \\
& = \frac{2^{-4}}{(2!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2]})^{\lambda\mu} (C^+ \gamma_{[a'_1} \gamma_{a'_2]})^{+\lambda'\mu'} [\psi_{\lambda\mu}(x), \psi_{\lambda'\mu'}^+(x')] \\
& = \frac{2^{-4}}{(2!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2]})^{\lambda\mu} (\gamma_{[a'_2} \gamma_{a'_1]} C)^{\lambda'\mu'} [\psi_{\lambda\mu}(x), \psi_{\lambda'\mu'}^+(x')] \\
& = -\frac{2^{-4}}{(2!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2]})^{\lambda\mu} (\gamma_{[a'_1} \gamma_{a'_2]} C)^{\lambda'\mu'} [\psi_{\lambda\mu}(x), \psi_{\lambda'\mu'}^+(x')] \\
& = -\frac{2^{-4}}{(2!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2]})^{\lambda\mu} (\gamma_{[a'_1} \gamma_{a'_2]} C)^{\lambda'\mu'} \\
& \frac{i}{2^2} \{ [(m-\gamma^a \partial_a)\gamma^0]_{\lambda\lambda'} [(-m-\gamma^b \partial_b)\gamma^0]_{\mu\mu'} + [(m-\gamma^a \partial_a)\gamma^0 \gamma^5]_{\mu\lambda'} [(-m-\gamma^b \partial_b)\gamma^0 \gamma^5]_{\lambda\mu'} \} \Delta(x-x') \\
& = -i \frac{2^{-6}}{(2!)^2} (C^+ \gamma_{[a_1} \gamma_{a_2]})^{\lambda\mu} (\gamma_{[a'_1} \gamma_{a'_2]} C)^{\lambda'\mu'} \\
& \{ [(m-\gamma^a \partial_a)\gamma^0]_{\lambda\lambda'} [(-m-\gamma^b \partial_b)\gamma^0]_{\mu\mu'} + [(m-\gamma^a \partial_a)\gamma^0 \gamma^5]_{\lambda\lambda'} [(-m-\gamma^b \partial_b)\gamma^0 \gamma^5]_{\mu\mu'} \} \Delta(x-x') \\
& = -i \frac{2^{-6}}{(2!)^2} \{ (C^+ \gamma_{[a_1} \gamma_{a_2]})^{\lambda\mu} [(m-\gamma^a \partial_a)\gamma^0]_{\lambda\lambda'} (\gamma_{[a'_1} \gamma_{a'_2]} C)^{\lambda'\mu'} [(-m-\gamma^b \partial_b)\gamma^0]_{\mu'\mu}^T \Delta(x-x') \\
& + (C^+ \gamma_{[a_1} \gamma_{a_2]})^{\lambda\mu} [(m-\gamma^a \partial_a)\gamma^0 \gamma^5]_{\lambda\lambda'} (\gamma_{[a'_1} \gamma_{a'_2]} C)^{\lambda'\mu'} [(-m-\gamma^b \partial_b)\gamma^0 \gamma^5]_{\mu'\mu}^T \Delta(x-x') \} \\
& = -i \frac{2^{-6}}{(2!)^2} tr \{ (C^+ \gamma_{[a_1} \gamma_{a_2]}) [(m-\gamma^a \partial_a)\gamma^0] (\gamma_{[a'_1} \gamma_{a'_2]} C) [(-m-\gamma^b \partial_b)\gamma^0]^T \\
& + (C^+ \gamma_{[a_1} \gamma_{a_2]}) [(m-\gamma^a \partial_a)\gamma^0 \gamma^5] (\gamma_{[a'_1} \gamma_{a'_2]} C) [(-m-\gamma^b \partial_b)\gamma^0 \gamma^5]^T \} \Delta(x-x') \\
& = -i \frac{2^{-6}}{(2!)^2} tr \{ \gamma_{[a_1} \gamma_{a_2]} [(m-\gamma^a \partial_a)\gamma^0] \gamma_{[a'_1} \gamma_{a'_2]} C [(-m-\gamma^b \partial_b)\gamma^0]^T C^+ \\
& + \gamma_{[a_1} \gamma_{a_2]} [(m-\gamma^a \partial_a)\gamma^0 \gamma^5] \gamma_{[a'_1} \gamma_{a'_2]} C [(-m-\gamma^b \partial_b)\gamma^0 \gamma^5]^T C^+ \} \Delta(x-x') \\
& = i \frac{2^{-6}}{(2!)^2} tr \{ \gamma_{[a_1} \gamma_{a_2]} [(m-\gamma^a \partial_a)\gamma^0] \gamma_{[a'_1} \gamma_{a'_2]} [\gamma^0 (-m+\gamma^b \partial_b)] + \gamma_{[a_1} \gamma_{a_2]} [(m-\gamma^a \partial_a)\gamma^0 \gamma^5] \gamma_{[a'_1} \gamma_{a'_2]} [\gamma^5 \gamma^0 (-m+\gamma^b \partial_b)] \} \Delta(x-x') \\
& = i \frac{2^{-5}}{(2!)^2} tr \{ \gamma_{[a_1} \gamma_{a_2]} [(m-\gamma^a \partial_a)\gamma^0] \gamma_{[a'_1} \gamma_{a'_2]} [(-m-\gamma^b \partial_b^+) \gamma^0] \} \Delta(x-x') \\
& = i \frac{2^{-5}}{(2!)^2} tr \{ -m^2 (\gamma_{[a_1} \gamma_{a_2]}) \gamma^0 (\gamma_{[a'_1} \gamma_{a'_2]}) \gamma^0 \} \Delta(x-x') + i \frac{2^{-5}}{(2!)^2} tr \{ (\gamma_{[a_1} \gamma_{a_2]}) \gamma_a \gamma_0 (\gamma_{[a'_1} \gamma_{a'_2]}) \gamma_b \gamma_0 \} \partial^a \partial^b \Delta(x-x') \\
& = -i \frac{2^{-5}}{(2!)^2} i^{2*3} 2^2 (2!)^2 m^2 \frac{1}{2!} \eta_{[a'_1}^{[a_1} \eta_{a'_2]}^{a_2]} \Delta(x-x')
\end{aligned}$$

$$\begin{aligned}
& + i \frac{2^{-5}}{(2!)^2} i^{3*4} 2^2 (2!)^2 \left\{ \frac{1}{3!} \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \eta_{b]}^{a]} - \frac{1}{1!} \eta_{[a_1}^{[a_1} \delta^{a_2]a} \delta_{a_2]b} \right\} \partial_a \partial^{+b} \Delta(x-x') \\
& = \frac{i}{2^3} \left\{ \frac{1}{2!} \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2]} m^2 + \left( \frac{1}{3!} \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \eta_{b]}^{a]} - \frac{1}{1!} \eta_{[a_1}^{[a_1} \delta^{a_2]a} \delta_{a_2]b} \right) \partial_a \partial^{+b} \right\} \Delta(x-x') \\
& = \frac{i}{2^2} \left\{ \frac{1}{2!} \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2]} m^2 - \frac{1}{1!} \eta_{[a_1}^{[a_1} \partial^{a_2]} \partial_{a_2}^+ \right\} \Delta(x-x') \\
& = \frac{i}{2^2} \frac{1}{3!} \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \eta_{b]}^{a]} \partial_a \partial^{+b} \Delta(x-x')
\end{aligned}$$

□

$$\text{推论10.4.1. } [F^{a_1 a_2 a_3}(x), F_{a'_1 a'_2 a'_3}^+(x')] = -\frac{i}{2^2} \frac{1}{2!} \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \partial^{a_3]} \partial_{a_3}^+ \Delta(x-x')$$

定理10.4.2.

$$\begin{cases} [F^{a_1 a_2 a_3}(x), F_{a'_1 a'_2 a'_3}^+(x')] = -\frac{i}{2^2} \frac{1}{2!} \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \partial^{a_3]} \partial_{a_3}^+ \Delta(x-x') \Leftrightarrow [*F_{a_0}, *F_{a_0}^+] = \frac{i}{2^2} \frac{1}{2!} \eta_{[a_1}^{[a_1} \eta_{a_1}^{a_1]} \partial_a \partial^{+a'} \Delta(x-x') \\ [F_{a_1 a_2}(x), F_{a'_1 a'_2}^+(x')] = \frac{i}{2^2} \frac{1}{3!} \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \eta_{b]}^{a]} \partial_a \partial^{+b} \Delta(x-x') \Leftrightarrow [*F_{a_1 a_2}(x), *F_{a'_1 a'_2}^+(x')] = -\frac{i}{2^2} \eta_{[a_1}^{[a_1} \partial_{a_2]} \partial_{a_2}^+ \Delta(x-x') \end{cases}$$

证明:  $[\frac{1}{3!} \varepsilon_{a_0 a_1 a_2 a_3} F^{a_1 a_2 a_3}(x), \frac{1}{3!} \varepsilon^{a'_0 a'_1 a'_2 a'_3} F_{a'_1 a'_2 a'_3}^+(x')]$ 

$$\begin{aligned}
& = -\frac{1}{(3!)^2} \varepsilon_{a_0 a_1 a_2 a_3} \varepsilon^{a'_0 a'_1 a'_2 a'_3} \frac{i}{2^2} \frac{1}{2!} \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \partial^{a_3]} \partial_{a_3}^+ \Delta(x-x') \\
& = -\frac{i}{2^2} \frac{1}{2!} \varepsilon_{a_0 a_1 a_2 a_3} \varepsilon^{a'_0 a'_1 a'_2 a'_3} \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \partial^{a_3]} \partial_{a_3}^+ \\
& = -\frac{i}{2^2} \frac{1}{2!} \delta_{[a_0}^{a'_0} \delta_{a_1}^{a'_1} \delta_{a_2}^{a'_2} \delta_{a_3}^{a'_3}] \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \partial^{a_3]} \partial_{a_3}^+ \Delta(x-x') \\
& = -\frac{i}{2^2} \frac{1}{2!} (\delta_{a_0}^{a'_0} \delta_{[a_1}^{a'_1} \delta_{a_2}^{a'_2} \delta_{a_3}^{a'_3]} - \delta_{a_1}^{a'_1} \delta_{[a_0}^{a'_0} \delta_{a_2}^{a'_2} \delta_{a_3}^{a'_3]} + \delta_{a_2}^{a'_2} \delta_{[a_0}^{a'_0} \delta_{a_1}^{a'_1} \delta_{a_3}^{a'_3]} - \delta_{a_3}^{a'_3} \delta_{[a_0}^{a'_0} \delta_{a_1}^{a'_1} \delta_{a_2}^{a'_2}]) \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \partial^{a_3]} \partial_{a_3}^+ \Delta(x-x') \\
& = -\frac{i}{2^2} \frac{1}{2!} \{ \delta_{a_0}^{a'_0} (\delta_{a_1}^{a'_1} \delta_{a_2}^{a'_2} \delta_{a_3}^{a'_3} - \delta_{a_1}^{a'_1} \delta_{a_2}^{a'_2} \delta_{a_3}^{a'_3} + \delta_{a_2}^{a'_2} \delta_{a_3}^{a'_3} \delta_{a_1}^{a'_1} - \delta_{a_2}^{a'_2} \delta_{a_1}^{a'_1} \delta_{a_3}^{a'_3} + \delta_{a_3}^{a'_3} \delta_{a_1}^{a'_1} \delta_{a_2}^{a'_2} - \delta_{a_3}^{a'_3} \delta_{a_2}^{a'_2} \delta_{a_1}^{a'_1}) \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \partial^{a_3]} \partial_{a_3}^+ \\
& \quad - \delta_{a_1}^{a'_1} (\delta_{a_0}^{a'_0} \delta_{a_2}^{a'_2} \delta_{a_3}^{a'_3} - \delta_{a_0}^{a'_0} \delta_{a_3}^{a'_3} \delta_{a_2}^{a'_2} + \delta_{a_2}^{a'_2} \delta_{a_3}^{a'_3} \delta_{a_0}^{a'_0} - \delta_{a_2}^{a'_2} \delta_{a_0}^{a'_0} \delta_{a_3}^{a'_3} + \delta_{a_3}^{a'_3} \delta_{a_0}^{a'_0} \delta_{a_2}^{a'_2} - \delta_{a_3}^{a'_3} \delta_{a_2}^{a'_2} \delta_{a_0}^{a'_0}) \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \partial^{a_3]} \partial_{a_3}^+ \\
& \quad - \delta_{a_2}^{a'_2} (\delta_{a_1}^{a'_1} \delta_{a_0}^{a'_0} \delta_{a_3}^{a'_3} - \delta_{a_1}^{a'_1} \delta_{a_3}^{a'_3} \delta_{a_0}^{a'_0} + \delta_{a_0}^{a'_0} \delta_{a_3}^{a'_3} \delta_{a_1}^{a'_1} - \delta_{a_0}^{a'_0} \delta_{a_1}^{a'_1} \delta_{a_3}^{a'_3} + \delta_{a_3}^{a'_3} \delta_{a_1}^{a'_1} \delta_{a_0}^{a'_0} - \delta_{a_3}^{a'_3} \delta_{a_0}^{a'_0} \delta_{a_1}^{a'_1}) \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \partial^{a_3]} \partial_{a_3}^+ \\
& \quad - \delta_{a_3}^{a'_3} (\delta_{a_1}^{a'_1} \delta_{a_2}^{a'_2} \delta_{a_0}^{a'_0} - \delta_{a_1}^{a'_1} \delta_{a_0}^{a'_0} \delta_{a_2}^{a'_2} + \delta_{a_2}^{a'_2} \delta_{a_0}^{a'_0} \delta_{a_1}^{a'_1} - \delta_{a_2}^{a'_2} \delta_{a_1}^{a'_1} \delta_{a_0}^{a'_0} + \delta_{a_0}^{a'_0} \delta_{a_1}^{a'_1} \delta_{a_2}^{a'_2} - \delta_{a_0}^{a'_0} \delta_{a_2}^{a'_2} \delta_{a_1}^{a'_1}) \eta_{[a_1}^{[a_1} \eta_{a_2}^{a_2} \partial^{a_3]} \partial_{a_3}^+ \} \Delta(x-x') \\
& = -\frac{i}{2^2} \frac{1}{2!} \{ \delta_{a_0}^{a'_0} (\eta_{a_1}^{a_1} \eta_{a_2}^{a_2} \partial^{a_3} \partial_{a_3}^+ - \eta_{a_1}^{a_1} \eta_{a_3}^{a_3} \partial^{a_2} \partial_{a_2}^+ + \eta_{a_2}^{a_2} \eta_{a_3}^{a_3} \partial^{a_1} \partial_{a_1}^+ - \eta_{a_1}^{a_1} \eta_{a_2}^{a_2} \partial^{a_3} \partial_{a_3}^+ + \eta_{a_3}^{a_3} \eta_{a_1}^{a_1} \partial^{a_2} \partial_{a_2}^+ - \eta_{a_3}^{a_3} \eta_{a_2}^{a_2} \partial^{a_1} \partial_{a_1}^+) \\
& \quad - \delta_{a_1}^{a'_1} (\eta_{a_0}^{a_0} \eta_{a_2}^{a_2} \partial^{a_3} \partial_{a_3}^+ - \eta_{a_0}^{a_0} \eta_{a_3}^{a_3} \partial^{a_2} \partial_{a_2}^+ + \eta_{a_2}^{a_2} \eta_{a_3}^{a_3} \partial^{a_0} \partial_{a_0}^+ - \eta_{a_2}^{a_2} \eta_{a_0}^{a_0} \partial^{a_3} \partial_{a_3}^+ + \eta_{a_3}^{a_3} \eta_{a_0}^{a_0} \partial^{a_2} \partial_{a_2}^+ - \eta_{a_3}^{a_3} \eta_{a_2}^{a_2} \partial^{a_1} \partial_{a_1}^+) \\
& \quad - \delta_{a_2}^{a'_2} (\eta_{a_1}^{a_1} \eta_{a_0}^{a_0} \partial^{a_3} \partial_{a_3}^+ - \eta_{a_1}^{a_1} \eta_{a_3}^{a_3} \partial^{a_2} \partial_{a_2}^+ + \eta_{a_0}^{a_0} \eta_{a_3}^{a_3} \partial^{a_1} \partial_{a_1}^+ - \eta_{a_0}^{a_0} \eta_{a_2}^{a_2} \partial^{a_3} \partial_{a_3}^+ + \eta_{a_3}^{a_3} \eta_{a_1}^{a_1} \partial^{a_2} \partial_{a_2}^+ - \eta_{a_3}^{a_3} \eta_{a_0}^{a_0} \partial^{a_1} \partial_{a_1}^+) \\
& \quad - \delta_{a_3}^{a'_3} (\eta_{a_1}^{a_1} \eta_{a_2}^{a_2} \partial^{a_3} \partial_{a_3}^+ - \eta_{a_1}^{a_1} \eta_{a_0}^{a_0} \partial^{a_2} \partial_{a_2}^+ + \eta_{a_2}^{a_2} \eta_{a_0}^{a_0} \partial^{a_1} \partial_{a_1}^+ - \eta_{a_2}^{a_2} \eta_{a_1}^{a_1} \partial^{a_3} \partial_{a_3}^+ + \eta_{a_0}^{a_0} \eta_{a_1}^{a_1} \partial^{a_2} \partial_{a_2}^+ - \eta_{a_0}^{a_0} \eta_{a_2}^{a_2} \partial^{a_1} \partial_{a_1}^+) \} \Delta(x-x') \\
& = -\frac{i}{2^2} \frac{1}{2!} \{ \delta_{a_0}^{a'_0} (4\partial^{a_3} \partial_{a_3}^+ - 2\partial^{a_3} \partial_{a_3} + \partial^{a_3} \partial_{a_3}^+ - 4\partial^{a_3} \partial_{a_3}^+ + \partial^{a_3} \partial_{a_3}^+ - 2\partial^{a_3} \partial_{a_3}) \\
& \quad - \delta_{a_1}^{a'_1} (2\eta_{a_0}^{a_0} \partial^{a_3} \partial_{a_3}^+ - \eta_{a_0}^{a_0} \partial^{a_3} \partial_{a_3} + \partial^{a_1} \partial_{a_0}^+ - \delta_{a_0}^{a'_0} \partial^{a_3} \partial_{a_3}^+ + \partial^{+a_1} \partial_{a_0} - 2\partial^{+a_1} \partial_{a_0}^+) \\
& \quad - \delta_{a_2}^{a'_2} (2\eta_{a_0}^{a_0} \partial^{a_3} \partial_{a_3}^+ - 2\partial^{+a_2} \partial_{a_0}^+ + \partial^{+a_2} \partial_{a_0} - \delta_{a_0}^{a'_0} \partial^{a_3} \partial_{a_3}^+ + \partial^{a_2} \partial_{a_0}^+ - \eta_{a_0}^{a_0} \partial^{+a_1} \partial_{a_1}^+) \\
& \quad - \delta_{a_3}^{a'_3} (4\partial^{a_3} \partial_{a_0}^+ - 2\partial^{a_3} \partial_{a_0} + \partial^{a_3} \partial_{a_0}^+ - 4\partial^{a_3} \partial_{a_0}^+ + \partial^{a_3} \partial_{a_0}^+ - 2\partial^{a_3} \partial_{a_0}) \} \Delta(x-x') \\
& = -\frac{i}{2^2} \frac{1}{2!} \{ \delta_{a_0}^{a'_0} (4\partial^{a_3} \partial_{a_3}^+ - 2\partial^{a_3} \partial_{a_3} + \partial^{a_3} \partial_{a_3}^+ - 4\partial^{a_3} \partial_{a_3}^+ + \partial^{a_3} \partial_{a_3}^+ - 2\partial^{a_3} \partial_{a_3}) \\
& \quad - (2\eta_{a_0}^{a_0} \partial^{a_3} \partial_{a_3}^+ - \eta_{a_0}^{a_0} \partial^{a_3} \partial_{a_3} + \partial^{a_0} \partial_{a_0}^+ - \delta_{a_0}^{a'_0} \partial^{a_3} \partial_{a_3}^+ + \partial^{+a_0} \partial_{a_0} - 2\partial^{+a_0} \partial_{a_0}^+) \\
& \quad - (2\eta_{a_0}^{a_0} \partial^{a_3} \partial_{a_3}^+ - 2\partial^{+a_0} \partial_{a_0}^+ + \partial^{+a_0} \partial_{a_0} - \delta_{a_0}^{a'_0} \partial^{a_3} \partial_{a_3}^+ + \partial^{a_0} \partial_{a_0}^+ - \eta_{a_0}^{a_0} \partial^{+a_1} \partial_{a_1}^+) \\
& \quad - (4\partial^{a_0} \partial_{a_0}^+ - 2\partial^{a_0} \partial_{a_0} + \partial^{a_0} \partial_{a_0}^+ - 4\partial^{a_0} \partial_{a_0}^+ + \partial^{a_0} \partial_{a_0}^+ - 2\partial^{a_0} \partial_{a_0}) \} \Delta(x-x') \\
& = -\frac{i}{2^2} \frac{1}{2!} \{ \delta_{a_0}^{a'_0} (4\partial^{a_3} \partial_{a_3}^+ - 4\partial^{a_3} \partial_{a_3}) - 2(2\eta_{a_0}^{a_0} \partial^{a_3} \partial_{a_3}^+ - \eta_{a_0}^{a_0} \partial^{a_3} \partial_{a_3} + \partial^{a_0} \partial_{a_0}^+ + \partial^{+a_0} \partial_{a_0} - 2\partial^{+a_0} \partial_{a_0}^+) \\
& \quad - (2\partial^{a_0} \partial_{a_0}^+ - 4\partial^{a_0} \partial_{a_0}) \} \Delta(x-x') \\
& = -\frac{i}{2^2} \frac{1}{2!} \{ -8\delta_{a_0}^{a'_0} \partial_{\pi}^2 - 2[\eta_{a_0}^{a_0} (\nabla^2 - 3\partial_{\pi}^2) + \partial^{a_0} \partial_{a_0}^+ + \partial^{+a_0} \partial_{a_0} - 2\partial^{+a_0} \partial_{a_0}^+] - (2\partial^{a_0} \partial_{a_0}^+ - 4\partial^{a_0} \partial_{a_0}) \} \\
& = \frac{i}{2^2} m^2 (\eta_{a_0}^{a_0} - \frac{\partial_{a_0} \partial^{+a_0}}{m^2}) \Delta(x-x') \\
& = \frac{i}{2^2} \frac{1}{2!} \eta_{[a_1}^{[a_1} \eta_{a_1}^{a_1]} \partial_a \partial^{+a'} \Delta(x-x')
\end{aligned}$$

□

## 11 相关性质的具体计算

### 11.1 四维时空中特殊的反对称张量场具体计算一

$$\text{性质11.1.1. } \begin{cases} u^T(\vec{p}, h) C^+ u(\vec{p}, h') = 0, u^T(\vec{p}, h) C^+ v(\vec{p}, h) = 0 \\ u^T(\vec{p}, \frac{1}{2}) C^+ v(\vec{p}, -\frac{1}{2}) = -\varsigma, u^T(\vec{p}, -\frac{1}{2}) C^+ v(\vec{p}, \frac{1}{2}) = \varsigma \end{cases}$$

证明:  $u^T(\vec{p}, h) C^+ u(\vec{p}, h) = 0$ 

$$u^T(\vec{p}, \frac{1}{2}) C^+ u(\vec{p}, -\frac{1}{2}) = -u^T(\vec{p}, -\frac{1}{2}) C^+ u(\vec{p}, \frac{1}{2})$$



$$\begin{aligned}
&= \frac{\lambda^T(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z) \frac{\lambda(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E - |\vec{p}| \end{bmatrix} \\
&= \frac{i\varsigma\lambda^T(\hat{p}, \frac{1}{2})\sigma_y\lambda(\hat{p}, -\frac{1}{2})}{2m^3} \otimes \left( \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T \begin{bmatrix} m \\ -\varsigma E + |\vec{p}| \end{bmatrix} \right) \\
&= 0
\end{aligned}$$

□

证明:  $u^T(\vec{p}, h)C^+v(\vec{p}, h) = 0$ 

$$\begin{aligned}
&u^T(\vec{p}, \frac{1}{2})C^+v(\vec{p}, -\frac{1}{2}) \\
&= \frac{\lambda^T(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z) \frac{\lambda(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E - |\vec{p}| \end{bmatrix} \\
&= \frac{i\varsigma\lambda^T(\hat{p}, \frac{1}{2})\sigma_y\lambda(\hat{p}, -\frac{1}{2})}{2m^2} \otimes \left( \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T \begin{bmatrix} -m \\ -\varsigma E + |\vec{p}| \end{bmatrix} \right) \\
&= -i\varsigma\lambda^T(\hat{p}, \frac{1}{2})\sigma_y\lambda(\hat{p}, -\frac{1}{2}) \\
&= -\varsigma\lambda^+(\hat{p}, -\frac{1}{2})\lambda(\hat{p}, -\frac{1}{2}) \\
&= -\varsigma
\end{aligned}$$

□

证明:

$$\begin{aligned}
&u^T(\vec{p}, -\frac{1}{2})C^+v(\vec{p}, \frac{1}{2}) \\
&= \frac{\lambda^T(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E - |\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z) \frac{\lambda(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + |\vec{p}| \end{bmatrix} \\
&= \frac{i\varsigma\lambda^T(\hat{p}, -\frac{1}{2})\sigma_y\lambda(\hat{p}, \frac{1}{2})}{2m^2} \otimes \left( \begin{bmatrix} m \\ \varsigma E - |\vec{p}| \end{bmatrix}^T \begin{bmatrix} -m \\ -\varsigma E - |\vec{p}| \end{bmatrix} \right) \\
&= -i\varsigma\lambda^T(\hat{p}, -\frac{1}{2})\sigma_y\lambda(\hat{p}, \frac{1}{2}) \\
&= \varsigma\lambda^+(\hat{p}, \frac{1}{2})\sigma_y\lambda(\hat{p}, \frac{1}{2}) \\
&= \varsigma
\end{aligned}$$

□

## 11.2 四维时空中特殊的反对称张量场具体计算二

$$\text{性质11.2.1. } \begin{cases} u^T(\vec{p}, h)C^+\gamma_5u(\vec{p}, h) = 0, u^T(\vec{p}, h)C^+\gamma_5v(\vec{p}, h') = 0 \\ u^T(\vec{p}, -\frac{1}{2})C^+\gamma_5u(\vec{p}, \frac{1}{2}) = -1, u^T(\vec{p}, \frac{1}{2})C^+\gamma_5u(\vec{p}, -\frac{1}{2}) = 1 \end{cases}$$

证明:

$$\begin{aligned}
&u^T(\vec{p}, \frac{1}{2})C^+\gamma_5v(\vec{p}, -\frac{1}{2}) \\
&= \frac{\lambda^T(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (i\sigma_y \otimes I) \frac{\lambda(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E - |\vec{p}| \end{bmatrix} \\
&= \frac{i\lambda^T(\hat{p}, \frac{1}{2})\sigma_y\lambda(\hat{p}, -\frac{1}{2})}{2m^2} \otimes \left( \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T \begin{bmatrix} -m \\ \varsigma E - |\vec{p}| \end{bmatrix} \right) \\
&= 0
\end{aligned}$$

□

证明:

$$\begin{aligned}
&u^T(\vec{p}, \frac{1}{2})C^+\gamma_5u(\vec{p}, -\frac{1}{2}) = -u^T(\vec{p}, -\frac{1}{2})C^+\gamma_5u(\vec{p}, \frac{1}{2}) \\
&= \frac{\lambda^T(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (i\sigma_y \otimes I) \frac{\lambda(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E - |\vec{p}| \end{bmatrix} \\
&= \frac{i\lambda^T(\hat{p}, \frac{1}{2})\sigma_y\lambda(\hat{p}, -\frac{1}{2})}{2m^2} \otimes \left( \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T \begin{bmatrix} m \\ \varsigma E - |\vec{p}| \end{bmatrix} \right) \\
&= 1
\end{aligned}$$

□

## 11.3 四维时空中特殊的反对称张量场具体计算三

$$\text{性质11.3.1. } \begin{cases} u^T(\hat{p}, \frac{\kappa}{2})\bar{C}\gamma_a u(\hat{p}, \frac{\kappa}{2}) = i\sqrt{2}\varepsilon_a(\vec{p}, \kappa), u^T(\hat{p}, \frac{\kappa}{2})\bar{C}\gamma_a u(\hat{p}, -\frac{\kappa}{2}) = i\varepsilon_a(\vec{p}, 0) \\ u^T(\vec{p}, \frac{\kappa}{2})C^+\gamma_a v(\vec{p}, \frac{\kappa}{2}) = 0, u^T(\vec{p}, \frac{1}{2})C^+\gamma_a v(\vec{p}, -\frac{1}{2}) = \frac{-i\kappa p_a}{m}, u^T(\vec{p}, -\frac{1}{2})C^+\gamma_a v(\vec{p}, \frac{1}{2}) = \frac{i\kappa p_a}{m} \end{cases}$$

证明:

$$\begin{aligned} & u^T(\vec{p}, \frac{1}{2})C^+\gamma_a v(\vec{p}, -\frac{1}{2}) \\ &= \frac{\lambda^T(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z)\gamma_a \frac{\lambda(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E - |\vec{p}| \end{bmatrix} \\ &= \frac{\lambda^+(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (\varsigma I \otimes \sigma_z)(\sigma_x \otimes \sigma_y, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_y, \varsigma I \otimes \sigma_x) \frac{\lambda(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E - |\vec{p}| \end{bmatrix} \\ &= \frac{\lambda^+(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (-i\varsigma\sigma \otimes \sigma_x, iI \otimes \sigma_y) \frac{\lambda(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E - |\vec{p}| \end{bmatrix} \\ &= \begin{pmatrix} -i\varsigma\vec{p} & \varsigma E \\ m & m \end{pmatrix} \\ &= \frac{-i\kappa p_a}{m} \end{aligned} \quad \square$$

证明:

$$\begin{aligned} & u^T(\vec{p}, -\frac{1}{2})C^+\gamma_a v(\vec{p}, \frac{1}{2}) \\ &= \frac{\lambda^T(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E - |\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z)\gamma_a \frac{\lambda(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + |\vec{p}| \end{bmatrix} \\ &= -\frac{\lambda^+(\hat{p}, \frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E - |\vec{p}| \end{bmatrix}^T (\varsigma I \otimes \sigma_z)(\sigma_x \otimes \sigma_y, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_y, \varsigma I \otimes \sigma_x) \frac{\lambda(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + |\vec{p}| \end{bmatrix} \\ &= -\frac{\lambda^+(\hat{p}, \frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E - |\vec{p}| \end{bmatrix}^T (-i\varsigma\sigma \otimes \sigma_x, iI \otimes \sigma_y) \frac{\lambda(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + |\vec{p}| \end{bmatrix} \\ &= \begin{pmatrix} i\varsigma\vec{p} & -\varsigma E \\ m & m \end{pmatrix} \\ &= \frac{i\kappa p_a}{m} \end{aligned} \quad \square$$

证明:

$$\begin{aligned} & u^T(\vec{p}, \frac{1}{2})C^+\gamma_a v(\vec{p}, \frac{1}{2}) \\ &= \frac{\lambda^T(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z)\gamma_a \frac{\lambda(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + |\vec{p}| \end{bmatrix} \\ &= \frac{\lambda^+(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (\varsigma I \otimes \sigma_z)(\sigma_x \otimes \sigma_y, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_y, \varsigma I \otimes \sigma_x) \frac{\lambda(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + |\vec{p}| \end{bmatrix} \\ &= \frac{\lambda^+(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (-i\varsigma\sigma \otimes \sigma_x, iI \otimes \sigma_y) \frac{\lambda(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E + |\vec{p}| \end{bmatrix} \\ &= 0 \end{aligned} \quad \square$$

## 11.4 四维时空中特殊的反对称张量场具体计算四

$$\text{性质11.4.1. } u^T(\hat{p}, \frac{\kappa}{2})\bar{C}\gamma_{[a}\gamma_{b]}u(\hat{p}, \frac{\kappa}{2}) = i\sqrt{2}p_a\varepsilon_b(\vec{p}, \kappa)$$

证明:

$$\begin{aligned} & u^T(\vec{p}, \frac{1}{2})C^+\gamma_{[a}\gamma_{b]}v(\vec{p}, -\frac{1}{2}) \\ &= \frac{\lambda^T(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T (i\varsigma\sigma_y \otimes \sigma_z)\gamma_{[a}\gamma_{b]} \frac{\lambda(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \varsigma E - |\vec{p}| \end{bmatrix} \\ &= \frac{\lambda^+(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E+\varsigma|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \varsigma E + |\vec{p}| \end{bmatrix}^T \end{aligned}$$

$$\begin{aligned}
 & (\zeta I \otimes \sigma_z)(i\sigma_z \otimes I, i\sigma_x \otimes I, i\sigma_y \otimes I, -i\zeta\sigma_x \otimes \sigma_z, -i\zeta\sigma_y \otimes \sigma_z, -i\zeta\sigma_z \otimes \sigma_z) \\
 & \frac{\lambda(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\zeta|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \zeta E - |\vec{p}| \end{bmatrix} \\
 & = \frac{\lambda^+(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E+\zeta|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \zeta E + |\vec{p}| \end{bmatrix}^T \\
 & (i\zeta\sigma_z \otimes \sigma_z, i\zeta\sigma_x \otimes \sigma_z, i\zeta\sigma_y \otimes \sigma_z, -i\sigma_x \otimes I, -i\sigma_y \otimes I, -i\sigma_z \otimes I) \\
 & \frac{\lambda(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E-\zeta|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \zeta E - |\vec{p}| \end{bmatrix} \\
 & = (i\zeta\hat{p}_z, i\zeta\hat{p}_x, i\zeta\hat{p}_y, 0, 0, 0)
 \end{aligned}$$

□

证明:

$$\begin{aligned}
 & u^T(\vec{p}, \frac{1}{2})C^+\gamma_{[a}\gamma_{b]}v(\vec{p}, \frac{1}{2}) \\
 & = \frac{\lambda^T(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\zeta|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \zeta E + |\vec{p}| \end{bmatrix}^T (i\zeta\sigma_y \otimes \sigma_z)\gamma_{[a}\gamma_{b]}\frac{\lambda(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\zeta|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \zeta E + |\vec{p}| \end{bmatrix} \\
 & = \frac{\lambda^+(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E+\zeta|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \zeta E + |\vec{p}| \end{bmatrix}^T \\
 & (\zeta I \otimes \sigma_z)(i\sigma_z \otimes I, i\sigma_x \otimes I, i\sigma_y \otimes I, -i\zeta\sigma_x \otimes \sigma_z, -i\zeta\sigma_y \otimes \sigma_z, -i\zeta\sigma_z \otimes \sigma_z) \\
 & \frac{\lambda(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\zeta|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \zeta E + |\vec{p}| \end{bmatrix} \\
 & = \frac{\lambda^+(\hat{p}, -\frac{1}{2})}{\sqrt{2m(E+\zeta|\vec{p}|)}} \otimes \begin{bmatrix} m \\ \zeta E + |\vec{p}| \end{bmatrix}^T \\
 & (i\zeta\sigma_z \otimes \sigma_z, i\zeta\sigma_x \otimes \sigma_z, i\zeta\sigma_y \otimes \sigma_z, -i\sigma_x \otimes I, -i\sigma_y \otimes I, -i\sigma_z \otimes I) \\
 & \frac{\lambda(\hat{p}, \frac{1}{2})}{\sqrt{2m(E+\zeta|\vec{p}|)}} \otimes \begin{bmatrix} -m \\ \zeta E + |\vec{p}| \end{bmatrix} \\
 & =
 \end{aligned}$$

□

## 12 N+1维时空中的基本反对称张量场高阶推广的猜想

### 12.1 N+1维时空中有质量自旋 $s = n$ 的 Bargmann-Wigner 方程的猜想

定义12.1.1.  $\mathbb{X}^{a_1 \dots a_l} := \{\frac{1}{l!}\gamma^{[a_1} \dots \gamma^{a_l]} + \frac{(-1)^l}{(l+1)!m}\gamma^{[a_1} \dots \gamma^{a_{l+1}}]\partial_{a_{l+1}}\}C^{\frac{1}{l!}}$

猜想12.1.1.

$$\begin{cases}
 [\gamma^a(\zeta)\partial_a + m]\psi_{\underbrace{[\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \zeta_\zeta]_{2n}}_{2n}} = 0, \psi_{\underbrace{[\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \zeta_\zeta]_{2n}}_{2n}} = \frac{1}{(2n)!}\psi_{\{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \zeta_\zeta\}_{2n}}^\sigma \\
 \psi_{\underbrace{[\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \dots \zeta_\zeta]_{2n}}_{2n}}^\sigma = \frac{1}{(l!)^2}F_{a_1 \dots a_l} \underbrace{\eta_\zeta \xi_\zeta \dots \zeta_\zeta}_{2n-2}^\sigma \gamma^{[a_1} \dots \gamma^{a_l]}C + \frac{1}{[(l+1)!]^2}F_{a_1 \dots a_{l+1}} \underbrace{\eta_\zeta \xi_\zeta \dots \zeta_\zeta}_{2n-2}^\sigma \gamma^{[a_1} \dots \gamma^{a_{l+1}}]C \\
 \Leftrightarrow \begin{cases}
 (-\partial^d \partial_d + m^2)A_{\underbrace{a_1 \dots a_l | b_1 \dots b_l | c_1 \dots c_l}_{n}}^\sigma = 0 \\
 \delta^{a_1 b_1} A_{\underbrace{a_1 \dots a_l | b_1 \dots b_l | c_1 \dots c_l}_{n}}^\sigma = 0, \partial^{a_1} A_{\underbrace{a_1 \dots a_l | b_1 \dots b_l | c_1 \dots c_l}_{n}}^\sigma = 0 \\
 A_{\underbrace{a_1 \dots a_l | b_1 \dots b_l | c_1 \dots c_l}_{n}}^\sigma \text{ 关于 } a_1 b_1 c_1 \dots \text{ 全对称, 关于 } a_1 \dots a_l, b_1 \dots b_l, c_1 \dots c_l, \dots \text{ 全反对称} \\
 \psi_{\underbrace{[\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \rho_\zeta \tau_\zeta]_{2n}}_{2n}}^\sigma = \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^{a_1 \dots a_l} \mathbb{X}_{\eta_\zeta \xi_\zeta}^{b_1 \dots b_l} \mathbb{X}_{\rho_\zeta \tau_\zeta}^{c_1 \dots c_l}}_n \dots A_{\underbrace{a_1 \dots a_l | b_1 \dots b_l | c_1 \dots c_l}_{n}}^\sigma
 \end{cases}
 \end{cases}$$

## 12.2 N+1维时空中有质量自旋 $s = n + \frac{1}{2}$ 的 Bargmann-Wigner 方程的猜想

猜想12.2.1.

$$\begin{cases} [\gamma^a(\zeta)\partial_a + m]\psi_{\underbrace{[\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\cdots\zeta_\zeta]_{2n+1}}^\sigma} = 0, \psi_{\underbrace{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\cdots\zeta_\zeta}_{2n+1}}^\sigma = \frac{1}{(2n+1)!}\psi_{\underbrace{\{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\cdots\zeta_\zeta\}_{2n+1}}^\sigma} \\ \psi_{\underbrace{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\cdots\zeta_\zeta}_{2n+1}}^\sigma = \frac{1}{(l!)^2}F_{a_1\cdots a_l}\underbrace{\eta_\zeta\xi_\zeta\cdots\zeta_\zeta}_{2n-1}^\sigma\gamma^{[a_1\cdots a_l]}C + \frac{1}{[(l+1)!]^2}F_{a_1\cdots a_{l+1}}\underbrace{\eta_\zeta\xi_\zeta\cdots\zeta_\zeta}_{2n-1}^\sigma\gamma^{[a_1\cdots a_{l+1}]}C \\ \Leftrightarrow \begin{cases} [\gamma^d(\zeta)\partial_d + m]A_{\underbrace{a_1\cdots a_l|b_1\cdots b_l|c_1\cdots c_l|\cdots[\zeta_\zeta]}_n}^\sigma = 0 \\ \delta^{a_1b_1}A_{\underbrace{a_1\cdots a_l|b_1\cdots b_l|c_1\cdots c_l|\cdots[\zeta_\zeta]}_n}^\sigma = 0, \gamma^{a_1}(\zeta)A_{\underbrace{a_1\cdots a_l|b_1\cdots b_l|c_1\cdots c_l|\cdots[\zeta_\zeta]}_n}^\sigma = 0 \\ A_{\underbrace{a_1\cdots a_l|b_1\cdots b_l|c_1\cdots c_l|\cdots[\zeta_\zeta]}_n}^\sigma \text{ 关于 } a_1b_1c_1\cdots \text{ 全对称, 关于 } a_1\cdots a_l, b_1\cdots b_l, c_1\cdots c_l, \cdots \text{ 全反对称} \\ \psi_{\underbrace{\lambda_\zeta\mu_\zeta\eta_\zeta\xi_\zeta\rho_\zeta\tau_\zeta\cdots\zeta_\zeta}_{2n+1}}^\sigma = \underbrace{\mathbb{X}_{\lambda_\zeta\mu_\zeta}^{a_1\cdots a_l}\mathbb{X}_{\eta_\zeta\xi_\zeta}^{b_1\cdots b_l}\mathbb{X}_{\rho_\zeta\tau_\zeta}^{c_1\cdots c_l}}_n \cdots A_{\underbrace{a_1\cdots a_l|b_1\cdots b_l|c_1\cdots c_l|\cdots[\zeta_\zeta]}_n}^\sigma \end{cases} \end{cases}$$

自我评述：采用数学归纳法应该可以严格证明得到以上两个猜想，以后有时间再说。同时也启示物理学远远没有完成，因为还可以构造很多很多有意思的物理方程，原则上可以是无穷个，所以物理学发展是永无止境的，让人无限向往，但也让人绝望至极！

## 13 N+1维时空中的电磁场方程分析

### 13.1 N+1维时空中的 Bargmann-Wigner 矢量场方程

定义13.1.1.  $\mathbb{X}_a := [im\gamma_a - 2S_{ab}(e)\partial^b]C, \mathbb{X}^a := [im\gamma^a - 2S^{ab}(e)\partial_b]C$

$$\text{引理13.1.1.} \begin{cases} (\gamma^a\partial_a + m)X = 0 \\ X = \left\{ \frac{1}{(1!)^2}F^a\gamma_a + \frac{1}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]} \right\} C \end{cases} \Leftrightarrow \begin{cases} \partial^{[a}F^{b]} + mF^{ab} = 0, \partial_a F^a = 0 \\ \partial^{[a}F^{bc]} = 0, \partial_a F^{ab} + mF^b = 0 \end{cases}$$

$$\text{推论13.1.1.} \begin{cases} (\gamma^a\partial_a + m)\psi = 0 \\ \psi = \left\{ \frac{1}{(1!)^2}imA^a\gamma_a - \frac{i}{(2!)^2}F^{ab}\gamma_{[a}\gamma_{b]} \right\} C \end{cases} \Leftrightarrow \begin{cases} \partial_a F^{ab} - m^2 A^b = 0 \\ F^{ab} = \partial^{[a}A^{b]}, \partial_a A^a = 0 \end{cases}$$

引理13.1.2.  $\frac{2^{-5}}{im}tr(\bar{C}\gamma_{a'}\mathbb{X}^a)A_a = \frac{2^{-5}}{im}(\bar{C}\gamma_{a'})^{\lambda_\zeta\mu_\zeta}\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a A_a = A_{a'}, (\bar{C}\gamma_{a'})^{\lambda_\zeta\mu_\zeta}\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a = im2^5\delta_a^a$

引理13.1.3.  $\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a \mathbb{X}_{\eta_\zeta\xi_\zeta}^b A_{ab} = \mathbb{X}_{\eta_\zeta\xi_\zeta}^a \mathbb{X}_{\lambda_\zeta\mu_\zeta}^b A_{ab} \Leftrightarrow A_{ab} = A_{ba}$

推论13.1.2.  $(\gamma^b\partial_b + m)\mathbb{X}^a A_a \Leftrightarrow (\partial^b\partial_b - m^2)A_a = 0, \partial^a A_a = 0$

证明:  $(\gamma^c\partial_c + m)\mathbb{X}^a A_a = 0$

$$\Leftrightarrow (\gamma^c\partial_c + m)[im\gamma^a - 2S^{ab}(e)\partial_b]CA_a = 0$$

$$\Leftrightarrow (\gamma^c\partial_c + m)[im\gamma^a - 2S^{ab}(e)\partial_b]A_a = 0$$

$$\Leftrightarrow im\gamma^c\gamma^a\partial_c A_a + im^2\gamma^a A_a + \frac{i}{2}\gamma^c\gamma^{[a}\gamma^{b]}\partial_c\partial_b A_a + \frac{i}{2}m\gamma^{[a}\gamma^{b]}\partial_b A_a = 0$$

$$\Leftrightarrow -\frac{i}{2}\gamma^a\gamma^{[b}\gamma^{c]}\partial_a\partial_b A_c + im^2\gamma^a A_a + \frac{i}{2}m\gamma^{[a}\gamma^{b]}\partial_b A_a = 0$$

$$\Leftrightarrow -\frac{i}{2}(\frac{1}{3}\gamma_{[a}\gamma_b\gamma_{c]} + 2\delta_{a[b}\gamma_{c]})\partial_a\partial_b A_c + im^2\gamma^a A_a + im\delta^{ab}\partial_b A_a = 0$$

$$\Leftrightarrow -i\delta_{a[b}\gamma_{c]}\partial_a\partial_b A_c + im^2\gamma^a A_a + im\partial^a A_a = 0$$

$$\Leftrightarrow -i\gamma^c\partial_a\partial^a A_c + i\gamma^b\partial_b(\partial^a A_a) + im^2\gamma^a A_a + im\partial^a A_a = 0$$

$$\Leftrightarrow \gamma^a[-i\partial_b\partial^b A_a + i\partial_a(\partial^b A_b) + im^2 A_a] + im\partial^a A_a = 0$$

$$\Leftrightarrow -i\partial_b\partial^b A_a + i\partial_a(\partial^b A_b) + im^2 A_a = 0, im\partial^a A_a = 0$$

$$\Leftrightarrow (\partial_b\partial^b - m^2)A_a = 0, \partial^a A_a = 0$$

□

推论13.1.3.  $(\gamma^c\partial_c + m)_{\kappa_\zeta}^{\lambda_\zeta}\mathbb{X}_{\lambda_\zeta\mu_\zeta}^a \mathbb{X}_{\eta_\zeta\xi_\zeta}^b A_{ab} = 0 \Leftrightarrow (\partial^c\partial_c - m^2)A_{ab} = 0, \partial^a A_{ab} = 0$

证明:  $(\gamma^c \partial_c + m)_{\kappa_\zeta} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \mathbb{X}_{\eta_\zeta \xi_\zeta}^b A_{ab} = 0$

$$\Leftrightarrow (\gamma^c \partial_c + m) \mathbb{X}^a \mathbb{X}^b A_{ab} = 0$$

$$\Leftrightarrow (\partial_c \partial^c - m^2) \mathbb{X}_{\eta_\zeta \xi_\zeta}^b A_{ab} = 0, \partial^a \mathbb{X}_{\eta_\zeta \xi_\zeta}^b A_{ab} = 0$$

$$\Leftrightarrow (\partial_c \partial^c - m^2) \mathbb{X}^b A_{ab} = 0, \partial^a \mathbb{X}^b A_{ab} = 0$$

$$\Leftrightarrow (\partial_a \partial^d - m^2) [im\gamma^b - 2S^{bc}(e)\partial_c] C A_{ab} = 0, \partial^a [im\gamma^b - 2S^{bc}(e)\partial_c] C A_{ab} = 0$$

$$\Leftrightarrow (\partial_d \partial^d - m^2) A_{ab} = 0, (\partial_d \partial^d - m^2) (\partial_c A_{ab} - \partial_b A_{ac}) = 0, \partial^a A_{ab} = 0, \partial^a (\partial_c A_{ab} - \partial_b A_{ac}) = 0$$

$$\Leftrightarrow (\partial_c \partial^c - m^2) A_{ab} = 0, \partial^a A_{ab} = 0 \quad \square$$

$$\text{推论13.1.4.} \quad \begin{cases} (\gamma^c \partial_c + m)_{\kappa_\zeta} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \mathbb{X}_{\eta_\zeta \xi_\zeta}^b A_{ab} = 0 \\ \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \mathbb{X}_{\eta_\zeta \xi_\zeta}^b A_{ab} = \mathbb{X}_{\eta_\zeta \xi_\zeta}^b \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a A_{ab} \end{cases} \Leftrightarrow \begin{cases} (\partial^c \partial_c - m^2) A_{ab} = 0 \\ \partial^a A_{ab} = 0, A_{ab} = A_{ba} \end{cases}$$

推论13.1.5.  $\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \mathbb{X}_{\eta_\zeta \xi_\zeta}^b A_{ab} = \mathbb{X}_{\eta_\zeta \xi_\zeta}^b \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a A_{ab} \Leftrightarrow A_{ab} = A_{ba}$

证明:  $\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \mathbb{X}_{\eta_\zeta \xi_\zeta}^b A_{ab} = \mathbb{X}_{\eta_\zeta \xi_\zeta}^b \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a A_{ab}$

$$\Leftrightarrow [im\gamma^a C - 2S^{ac}(e)C\partial_c]_{\lambda_\zeta \mu_\zeta} [im\gamma^b C - 2S^{bd}(e)C\partial_d]_{\eta_\zeta \xi_\zeta} A_{ab} = [im\gamma^a C - 2S^{ac}(e)C\partial_c]_{\lambda_\zeta \mu_\zeta} [im\gamma^b C - 2S^{bd}(e)C\partial_d]_{\eta_\zeta \xi_\zeta} A_{ba}$$

$$\Leftrightarrow (im)^2 A_{ab} [\gamma^a C]_{\lambda_\zeta \mu_\zeta} [\gamma^b C]_{\eta_\zeta \xi_\zeta} + 4\partial_c \partial_d A_{ab} [S^{ac}(e)C]_{\lambda_\zeta \mu_\zeta} [S^{bd}(e)C]_{\eta_\zeta \xi_\zeta}$$

$$- 2im\partial_d A_{ab} [\gamma^a C]_{\lambda_\zeta \mu_\zeta} [S^{bd}(e)C]_{\eta_\zeta \xi_\zeta} - 2im\partial_c A_{ab} [S^{ac}(e)C]_{\lambda_\zeta \mu_\zeta} [\gamma^b C]_{\eta_\zeta \xi_\zeta}$$

$$= (im)^2 A_{ba} [\gamma^a C]_{\lambda_\zeta \mu_\zeta} [\gamma^b C]_{\eta_\zeta \xi_\zeta} + 4\partial_c \partial_d A_{ba} [S^{ac}(e)C]_{\lambda_\zeta \mu_\zeta} [S^{bd}(e)C]_{\eta_\zeta \xi_\zeta}$$

$$- 2im\partial_d A_{ba} [\gamma^a C]_{\lambda_\zeta \mu_\zeta} [S^{bd}(e)C]_{\eta_\zeta \xi_\zeta} - 2im\partial_c A_{ba} [S^{ac}(e)C]_{\lambda_\zeta \mu_\zeta} [\gamma^b C]_{\eta_\zeta \xi_\zeta}$$

$$\Leftrightarrow A_{ab} = A_{ba}, \partial_c \partial_d A_{ab} + \partial_a \partial_b A_{cd} - \partial_a \partial_d A_{cb} - \partial_c \partial_d A_{ab} = \partial_c \partial_d A_{ba} + \partial_a \partial_b A_{dc} - \partial_a \partial_d A_{bc} - \partial_c \partial_d A_{ba},$$

$$\partial_d A_{ab} - \partial_b A_{ad} = \partial_d A_{ba} - \partial_b A_{da}, \partial_c A_{ab} - \partial_a A_{cb} = \partial_c A_{ba} - \partial_a A_{bc}$$

$$\Leftrightarrow A_{ab} = A_{ba} \quad \square$$

$$\text{推论13.1.6.} \quad \begin{cases} (\gamma^c \partial_c + m)_{\kappa_\zeta} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \mathbb{X}_{\eta_\zeta \xi_\zeta}^b A_{ab} = 0 \\ \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \mathbb{X}_{\eta_\zeta \xi_\zeta}^b A_{ab} = \mathbb{X}_{\eta_\zeta \xi_\zeta}^b \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a A_{ab} \end{cases} \quad ?? \Leftrightarrow \begin{cases} (\partial^c \partial_c - m^2) A_{ab} = 0 \\ \partial^a A_{ab} = 0, A_{ab} = A_{ba}, \delta^{ab} A_{ab} = 0 \end{cases}$$

$$\text{推论13.1.7.} \quad \begin{cases} [\gamma^a(\zeta)\partial_a + m]_{\kappa_\zeta} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \overbrace{\mathbb{X}_{\eta_\zeta \xi_\zeta}^b \cdots A_{abc \cdots}^\sigma}^n = 0 \Leftrightarrow (-\partial^d \partial_d + m^2) \overbrace{A_{abc \cdots}^\sigma}^n = 0 \\ \overbrace{A_{abc \cdots}^\sigma}^n = \frac{1}{n!} A_{\{abc \cdots\}^\sigma}, \delta^{ab} \overbrace{A_{abc \cdots}^\sigma}^n = 0, \partial^a \overbrace{A_{abc \cdots}^\sigma}^n = 0 \end{cases}$$

$$\text{推论13.1.8.} \quad \begin{cases} [\gamma^a(\zeta)\partial_a + m]_{\kappa_\zeta} \mathbb{X}_{\lambda_\zeta \mu_\zeta}^a \overbrace{\mathbb{X}_{\eta_\zeta \xi_\zeta}^b \cdots A_{abc \cdots}^{[\zeta]^\sigma}}^n = 0 \Leftrightarrow [\gamma^d(\zeta)\partial_d + m] \overbrace{A_{abc \cdots}^{[\zeta]^\sigma}}^n = 0 \\ \overbrace{A_{abc \cdots}^{[\zeta]^\sigma}}^n = \frac{1}{n!} A_{\{abc \cdots\}^{[\zeta]^\sigma}}, \delta^{ab} \overbrace{A_{abc \cdots}^{[\zeta]^\sigma}}^n = 0, \gamma^a(\zeta) \overbrace{A_{abc \cdots}^{[\zeta]^\sigma}}^n = 0 \end{cases}$$

猜想13.1.1.  $(\gamma^c \partial_c + m) \mathbb{X}^a \mathbb{X}^b A_{ab} = 0 \Leftrightarrow ?? (\partial^c \partial_c - m^2) A_{ab} = 0, \partial^a A_{ab} = 0, \delta^{ab} A_{ab} = 0$

证明:  $(\gamma^c \partial_c + m) \mathbb{X}^a \mathbb{X}^b A_{ab} = 0$

$$\Leftrightarrow (\gamma^c \partial_c + m) [im\gamma^a - 2S^{ac}(e)\partial_c] C [im\gamma^b - 2S^{bd}(e)\partial_d] C A_{ab} = 0$$

$$\Leftrightarrow (\gamma^c \partial_c + m) [-m^2 \gamma^a C \gamma^b C + 4S^{ac}(e)CS^{bd}(e)C\partial_c \partial_d - 2im\gamma^a CS^{bd}(e)C\partial_d - 2imS^{ac}(e)C\gamma^b C\partial_c] A_{ab} = 0 \quad \square$$

# 第三十八章 二阶完美常数不变张量

自我评述：本章主要是二阶完美常数不变张量的具体介绍，是对更一般完美常数不变张量的深入具体了解和具体扩展。若将全对称自旋张量提取高自旋的程序反复利用，就可以得到越来越高阶的完美常数不变张量。同时它是之前完美常数不变张量一般情形的特殊运用而已，故直接写出其结论即可，而无需再作证明。通过本章具体化的例子，更可以明确以下事实：完美常数不变张量只是反映了全对称代数结构的内在固有性质，只与全对称有关、与其它无关，因此表现出了一系列完美的代数性质和变换性质。本章符号约定符合以往规范，出了本章也不会混淆。本章节的完美常数不变张量与  $w = 1$  情形不同构。

## 1 二阶完美常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)$

### 1.1 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)$

定义1.1.1.  $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j) = \frac{1}{(2s)!} \Gamma_{(A_\zeta B_\zeta C_\zeta \dots)}^{k_\zeta}(s; 2j), w = 2j$

$$\Gamma_{\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}}^{k_\zeta}(s; 2j) = \sqrt{\frac{l_0! l_1! \dots l_w!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^{[s]} (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+2j-k}^{2l} - C_{2l+2j-\lambda_{2l}}^{2l})\}, l_0 + l_1 + \dots + l_w = 2s$$

定义1.1.2.  $\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j) = \frac{1}{(2s)!} \Gamma_{k_\zeta}^{(A_\zeta B_\zeta C_\zeta \dots)}(s; 2j), w = 2j$

$$\Gamma_{k_\zeta}^{\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}}(s; 2j) = \sqrt{\frac{l_0! l_1! \dots l_w!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^{[s]} (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+2j-k}^{2l} - C_{2l+2j-\lambda_{2l}}^{2l})\}, l_0 + l_1 + \dots + l_w = 2s$$

### 1.2 常数矩阵 $\Gamma(s; 2j), \bar{\Gamma}(s; 2j)$

定义1.2.1.  $\Gamma(s; 2j) \succ \Gamma_{\underbrace{A_\zeta \otimes B_\zeta \otimes C_\zeta \otimes \dots}_{2s}}^{k_\zeta}(s; 2j), \bar{\Gamma}(s; 2j) \succ \Gamma_{k_\zeta}^{\underbrace{A_\zeta \otimes B_\zeta \otimes C_\zeta \otimes \dots}_{2s}}(s; 2j) \simeq \Gamma^T(s; 2j)$

推论1.2.1.  $[\Gamma(s; 2j)] = (2j+1)^{2s} \times C_{2s+2j}^{2s}, [\bar{\Gamma}(s; 2j)] = C_{2s+2j}^{2s} \times (2j+1)^{2s}, [A_\zeta] = 2j+1, [k_\zeta] = C_{2s+2j}^{2s}$

### 1.3 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)$ 的基本性质

相等性：

性质1.3.1.  $\Gamma_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}^{k'_\zeta}(s; 2j) \simeq \Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; 2j) \simeq \Gamma_{k_\zeta}^{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}(s; 2j) \simeq \Gamma_{k'_\zeta}^{\underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}(s; 2j)$

性质1.3.2.  $[\Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; 2j)]^* \simeq \Gamma_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}^{k'_\zeta}(s; 2j), [\Gamma_{k_\zeta}^{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}(s; 2j)]^* \simeq \Gamma_{k'_\zeta}^{\underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}(s; 2j)$

推论1.3.1.  $\Gamma(s; 2j) = \Gamma^*(s; 2j), \bar{\Gamma}(s; 2j) = \bar{\Gamma}^*(s; 2j), \bar{\Gamma}(s; 2j) = \Gamma^+(s; 2j), \Gamma(s; 2j) = \bar{\Gamma}^+(s; 2j)$

正交性：

性质1.3.3.  $\Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; 2j) \Gamma_{l_\zeta}^{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}(s; 2j) = \delta^{k_\zeta l_\zeta} [\Leftrightarrow] \bar{\Gamma}(s; 2j) \Gamma(s; 2j) = I$

性质1.3.4.  $\Gamma_{A_{1\zeta} A_{2\zeta} \dots A_{2s\zeta}}^{k_\zeta}(s; 2j) \Gamma_{k_\zeta}^{B_{1\zeta} B_{2\zeta} \dots B_{2s\zeta}}(s; 2j) = \frac{1}{(2s)!} \delta_{A_{1\zeta}}^{(B_{1\zeta}} \delta_{A_{2\zeta}}^{B_{2\zeta}} \dots \delta_{A_{2s\zeta}}^{B_{2s\zeta}}) = \frac{1}{(2s)!} \delta_{(A_{1\zeta}}^{B_{1\zeta}} \delta_{A_{2\zeta}}^{B_{2\zeta}} \dots \delta_{A_{2s\zeta}}^{B_{2s\zeta}})$

对比性：

性质1.3.5.  $\varepsilon_{a_1 a_2 \dots a_n} \varepsilon^{b_1 b_2 \dots b_n} = \delta_{a_1}^{[b_1} \delta_{a_2}^{b_2} \dots \delta_{a_n}^{b_n]} = \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_n}^{b_n]}$

其它性质：

$$\text{性质1.3.6. } \Gamma_{A_\zeta}^{k_\zeta}(\frac{1}{2}; 2j) = \delta_{A_\zeta}^{k_\zeta}, \Gamma_{k_\zeta}^{A_\zeta}(\frac{1}{2}; 2j) = \delta_{k_\zeta}^{A_\zeta}; \Gamma(0; 2j) = 1, \bar{\Gamma}(0; 2j) = 1$$

#### 1.4 度规常数不变张量 $\varepsilon_{k_\zeta l_\zeta}(s; 2j)$ 的引入及其性质

度规定义：

$$\text{定义1.4.1. } \left\{ \begin{array}{l} \varepsilon_{k_\zeta l_\zeta}(s; 2j) := \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)}_{2s} \underbrace{\varepsilon_{A_\zeta E_\zeta}(j) \varepsilon_{B_\zeta F_\zeta}(j) \varepsilon_{C_\zeta G_\zeta}(j) \dots}_{2s} \underbrace{\Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; 2j)}_{2s} \\ \varepsilon^{k_\zeta l_\zeta}(s; 2j) := \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j)}_{2s} \underbrace{\varepsilon^{A_\zeta E_\zeta}(j) \varepsilon^{B_\zeta F_\zeta}(j) \varepsilon^{C_\zeta G_\zeta}(j) \dots}_{2s} \underbrace{\Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; 2j)}_{2s} \end{array} \right.$$

$$\text{性质1.4.1. } \left\{ \begin{array}{l} \underbrace{\varepsilon_{A_\zeta E_\zeta}(j) \varepsilon_{B_\zeta F_\zeta}(j) \varepsilon_{C_\zeta G_\zeta}(j) \dots}_{2s} \underbrace{\Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; 2j)}_{2s} \text{ 关于 } ABC\dots \text{ 全对称} \\ \underbrace{\varepsilon^{A_\zeta E_\zeta}(j) \varepsilon^{B_\zeta F_\zeta}(j) \varepsilon^{C_\zeta G_\zeta}(j) \dots}_{2s} \underbrace{\Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; 2j)}_{2s} \text{ 关于 } ABC\dots \text{ 全对称} \end{array} \right.$$

$$\text{推论1.4.1. } \varepsilon(s; 2j) := \bar{\Gamma}(s; 2j) \underbrace{\varepsilon(j) \otimes \dots \otimes \varepsilon(j)}_{2s} \Gamma(s; 2j); \varepsilon(j) = \varepsilon(j; 1) = \varepsilon(\frac{1}{2}; 2j)$$

$$\text{推论1.4.2. } \varepsilon(s; 2j) \varepsilon^+(s; 2j) = \varepsilon^+(s; 2j) \varepsilon(s; 2j) = 1; \varepsilon(s; 2j) = \varepsilon^*(s; 2j), \varepsilon(s; 2j) = (-1)^{4sj} \varepsilon(s; 2j)$$

升降指标：

$$\text{性质1.4.2. } \left\{ \begin{array}{l} \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j)}_{2s} = \varepsilon^{k_\zeta l_\zeta}(s; 2j) \underbrace{\varepsilon_{A_\zeta E_\zeta}(j) \varepsilon_{B_\zeta F_\zeta}(j) \varepsilon_{C_\zeta G_\zeta}(j) \dots}_{2s} \underbrace{\Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; 2j)}_{2s} \\ \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)}_{2s} = \varepsilon_{k_\zeta l_\zeta}(s; 2j) \underbrace{\varepsilon^{A_\zeta E_\zeta}(j) \varepsilon^{B_\zeta F_\zeta}(j) \varepsilon^{C_\zeta G_\zeta}(j) \dots}_{2s} \underbrace{\Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; 2j)}_{2s} \end{array} \right.$$

$$\text{推论1.4.3. } \Gamma(s; 2j) \varepsilon(s; 2j) = \underbrace{\varepsilon(j) \otimes \dots \otimes \varepsilon(j)}_{2s} \Gamma(s; 2j), \varepsilon(s; 2j) \bar{\Gamma}(s; 2j) = \bar{\Gamma}(s; 2j) \underbrace{\varepsilon(j) \otimes \dots \otimes \varepsilon(j)}_{2s}$$

Penrose标准升降规则：

性质1.4.3.

$$\left\{ \begin{array}{l} \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j)}_{2s} = (-1)^{2s} [\zeta^{2s} \varepsilon^{k_\zeta l_\zeta}(s; 2j)] \underbrace{[-\zeta \varepsilon_{A_\zeta E_\zeta}(j)] [-\zeta \varepsilon_{B_\zeta F_\zeta}(j)] [-\zeta \varepsilon_{C_\zeta G_\zeta}(j)] \dots}_{2s} \underbrace{\Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; 2j)}_{2s} \\ \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)}_{2s} = (-1)^{2s} [(-\zeta)^{2s} \varepsilon_{k_\zeta l_\zeta}(s; 2j)] \underbrace{[\zeta \varepsilon^{A_\zeta E_\zeta}(j)] [\zeta \varepsilon^{B_\zeta F_\zeta}(j)] [\zeta \varepsilon^{C_\zeta G_\zeta}(j)] \dots}_{2s} \underbrace{\Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; 2j)}_{2s} \end{array} \right.$$

#### 1.5 自旋常数不变张量 $\sigma^{\alpha_\zeta l_\zeta}(s; 2j), S_{ab}(s, \zeta; 2j)$

定义1.5.1.

$$\left\{ \begin{array}{l} \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)}_{2s} \sigma^{\alpha_\zeta A_\zeta Z_\zeta}(j) \underbrace{\Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta}(s; 2j)}_{2s} = \frac{1}{2s} \sigma^{\alpha_\zeta k_\zeta l_\zeta}(s; 2j) [\Leftrightarrow] \bar{\Gamma}(s; 2j) \sigma^{\alpha_\zeta}(j) \Gamma(s; 2j) = \frac{1}{2s} \sigma^{\alpha_\zeta}(s; 2j) \\ \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j)}_{2s} S_{ab A_\zeta Z_\zeta}(j) \underbrace{\Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta}(s; 2j)}_{2s} = \frac{1}{2s} S_{ab k_\zeta l_\zeta}(s; 2j) [\Leftrightarrow] \bar{\Gamma}(s; 2j) S_{ab}(j, \zeta) \Gamma(s; 2j) = \frac{1}{2s} S_{ab}(s, \zeta; 2j) \end{array} \right.$$

## 1.6 常数不变张量 $\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j), \Omega(s; 2j)$ 的引入及其性质

定义1.6.1.

$$\underbrace{\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j)}_{2s} := \underbrace{\sigma_{A_\zeta}^{A'_\zeta}(j) \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \sigma_{B_\zeta}^{B'_\zeta}(j) \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \sigma_{C_\zeta}^{C'_\zeta}(j) \dots}_{2s} + \dots$$

[⇔] [⇔]

定义1.6.2.  $\Omega(s; 2j) := \sigma(j) \otimes I_{(2j+1)^{2s-1}} + I_{2j+1} \otimes \sigma(j) \otimes I_{(2j+1)^{2s-2}} + \dots + I_{(2j+1)^{2s-1}} \otimes \sigma(j)$

推论1.6.1.  $\underbrace{\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j)}_{2s} := \sigma_{A_\zeta}^{A'_\zeta}(j) \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s-1} + \delta_{A_\zeta}^{A'_\zeta} \underbrace{\Omega_{B_\zeta C_\zeta \dots}^{B'_\zeta C'_\zeta \dots}(s - \frac{1}{2}; 2j)}_{2s-1}$

[⇔] [⇔]

推论1.6.2.  $\Omega(s; 2j) = \sigma(j) \otimes I_{(2j+1)^{2s-1}} + I_{2j+1} \otimes \Omega(s - \frac{1}{2}; 2j)$

推论1.6.3.  $\Omega(s; 2j) = \Omega(s - s'; 2j) \otimes I_{(2j+1)^{2s'}} + I_{(2j+1)^{2(s-s')}} \otimes \Omega(s'; 2j)$

## 1.7 常数不变张量 $\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j), \Omega_{ab}(s, \varsigma; 2j)$ 的引入及其基本性质

定义1.7.1.

$$\underbrace{\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j)}_{2s} := \underbrace{S_{ab A_\zeta}^{A'_\zeta}(j) \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} S_{ab B_\zeta}^{B'_\zeta}(j) \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} S_{ab C_\zeta}^{C'_\zeta}(j) \dots}_{2s} + \dots$$

[⇔] [⇔]

定义1.7.2.  $\Omega_{ab}(s, \varsigma; 2j) := S_{ab}(j, \varsigma) \otimes I_{(w+1)^{2s-1}} + I_4 \otimes S_{ab}(j, \varsigma) \otimes I_{(w+1)^{2s-2}} + \dots + I_{(w+1)^{2s-1}} \otimes S_{ab}(j, \varsigma)$

推论1.7.1.  $\underbrace{\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j)}_{2s} := S_{ab A_\zeta}^{A'_\zeta}(j) \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s-1} + \delta_{A_\zeta}^{A'_\zeta} \underbrace{\Omega_{ab B_\zeta C_\zeta \dots}^{B'_\zeta C'_\zeta \dots}(s - \frac{1}{2}; 2j)}_{2s-1}$

[⇔] [⇔]

推论1.7.2.  $\Omega_{ab}(s, \varsigma; 2j) = S_{ab}(j, \varsigma) \otimes I_{(w+1)^{2s-1}} + I_4 \otimes \Omega_{ab}(s - \frac{1}{2}, \varsigma; 2j)$

推论1.7.3.  $\Omega_{ab}(s, \varsigma; 2j) = \Omega_{ab}(s - s', \varsigma; 2j) \otimes I_{(w+1)^{2s'}} + I_{(w+1)^{2(s-s')}} \otimes \Omega_{ab}(s', \varsigma; 2j)$

## 1.8 自旋常数不变张量 $\sigma^{\alpha_\zeta}{}_{k_\zeta}{}^{l_\zeta}(s; 2j), \sigma(s; 2j)$ 的引入

定义1.8.1.  $\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j) \sigma^{\alpha_\zeta}{}_{A_\zeta}{}^{Z_\zeta}(j) \Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta}(s; 2j) := \frac{1}{2s} \sigma^{\alpha_\zeta}{}_{k_\zeta}{}^{l_\zeta}(s; 2j) [\Leftrightarrow] \sigma(s; 2j) := \bar{\Gamma}(s; 2j) \Omega(s; 2j) \Gamma(s; 2j)$

## 1.9 自旋常数不变张量 $S_{ab k_\zeta}{}^{l_\zeta}(s; 2j), S_{ab}(s, \varsigma; 2j)$ 的引入

定义1.9.1.

$$\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j) S_{ab A_\zeta}{}^{Z_\zeta}(j) \Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta}(s; 2j) := \frac{1}{2s} S_{ab k_\zeta}{}^{l_\zeta}(s; 2j) [\Leftrightarrow] S_{ab}(s, \varsigma; 2j) := \bar{\Gamma}(s; 2j) \Omega_{ab}(s, \varsigma; 2j) \Gamma(s; 2j)$$

## 1.10 常数矩阵 $\Omega_{ab}(s, \varsigma; 2j), S_{ab}(s, \varsigma; 2j)$ 与 $\Omega_{\alpha_\zeta}(s; 2j), \sigma_{\alpha_\zeta}(s; 2j)$ 之间关系

推论1.10.1.  $S_{ab}(j, \varsigma) = \sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta}(j) [\Rightarrow] \Omega_{ab}(s, \varsigma; 2j) = \sigma_{\zeta ab}^{\alpha_\zeta} \Omega_{\alpha_\zeta}(s; 2j) [\Rightarrow] S_{ab}(s, \varsigma; 2j) = \sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta}(s; 2j)$

## 1.11 常数矩阵 $\Omega(s; 2j)$ 的两个重要引理

引理1.11.1.  $\Gamma(s; 2j) \bar{\Gamma}(s; 2j) \Omega(s; 2j) \Gamma(s; 2j) = \Omega(s; 2j) \Gamma(s; 2j)$

引理1.11.2.  $\bar{\Gamma}(s; 2j) \Omega(s; 2j) \Gamma(s; 2j) \bar{\Gamma}(s; 2j) = \bar{\Gamma}(s; 2j) \Omega(s; 2j)$



### 1.12 常数矩阵 $\Omega_{ab}(s, \varsigma; 2j)$ 的两个重要引理

$$\text{引理1.12.1. } \Gamma(s; 2j)\bar{\Gamma}(s; 2j)\Omega_{ab}(s, \varsigma; 2j)\Gamma(s; 2j) = \Omega_{ab}(s, \varsigma; 2j)\Gamma(s; 2j)$$

$$\text{引理1.12.2. } \bar{\Gamma}(s; 2j)\Omega_{ab}(s, \varsigma; 2j)\Gamma(s; 2j)\bar{\Gamma}(s; 2j) = \bar{\Gamma}(s; 2j)\Omega_{ab}(s, \varsigma; 2j)$$

### 1.13 关于常数矩阵 $\bar{\Gamma}(s; 3), \Omega(s; 3), \sigma(s; 3), \Gamma(s; 3)$ 的置换性质及其推论

$$\text{定理1.13.1. } \underbrace{\Omega_{A_\varsigma B_\varsigma C_\varsigma \dots}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; 2j)\underbrace{\Gamma_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}^{l_\varsigma}}_{2s}(s; 2j) = \underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}}_{2s}(s; 2j)\sigma_{k_\varsigma}^{l_\varsigma}(s; 2j)$$

$$[\Leftrightarrow]\Omega(s; 2j)\Gamma(s; 2j) = \Gamma(s; 2j)\sigma(s; 2j)$$

$$\text{定理1.13.2. } \underbrace{\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma \dots}}_{2s}(s; 2j)\underbrace{\Omega_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}^{l_\varsigma}}_{2s}(s; 2j) = \sigma_{k_\varsigma}^{l_\varsigma}(s; 2j)\underbrace{\Gamma_{l_\varsigma}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; 2j)$$

$$[\Leftrightarrow]\bar{\Gamma}(s; 2j)\Omega(s; 2j) = \sigma(s; 2j)\bar{\Gamma}(s; 2j)$$

$$\text{推论1.13.1. } \bar{\Gamma}(s; 2j)\Omega(s; 2j)\Gamma(s; 2j) = \sigma(s; 2j)$$

$$\Leftrightarrow \Omega(s; 2j)\Gamma(s; 2j) = \Gamma(s; 2j)\sigma(s; 2j) \Leftrightarrow \bar{\Gamma}(s; 2j)\Omega(s; 2j) = \sigma(s; 2j)\bar{\Gamma}(s; 2j)$$

$$\text{推论1.13.2. } \Omega^2(s; 2j)\Gamma(s; 2j) = \Gamma(s; 2j)\sigma^2(s; 2j)$$

$$\text{推论1.13.3. } \bar{\Gamma}(s; 2j)\Omega^2(s; 2j) = \sigma^2(s; 2j)\bar{\Gamma}(s; 2j)$$

$$\text{推论1.13.4. } [\sigma(\frac{1}{2}) \otimes I_j]_{(A_\varsigma \underbrace{A'_\varsigma \Gamma_{B_\varsigma C_\varsigma \dots}^{l_\varsigma}}_{2s})A'_\varsigma}(s; 2j) = \frac{1}{(2s-1)!}\underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}}_{2s}(s; 2j)\sigma_{k_\varsigma}^{l_\varsigma}(s; 2j)$$

### 1.14 关于常数矩阵 $\bar{\Gamma}(s; 2j), \Omega_{ab}(s, \varsigma; 2j), S_{ab}(s, \varsigma; 2j), \Gamma(s; 2j)$ 的置换性质及其推论

$$\text{定理1.14.1. } \underbrace{\Omega_{ab}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; 2j)\underbrace{\Gamma_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}^{l_\varsigma}}_{2s}(s; 2j) = \underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}}_{2s}(s; 2j)S_{abk_\varsigma}^{l_\varsigma}(s; 2j)$$

$$[\Leftrightarrow]\Omega_{ab}(s, \varsigma; 2j)\Gamma(s; 2j) = \Gamma(s; 2j)S_{ab}(s, \varsigma; 2j)$$

$$\text{定理1.14.2. } \underbrace{\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma \dots}}_{2s}(s; 2j)\underbrace{\Omega_{ab}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; 2j) = S_{abk_\varsigma}^{l_\varsigma}(s; 2j)\underbrace{\Gamma_{l_\varsigma}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; 2j)$$

$$[\Leftrightarrow]\bar{\Gamma}(s; 2j)\Omega_{ab}(s, \varsigma; 2j) = S_{ab}(s, \varsigma; 2j)\bar{\Gamma}(s; 2j)$$

$$\text{推论1.14.1. } \bar{\Gamma}(s; 2j)\Omega_{ab}(s, \varsigma; 2j)\Gamma(s; 2j) = S_{ab}(s, \varsigma; 2j)$$

$$\Leftrightarrow \Omega_{ab}(s, \varsigma; 2j)\Gamma(s; 2j) = \Gamma(s; 2j)S_{ab}(s, \varsigma; 2j) \Leftrightarrow \bar{\Gamma}(s; 2j)\Omega_{ab}(s, \varsigma; 2j) = S_{ab}(s, \varsigma; 2j)\bar{\Gamma}(s; 2j)$$

$$\text{推论1.14.2. } [S_{ab}(\frac{1}{2}, \varsigma) \otimes I_j]_{(A_\varsigma \underbrace{A'_\varsigma \Gamma_{B_\varsigma C_\varsigma \dots}^{l_\varsigma}}_{2s})A'_\varsigma}(s; 2j) = \frac{1}{(2s-1)!}\underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}}_{2s}(s; 2j)S_{abk_\varsigma}^{l_\varsigma}(s; 2j)$$

### 1.15 常数矩阵 $\Omega(s; 2j), \sigma(s; 2j)$ 的洛伦兹群表示

$$\text{推论1.15.1. } \Omega(s; 2j) \times \Omega(s; 2j) = i\Omega(s; 2j)[\Rightarrow]\sigma(s; 2j) \times \sigma(s; 2j) = i\sigma(s; 2j)$$

### 1.16 常数矩阵 $\Omega_{ab}(s, \varsigma; 2j), S_{ab}(s, \varsigma; 2j)$ 的洛伦兹群表示

$$\text{推论1.16.1. } i[S_{ab}(j, \varsigma), S_{cd}(j, \varsigma)] = \delta_{ad}S_{bc}(j, \varsigma) - \delta_{ac}S_{bd}(j, \varsigma) + \delta_{bc}S_{ad}(j, \varsigma) - \delta_{bd}S_{ac}(j, \varsigma)$$

$$[\Rightarrow]i[\Omega_{ab}(s, \varsigma; 2j), \Omega_{cd}(s, \varsigma; 2j)] = \delta_{ad}\Omega_{bc}(s, \varsigma; 2j) - \delta_{ac}\Omega_{bd}(s, \varsigma; 2j) + \delta_{bc}\Omega_{ad}(s, \varsigma; 2j) - \delta_{bd}\Omega_{ac}(s, \varsigma; 2j)$$

$$[\Rightarrow]i[S_{ab}(s, \varsigma; 2j), S_{cd}(s, \varsigma; 2j)] = \delta_{ad}S_{bc}(s, \varsigma; 2j) - \delta_{ac}S_{bd}(s, \varsigma; 2j) + \delta_{bc}S_{ad}(s, \varsigma; 2j) - \delta_{bd}S_{ac}(s, \varsigma; 2j)$$

$$\text{引理1.16.1. } \Gamma(s; 2j)\bar{\Gamma}(s; 2j)\Omega(s; 2j)\Gamma(s; 2j) = \Omega(s; 2j)\Gamma(s; 2j)$$

$$\text{定义1.16.1. } \sigma(s; 2j) := \bar{\Gamma}(s; 2j)\Omega(s; 2j)\Gamma(s; 2j)$$

推论1.16.2.  $\Omega(s; 2j) \times \Omega(s; 2j) = i\Omega(s; 2j)[\Rightarrow]\sigma(s; 2j) \times \sigma(s; 2j) = i\sigma(s; 2j)$

定理1.16.1.  $\Omega_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{\underbrace{A'_{\zeta} B'_{\zeta} C'_{\zeta} \dots}_{2s}}(s; 2j) \Gamma_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{l_{\zeta}}(s; 2j) = \Gamma_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{k_{\zeta}}(s; 2j) \sigma_{k_{\zeta}}^{l_{\zeta}}(s; 2j) [\Leftrightarrow] \Omega(s; 2j) \Gamma(s; 2j) = \Gamma(s; 2j) \sigma(s; 2j)$

引理1.16.2.  $\bar{\Gamma}(s; 2j) \Omega(s; 2j) \Gamma(s; 2j) \bar{\Gamma}(s; 2j) = \bar{\Gamma}(s; 2j) \Omega(s; 2j)$

定理1.16.2.  $\Gamma_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}(s; 2j) \Omega_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{\underbrace{A'_{\zeta} B'_{\zeta} C'_{\zeta} \dots}_{2s}}(s; 2j) = \sigma_{k_{\zeta}}^{l_{\zeta}}(s; 2j) \Gamma_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{\underbrace{A'_{\zeta} B'_{\zeta} C'_{\zeta} \dots}_{2s}}(s; 2j) [\Leftrightarrow] \bar{\Gamma}(s; 2j) \Omega(s; 2j) = \sigma(s; 2j) \bar{\Gamma}(s; 2j)$

推论1.16.3.

$\bar{\Gamma}(s; 2j) \Omega(s; 2j) \Gamma(s; 2j) = \sigma(s; 2j) \Leftrightarrow \Omega(s; 2j) \Gamma(s; 2j) = \Gamma(s; 2j) \sigma(s; 2j) \Leftrightarrow \bar{\Gamma}(s; 2j) \Omega(s; 2j) = \sigma(s; 2j) \bar{\Gamma}(s; 2j)$

推论1.16.4.  $\Omega^2(s; 2j) \Gamma(s; 2j) = \Gamma(s; 2j) \sigma^2(s; 2j)$

推论1.16.5.  $\bar{\Gamma}(s; 2j) \Omega^2(s; 2j) = \sigma^2(s; 2j) \bar{\Gamma}(s; 2j)$

推论1.16.6.  $\Omega_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{\underbrace{A'_{\zeta} B'_{\zeta} C'_{\zeta} \dots}_{2s}}(s; 2j) \Gamma_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{l_{\zeta}}(s; 2j) = \Gamma_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{k_{\zeta}}(s; 2j) \sigma_{k_{\zeta}}^{l_{\zeta}}(s; 2j)$   
 $\Leftrightarrow \sigma(j)_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{\underbrace{A'_{\zeta} B'_{\zeta} C'_{\zeta} \dots}_{2s}} \Gamma_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{l_{\zeta}}(s; 2j) = \Gamma_{\underbrace{A_{\zeta} B_{\zeta} C_{\zeta} \dots}_{2s}}^{k_{\zeta}}(s; 2j) \sigma_{k_{\zeta}}^{l_{\zeta}}(s; 2j)$

### 1.17 推论：常数矩阵 $\Gamma(s; 2j)$ , $\bar{\Gamma}(s; 2j)$ 的几个恒等式

性质1.17.1.  $\begin{cases} \bar{\Gamma}(s; 2j) \Omega(s; 2j) \Gamma(s; 2j) = \sigma(s; 2j), [\Gamma(s; 2j) \bar{\Gamma}(s; 2j), \Omega(s; 2j)] = 0 \\ \Gamma(s; 2j) \sigma(s; 2j) \bar{\Gamma}(s; 2j) = \Omega(s; 2j) \Gamma(s; 2j) \bar{\Gamma}(s; 2j) = \Gamma(s; 2j) \bar{\Gamma}(s; 2j) \Omega(s; 2j) \end{cases}$

性质1.17.2.  $\begin{cases} \bar{\Gamma}(s; 2j) \Omega_{ab}(s, \zeta; 2j) \Gamma(s; 2j) = S_{ab}(s, \zeta; 2j), [\Gamma(s; 2j) \bar{\Gamma}(s; 2j), \Omega_{ab}(s, \zeta; 2j)] = 0 \\ \Gamma(s; 2j) S_{ab}(s, \zeta; 2j) \bar{\Gamma}(s; 2j) = \Omega_{ab}(s, \zeta; 2j) \Gamma(s; 2j) \bar{\Gamma}(s; 2j) = \Gamma(s; 2j) \bar{\Gamma}(s; 2j) \Omega_{ab}(s, \zeta; 2j) \end{cases}$

性质1.17.3.  $\begin{cases} \bar{\Gamma}(s; 2j) [\vartheta \cdot \Omega(s; 2j)]^n \Gamma(s; 2j) = [\vartheta \cdot \sigma(s; 2j)]^n, [\Gamma(s; 2j) \bar{\Gamma}(s; 2j), [\vartheta \cdot \Omega(s; 2j)]^n] = 0 \\ \Gamma(s; 2j) [\vartheta \cdot \sigma(s; 2j)]^n \bar{\Gamma}(s; 2j) = [\vartheta \cdot \Omega(s; 2j)]^n \Gamma(s; 2j) \bar{\Gamma}(s; 2j) = \Gamma(s; 2j) \bar{\Gamma}(s; 2j) [\vartheta \cdot \Omega(s; 2j)]^n \end{cases}$

性质1.17.4.  $\begin{cases} \bar{\Gamma}(s; 2j) [\vartheta^{ab} \Omega_{ab}(s, \zeta; 2j)]^n \Gamma(s; 2j) = [\vartheta^{ab} S_{ab}(s, \zeta; 2j)]^n, [\Gamma(s; 2j) \bar{\Gamma}(s; 2j), [\vartheta^{ab} \Omega_{ab}(s, \zeta; 2j)]^n] = 0 \\ \Gamma(s; 2j) [\vartheta^{ab} S_{ab}(s, \zeta; 2j)]^n \bar{\Gamma}(s; 2j) = [\vartheta^{ab} \Omega_{ab}(s, \zeta; 2j)]^n \Gamma(s; 2j) \bar{\Gamma}(s; 2j) = \Gamma(s; 2j) \bar{\Gamma}(s; 2j) [\vartheta^{ab} \Omega_{ab}(s, \zeta; 2j)]^n \end{cases}$

推论1.17.1.  $\begin{cases} \bar{\Gamma}(s; 2j) e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s, \zeta; 2j)} \Gamma(s; 2j) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \zeta; 2j)}, [\Gamma(s; 2j) \bar{\Gamma}(s; 2j), e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s, \zeta; 2j)}] = 0 \\ \Gamma(s; 2j) e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \zeta; 2j)} \bar{\Gamma}(s; 2j) = e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s, \zeta; 2j)} \Gamma(s; 2j) \bar{\Gamma}(s; 2j) = \Gamma(s; 2j) \bar{\Gamma}(s; 2j) e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s, \zeta; 2j)} \end{cases}$

### 1.18 矩阵 $\Gamma(s; 2j)$ , $\bar{\Gamma}(s; 2j)$ 的常数不变张量性质

定理1.18.1.  $\Gamma(s; 2j) = e^{(i\omega + \zeta\epsilon) \cdot \Omega(s; 2j)} \Gamma(s; 2j) e^{-(i\omega + \zeta\epsilon) \cdot \sigma(s; 2j)}$

定理1.18.2.  $\bar{\Gamma}(s; 2j) = e^{(i\omega + \zeta\epsilon) \cdot \sigma(s; 2j)} \bar{\Gamma}(s; 2j) e^{-(i\omega + \zeta\epsilon) \cdot \Omega(s; 2j)}$

推论1.18.1.

$$\begin{cases} S(s; 2j) = e^{(i\omega + \zeta\epsilon) \cdot \Omega(s; 2j)} S(s; 2j) e^{-(i\omega + \zeta\epsilon) \cdot S^+(s; 2j) \Omega(s; 2j) S(s; 2j)} \\ S(s; 2j) = e^{(i\omega + \zeta\epsilon) \cdot S(s; 2j) \Omega(s; 2j) S^+(s; 2j)} S(s; 2j) e^{-(i\omega + \zeta\epsilon) \cdot \Omega(s; 2j)} \\ S^+(s; 2j) = e^{(i\omega + \zeta\epsilon) \cdot \Omega(s; 2j)} S^+(s; 2j) e^{-(i\omega + \zeta\epsilon) \cdot S(s; 2j) \Omega(s; 2j) S^+(s; 2j)} \\ S^+(s; 2j) = e^{(i\omega + \zeta\epsilon) \cdot S^+(s; 2j) \Omega(s; 2j) S(s; 2j)} S^+(s; 2j) e^{-(i\omega + \zeta\epsilon) \cdot \Omega(s; 2j)} \end{cases}$$

### 1.19 常数矩阵 $I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)$ , $I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)$ 的置换性质

$$\text{推论1.19.1. } \Omega(s; 2j)[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] = [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)]$$

$$\text{推论1.19.2. } [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]\Omega(s; 2j) = [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]$$

### 1.20 推论: 常数矩阵 $\Gamma(s - \frac{1}{2}; 2j)$ , $\bar{\Gamma}(s - \frac{1}{2}; 2j)$ 的几个恒等式

性质1.20.1.

$$\begin{cases} [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]\Omega(s; 2j)[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] = [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)] \\ [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] \\ = \Omega(s; 2j)[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] = [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]\Omega(s; 2j) \\ [[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)], \Omega(s; 2j)] = 0 \end{cases}$$

性质1.20.2.

$$\begin{cases} [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]\Omega_{ab}(s, \varsigma; 2j)[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] = [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)] \\ [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] \\ = \Omega_{ab}(s, \varsigma; 2j)[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] = [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]\Omega_{ab}(s, \varsigma; 2j) \\ [[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)], \Omega_{ab}(s, \varsigma; 2j)] = 0 \end{cases}$$

性质1.20.3.

$$\begin{cases} [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)][\vartheta \cdot \Omega(s; 2j)]^n [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] = \{\vartheta \cdot [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)]\}^n \\ [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]\{\vartheta \cdot [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)]\}^n [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] \\ = [\vartheta \cdot \Omega(s; 2j)]^n [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] = [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)][\vartheta \cdot \Omega(s; 2j)]^n \\ [[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)], [\vartheta \cdot \Omega(s; 2j)]^n] = 0 \end{cases}$$

性质1.20.4.

$$\begin{cases} [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)][\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j)]^n [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] = \{\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]\}^n \\ [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]\{\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]\}^n [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] \\ = [\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j)]^n [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] = [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)][\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j)]^n \\ [[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)], [\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j)]^n] = 0 \end{cases}$$

推论1.20.1.

$$\begin{cases} [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j)} [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] = e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]} \\ [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]} [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] \\ = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j)} [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] = [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j)} \\ [[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)], e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j)}] = 0 \end{cases}$$

推论1.20.2.

$$\begin{cases} I_{(2j+1)2s-1}\Gamma(s - \frac{1}{2}; 2j) = \Gamma(s - \frac{1}{2}; 2j)I_{C_{2s-1+2j}^{2s-1}}, \bar{\Gamma}(s - \frac{1}{2}; 2j)I_{(2j+1)2s-1} = I_{C_{2s-1+2j}^{2s-1}}\bar{\Gamma}(s - \frac{1}{2}; 2j) \\ [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]\sigma(j) \otimes I_{(2j+1)2s-1}[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] = \sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} \\ [\sigma(j) \otimes I_{(2j+1)2s-1}[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] = [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)][\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}}] \\ [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)][\sigma(j) \otimes I_{(2j+1)2s-1}] = [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}}][I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] \end{cases}$$

### 1.21 常数矩阵 $\Gamma(s; 2j)$ , $\bar{\Gamma}(s; 2j)$ 的对称置换性质

$$\text{定义1.21.1. } S_{ex}(s, n) = (\overbrace{I_{2j+1} \otimes \cdots \otimes I_{2j+1}}^{n-1}) S_{ex} (\overbrace{\otimes I_{2j+1} \otimes \cdots \otimes I}^{2s-n-1})$$

推论1.21.1.  $\Gamma(s; 2j) = S_{ex}(s, n)\Gamma(s; 2j), \bar{\Gamma}(s; 2j) = \bar{\Gamma}(s; 2j)S_{ex}(s, n)$

推论1.21.2.  $S_{ex}(s, n)\Omega(s; 2j)S_{ex}(s, n) = \Omega(s; 2j)$

推论1.21.3.  $\hat{\psi}(s, \varsigma; 2j) = S_{ex}(s, n)\hat{\psi}(s, \varsigma; 2j), \forall n \in \{1, 2, \dots, 2s+1\}$

## 1.22 矩阵 $I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j), I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)$ 的常数不变张量性质

定理1.22.1.  $[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] = e^{(i\omega+\varsigma\epsilon)\cdot\Omega(s; 2j)} [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(j)} \otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2}; 2j)}$

定理1.22.2.  $[I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(j)} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2}; 2j)} [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] e^{-(i\omega+\varsigma\epsilon)\cdot\Omega(s; 2j)}$

## 2 二阶完美常数不变张量 $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$

### 2.1 完美常数不变张量 $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$

定义2.1.1.  $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s) := \Gamma_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}}^{k_\varsigma}(s) \Gamma_{l_\varsigma}^{B_\varsigma C_\varsigma \dots}(s - \frac{1}{2}), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s) := \Gamma_{k_\varsigma}^{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}}(s) \Gamma_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}}^{l_\varsigma}(s - \frac{1}{2})$

定义2.1.2.  $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j) := \Gamma_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}}^{k_\varsigma}(s; 2j) \Gamma_{l_\varsigma}^{B_\varsigma C_\varsigma \dots}(s - \frac{1}{2}; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) := \Gamma_{k_\varsigma}^{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}}(s; 2j) \Gamma_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}}^{l_\varsigma}(s - \frac{1}{2}; 2j)$

推论2.1.1.  $N(s; 2j) = [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] \Gamma(s; 2j), \bar{N}(s; 2j) = \bar{\Gamma}(s; 2j) [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]$

性质2.1.1.  $\Gamma_{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}}^{k_\varsigma}(s; 2j) = N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j) \Gamma_{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}}^{l_\varsigma}(s - \frac{1}{2}; 2j), \Gamma_{k_\varsigma}^{\underbrace{A_\varsigma B_\varsigma C_\varsigma \dots}_{2s}}(s; 2j) = N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) \Gamma_{l_\varsigma}^{\underbrace{B_\varsigma C_\varsigma \dots}_{2s-1}}(s - \frac{1}{2}; 2j)$

推论2.1.2.  $\Gamma(s; 2j) = [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] N(s; 2j), \bar{\Gamma}(s; 2j) = \bar{N}(s; 2j) [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)],$

性质2.1.2.  $\Gamma(s; 2j) = [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)] [I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] \Gamma(s; 2j)$

### 2.2 常数矩阵 $N_{A_\varsigma}(s; 2j), N^{A_\varsigma}(s; 2j); \bar{N}_{A_\varsigma}(s; 2j), \bar{N}^{A_\varsigma}(s; 2j); N(s; 2j), \bar{N}(s; 2j)$

定义2.2.1.

$$\begin{cases} N_{A_\varsigma}(s; 2j) \prec N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N^{A_\varsigma}(s; 2j) \prec N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) |_{I_{C_{2s+2j}^{2s}} \times I_{C_{2s-1+2j}^{2s-1}}} \\ \bar{N}_{A_\varsigma}(s; 2j) := N_{A_\varsigma}^+(s; 2j) \succ N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), \bar{N}^{A_\varsigma}(s; 2j) := N^{+A_\varsigma}(s; 2j) \succ N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) |_{I_{C_{2s-1+2j}^{2s-1}} \times I_{C_{2s+2j}^{2s}}} \\ N(s; 2j) \prec N_{A_\varsigma \otimes l_\varsigma}^{k_\varsigma}(s; 2j) |_{(2j+1)I_{C_{2s-1+2j}^{2s-1}} \times I_{C_{2s+2j}^{2s}}}, \bar{N}(s; 2j) = N^+(s; 2j) \prec N_{k_\varsigma}^{A_\varsigma \otimes l_\varsigma}(s; 2j) |_{I_{C_{2s+2j}^{2s}} \times (2j+1)I_{C_{2s-1+2j}^{2s-1}}} \end{cases}$$

### 2.3 常数不变张量 $N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j), N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ 的基本性质

相等性:

性质2.3.1.

$$\begin{cases} N_{A'_\varsigma l'_\varsigma}^{k'_\varsigma}(s; 2j) \simeq N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j) \simeq N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) \simeq N_{k'_\varsigma}^{A'_\varsigma l'_\varsigma}(s; 2j) \\ [N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j)]^* \simeq N_{A'_\varsigma l'_\varsigma}^{k'_\varsigma}(s; 2j), [N_{k_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)]^* \simeq N_{k'_\varsigma}^{A'_\varsigma l'_\varsigma}(s; 2j) \end{cases}$$

推论2.3.1.

$$\begin{cases} N_{A_\varsigma}(s; 2j) \simeq N^{A_\varsigma}(s; 2j) \simeq N_{A'_\varsigma}(s; 2j) \simeq N^{A'_\varsigma}(s; 2j); \bar{N}_{A_\varsigma}(s; 2j) \simeq \bar{N}^{A_\varsigma}(s; 2j) \simeq \bar{N}_{A'_\varsigma}(s; 2j) \simeq \bar{N}^{A'_\varsigma}(s; 2j) \\ N_{A_\varsigma}(s; 2j) = N_{A_\varsigma}^*(s; 2j), \bar{N}_{A_\varsigma}(s; 2j) = \bar{N}_{A_\varsigma}^*(s; 2j); N(s; 2j) = N^*(s; 2j), \bar{N}(s; 2j) = \bar{N}^*(s; 2j) \end{cases}$$

## 2.4 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j)$ 的正交性质

正交性:

引理2.4.1. 
$$\sum_{k=0}^{2s-1} C_{2j+k}^{2j} = C_{2j+2s}^{2j+1}$$

引理2.4.2. 
$$N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j) N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; 2j) = \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} \underbrace{\phantom{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}}_{2s}(s; 2j) \Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots} \overbrace{\phantom{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}}(2s)(s; 2j)$$

定理2.4.1.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j) N_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j) = \delta_{m_\zeta}^{k_\zeta} [\Leftrightarrow] N^{A_\zeta}(s; 2j) \bar{N}_{A_\zeta}(s; 2j) = I_{C_{2s+2j}^{2s}} [\Leftrightarrow] \bar{N}(s; 2j) N(s; 2j) = I_{C_{2s+2j}^{2s}} \\ N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j) N_{k_\zeta}^{A_\zeta m_\zeta}(s; 2j) = (1 + \frac{j}{s}) \delta_{l_\zeta}^{m_\zeta} [\Leftrightarrow] \bar{N}_{A_\zeta}(s; 2j) N^{A_\zeta}(s; 2j) = (1 + \frac{j}{s}) I_{C_{2s-1+2j}^{2s-1}} \\ N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j) N_{k_\zeta}^{B_\zeta l_\zeta}(s; 2j) = \frac{1}{2j+1} C_{2s+2j}^{2s} \delta_{A_\zeta}^{B_\zeta} [\Leftrightarrow] \text{tr}[\bar{N}_{A_\zeta}(s; 2j) N^{B_\zeta}(s; 2j)] = \frac{1}{2j+1} C_{2s+2j}^{2s} \delta_{A_\zeta}^{B_\zeta} \end{cases}$$

性质2.4.1.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j) N_{k_\zeta}^{B_\zeta m_\zeta}(s; 2j) = \frac{1}{2s} [\delta_{A_\zeta}^{B_\zeta} \delta_{l_\zeta}^{m_\zeta} + (2s-1) N_{A_\zeta n_\zeta}^{m_\zeta}(s - \frac{1}{2}; 2j) N_{l_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2}; 2j)] \\ \bar{N}_{A_\zeta}(s; 2j) N^{B_\zeta}(s; 2j) = \frac{1}{2s} [\delta_{A_\zeta}^{B_\zeta} I_{C_{2s-1+2j}^{2s-1}} + (2s-1) N^{B_\zeta}(s - \frac{1}{2}; 2j) \bar{N}_{A_\zeta}(s - \frac{1}{2}; 2j)] \end{cases}$$

## 2.5 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j)$ 的升降指标

升降指标:

性质2.5.1.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j) = \varepsilon^{k_\zeta m_\zeta}(s; 2j) \varepsilon_{A_\zeta B_\zeta}(j) \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; 2j) N_{m_\zeta}^{B_\zeta n_\zeta}(s; 2j) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j) = \varepsilon_{k_\zeta m_\zeta}(s; 2j) \varepsilon^{A_\zeta B_\zeta}(j) \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; 2j) N_{B_\zeta n_\zeta}^{m_\zeta}(s; 2j) \end{cases}$$

推论2.5.1.

$$\begin{cases} N_{A_\zeta}(s; 2j) \varepsilon(s - \frac{1}{2}; 2j) = \varepsilon_{A_\zeta B_\zeta}(j) \varepsilon(s; 2j) N^{B_\zeta}(s; 2j), \varepsilon(s - \frac{1}{2}; 2j) \bar{N}_{A_\zeta}(s; 2j) = \bar{N}^{B_\zeta}(s; 2j) \varepsilon_{B_\zeta A_\zeta}(j) \varepsilon(s; 2j) \\ N^{A_\zeta}(s; 2j) \varepsilon(s - \frac{1}{2}; 2j) = \varepsilon^{A_\zeta B_\zeta}(j) \varepsilon(s; 2j) N_{B_\zeta}(s; 2j), \varepsilon(s - \frac{1}{2}; 2j) \bar{N}^{A_\zeta}(s; 2j) = \bar{N}_{B_\zeta}(s; 2j) \varepsilon^{B_\zeta A_\zeta}(j) \varepsilon(s; 2j) \\ N(s; 2j) \varepsilon(s; 2j) = [\varepsilon(j) \otimes \varepsilon(s - \frac{1}{2}; 2j)] N(s; 2j), \varepsilon(s; 2j) \bar{N}(s; 2j) = \bar{N}(s; 2j) [\varepsilon(j) \otimes \varepsilon(s - \frac{1}{2}; 2j)] \end{cases}$$

Penrose标准升降规则:

性质2.5.2.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j) = (-1)^{2s} [\zeta^{2s} \varepsilon^{k_\zeta m_\zeta}(s; 2j)] [-\zeta \varepsilon_{A_\zeta B_\zeta}(j)] [(-\zeta)^{2s-1} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; 2j)] N_{m_\zeta}^{B_\zeta n_\zeta}(s; 2j) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j) = (-1)^{2s} [(-\zeta)^{2s} \varepsilon_{k_\zeta m_\zeta}(s; 2j)] [\zeta \varepsilon^{A_\zeta B_\zeta}(j)] [\zeta^{2s-1} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; 2j)] N_{B_\zeta n_\zeta}^{m_\zeta}(s; 2j) \end{cases}$$

## 2.6 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j)$ 的自旋矩阵变换

性质2.6.1.

$$\begin{cases} N_{k_\zeta}^{A_\zeta m_\zeta}(s; 2j) \sigma^{\alpha_\zeta} \varepsilon_{A_\zeta B_\zeta}(j) N_{B_\zeta m_\zeta}^{l_\zeta}(s; 2j) = \frac{1}{2s} \sigma^{\alpha_\zeta} \varepsilon_{k_\zeta l_\zeta}(s; 2j) [\Leftrightarrow] N^{A_\zeta}(s; 2j) \sigma^{\alpha_\zeta} \varepsilon_{A_\zeta B_\zeta}(j) \bar{N}_{B_\zeta}(s; 2j) = \frac{1}{2s} \sigma^{\alpha_\zeta}(s; 2j) \\ [\Leftrightarrow] \bar{N}(s; 2j) \sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} N(s; 2j) = \frac{1}{2s} \sigma(s; 2j) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j) \sigma^{\alpha_\zeta} \varepsilon_{A_\zeta B_\zeta}(j) N_{B_\zeta m_\zeta}^{k_\zeta}(s; 2j) = \frac{1}{2s} \sigma^{\alpha_\zeta} \varepsilon_{m_\zeta l_\zeta}(s - \frac{1}{2}; 2j) [\Leftrightarrow] \bar{N}_{B_\zeta}(s; 2j) \sigma^{\alpha_\zeta} \varepsilon_{A_\zeta B_\zeta}(j) N^{A_\zeta}(s; 2j) = \frac{1}{2s} \sigma^{\alpha_\zeta}(s - \frac{1}{2}; 2j) \end{cases}$$

性质2.6.2.

$$\begin{cases} N_{k_\zeta}^{A_\zeta m_\zeta}(s; 2j) S_{ab A_\zeta B_\zeta}(j) N_{B_\zeta m_\zeta}^{l_\zeta}(s; 2j) = \frac{1}{2s} S_{ab k_\zeta l_\zeta}(s; 2j) [\Leftrightarrow] N^{A_\zeta}(s; 2j) S_{ab A_\zeta B_\zeta}(j) \bar{N}_{B_\zeta}(s; 2j) = \frac{1}{2s} S_{ab}(s, \zeta; 2j) \\ [\Leftrightarrow] \bar{N}(s; 2j) S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j}^{2s-1}} N(s; 2j) = \frac{1}{2s} S_{ab}(s, \zeta; 2j) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j) S_{ab A_\zeta B_\zeta}(j) N_{B_\zeta m_\zeta}^{k_\zeta}(s; 2j) = \frac{1}{2s} S_{ab m_\zeta l_\zeta}(s - \frac{1}{2}; 2j) [\Leftrightarrow] \bar{N}_{B_\zeta}(s; 2j) S_{ab A_\zeta B_\zeta}(j) N^{A_\zeta}(s; 2j) = \frac{1}{2s} S_{ab}(s - \frac{1}{2}, \zeta; 2j) \end{cases}$$

## 2.7 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j)$ 的置换性质

定理2.7.1.

$$\begin{cases} \sigma^{\alpha_\zeta}_{A_\zeta}{}^{B_\zeta}(j)N_{B_\zeta l_\zeta}^{k_\zeta}(s; 2j) + \sigma^{\alpha_\zeta}_{l_\zeta}{}^{m_\zeta}(s - \frac{1}{2}; 2j)N_{A_\zeta m_\zeta}^{k_\zeta}(s; 2j) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; 2j)\sigma^{\alpha_\zeta}_{j_\zeta}{}^{k_\zeta}(s; 2j) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j)\sigma^{\alpha_\zeta}_{A_\zeta}{}^{B_\zeta}(j) + N_{k_\zeta}^{B_\zeta m_\zeta}(s; 2j)\sigma^{\alpha_\zeta}_{m_\zeta}{}^{l_\zeta}(s - \frac{1}{2}; 2j) = \sigma^{\alpha_\zeta}_{k_\zeta}{}^{j_\zeta}(s; 2j)N_{j_\zeta}^{B_\zeta l_\zeta}(s; 2j) \\ \sigma^{\alpha_\zeta}_{A_\zeta}{}^{B_\zeta}(j)\bar{N}_{B_\zeta}(s; 2j) + \sigma^{\alpha_\zeta}(s - \frac{1}{2}; 2j)\bar{N}_{A_\zeta}(s; 2j) = \bar{N}_{A_\zeta}(s; 2j)\sigma^{\alpha_\zeta}(s; 2j) \\ N^{A_\zeta}(s; 2j)\sigma^{\alpha_\zeta}_{A_\zeta}{}^{B_\zeta}(j) + N^{B_\zeta}(s; 2j)\sigma^{\alpha_\zeta}(s - \frac{1}{2}; 2j) = \sigma^{\alpha_\zeta}(s; 2j)N^{B_\zeta}(s; 2j) \\ [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma^{\alpha_\zeta}(s - \frac{1}{2}; 2j)]N(s; 2j) = N(s; 2j)\sigma^{\alpha_\zeta}(s; 2j) \\ \bar{N}(s; 2j)[\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma^{\alpha_\zeta}(s - \frac{1}{2}; 2j)] = \sigma^{\alpha_\zeta}(s; 2j)\bar{N}(s; 2j) \end{cases}$$

定理2.7.2.

$$\begin{cases} S_{abA_\zeta}{}^{B_\zeta}(j)N_{B_\zeta l_\zeta}^{k_\zeta}(s; 2j) + S_{abl_\zeta}{}^{m_\zeta}(s - \frac{1}{2}; 2j)N_{A_\zeta m_\zeta}^{k_\zeta}(s; 2j) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; 2j)S_{abj_\zeta}{}^{k_\zeta}(s; 2j) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j)S_{abA_\zeta}{}^{B_\zeta}(j) + N_{k_\zeta}^{B_\zeta m_\zeta}(s; 2j)S_{abm_\zeta}{}^{l_\zeta}(s - \frac{1}{2}; 2j) = S_{abk_\zeta}{}^{j_\zeta}(s; 2j)N_{j_\zeta}^{B_\zeta l_\zeta}(s; 2j) \\ S_{abA_\zeta}{}^{B_\zeta}(j)\bar{N}_{B_\zeta}(s; 2j) + S_{ab}(s - \frac{1}{2}, \zeta; 2j)\bar{N}_{A_\zeta}(s; 2j) = \bar{N}_{A_\zeta}(s; 2j)S_{ab}(s, \zeta; 2j) \\ N^{A_\zeta}(s; 2j)S_{abA_\zeta}{}^{B_\zeta}(j) + N^{B_\zeta}(s; 2j)S_{ab}(s - \frac{1}{2}, \zeta; 2j) = S_{ab}(s, \zeta; 2j)N^{B_\zeta}(s; 2j) \\ [S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j)]N(s; 2j) = N(s; 2j)S_{ab}(s, \zeta; 2j) \\ \bar{N}(s; 2j)[S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j)] = S_{ab}(s, \zeta; 2j)\bar{N}(s; 2j) \end{cases}$$

## 2.8 常数不变张量 $N_{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j)$ , $N_{k_\zeta}^{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}(s; 2j)$ 的性质

$$\text{定义2.8.1. } \begin{cases} N_{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j) := \Gamma_{A_{\zeta 1} \cdots A_{\zeta 2s}}^{k_\zeta}(s; 2j)\Gamma_{l_\zeta}^{A_{\zeta n+1} \cdots A_{\zeta 2s}}(s - \frac{n}{2}; 2j) \\ N_{k_\zeta}^{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}(s; 2j) := \Gamma_{k_\zeta}^{A_{\zeta 1} \cdots A_{\zeta 2s}}(s; 2j)\Gamma_{A_{\zeta n+1} \cdots A_{\zeta 2s}}^{l_\zeta}(s - \frac{n}{2}; 2j) \end{cases}$$

相等性:

$$\text{性质2.8.1. } N_{A'_{\zeta 1} \cdots A'_{\zeta n} l'_{\zeta n}}^{k'_{\zeta}}(s; 2j) \simeq N_{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j) \simeq N_{k_\zeta}^{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}(s; 2j) \simeq N_{k'_{\zeta}}^{A'_{\zeta 1} \cdots A'_{\zeta n} l'_{\zeta n}}(s; 2j)$$

$$\text{性质2.8.2. } [N_{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j)]^* \simeq N_{A'_{\zeta 1} \cdots A'_{\zeta n} l'_{\zeta n}}^{k'_{\zeta}}(s; 2j), [N_{k_\zeta}^{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}(s; 2j)]^* \simeq N_{k'_{\zeta}}^{A'_{\zeta 1} \cdots A'_{\zeta n} l'_{\zeta n}}(s; 2j)$$

展开性:

性质2.8.3.

$$\begin{cases} N_{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j) = N_{A_{\zeta 1} l_{\zeta 1}}^{k_\zeta}(s; 2j)N_{A_{\zeta 2} l_{\zeta 2}}^{l_{\zeta 1}}(s - \frac{1}{2}; 2j) \cdots N_{A_{\zeta n} l_{\zeta n}}^{l_{\zeta n-1}}(s - \frac{n-1}{2}; 2j) \\ N_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta n} l_\zeta}(s; 2j) = N_{k_\zeta}^{A_{\zeta 1} l_{\zeta 1}}(s; 2j)N_{l_{\zeta 1}}^{A_{\zeta 2} l_{\zeta 2}}(s - \frac{1}{2}; 2j) \cdots N_{l_{\zeta n-1}}^{A_{\zeta n} l_{\zeta n}}(s - \frac{n-1}{2}; 2j) \end{cases}$$

性质2.8.4.

$$\begin{cases} \Gamma_{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}^{k_\zeta}(s; 2j) = N_{A_{\zeta 1} l_{\zeta 1}}^{k_\zeta}(s; 2j)N_{A_{\zeta 2} l_{\zeta 2}}^{l_{\zeta 1}}(s - \frac{1}{2}; 2j) \cdots N_{A_{\zeta 2s} l_{\zeta 2s}}^{l_{\zeta 2s-1}}(\frac{1}{2}; 2j) \\ \Gamma_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}(s; 2j) = N_{k_\zeta}^{A_{\zeta 1} l_{\zeta 1}}(s; 2j)N_{l_{\zeta 1}}^{A_{\zeta 2} l_{\zeta 2}}(s - \frac{1}{2}; 2j) \cdots N_{l_{\zeta 2s-1}}^{A_{\zeta 2s} l_{\zeta 2s}}(\frac{1}{2}; 2j) \\ \Gamma_{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}^{k_\zeta}(s; 2j) \succ \Gamma_{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}(s; 2j) = N_{A_{\zeta 1}}(s; 2j)N_{A_{\zeta 2}}(s - \frac{1}{2}; 2j) \cdots N_{A_{\zeta 2s}}(\frac{1}{2}; 2j) \\ \Gamma_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}(s; 2j) \succ \Gamma_{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}(s; 2j) = N^{A_{\zeta 1}}(s; 2j)N^{A_{\zeta 2}}(s - \frac{1}{2}; 2j) \cdots N^{A_{\zeta 2s}}(\frac{1}{2}; 2j) \\ \bar{\Gamma}(s; 2j) = \bar{N}(s; 2j)[I_{2j+1} \otimes \bar{N}(s - \frac{1}{2}; 2j)] \cdots [I_{(2j+1)2s-2} \otimes \bar{N}(1)][I_{(2j+1)2s-1} \otimes \bar{N}(\frac{1}{2}; 2j)] \\ \Gamma(s; 2j) = [I_{(2j+1)2s-1} \otimes N(\frac{1}{2}; 2j)][I_{(2j+1)2s-2} \otimes N(1)] \cdots [I_{2j+1} \otimes N(s - \frac{1}{2}; 2j)]N(s; 2j) \end{cases}$$

## 2.9 推论1: 常数矩阵 $N(s; 2j)$ , $\bar{N}(s; 2j)$ 的几个恒等式

性质2.9.1.

$$\begin{cases} \bar{N}(s; 2j)[\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)]N(s; 2j) = \sigma(s; 2j) \\ N(s; 2j)\sigma(s; 2j)\bar{N}(s; 2j) = [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)]N(s; 2j)\bar{N}(s; 2j) \\ N(s; 2j)\sigma(s; 2j)\bar{N}(s; 2j) = N(s; 2j)\bar{N}(s; 2j)\{\vartheta \cdot [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)]\}^n \\ [N(s; 2j)\bar{N}(s; 2j), \sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)] = 0 \end{cases}$$

性质2.9.2.

$$\begin{cases} \bar{N}(s; 2j)[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]N(s; 2j) = S_{ab}(s, \varsigma; 2j) \\ N(s; 2j)S_{ab}(s, \varsigma; 2j)\bar{N}(s; 2j) = [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]N(s; 2j)\bar{N}(s; 2j) \\ N(s; 2j)S_{ab}(s, \varsigma; 2j)\bar{N}(s; 2j) = N(s; 2j)\bar{N}(s; 2j)[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)] \\ [N(s; 2j)\bar{N}(s; 2j), S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)] = 0 \end{cases}$$

性质2.9.3.

$$\begin{cases} \bar{N}(s; 2j)\{\vartheta \cdot [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)]\}^n N(s; 2j) = [\vartheta \cdot \sigma(s; 2j)]^n \\ N(s; 2j)\{\vartheta \cdot \sigma(s; 2j)\}^n \bar{N}(s; 2j) = \{\vartheta \cdot [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)]\}^n N(s; 2j)\bar{N}(s; 2j) \\ N(s; 2j)\{\vartheta \cdot \sigma(s; 2j)\}^n \bar{N}(s; 2j) = N(s; 2j)\bar{N}(s; 2j)\{\vartheta \cdot [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)]\}^n \\ [N(s; 2j)\bar{N}(s; 2j), \{\vartheta \cdot [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)]\}^n] = 0 \end{cases}$$

性质2.9.4.

$$\begin{cases} \bar{N}(s; 2j)\{\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]\}^n N(s; 2j) = [\vartheta^{ab}S_{ab}(s, \varsigma; 2j)]^n \\ N(s; 2j)[\vartheta^{ab}S_{ab}(s, \varsigma; 2j)]^n \bar{N}(s; 2j) = \{\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]\}^n N(s; 2j)\bar{N}(s; 2j) \\ N(s; 2j)[\vartheta^{ab}S_{ab}(s, \varsigma; 2j)]^n \bar{N}(s; 2j) = N(s; 2j)\bar{N}(s; 2j)\{\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]\}^n \\ [N(s; 2j)\bar{N}(s; 2j), \{\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]\}^n] = 0 \end{cases}$$

推论2.9.1.

$$\begin{cases} \bar{N}(s; 2j)e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]} N(s; 2j) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; 2j)} \\ N(s; 2j)e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; 2j)} \bar{N}(s; 2j) = e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]} N(s; 2j)\bar{N}(s; 2j) \\ N(s; 2j)e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; 2j)} \bar{N}(s; 2j) = N(s; 2j)\bar{N}(s; 2j)e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]} \\ [N(s; 2j)\bar{N}(s; 2j), e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]}] = 0 \end{cases}$$

## 2.10 推论2: 常数矩阵 $N(s; 2j)$ , $\bar{N}(s; 2j)$ 的另外几个恒等式

推论2.10.1.

$$\begin{cases} \bar{N}(s; 2j)\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} N(s; 2j) = \frac{1}{2s}\sigma(s; 2j) \\ \bar{N}(s; 2j)I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)N(s; 2j) = (1 - \frac{1}{2s})\sigma(s; 2j) \\ N^{A_\varsigma}(s; 2j)\sigma(s - \frac{1}{2}; 2j)\bar{N}_{A_\varsigma}(s; 2j) = (1 - \frac{1}{2s})\sigma(s; 2j) \\ \bar{N}_{A_\varsigma}(s; 2j)\sigma(s; 2j)N^{A_\varsigma}(s; 2j) = (1 + \frac{2j+1}{2s})\sigma(s - \frac{1}{2}; 2j) \end{cases}$$

推论2.10.2.

$$\begin{cases} \bar{N}(s; 2j)S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} N(s; 2j) = \frac{1}{2s}S_{ab}(s, \varsigma; 2j) \\ \bar{N}(s; 2j)I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)N(s; 2j) = (1 - \frac{1}{2s})S_{ab}(s, \varsigma; 2j) \\ N^{A_\varsigma}(s; 2j)S_{ab}(s - \frac{1}{2}, \varsigma; 2j)\bar{N}_{A_\varsigma}(s; 2j) = (1 - \frac{1}{2s})S_{ab}(s, \varsigma; 2j) \\ \bar{N}_{A_\varsigma}(s; 2j)S_{ab}(s, \varsigma; 2j)N^{A_\varsigma}(s; 2j) = (1 + \frac{2j+1}{2s})S_{ab}(s - \frac{1}{2}, \varsigma; 2j) \end{cases}$$

推论2.10.3.

$$\begin{cases} \bar{N}(1)[\sigma(j) \otimes I_2 + I_{2j+1} \otimes \sigma(j)]N(1) = \sigma(1) \\ \bar{N}(\frac{3}{2})\{\sigma(j) \otimes I_3 + I_{2j+1} \otimes \{\bar{N}(1)[\sigma(j) \otimes I_2 + I_{2j+1} \otimes \sigma(j)]N(1)\}\}N(\frac{3}{2}) = \sigma(\frac{3}{2}) \\ \bar{N}(s; 2j) \cdots \bar{N}(\frac{3}{2})\{\sigma(j) \otimes I_3 + I_{2j+1} \otimes \{\bar{N}(1)[\sigma(j) \otimes I_2 + I_{2j+1} \otimes \sigma(j)]N(1)\}\}N(\frac{3}{2}) \cdots N(s; 2j) = \sigma(s; 2j) \end{cases}$$

### 2.11 矩阵 $N(s; 2j)$ , $\bar{N}(s; 2j)$ 的常数不变张量性质

定理2.11.1.  $N(s; 2j) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(j)} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2}; 2j)} N(s; 2j) e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s; 2j)}$

定理2.11.2.  $\bar{N}(s; 2j) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s; 2j)} \bar{N}(s; 2j) e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(j)} \otimes e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2}; 2j)}$

## 3 二阶完美常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ , $X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$

只有满足 $\varepsilon_{A_\varsigma B_\varsigma}(s; 2j) = -\varepsilon_{B_\varsigma A_\varsigma}(s; 2j)$ 反对称条件时, 本章节内容才全部成立( $j$ =半整数), 否则只有部分成立( $j$ =整数)。对于 $j$ =整数的 $\varepsilon_{A_\varsigma B_\varsigma}(s; 2n) = \varepsilon_{B_\varsigma A_\varsigma}(s; 2n)$ 对称情形还需重新研究考虑, 先放一放, 等有时间了再说。

### 3.1 完美常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ , $X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$

定义3.1.1.  $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) := \frac{\sqrt{2s-1}}{\sqrt{2s-1+2j}} \varepsilon^{A_\varsigma B_\varsigma}(j) N_{B_\varsigma m_\varsigma}^{l_\varsigma}(s - \frac{1}{2}; 2j)$ ,  $X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j) := \frac{\sqrt{2s-1}}{\sqrt{2s-1+2j}} \varepsilon_{A_\varsigma B_\varsigma}(j) N_{l_\varsigma}^{B_\varsigma m_\varsigma}(s - \frac{1}{2}; 2j)$

性质3.1.1.  $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) \simeq X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$

### 3.2 常数矩阵 $X(s; 2j)$ , $\bar{X}(s; 2j)$

定义3.2.1.

$$\begin{cases} X^{A_\varsigma}(s; 2j) \prec X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j), X_{A_\varsigma}(s; 2j) \prec X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j) \\ \bar{X}_{A_\varsigma}(s; 2j) \prec X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j), \bar{X}^{A_\varsigma}(s; 2j) \prec X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) \\ X(s; 2j) \prec X_{A_\varsigma \otimes l_\varsigma}^{m_\varsigma}(s; 2j), \bar{X}(s; 2j) \prec X_{m_\varsigma}^{A_\varsigma \otimes l_\varsigma}(s; 2j) = X^+(s; 2j) \end{cases}$$

### 3.3 常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ , $X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$ 的升降指标

性质3.3.1.

$$\begin{cases} X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) = \varepsilon^{A_\varsigma B_\varsigma} \varepsilon^{l_\varsigma n_\varsigma}(s - \frac{1}{2}; 2j) \varepsilon_{m_\varsigma r_\varsigma}(s - 1; 2j) X_{B_\varsigma n_\varsigma}^{r_\varsigma}(s - \frac{1}{2}; 2j) \\ X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j) = \varepsilon_{A_\varsigma B_\varsigma} \varepsilon_{l_\varsigma n_\varsigma}(s - \frac{1}{2}; 2j) \varepsilon^{m_\varsigma r_\varsigma}(s - 1; 2j) X_{r_\varsigma}^{B_\varsigma n_\varsigma}(s - \frac{1}{2}; 2j) \end{cases}$$

### 3.4 常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ , $X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$ 的正交性

性质3.4.1.  $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j) = \delta_{m_\varsigma}^{n_\varsigma} [\Leftrightarrow] X^{A_\varsigma}(s; 2j) \bar{X}_{A_\varsigma}(s; 2j) = I_{C_{2s-2+2j}^{2s-2}} [\Leftrightarrow] \bar{X}(s; 2j) X(s; 2j) = I_{C_{2s-2+2j}^{2s-2}}$

性质3.4.2.  $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) N_{A_\varsigma l_\varsigma}^{k_\varsigma}(s; 2j) = 0$

$[\Leftrightarrow] X^{A_\varsigma}(s; 2j) \bar{N}_{A_\varsigma}(s; 2j) = 0, N_{A_\varsigma}(s; 2j) \bar{X}^{A_\varsigma}(s; 2j) = 0 [\Leftrightarrow] \bar{X}(s; 2j) N(s; 2j) = 0, \bar{N}(s; 2j) X(s; 2j) = 0$

性质3.4.3.  $X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j) X_{m_\varsigma}^{A_\varsigma k_\varsigma}(s; 2j) = \frac{2s-1}{2s-1+2j} \delta_{l_\varsigma}^{k_\varsigma} [\Leftrightarrow] \bar{X}_{A_\varsigma}(s; 2j) X^{A_\varsigma}(s; 2j) = \frac{2s-1}{2s-1+2j} I_{C_{2s-1+2j}^{2s-1}}$

性质3.4.4.  $X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j) X_{m_\varsigma}^{B_\varsigma l_\varsigma}(s; 2j) = \frac{1}{2j+1} C_{2s-2+2j}^{2s-2} \delta_{A_\varsigma}^{B_\varsigma} [\Leftrightarrow] tr[\bar{X}_{A_\varsigma}(s; 2j) X^{B_\varsigma}(s; 2j)] = \frac{1}{2j+1} C_{2s-2+2j}^{2s-2} \delta_{A_\varsigma}^{B_\varsigma}$

推论3.4.1.  $\bar{N}(s; 2j) N(s; 2j) = I_{C_{2s+2j}^{2s}}, \bar{X}(s; 2j) X(s; 2j) = I_{C_{2s-2+2j}^{2s-2}}, \bar{N}(s; 2j) X(s; 2j) = 0, \bar{X}(s; 2j) N(s; 2j) = 0$

### 3.5 常数不变张量 $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j)$ , $X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j)$ 的自旋变换

推论3.5.1.  $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) \sigma^{\alpha_\varsigma}_{A_\varsigma}{}^{B_\varsigma}(j) X_{B_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j) = -\frac{1}{2s-1+2j} \sigma^{\alpha_\varsigma}_{m_\varsigma}{}^{n_\varsigma}(s-1; 2j)$

$[\Leftrightarrow] X^{A_\varsigma}(s; 2j) \sigma_{A_\varsigma}{}^{B_\varsigma}(j) \bar{X}_{B_\varsigma}(s; 2j) = -\frac{1}{2s-1+2j} \sigma(s-1; 2j)$

$[\Leftrightarrow] \bar{X}(s; 2j) \sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} X(s; 2j) = -\frac{1}{2s-1+2j} \sigma(s-1; 2j)$

推论3.5.2.  $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j) \sigma^{\alpha_\varsigma}_{A_\varsigma}{}^{B_\varsigma}(j) X_{B_\varsigma k_\varsigma}^{m_\varsigma}(s; 2j) = -\frac{1}{2s-1+2j} \sigma^{\alpha_\varsigma}_{k_\varsigma}{}^{l_\varsigma}(s-\frac{1}{2}; 2j)$

$[\Leftrightarrow] \bar{X}^{A_\varsigma}(s; 2j) \sigma^{\alpha_\varsigma}_{A_\varsigma}{}^{B_\varsigma}(j) X_{B_\varsigma}(s; 2j) = -\frac{1}{2s-1+2j} \sigma^{\alpha_\varsigma}(s-\frac{1}{2}; 2j)$



推论3.5.3.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j)S_{abA_\zeta}^{B_\zeta}(j)X_{B_\zeta l_\zeta}^{n_\zeta}(s; 2j) = -\frac{1}{2s-1+2j}S_{abm_\zeta}^{n_\zeta}(s-1; 2j)$

$[\Leftrightarrow] X^{A_\zeta}(s; 2j)S_{ab}(j, \varsigma) \otimes I_{C_{2s-2+2j}^{2s-2}} \bar{X}_{A_\zeta}(s; 2j) = -\frac{1}{2s-1+2j}S_{ab}(s-1, \varsigma; 2j)$

$[\Leftrightarrow] \bar{X}(s; 2j)S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} X(s; 2j) = -\frac{1}{2s-1+2j}S_{ab}(s-1, \varsigma; 2j)$

推论3.5.4.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j)S_{abA_\zeta}^{B_\zeta}(j)X_{B_\zeta k_\zeta}^{m_\zeta}(s; 2j) = -\frac{1}{2s}S_{abk_\zeta}^{l_\zeta}(s-\frac{1}{2}; 2j)$

$[\Leftrightarrow] \bar{X}^{A_\zeta}(s; 2j)S_{abA_\zeta}^{B_\zeta}(j)X_{B_\zeta}(s; 2j) = -\frac{1}{2s-1+2j}S_{ab}(s-\frac{1}{2}, \varsigma; 2j)$

### 3.6 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j)$ , $X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j)$ 的置换性质

定理3.6.1.

$\left\{ X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j)[\sigma_{A_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{l_\zeta}^{k_\zeta}(s-\frac{1}{2}; 2j)] = \sigma_{m_\zeta}^{n_\zeta}(s-1; 2j)X_{n_\zeta}^{B_\zeta k_\zeta}(s; 2j) \right.$

$\left. \left\{ [\sigma_{A_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{l_\zeta}^{k_\zeta}(s-\frac{1}{2}; 2j)]X_{B_\zeta k_\zeta}^{n_\zeta}(s; 2j) = X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j)\sigma_{m_\zeta}^{n_\zeta}(s-1; 2j) \right. \right.$

$\left. \left\{ X^{A_\zeta}(s; 2j)[\sigma_{A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta}\sigma(s-\frac{1}{2}; 2j)] = \sigma(s-1; 2j)X^{B_\zeta}(s; 2j) \right. \right.$

$\left. \left\{ [\sigma_{A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta}\sigma(s-\frac{1}{2}; 2j)]\bar{X}_{B_\zeta}(s; 2j) = \bar{X}_{A_\zeta}(s; 2j)\sigma(s-1; 2j) \right. \right.$

$\left. \left\{ \bar{X}(s; 2j)[\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s-\frac{1}{2}; 2j)] = \sigma(s-1; 2j)\bar{X}(s; 2j) \right. \right.$

$\left. \left\{ [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s-\frac{1}{2}; 2j)]X(s; 2j) = X(s; 2j)\sigma(s-1; 2j) \right. \right.$

定理3.6.2.

$\left\{ X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j)[S_{abA_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abl_\zeta}^{k_\zeta}(s-\frac{1}{2}; 2j)] = S_{abm_\zeta}^{n_\zeta}(s-1; 2j)X_{n_\zeta}^{B_\zeta \otimes k_\zeta}(s; 2j) \right.$

$\left. \left\{ [S_{abA_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abl_\zeta}^{k_\zeta}(s-\frac{1}{2}; 2j)]X_{B_\zeta k_\zeta}^{n_\zeta}(s; 2j) = X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j)S_{abm_\zeta}^{n_\zeta}(s-1; 2j) \right. \right.$

$\left. \left\{ X^{A_\zeta}(s; 2j)[S_{abA_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta}S_{ab}(s-\frac{1}{2}, \varsigma; 2j)] = S_{ab}(s-1, \varsigma; 2j)X^{B_\zeta}(s; 2j) \right. \right.$

$\left. \left\{ [S_{abA_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta}S_{ab}(s-\frac{1}{2}, \varsigma; 2j)]\bar{X}_{B_\zeta}(s; 2j) = \bar{X}_{A_\zeta}(s; 2j)S_{ab}(s-1, \varsigma; 2j) \right. \right.$

$\left. \left\{ \bar{X}(s; 2j)[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s-\frac{1}{2}, \varsigma; 2j)] = S_{ab}(s-1, \varsigma; 2j)\bar{X}(s; 2j) \right. \right.$

$\left. \left\{ [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s-\frac{1}{2}, \varsigma; 2j)]X(s; 2j) = X(s; 2j)S_{ab}(s-1, \varsigma; 2j) \right. \right.$

推论3.6.1.

$\left\{ N_{C_\zeta m_\zeta}^{l_\zeta}(s-\frac{1}{2}; 2j)\varepsilon^{C_\zeta A_\zeta}(j)[\sigma_{A_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{l_\zeta}^{k_\zeta}(s-\frac{1}{2}; 2j)] = \sigma_{m_\zeta}^{n_\zeta}(s-1; 2j)N_{D_\zeta n_\zeta}^{k_\zeta}(s-\frac{1}{2}; 2j)\varepsilon^{D_\zeta B_\zeta}(j) \right.$

$\left. \left\{ [\sigma_{A_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{l_\zeta}^{k_\zeta}(s-\frac{1}{2}; 2j)]\varepsilon_{B_\zeta C_\zeta}(j)N_{k_\zeta}^{C_\zeta n_\zeta}(s-\frac{1}{2}; 2j) = \varepsilon_{A_\zeta D_\zeta}(j)N_{l_\zeta}^{D_\zeta m_\zeta}(s-\frac{1}{2}; 2j)\sigma_{m_\zeta}^{n_\zeta}(s-1; 2j) \right. \right.$

$\left. \left\{ N_{C_\zeta}(s-\frac{1}{2}; 2j)\varepsilon^{C_\zeta A_\zeta}(j)[\sigma_{A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta}\sigma(s-\frac{1}{2}; 2j)] = \sigma(s-1; 2j)N_{D_\zeta}(s-\frac{1}{2}; 2j)\varepsilon^{D_\zeta B_\zeta}(j) \right. \right.$

$\left. \left\{ [\sigma_{A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta}\sigma(s-\frac{1}{2}; 2j)]\varepsilon_{B_\zeta C_\zeta}(j)N^{C_\zeta}(s-\frac{1}{2}; 2j) = \varepsilon_{A_\zeta D_\zeta}(j)N^{D_\zeta}(s-\frac{1}{2}; 2j)\sigma(s-1; 2j) \right. \right.$

推论3.6.2.

$\left\{ N_{C_\zeta m_\zeta}^{l_\zeta}(s-\frac{1}{2}; 2j)\varepsilon^{C_\zeta A_\zeta}(j)[S_{abA_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abl_\zeta}^{k_\zeta}(s-\frac{1}{2}; 2j)] = S_{abm_\zeta}^{n_\zeta}(s-1; 2j)N_{D_\zeta n_\zeta}^{k_\zeta}(s-\frac{1}{2}; 2j)\varepsilon^{D_\zeta B_\zeta}(j) \right.$

$\left. \left\{ [S_{abA_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abl_\zeta}^{k_\zeta}(s-\frac{1}{2}; 2j)]\varepsilon_{B_\zeta C_\zeta}(j)N_{k_\zeta}^{C_\zeta n_\zeta}(s-\frac{1}{2}; 2j) = \varepsilon_{A_\zeta D_\zeta}(j)N_{l_\zeta}^{D_\zeta m_\zeta}(s-\frac{1}{2}; 2j)S_{abm_\zeta}^{n_\zeta}(s-1; 2j) \right. \right.$

$\left. \left\{ N_{C_\zeta}(s-\frac{1}{2}; 2j)\varepsilon^{C_\zeta A_\zeta}(j)[S_{abA_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta}S_{ab}(s-\frac{1}{2}, \varsigma; 2j)] = S_{ab}(s-1, \varsigma; 2j)N_{D_\zeta}(s-\frac{1}{2}; 2j)\varepsilon^{D_\zeta B_\zeta}(j) \right. \right.$

$\left. \left\{ [S_{abA_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta}S_{ab}(s-\frac{1}{2}, \varsigma; 2j)]\varepsilon_{B_\zeta C_\zeta}(j)N^{C_\zeta}(s-\frac{1}{2}; 2j) = \varepsilon_{A_\zeta D_\zeta}(j)N^{D_\zeta}(s-\frac{1}{2}; 2j)S_{ab}(s-1, \varsigma; 2j) \right. \right.$

### 3.7 推论：关于常数矩阵 $X(s; 2j)$ , $\bar{X}(s; 2j)$ 的重要性质

推论3.7.1.

$\left\{ \bar{X}(s; 2j)[\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s-\frac{1}{2}; 2j)]X(s; 2j) = \sigma(s-1; 2j) \right.$

$\left. \left\{ X(s; 2j)\sigma(s-1; 2j)\bar{X}(s; 2j) = [\sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s-\frac{1}{2}; 2j)]X(s; 2j)\bar{X}(s; 2j) \right. \right.$

$\left. \left\{ [X(s; 2j)\bar{X}(s; 2j), \sigma(j) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes \sigma(s-\frac{1}{2}; 2j)] = 0 \right. \right.$

推论3.7.2.

$$\begin{cases} \bar{X}(s; 2j)[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]X(s; 2j) = S_{ab}(s - 1, \varsigma; 2j) \\ X(s; 2j)S_{ab}(s, \varsigma - 1, \varsigma; 2j)\bar{X}(s; 2j) = [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)]X(s; 2j)\bar{X}(s; 2j) \\ [X(s; 2j)\bar{X}(s; 2j), S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j}^{2s-1}} + I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)] = 0 \end{cases}$$

推论3.7.3.  $X^{A_\varsigma}(s; 2j)\sigma(s - \frac{1}{2}; 2j)\bar{X}_{A_\varsigma}(s; 2j) = \frac{2s+2j}{2s-1+2j}\sigma(s - 1; 2j)$

$[\Leftrightarrow]\bar{X}(s; 2j)I_{2j+1} \otimes \sigma(s - \frac{1}{2}; 2j)X(s; 2j) = \frac{2s+2j}{2s-1+2j}\sigma(s - 1; 2j)$

推论3.7.4.  $X^{A_\varsigma}(s; 2j)I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)\bar{X}_{A_\varsigma}(s; 2j) = \frac{2s+2j}{2s-1+2j}S_{ab}(s - 1, \varsigma; 2j)$

$[\Leftrightarrow]\bar{X}(s; 2j)I_{2j+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j)X(s; 2j) = \frac{2s+2j}{2s-1+2j}S_{ab}(s - 1, \varsigma; 2j)$

### 3.8 矩阵 $X(s; 2j)$ , $\bar{X}(s; 2j)$ 的常数不变张量性质

定理3.8.1.  $X(s; 2j) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(j)} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2}; 2j)} X(s; 2j) e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s-1; 2j)}$

定理3.8.2.  $\bar{X}(s; 2j) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-1; 2j)} \bar{X}(s; 2j) e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(j)} \otimes e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s-\frac{1}{2}; 2j)}$

### 3.9 常数矩阵 $\Omega(s; 2j)$ , $\sigma(s - 1; 2j)$ 的置换性质

推论3.9.1.  $\begin{cases} \Omega(s; 2j)[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]X(s; 2j) = [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]X(s; 2j)\sigma(s - 1; 2j) \\ \bar{X}(s; 2j)[I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]\Omega(s; 2j) = \sigma(s - 1; 2j)\bar{X}(s; 2j)[I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)] \end{cases}$

推论3.9.2.  $\begin{cases} \sigma(s; 2j) = \bar{N}(s; 2j)[I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]\Omega(s; 2j)[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]N(s; 2j) \\ \sigma(s - 1; 2j) = \bar{X}(s; 2j)[I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]\Omega(s; 2j)[I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]X(s; 2j) \end{cases}$

推论3.9.3.  $\begin{cases} [\vec{\vartheta} \cdot \sigma(s; 2j)]^n = \bar{N}(s; 2j)[I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)][\vec{\vartheta} \cdot \Omega(s; 2j)]^n [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]N(s; 2j) \\ [\vec{\vartheta} \cdot \sigma(s - 1; 2j)]^n = \bar{X}(s; 2j)[I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)][\vec{\vartheta} \cdot \Omega(s; 2j)]^n [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]X(s; 2j) \end{cases}$

推论3.9.4.  $\begin{cases} e^{\vec{\vartheta} \cdot \sigma(s; 2j)} = \bar{N}(s; 2j)[I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]e^{\vec{\vartheta} \cdot \Omega(s; 2j)} [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]N(s; 2j) \\ e^{\vec{\vartheta} \cdot \sigma(s-1; 2j)} = \bar{X}(s; 2j)[I_{2j+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j)]e^{\vec{\vartheta} \cdot \Omega(s; 2j)} [I_{2j+1} \otimes \Gamma(s - \frac{1}{2}; 2j)]X(s; 2j) \end{cases}$

### 3.10 常数矩阵 $\Omega(s - l; 2j)$ , $[\vec{\vartheta} \cdot \Omega(s - l; 2j)]^n$ , $e^{\vec{\vartheta} \cdot \Omega(s-l; 2j)}$ 的同构性表示

推论3.10.1.  $\Omega(s; 2j) = \Omega(s - 1; 2j) \otimes I_{(2j+1)^2} + I_{(2j+1)^{2s-2}} \otimes \Omega(1; 2j)$

推论3.10.2.

$$\begin{cases} \Omega(s; 2j)I_{(2j+1)^{2s-2}} \otimes \{[I_{2j+1} \otimes \Gamma(\frac{1}{2}; 2j)]X(1; 2j)\} = I_{(2j+1)^{2s-2}} \otimes \{[I_{2j+1} \otimes \Gamma(\frac{1}{2}; 2j)]X(1; 2j)\}\Omega(s - 1; 2j) \\ I_{(2j+1)^{2s-2}} \otimes \{\bar{X}(1; 2j)[I_{2j+1} \otimes \bar{\Gamma}(\frac{1}{2}; 2j)]\}\Omega(s; 2j) = \Omega(s - 1; 2j)I_{(2j+1)^{2s-2}} \otimes \{\bar{X}(1; 2j)[I_{2j+1} \otimes \bar{\Gamma}(\frac{1}{2}; 2j)]\} \end{cases}$$

推论3.10.3.

$$\begin{cases} \Omega(s - 1; 2j) = I_{(2j+1)^{2s-2}} \otimes \{\bar{X}(1; 2j)[I_{2j+1} \otimes \bar{\Gamma}(\frac{1}{2}; 2j)]\}\Omega(s; 2j)I_{(2j+1)^{2s-2}} \otimes \{[I_{2j+1} \otimes \Gamma(\frac{1}{2}; 2j)]X(1; 2j)\} \\ [\vec{\vartheta} \cdot \Omega(s - 1; 2j)]^n = I_{(2j+1)^{2s-2}} \otimes \{\bar{X}(1; 2j)[I_{2j+1} \otimes \bar{\Gamma}(\frac{1}{2}; 2j)]\}[\vec{\vartheta} \cdot \Omega(s; 2j)]^n I_{(2j+1)^{2s-2}} \otimes \{[I_{2j+1} \otimes \Gamma(\frac{1}{2}; 2j)]X(1; 2j)\} \\ e^{\vec{\vartheta} \cdot \Omega(s-1; 2j)} = I_{(2j+1)^{2s-2}} \otimes \{\bar{X}(1; 2j)[I_{2j+1} \otimes \bar{\Gamma}(\frac{1}{2}; 2j)]\}e^{\vec{\vartheta} \cdot \Omega(s; 2j)} I_{(2j+1)^{2s-2}} \otimes \{[I_{2j+1} \otimes \Gamma(\frac{1}{2}; 2j)]X(1; 2j)\} \end{cases}$$

定义3.10.1.

$$\begin{cases} T(s; 2j) := I_{(2j+1)^{2s-2}} \otimes \{[I_{2j+1} \otimes \Gamma(\frac{1}{2}; 2j)]X(1; 2j)\} \\ \bar{T}(s; 2j) := I_{(2j+1)^{2s-2}} \otimes \{\bar{X}(1; 2j)[I_{2j+1} \otimes \bar{\Gamma}(\frac{1}{2}; 2j)]\} = T^+(s; 2j) \end{cases}$$

推论3.10.4.

$$\begin{cases} \Omega(s - l; 2j) = \bar{T}(s - l + 1; 2j) \cdots \bar{T}(s - 1; 2j)\bar{T}(s; 2j)\Omega(s; 2j)T(s; 2j)T(s - 1; 2j) \cdots T(s - l + 1; 2j) \\ [\vec{\vartheta} \cdot \Omega(s - l; 2j)]^n = \bar{T}(s - l + 1; 2j) \cdots \bar{T}(s - 1; 2j)\bar{T}(s; 2j)[\vec{\vartheta} \cdot \Omega(s; 2j)]^n T(s; 2j)T(s - 1; 2j) \cdots T(s - l + 1; 2j) \\ e^{\vec{\vartheta} \cdot \Omega(s-l; 2j)} = \bar{T}(s - l + 1; 2j) \cdots \bar{T}(s - 1; 2j)\bar{T}(s; 2j)e^{\vec{\vartheta} \cdot \Omega(s; 2j)} T(s; 2j)T(s - 1; 2j) \cdots T(s - l + 1; 2j) \end{cases}$$

推论3.10.5.

$$\begin{cases} \sigma(s-l; 2j) = \bar{\Gamma}(s-l; 2j)\bar{T}(s-l+1; 2j) \cdots \bar{T}(s; 2j)\Omega(s; 2j)T(s; 2j) \cdots T(s-l+1; 2j)\Gamma(s-l; 2j) \\ [\vec{\vartheta} \cdot \sigma(s-l; 2j)]^n = \bar{\Gamma}(s-l; 2j)\bar{T}(s-l+1; 2j) \cdots \bar{T}(s; 2j)[\vec{\vartheta} \cdot \Omega(s; 2j)]^n T(s; 2j) \cdots T(s-l+1; 2j)\Gamma(s-l; 2j) \\ e^{\vec{\vartheta} \cdot \sigma(s-l; 2j)} = \bar{\Gamma}(s-l; 2j)\bar{T}(s-l+1; 2j) \cdots \bar{T}(s; 2j)e^{\vec{\vartheta} \cdot \Omega(s; 2j)} T(s; 2j) \cdots T(s-l+1; 2j)\Gamma(s-l; 2j) \end{cases}$$

## 4 几个例子

### 4.1 例子一

推论4.1.1.  $\Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta}^{k_\zeta}(2; 1)\sigma_{\alpha_\zeta}^{A_\zeta B_\zeta}\sigma_{\beta_\zeta}^{C_\zeta D_\zeta} = \Gamma_{\alpha_\zeta \beta_\zeta}^{k_\zeta}(2; 1)\Gamma_{\alpha_\zeta \beta_\zeta}^{\rho_\zeta}(1; 2)$

### 4.2 具体例子

定义4.2.1.  $\Gamma_{A_\zeta B_\zeta}^{k_\zeta}(1; 2) = \frac{1}{2!}\Gamma_{(A_\zeta B_\zeta)}^{k_\zeta}(1; 2)$

$$\Gamma_{\underbrace{0_\zeta \cdots 0_\zeta}_{l_0} \underbrace{1_\zeta \cdots 1_\zeta}_{l_1} \underbrace{2_\zeta \cdots 2_\zeta}_{l_2}}^{k_\zeta}(1; 2) = \sqrt{\frac{l_0!l_1!l_2!}{2!}}\delta\{k_\zeta, \sum_{l=0}^1 \sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+2-k}^{2l} - C_{2l+2-\lambda_{2l}}^{2l}\}, l_0 + l_1 + l_2 = 2$$

定义4.2.2.  $\Gamma_{k_\zeta}^{A_\zeta B_\zeta}(1; 2) = \frac{1}{2!}\Gamma_{k_\zeta}^{(A_\zeta B_\zeta)}(1; 2)$

$$\Gamma_{k_\zeta}^{\underbrace{0_\zeta \cdots 0_\zeta}_{l_0} \underbrace{1_\zeta \cdots 1_\zeta}_{l_1} \underbrace{2_\zeta \cdots 2_\zeta}_{l_2}}(1; 2) = \sqrt{\frac{l_0!l_1!l_2!}{2!}}\delta\{k_\zeta, \sum_{l=0}^1 \sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+2-k}^{2l} - C_{2l+2-\lambda_{2l}}^{2l}\}, l_0 + l_1 + l_2 = 2$$

性质4.2.1.  $\bar{\Gamma}(1; 2)\Gamma(1; 2) = I_6$

$$\bar{\Gamma}(1; 2) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \Gamma(1; 2) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}, \Gamma(1; 2)\bar{\Gamma}(1; 2) = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

性质4.2.2.

$$\bar{\Gamma}(1; 2)[\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]\Gamma(1; 2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 1 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \sqrt{2} & 0 & \sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2\sqrt{2} & 0 & 0 \\ 0 & 0 & 2\sqrt{2} & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & -1 & -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & \sqrt{2} & 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 1 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$[\bar{\Gamma}(1; 2)[\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]\Gamma(1; 2)]^2 = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2\sqrt{2} & 0 & 0 \\ 0 & 0 & 2\sqrt{2} & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

性质4.2.3.

$$\bar{\Gamma}(1; 2)(\gamma \otimes I_3 + I_3 \otimes \gamma)\Gamma(1; 2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\sqrt{2} & 0 \\ 0 & 0 & 0 & i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & 0 & 0 & 0 & i\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & i\sqrt{2} & 0 & 0 & 0 \\ -i\sqrt{2} & 0 & 0 & 0 & 0 & i\sqrt{2} \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{2} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i\sqrt{2} & 0 & 0 & 0 & 0 \\ i\sqrt{2} & 0 & 0 & -i\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & i\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\bar{\Gamma}(1; 2)(\gamma \otimes I_3 + I_3 \otimes \gamma)\Gamma(1; 2)]^2 = \begin{bmatrix} 4 & 0 & 0 & -2 & 0 & -2 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ -2 & 0 & 0 & 4 & 0 & -2 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ -2 & 0 & -2 & 0 & 0 & 4 \end{bmatrix}$$

## 5 高阶完美常数不变张量的一般推广

### 5.1 无穷扩展和推广(“二阶”名字的由来)

推论5.1.1.

$$\underbrace{\sigma(\frac{1}{2}), \varepsilon(\frac{1}{2})}_{\text{零阶}} \rightarrow \underbrace{\sigma(j), \varepsilon(j)}_{\text{一阶}} \rightarrow \underbrace{\sigma(s_1; 2j), \varepsilon(s_1; 2j)}_{\text{二阶}} \rightarrow \underbrace{\sigma(s_2, s_1; 2j), \varepsilon(s_2, s_1; 2j)}_{\text{三阶}} \rightarrow \underbrace{\sigma(s_3, s_2, s_1; 2j), \varepsilon(s_3, s_2, s_1; 2j)}_{\text{四阶}} \rightarrow \cdots$$

推论5.1.2.  $\underbrace{\sigma(\frac{1}{2}), \varepsilon(\frac{1}{2})}_{\text{零阶}} \rightarrow \underbrace{\sigma(s_1), \varepsilon(s_1)}_{\text{一阶}} \rightarrow \underbrace{\sigma(s_2, s_1), \varepsilon(s_2, s_1)}_{\text{二阶}} \rightarrow \underbrace{\sigma(s_3, s_2, s_1), \varepsilon(s_3, s_2, s_1)}_{\text{三阶}} \rightarrow \cdots$

$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$

$$\text{推论5.1.3. } \underbrace{\Gamma(s_1; 1)}_{\text{一阶}} \rightarrow \rightarrow \underbrace{\Gamma(s_2; 2s_1)}_{\text{二阶}} \rightarrow \rightarrow \underbrace{\Gamma(s_3; w < s_2, s_1 >)}_{\text{三阶}} \rightarrow \rightarrow \underbrace{\Gamma(s_4; w < s_3, s_2, s_1 >)}_{\text{四阶}} \rightarrow \rightarrow \dots$$

## 5.2 高阶完美常数不变张量的表示

推论5.2.1.

$$\bar{N}(s_0; w_0) [\tau(\frac{1}{2}) \otimes 1] N(s_0; w_0) = \frac{1}{2s_0} \tau(s_0; w_0); s = s_0, w_0 = 1; \tau(s_0) \equiv \tau(s_0; w_0)$$

$$\bar{N}(s_1, s_0; w_1) [\tau(s_0; w_0) \otimes I_{*0}] N(s_1, s_0; w_1) = \frac{1}{2s_1} \tau(s_1, s_0; w_1); s = 2s_1 s_0, w_1 = 2s_0$$

$$\bar{N}(s_2, s_1, s_0; w_2) [\tau(s_1, s_0; w_1) \otimes I_{*1}] N(s_2, s_1, s_0; w_2) = \frac{1}{2s_2} \tau(s_2, s_1, s_0; w_2); s = 2s_2 2s_1 s_0, w_2 = ?$$

$$\bar{N}(s_3, s_2, s_1, s_0; w_3) [\tau(s_2, s_1, s_0; w_1) \otimes I_{*2}] N(s_3, s_2, s_1, s_0; w_3) = \frac{1}{2s_3} \tau(s_3, s_2, s_1, s_0; w_3); s = 2s_3 2s_2 2s_1 s_0, w_3 = ??$$

推论5.2.2.

$$\bar{N}(s_0; w_0) [\tau(\frac{1}{2}) \otimes 1] N(s_0; w_0) = \frac{1}{2s_0} \tau(s_0; w_0); s = s_0, w_0 = 1; \tau(s_0) \equiv \tau(s_0; w_0)$$

$$\bar{N}(s_1; w_1) [\tau(s_0; w_0) \otimes I_{*0}] N(s_1; w_1) = \frac{1}{2s_1} \tau(s_1, s_0; w_1); s = 2s_1 s_0, w_1 = 2s_0, [\tau(s_1, s_0; w_1)] = C_{2s_1+w_1}^{2s_1}$$

$$\bar{N}(s_2; w_2) [\tau(s_1, s_0; w_1) \otimes I_{*1}] N(s_2; w_2) = \frac{1}{2s_2} \tau(s_2, s_1, s_0; w_2)$$

$$; s = 2s_2 2s_1 s_0, w_2 = C_{2s_1+w_1}^{2s_1} - 1, [\tau(s_2, s_1, s_0; w_2)] = C_{2s_2+w_2}^{2s_2}$$

$$\bar{N}(s_3; w_3) [\tau(s_2, s_1, s_0; w_2) \otimes I_{*2}] N(s_3; w_3) = \frac{1}{2s_3} \tau(s_3, s_2, s_1, s_0; w_3); s = 2s_3 2s_2 2s_1 s_0, w_3 = C_{2s_2+w_2}^{2s_2} - 1$$

$$\text{推论5.2.3. } [N(s; w)] = (w+1) C_{2s-1+w}^{2s-1} \times C_{2s+w}^{2s}, [\bar{N}(s; w)] = C_{2s+w}^{2s} \times (w+1) C_{2s-1+w}^{2s-1}$$

# 第三十九章 Dirac型完美常数不变张量

自我评述：本章主要是Dirac型完美常数不变张量的具体介绍，是对更一般完美常数不变张量的深入具体了解 and 具体扩展。若将中微子自旋矩阵替换为Dirac矩阵，就可以得到Dirac型完美常数不变张量。同时它是之前完美常数不变张量一般情形的特殊运用而已，故直接写出其结论即可，而无需再作证明。本章符号约定符合以往规范，出了本章也不会混淆。本章节的完美常数不变张量与 $w = 1$ 情形同构，所以同样满足重要的复合常数不变张量那一章的展开性质，并具有完全相同的展开系数，为了节省篇幅，不再重复写出。

## 1 Dirac型完美常数不变张量 $\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s; 3), \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}(s; 3)$

### 1.1 常数不变张量 $\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s; 3), \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}(s; 3)$

定义1.1.1.  $\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s; 3) = \frac{1}{(2s)!} \Gamma_{(\lambda_\zeta \mu_\zeta \eta_\zeta \dots)}^{k_\zeta}(s; 3)$

$$\Gamma_{\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{3_\zeta \dots 3_\zeta}_{l_3}}^{k_\zeta}(s; 3) = \sqrt{\frac{l_0! l_1! \dots l_3!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^3 (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+3-k}^{2l} - C_{2l+3-\lambda_{2l}}^{2l})\}, l_0 + l_1 + \dots + l_3 = 2s$$

定义1.1.2.  $\Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}(s; 3) = \frac{1}{(2s)!} \Gamma_{k_\zeta}^{(\lambda_\zeta \mu_\zeta \eta_\zeta \dots)}(s; 3)$

$$\Gamma_{k_\zeta}^{\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{3_\zeta \dots 3_\zeta}_{l_3}}(s; 3) = \sqrt{\frac{l_0! l_1! \dots l_3!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^3 (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+3-k}^{2l} - C_{2l+3-\lambda_{2l}}^{2l})\}, l_0 + l_1 + \dots + l_3 = 2s$$

### 1.2 常数矩阵 $\Gamma(s; 3), \bar{\Gamma}(s; 3)$

定义1.2.1.  $\Gamma(s; 3) \succ \Gamma_{\lambda_\zeta \otimes \mu_\zeta \otimes \eta_\zeta \otimes \dots}^{k_\zeta}(s; 3), \bar{\Gamma}(s; 3) \succ \Gamma_{k_\zeta}^{\lambda_\zeta \otimes \mu_\zeta \otimes \eta_\zeta \otimes \dots}(s; 3) \simeq \Gamma^T(s; 3)$

推论1.2.1.  $[\Gamma(s; 3)] = 4^{2s} \times C_{2s+3}^{2s}, [\bar{\Gamma}(s; 3)] = C_{2s+3}^{2s} \times 4^{2s}, [A_\zeta] = 3 + 1, [k_\zeta] = C_{2s+3}^{2s}$

### 1.3 常数不变张量 $\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s; 3), \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}(s; 3)$ 的基本性质

相等性：

性质1.3.1.  $\Gamma_{\lambda_\zeta' \mu_\zeta' \eta_\zeta' \dots}^{k_\zeta'}(s; 3) \simeq \Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s; 3) \simeq \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}(s; 3) \simeq \Gamma_{k_\zeta'}^{\lambda_\zeta' \mu_\zeta' \eta_\zeta' \dots}(s; 3)$

性质1.3.2.  $[\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s; 3)]^* \simeq \Gamma_{\lambda_\zeta' \mu_\zeta' \eta_\zeta' \dots}^{k_\zeta'}(s; 3), [\Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}(s; 3)]^* \simeq \Gamma_{k_\zeta'}^{\lambda_\zeta' \mu_\zeta' \eta_\zeta' \dots}(s; 3)$

推论1.3.1.  $\Gamma(s; 3) = \Gamma^*(s; 3), \bar{\Gamma}(s; 3) = \bar{\Gamma}^*(s; 3), \bar{\Gamma}(s; 3) = \bar{\Gamma}^+(s; 3), \Gamma(s; 3) = \bar{\Gamma}^+(s; 3)$

正交性：

性质1.3.3.  $\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s; 3) \Gamma_{l_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}(s; 3) = \delta^{k_\zeta l_\zeta} [\Leftrightarrow] \bar{\Gamma}(s; 3) \Gamma(s; 3) = I$

性质1.3.4.  $\Gamma_{\lambda_{1\zeta} \lambda_{2\zeta} \dots \lambda_{2s\zeta}}^{k_\zeta}(s; 3) \Gamma_{k_\zeta}^{\mu_{1\zeta} \mu_{2\zeta} \dots \mu_{2s\zeta}}(s; 3) = \frac{1}{(2s)!} \delta_{\lambda_{1\zeta}}^{(\mu_{1\zeta}} \delta_{\lambda_{2\zeta}}^{\mu_{2\zeta}} \dots \delta_{\lambda_{2s\zeta}}^{\mu_{2s\zeta})} = \frac{1}{(2s)!} \delta_{(\lambda_{1\zeta}}^{\mu_{1\zeta}} \delta_{\lambda_{2\zeta}}^{\mu_{2\zeta}} \dots \delta_{\lambda_{2s\zeta}}^{\mu_{2s\zeta}})$

对比性：

性质1.3.5.  $\varepsilon_{a_1 a_2 \dots a_n} \varepsilon^{b_1 b_2 \dots b_n} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_n}^{b_n} = \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_n}^{b_n]}$

其它性质：

性质1.3.6.  $\Gamma_{\lambda_\zeta}^{k_\zeta}(\frac{1}{2}; 3) = \delta_{\lambda_\zeta}^{k_\zeta}, \Gamma_{k_\zeta}^{\lambda_\zeta}(\frac{1}{2}; 3) = \delta_{k_\zeta}^{\lambda_\zeta}; \Gamma(0; 3) = 1, \bar{\Gamma}(0; 3) = 1$





### 1.12 关于常数不变张量 $\gamma^a{}_{k_\zeta}{}^{l_\zeta}(s; 3)$ , $S_{abk_\zeta}{}^{l_\zeta}(s; 3)$ 的重要推论

$$\text{推论1.12.1. } \underbrace{\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}}_{2s}(s; 3) \gamma^a{}_{k_\zeta}{}^{l_\zeta}(s; 3) = \frac{1}{(2s-1)!} \gamma^a(\lambda_\zeta \underbrace{\lambda'_\zeta \Gamma^{l_\zeta}_{\mu_\zeta \eta_\zeta \dots \lambda'_\zeta}}_{2s})(s; 3)$$

$$\text{推论1.12.2. } \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; 3) S_{abk_\zeta}{}^{l_\zeta}(s; 3) = \frac{1}{(2s-1)!} S_{ab}(\frac{1}{2}; 3) \underbrace{A'_\zeta \Gamma^{l_\zeta}_{B_\zeta C_\zeta \dots A'_\zeta}}_{2s}(s; 3)$$

### 1.13 常数矩阵 $\Omega_{ab}(s; 3)$ , $S_{ab}(s; 3)$ 的洛伦兹群表示

$$\text{推论1.13.1. } i[S_{ab}(\frac{1}{2}; 3), S_{cd}(\frac{1}{2}; 3)] = \delta_{ad} S_{bc}(\frac{1}{2}; 3) - \delta_{ac} S_{bd}(\frac{1}{2}; 3) + \delta_{bc} S_{ad}(\frac{1}{2}; 3) - \delta_{bd} S_{ac}(\frac{1}{2}; 3)$$

$$\Rightarrow i[\Omega_{ab}(s; 3), \Omega_{cd}(s; 3)] = \delta_{ad} \Omega_{bc}(s; 3) - \delta_{ac} \Omega_{bd}(s; 3) + \delta_{bc} \Omega_{ad}(s; 3) - \delta_{bd} \Omega_{ac}(s; 3)$$

$$\Rightarrow i[S_{ab}(s; 3), S_{cd}(s; 3)] = \delta_{ad} S_{bc}(s; 3) - \delta_{ac} S_{bd}(s; 3) + \delta_{bc} S_{ad}(s; 3) - \delta_{bd} S_{ac}(s; 3)$$

### 1.14 推论：常数矩阵 $\Gamma(s; 3)$ , $\bar{\Gamma}(s; 3)$ 的几个恒等式

$$\text{性质1.14.1. } \begin{cases} \bar{\Gamma}(s; 3) \Omega^a(s; 3) \Gamma(s; 3) = \gamma^a(s; 3), [\Gamma(s; 3) \bar{\Gamma}(s; 3), \Omega^a(s; 3)] = 0 \\ \Gamma(s; 3) \gamma^a(s; 3) \bar{\Gamma}(s; 3) = \Omega^a(s; 3) \Gamma(s; 3) \bar{\Gamma}(s; 3) = \Gamma(s; 3) \bar{\Gamma}(s; 3) \Omega^a(s; 3) \end{cases}$$

$$\text{性质1.14.2. } \begin{cases} \bar{\Gamma}(s; 3) \Omega_{ab}(s; 3) \Gamma(s; 3) = S_{ab}(s; 3), [\Gamma(s; 3) \bar{\Gamma}(s; 3), \Omega_{ab}(s; 3)] = 0 \\ \Gamma(s; 3) S_{ab}(s; 3) \bar{\Gamma}(s; 3) = \Omega_{ab}(s; 3) \Gamma(s; 3) \bar{\Gamma}(s; 3) = \Gamma(s; 3) \bar{\Gamma}(s; 3) \Omega_{ab}(s; 3) \end{cases}$$

$$\text{性质1.14.3. } \begin{cases} \bar{\Gamma}(s; 3) [\vartheta^{ab} \Omega_{ab}(s; 3)]^n \Gamma(s; 3) = [\vartheta^{ab} S_{ab}(s; 3)]^n, [\Gamma(s; 3) \bar{\Gamma}(s; 3), [\vartheta^{ab} \Omega_{ab}(s; 3)]^n] = 0 \\ \Gamma(s; 3) [\vartheta^{ab} S_{ab}(s; 3)]^n \bar{\Gamma}(s; 3) = [\vartheta^{ab} \Omega_{ab}(s; 3)]^n \Gamma(s; 3) \bar{\Gamma}(s; 3) = \Gamma(s; 3) \bar{\Gamma}(s; 3) [\vartheta^{ab} \Omega_{ab}(s; 3)]^n \end{cases}$$

$$\text{推论1.14.1. } \begin{cases} \bar{\Gamma}(s; 3) e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s; 3)} \Gamma(s; 3) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s; 3)}, [\Gamma(s; 3) \bar{\Gamma}(s; 3), e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s; 3)}] = 0 \\ \Gamma(s; 3) e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s; 3)} \bar{\Gamma}(s; 3) = e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s; 3)} \Gamma(s; 3) \bar{\Gamma}(s; 3) = \Gamma(s; 3) \bar{\Gamma}(s; 3) e^{\frac{i}{2} \vartheta^{ab} \Omega_{ab}(s; 3)} \end{cases}$$

### 1.15 常数矩阵 $\Gamma(s; 3)$ , $\bar{\Gamma}(s; 3)$ 的对称置换性质

$$\text{定义1.15.1. } S_{ex}(s, n) = \overbrace{(I_4 \otimes \dots \otimes I_4)}^{n-1} \otimes \overbrace{S_{ex}}^{2s-n-1} \otimes \dots \otimes I$$

$$\text{推论1.15.1. } \Gamma(s; 3) = S_{ex}(s, n) \Gamma(s; 3), \bar{\Gamma}(s; 3) = \bar{\Gamma}(s; 3) S_{ex}(s, n)$$

$$\text{推论1.15.2. } S_{ex}(s, n) \Omega^a(s; 3) S_{ex}(s, n) = \Omega^a(s; 3)$$

$$\text{推论1.15.3. } \hat{\psi}(s; 3) = S_{ex}(s, n) \hat{\psi}(s; 3), \forall n \in \{1, 2, \dots, 2s+1\}$$

### 1.16 矩阵 $\Gamma(s; 3)$ , $\bar{\Gamma}(s; 3)$ 的常数不变张量性质

$$\text{定理1.16.1. } \Gamma(s; 3) = e^{(i\Omega^a + \varsigma \epsilon) \cdot \Omega^a(s; 3)} \Gamma(s; 3) e^{-(i\Omega^a + \varsigma \epsilon) \cdot \gamma^a(s; 3)}$$

$$\text{定理1.16.2. } \bar{\Gamma}(s; 3) = e^{(i\Omega^a + \varsigma \epsilon) \cdot \gamma^a(s; 3)} \bar{\Gamma}(s; 3) e^{-(i\Omega^a + \varsigma \epsilon) \cdot \Omega^a(s; 3)}$$

推论1.16.1.

$$\begin{cases} S(s; 3) = e^{(i\Omega^a + \varsigma \epsilon) \cdot \Omega^a(s; 3)} S(s; 3) e^{-(i\Omega^a + \varsigma \epsilon) \cdot S^+(s; 3) \Omega^a(s; 3) S(s; 3)} \\ S(s; 3) = e^{(i\Omega^a + \varsigma \epsilon) \cdot S(s; 3) \Omega^a(s; 3) S^+(s; 3)} S(s; 3) e^{-(i\Omega^a + \varsigma \epsilon) \cdot \Omega^a(s; 3)} \\ S^+(s; 3) = e^{(i\Omega^a + \varsigma \epsilon) \cdot \Omega^a(s; 3)} S^+(s; 3) e^{-(i\Omega^a + \varsigma \epsilon) \cdot S(s; 3) \Omega^a(s; 3) S^+(s; 3)} \\ S^+(s; 3) = e^{(i\Omega^a + \varsigma \epsilon) \cdot S^+(s; 3) \Omega^a(s; 3) S(s; 3)} S^+(s; 3) e^{-(i\Omega^a + \varsigma \epsilon) \cdot \Omega^a(s; 3)} \end{cases}$$

### 1.17 常数矩阵 $I_4 \otimes \Gamma(s - \frac{1}{2}; 3)$ , $I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)$ 的置换性质

$$\text{推论1.17.1. } \Omega^a(s; 3) [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] = [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)]$$

$$\text{推论1.17.2. } [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] \Omega^a(s; 3) = [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)] [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]$$



1.18 推论：常数矩阵 $\Gamma(s - \frac{1}{2}; 3), \bar{\Gamma}(s - \frac{1}{2}; 3)$ 的几个恒等式

性质1.18.1.

$$\begin{cases} [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]\Omega^a(s; 3)[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] = [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)] \\ [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] \\ = \Omega^a(s; 3)[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] = [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]\Omega^a(s; 3) \\ [[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)], \Omega^a(s; 3)] = 0 \end{cases}$$

性质1.18.2.

$$\begin{cases} [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]\Omega_{ab}(s; 3)[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] = [S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)] \\ [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] \\ = \Omega_{ab}(s; 3)[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] = [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]\Omega_{ab}(s; 3) \\ [[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)], \Omega_{ab}(s; 3)] = 0 \end{cases}$$

性质1.18.3.

$$\begin{cases} [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)][\vartheta^{ab}\Omega_{ab}(s; 3)]^n[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] = \{\vartheta^{ab}[S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)]\}^n \\ [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)]\{\vartheta^{ab}[S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)]\}^n[I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] \\ = [\vartheta^{ab}\Omega_{ab}(s; 3)]^n[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] = [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)][\vartheta^{ab}\Omega_{ab}(s; 3)]^n \\ [[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)], [\vartheta^{ab}\Omega_{ab}(s; 3)]^n] = 0 \end{cases}$$

推论1.18.1.

$$\begin{cases} [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s; 3)}[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] = e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)]} \\ [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)]e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)]}[I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] \\ = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s; 3)}[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] = [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s; 3)} \\ [[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)], e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s; 3)}] = 0 \end{cases}$$

推论1.18.2.

$$\begin{cases} I_{4^{2s-1}}\Gamma(s - \frac{1}{2}; 3) = \Gamma(s - \frac{1}{2}; 3)I_{C_{2s-1+3}^{2s-1}}, \bar{\Gamma}(s - \frac{1}{2}; 3)I_{4^{2s-1}} = I_{C_{2s-1+3}^{2s-1}}\bar{\Gamma}(s - \frac{1}{2}; 3) \\ [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]\gamma^a \otimes I_{4^{2s-1}}[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] = \gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} \\ [\gamma^a \otimes I_{4^{2s-1}}][I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] = [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}}] \\ [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)][\gamma^a \otimes I_{4^{2s-1}}] = [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}}][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] \end{cases}$$

1.19 矩阵 $I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3), I_4 \otimes \Gamma(s - \frac{1}{2}; 3)$ 的常数不变张量性质

$$\text{定理1.19.1. } [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] = e^{(i\Omega^a + \varsigma\epsilon) \cdot \Omega^a(s; 3)}[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)]e^{-(i\Omega^a + \varsigma\epsilon) \cdot \gamma^a} \otimes e^{-(i\Omega^a + \varsigma\epsilon) \cdot \gamma^a(s - \frac{1}{2}; 3)}$$

$$\text{定理1.19.2. } [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] = e^{(i\Omega^a + \varsigma\epsilon) \cdot \gamma^a} \otimes e^{(i\Omega^a + \varsigma\epsilon) \cdot \gamma^a(s - \frac{1}{2}; 3)}[I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]e^{-(i\Omega^a + \varsigma\epsilon) \cdot \Omega^a(s; 3)}$$

2 Dirac型完美常数不变张量 $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ 2.1 完美常数不变张量 $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ 

$$\text{定义2.1.1. } N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s) := \underbrace{\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}}_{2s} \cdot (s) \underbrace{\Gamma_{l_\zeta}^{\mu_\zeta \eta_\zeta \dots}}_{2s-1} \cdot (s - \frac{1}{2}), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s) := \underbrace{\Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}}_{2s} \cdot (s) \underbrace{\Gamma_{\mu_\zeta \eta_\zeta \dots}^{l_\zeta}}_{2s-1} \cdot (s - \frac{1}{2})$$

$$\text{定义2.1.2. } N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) := \underbrace{\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}}_{2s} \cdot (s; 3) \underbrace{\Gamma_{l_\zeta}^{\mu_\zeta \eta_\zeta \dots}}_{2s-1} \cdot (s - \frac{1}{2}; 3), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) := \underbrace{\Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}}_{2s} \cdot (s; 3) \underbrace{\Gamma_{\mu_\zeta \eta_\zeta \dots}^{l_\zeta}}_{2s-1} \cdot (s - \frac{1}{2}; 3)$$

推论2.1.1.  $N(s; 3) = [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]\Gamma(s; 3), \bar{N}(s; 3) = \bar{\Gamma}(s; 3)[I_4 \otimes \Gamma(s - \frac{1}{2}; 3)]$

性质2.1.1.  $\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s; 3) = N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) \Gamma_{\mu_\zeta \eta_\zeta \dots}^{l_\zeta}(s - \frac{1}{2}; 3), \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}(s; 3) = N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) \Gamma_{l_\zeta}^{\mu_\zeta \eta_\zeta \dots}(s - \frac{1}{2}; 3)$

推论2.1.2.  $\Gamma(s; 3) = [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)]N(s; 3), \bar{\Gamma}(s; 3) = \bar{N}(s; 3)[I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)],$

性质2.1.2.  $\Gamma(s; 3) = [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)][I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)]\Gamma(s; 3)$

推论2.1.3.  $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s) = N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 1), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s) = N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 1)$

自我评述：以上表明  $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$  是常数不变张量  $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s)$  的推广。

## 2.2 常数矩阵 $N_{\lambda_\zeta}(s; 3), N^{\lambda_\zeta}(s; 3); \bar{N}_{\lambda_\zeta}(s; 3), \bar{N}^{\lambda_\zeta}(s; 3); N(s; 3), \bar{N}(s; 3)$

定义2.2.1.

$$\begin{cases} N_{\lambda_\zeta}(s; 3) \prec N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3), N^{\lambda_\zeta}(s; 3) \prec N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) |_{I_{C_{2s+3}^{2s}} \times I_{C_{2s-1+3}^{2s-1}}} \\ \bar{N}_{\lambda_\zeta}(s; 3) := N_{\lambda_\zeta}^+(s; 3) \succ N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3), \bar{N}^{\lambda_\zeta}(s; 3) := N^{+A_\zeta}(s; 3) \succ N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) |_{I_{C_{2s-1+3}^{2s-1}} \times I_{C_{2s+3}^{2s}}} \\ N(s; 3) \prec N_{\lambda_\zeta \otimes l_\zeta}^{k_\zeta}(s; 3) |_{(3+1)I_{C_{2s-1+3}^{2s-1}} \times I_{C_{2s+3}^{2s}}}, \bar{N}(s; 3) = N^+(s; 3) \prec N_{k_\zeta}^{\lambda_\zeta \otimes l_\zeta}(s; 3) |_{I_{C_{2s+3}^{2s}} \times (3+1)I_{C_{2s-1+3}^{2s-1}}} \end{cases}$$

## 2.3 常数不变张量 $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ 的基本性质

相等性：

性质2.3.1.

$$\begin{cases} N_{\lambda_\zeta l_\zeta}^{k_\zeta'}(s; 3) \simeq N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) \simeq N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) \simeq N_{k_\zeta'}^{\lambda_\zeta l_\zeta'}(s; 3) \\ [N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3)]^* \simeq N_{\lambda_\zeta l_\zeta}^{k_\zeta'}(s; 3), [N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)]^* \simeq N_{k_\zeta'}^{\lambda_\zeta l_\zeta'}(s; 3) \end{cases}$$

推论2.3.1.

$$\begin{cases} N_{\lambda_\zeta}(s; 3) \simeq N^{\lambda_\zeta}(s; 3) \simeq N_{\lambda_\zeta}(s; 3) \simeq N^{\lambda_\zeta'}(s; 3); \bar{N}_{\lambda_\zeta}(s; 3) \simeq \bar{N}^{\lambda_\zeta}(s; 3) \simeq \bar{N}_{\lambda_\zeta}(s; 3) \simeq \bar{N}^{\lambda_\zeta'}(s; 3) \\ N_{\lambda_\zeta}(s; 3) = N_{\lambda_\zeta}^*(s; 3), \bar{N}_{\lambda_\zeta}(s; 3) = \bar{N}_{\lambda_\zeta}^*(s; 3); N(s; 3) = N^*(s; 3), \bar{N}(s; 3) = \bar{N}^*(s; 3) \end{cases}$$

## 2.4 常数不变张量 $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3), N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ 的正交性质

正交性：

引理2.4.1.  $\sum_{k=0}^{2s-1} C_{3+k}^3 = C_{3+2s}^{3+1}$

引理2.4.2.  $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) N_{k_\zeta'}^{\lambda_\zeta l_\zeta'}(s; 3) = \Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}^{k_\zeta}(s; 3) \Gamma_{k_\zeta'}^{\lambda_\zeta \mu_\zeta \eta_\zeta \dots}(s; 3)$

定理2.4.1.

$$\begin{cases} N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) N_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) = \delta_{m_\zeta}^{k_\zeta} [\Leftrightarrow] N^{\lambda_\zeta}(s; 3) \bar{N}_{\lambda_\zeta}(s; 3) = I_{C_{2s+3}^{2s}} [\Leftrightarrow] \bar{N}(s; 3) N(s; 3) = I_{C_{2s+3}^{2s}} \\ N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) N_{k_\zeta}^{\lambda_\zeta m_\zeta}(s; 3) = (1 + \frac{i}{s}) \delta_{l_\zeta}^{m_\zeta} [\Leftrightarrow] \bar{N}_{\lambda_\zeta}(s; 3) N^{\lambda_\zeta}(s; 3) = (1 + \frac{i}{s}) I_{C_{2s-1+3}^{2s-1}} \\ N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) N_{k_\zeta}^{\lambda_\zeta m_\zeta}(s; 3) = \frac{1}{3+1} C_{2s+3}^{2s} \delta_{\lambda_\zeta}^{\mu_\zeta} [\Leftrightarrow] \text{tr}[\bar{N}_{\lambda_\zeta}(s; 3) N^{\mu_\zeta}(s; 3)] = \frac{1}{3+1} C_{2s+3}^{2s} \delta_{\lambda_\zeta}^{\mu_\zeta} \end{cases}$$

性质2.4.1.

$$\begin{cases} N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) N_{k_\zeta}^{\mu_\zeta m_\zeta}(s; 3) = \frac{1}{2s} [\delta_{\lambda_\zeta}^{\mu_\zeta} \delta_{l_\zeta}^{m_\zeta} + (2s-1) N_{\lambda_\zeta n_\zeta}^{m_\zeta}(s - \frac{1}{2}; 3) N_{l_\zeta}^{\mu_\zeta n_\zeta}(s - \frac{1}{2}; 3)] \\ \bar{N}_{\lambda_\zeta}(s; 3) N^{\mu_\zeta}(s; 3) = \frac{1}{2s} [\delta_{\lambda_\zeta}^{\mu_\zeta} I_{C_{2s-1+3}^{2s-1}} + (2s-1) N^{\mu_\zeta}(s - \frac{1}{2}; 3) \bar{N}_{\lambda_\zeta}(s - \frac{1}{2}; 3)] \end{cases}$$

## 2.5 常数不变张量 $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3)$ , $N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ 的升降指标

升降指标:

性质2.5.1.

$$\begin{cases} N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) = \varepsilon^{k_\zeta m_\zeta}(s; 3) \varepsilon_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; 3) N_{m_\zeta}^{\mu_\zeta n_\zeta}(s; 3) \\ N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) = \varepsilon_{k_\zeta m_\zeta}(s; 3) \varepsilon^{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; 3) N_{\mu_\zeta n_\zeta}^{m_\zeta}(s; 3) \end{cases}$$

推论2.5.1.

$$\begin{cases} N_{\lambda_\zeta}(s; 3) \varepsilon(s - \frac{1}{2}; 3) = \varepsilon_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) \varepsilon(s; 3) N^{\mu_\zeta}(s; 3), \varepsilon(s - \frac{1}{2}; 3) \bar{N}_{\lambda_\zeta}(s; 3) = \bar{N}^{\mu_\zeta}(s; 3) \varepsilon_{\mu_\zeta \lambda_\zeta}(\frac{1}{2}; 3) \varepsilon(s; 3) \\ N^{\lambda_\zeta}(s; 3) \varepsilon(s - \frac{1}{2}; 3) = \varepsilon^{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) \varepsilon(s; 3) N_{\mu_\zeta}(s; 3), \varepsilon(s - \frac{1}{2}; 3) \bar{N}^{\lambda_\zeta}(s; 3) = \bar{N}_{\mu_\zeta}(s; 3) \varepsilon^{\mu_\zeta \lambda_\zeta}(\frac{1}{2}; 3) \varepsilon(s; 3) \\ N(s; 3) \varepsilon(s; 3) = [\varepsilon(\frac{1}{2}; 3) \otimes \varepsilon(s - \frac{1}{2}; 3)] N(s; 3), \varepsilon(s; 3) \bar{N}(s; 3) = \bar{N}(s; 3) [\varepsilon(\frac{1}{2}; 3) \otimes \varepsilon(s - \frac{1}{2}; 3)] \end{cases}$$

Penrose标准升降规则:

性质2.5.2.

$$\begin{cases} N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) = (-1)^{2s} [\zeta^{2s} \varepsilon^{k_\zeta m_\zeta}(s; 3)] [-\zeta \varepsilon_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)] [(-\zeta)^{2s-1} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; 3)] N_{m_\zeta}^{\mu_\zeta n_\zeta}(s; 3) \\ N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) = (-1)^{2s} [(-\zeta)^{2s} \varepsilon_{k_\zeta m_\zeta}(s; 3)] [\zeta \varepsilon^{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)] [\zeta^{2s-1} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; 3)] N_{\mu_\zeta n_\zeta}^{m_\zeta}(s; 3) \end{cases}$$

## 2.6 常数不变张量 $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3)$ , $N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ 的自旋矩阵变换

性质2.6.1.

$$\begin{cases} N_{k_\zeta}^{\lambda_\zeta m_\zeta}(s; 3) \gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) N_{\mu_\zeta m_\zeta}^{l_\zeta}(s; 3) = \frac{1}{2s} \gamma^a_{k_\zeta l_\zeta}(s; 3) [\Leftrightarrow] N^{\lambda_\zeta}(s; 3) \gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) \bar{N}_{\mu_\zeta}(s; 3) = \frac{1}{2s} \gamma^a(s; 3) \\ [\Leftrightarrow] \bar{N}(s; 3) \gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} N(s; 3) = \frac{1}{2s} \gamma^a(s; 3) \\ N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) \gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) N_{\mu_\zeta m_\zeta}^{k_\zeta}(s; 3) = \frac{1}{2s} \gamma^a_{m_\zeta l_\zeta}(s - \frac{1}{2}; 3) [\Leftrightarrow] \bar{N}_{\mu_\zeta}(s; 3) \gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) N^{\lambda_\zeta}(s; 3) = \frac{1}{2s} \gamma^a(s - \frac{1}{2}; 3) \end{cases}$$

性质2.6.2.

$$\begin{cases} N_{k_\zeta}^{\lambda_\zeta m_\zeta}(s; 3) S_{ab \lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) N_{\mu_\zeta m_\zeta}^{l_\zeta}(s; 3) = \frac{1}{2s} S_{ab k_\zeta l_\zeta}(s; 3) [\Leftrightarrow] N^{\lambda_\zeta}(s; 3) S_{ab \lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) \bar{N}_{\mu_\zeta}(s; 3) = \frac{1}{2s} S_{ab}(s; 3) \\ [\Leftrightarrow] \bar{N}(s; 3) S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} N(s; 3) = \frac{1}{2s} S_{ab}(s; 3) \\ N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) S_{ab \lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) N_{\mu_\zeta m_\zeta}^{k_\zeta}(s; 3) = \frac{1}{2s} S_{ab m_\zeta l_\zeta}(s - \frac{1}{2}; 3) [\Leftrightarrow] \bar{N}_{\mu_\zeta}(s; 3) S_{ab \lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) N^{\lambda_\zeta}(s; 3) = \frac{1}{2s} S_{ab}(s - \frac{1}{2}; 3) \end{cases}$$

## 2.7 常数不变张量 $N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3)$ , $N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ 的置换性质

定理2.7.1.

$$\begin{cases} \gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) N_{\mu_\zeta l_\zeta}^{k_\zeta}(s; 3) + \gamma^a_{l_\zeta m_\zeta}(s - \frac{1}{2}; 3) N_{\lambda_\zeta m_\zeta}^{k_\zeta}(s; 3) = N_{\lambda_\zeta l_\zeta}^{j_\zeta}(s; 3) \gamma^a_{j_\zeta k_\zeta}(s; 3) \\ N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) \gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) + N_{k_\zeta}^{\mu_\zeta m_\zeta}(s; 3) \gamma^a_{m_\zeta l_\zeta}(s - \frac{1}{2}; 3) = \gamma^a_{k_\zeta j_\zeta}(s; 3) N_{j_\zeta}^{\mu_\zeta l_\zeta}(s; 3) \\ \gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) \bar{N}_{\mu_\zeta}(s; 3) + \gamma^a(s - \frac{1}{2}; 3) \bar{N}_{\lambda_\zeta}(s; 3) = \bar{N}_{\lambda_\zeta}(s; 3) \gamma^a(s; 3) \\ N^{\lambda_\zeta}(s; 3) \gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) + N^{\mu_\zeta}(s; 3) \gamma^a(s - \frac{1}{2}; 3) = \gamma^a(s; 3) N^{\mu_\zeta}(s; 3) \\ [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)] N(s; 3) = N(s; 3) \gamma^a(s; 3) \\ \bar{N}(s; 3) [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)] = \gamma^a(s; 3) \bar{N}(s; 3) \end{cases}$$

定理2.7.2.

$$\begin{cases} S_{ab \lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) N_{\mu_\zeta l_\zeta}^{k_\zeta}(s; 3) + S_{ab l_\zeta m_\zeta}(s - \frac{1}{2}; 3) N_{\lambda_\zeta m_\zeta}^{k_\zeta}(s; 3) = N_{\lambda_\zeta l_\zeta}^{j_\zeta}(s; 3) S_{ab j_\zeta k_\zeta}(s; 3) \\ N_{k_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) S_{ab \lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) + N_{k_\zeta}^{\mu_\zeta m_\zeta}(s; 3) S_{ab m_\zeta l_\zeta}(s - \frac{1}{2}; 3) = S_{ab k_\zeta j_\zeta}(s; 3) N_{j_\zeta}^{\mu_\zeta l_\zeta}(s; 3) \\ S_{ab \lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) \bar{N}_{\mu_\zeta}(s; 3) + S_{ab}(s - \frac{1}{2}; 3) \bar{N}_{\lambda_\zeta}(s; 3) = \bar{N}_{\lambda_\zeta}(s; 3) S_{ab}(s; 3) \\ N^{\lambda_\zeta}(s; 3) S_{ab \lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) + N^{\mu_\zeta}(s; 3) S_{ab}(s - \frac{1}{2}; 3) = S_{ab}(s; 3) N^{\mu_\zeta}(s; 3) \\ [S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)] N(s; 3) = N(s; 3) S_{ab}(s; 3) \\ \bar{N}(s; 3) [S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)] = S_{ab}(s; 3) \bar{N}(s; 3) \end{cases}$$

## 2.8 常数不变张量 $N_{A_{\zeta_1} \cdots A_{\zeta_n} l_{\zeta}}^{k_{\zeta}}(s; \mathfrak{Z})$ , $N_{k_{\zeta}}^{A_{\zeta_1} \cdots A_{\zeta_n} l_{\zeta}}(s; \mathfrak{Z})$ 的性质

$$\text{定义2.8.1. } \begin{cases} N_{A_{\zeta_1} \cdots A_{\zeta_n} l_{\zeta}}^{k_{\zeta}}(s; \mathfrak{Z}) := \Gamma_{A_{\zeta_1} \cdots A_{\zeta_{2s}}}^{k_{\zeta}}(s; \mathfrak{Z}) \Gamma_{l_{\zeta}}^{A_{\zeta_{n+1}} \cdots A_{\zeta_{2s}}}(s - \frac{n}{2}; \mathfrak{Z}) \\ N_{k_{\zeta}}^{A_{\zeta_1} \cdots A_{\zeta_n} l_{\zeta}}(s; \mathfrak{Z}) := \Gamma_{k_{\zeta}}^{A_{\zeta_1} \cdots A_{\zeta_{2s}}}(s; \mathfrak{Z}) \Gamma_{A_{\zeta_{n+1}} \cdots A_{\zeta_{2s}}}^{l_{\zeta}}(s - \frac{n}{2}; \mathfrak{Z}) \end{cases}$$

相等性:

$$\text{性质2.8.1. } N_{A'_{\zeta_1} \cdots A'_{\zeta_n} l'_{\zeta}}^{k'_{\zeta}}(s; \mathfrak{Z}) \simeq N_{A_{\zeta_1} \cdots A_{\zeta_n} l_{\zeta}}^{k_{\zeta}}(s; \mathfrak{Z}) \simeq N_{k_{\zeta}}^{A_{\zeta_1} \cdots A_{\zeta_n} l_{\zeta}}(s; \mathfrak{Z}) \simeq N_{k'_{\zeta}}^{A'_{\zeta_1} \cdots A'_{\zeta_n} l'_{\zeta}}(s; \mathfrak{Z})$$

$$\text{性质2.8.2. } [N_{A_{\zeta_1} \cdots A_{\zeta_n} l_{\zeta}}^{k_{\zeta}}(s; \mathfrak{Z})]^* \simeq N_{A'_{\zeta_1} \cdots A'_{\zeta_n} l'_{\zeta}}^{k'_{\zeta}}(s; \mathfrak{Z}), [N_{k_{\zeta}}^{A_{\zeta_1} \cdots A_{\zeta_n} l_{\zeta}}(s; \mathfrak{Z})]^* \simeq N_{k'_{\zeta}}^{A'_{\zeta_1} \cdots A'_{\zeta_n} l'_{\zeta}}(s; \mathfrak{Z})$$

展开性:

性质2.8.3.

$$\begin{cases} N_{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_n} l_{\zeta}}^{k_{\zeta}}(s; \mathfrak{Z}) = N_{A_{\zeta_1} l_{\zeta_1}}^{k_{\zeta}}(s; \mathfrak{Z}) N_{A_{\zeta_2} l_{\zeta_2}}^{l_{\zeta_1}}(s - \frac{1}{2}; \mathfrak{Z}) \cdots N_{A_{\zeta_n} l_{\zeta_n}}^{l_{\zeta_{n-1}}}(s - \frac{n-1}{2}; \mathfrak{Z}) \\ N_{k_{\zeta}}^{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_n} l_{\zeta}}(s; \mathfrak{Z}) = N_{k_{\zeta}}^{A_{\zeta_1} l_{\zeta_1}}(s; \mathfrak{Z}) N_{l_{\zeta_1}}^{A_{\zeta_2} l_{\zeta_2}}(s - \frac{1}{2}; \mathfrak{Z}) \cdots N_{l_{\zeta_{n-1}}}^{A_{\zeta_n} l_{\zeta_n}}(s - \frac{n-1}{2}; \mathfrak{Z}) \end{cases}$$

性质2.8.4.

$$\begin{cases} \Gamma_{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_{2s}}}^{k_{\zeta}}(s; \mathfrak{Z}) = N_{A_{\zeta_1} l_{\zeta_1}}^{k_{\zeta}}(s; \mathfrak{Z}) N_{A_{\zeta_2} l_{\zeta_2}}^{l_{\zeta_1}}(s - \frac{1}{2}; \mathfrak{Z}) \cdots N_{A_{\zeta_{2s-1}} l_{\zeta_{2s}}}^{l_{\zeta_{2s-2}}}(s - \frac{1}{2}; \mathfrak{Z}) \\ \Gamma_{k_{\zeta}}^{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_{2s}}}(s; \mathfrak{Z}) = N_{k_{\zeta}}^{A_{\zeta_1} l_{\zeta_1}}(s; \mathfrak{Z}) N_{l_{\zeta_1}}^{A_{\zeta_2} l_{\zeta_2}}(s - \frac{1}{2}; \mathfrak{Z}) \cdots N_{l_{\zeta_{2s-1}}}^{A_{\zeta_{2s}} l_{\zeta_{2s}}}(s - \frac{1}{2}; \mathfrak{Z}) \\ \Gamma_{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_{2s}}}^{k_{\zeta}}(s; \mathfrak{Z}) \succ \Gamma_{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_{2s}}}(s; \mathfrak{Z}) = N_{A_{\zeta_1}}(s; \mathfrak{Z}) N_{A_{\zeta_2}}(s - \frac{1}{2}; \mathfrak{Z}) \cdots N_{A_{\zeta_{2s}}}(s - \frac{1}{2}; \mathfrak{Z}) \\ \Gamma_{k_{\zeta}}^{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_{2s}}}(s; \mathfrak{Z}) \succ \Gamma_{A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_{2s}}}(s; \mathfrak{Z}) = N^{A_{\zeta_1}}(s; \mathfrak{Z}) N^{A_{\zeta_2}}(s - \frac{1}{2}; \mathfrak{Z}) \cdots N^{A_{\zeta_{2s}}}(s - \frac{1}{2}; \mathfrak{Z}) \\ \bar{\Gamma}(s; \mathfrak{Z}) = \bar{N}(s; \mathfrak{Z}) [I_4 \otimes \bar{N}(s - \frac{1}{2}; \mathfrak{Z})] \cdots [I_{4^{2s-2}} \otimes \bar{N}(1)] [I_{4^{2s-1}} \otimes \bar{N}(\frac{1}{2}; \mathfrak{Z})] \\ \Gamma(s; \mathfrak{Z}) = [I_{4^{2s-1}} \otimes N(\frac{1}{2}; \mathfrak{Z})] [I_{4^{2s-2}} \otimes N(1)] \cdots [I_4 \otimes N(s - \frac{1}{2}; \mathfrak{Z})] N(s; \mathfrak{Z}) \end{cases}$$

## 2.9 推论1: 常数矩阵 $N(s; \mathfrak{Z})$ , $\bar{N}(s; \mathfrak{Z})$ 的几个恒等式

性质2.9.1.

$$\begin{cases} \bar{N}(s; \mathfrak{Z}) [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; \mathfrak{Z})] N(s; \mathfrak{Z}) = \gamma^a(s; \mathfrak{Z}) \\ N(s; \mathfrak{Z}) \gamma^a(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}) = [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; \mathfrak{Z})] N(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}) \\ N(s; \mathfrak{Z}) \gamma^a(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}) = N(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}) \{ \vartheta \cdot [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; \mathfrak{Z})] \}^n \\ [N(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}), \gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; \mathfrak{Z})] = 0 \end{cases}$$

性质2.9.2.

$$\begin{cases} \bar{N}(s; \mathfrak{Z}) [S_{ab}(\frac{1}{2}; \mathfrak{Z}) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; \mathfrak{Z})] N(s; \mathfrak{Z}) = S_{ab}(s; \mathfrak{Z}) \\ N(s; \mathfrak{Z}) S_{ab}(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}) = [S_{ab}(\frac{1}{2}; \mathfrak{Z}) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; \mathfrak{Z})] N(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}) \\ N(s; \mathfrak{Z}) S_{ab}(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}) = N(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}) [S_{ab}(\frac{1}{2}; \mathfrak{Z}) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; \mathfrak{Z})] \\ [N(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}), S_{ab}(\frac{1}{2}; \mathfrak{Z}) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; \mathfrak{Z})] = 0 \end{cases}$$

性质2.9.3.

$$\begin{cases} \bar{N}(s; \mathfrak{Z}) \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}; \mathfrak{Z}) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; \mathfrak{Z})] \}^n N(s; \mathfrak{Z}) = [\vartheta^{ab} S_{ab}(s; \mathfrak{Z})]^n \\ N(s; \mathfrak{Z}) [\vartheta^{ab} S_{ab}(s; \mathfrak{Z})]^n \bar{N}(s; \mathfrak{Z}) = \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}; \mathfrak{Z}) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; \mathfrak{Z})] \}^n N(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}) \\ N(s; \mathfrak{Z}) [\vartheta^{ab} S_{ab}(s; \mathfrak{Z})]^n \bar{N}(s; \mathfrak{Z}) = N(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}) \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}; \mathfrak{Z}) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; \mathfrak{Z})] \}^n \\ [N(s; \mathfrak{Z}) \bar{N}(s; \mathfrak{Z}), \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}; \mathfrak{Z}) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; \mathfrak{Z})] \}^n] = 0 \end{cases}$$

推论2.9.1.

$$\begin{cases} \bar{N}(s; 3) e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s-\frac{1}{2}; 3)]} N(s; 3) = e^{\frac{i}{2}\vartheta^{ab} S_{ab}(s; 3)} \\ N(s; 3) e^{\frac{i}{2}\vartheta^{ab} S_{ab}(s; 3)} \bar{N}(s; 3) = e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s-\frac{1}{2}; 3)]} N(s; 3) \bar{N}(s; 3) \\ N(s; 3) e^{\frac{i}{2}\vartheta^{ab} S_{ab}(s; 3)} \bar{N}(s; 3) = N(s; 3) \bar{N}(s; 3) e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s-\frac{1}{2}; 3)]} \\ [N(s; 3) \bar{N}(s; 3), e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s-\frac{1}{2}; 3)]}] = 0 \end{cases}$$

## 2.10 推论2: 常数矩阵 $N(s; 3)$ , $\bar{N}(s; 3)$ 的另外几个恒等式

推论2.10.1.

$$\begin{cases} \bar{N}(s; 3) \gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} N(s; 3) = \frac{1}{2s} \gamma^a(s; 3) \\ \bar{N}(s; 3) I_4 \otimes \gamma^a(s - \frac{1}{2}; 3) N(s; 3) = (1 - \frac{1}{2s}) \gamma^a(s; 3) \\ N^{\lambda_\zeta}(s; 3) \gamma^a(s - \frac{1}{2}; 3) \bar{N}_{\lambda_\zeta}(s; 3) = (1 - \frac{1}{2s}) \gamma^a(s; 3) \\ \bar{N}_{\lambda_\zeta}(s; 3) \gamma^a(s; 3) N^{\lambda_\zeta}(s; 3) = (1 + \frac{3+1}{2s}) \gamma^a(s - \frac{1}{2}; 3) \end{cases}$$

推论2.10.2.

$$\begin{cases} \bar{N}(s; 3) S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} N(s; 3) = \frac{1}{2s} S_{ab}(s; 3) \\ \bar{N}(s; 3) I_4 \otimes S_{ab}(s - \frac{1}{2}; 3) N(s; 3) = (1 - \frac{1}{2s}) S_{ab}(s; 3) \\ N^{\lambda_\zeta}(s; 3) S_{ab}(s - \frac{1}{2}; 3) \bar{N}_{\lambda_\zeta}(s; 3) = (1 - \frac{1}{2s}) S_{ab}(s; 3) \\ \bar{N}_{\lambda_\zeta}(s; 3) S_{ab}(s; 3) N^{\lambda_\zeta}(s; 3) = (1 + \frac{3+1}{2s}) S_{ab}(s - \frac{1}{2}; 3) \end{cases}$$

推论2.10.3.

$$\begin{cases} \bar{N}(1) [\gamma^a \otimes I_2 + I_4 \otimes \gamma^a] N(1) = \gamma^a(1) \\ \bar{N}(\frac{3}{2}) \{ \gamma^a \otimes I_3 + I_4 \otimes \{ \bar{N}(1) [\gamma^a \otimes I_2 + I_4 \otimes \gamma^a] N(1) \} \} N(\frac{3}{2}) = \gamma^a(\frac{3}{2}) \\ \bar{N}(s; 3) \cdot \bar{N}(\frac{3}{2}) \{ \gamma^a \otimes I_3 + I_4 \otimes \{ \bar{N}(1) [\gamma^a \otimes I_2 + I_4 \otimes \gamma^a] N(1) \} \} N(\frac{3}{2}) \cdot N(s; 3) = \gamma^a(s; 3) \end{cases}$$

## 2.11 矩阵 $N(s; 3)$ , $\bar{N}(s; 3)$ 的常数不变张量性质

定理2.11.1.  $N(s; 3) = e^{(i\Omega^a + \zeta\epsilon) \cdot \gamma^a} \otimes e^{(i\Omega^a + \zeta\epsilon) \cdot \gamma^a(s - \frac{1}{2}; 3)} N(s; 3) e^{-(i\Omega^a + \zeta\epsilon) \cdot \gamma^a(s; 3)}$

定理2.11.2.  $\bar{N}(s; 3) = e^{(i\Omega^a + \zeta\epsilon) \cdot \gamma^a(s; 3)} \bar{N}(s; 3) e^{-(i\Omega^a + \zeta\epsilon) \cdot \gamma^a} \otimes e^{-(i\Omega^a + \zeta\epsilon) \cdot \gamma^a(s - \frac{1}{2}; 3)}$

## 3 Dirac型完美常数不变张量 $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ , $X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)$

### 3.1 完美常数不变张量 $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ , $X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)$

定义3.1.1.  $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) := \frac{\sqrt{2s-1}}{\sqrt{2s-1+3}} \varepsilon^{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) N_{\mu_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; 3)$ ,  $X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3) := \frac{\sqrt{2s-1}}{\sqrt{2s-1+3}} \varepsilon_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) N_{l_\zeta}^{\mu_\zeta m_\zeta}(s - \frac{1}{2}; 3)$

性质3.1.1.  $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) \simeq X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)$

### 3.2 常数矩阵 $X(s; 3)$ , $\bar{X}(s; 3)$

定义3.2.1.

$$\begin{cases} X^{\lambda_\zeta}(s; 3) \prec X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3), X_{\lambda_\zeta}(s; 3) \prec X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3) \\ \bar{X}_{\lambda_\zeta}(s; 3) \prec X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3), \bar{X}^{\lambda_\zeta}(s; 3) \prec X^{\lambda_\zeta l_\zeta}_{m_\zeta}(s; 3) \\ X(s; 3) \prec X_{\lambda_\zeta \otimes l_\zeta}^{m_\zeta}(s; 3), \bar{X}(s; 3) \prec X_{m_\zeta}^{\lambda_\zeta \otimes l_\zeta}(s; 3) = X^+(s; 3) \end{cases}$$

### 3.3 常数不变张量 $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ , $X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)$ 的升降指标

性质3.3.1.

$$\begin{cases} X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3) = \varepsilon^{\lambda_\zeta \mu_\zeta} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; 3) \varepsilon_{m_\zeta r_\zeta}(s - 1; 3) X_{\mu_\zeta n_\zeta}^{r_\zeta}(s - \frac{1}{2}; 3) \\ X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3) = \varepsilon_{\lambda_\zeta \mu_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; 3) \varepsilon^{m_\zeta r_\zeta}(s - 1; 3) X_{r_\zeta}^{\mu_\zeta n_\zeta}(s - \frac{1}{2}; 3) \end{cases}$$

### 3.4 常数不变张量 $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ , $X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)$ 的正交性

性质3.4.1.  $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3) = \delta_{m_\zeta}^{n_\zeta} [\Leftrightarrow] X^{\lambda_\zeta}(s; 3)\bar{X}_{\lambda_\zeta}(s; 3) = I_{C_{2s-2+3}^{2s-2}} [\Leftrightarrow] \bar{X}(s; 3)X(s; 3) = I_{C_{2s-2+3}^{2s-2}}$

性质3.4.2.  $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) = 0$

$[\Leftrightarrow] X^{\lambda_\zeta}(s; 3)\bar{N}_{\lambda_\zeta}(s; 3) = 0, N_{\lambda_\zeta}(s; 3)\bar{X}^{\lambda_\zeta}(s; 3) = 0 [\Leftrightarrow] \bar{X}(s; 3)N(s; 3) = 0, \bar{N}(s; 3)X(s; 3) = 0$

性质3.4.3.  $X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)X_{m_\zeta}^{\lambda_\zeta k_\zeta}(s; 3) = \frac{2s-1}{2s-1+3}\delta_{l_\zeta}^{k_\zeta} [\Leftrightarrow] \bar{X}_{\lambda_\zeta}(s; 3)X^{\lambda_\zeta}(s; 3) = \frac{2s-1}{2s-1+3}I_{C_{2s-1+3}^{2s-1}}$

性质3.4.4.  $X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)X_{m_\zeta}^{\mu_\zeta l_\zeta}(s; 3) = \frac{1}{3+1}C_{2s-2+3}^{2s-2}\delta_{\lambda_\zeta}^{\mu_\zeta} [\Leftrightarrow] tr[\bar{X}_{\lambda_\zeta}(s; 3)X^{\mu_\zeta}(s; 3)] = \frac{1}{3+1}C_{2s-2+3}^{2s-2}\delta_{\lambda_\zeta}^{\mu_\zeta}$

推论3.4.1.  $\bar{N}(s; 3)N(s; 3) = I_{C_{2s+3}^{2s}}, \bar{X}(s; 3)X(s; 3) = I_{C_{2s-2+3}^{2s-2}}, \bar{N}(s; 3)X(s; 3) = 0, \bar{X}(s; 3)N(s; 3) = 0$

### 3.5 常数不变张量 $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ , $X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)$ 的自旋变换

推论3.5.1.  $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)X_{\mu_\zeta l_\zeta}^{n_\zeta}(s; 3) = -\frac{1}{2s-1+3}\gamma^a_{m_\zeta n_\zeta}(s-1; 3)$

$[\Leftrightarrow] X^{\lambda_\zeta}(s; 3)\sigma_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)\bar{X}_{\mu_\zeta}(s; 3) = -\frac{1}{2s-1+3}\gamma^a(s-1; 3)$

$[\Leftrightarrow] \bar{X}(s; 3)\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} X(s; 3) = -\frac{1}{2s-1+3}\gamma^a(s-1; 3)$

推论3.5.2.  $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)X_{\mu_\zeta k_\zeta}^{m_\zeta}(s; 3) = -\frac{1}{2s-1+3}\gamma^a_{k_\zeta l_\zeta}(s-\frac{1}{2}; 3)$

$[\Leftrightarrow] \bar{X}^{\lambda_\zeta}(s; 3)\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)X_{\mu_\zeta}(s; 3) = -\frac{1}{2s-1+3}\gamma^a(s-\frac{1}{2}; 3)$

推论3.5.3.  $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)S_{ab\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)X_{\mu_\zeta l_\zeta}^{n_\zeta}(s; 3) = -\frac{1}{2s-1+3}S_{abm_\zeta n_\zeta}(s-1; 3)$

$[\Leftrightarrow] X^{\lambda_\zeta}(s; 3)S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-2+3}^{2s-2}} \bar{X}_{\lambda_\zeta}(s; 3) = -\frac{1}{2s-1+3}S_{ab}(s-1; 3)$

$[\Leftrightarrow] \bar{X}(s; 3)S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} X(s; 3) = -\frac{1}{2s-1+3}S_{ab}(s-1; 3)$

推论3.5.4.  $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)S_{ab\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)X_{\mu_\zeta k_\zeta}^{m_\zeta}(s; 3) = -\frac{1}{2s}S_{abk_\zeta l_\zeta}(s-\frac{1}{2}; 3)$

$[\Leftrightarrow] \bar{X}^{\lambda_\zeta}(s; 3)S_{ab\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)X_{\mu_\zeta}(s; 3) = -\frac{1}{2s-1+3}S_{ab}(s-\frac{1}{2}; 3)$

### 3.6 常数不变张量 $X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)$ , $X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)$ 的置换性质

定理3.6.1.

$$\left\{ \begin{aligned} X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)[\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)\delta_{l_\zeta}^{k_\zeta} + \delta_{\lambda_\zeta}^{\mu_\zeta}\gamma^a_{l_\zeta k_\zeta}(s-\frac{1}{2}; 3)] &= \gamma^a_{m_\zeta n_\zeta}(s-1; 3)X_{n_\zeta}^{\mu_\zeta k_\zeta}(s; 3) \\ [\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)\delta_{l_\zeta}^{k_\zeta} + \delta_{\lambda_\zeta}^{\mu_\zeta}\gamma^a_{l_\zeta k_\zeta}(s-\frac{1}{2}; 3)]X_{\mu_\zeta k_\zeta}^{n_\zeta}(s; 3) &= X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)\gamma^a_{m_\zeta n_\zeta}(s-1; 3) \end{aligned} \right.$$

$$\left\{ \begin{aligned} X^{\lambda_\zeta}(s; 3)[\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) + \delta_{\lambda_\zeta}^{\mu_\zeta}\gamma^a(s-\frac{1}{2}; 3)] &= \gamma^a(s-1; 3)X^{\mu_\zeta}(s; 3) \\ [\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) + \delta_{\lambda_\zeta}^{\mu_\zeta}\gamma^a(s-\frac{1}{2}; 3)]\bar{X}_{\mu_\zeta}(s; 3) &= \bar{X}_{\lambda_\zeta}(s; 3)\gamma^a(s-1; 3) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \bar{X}(s; 3)[\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s-\frac{1}{2}; 3)] &= \gamma^a(s-1; 3)\bar{X}(s; 3) \\ [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s-\frac{1}{2}; 3)]X(s; 3) &= X(s; 3)\gamma^a(s-1; 3) \end{aligned} \right.$$

定理3.6.2.

$$\left\{ \begin{aligned} X_{m_\zeta}^{\lambda_\zeta l_\zeta}(s; 3)[S_{ab\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)\delta_{l_\zeta}^{k_\zeta} + \delta_{\lambda_\zeta}^{\mu_\zeta}S_{abl_\zeta}^{k_\zeta}(s-\frac{1}{2}; 3)] &= S_{abm_\zeta n_\zeta}(s-1; 3)X_{n_\zeta}^{\mu_\zeta k_\zeta}(s; 3) \\ [S_{ab\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)\delta_{l_\zeta}^{k_\zeta} + \delta_{\lambda_\zeta}^{\mu_\zeta}S_{abl_\zeta}^{k_\zeta}(s-\frac{1}{2}; 3)]X_{\mu_\zeta k_\zeta}^{n_\zeta}(s; 3) &= X_{\lambda_\zeta l_\zeta}^{m_\zeta}(s; 3)S_{abm_\zeta n_\zeta}(s-1; 3) \end{aligned} \right.$$

$$\left\{ \begin{aligned} X^{\lambda_\zeta}(s; 3)[S_{ab\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) + \delta_{\lambda_\zeta}^{\mu_\zeta}S_{ab}(s-\frac{1}{2}; 3)] &= S_{ab}(s-1; 3)X^{\mu_\zeta}(s; 3) \\ [S_{ab\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) + \delta_{\lambda_\zeta}^{\mu_\zeta}S_{ab}(s-\frac{1}{2}; 3)]\bar{X}_{\mu_\zeta}(s; 3) &= \bar{X}_{\lambda_\zeta}(s; 3)S_{ab}(s-1; 3) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \bar{X}(s; 3)[S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s-\frac{1}{2}; 3)] &= S_{ab}(s-1; 3)\bar{X}(s; 3) \\ [S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s-\frac{1}{2}; 3)]X(s; 3) &= X(s; 3)S_{ab}(s-1; 3) \end{aligned} \right.$$

推论3.6.1.

$$\left\{ \begin{aligned} N_{\eta_\zeta m_\zeta}^{\lambda_\zeta l_\zeta}(s-\frac{1}{2}; 3)\varepsilon^{\eta_\zeta \lambda_\zeta}(\frac{1}{2}; 3)[\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)\delta_{l_\zeta}^{k_\zeta} + \delta_{\lambda_\zeta}^{\mu_\zeta}\gamma^a_{l_\zeta k_\zeta}(s-\frac{1}{2}; 3)] &= \gamma^a_{m_\zeta n_\zeta}(s-1; 3)N_{\xi_\zeta n_\zeta}^{k_\zeta}(s-\frac{1}{2}; 3)\varepsilon^{\xi_\zeta \mu_\zeta}(\frac{1}{2}; 3) \\ [\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3)\delta_{l_\zeta}^{k_\zeta} + \delta_{\lambda_\zeta}^{\mu_\zeta}\gamma^a_{l_\zeta k_\zeta}(s-\frac{1}{2}; 3)]\varepsilon_{\mu_\zeta \eta_\zeta}(\frac{1}{2}; 3)N_{k_\zeta}^{\eta_\zeta n_\zeta}(s-\frac{1}{2}; 3) &= \varepsilon_{\lambda_\zeta \xi_\zeta}(\frac{1}{2}; 3)N_{l_\zeta}^{\xi_\zeta m_\zeta}(s-\frac{1}{2}; 3)\gamma^a_{m_\zeta n_\zeta}(s-1; 3) \end{aligned} \right.$$

$$\left\{ \begin{aligned} N_{\eta_\zeta}(s-\frac{1}{2}; 3)\varepsilon^{\eta_\zeta \lambda_\zeta}(\frac{1}{2}; 3)[\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) + \delta_{\lambda_\zeta}^{\mu_\zeta}\gamma^a(s-\frac{1}{2}; 3)] &= \gamma^a(s-1; 3)N_{\xi_\zeta}(s-\frac{1}{2}; 3)\varepsilon^{\xi_\zeta \mu_\zeta}(\frac{1}{2}; 3) \\ [\gamma^a_{\lambda_\zeta \mu_\zeta}(\frac{1}{2}; 3) + \delta_{\lambda_\zeta}^{\mu_\zeta}\gamma^a(s-\frac{1}{2}; 3)]\varepsilon_{\mu_\zeta \eta_\zeta}(\frac{1}{2}; 3)N^{\eta_\zeta}(s-\frac{1}{2}; 3) &= \varepsilon_{\lambda_\zeta \xi_\zeta}(\frac{1}{2}; 3)N^{\xi_\zeta}(s-\frac{1}{2}; 3)\gamma^a(s-1; 3) \end{aligned} \right.$$

## 推论3.6.2.

$$\begin{cases} N_{\eta_\zeta m_\zeta}^{\lambda_\zeta}(s - \frac{1}{2}; 3) \varepsilon^{\eta_\zeta \lambda_\zeta}(\frac{1}{2}; 3) [S_{ab\lambda_\zeta}^{\mu_\zeta}(\frac{1}{2}; 3) \delta_{l_\zeta}^{k_\zeta} + \delta_{\lambda_\zeta}^{\mu_\zeta} S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; 3)] = S_{abm_\zeta}^{n_\zeta}(s - 1; 3) N_{\xi_\zeta n_\zeta}^{k_\zeta}(s - \frac{1}{2}; 3) \varepsilon^{\xi_\zeta \mu_\zeta}(\frac{1}{2}; 3) \\ [S_{ab\lambda_\zeta}^{\mu_\zeta}(\frac{1}{2}; 3) \delta_{l_\zeta}^{k_\zeta} + \delta_{\lambda_\zeta}^{\mu_\zeta} S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; 3)] \varepsilon_{\mu_\zeta \eta_\zeta}(\frac{1}{2}; 3) N_{k_\zeta}^{\eta_\zeta n_\zeta}(s - \frac{1}{2}; 3) = \varepsilon_{\lambda_\zeta \xi_\zeta}(\frac{1}{2}; 3) N_{l_\zeta}^{\xi_\zeta m_\zeta}(s - \frac{1}{2}; 3) S_{abm_\zeta}^{n_\zeta}(s - 1; 3) \\ N_{\eta_\zeta}(s - \frac{1}{2}; 3) \varepsilon^{\eta_\zeta \lambda_\zeta}(\frac{1}{2}; 3) [S_{ab\lambda_\zeta}^{\mu_\zeta}(\frac{1}{2}; 3) + \delta_{\lambda_\zeta}^{\mu_\zeta} S_{ab}(s - \frac{1}{2}; 3)] = S_{ab}(s - 1; 3) N_{\xi_\zeta}(s - \frac{1}{2}; 3) \varepsilon^{\xi_\zeta \mu_\zeta}(\frac{1}{2}; 3) \\ [S_{ab\lambda_\zeta}^{\mu_\zeta}(\frac{1}{2}; 3) + \delta_{\lambda_\zeta}^{\mu_\zeta} S_{ab}(s - \frac{1}{2}; 3)] \varepsilon_{\mu_\zeta \eta_\zeta}(\frac{1}{2}; 3) N^{\eta_\zeta}(s - \frac{1}{2}; 3) = \varepsilon_{\lambda_\zeta \xi_\zeta}(\frac{1}{2}; 3) N^{\xi_\zeta}(s - \frac{1}{2}; 3) S_{ab}(s - 1; 3) \end{cases}$$

3.7 推论: 关于常数矩阵  $X(s; 3)$ ,  $\bar{X}(s; 3)$  的重要性质

## 推论3.7.1.

$$\begin{cases} \bar{X}(s; 3) [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)] X(s; 3) = \gamma^a(s - 1; 3) \\ X(s; 3) \gamma^a(s - 1; 3) \bar{X}(s; 3) = [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)] X(s; 3) \bar{X}(s; 3) \\ [X(s; 3) \bar{X}(s; 3), \gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)] = 0 \end{cases}$$

## 推论3.7.2.

$$\begin{cases} \bar{X}(s; 3) [S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)] X(s; 3) = S_{ab}(s - 1; 3) \\ X(s; 3) S_{ab}(s, s - 1; 3) \bar{X}(s; 3) = [S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)] X(s; 3) \bar{X}(s; 3) \\ [X(s; 3) \bar{X}(s; 3), S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)] = 0 \end{cases}$$

推论3.7.3.  $X^{\lambda_\zeta}(s; 3) \gamma^a(s - \frac{1}{2}; 3) \bar{X}_{\lambda_\zeta}(s; 3) = \frac{2s+3}{2s-1+3} \gamma^a(s - 1; 3)$

$$[\Leftrightarrow] \bar{X}(s; 3) I_4 \otimes \gamma^a(s - \frac{1}{2}; 3) X(s; 3) = \frac{2s+3}{2s-1+3} \gamma^a(s - 1; 3)$$

推论3.7.4.  $X^{\lambda_\zeta}(s; 3) I_4 \otimes S_{ab}(s - \frac{1}{2}; 3) \bar{X}_{\lambda_\zeta}(s; 3) = \frac{2s+3}{2s-1+3} S_{ab}(s - 1; 3)$

$$[\Leftrightarrow] \bar{X}(s; 3) I_4 \otimes S_{ab}(s - \frac{1}{2}; 3) X(s; 3) = \frac{2s+3}{2s-1+3} S_{ab}(s - 1; 3)$$

3.8 矩阵  $X(s; 3)$ ,  $\bar{X}(s; 3)$  的常数不变张量性质

定理3.8.1.  $X(s; 3) = e^{(i\Omega^a + \varsigma \epsilon) \cdot \gamma^a} \otimes e^{(i\Omega^a + \varsigma \epsilon) \cdot \gamma^a(s - \frac{1}{2}; 3)} X(s; 3) e^{-(i\Omega^a + \varsigma \epsilon) \cdot \gamma^a(s - 1; 3)}$

定理3.8.2.  $\bar{X}(s; 3) = e^{(i\Omega^a + \varsigma \epsilon) \cdot \gamma^a(s - 1; 3)} \bar{X}(s; 3) e^{-(i\Omega^a + \varsigma \epsilon) \cdot \gamma^a} \otimes e^{(i\Omega^a + \varsigma \epsilon) \cdot \gamma^a(s - \frac{1}{2}; 3)}$

3.9 常数矩阵  $\Omega^a(s; 3)$ ,  $\gamma^a(s - 1; 3)$  的置换性质

$$\text{推论3.9.1. } \begin{cases} \Omega^a(s; 3) [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] X(s; 3) = [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] X(s; 3) \gamma^a(s - 1; 3) \\ \bar{X}(s; 3) [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] \Omega^a(s; 3) = \gamma^a(s - 1; 3) \bar{X}(s; 3) [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] \end{cases}$$

$$\text{推论3.9.2. } \begin{cases} \gamma^a(s; 3) = \bar{N}(s; 3) [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] \Omega^a(s; 3) [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] N(s; 3) \\ \gamma^a(s - 1; 3) = \bar{X}(s; 3) [I_4 \otimes \bar{\Gamma}(s - \frac{1}{2}; 3)] \Omega^a(s; 3) [I_4 \otimes \Gamma(s - \frac{1}{2}; 3)] X(s; 3) \end{cases}$$

3.10 常数矩阵  $\Omega^a(s - l; 3)$ ,  $[\vec{\vartheta} \cdot \Omega^a(s - l; 3)]^n$ ,  $e^{\vec{\vartheta} \cdot \Omega^a(s - l; 3)}$  的同构性表示

推论3.10.1.  $\Omega^a(s; 3) = \Omega^a(s - 1; 3) \otimes I_{4^2} + I_{4^{2s-2}} \otimes \Omega^a(1; 3)$

## 推论3.10.2.

$$\begin{cases} \Omega^a(s; 3) I_{4^{2s-2}} \otimes \{[I_4 \otimes \Gamma(\frac{1}{2}; 3)] X(1; 3)\} = I_{4^{2s-2}} \otimes \{[I_4 \otimes \Gamma(\frac{1}{2}; 3)] X(1; 3)\} \Omega^a(s - 1; 3) \\ I_{4^{2s-2}} \otimes \{\bar{X}(1; 3) [I_4 \otimes \bar{\Gamma}(\frac{1}{2}; 3)]\} \Omega^a(s; 3) = \Omega^a(s - 1; 3) I_{4^{2s-2}} \otimes \{\bar{X}(1; 3) [I_4 \otimes \bar{\Gamma}(\frac{1}{2}; 3)]\} \end{cases}$$

## 推论3.10.3.

$$\begin{cases} \Omega^a(s - 1; 3) = I_{4^{2s-2}} \otimes \{\bar{X}(1; 3) [I_4 \otimes \bar{\Gamma}(\frac{1}{2}; 3)]\} \Omega^a(s; 3) I_{4^{2s-2}} \otimes \{[I_4 \otimes \Gamma(\frac{1}{2}; 3)] X(1; 3)\} \\ [\vec{\vartheta} \cdot \Omega^a(s - 1; 3)]^n = I_{4^{2s-2}} \otimes \{\bar{X}(1; 3) [I_4 \otimes \bar{\Gamma}(\frac{1}{2}; 3)]\} [\vec{\vartheta} \cdot \Omega^a(s; 3)]^n I_{4^{2s-2}} \otimes \{[I_4 \otimes \Gamma(\frac{1}{2}; 3)] X(1; 3)\} \\ e^{\vec{\vartheta} \cdot \Omega^a(s - 1; 3)} = I_{4^{2s-2}} \otimes \{\bar{X}(1; 3) [I_4 \otimes \bar{\Gamma}(\frac{1}{2}; 3)]\} e^{\vec{\vartheta} \cdot \Omega^a(s; 3)} I_{4^{2s-2}} \otimes \{[I_4 \otimes \Gamma(\frac{1}{2}; 3)] X(1; 3)\} \end{cases}$$

定义3.10.1.

$$\begin{cases} T(s; 3) := I_{4^{2s-2}} \otimes \{[I_4 \otimes \Gamma(\frac{1}{2}; 3)]X(1; 3)\} \\ \bar{T}(s; 3) := I_{4^{2s-2}} \otimes \{\bar{X}(1; 3)[I_4 \otimes \bar{\Gamma}(\frac{1}{2}; 3)]\} = T^+(s; 3) \end{cases}$$

推论3.10.4.

$$\begin{cases} \Omega^\alpha(s-l; 3) = \bar{T}(s-l+1; 3) \cdots \bar{T}(s-1; 3) \bar{T}(s; 3) \Omega^\alpha(s; 3) T(s; 3) T(s-1; 3) \cdots T(s-l+1; 3) \\ [\vec{\nu} \cdot \Omega^\alpha(s-l; 3)]^n = \bar{T}(s-l+1; 3) \cdots \bar{T}(s-1; 3) \bar{T}(s; 3) [\vec{\nu} \cdot \Omega^\alpha(s; 3)]^n T(s; 3) T(s-1; 3) \cdots T(s-l+1; 3) \\ e^{\vec{\nu} \cdot \Omega^\alpha(s-l; 3)} = \bar{T}(s-l+1; 3) \cdots \bar{T}(s-1; 3) \bar{T}(s; 3) e^{\vec{\nu} \cdot \Omega^\alpha(s; 3)} T(s; 3) T(s-1; 3) \cdots T(s-l+1; 3) \end{cases}$$

推论3.10.5.

$$\begin{cases} \gamma^\alpha(s-l; 3) = \bar{\Gamma}(s-l; 3) \bar{T}(s-l+1; 3) \cdots \bar{T}(s; 3) \Omega^\alpha(s; 3) T(s; 3) \cdots T(s-l+1; 3) \Gamma(s-l; 3) \\ [\vec{\nu} \cdot \gamma^\alpha(s-l; 3)]^n = \bar{\Gamma}(s-l; 3) \bar{T}(s-l+1; 3) \cdots \bar{T}(s; 3) [\vec{\nu} \cdot \Omega^\alpha(s; 3)]^n T(s; 3) \cdots T(s-l+1; 3) \Gamma(s-l; 3) \\ e^{\vec{\nu} \cdot \gamma^\alpha(s-l; 3)} = \bar{\Gamma}(s-l; 3) \bar{T}(s-l+1; 3) \cdots \bar{T}(s; 3) e^{\vec{\nu} \cdot \Omega^\alpha(s; 3)} T(s; 3) \cdots T(s-l+1; 3) \Gamma(s-l; 3) \end{cases}$$

## 4 高阶Dirac型完美常数不变张量

### 4.1 无穷扩展和推广

推论4.1.1.

$$\underbrace{\gamma^\alpha, \varepsilon(\frac{1}{2}; 3)}_{\text{零阶}} \rightarrow \underbrace{\gamma^\alpha(s_1; 3), \varepsilon(s_1; 3)}_{\text{一阶}} \rightarrow \underbrace{\gamma^\alpha(s_2, s_1; 3), \varepsilon(s_2, s_1; 3)}_{\text{二阶}} \rightarrow \underbrace{\gamma^\alpha(s_3, s_2, s_1; 3), \varepsilon(s_3, s_2, s_1; 3)}_{\text{三阶}} \rightarrow \cdots$$

$$\text{推论4.1.2. } \underbrace{\gamma^\alpha, \varepsilon(\frac{1}{2}; 3)}_{\text{零阶}} \rightarrow \underbrace{\gamma^\alpha(s_1), \varepsilon(s_1)}_{\text{一阶}} \rightarrow \underbrace{\gamma^\alpha(s_2, s_1), \varepsilon(s_2, s_1)}_{\text{二阶}} \rightarrow \underbrace{\gamma^\alpha(s_3, s_2, s_1), \varepsilon(s_3, s_2, s_1)}_{\text{三阶}} \rightarrow \cdots$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{matrix}$$

$$\text{推论4.1.3. } \underbrace{\Gamma(s_1; 3)}_{\text{一阶}} \rightarrow \rightarrow \underbrace{\Gamma(s_2; w < s_1 >)}_{\text{二阶}} \rightarrow \rightarrow \underbrace{\Gamma(s_3; w < s_2, s_1 >)}_{\text{三阶}} \rightarrow \rightarrow \underbrace{\Gamma(s_4; w < s_3, s_2, s_1 >)}_{\text{四阶}} \rightarrow \rightarrow \cdots$$

## 5 Dirac型常数不变张量的具体计算

### 5.1 Dirac型常数不变张量 $\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \cdots}^{k_\zeta}(s; 3)$ , $\Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \cdots}(s; 3)$ 的引入

定义5.1.1.

$$\Gamma_{\lambda_\zeta \mu_\zeta \eta_\zeta \cdots}^{k_\zeta}(s; 3) = \frac{1}{(2s)!} \Gamma_{(\lambda_\zeta \mu_\zeta \eta_\zeta \cdots)}^{k_\zeta}(s; 3)$$

$$\Gamma_{0_\zeta \cdots 0_\zeta 1_\zeta \cdots 1_\zeta 2_\zeta \cdots 2_\zeta 3_\zeta \cdots 3_\zeta}^{k_\zeta}(s; 3) = \sqrt{\frac{l_0! l_1! l_2! l_3!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^{[s]} (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+3-k}^{2l} - C_{2l+3-\lambda_{2l}}^{2l})\}, l_0 + l_1 + l_2 + l_3 = 2s$$

定义5.1.2.

$$\Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \cdots}(s; 3) = \frac{1}{(2s)!} \Gamma_{k_\zeta}^{(\lambda_\zeta \mu_\zeta \eta_\zeta \cdots)}(s; 3)$$

$$\Gamma_{k_\zeta}^{0_\zeta \cdots 0_\zeta 1_\zeta \cdots 1_\zeta 2_\zeta \cdots 2_\zeta 3_\zeta \cdots 3_\zeta}(s; 3) = \sqrt{\frac{l_0! l_1! l_2! l_3!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^{[s]} (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+3-k}^{2l} - C_{2l+3-\lambda_{2l}}^{2l})\}, l_0 + l_1 + l_2 + l_3 = 2s$$

### 5.2 Dirac型常数不变张量 $\Gamma_{\lambda_\zeta \mu_\zeta}^{k_\zeta}(1; 3)$ , $\Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta}(1; 3)$ 的性质

定义5.2.1.

$$\begin{aligned} \Gamma_{0_\zeta \cdots 0_\zeta 1_\zeta \cdots 1_\zeta 2_\zeta \cdots 2_\zeta 3_\zeta \cdots 3_\zeta}^{k_\zeta}(\frac{1}{2}; 3) &= \delta\{k_\zeta, \sum_{l=0}^0 (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+3-k}^{2l} - C_{2l+3-\lambda_{2l}}^{2l})\}, l_0 + l_1 + l_2 + l_3 = 1 \\ &= \delta\{k_\zeta, \lambda_1\}, l_0 + l_1 + l_2 + l_3 = 1 \end{aligned}$$











$$= \frac{i}{2} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

推论5.5.8.  $[\gamma_x(1;3), \gamma_y(1;3)] = i \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}, \{\gamma_x(1;3), \gamma_y(1;3)\} = i \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

# 第四十章 多重完美常数不变张量

自我评述：本章主要是多重完美常数不变张量的具体介绍，是对更一般完美常数不变张量的深入具体了解和具体扩展。若将中微子自旋矩阵替换为多重矩阵，就可以得到多重完美常数不变张量。同时它是之前完美常数不变张量一般情形的特殊运用而已，故直接写出其结论即可，也无需再作证明。在此做出特别说明，本章的符号约定虽然尽量符合规范，但有些约定只限于本章，出了本章之外需重新约定，只有这样才不会出现混淆。本章节的完美常数不变张量与 $w = 1$ 情形同构，所以同样满足重要的复合常数不变张量那一章的展开性质，并具有完全相同的展开系数，为了节省篇幅，不再重复写出。

## 1 多重完美常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1)$

### 1.1 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1)$

定义1.1.1.  $\Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; 2j-1) = \frac{1}{(2s)!} \Gamma_{\underbrace{(A_\zeta B_\zeta C_\zeta \dots)}_{2s}}^{k_\zeta}(s; 2j-1), w = 2j-1$

$$\Gamma_{\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}}^{k_\zeta}(s; 2j-1) = \sqrt{\frac{l_0! l_1! \dots l_w!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^{[s]} (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+2j-1-k}^{2l} - C_{2l+2j-1-\lambda_{2l}}^{2l})\}, l_0 + l_1 + \dots + l_w = 2s$$

定义1.1.2.  $\Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}^{2s}}(s; 2j-1) = \frac{1}{(2s)!} \Gamma_{k_\zeta}^{\overbrace{(A_\zeta B_\zeta C_\zeta \dots)}^{2s}}(s; 2j-1), w = 2j-1$

$$\Gamma_{k_\zeta}^{\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}}(s; 2j-1) = \sqrt{\frac{l_0! l_1! \dots l_w!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^{[s]} (\sum_{k=\lambda_{2l+2}}^{\lambda_{2l+1}} C_{2l+2j-1-k}^{2l} - C_{2l+2j-1-\lambda_{2l}}^{2l})\}, l_0 + l_1 + \dots + l_w = 2s$$

### 1.2 常数矩阵 $\Gamma(s; 2j-1), \bar{\Gamma}(s; 2j-1)$

定义1.2.1.  $\Gamma(s; 2j-1) \succ \Gamma_{\underbrace{A_\zeta \otimes B_\zeta \otimes C_\zeta \otimes \dots}_{2s}}^{k_\zeta}(s; 2j-1), \bar{\Gamma}(s; 2j-1) \succ \Gamma_{k_\zeta}^{\overbrace{A_\zeta \otimes B_\zeta \otimes C_\zeta \otimes \dots}^{2s}}(s; 2j-1) \simeq \Gamma^T(s; 2j-1)$

推论1.2.1.  $[\Gamma(s; 2j-1)] = (2j)^{2s} \times C_{2s+2j-1}^{2s}, [\bar{\Gamma}(s; 2j-1)] = C_{2s+2j-1}^{2s} \times (2j)^{2s}, [A_\zeta] = 2j, [k_\zeta] = C_{2s+2j-1}^{2s}$

### 1.3 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1)$ 的基本性质

相等性：

性质1.3.1.  $\Gamma_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}^{k'_\zeta}(s; 2j-1) \simeq \Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; 2j-1) \simeq \Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}^{2s}}(s; 2j-1) \simeq \Gamma_{k'_\zeta}^{\overbrace{A'_\zeta B'_\zeta C'_\zeta \dots}^{2s}}(s; 2j-1)$

性质1.3.2.  $[\Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; 2j-1)]^* \simeq \Gamma_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}^{k'_\zeta}(s; 2j-1), [\Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}^{2s}}(s; 2j-1)]^* \simeq \Gamma_{k'_\zeta}^{\overbrace{A'_\zeta B'_\zeta C'_\zeta \dots}^{2s}}(s; 2j-1)$

推论1.3.1.  $\Gamma(s; 2j-1) = \Gamma^*(s; 2j-1), \bar{\Gamma}(s; 2j-1) = \bar{\Gamma}^*(s; 2j-1), \bar{\Gamma}(s; 2j-1) = \Gamma^+(s; 2j-1), \Gamma(s; 2j-1) = \bar{\Gamma}^+(s; 2j-1)$

正交性：

性质1.3.3.  $\Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; 2j-1) \Gamma_{l_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}^{2s}}(s; 2j-1) = \delta^{k_\zeta l_\zeta} [\Leftrightarrow] \bar{\Gamma}(s; 2j-1) \Gamma(s; 2j-1) = I$

性质1.3.4.  $\Gamma_{A_{1\zeta} A_{2\zeta} \dots A_{2s\zeta}}^{k_\zeta}(s; 2j-1) \Gamma_{k_\zeta}^{B_{1\zeta} B_{2\zeta} \dots B_{2s\zeta}}(s; 2j-1) = \frac{1}{(2s)!} \delta_{A_{1\zeta}}^{(B_{1\zeta}} \delta_{A_{2\zeta}}^{B_{2\zeta}} \dots \delta_{A_{2s\zeta}}^{B_{2s\zeta}}) = \frac{1}{(2s)!} \delta_{(A_{1\zeta} A_{2\zeta} \dots A_{2s\zeta})}^{B_{1\zeta} B_{2\zeta} \dots B_{2s\zeta}}$

对比性：

性质1.3.5.  $\varepsilon_{a_1 a_2 \dots a_n} \varepsilon^{b_1 b_2 \dots b_n} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_n}^{b_n} = \delta_{[a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_n}^{b_n]}$

其它性质：

性质1.3.6.  $\Gamma_{A_\zeta}^{k_\zeta}(\frac{1}{2}; 2j-1) = \delta_{A_\zeta}^{k_\zeta}, \Gamma_{k_\zeta}^{A_\zeta}(\frac{1}{2}; 2j-1) = \delta_{k_\zeta}^{A_\zeta}; \Gamma(0; 2j-1) = 1, \bar{\Gamma}(0; 2j-1) = 1$

### 1.4 度规常数不变张量 $\varepsilon_{k_\zeta l_\zeta}(s; 2j-1)$ 的引入及其性质

度规定义： $(\varepsilon \otimes I_j)_{A_\zeta B_\zeta} := (\varepsilon \otimes I_j)_{A_\zeta B_\zeta}$

定义1.4.1. 
$$\left\{ \begin{aligned} \varepsilon_{k_\zeta l_\zeta}(s; 2j-1) &:= \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1) \underbrace{(\varepsilon \otimes I_j)_{A_\zeta E_\zeta} (\varepsilon \otimes I_j)_{B_\zeta F_\zeta} (\varepsilon \otimes I_j)_{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; 2j-1) \\ \varepsilon^{k_\zeta l_\zeta}(s; 2j-1) &:= \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1) \underbrace{(\varepsilon \otimes I_j)^{A_\zeta E_\zeta} (\varepsilon \otimes I_j)^{B_\zeta F_\zeta} (\varepsilon \otimes I_j)^{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; 2j-1) \end{aligned} \right.$$

性质1.4.1. 
$$\left\{ \begin{aligned} &\underbrace{(\varepsilon \otimes I_j)_{A_\zeta E_\zeta} (\varepsilon \otimes I_j)_{B_\zeta F_\zeta} (\varepsilon \otimes I_j)_{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; 2j-1) \text{ 关于 } ABC\dots \text{ 全对称} \\ &\underbrace{(\varepsilon \otimes I_j)^{A_\zeta E_\zeta} (\varepsilon \otimes I_j)^{B_\zeta F_\zeta} (\varepsilon \otimes I_j)^{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; 2j-1) \text{ 关于 } ABC\dots \text{ 全对称} \end{aligned} \right.$$

推论1.4.1.  $\varepsilon(s; 2j-1) := \bar{\Gamma}(s; 2j-1) \underbrace{(\varepsilon \otimes I_j) \otimes \dots \otimes (\varepsilon \otimes I_j)}_{2s} \Gamma(s; 2j-1); (\varepsilon \otimes I_j) = \varepsilon(\frac{1}{2}; 2j-1)$

推论1.4.2.  $\varepsilon(s; 2j-1)\varepsilon^+(s; 2j-1) = \varepsilon^+(s; 2j-1)\varepsilon(s; 2j-1) = 1; \varepsilon(s; 2j-1) = \varepsilon^*(s; 2j-1), \varepsilon(s; 2j-1) = (-1)^{4sj}\varepsilon(s; 2j-1)$

升降指标：

性质1.4.2. 
$$\left\{ \begin{aligned} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1) &= \varepsilon^{k_\zeta l_\zeta}(s; 2j-1) \underbrace{(\varepsilon \otimes I_j)_{A_\zeta E_\zeta} (\varepsilon \otimes I_j)_{B_\zeta F_\zeta} (\varepsilon \otimes I_j)_{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; 2j-1) \\ \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1) &= \varepsilon_{k_\zeta l_\zeta}(s; 2j-1) \underbrace{(\varepsilon \otimes I_j)^{A_\zeta E_\zeta} (\varepsilon \otimes I_j)^{B_\zeta F_\zeta} (\varepsilon \otimes I_j)^{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; 2j-1) \end{aligned} \right.$$

推论1.4.3. 
$$\left\{ \begin{aligned} \Gamma(s; 2j-1)\varepsilon(s; 2j-1) &= \underbrace{(\varepsilon \otimes I_j) \otimes \dots \otimes (\varepsilon \otimes I_j)}_{2s} \Gamma(s; 2j-1) \\ \varepsilon(s; 2j-1)\bar{\Gamma}(s; 2j-1) &= \bar{\Gamma}(s; 2j-1) \underbrace{(\varepsilon \otimes I_j) \otimes \dots \otimes (\varepsilon \otimes I_j)}_{2s} \end{aligned} \right.$$

Penrose标准升降规则：

性质1.4.3. 
$$\left\{ \begin{aligned} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1) &= (-1)^{2s} [\varsigma^{2s} \varepsilon^{k_\zeta l_\zeta}(s; 2j-1)] \underbrace{[-\varsigma(\varepsilon \otimes I_j)_{A_\zeta E_\zeta}] [-\varsigma(\varepsilon \otimes I_j)_{B_\zeta F_\zeta}] [-\varsigma(\varepsilon \otimes I_j)_{C_\zeta G_\zeta}] \dots}_{2s} \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; 2j-1) \\ \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1) &= (-1)^{2s} [(-\varsigma)^{2s} \varepsilon_{k_\zeta l_\zeta}(s; 2j-1)] \underbrace{[\varsigma(\varepsilon \otimes I_j)^{A_\zeta E_\zeta}] [\varsigma(\varepsilon \otimes I_j)^{B_\zeta F_\zeta}] [\varsigma(\varepsilon \otimes I_j)^{C_\zeta G_\zeta}] \dots}_{2s} \Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; 2j-1) \end{aligned} \right.$$

### 1.5 自旋常数不变张量 $\sigma^{\alpha\zeta}_{k_\zeta}{}^{l_\zeta}(s; 2j-1), S_{ab}(s, \varsigma; 2j-1)$

定义1.5.1.

$$\left\{ \begin{aligned} \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1) \sigma^{\alpha\zeta}_{A_\zeta}{}^{Z_\zeta}(j) \Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta}(s; 2j-1) &= \frac{1}{2s} \sigma^{\alpha\zeta}_{k_\zeta}{}^{l_\zeta}(s; 2j-1) [\Leftrightarrow] \bar{\Gamma}(s; 2j-1) \sigma^{\alpha\zeta}(j) \Gamma(s; 2j-1) = \frac{1}{2s} \sigma^{\alpha\zeta}(s; 2j-1) \\ \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1) S_{ab A_\zeta}{}^{Z_\zeta}(j) \Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta}(s; 2j-1) &= \frac{1}{2s} S_{ab k_\zeta}{}^{l_\zeta}(s; 2j-1) [\Leftrightarrow] \bar{\Gamma}(s; 2j-1) S_{ab}(j, \varsigma) \Gamma(s; 2j-1) = \frac{1}{2s} S_{ab}(s, \varsigma; 2j-1) \end{aligned} \right.$$

### 1.6 常数不变张量 $\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j-1), \Omega(s; 2j-1)$ 的引入及其基本性质

定义1.6.1.

$$\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j-1) := \underbrace{[\sigma(\frac{1}{2}) \otimes I_j]_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} [\sigma(\frac{1}{2}) \otimes I_j]_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} [\sigma(\frac{1}{2}) \otimes I_j]_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \dots$$

[⇔] [⇔]

定义1.6.2.  $\Omega(s; 2j-1) := [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{(2j)^{2s-1}} + I_{2j} \otimes [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{(2j)^{2s-2}} + \dots + I_{(2j)^{2s-1}} \otimes [\sigma(\frac{1}{2}) \otimes I_j]$

推论1.6.1.  $\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j-1) := [\sigma(\frac{1}{2}) \otimes I_j]_{A_\zeta}^{A'_\zeta} \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s-1} + \delta_{A_\zeta}^{A'_\zeta} \Omega_{B_\zeta C_\zeta \dots}^{B'_\zeta C'_\zeta \dots}(s - \frac{1}{2}; 2j-1)$

[⇔] [⇔]

推论1.6.2.  $\Omega(s; 2j-1) = [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{(2j)^{2s-1}} + I_{2j} \otimes \Omega(s - \frac{1}{2}; 2j-1)$

推论1.6.3.  $\Omega(s; 2j-1) = \Omega(s - s'; 2j-1) \otimes I_{(2j)^{2s'}} + I_{(2j)^{2(s-s')}} \otimes \Omega(s'; 2j-1)$

### 1.7 常数不变张量 $\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j-1), \Omega_{ab}(s, \varsigma; 2j-1)$ 的引入及其基本性质

定义1.7.1.

$$\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j-1) := \underbrace{S_{ab A_\zeta}^{A'_\zeta}(\frac{1}{2}; 2j-1) \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} S_{ab B_\zeta}^{B'_\zeta}(\frac{1}{2}; 2j-1) \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} S_{ab C_\zeta}^{C'_\zeta}(\frac{1}{2}; 2j-1)}_{2s} + \dots$$

[⇔] [⇔]

定义1.7.2.  $\Omega_{ab}(s, \varsigma; 2j-1) := S_{ab}(\frac{1}{2}, \varsigma; 2j-1) \otimes I_{(w+1)^{2s-1}} + I_4 \otimes S_{ab}(\frac{1}{2}, \varsigma; 2j-1) \otimes I_{(w+1)^{2s-2}} + \dots + I_{(w+1)^{2s-1}} \otimes S_{ab}(\frac{1}{2}, \varsigma; 2j-1)$

推论1.7.1.  $\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots}(s; 2j-1) := S_{ab A_\zeta}^{A'_\zeta}(\frac{1}{2}; 2j-1) \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s-1} + \delta_{A_\zeta}^{A'_\zeta} \Omega_{ab B_\zeta C_\zeta \dots}^{B'_\zeta C'_\zeta \dots}(s - \frac{1}{2}; 2j-1)$

[⇔] [⇔]

推论1.7.2.  $\Omega_{ab}(s, \varsigma; 2j-1) = S_{ab}(\frac{1}{2}, \varsigma; 2j-1) \otimes I_{(w+1)^{2s-1}} + I_4 \otimes \Omega_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)$

推论1.7.3.  $\Omega_{ab}(s, \varsigma; 2j-1) = \Omega_{ab}(s - s', \varsigma; 2j-1) \otimes I_{(w+1)^{2s'}} + I_{(w+1)^{2(s-s')}} \otimes \Omega_{ab}(s', \varsigma; 2j-1)$

### 1.8 自旋常数不变张量 $\sigma^{\alpha_\zeta k_\zeta l_\zeta}(s; 2j-1), \sigma(s; 2j-1)$ 的引入

定义1.8.1.  $\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1) \sigma^{\alpha_\zeta A_\zeta Z_\zeta}(\frac{1}{2}; 2j-1) \Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta}(s; 2j-1) := \frac{1}{2s} \sigma^{\alpha_\zeta k_\zeta l_\zeta}(s; 2j-1)$

[⇔]  $\sigma(s; 2j-1) := \bar{\Gamma}(s; 2j-1) \Omega(s; 2j-1) \Gamma(s; 2j-1)$

### 1.9 自旋常数不变张量 $S_{ab k_\zeta l_\zeta}(s; 2j-1), S_{ab}(s, \varsigma; 2j-1)$ 的引入

定义1.9.1.  $\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; 2j-1) S_{ab A_\zeta Z_\zeta}(\frac{1}{2}; 2j-1) \Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta}(s; 2j-1) := \frac{1}{2s} S_{ab k_\zeta l_\zeta}(s; 2j-1)$

[⇔]  $S_{ab}(s, \varsigma; 2j-1) := \bar{\Gamma}(s; 2j-1) \Omega_{ab}(s, \varsigma; 2j-1) \Gamma(s; 2j-1)$

### 1.10 常数矩阵 $\Omega_{ab}(s, \varsigma; 2j-1), S_{ab}(s, \varsigma; 2j-1)$ 与 $\Omega_{\alpha_\zeta}(s; 2j-1), \sigma_{\alpha_\zeta}(s; 2j-1)$ 之间关系

推论1.10.1.  $S_{ab}(\frac{1}{2}, \varsigma; 2j-1) = \sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta}(\frac{1}{2}; 2j-1)$

[⇒]  $\Omega_{ab}(s, \varsigma; 2j-1) = \sigma_{\zeta ab}^{\alpha_\zeta} \Omega_{\alpha_\zeta}(s; 2j-1)$  [⇒]  $S_{ab}(s, \varsigma; 2j-1) = \sigma_{\zeta ab}^{\alpha_\zeta} \sigma_{\alpha_\zeta}(s; 2j-1)$



### 1.11 常数矩阵 $\Omega(s; 2j-1)$ 的两个重要引理

引理1.11.1.  $\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1)\Omega(s; 2j-1)\Gamma(s; 2j-1) = \Omega(s; 2j-1)\Gamma(s; 2j-1)$

引理1.11.2.  $\bar{\Gamma}(s; 2j-1)\Omega(s; 2j-1)\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1) = \bar{\Gamma}(s; 2j-1)\Omega(s; 2j-1)$

### 1.12 常数矩阵 $\Omega_{ab}(s, \varsigma; 2j-1)$ 的两个重要引理

引理1.12.1.  $\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1)\Omega_{ab}(s, \varsigma; 2j-1)\Gamma(s; 2j-1) = \Omega_{ab}(s, \varsigma; 2j-1)\Gamma(s; 2j-1)$

引理1.12.2.  $\bar{\Gamma}(s; 2j-1)\Omega_{ab}(s, \varsigma; 2j-1)\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1) = \bar{\Gamma}(s; 2j-1)\Omega_{ab}(s, \varsigma; 2j-1)$

### 1.13 关于常数矩阵 $\bar{\Gamma}(s; 3), \Omega(s; 3), \sigma(s; 3), \Gamma(s; 3)$ 的置换性质及其推论

定理1.13.1.  $\underbrace{\Omega_{A_\varsigma B_\varsigma C_\varsigma \dots}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; 2j-1)\underbrace{\Gamma_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}^{l_\varsigma}}_{2s}(s; 2j-1) = \underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}}_{2s}(s; 2j-1)\underbrace{\sigma_{k_\varsigma}^{l_\varsigma}}_{2s}(s; 2j-1)$   
 $[\Leftrightarrow]\Omega(s; 2j-1)\Gamma(s; 2j-1) = \Gamma(s; 2j-1)\sigma(s; 2j-1)$

定理1.13.2.  $\underbrace{\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma \dots}}_{2s}(s; 2j-1)\underbrace{\Omega_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}^{l_\varsigma}}_{2s}(s; 2j-1) = \underbrace{\sigma_{k_\varsigma}^{l_\varsigma}}_{2s}(s; 2j-1)\underbrace{\Gamma_{l_\varsigma}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; 2j-1)$   
 $[\Leftrightarrow]\bar{\Gamma}(s; 2j-1)\Omega(s; 2j-1) = \sigma(s; 2j-1)\bar{\Gamma}(s; 2j-1)$

推论1.13.1.  $\bar{\Gamma}(s; 2j-1)\Omega(s; 2j-1)\Gamma(s; 2j-1) = \sigma(s; 2j-1)$

$\Leftrightarrow \Omega(s; 2j-1)\Gamma(s; 2j-1) = \Gamma(s; 2j-1)\sigma(s; 2j-1) \Leftrightarrow \bar{\Gamma}(s; 2j-1)\Omega(s; 2j-1) = \sigma(s; 2j-1)\bar{\Gamma}(s; 2j-1)$

推论1.13.2.  $\Omega^2(s; 2j-1)\Gamma(s; 2j-1) = \Gamma(s; 2j-1)\sigma^2(s; 2j-1)$

推论1.13.3.  $\bar{\Gamma}(s; 2j-1)\Omega^2(s; 2j-1) = \sigma^2(s; 2j-1)\bar{\Gamma}(s; 2j-1)$

推论1.13.4.  $[\sigma(\frac{1}{2}) \otimes I_j]_{(A_\varsigma \underbrace{A'_\varsigma \Gamma_{B_\varsigma C_\varsigma \dots}^{l_\varsigma}}_{2s} A'_\varsigma)}(s; 2j-1) = \frac{1}{(2s-1)!} \underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}}_{2s}(s; 2j-1)\underbrace{\sigma_{k_\varsigma}^{l_\varsigma}}_{2s}(s; 2j-1)$

### 1.14 关于常数矩阵 $\bar{\Gamma}(s; 2j-1), \Omega_{ab}(s, \varsigma; 2j-1), S_{ab}(s, \varsigma; 2j-1), \Gamma(s; 2j-1)$ 的置换性质及其推论

定理1.14.1.  $\underbrace{\Omega_{ab}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; 2j-1)\underbrace{\Gamma_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}^{l_\varsigma}}_{2s}(s; 2j-1) = \underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}}_{2s}(s; 2j-1)\underbrace{S_{abk_\varsigma}^{l_\varsigma}}_{2s}(s; 2j-1)$   
 $[\Leftrightarrow]\Omega_{ab}(s, \varsigma; 2j-1)\Gamma(s; 2j-1) = \Gamma(s; 2j-1)S_{ab}(s, \varsigma; 2j-1)$

定理1.14.2.  $\underbrace{\Gamma_{k_\varsigma}^{A_\varsigma B_\varsigma C_\varsigma \dots}}_{2s}(s; 2j-1)\underbrace{\Omega_{ab}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; 2j-1) = \underbrace{S_{abk_\varsigma}^{l_\varsigma}}_{2s}(s; 2j-1)\underbrace{\Gamma_{l_\varsigma}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; 2j-1)$   
 $[\Leftrightarrow]\bar{\Gamma}(s; 2j-1)\Omega_{ab}(s, \varsigma; 2j-1) = S_{ab}(s, \varsigma; 2j-1)\bar{\Gamma}(s; 2j-1)$

推论1.14.1.  $\bar{\Gamma}(s; 2j-1)\Omega_{ab}(s, \varsigma; 2j-1)\Gamma(s; 2j-1) = S_{ab}(s, \varsigma; 2j-1)$

$\Leftrightarrow \Omega_{ab}(s, \varsigma; 2j-1)\Gamma(s; 2j-1) = \Gamma(s; 2j-1)S_{ab}(s, \varsigma; 2j-1) \Leftrightarrow \bar{\Gamma}(s; 2j-1)\Omega_{ab}(s, \varsigma; 2j-1) = S_{ab}(s, \varsigma; 2j-1)\bar{\Gamma}(s; 2j-1)$

推论1.14.2.  $[S_{ab}(\frac{1}{2}, \varsigma) \otimes I_j]_{(A_\varsigma \underbrace{A'_\varsigma \Gamma_{B_\varsigma C_\varsigma \dots}^{l_\varsigma}}_{2s} A'_\varsigma)}(s; 2j-1) = \frac{1}{(2s-1)!} \underbrace{\Gamma_{A_\varsigma B_\varsigma C_\varsigma \dots}^{k_\varsigma}}_{2s}(s; 2j-1)\underbrace{S_{abk_\varsigma}^{l_\varsigma}}_{2s}(s; 2j-1)$

### 1.15 常数矩阵 $\Omega(s; 2j-1), \sigma(s; 2j-1)$ 的洛伦兹群表示

推论1.15.1.  $\Omega(s; 2j-1) \times \Omega(s; 2j-1) = i\Omega(s; 2j-1)[\Rightarrow]\sigma(s; 2j-1) \times \sigma(s; 2j-1) = i\sigma(s; 2j-1)$

### 1.16 常数矩阵 $\Omega_{ab}(s, \varsigma; 2j-1)$ , $S_{ab}(s, \varsigma; 2j-1)$ 的洛伦兹群表示

推论1.16.1.  $i[S_{ab}(\frac{1}{2}, \varsigma; 2j-1), S_{cd}(\frac{1}{2}, \varsigma; 2j-1)]$   
 $= \delta_{ad}S_{bc}(\frac{1}{2}, \varsigma; 2j-1) - \delta_{ac}S_{bd}(\frac{1}{2}, \varsigma; 2j-1) + \delta_{bc}S_{ad}(\frac{1}{2}, \varsigma; 2j-1) - \delta_{bd}S_{ac}(\frac{1}{2}, \varsigma; 2j-1)$   
 $[\Rightarrow] i[\Omega_{ab}(s, \varsigma; 2j-1), \Omega_{cd}(s, \varsigma; 2j-1)]$   
 $= \delta_{ad}\Omega_{bc}(s, \varsigma; 2j-1) - \delta_{ac}\Omega_{bd}(s, \varsigma; 2j-1) + \delta_{bc}\Omega_{ad}(s, \varsigma; 2j-1) - \delta_{bd}\Omega_{ac}(s, \varsigma; 2j-1)$   
 $[\Rightarrow] i[S_{ab}(s, \varsigma; 2j-1), S_{cd}(s, \varsigma; 2j-1)]$   
 $= \delta_{ad}S_{bc}(s, \varsigma; 2j-1) - \delta_{ac}S_{bd}(s, \varsigma; 2j-1) + \delta_{bc}S_{ad}(s, \varsigma; 2j-1) - \delta_{bd}S_{ac}(s, \varsigma; 2j-1)$

### 1.17 推论：常数矩阵 $\Gamma(s; 2j-1)$ , $\bar{\Gamma}(s; 2j-1)$ 的几个恒等式

性质1.17.1.  $\begin{cases} \bar{\Gamma}(s; 2j-1)\Omega(s; 2j-1)\Gamma(s; 2j-1) = \sigma(s; 2j-1), [\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1), \Omega(s; 2j-1)] = 0 \\ \Gamma(s; 2j-1)\sigma(s; 2j-1)\bar{\Gamma}(s; 2j-1) = \Omega(s; 2j-1)\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1) = \Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1)\Omega(s; 2j-1) \end{cases}$

性质1.17.2.  $\begin{cases} \bar{\Gamma}(s; 2j-1)\Omega_{ab}(s, \varsigma; 2j-1)\Gamma(s; 2j-1) = S_{ab}(s, \varsigma; 2j-1), [\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1), \Omega_{ab}(s, \varsigma; 2j-1)] = 0 \\ \Gamma(s; 2j-1)S_{ab}(s, \varsigma; 2j-1)\bar{\Gamma}(s; 2j-1) = \Omega_{ab}(s, \varsigma; 2j-1)\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1) = \Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1)\Omega_{ab}(s, \varsigma; 2j-1) \end{cases}$

性质1.17.3.  $\begin{cases} \bar{\Gamma}(s; 2j-1)[\vartheta \cdot \Omega(s; 2j-1)]^n \Gamma(s; 2j-1) = [\vartheta \cdot \sigma(s; 2j-1)]^n, [\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1), [\vartheta \cdot \Omega(s; 2j-1)]^n] = 0 \\ \Gamma(s; 2j-1)[\vartheta \cdot \sigma(s; 2j-1)]^n \bar{\Gamma}(s; 2j-1) = [\vartheta \cdot \Omega(s; 2j-1)]^n \Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1) = \Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1)[\vartheta \cdot \Omega(s; 2j-1)]^n \end{cases}$

性质1.17.4.  $\begin{cases} \bar{\Gamma}(s; 2j-1)[\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)]^n \Gamma(s; 2j-1) = [\vartheta^{ab}S_{ab}(s, \varsigma; 2j-1)]^n, [\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1), [\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)]^n] = 0 \\ \Gamma(s; 2j-1)[\vartheta^{ab}S_{ab}(s, \varsigma; 2j-1)]^n \bar{\Gamma}(s; 2j-1) = [\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)]^n \Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1) = \Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1)[\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)]^n \end{cases}$

推论1.17.1.  $\begin{cases} \bar{\Gamma}(s; 2j-1)e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)}\Gamma(s; 2j-1) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; 2j-1)}, [\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1), e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)}] = 0 \\ \Gamma(s; 2j-1)e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; 2j-1)}\bar{\Gamma}(s; 2j-1) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)}\Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1) = \Gamma(s; 2j-1)\bar{\Gamma}(s; 2j-1)e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)} \end{cases}$

### 1.18 常数矩阵 $\Gamma(s; 2j-1)$ , $\bar{\Gamma}(s; 2j-1)$ 的对称置换性质

定义1.18.1.  $S_{ex}(s, n) = \overbrace{(I_{2j} \otimes \cdots \otimes I_{2j})}^{n-1} S_{ex} \overbrace{(I_{2j} \otimes \cdots \otimes I)}^{2s-n-1}$

推论1.18.1.  $\Gamma(s; 2j-1) = S_{ex}(s, n)\Gamma(s; 2j-1)$ ,  $\bar{\Gamma}(s; 2j-1) = \bar{\Gamma}(s; 2j-1)S_{ex}(s, n)$

推论1.18.2.  $S_{ex}(s, n)\Omega(s; 2j-1)S_{ex}(s, n) = \Omega(s; 2j-1)$

推论1.18.3.  $\hat{\psi}(s, \varsigma; 2j-1) = S_{ex}(s, n)\hat{\psi}(s, \varsigma; 2j-1)$ ,  $\forall n \in \{1, 2, \dots, 2s+1\}$

### 1.19 矩阵 $\Gamma(s; 2j-1)$ , $\bar{\Gamma}(s; 2j-1)$ 的常数不变张量性质

定理1.19.1.  $\Gamma(s; 2j-1) = e^{(i\omega+\varsigma\epsilon)\cdot\Omega(s; 2j-1)}\Gamma(s; 2j-1)e^{-(i\omega+\varsigma\epsilon)\cdot\sigma(s; 2j-1)}$

定理1.19.2.  $\bar{\Gamma}(s; 2j-1) = e^{(i\omega+\varsigma\epsilon)\cdot\sigma(s; 2j-1)}\bar{\Gamma}(s; 2j-1)e^{-(i\omega+\varsigma\epsilon)\cdot\Omega(s; 2j-1)}$

推论1.19.1.

$$\begin{cases} S(s; 2j-1) = e^{(i\omega+\varsigma\epsilon)\cdot\Omega(s; 2j-1)}S(s; 2j-1)e^{-(i\omega+\varsigma\epsilon)\cdot S^+(s; 2j-1)\Omega(s; 2j-1)S(s; 2j-1)} \\ S(s; 2j-1) = e^{(i\omega+\varsigma\epsilon)\cdot S(s; 2j-1)\Omega(s; 2j-1)S^+(s; 2j-1)}S(s; 2j-1)e^{-(i\omega+\varsigma\epsilon)\cdot\Omega(s; 2j-1)} \\ S^+(s; 2j-1) = e^{(i\omega+\varsigma\epsilon)\cdot\Omega(s; 2j-1)}S^+(s; 2j-1)e^{-(i\omega+\varsigma\epsilon)\cdot S(s; 2j-1)\Omega(s; 2j-1)S^+(s; 2j-1)} \\ S^+(s; 2j-1) = e^{(i\omega+\varsigma\epsilon)\cdot S^+(s; 2j-1)\Omega(s; 2j-1)S(s; 2j-1)}S^+(s; 2j-1)e^{-(i\omega+\varsigma\epsilon)\cdot\Omega(s; 2j-1)} \end{cases}$$

### 1.20 常数矩阵 $I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)$ , $I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)$ 的置换性质

推论1.20.1.  $\Omega(s; 2j-1)[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)] = [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)][[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)]$

推论1.20.2.  $[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)]\Omega(s; 2j-1) = [[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)]$

### 1.21 推论：常数矩阵 $\Gamma(s - \frac{1}{2}; 2j - 1), \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)$ 的几个恒等式

性质1.21.1.

$$\begin{cases} [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)]\Omega(s; 2j - 1)[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)] = [[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j - 1)] \\ [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] \\ = \Omega(s; 2j - 1)[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] = [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)]\Omega(s; 2j - 1) \\ [[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)], \Omega(s; 2j - 1)] = 0 \end{cases}$$

性质1.21.2.

$$\begin{cases} [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)]\Omega_{ab}(s, \varsigma; 2j - 1)[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)] = [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j - 1)] \\ [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] \\ = \Omega_{ab}(s, \varsigma; 2j - 1)[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] = [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)]\Omega_{ab}(s, \varsigma; 2j - 1) \\ [[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)], \Omega_{ab}(s, \varsigma; 2j - 1)] = 0 \end{cases}$$

性质1.21.3.

$$\begin{cases} [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)][\vartheta \cdot \Omega(s; 2j - 1)]^n [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)] = \{\vartheta \cdot [[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j - 1)]\}^n \\ [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)]\{\vartheta \cdot [[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j - 1)]\}^n [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] \\ = [\vartheta \cdot \Omega(s; 2j - 1)]^n [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] = [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)][\vartheta \cdot \Omega(s; 2j - 1)]^n \\ [[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)], [\vartheta \cdot \Omega(s; 2j - 1)]^n] = 0 \end{cases}$$

性质1.21.4.

$$\begin{cases} [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)][\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j - 1)]^n [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)] = \{\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j - 1)]\}^n \\ [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)]\{\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j - 1)]\}^n [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] \\ = [\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j - 1)]^n [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] = [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)][\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j - 1)]^n \\ [[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)], [\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j - 1)]^n] = 0 \end{cases}$$

推论1.21.1.

$$\begin{cases} [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)} [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)] = e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)]} \\ [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)]e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)]} [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] \\ = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)} [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] = [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)} \\ [[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)], e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; 2j-1)}] = 0 \end{cases}$$

推论1.21.2.

$$\begin{cases} I_{(2j)^{2s-1}}\Gamma(s - \frac{1}{2}; 2j - 1) = \Gamma(s - \frac{1}{2}; 2j - 1)I_{C_{2s-1+2j-1}^{2s-1}}, \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)I_{(2j)^{2s-1}} = I_{C_{2s-1+2j-1}^{2s-1}}\bar{\Gamma}(s - \frac{1}{2}; 2j - 1) \\ [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)][\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{(2j)^{2s-1}} [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)] = [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} \\ [[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{(2j)^{2s-1}} [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)]] = [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)][[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}}] \\ [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)][[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{(2j)^{2s-1}}] = [[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}}] [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] \end{cases}$$

### 1.22 矩阵 $I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1), I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)$ 的常数不变张量性质

定理1.22.1.  $[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)] = e^{(i\omega + \varsigma\epsilon) \cdot \Omega(s; 2j-1)} [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j - 1)] e^{-(i\omega + \varsigma\epsilon) \cdot [\sigma(\frac{1}{2}) \otimes I_j]} \otimes e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2}; 2j-1)}$

定理1.22.2.  $[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] = e^{(i\omega + \varsigma\epsilon) \cdot [\sigma(\frac{1}{2}) \otimes I_j]} \otimes e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2}; 2j-1)} [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j - 1)] e^{-(i\omega + \varsigma\epsilon) \cdot \Omega(s; 2j-1)}$

## 2 多重完美常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$

### 2.1 完美常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$

定义2.1.1.  $N_{A_\zeta l_\zeta}^{k_\zeta}(s) := \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s) \overbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}^{2s-1}(s - \frac{1}{2}), N_{k_\zeta}^{A_\zeta l_\zeta}(s) := \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}}^{2s}(s) \Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}(s - \frac{1}{2})$

定义2.1.2.  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1) := \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1) \overbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}^{2s-1}(s - \frac{1}{2}; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) := \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}}^{2s}(s; 2j-1) \Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}(s - \frac{1}{2}; 2j-1)$

推论2.1.1.  $N(s; 2j-1) = [I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)]\Gamma(s; 2j-1), \bar{N}(s; 2j-1) = \bar{\Gamma}(s; 2j-1)[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)]$

性质2.1.1.  $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1) = N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1) \Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta}(s - \frac{1}{2}; 2j-1), \overbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}}^{2s}(s; 2j-1) = N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) \overbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}^{2s-1}(s - \frac{1}{2}; 2j-1)$

推论2.1.2.  $\Gamma(s; 2j-1) = [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)]N(s; 2j-1), \bar{\Gamma}(s; 2j-1) = \bar{N}(s; 2j-1)[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)],$

性质2.1.2.  $\Gamma(s; 2j-1) = [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)][I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)]\Gamma(s; 2j-1)$

推论2.1.3.  $N_{A_\zeta l_\zeta}^{k_\zeta}(s) = N_{A_\zeta l_\zeta}^{k_\zeta}(s; 1), N_{k_\zeta}^{A_\zeta l_\zeta}(s) = N_{k_\zeta}^{A_\zeta l_\zeta}(s; 1)$

自我评述：以上表明  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$  是常数不变张量  $N_{A_\zeta l_\zeta}^{k_\zeta}(s), N_{k_\zeta}^{A_\zeta l_\zeta}(s)$  的推广。

### 2.2 常数矩阵 $N_{A_\zeta}(s; 2j-1), N^{A_\zeta}(s; 2j-1); \bar{N}_{A_\zeta}(s; 2j-1), \bar{N}^{A_\zeta}(s; 2j-1); N(s; 2j-1), \bar{N}(s; 2j-1)$

定义2.2.1.

$$\begin{cases} N_{A_\zeta}(s; 2j-1) \prec N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N^{A_\zeta}(s; 2j-1) \prec N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) |_{I_{C_{2s+2j-1}^{2s}} \times I_{C_{2s-1+2j-1}^{2s-1}}} \\ \bar{N}_{A_\zeta}(s; 2j-1) := N_{A_\zeta}^+(s; 2j-1) \succ N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), \bar{N}^{A_\zeta}(s; 2j-1) := N^{+A_\zeta}(s; 2j-1) \succ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) |_{I_{C_{2s-1+2j-1}^{2s-1}} \times I_{C_{2s+2j-1}^{2s}}} \\ N(s; 2j-1) \prec N_{A_\zeta \otimes l_\zeta}^{k_\zeta}(s; 2j-1) |_{(2j)I_{C_{2s-1+2j-1}^{2s-1}} \times I_{C_{2s+2j-1}^{2s}}}, \bar{N}(s; 2j-1) = N^+(s; 2j-1) \prec N_{k_\zeta}^{A_\zeta \otimes l_\zeta}(s; 2j-1) |_{I_{C_{2s+2j-1}^{2s}} \times I_{(2j)I_{C_{2s-1+2j-1}^{2s-1}}}} \end{cases}$$

### 2.3 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$ 的基本性质

相等性：

性质2.3.1.

$$\begin{cases} N_{A'_\zeta l'_\zeta}^{k'_\zeta}(s; 2j-1) \simeq N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1) \simeq N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) \simeq N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; 2j-1) \\ [N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1)]^* \simeq N_{A'_\zeta l'_\zeta}^{k'_\zeta}(s; 2j-1), [N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)]^* \simeq N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; 2j-1) \end{cases}$$

推论2.3.1.

$$\begin{cases} N_{A_\zeta}(s; 2j-1) \simeq N^{A_\zeta}(s; 2j-1) \simeq N_{A'_\zeta}(s; 2j-1) \simeq N^{A'_\zeta}(s; 2j-1); \bar{N}_{A_\zeta}(s; 2j-1) \simeq \bar{N}^{A_\zeta}(s; 2j-1) \simeq \bar{N}_{A'_\zeta}(s; 2j-1) \simeq \bar{N}^{A'_\zeta}(s; 2j-1) \\ N_{A_\zeta}(s; 2j-1) = N_{A_\zeta}^*(s; 2j-1), \bar{N}_{A_\zeta}(s; 2j-1) = \bar{N}_{A_\zeta}^*(s; 2j-1); N(s; 2j-1) = N^*(s; 2j-1), \bar{N}(s; 2j-1) = \bar{N}^*(s; 2j-1) \end{cases}$$

### 2.4 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1), N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$ 的正交性质

正交性：

引理2.4.1.  $\sum_{k=0}^{2s-1} C_{2j-1+k}^{2j-1} = C_{2j-1+2s}^{2j}$

引理2.4.2.  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1) N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; 2j-1) = \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; 2j-1) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}^{2s}(s; 2j-1)$

定理2.4.1.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1)N_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) = \delta_{m_\zeta}^{k_\zeta} [\Leftrightarrow] N^{A_\zeta}(s; 2j-1)\bar{N}_{A_\zeta}(s; 2j-1) = I_{C_{2s+2j-1}^{2s}} [\Leftrightarrow] \bar{N}(s; 2j-1)N(s; 2j-1) = I_{C_{2s+2j-1}^{2s}} \\ N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1)N_{k_\zeta}^{A_\zeta m_\zeta}(s; 2j-1) = (1 + \frac{j}{s})\delta_{l_\zeta}^{m_\zeta} [\Leftrightarrow] \bar{N}_{A_\zeta}(s; 2j-1)N^{A_\zeta}(s; 2j-1) = (1 + \frac{j}{s})I_{C_{2s-1+2j-1}^{2s-1}} \\ N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1)N_{k_\zeta}^{B_\zeta l_\zeta}(s; 2j-1) = \frac{1}{2j}C_{2s+2j-1}^{2s}\delta_{A_\zeta}^{B_\zeta} [\Leftrightarrow] \text{tr}[\bar{N}_{A_\zeta}(s; 2j-1)N^{B_\zeta}(s; 2j-1)] = \frac{1}{2j}C_{2s+2j-1}^{2s}\delta_{A_\zeta}^{B_\zeta} \end{cases}$$

性质2.4.1.  $\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1)N_{k_\zeta}^{B_\zeta m_\zeta}(s; 2j-1) = \frac{1}{2s}[\delta_{A_\zeta}^{B_\zeta}\delta_{l_\zeta}^{m_\zeta} + (2s-1)N_{A_\zeta n_\zeta}^{m_\zeta}(s-\frac{1}{2}; 2j-1)N_{l_\zeta}^{B_\zeta n_\zeta}(s-\frac{1}{2}; 2j-1)] \\ \bar{N}_{A_\zeta}(s; 2j-1)N^{B_\zeta}(s; 2j-1) = \frac{1}{2s}[\delta_{A_\zeta}^{B_\zeta}I_{C_{2s-1+2j-1}^{2s-1}} + (2s-1)N^{B_\zeta}(s-\frac{1}{2}; 2j-1)\bar{N}_{A_\zeta}(s-\frac{1}{2}; 2j-1)] \end{cases}$

## 2.5 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$ 的升降指标

升降指标:

性质2.5.1.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1) = \varepsilon^{k_\zeta m_\zeta}(s; 2j-1)(\varepsilon \otimes I_j)_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s-\frac{1}{2}; 2j-1)N_{m_\zeta}^{B_\zeta n_\zeta}(s; 2j-1) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) = \varepsilon_{k_\zeta m_\zeta}(s; 2j-1)(\varepsilon \otimes I_j)^{A_\zeta B_\zeta} \varepsilon^{l_\zeta n_\zeta}(s-\frac{1}{2}; 2j-1)N_{B_\zeta n_\zeta}^{m_\zeta}(s; 2j-1) \end{cases}$$

推论2.5.1.

$$\begin{cases} N_{A_\zeta}(s; 2j-1)\varepsilon(s-\frac{1}{2}; 2j-1) = (\varepsilon \otimes I_j)_{A_\zeta B_\zeta} \varepsilon(s; 2j-1)N^{B_\zeta}(s; 2j-1), \varepsilon(s-\frac{1}{2}; 2j-1)\bar{N}_{A_\zeta}(s; 2j-1) = \bar{N}^{B_\zeta}(s; 2j-1)\varepsilon \\ N^{A_\zeta}(s; 2j-1)\varepsilon(s-\frac{1}{2}; 2j-1) = (\varepsilon \otimes I_j)^{A_\zeta B_\zeta} \varepsilon(s; 2j-1)N_{B_\zeta}(s; 2j-1), \varepsilon(s-\frac{1}{2}; 2j-1)\bar{N}^{A_\zeta}(s; 2j-1) = \bar{N}_{B_\zeta}(s; 2j-1)\varepsilon \\ N(s; 2j-1)\varepsilon(s; 2j-1) = [(\varepsilon \otimes I_j) \otimes \varepsilon(s-\frac{1}{2}; 2j-1)]N(s; 2j-1), \varepsilon(s; 2j-1)\bar{N}(s; 2j-1) = \bar{N}(s; 2j-1)[(\varepsilon \otimes I_j) \otimes \varepsilon(s; 2j-1)] \end{cases}$$

Penrose标准升降规则:

性质2.5.2.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1) = (-1)^{2s}[\zeta^{2s}\varepsilon^{k_\zeta m_\zeta}(s; 2j-1)][-\zeta(\varepsilon \otimes I_j)_{A_\zeta B_\zeta}][(-\zeta)^{2s-1}\varepsilon_{l_\zeta n_\zeta}(s-\frac{1}{2}; 2j-1)]N_{m_\zeta}^{B_\zeta n_\zeta}(s; 2j-1) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) = (-1)^{2s}[(-\zeta)^{2s}\varepsilon_{k_\zeta m_\zeta}(s; 2j-1)][\zeta(\varepsilon \otimes I_j)^{A_\zeta B_\zeta}][\zeta^{2s-1}\varepsilon^{l_\zeta n_\zeta}(s-\frac{1}{2}; 2j-1)]N_{B_\zeta n_\zeta}^{m_\zeta}(s; 2j-1) \end{cases}$$

## 2.6 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$ 的自旋矩阵变换

性质2.6.1.

$$\begin{cases} N_{k_\zeta}^{A_\zeta m_\zeta}(s; 2j-1)\sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(j)N_{B_\zeta m_\zeta}^{l_\zeta}(s; 2j-1) = \frac{1}{2s}\sigma^{\alpha_\zeta}_{k_\zeta l_\zeta}(s; 2j-1)[\Leftrightarrow]N^{A_\zeta}(s; 2j-1)\sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(j)\bar{N}_{B_\zeta}(s; 2j-1) = \frac{1}{2s}\sigma^{\alpha_\zeta}(s; 2j-1) \\ [\Leftrightarrow]\bar{N}(s; 2j-1)[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}}N(s; 2j-1) = \frac{1}{2s}\sigma(s; 2j-1) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)\sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(j)N_{B_\zeta m_\zeta}^{k_\zeta}(s; 2j-1) = \frac{1}{2s}\sigma^{\alpha_\zeta}_{m_\zeta l_\zeta}(s-\frac{1}{2}; 2j-1)[\Leftrightarrow]\bar{N}_{B_\zeta}(s; 2j-1)\sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(j)N^{A_\zeta}(s; 2j-1) = \frac{1}{2s}\sigma^{\alpha_\zeta}(s; 2j-1) \end{cases}$$

性质2.6.2.

$$\begin{cases} N_{k_\zeta}^{A_\zeta m_\zeta}(s; 2j-1)S_{ab A_\zeta}^{B_\zeta}(j)N_{B_\zeta m_\zeta}^{l_\zeta}(s; 2j-1) = \frac{1}{2s}S_{ab k_\zeta}^{l_\zeta}(s; 2j-1)[\Leftrightarrow]N^{A_\zeta}(s; 2j-1)S_{ab A_\zeta}^{B_\zeta}(j)\bar{N}_{B_\zeta}(s; 2j-1) = \frac{1}{2s}S_{ab}(s; 2j-1) \\ [\Leftrightarrow]\bar{N}(s; 2j-1)S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}}N(s; 2j-1) = \frac{1}{2s}S_{ab}(s, \zeta; 2j-1) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)S_{ab A_\zeta}^{B_\zeta}(j)N_{B_\zeta m_\zeta}^{k_\zeta}(s; 2j-1) = \frac{1}{2s}S_{ab m_\zeta}^{l_\zeta}(s-\frac{1}{2}; 2j-1)[\Leftrightarrow]\bar{N}_{B_\zeta}(s; 2j-1)S_{ab A_\zeta}^{B_\zeta}(j)N^{A_\zeta}(s; 2j-1) = \frac{1}{2s}S_{ab}(s; 2j-1) \end{cases}$$

## 2.7 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)$ 的置换性质

定理2.7.1.

$$\begin{cases} \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(j)N_{B_\zeta l_\zeta}^{k_\zeta}(s; 2j-1) + \sigma^{\alpha_\zeta}_{l_\zeta m_\zeta}(s-\frac{1}{2}; 2j-1)N_{A_\zeta m_\zeta}^{k_\zeta}(s; 2j-1) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; 2j-1)\sigma^{\alpha_\zeta}_{j_\zeta k_\zeta}(s; 2j-1) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)\sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(j) + N_{k_\zeta}^{B_\zeta m_\zeta}(s; 2j-1)\sigma^{\alpha_\zeta}_{m_\zeta l_\zeta}(s-\frac{1}{2}; 2j-1) = \sigma^{\alpha_\zeta}_{k_\zeta j_\zeta}(s; 2j-1)N_{j_\zeta}^{B_\zeta l_\zeta}(s; 2j-1) \\ \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(j)\bar{N}_{B_\zeta}(s; 2j-1) + \sigma^{\alpha_\zeta}(s-\frac{1}{2}; 2j-1)\bar{N}_{A_\zeta}(s; 2j-1) = \bar{N}_{A_\zeta}(s; 2j-1)\sigma^{\alpha_\zeta}(s; 2j-1) \\ N^{A_\zeta}(s; 2j-1)\sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(j) + N^{B_\zeta}(s; 2j-1)\sigma^{\alpha_\zeta}(s-\frac{1}{2}; 2j-1) = \sigma^{\alpha_\zeta}(s; 2j-1)N^{B_\zeta}(s; 2j-1) \\ [([\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma^{\alpha_\zeta}(s-\frac{1}{2}; 2j-1))]N(s; 2j-1) = N(s; 2j-1)\sigma^{\alpha_\zeta}(s; 2j-1) \\ \bar{N}(s; 2j-1)[([\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma^{\alpha_\zeta}(s-\frac{1}{2}; 2j-1))] = \sigma^{\alpha_\zeta}(s; 2j-1)\bar{N}(s; 2j-1) \end{cases}$$

定理2.7.2.

$$\begin{cases} S_{abA_\zeta}^{B_\zeta}(j)N_{B_\zeta l_\zeta}^{k_\zeta}(s; 2j-1) + S_{abl_\zeta}^{m_\zeta}(s - \frac{1}{2}; 2j-1)N_{A_\zeta m_\zeta}^{k_\zeta}(s; 2j-1) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; 2j-1)S_{abj_\zeta}^{k_\zeta}(s; 2j-1) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)S_{abA_\zeta}^{B_\zeta}(j) + N_{k_\zeta}^{B_\zeta m_\zeta}(s; 2j-1)S_{abm_\zeta}^{l_\zeta}(s - \frac{1}{2}; 2j-1) = S_{abk_\zeta}^{j_\zeta}(s; 2j-1)N_{j_\zeta}^{B_\zeta l_\zeta}(s; 2j-1) \\ S_{abA_\zeta}^{B_\zeta}(j)\bar{N}_{B_\zeta}(s; 2j-1) + S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)\bar{N}_{A_\zeta}(s; 2j-1) = \bar{N}_{A_\zeta}(s; 2j-1)S_{ab}(s, \zeta; 2j-1) \\ N^{A_\zeta}(s; 2j-1)S_{abA_\zeta}^{B_\zeta}(j) + N^{B_\zeta}(s; 2j-1)S_{ab}(s - \frac{1}{2}, \zeta; 2j-1) = S_{ab}(s, \zeta; 2j-1)N^{B_\zeta}(s; 2j-1) \\ [S_{ab}(j, \zeta) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)N(s; 2j-1) = N(s; 2j-1)S_{ab}(s, \zeta; 2j-1) \\ \bar{N}(s; 2j-1)[S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)] = S_{ab}(s, \zeta; 2j-1)\bar{N}(s; 2j-1) \end{cases}$$

2.8 常数不变张量 $N_{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j-1)$ ,  $N_{k_\zeta}^{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}(s; 2j-1)$ 的性质

$$\text{定义2.8.1. } \begin{cases} N_{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j-1) := \Gamma_{A_{\zeta 1} \cdots A_{\zeta 2s}}^{k_\zeta}(s; 2j-1)\Gamma_{l_\zeta}^{A_{\zeta n+1} \cdots A_{\zeta 2s}}(s - \frac{n}{2}; 2j-1) \\ N_{k_\zeta}^{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}(s; 2j-1) := \Gamma_{k_\zeta}^{A_{\zeta 1} \cdots A_{\zeta 2s}}(s; 2j-1)\Gamma_{A_{\zeta n+1} \cdots A_{\zeta 2s}}^{l_\zeta}(s - \frac{n}{2}; 2j-1) \end{cases}$$

相等性:

$$\text{性质2.8.1. } N_{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}^{k'_\zeta}(s; 2j-1) \simeq N_{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j-1) \simeq N_{k_\zeta}^{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}(s; 2j-1) \simeq N_{k'_\zeta}^{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}(s; 2j-1)$$

$$\text{性质2.8.2. } [N_{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j-1)]^* \simeq N_{A'_{\zeta 1} \cdots A'_{\zeta n} l'_\zeta}^{k'_\zeta}(s; 2j-1), [N_{k_\zeta}^{A_{\zeta 1} \cdots A_{\zeta n} l_\zeta}(s; 2j-1)]^* \simeq N_{k'_\zeta}^{A'_{\zeta 1} \cdots A'_{\zeta n} l'_\zeta}(s; 2j-1)$$

展开性:

性质2.8.3.

$$\begin{cases} \Gamma_{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta n} l_\zeta}^{k_\zeta}(s; 2j-1) = N_{A_{\zeta 1} l_{\zeta 1}}^{k_\zeta}(s; 2j-1)N_{A_{\zeta 2} l_{\zeta 2}}^{l_{\zeta 1}}(s - \frac{1}{2}; 2j-1) \cdots N_{A_{\zeta n-1} l_{\zeta n-1}}^{l_{\zeta n-2}}(s - \frac{n-1}{2}; 2j-1) \\ N_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta n} l_\zeta}(s; 2j-1) = N_{k_\zeta}^{A_{\zeta 1} l_{\zeta 1}}(s; 2j-1)N_{l_{\zeta 1}}^{A_{\zeta 2} l_{\zeta 2}}(s - \frac{1}{2}; 2j-1) \cdots N_{l_{\zeta n-1}}^{A_{\zeta n} l_{\zeta n}}(s - \frac{n-1}{2}; 2j-1) \end{cases}$$

性质2.8.4.

$$\begin{cases} \Gamma_{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}^{k_\zeta}(s; 2j-1) = N_{A_{\zeta 1} l_{\zeta 1}}^{k_\zeta}(s; 2j-1)N_{A_{\zeta 2} l_{\zeta 2}}^{l_{\zeta 1}}(s - \frac{1}{2}; 2j-1) \cdots N_{A_{\zeta 2s-1} l_{\zeta 2s-1}}^{l_{\zeta 2s-2}}(\frac{1}{2}; 2j-1) \\ \Gamma_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}(s; 2j-1) = N_{k_\zeta}^{A_{\zeta 1} l_{\zeta 1}}(s; 2j-1)N_{l_{\zeta 1}}^{A_{\zeta 2} l_{\zeta 2}}(s - \frac{1}{2}; 2j-1) \cdots N_{l_{\zeta 2s-1}}^{A_{\zeta 2s} l_{\zeta 2s}}(\frac{1}{2}; 2j-1) \\ \Gamma_{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}^{k_\zeta}(s; 2j-1) \succ \Gamma_{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}(s; 2j-1) = N_{A_{\zeta 1}}(s; 2j-1)N_{A_{\zeta 2}}(s - \frac{1}{2}; 2j-1) \cdots N_{A_{\zeta 2s}}(\frac{1}{2}; 2j-1) \\ \Gamma_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}(s; 2j-1) \succ \Gamma_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \cdots A_{\zeta 2s}}(s; 2j-1) = N^{A_{\zeta 1}}(s; 2j-1)N^{A_{\zeta 2}}(s - \frac{1}{2}; 2j-1) \cdots N^{A_{\zeta 2s}}(\frac{1}{2}; 2j-1) \\ \bar{\Gamma}(s; 2j-1) = \bar{N}(s; 2j-1)[I_{2j} \otimes \bar{N}(s - \frac{1}{2}; 2j-1)] \cdots [I_{(2j)^{2s-2}} \otimes \bar{N}(1)][I_{(2j)^{2s-1}} \otimes \bar{N}(\frac{1}{2}; 2j-1)] \\ \Gamma(s; 2j-1) = [I_{(2j)^{2s-1}} \otimes N(\frac{1}{2}; 2j-1)][I_{(2j)^{2s-2}} \otimes N(1)] \cdots [I_{2j} \otimes N(s - \frac{1}{2}; 2j-1)]N(s; 2j-1) \end{cases}$$

2.9 推论1: 常数矩阵 $N(s; 2j-1)$ ,  $\bar{N}(s; 2j-1)$ 的几个恒等式

性质2.9.1.

$$\begin{cases} \bar{N}(s; 2j-1)[[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)]N(s; 2j-1) = \sigma(s; 2j-1) \\ N(s; 2j-1)\sigma(s; 2j-1)\bar{N}(s; 2j-1) = [[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)]N(s; 2j-1)\bar{N}(s; 2j-1) \\ N(s; 2j-1)\sigma(s; 2j-1)\bar{N}(s; 2j-1) = N(s; 2j-1)\bar{N}(s; 2j-1)\{\vartheta \cdot [[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)]\}^n \\ [N(s; 2j-1)\bar{N}(s; 2j-1), [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)] = 0 \end{cases}$$

性质2.9.2.

$$\begin{cases} \bar{N}(s; 2j-1)[S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)]N(s; 2j-1) = S_{ab}(s, \zeta; 2j-1) \\ N(s; 2j-1)S_{ab}(s, \zeta; 2j-1)\bar{N}(s; 2j-1) = [S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)]N(s; 2j-1)\bar{N}(s; 2j-1) \\ N(s; 2j-1)S_{ab}(s, \zeta; 2j-1)\bar{N}(s; 2j-1) = N(s; 2j-1)\bar{N}(s; 2j-1)[S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)] \\ [N(s; 2j-1)\bar{N}(s; 2j-1), S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)] = 0 \end{cases}$$

## 性质2.9.3.

$$\begin{cases} \bar{N}(s; 2j-1) \{ \vartheta \cdot [ [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1) ] \}^n N(s; 2j-1) = [\vartheta \cdot \sigma(s; 2j-1)]^n \\ N(s; 2j-1) [\vartheta \cdot \sigma(s; 2j-1)]^n \bar{N}(s; 2j-1) = \{ \vartheta \cdot [ [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1) ] \}^n N(s; 2j-1) \bar{N}(s; 2j-1) \\ N(s; 2j-1) [\vartheta \cdot \sigma(s; 2j-1)]^n \bar{N}(s; 2j-1) = N(s; 2j-1) \bar{N}(s; 2j-1) \{ \vartheta \cdot [ [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1) ] \}^n \\ [N(s; 2j-1) \bar{N}(s; 2j-1), \{ \vartheta \cdot [ [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1) ] \}^n] = 0 \end{cases}$$

## 性质2.9.4.

$$\begin{cases} \bar{N}(s; 2j-1) \{ \vartheta^{ab} [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)] \}^n N(s; 2j-1) = [\vartheta^{ab} S_{ab}(s, \varsigma; 2j-1)]^n \\ N(s; 2j-1) [\vartheta^{ab} S_{ab}(s, \varsigma; 2j-1)]^n \bar{N}(s; 2j-1) = \{ \vartheta^{ab} [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)] \}^n N(s; 2j-1) \bar{N}(s; 2j-1) \\ N(s; 2j-1) [\vartheta^{ab} S_{ab}(s, \varsigma; 2j-1)]^n \bar{N}(s; 2j-1) = N(s; 2j-1) \bar{N}(s; 2j-1) \{ \vartheta^{ab} [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)] \}^n \\ [N(s; 2j-1) \bar{N}(s; 2j-1), \{ \vartheta^{ab} [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)] \}^n] = 0 \end{cases}$$

## 推论2.9.1.

$$\begin{cases} \bar{N}(s; 2j-1) e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)]} N(s; 2j-1) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; 2j-1)} \\ N(s; 2j-1) e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; 2j-1)} \bar{N}(s; 2j-1) = e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)]} N(s; 2j-1) \bar{N}(s; 2j-1) \\ N(s; 2j-1) e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; 2j-1)} \bar{N}(s; 2j-1) = N(s; 2j-1) \bar{N}(s; 2j-1) e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)]} \\ [N(s; 2j-1) \bar{N}(s; 2j-1), e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)]}] = 0 \end{cases}$$

2.10 推论2: 常数矩阵  $N(s; 2j-1)$ ,  $\bar{N}(s; 2j-1)$  的另外几个恒等式

## 推论2.10.1.

$$\begin{cases} \bar{N}(s; 2j-1) [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} N(s; 2j-1) = \frac{1}{2s} \sigma(s; 2j-1) \\ \bar{N}(s; 2j-1) I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1) N(s; 2j-1) = (1 - \frac{1}{2s}) \sigma(s; 2j-1) \\ N^{A_\varsigma}(s; 2j-1) \sigma(s - \frac{1}{2}; 2j-1) \bar{N}_{A_\varsigma}(s; 2j-1) = (1 - \frac{1}{2s}) \sigma(s; 2j-1) \\ \bar{N}_{A_\varsigma}(s; 2j-1) \sigma(s; 2j-1) N^{A_\varsigma}(s; 2j-1) = (1 + \frac{2j}{2s}) \sigma(s - \frac{1}{2}; 2j-1) \end{cases}$$

## 推论2.10.2.

$$\begin{cases} \bar{N}(s; 2j-1) S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} N(s; 2j-1) = \frac{1}{2s} S_{ab}(s, \varsigma; 2j-1) \\ \bar{N}(s; 2j-1) I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1) N(s; 2j-1) = (1 - \frac{1}{2s}) S_{ab}(s, \varsigma; 2j-1) \\ N^{A_\varsigma}(s; 2j-1) S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1) \bar{N}_{A_\varsigma}(s; 2j-1) = (1 - \frac{1}{2s}) S_{ab}(s, \varsigma; 2j-1) \\ \bar{N}_{A_\varsigma}(s; 2j-1) S_{ab}(s, \varsigma; 2j-1) N^{A_\varsigma}(s; 2j-1) = (1 + \frac{2j}{2s}) S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1) \end{cases}$$

## 推论2.10.3.

$$\begin{cases} \bar{N}(1) [ [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_2 + I_{2j} \otimes [ [\sigma(\frac{1}{2}) \otimes I_j] ] ] N(1) = \sigma(1) \\ \bar{N}(\frac{3}{2}) \{ [ [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_3 + I_{2j} \otimes \{ \bar{N}(1) [ [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_2 + I_{2j} \otimes [ [\sigma(\frac{1}{2}) \otimes I_j] ] ] N(1) \} \} \} N(\frac{3}{2}) = \sigma(\frac{3}{2}) \\ \bar{N}(s; 2j-1) \cdot \bar{N}(\frac{3}{2}) \{ [ [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_3 + I_{2j} \otimes \{ \bar{N}(1) [ [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_2 + I_{2j} \otimes [ [\sigma(\frac{1}{2}) \otimes I_j] ] ] N(1) \} \} \} N(\frac{3}{2}) \cdot N(s; 2j-1) = \sigma(s) \end{cases}$$

2.11 矩阵  $N(s; 2j-1)$ ,  $\bar{N}(s; 2j-1)$  的常数不变张量性质

$$\text{定理2.11.1. } N(s; 2j-1) = e^{(i\omega + \varsigma\epsilon) \cdot [ [\sigma(\frac{1}{2}) \otimes I_j] ]} \otimes e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2}; 2j-1)} N(s; 2j-1) e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(s; 2j-1)}$$

$$\text{定理2.11.2. } \bar{N}(s; 2j-1) = e^{(i\omega + \varsigma\epsilon) \cdot \sigma(s; 2j-1)} \bar{N}(s; 2j-1) e^{-(i\omega + \varsigma\epsilon) \cdot [ [\sigma(\frac{1}{2}) \otimes I_j] ]} \otimes e^{-(i\omega + \varsigma\epsilon) \cdot \sigma(s - \frac{1}{2}; 2j-1)}$$

3 多重完美常数不变张量  $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j-1)$ ,  $X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j-1)$ 3.1 完美常数不变张量  $X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j-1)$ ,  $X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j-1)$ 

$$\text{定义3.1.1. } X_{m_\varsigma}^{A_\varsigma l_\varsigma}(s; 2j-1) := \frac{\sqrt{2s-1}}{\sqrt{2s-1+2j-1}} (\epsilon \otimes I_j)^{A_\varsigma B_\varsigma} N_{B_\varsigma m_\varsigma}^{l_\varsigma}(s - \frac{1}{2}; 2j-1), X_{A_\varsigma l_\varsigma}^{m_\varsigma}(s; 2j-1) := \frac{\sqrt{2s-1}}{\sqrt{2s-1+2j-1}} (\epsilon \otimes I_j)_{A_\varsigma B_\varsigma} N_{l_\varsigma}^{B_\varsigma m_\varsigma}(s - \frac{1}{2}; 2j-1)$$

性质3.1.1.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) \simeq X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$

### 3.2 常数矩阵 $X(s; 2j-1), \bar{X}(s; 2j-1)$

定义3.2.1.

$$\begin{cases} X^{A_\zeta}(s; 2j-1) \prec X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta}(s; 2j-1) \prec X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1) \\ \bar{X}_{A_\zeta}(s; 2j-1) \prec X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1), \bar{X}^{A_\zeta}(s; 2j-1) \prec X^{A_\zeta l_\zeta m_\zeta}(s; 2j-1) \\ X(s; 2j-1) \prec X_{A_\zeta \otimes l_\zeta}^{m_\zeta}(s; 2j-1), \bar{X}(s; 2j-1) \prec X_{m_\zeta}^{A_\zeta \otimes l_\zeta}(s; 2j-1) = X^+(s; 2j-1) \end{cases}$$

### 3.3 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$ 的升降指标

性质3.3.1.

$$\begin{cases} X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) = (\varepsilon \otimes I_j)^{A_\zeta B_\zeta} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; 2j-1) \varepsilon_{m_\zeta r_\zeta}(s-1; 2j-1) X_{B_\zeta n_\zeta}^{r_\zeta}(s - \frac{1}{2}; 2j-1) \\ X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1) = (\varepsilon \otimes I_j)_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; 2j-1) \varepsilon^{m_\zeta r_\zeta}(s-1; 2j-1) X_{r_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2}; 2j-1) \end{cases}$$

### 3.4 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$ 的正交性

性质3.4.1.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) X_{A_\zeta l_\zeta}^{n_\zeta}(s; 2j-1) = \delta_{m_\zeta}^{n_\zeta} [\Leftrightarrow] X^{A_\zeta}(s; 2j-1) \bar{X}_{A_\zeta}(s; 2j-1) = I_{C_{2s-2+2j-1}^{2s-2}} [\Leftrightarrow] \bar{X}(s; 2j-1) X(s; 2j-1) = I_{C_{2s-2+2j-1}^{2s-2}}$

性质3.4.2.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) N_{A_\zeta l_\zeta}^{k_\zeta}(s; 2j-1) = 0$

$[\Leftrightarrow] X^{A_\zeta}(s; 2j-1) \bar{N}_{A_\zeta}(s; 2j-1) = 0, N_{A_\zeta}(s; 2j-1) \bar{X}^{A_\zeta}(s; 2j-1) = 0 [\Leftrightarrow] \bar{X}(s; 2j-1) N(s; 2j-1) = 0, \bar{N}(s; 2j-1) X(s; 2j-1) = 0$

性质3.4.3.  $X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1) X_{m_\zeta}^{A_\zeta k_\zeta}(s; 2j-1) = \frac{2s-1}{2s-1+2j-1} \delta_{l_\zeta}^{k_\zeta} [\Leftrightarrow] \bar{X}_{A_\zeta}(s; 2j-1) X^{A_\zeta}(s; 2j-1) = \frac{2s-1}{2s-1+2j-1} I_{C_{2s-1+2j-1}^{2s-1}}$

性质3.4.4.  $X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1) X_{m_\zeta}^{B_\zeta l_\zeta}(s; 2j-1) = \frac{1}{2j} C_{2s-2+2j-1}^{2s-2} \delta_{A_\zeta}^{B_\zeta} [\Leftrightarrow] \text{tr}[\bar{X}_{A_\zeta}(s; 2j-1) X^{B_\zeta}(s; 2j-1)] = \frac{1}{2j} C_{2s-2+2j-1}^{2s-2} \delta_{A_\zeta}^{B_\zeta}$

推论3.4.1.  $\bar{N}(s; 2j-1) N(s; 2j-1) = I_{C_{2s+2j-1}^{2s-2}}, \bar{X}(s; 2j-1) X(s; 2j-1) = I_{C_{2s-2+2j-1}^{2s-2}}, \bar{N}(s; 2j-1) X(s; 2j-1) = 0, \bar{X}(s; 2j-1) N(s; 2j-1) = 0$

### 3.5 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$ 的自旋变换

推论3.5.1.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) \sigma_{A_\zeta}^{\alpha_\zeta} X_{B_\zeta l_\zeta}^{n_\zeta}(s; 2j-1) = -\frac{1}{2s-1+2j-1} \sigma_{m_\zeta}^{\alpha_\zeta} X_{B_\zeta n_\zeta}^{n_\zeta}(s-1; 2j-1)$

$[\Leftrightarrow] X^{A_\zeta}(s; 2j-1) \sigma_{A_\zeta}^{B_\zeta}(j) \bar{X}_{B_\zeta}(s; 2j-1) = -\frac{1}{2s-1+2j-1} \sigma(s-1; 2j-1)$

$[\Leftrightarrow] \bar{X}(s; 2j-1) [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} X(s; 2j-1) = -\frac{1}{2s-1+2j-1} \sigma(s-1; 2j-1)$

推论3.5.2.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) \sigma_{A_\zeta}^{\alpha_\zeta} X_{B_\zeta k_\zeta}^{m_\zeta}(s; 2j-1) = -\frac{1}{2s-1+2j-1} \sigma_{k_\zeta}^{\alpha_\zeta} X_{B_\zeta l_\zeta}^{l_\zeta}(s - \frac{1}{2}; 2j-1)$

$[\Leftrightarrow] \bar{X}^{A_\zeta}(s; 2j-1) \sigma_{A_\zeta}^{B_\zeta}(j) X_{B_\zeta}(s; 2j-1) = -\frac{1}{2s-1+2j-1} \sigma_{k_\zeta}^{\alpha_\zeta}(s - \frac{1}{2}; 2j-1)$

推论3.5.3.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) S_{ab A_\zeta}^{B_\zeta}(j) X_{B_\zeta l_\zeta}^{n_\zeta}(s; 2j-1) = -\frac{1}{2s-1+2j-1} S_{ab m_\zeta}^{n_\zeta}(s-1; 2j-1)$

$[\Leftrightarrow] X^{A_\zeta}(s; 2j-1) S_{ab}(j, \varsigma) \otimes I_{C_{2s-2+2j-1}^{2s-2}} \bar{X}_{A_\zeta}(s; 2j-1) = -\frac{1}{2s-1+2j-1} S_{ab}(s-1, \varsigma; 2j-1)$

$[\Leftrightarrow] \bar{X}(s; 2j-1) S_{ab}(j, \varsigma) \otimes I_{C_{2s-1+2j-1}^{2s-1}} X(s; 2j-1) = -\frac{1}{2s-1+2j-1} S_{ab}(s-1, \varsigma; 2j-1)$

推论3.5.4.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) S_{ab A_\zeta}^{B_\zeta}(j) X_{B_\zeta k_\zeta}^{m_\zeta}(s; 2j-1) = -\frac{1}{2s} S_{ab k_\zeta}^{l_\zeta}(s - \frac{1}{2}; 2j-1)$

$[\Leftrightarrow] \bar{X}^{A_\zeta}(s; 2j-1) S_{ab A_\zeta}^{B_\zeta}(j) X_{B_\zeta}(s; 2j-1) = -\frac{1}{2s-1+2j-1} S_{ab}(s - \frac{1}{2}, \varsigma; 2j-1)$

### 3.6 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1), X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)$ 的置换性质

定理3.6.1.

$$\begin{cases} X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1) [\sigma_{A_\zeta}^{B_\zeta}(j) \delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{l_\zeta}^{k_\zeta}(s - \frac{1}{2}; 2j-1)] = \sigma_{m_\zeta}^{n_\zeta}(s-1; 2j-1) X_{n_\zeta}^{B_\zeta k_\zeta}(s; 2j-1) \\ [\sigma_{A_\zeta}^{B_\zeta}(j) \delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{l_\zeta}^{k_\zeta}(s - \frac{1}{2}; 2j-1)] X_{B_\zeta k_\zeta}^{n_\zeta}(s; 2j-1) = X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1) \sigma_{m_\zeta}^{n_\zeta}(s-1; 2j-1) \end{cases}$$



$$\begin{cases} X^{A_\zeta}(s; 2j-1)[\sigma_{A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta} \sigma(s - \frac{1}{2}; 2j-1)] = \sigma(s-1; 2j-1)X^{B_\zeta}(s; 2j-1) \\ [\sigma_{A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta} \sigma(s - \frac{1}{2}; 2j-1)]\bar{X}_{B_\zeta}(s; 2j-1) = \bar{X}_{A_\zeta}(s; 2j-1)\sigma(s-1; 2j-1) \\ \bar{X}(s; 2j-1)[[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)] = \sigma(s-1; 2j-1)\bar{X}(s; 2j-1) \\ [[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)]X(s; 2j-1) = X(s; 2j-1)\sigma(s-1; 2j-1) \end{cases}$$

**定理3.6.2.**

$$\begin{cases} X_{m_\zeta}^{A_\zeta l_\zeta}(s; 2j-1)[S_{ab A_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; 2j-1)] = S_{ab m_\zeta}^{n_\zeta}(s-1; 2j-1)X_{n_\zeta}^{B_\zeta \otimes k_\zeta}(s; 2j-1) \\ [S_{ab A_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; 2j-1)]X_{B_\zeta k_\zeta}^{n_\zeta}(s; 2j-1) = X_{A_\zeta l_\zeta}^{m_\zeta}(s; 2j-1)S_{ab m_\zeta}^{n_\zeta}(s-1; 2j-1) \\ X^{A_\zeta}(s; 2j-1)[S_{ab A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta} S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)] = S_{ab}(s-1, \zeta; 2j-1)X^{B_\zeta}(s; 2j-1) \\ [S_{ab A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta} S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)]\bar{X}_{B_\zeta}(s; 2j-1) = \bar{X}_{A_\zeta}(s; 2j-1)S_{ab}(s-1, \zeta; 2j-1) \\ \bar{X}(s; 2j-1)[S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)] = S_{ab}(s-1, \zeta; 2j-1)\bar{X}(s; 2j-1) \\ [S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)]X(s; 2j-1) = X(s; 2j-1)S_{ab}(s-1, \zeta; 2j-1) \end{cases}$$

**推论3.6.1.**

$$\begin{cases} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; 2j-1)\varepsilon^{C_\zeta A_\zeta}(j)[\sigma_{A_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{l_\zeta}^{k_\zeta}(s - \frac{1}{2}; 2j-1)] = \sigma_{m_\zeta}^{n_\zeta}(s-1; 2j-1)N_{D_\zeta n_\zeta}^{k_\zeta}(s - \frac{1}{2}; 2j-1)\varepsilon^{D_\zeta B_\zeta}(j) \\ [\sigma_{A_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} \sigma_{l_\zeta}^{k_\zeta}(s - \frac{1}{2}; 2j-1)]\varepsilon_{B_\zeta C_\zeta}(j)N_{k_\zeta}^{C_\zeta n_\zeta}(s - \frac{1}{2}; 2j-1) = \varepsilon_{A_\zeta D_\zeta}(j)N_{l_\zeta}^{D_\zeta m_\zeta}(s - \frac{1}{2}; 2j-1)\sigma_{m_\zeta}^{n_\zeta}(s-1; 2j-1) \\ N_{C_\zeta}(s - \frac{1}{2}; 2j-1)\varepsilon^{C_\zeta A_\zeta}(j)[\sigma_{A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta} \sigma(s - \frac{1}{2}; 2j-1)] = \sigma(s-1; 2j-1)N_{D_\zeta}(s - \frac{1}{2}; 2j-1)\varepsilon^{D_\zeta B_\zeta}(j) \\ [\sigma_{A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta} \sigma(s - \frac{1}{2}; 2j-1)]\varepsilon_{B_\zeta C_\zeta}(j)N^{C_\zeta}(s - \frac{1}{2}; 2j-1) = \varepsilon_{A_\zeta D_\zeta}(j)N^{D_\zeta}(s - \frac{1}{2}; 2j-1)\sigma(s-1; 2j-1) \end{cases}$$

**推论3.6.2.**

$$\begin{cases} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; 2j-1)\varepsilon^{C_\zeta A_\zeta}(j)[S_{ab A_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; 2j-1)] = S_{ab m_\zeta}^{n_\zeta}(s-1; 2j-1)N_{D_\zeta n_\zeta}^{k_\zeta}(s - \frac{1}{2}; 2j-1) \\ [S_{ab A_\zeta}^{B_\zeta}(j)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta} S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; 2j-1)]\varepsilon_{B_\zeta C_\zeta}(j)N_{k_\zeta}^{C_\zeta n_\zeta}(s - \frac{1}{2}; 2j-1) = \varepsilon_{A_\zeta D_\zeta}(j)N_{l_\zeta}^{D_\zeta m_\zeta}(s - \frac{1}{2}; 2j-1)S_{ab m_\zeta}^{n_\zeta}(s-1; 2j-1) \\ N_{C_\zeta}(s - \frac{1}{2}; 2j-1)\varepsilon^{C_\zeta A_\zeta}(j)[S_{ab A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta} S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)] = S_{ab}(s-1, \zeta; 2j-1)N_{D_\zeta}(s - \frac{1}{2}; 2j-1)\varepsilon^{D_\zeta B_\zeta}(j) \\ [S_{ab A_\zeta}^{B_\zeta}(j) + \delta_{A_\zeta}^{B_\zeta} S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)]\varepsilon_{B_\zeta C_\zeta}(j)N^{C_\zeta}(s - \frac{1}{2}; 2j-1) = \varepsilon_{A_\zeta D_\zeta}(j)N^{D_\zeta}(s - \frac{1}{2}; 2j-1)S_{ab}(s-1, \zeta; 2j-1) \end{cases}$$

### 3.7 推论: 关于常数矩阵 $X(s; 2j-1)$ , $\bar{X}(s; 2j-1)$ 的重要性质

**推论3.7.1.**

$$\begin{cases} \bar{X}(s; 2j-1)[[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)]X(s; 2j-1) = \sigma(s-1; 2j-1) \\ X(s; 2j-1)\sigma(s-1; 2j-1)\bar{X}(s; 2j-1) = [[\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)]X(s; 2j-1)\bar{X}(s; 2j-1) \\ [X(s; 2j-1)\bar{X}(s; 2j-1), [\sigma(\frac{1}{2}) \otimes I_j] \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)] = 0 \end{cases}$$

**推论3.7.2.**

$$\begin{cases} \bar{X}(s; 2j-1)[S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)]X(s; 2j-1) = S_{ab}(s-1, \zeta; 2j-1) \\ X(s; 2j-1)S_{ab}(s, \zeta-1, \zeta; 2j-1)\bar{X}(s; 2j-1) = [S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)]X(s; 2j-1)\bar{X}(s; 2j-1) \\ [X(s; 2j-1)\bar{X}(s; 2j-1), S_{ab}(j, \zeta) \otimes I_{C_{2s-1+2j-1}^{2s-1}} + I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)] = 0 \end{cases}$$

**推论3.7.3.**  $X^{A_\zeta}(s; 2j-1)\sigma(s - \frac{1}{2}; 2j-1)\bar{X}_{A_\zeta}(s; 2j-1) = \frac{2s+2j-1}{2s-1+2j-1}\sigma(s-1; 2j-1)$

$[\Leftrightarrow]\bar{X}(s; 2j-1)I_{2j} \otimes \sigma(s - \frac{1}{2}; 2j-1)X(s; 2j-1) = \frac{2s+2j-1}{2s-1+2j-1}\sigma(s-1; 2j-1)$

**推论3.7.4.**  $X^{A_\zeta}(s; 2j-1)I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)\bar{X}_{A_\zeta}(s; 2j-1) = \frac{2s+2j-1}{2s-1+2j-1}S_{ab}(s-1, \zeta; 2j-1)$

$[\Leftrightarrow]\bar{X}(s; 2j-1)I_{2j} \otimes S_{ab}(s - \frac{1}{2}, \zeta; 2j-1)X(s; 2j-1) = \frac{2s+2j-1}{2s-1+2j-1}S_{ab}(s-1, \zeta; 2j-1)$

### 3.8 矩阵 $X(s; 2j-1)$ , $\bar{X}(s; 2j-1)$ 的常数不变张量性质

**定理3.8.1.**  $X(s; 2j-1) = e^{(i\omega+\zeta\varepsilon)\cdot[\sigma(\frac{1}{2})\otimes I_j]} \otimes e^{(i\omega+\zeta\varepsilon)\cdot\sigma(s-\frac{1}{2}; 2j-1)} X(s; 2j-1) e^{-(i\omega+\zeta\varepsilon)\cdot\sigma(s-1; 2j-1)}$

**定理3.8.2.**  $\bar{X}(s; 2j-1) = e^{(i\omega+\zeta\varepsilon)\cdot\sigma(s-1; 2j-1)} \bar{X}(s; 2j-1) e^{-(i\omega+\zeta\varepsilon)\cdot[\sigma(\frac{1}{2})\otimes I_j]} \otimes e^{(i\omega+\zeta\varepsilon)\cdot\sigma(s-\frac{1}{2}; 2j-1)}$

### 3.9 常数矩阵 $\Omega(s; 2j-1), \sigma(s-1; 2j-1)$ 的置换性质

$$\text{推论3.9.1. } \begin{cases} \Omega(s; 2j-1)[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)]X(s; 2j-1) = [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)]X(s; 2j-1)\sigma(s-1; 2j-1) \\ \bar{X}(s; 2j-1)[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)]\Omega(s; 2j-1) = \sigma(s-1; 2j-1)\bar{X}(s; 2j-1)[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)] \end{cases}$$

$$\text{推论3.9.2. } \begin{cases} \sigma(s; 2j-1) = \bar{N}(s; 2j-1)[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)]\Omega(s; 2j-1)[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)]N(s; 2j-1) \\ \sigma(s-1; 2j-1) = \bar{X}(s; 2j-1)[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)]\Omega(s; 2j-1)[I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)]X(s; 2j-1) \end{cases}$$

$$\text{推论3.9.3. } \begin{cases} [\vec{\vartheta} \cdot \sigma(s; 2j-1)]^n = \bar{N}(s; 2j-1)[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)][\vec{\vartheta} \cdot \Omega(s; 2j-1)]^n [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)]N(s; 2j-1) \\ [\vec{\vartheta} \cdot \sigma(s-1; 2j-1)]^n = \bar{X}(s; 2j-1)[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)][\vec{\vartheta} \cdot \Omega(s; 2j-1)]^n [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)]X(s; 2j-1) \end{cases}$$

$$\text{推论3.9.4. } \begin{cases} e^{\vec{\vartheta} \cdot \sigma(s; 2j-1)} = \bar{N}(s; 2j-1)[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)]e^{\vec{\vartheta} \cdot \Omega(s; 2j-1)} [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)]N(s; 2j-1) \\ e^{\vec{\vartheta} \cdot \sigma(s-1; 2j-1)} = \bar{X}(s; 2j-1)[I_{2j} \otimes \bar{\Gamma}(s - \frac{1}{2}; 2j-1)]e^{\vec{\vartheta} \cdot \Omega(s; 2j-1)} [I_{2j} \otimes \Gamma(s - \frac{1}{2}; 2j-1)]X(s; 2j-1) \end{cases}$$

### 3.10 常数矩阵 $\Omega(s-l; 2j-1), [\vec{\vartheta} \cdot \Omega(s-l; 2j-1)]^n, e^{\vec{\vartheta} \cdot \Omega(s-l; 2j-1)}$ 的同构性表示

$$\text{推论3.10.1. } \Omega(s; 2j-1) = \Omega(s-1; 2j-1) \otimes I_{(2j)^2} + I_{(2j)^{2s-2}} \otimes \Omega(1; 2j-1)$$

推论3.10.2.

$$\begin{cases} \Omega(s; 2j-1)I_{(2j)^{2s-2}} \otimes \{[I_{2j} \otimes \Gamma(\frac{1}{2}; 2j-1)]X(1; 2j-1)\} = I_{(2j)^{2s-2}} \otimes \{[I_{2j} \otimes \Gamma(\frac{1}{2}; 2j-1)]X(1; 2j-1)\}\Omega(s-1; 2j-1) \\ I_{(2j)^{2s-2}} \otimes \{\bar{X}(1; 2j-1)[I_{2j} \otimes \bar{\Gamma}(\frac{1}{2}; 2j-1)]\}\Omega(s; 2j-1) = \Omega(s-1; 2j-1)I_{(2j)^{2s-2}} \otimes \{\bar{X}(1; 2j-1)[I_{2j} \otimes \bar{\Gamma}(\frac{1}{2}; 2j-1)]\} \end{cases}$$

推论3.10.3.

$$\begin{cases} \Omega(s-1; 2j-1) = I_{(2j)^{2s-2}} \otimes \{\bar{X}(1; 2j-1)[I_{2j} \otimes \bar{\Gamma}(\frac{1}{2}; 2j-1)]\}\Omega(s; 2j-1)I_{(2j)^{2s-2}} \otimes \{[I_{2j} \otimes \Gamma(\frac{1}{2}; 2j-1)]X(1; 2j-1)\} \\ [\vec{\vartheta} \cdot \Omega(s-1; 2j-1)]^n = I_{(2j)^{2s-2}} \otimes \{\bar{X}(1; 2j-1)[I_{2j} \otimes \bar{\Gamma}(\frac{1}{2}; 2j-1)]\}[\vec{\vartheta} \cdot \Omega(s; 2j-1)]^n I_{(2j)^{2s-2}} \otimes \{[I_{2j} \otimes \Gamma(\frac{1}{2}; 2j-1)]X(1; 2j-1)\} \\ e^{\vec{\vartheta} \cdot \Omega(s-1; 2j-1)} = I_{(2j)^{2s-2}} \otimes \{\bar{X}(1; 2j-1)[I_{2j} \otimes \bar{\Gamma}(\frac{1}{2}; 2j-1)]\}e^{\vec{\vartheta} \cdot \Omega(s; 2j-1)} I_{(2j)^{2s-2}} \otimes \{[I_{2j} \otimes \Gamma(\frac{1}{2}; 2j-1)]X(1; 2j-1)\} \end{cases}$$

定义3.10.1.

$$\begin{cases} T(s; 2j-1) := I_{(2j)^{2s-2}} \otimes \{[I_{2j} \otimes \Gamma(\frac{1}{2}; 2j-1)]X(1; 2j-1)\} \\ \bar{T}(s; 2j-1) := I_{(2j)^{2s-2}} \otimes \{\bar{X}(1; 2j-1)[I_{2j} \otimes \bar{\Gamma}(\frac{1}{2}; 2j-1)]\} = T^+(s; 2j-1) \end{cases}$$

推论3.10.4.

$$\begin{cases} \Omega(s-l; 2j-1) = \bar{T}(s-l+1; 2j-1) \cdots \bar{T}(s-1; 2j-1)\bar{T}(s; 2j-1)\Omega(s; 2j-1)T(s; 2j-1)T(s-1; 2j-1) \cdots T(s-l+1; 2j-1) \\ [\vec{\vartheta} \cdot \Omega(s-l; 2j-1)]^n = \bar{T}(s-l+1; 2j-1) \cdots \bar{T}(s-1; 2j-1)\bar{T}(s; 2j-1)[\vec{\vartheta} \cdot \Omega(s; 2j-1)]^n T(s; 2j-1)T(s-1; 2j-1) \cdots T(s-l+1; 2j-1) \\ e^{\vec{\vartheta} \cdot \Omega(s-l; 2j-1)} = \bar{T}(s-l+1; 2j-1) \cdots \bar{T}(s-1; 2j-1)\bar{T}(s; 2j-1)e^{\vec{\vartheta} \cdot \Omega(s; 2j-1)} T(s; 2j-1)T(s-1; 2j-1) \cdots T(s-l+1; 2j-1) \end{cases}$$

推论3.10.5.

$$\begin{cases} \sigma(s-l; 2j-1) = \bar{\Gamma}(s-l; 2j-1)\bar{T}(s-l+1; 2j-1) \cdots \bar{T}(s; 2j-1)\Omega(s; 2j-1)T(s; 2j-1) \cdots T(s-l+1; 2j-1)\Gamma(s-l; 2j-1) \\ [\vec{\vartheta} \cdot \sigma(s-l; 2j-1)]^n = \bar{\Gamma}(s-l; 2j-1)\bar{T}(s-l+1; 2j-1) \cdots \bar{T}(s; 2j-1)[\vec{\vartheta} \cdot \Omega(s; 2j-1)]^n T(s; 2j-1) \cdots T(s-l+1; 2j-1)\Gamma(s-l; 2j-1) \\ e^{\vec{\vartheta} \cdot \sigma(s-l; 2j-1)} = \bar{\Gamma}(s-l; 2j-1)\bar{T}(s-l+1; 2j-1) \cdots \bar{T}(s; 2j-1)e^{\vec{\vartheta} \cdot \Omega(s; 2j-1)} T(s; 2j-1) \cdots T(s-l+1; 2j-1)\Gamma(s-l; 2j-1) \end{cases}$$

# 第四十一章 反对称完美常数不变张量

自我评述：本章类比前面章节全对称常数不变张量，建立了类似的反对称常数不变张量，得到很多类似的性质。并在最后给出它的一个具体应用，即反对称电磁张量的高旋量描述，从而验证了反对称常数不变张量法的正确性。

## 1 反对称指标排序

### 1.1 $w + 1$ 阶反对称指标正向排序规律的严格证明(从0排起)

定理1.1.1.  $\pi(\lambda_{2s} \leq \lambda_{2s-1} \cdots \leq \lambda_1; w) = \sum_{l=0}^{2s-1} (C_{w-\lambda_{l+2}}^{l+1} - C_{w+1-\lambda_{l+1}}^{l+1}), \lambda_{2s+1} := -1$

$$\begin{aligned}
 \text{证明: } \pi(\lambda_{2s} \leq \lambda_{2s-1} \cdots \leq \lambda_1; w) &= \pi(\lambda_{2s-1} \cdots \lambda_1 - (\lambda_{2s} + 1) \cdots (\lambda_{2s} + 1); w - \lambda_{2s} - 1) + \sum_{k=0}^{\lambda_{2s}-1} C_{w-k}^{2s-1} \\
 &= \pi(\lambda_{2s-2} \cdots \lambda_1 - (\lambda_{2s-1} + 1) \cdots (\lambda_{2s-1} + 1); w - \lambda_{2s-1} - 1) + \sum_{k=0}^{\lambda_{2s}-1} C_{w-k}^{2s-1} + \sum_{k=0}^{\lambda_{2s-1}-\lambda_{2s}-2} C_{w-\lambda_{2s}-1-k}^{2s-2} \\
 &= \pi(\lambda_1 - (\lambda_2 + 1); w - \lambda_2 - 1) + \sum_{k=0}^{\lambda_{2s}-1} C_{w-k}^{2s-1} + \sum_{k=0}^{\lambda_{2s-1}-\lambda_{2s}-2} C_{w-\lambda_{2s}-1-k}^{2s-2} \cdots + \sum_{k=0}^{\lambda_2-\lambda_3-2} C_{w-\lambda_3-1-k}^1 \\
 &= \sum_{k=0}^{\lambda_{2s}-1} C_{w-k}^{2s-1} + \sum_{k=0}^{\lambda_{2s-1}-\lambda_{2s}-2} C_{w-\lambda_{2s}-1-k}^{2s-2} \cdots + \sum_{k=0}^{\lambda_2-\lambda_3-2} C_{w-\lambda_3-1-k}^1 + \lambda_1 - \lambda_2 - 1 \\
 &= \sum_{l=1}^{2s-1} \sum_{k=0}^{\lambda_{l+1}-\lambda_{l+2}-2} C_{w-\lambda_{l+2}-1-k}^l + (\lambda_1 - \lambda_2 - 1); \lambda_{2s+1} := -1 \\
 &= \sum_{l=1}^{2s-1} \left( \sum_{k=0}^{\lambda_{l+1}-\lambda_{l+2}-1} C_{w-\lambda_{l+2}-1-k}^l - C_{w-\lambda_{l+1}}^l \right) + (\lambda_1 - \lambda_2 - 1) \\
 &= \sum_{l=0}^{2s-1} \left( \sum_{k=0}^{\lambda_{l+1}-\lambda_{l+2}-1} C_{w-\lambda_{l+2}-1-k}^l - C_{w-\lambda_{l+1}}^l \right) \\
 &= \sum_{l=0}^{2s-1} (C_{w-\lambda_{l+2}}^{l+1} - C_{w+1-\lambda_{l+1}}^{l+1}) \quad \square
 \end{aligned}$$

$$\begin{aligned}
 \text{证明: } \pi[(w - 2s + 1) \leq (w - 2s + 2) \cdots \leq w; w] &= \sum_{l=0}^{2s-1} (C_{w-\lambda_{l+2}}^{l+1} - C_{w+1-\lambda_{l+1}}^{l+1}) \\
 &= (C_{w-\lambda_{2s+1}}^{2s} - C_{w+1-\lambda_{2s}}^{2s}) + \sum_{l=0}^{2s-2} (C_{w-(w-l-1)}^{l+1} - C_{w+1-(w-l)}^{l+1}) \\
 &= C_{w-\lambda_{2s+1}}^{2s} - C_{w+1-\lambda_{2s}}^{2s} \\
 &= C_{w+1}^{2s} - C_{w+1-(w-2s+1)}^{2s} \\
 &= C_{w+1}^{2s} - 1 \quad \square
 \end{aligned}$$

#### 1.1.1 四阶反对称指标正向排序规律的验证

性质1.1.1.  $0 : 0 \rightarrow 1 : 1 \rightarrow 2 : 2 \rightarrow 3 : 3$

性质1.1.2.  $0 : 01 \rightarrow 1 : 02 \rightarrow 2 : 03 \rightarrow 3 : 12 \rightarrow 4 : 13 \rightarrow 5 : 23$

性质1.1.3.  $0 : 012 \rightarrow 1 : 013 \rightarrow 2 : 023 \rightarrow 3 : 123$

性质1.1.4.  $0 : 0123$

#### 1.1.2 五阶反对称指标正向排序规律的验证

性质1.1.5.  $0 : 0 \rightarrow 1 : 1 \rightarrow 2 : 2 \rightarrow 3 : 3 \rightarrow 4 : 4 \rightarrow$

性质1.1.6.  $0 : 01 \rightarrow 1 : 02 \rightarrow 2 : 03 \rightarrow 3 : 04 \rightarrow 4 : 12 \rightarrow 5 : 13 \rightarrow 6 : 14 \rightarrow 7 : 23 \rightarrow 8 : 24 \rightarrow 9 : 34$

性质1.1.7.  $0 : 012 \rightarrow 1 : 013 \rightarrow 2 : 014 \rightarrow 3 : 023 \rightarrow 4 : 024 \rightarrow 5 : 034 \rightarrow 6 : 123 \rightarrow 7 : 124 \rightarrow 8 : 134 \rightarrow 9 : 234$

性质1.1.8.  $0 : 0123 \rightarrow 1 : 0124 \rightarrow 2 : 0134 \rightarrow 3 : 0234 \rightarrow 4 : 1234$

性质1.1.9.  $0 : 01234$

## 2 完美常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)$

### 2.1 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)$ 的引入

定义2.1.1.  $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) = \frac{1}{(2s)!} \Gamma_{[A_\zeta B_\zeta C_\zeta \dots]}^{k_\zeta}(s; w)$

$$\Gamma_{\underbrace{0_\zeta}_{l_0} \underbrace{1_\zeta}_{l_1} \dots \underbrace{w_\zeta}_{l_w}}^{k_\zeta}(s; w) = \sqrt{\frac{l_0! l_1! \dots l_w!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^{2s-1} (C_{w-\lambda_{l+2}}^{l+1} - C_{w+1-\lambda_{l+1}}^{l+1})\}, l_0 + \dots + l_w = 2s, l_j = 0|1$$

定义2.1.2.  $\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w) = \frac{1}{(2s)!} \Gamma_{[A_\zeta B_\zeta C_\zeta \dots]}^{k_\zeta}(s; w)$

$$\Gamma_{k_\zeta}^{\underbrace{l_0}_{0_\zeta} \underbrace{l_1}_{1_\zeta} \dots \underbrace{l_w}_{w_\zeta}}(s; w) = \sqrt{\frac{l_0! l_1! \dots l_w!}{(2s)!}} \delta\{k_\zeta, \sum_{l=0}^{2s-1} (C_{w-\lambda_{l+2}}^{l+1} - C_{w+1-\lambda_{l+1}}^{l+1})\}, l_0 + \dots + l_w = 2s, l_j = 0|1$$

推论2.1.1.  $[A_\zeta] = w + 1, [k_\zeta(s)] = C_{w+1}^{2s}, \text{Max}(C_{w+1}^{2s}) = C_{w+1}^{[(w+1)/2]}$

### 2.2 定义的解读

上述定义本质上是对所有  $\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}$  先进行排序编号。然后对每一个  $\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}$  赋值，即先排成  $\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}$  的形式，并与之前所有编号进行对比，如果与编号不一致的为零；与编号一致的不为零，并归一化，归一化系数是  $\underbrace{0_\zeta \dots 0_\zeta}_{l_0} \underbrace{1_\zeta \dots 1_\zeta}_{l_1} \dots \underbrace{w_\zeta \dots w_\zeta}_{l_w}$  全反对称化后所有种类数的倒数开方。以上常数不变张量只与旋量全反对称代数性质相关，与旋量变换性质无直接相关，是全反对称性张量内在固有的代数性质，与  $A_\zeta$  指标阶数直接本质相关。

### 2.3 常数矩阵 $\Gamma(s; w), \bar{\Gamma}(s; w)$ 的引入

定义2.3.1.  $\Gamma(s; w) \succ \Gamma_{A_\zeta \otimes B_\zeta \otimes C_\zeta \otimes \dots}^{k_\zeta}(s; w), \bar{\Gamma}(s; w) \succ \Gamma_{k_\zeta}^{\overbrace{A_\zeta \otimes B_\zeta \otimes C_\zeta \otimes \dots}^{2s}}(s; w) \simeq \Gamma^T(s; w)$

推论2.3.1.  $[\Gamma(s; w)] = (w + 1)^{2s} \times C_{w+1}^{2s}, [\bar{\Gamma}(s; w)] = C_{w+1}^{2s} \times (w + 1)^{2s}$

### 2.4 常数不变张量 $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w), \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w)$ 的基本性质

相等性：

性质2.4.1.  $\Gamma_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}^{k'_\zeta}(s; w) \simeq \Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; w) \simeq \Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}(s; w) \simeq \Gamma_{k'_\zeta}^{\overbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}(s; w)$

性质2.4.2.  $[\Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; w)]^* \simeq \Gamma_{\underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}^{k'_\zeta}(s; w), [\Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}(s; w)]^* \simeq \Gamma_{k'_\zeta}^{\overbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s}}(s; w)$

推论2.4.1.  $\Gamma(s; w) = \Gamma^*(s; w), \bar{\Gamma}(s; w) = \bar{\Gamma}^*(s; w), \bar{\Gamma}(s; w) = \bar{\Gamma}^+(s; w), \Gamma(s; w) = \bar{\Gamma}^+(s; w)$

正交性：

性质2.4.3.  $\Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}^{k_\zeta}(s; w) \Gamma_{l_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \dots}_{2s}}(s; w) = \delta^{k_\zeta l_\zeta} [\Leftrightarrow] \bar{\Gamma}(s; w) \Gamma(s; w) = I$

性质2.4.4.  $\Gamma_{A_{1\zeta} A_{2\zeta} \dots A_{2s\zeta}}^{k_\zeta}(s; w) \Gamma_{k_\zeta}^{B_{1\zeta} B_{2\zeta} \dots B_{2s\zeta}}(s; w) = \frac{1}{(2s)!} \delta_{A_{1\zeta}}^{B_{1\zeta}} \delta_{A_{2\zeta}}^{B_{2\zeta}} \dots \delta_{A_{2s\zeta}}^{B_{2s\zeta}} = \frac{1}{(2s)!} \delta_{[A_{1\zeta} \delta_{A_{2\zeta}} \dots \delta_{A_{2s\zeta}}]^{B_{1\zeta} B_{2\zeta} \dots B_{2s\zeta}}}$

其它性质：

性质2.4.5.  $\Gamma_{A_\zeta}^{k_\zeta}(\frac{1}{2}; w) = \delta_{A_\zeta}^{k_\zeta}, \Gamma_{k_\zeta}^{A_\zeta}(\frac{1}{2}; w) = \delta_{k_\zeta}^{A_\zeta}; \Gamma(0; w) = 1, \bar{\Gamma}(0; w) = 1$

2.5 度规常数不变张量  $\varepsilon_{k_\zeta l_\zeta}(s; w)$  的引入及其性质 (存在  $\varepsilon_{A_\zeta B_\zeta}$  为前提条件)

定义2.5.1.  $\varepsilon(\frac{1}{2}; w)\varepsilon^+(\frac{1}{2}; w) = \varepsilon^+(\frac{1}{2}; w)\varepsilon(\frac{1}{2}; w) = 1; \varepsilon(\frac{1}{2}; w) = \varepsilon^*(\frac{1}{2}; w)$

度规定义:

$$\text{定义2.5.2. } \left\{ \begin{array}{l} \varepsilon_{k_\zeta l_\zeta}(s; w) := \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w) \underbrace{\varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; w) \\ \varepsilon^{k_\zeta l_\zeta}(s; w) := \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) \underbrace{\varepsilon^{A_\zeta E_\zeta} \varepsilon^{B_\zeta F_\zeta} \varepsilon^{C_\zeta G_\zeta} \dots}_{2s} \Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; w) \end{array} \right.$$

$$\text{性质2.5.1. } \left\{ \begin{array}{l} \varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; w) \text{ 关于 } ABC\dots \text{ 全反对称} \\ \varepsilon^{A_\zeta E_\zeta} \varepsilon^{B_\zeta F_\zeta} \varepsilon^{C_\zeta G_\zeta} \dots \Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; w) \text{ 关于 } ABC\dots \text{ 全反对称} \end{array} \right.$$

推论2.5.1.  $\varepsilon(s; w) := \bar{\Gamma}(s; w) \underbrace{\varepsilon(\frac{1}{2}; w) \otimes \dots \otimes \varepsilon(\frac{1}{2}; w)}_{2s} \Gamma(s; w)$

推论2.5.2.  $\varepsilon(s; w)\varepsilon^+(s; w) = \varepsilon^+(s; w)\varepsilon(s; w) = 1; \varepsilon(s; w) = \varepsilon^*(s; w)$

升降指标:

$$\text{性质2.5.2. } \left\{ \begin{array}{l} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) = \varepsilon^{k_\zeta l_\zeta}(s; w) \varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; w) \\ \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}(s; w) = \varepsilon_{k_\zeta l_\zeta}(s; w) \varepsilon^{A_\zeta E_\zeta} \varepsilon^{B_\zeta F_\zeta} \varepsilon^{C_\zeta G_\zeta} \dots \Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta}(s; w) \end{array} \right.$$

推论2.5.3.  $\Gamma(s; w)\varepsilon(s; w) = \varepsilon(\frac{1}{2}; w) \otimes \dots \otimes \varepsilon(\frac{1}{2}; w) \Gamma(s; w), \varepsilon(s; w)\bar{\Gamma}(s; w) = \bar{\Gamma}(s; w) \underbrace{\varepsilon(\frac{1}{2}; w) \otimes \dots \otimes \varepsilon(\frac{1}{2}; w)}_{2s}$

$$\begin{aligned} \text{证明: } & \varepsilon^{k_\zeta l_\zeta}(s; w) \varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; w) \\ &= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}(s; w) \varepsilon^{A'_\zeta E'_\zeta} \varepsilon^{B'_\zeta F'_\zeta} \varepsilon^{C'_\zeta G'_\zeta} \dots \Gamma_{E'_\zeta F'_\zeta G'_\zeta \dots}^{l_\zeta}(s; w) \varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots}(s; w) \\ &= \frac{1}{(2s)!} \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}(s; w) \varepsilon^{A'_\zeta E'_\zeta} \varepsilon^{B'_\zeta F'_\zeta} \varepsilon^{C'_\zeta G'_\zeta} \dots \delta_{[E'_\zeta}^{E_\zeta} \delta_{F'_\zeta}^{F_\zeta} \delta_{G'_\zeta}^{G_\zeta} \dots]} \varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots \\ &= \frac{1}{(2s)!} \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}(s; w) \varepsilon^{A'_\zeta [E_\zeta} \varepsilon^{B'_\zeta F_\zeta} \varepsilon^{C'_\zeta G_\zeta]} \dots \varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots \\ &= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}(s; w) \varepsilon^{A'_\zeta E_\zeta} \varepsilon^{B'_\zeta F_\zeta} \varepsilon^{C'_\zeta G_\zeta} \dots \varepsilon_{A_\zeta E_\zeta} \varepsilon_{B_\zeta F_\zeta} \varepsilon_{C_\zeta G_\zeta} \dots \\ &= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{k_\zeta}(s; w) \delta_{A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots \\ &= \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) \end{aligned}$$

Penrose标准升降规则:

$$\text{性质2.5.3.} \left\{ \begin{array}{l} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; w) = (-1)^{2s} [\zeta^{2s} \varepsilon^{k_\zeta l_\zeta} (s; w)] \underbrace{(-\zeta \varepsilon_{A_\zeta E_\zeta})(-\zeta \varepsilon_{B_\zeta F_\zeta})(-\zeta \varepsilon_{C_\zeta G_\zeta})}_{2s} \cdot \Gamma_{l_\zeta}^{E_\zeta F_\zeta G_\zeta \dots} (s; w) \\ \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots} (s; w) = (-1)^{2s} [(-\zeta)^{2s} \varepsilon_{k_\zeta l_\zeta} (s; w)] \underbrace{(\zeta \varepsilon^{A_\zeta E_\zeta})(\zeta \varepsilon^{B_\zeta F_\zeta})(\zeta \varepsilon^{C_\zeta G_\zeta})}_{2s} \cdot \Gamma_{E_\zeta F_\zeta G_\zeta \dots}^{l_\zeta} (s; w) \end{array} \right.$$

## 2.6 常数不变张量 $\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w), \Omega (s; w)$ 的引入及其基本性质

定义2.6.1.

$$\underbrace{\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w)}_{2s} := \underbrace{\sigma_{A_\zeta}^{A'_\zeta} (\frac{1}{2}; w) \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \sigma_{B_\zeta}^{B'_\zeta} (\frac{1}{2}; w) \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \sigma_{C_\zeta}^{C'_\zeta} (\frac{1}{2}; w) \dots}_{2s} + \dots$$

(⇔) (⇔)

定义2.6.2.  $\Omega (s; w) := \sigma (\frac{1}{2}; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes \sigma (\frac{1}{2}; w) \otimes I_{(w+1)^{2s-2}} + \dots + I_{(w+1)^{2s-1}} \otimes \sigma (\frac{1}{2}; w)$

定义2.6.3.  $S (s; w) := \overbrace{S_{w+1} \otimes S_{w+1} \otimes S_{w+1} \otimes \dots \otimes S_{w+1}}^{2s}, S_{w+1} S_{w+1}^+ = S_{w+1}^+ S_{w+1} = I_{w+1}$

推论2.6.1.  $\underbrace{\Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w)}_{2s} := \sigma_{A_\zeta}^{A'_\zeta} (\frac{1}{2}; w) \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s-1} + \delta_{A_\zeta}^{A'_\zeta} \underbrace{\Omega_{B_\zeta C_\zeta \dots}^{B'_\zeta C'_\zeta \dots} (s - \frac{1}{2}; w)}_{2s-1}$

(⇔) (⇔)

推论2.6.2.  $\Omega (s; w) = \sigma (\frac{1}{2}; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes \Omega (s - \frac{1}{2}; w)$

推论2.6.3.  $\Omega (s; w) = \Omega (s - s'; w) \otimes I_{(w+1)^{2s'}} + I_{(w+1)^{2(s-s')}} \otimes \Omega (s'; w)$

## 2.7 常数不变张量 $\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w), \Omega_{ab} (s, \zeta; w)$ 的引入及其基本性质

定义2.7.1.

$$\underbrace{\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w)}_{2s} := \underbrace{S_{ab A_\zeta}^{A'_\zeta} (\frac{1}{2}; w) \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} S_{ab B_\zeta}^{B'_\zeta} (\frac{1}{2}; w) \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s} + \underbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} S_{ab C_\zeta}^{C'_\zeta} (\frac{1}{2}; w) \dots}_{2s} + \dots$$

(⇔) (⇔)

定义2.7.2.  $\Omega_{ab} (s, \zeta; w) := S_{ab} (\frac{1}{2}, \zeta; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes S_{ab} (\frac{1}{2}, \zeta; w) \otimes I_{(w+1)^{2s-2}} + \dots + I_{(w+1)^{2s-1}} \otimes S_{ab} (\frac{1}{2}, \zeta; w)$

推论2.7.1.  $\underbrace{\Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w)}_{2s} := S_{ab A_\zeta}^{A'_\zeta} (\frac{1}{2}; w) \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s-1} + \delta_{A_\zeta}^{A'_\zeta} \underbrace{\Omega_{ab B_\zeta C_\zeta \dots}^{B'_\zeta C'_\zeta \dots} (s - \frac{1}{2}; w)}_{2s-1}$

(⇔) (⇔)

推论2.7.2.  $\Omega_{ab} (s, \zeta; w) = S_{ab} (\frac{1}{2}, \zeta; w) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes \Omega_{ab} (s - \frac{1}{2}, \zeta; w)$

推论2.7.3.  $\Omega_{ab} (s, \zeta; w) = \Omega_{ab} (s - s'; \zeta; w) \otimes I_{(w+1)^{2s'}} + I_{(w+1)^{2(s-s')}} \otimes \Omega_{ab} (s'; \zeta; w)$

## 2.8 自旋常数不变张量 $\sigma^{\alpha_\zeta}_{k_\zeta} l_\zeta (s; w), \sigma (s; w)$ 的引入

定义2.8.1.  $\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots} (s; w) \sigma^{\alpha_\zeta}_{A_\zeta} Z_\zeta (\frac{1}{2}; w) \Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta} (s; w) := \frac{1}{2s} \sigma^{\alpha_\zeta}_{k_\zeta} l_\zeta (s; w) [\Leftrightarrow] \sigma (s; w) := \bar{\Gamma} (s; w) \Omega (s; w) \Gamma (s; w)$

## 2.9 自旋常数不变张量 $S_{ab k_\zeta} l_\zeta (s; w), S_{ab} (s, \zeta; w)$ 的引入

定义2.9.1.

$$\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \dots} (s; w) S_{ab A_\zeta}^{Z_\zeta} (\frac{1}{2}; w) \Gamma_{Z_\zeta B_\zeta C_\zeta \dots}^{l_\zeta} (s; w) := \frac{1}{2s} S_{ab k_\zeta} l_\zeta (s; w) [\Leftrightarrow] S_{ab} (s, \zeta; w) := \bar{\Gamma} (s; w) \Omega_{ab} (s, \zeta; w) \Gamma (s; w)$$

## 2.10 常数矩阵 $\Omega(s; w)$ 的两个重要引理

引理2.10.1.  $\Gamma(s; w)\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \Omega(s; w)\Gamma(s; w)$

$$\begin{aligned} \text{证明: } & \underbrace{\Gamma_{A''_s B''_s C''_s \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{k_\zeta}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Omega_{A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Gamma_{A'_s B'_s C'_s \dots}^{l_\zeta}}_{2s}(s; w) = \underbrace{\Omega_{A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Gamma_{A'_s B'_s C'_s \dots}^{l_\zeta}}_{2s}(s; w) \\ \Leftrightarrow & \Gamma(s; w)\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \Omega(s; w)\Gamma(s; w) \quad \square \end{aligned}$$

引理2.10.2.  $\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w)\bar{\Gamma}(s; w) = \bar{\Gamma}(s; w)\Omega(s; w)$

$$\begin{aligned} \text{证明: } & \underbrace{\Gamma_{k_\zeta}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Omega_{A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Gamma_{A'_s B'_s C'_s \dots}^{l_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{A''_s B''_s C''_s \dots}^{l_\zeta}}_{2s}(s; w) = \underbrace{\Gamma_{k_\zeta}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Omega_{A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \\ \Leftrightarrow & \bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w)\bar{\Gamma}(s; w) = \bar{\Gamma}(s; w)\Omega(s; w) \quad \square \end{aligned}$$

## 2.11 常数矩阵 $\Omega_{ab}(s; w)$ 的两个重要引理

引理2.11.1.  $\Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s; w)\Gamma(s; w) = \Omega_{ab}(s; w)\Gamma(s; w)$

$$\begin{aligned} \text{证明: } & \underbrace{\Gamma_{A''_s B''_s C''_s \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{k_\zeta}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Omega_{ab A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Gamma_{A'_s B'_s C'_s \dots}^{l_\zeta}}_{2s}(s; w) = \underbrace{\Omega_{ab A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Gamma_{A'_s B'_s C'_s \dots}^{l_\zeta}}_{2s}(s; w) \\ \Leftrightarrow & \Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s; w)\Gamma(s; w) = \Omega_{ab}(s; w)\Gamma(s; w) \quad \square \end{aligned}$$

引理2.11.2.  $\bar{\Gamma}(s; w)\Omega_{ab}(s; w)\Gamma(s; w)\bar{\Gamma}(s; w) = \bar{\Gamma}(s; w)\Omega_{ab}(s; w)$

$$\begin{aligned} \text{证明: } & \underbrace{\Gamma_{k_\zeta}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Omega_{ab A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Gamma_{A'_s B'_s C'_s \dots}^{l_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{A''_s B''_s C''_s \dots}^{l_\zeta}}_{2s}(s; w) = \underbrace{\Gamma_{k_\zeta}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Omega_{ab A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \\ \Leftrightarrow & \bar{\Gamma}(s; w)\Omega_{ab}(s; w)\Gamma(s; w)\bar{\Gamma}(s; w) = \bar{\Gamma}(s; w)\Omega_{ab}(s; w) \quad \square \end{aligned}$$

## 2.12 关于常数矩阵 $\bar{\Gamma}(s; w)$ , $\Omega(s; w)$ , $\sigma(s; w)$ , $\Gamma(s; w)$ 的置换性质及其推论

$$\text{定理2.12.1. } \Omega(s; w)\Gamma(s; w) = \Gamma(s; w)\sigma(s; w) [\Leftrightarrow] \underbrace{\Omega_{A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Gamma_{A'_s B'_s C'_s \dots}^{l_\zeta}}_{2s}(s; w) = \underbrace{\Gamma_{A'_s B'_s C'_s \dots}^{k_\zeta}}_{2s}(s; w) \sigma_{k_\zeta}^{l_\zeta}(s; w)$$

证明:  $\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \sigma(s; w)$

$$\Leftrightarrow \Gamma(s; w)\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \Gamma(s; w)\sigma(s; w)$$

$$\Leftrightarrow \Omega(s; w)\Gamma(s; w) = \Gamma(s; w)\sigma(s; w)$$

$$\Leftrightarrow \underbrace{\Omega_{A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Gamma_{A'_s B'_s C'_s \dots}^{l_\zeta}}_{2s}(s; w) = \underbrace{\Gamma_{A'_s B'_s C'_s \dots}^{k_\zeta}}_{2s}(s; w) \sigma_{k_\zeta}^{l_\zeta}(s; w) \quad \square$$

$$\text{定理2.12.2. } \bar{\Gamma}(s; w)\Omega(s; w) = \sigma(s; w)\bar{\Gamma}(s; w) [\Leftrightarrow] \underbrace{\Gamma_{k_\zeta}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Omega_{A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) = \sigma_{k_\zeta}^{l_\zeta}(s; w) \underbrace{\Gamma_{l_\zeta}^{A'_s B'_s C'_s \dots}}_{2s}(s; w)$$

证明:  $\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \sigma(s; w)$

$$\Leftrightarrow \bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w)\bar{\Gamma}(s; w) = \sigma(s; w)\bar{\Gamma}(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w)\Omega(s; w) = \sigma(s; w)\bar{\Gamma}(s; w)$$

$$\Leftrightarrow \underbrace{\Gamma_{k_\zeta}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \underbrace{\Omega_{A'_s B'_s C'_s \dots}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) = \sigma_{k_\zeta}^{l_\zeta}(s; w) \underbrace{\Gamma_{l_\zeta}^{A'_s B'_s C'_s \dots}}_{2s}(s; w) \quad \square$$

推论2.12.1.  $\bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \sigma(s; w) \Leftrightarrow \Omega(s; w)\Gamma(s; w) = \Gamma(s; w)\sigma(s; w) \Leftrightarrow \bar{\Gamma}(s; w)\Omega(s; w) = \sigma(s; w)\bar{\Gamma}(s; w)$

推论2.12.2.  $\Omega^2(s; w)\Gamma(s; w) = \Gamma(s; w)\sigma^2(s; w)$ ,  $\bar{\Gamma}(s; w)\Omega^2(s; w) = \sigma^2(s; w)\bar{\Gamma}(s; w)$

证明:  $\Omega^2(s; w)\Gamma(s; w)$

$$\begin{aligned} &= \Omega(s; w) \cdot \Omega(s; w)\Gamma(s; w) = \Omega(s; w) \cdot \Gamma(s; w)\sigma(s; w) \\ &= \Omega(s; w)\Gamma(s; w) \cdot \sigma(s; w) = \Gamma(s; w)\sigma(s; w) \cdot \sigma(s; w) \\ &= \Gamma(s; w)\sigma^2(s; w) \end{aligned}$$

□

证明:  $\bar{\Gamma}(s; w)\Omega^2(s; w)$

$$\begin{aligned} &= \bar{\Gamma}(s; w)\Omega(s; w) \cdot \Omega(s; w) = \sigma(s; w)\bar{\Gamma}(s; w) \cdot \Omega(s; w) \\ &= \sigma(s; w) \cdot \bar{\Gamma}(s; w)\Omega(s; w) = \sigma(s; w) \cdot \sigma(s; w)\bar{\Gamma}(s; w) \\ &= \sigma^2(s; w)\bar{\Gamma}(s; w) \end{aligned}$$

□

推论2.12.3.  $\Omega^2(s)\Gamma(s) = s(s+1)\Gamma(s)$ ,  $\bar{\Gamma}(s)\Omega^2(s) = \bar{\Gamma}(s)s(s+1)$

## 2.13 关于常数矩阵 $\bar{\Gamma}(s; w)$ , $\Omega_{ab}(s, \varsigma; w)$ , $S_{ab}(s, \varsigma; w)$ , $\Gamma(s; w)$ 的置换性质及其推论

定理2.13.1.  $\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = \Gamma(s; w)S_{ab}(s, \varsigma; w)$

$$[\Leftrightarrow] \underbrace{\Omega_{ab}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w) \underbrace{\Gamma^{l_\varsigma}_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w) = \underbrace{\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma \dots}}_{2s}(s; w) S_{abk_\varsigma}^{l_\varsigma}(s; w)$$

证明:  $\bar{\Gamma}(s, \varsigma; w)\Omega_{ab}(s; w)\Gamma(s; w) = S_{ab}(s, \varsigma; w)$

$$\Leftrightarrow \Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = \Gamma(s; w)S_{ab}(s, \varsigma; w)$$

$$\Leftrightarrow \Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = \Gamma(s; w)S_{ab}(s, \varsigma; w)$$

$$\Leftrightarrow \underbrace{\Omega_{ab}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w) \underbrace{\Gamma^{l_\varsigma}_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w) = \underbrace{\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma \dots}}_{2s}(s; w) S_{abk_\varsigma}^{l_\varsigma}(s; w)$$

□

定理2.13.2.  $\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w) = S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$

$$[\Leftrightarrow] \underbrace{\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma \dots}}_{2s}(s; w) \underbrace{\Omega_{ab}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w) = S_{abk_\varsigma}^{l_\varsigma}(s; w) \underbrace{\Gamma^{l_\varsigma}_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w)$$

证明:  $\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = S_{ab}(s, \varsigma; w)$

$$\Leftrightarrow \bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)\bar{\Gamma}(s; w) = S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w) = S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$$

$$\Leftrightarrow \underbrace{\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma \dots}}_{2s}(s; w) \underbrace{\Omega_{ab}^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w) = S_{abk_\varsigma}^{l_\varsigma}(s; w) \underbrace{\Gamma^{l_\varsigma}_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w)$$

□

推论2.13.1.  $\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = S_{ab}(s, \varsigma; w)$

$$\Leftrightarrow \Omega_{ab}(s; w)\Gamma(s, \varsigma; w) = \Gamma(s; w)S_{ab}(s, \varsigma; w) \Leftrightarrow \bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w) = S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$$

## 2.14 关于常数矩阵 $\Omega(s; w)$ , $\sigma(s; w)$ 的重要推论及洛伦兹群表示

推论2.14.1.  $\underbrace{\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma \dots}}_{2s}(s; w)\sigma_{k_\varsigma}^{l_\varsigma}(s; w) = \frac{1}{(2s-1)!}\sigma(\frac{1}{2}; w)_{(A_\varsigma \underbrace{A'_\varsigma \Gamma^{l_\varsigma}_{B_\varsigma C_\varsigma \dots}}_{2s})A'_\varsigma}(s; w)$

$$\text{证明: } \underbrace{\Gamma^{k_\varsigma}_{A_\varsigma B_\varsigma C_\varsigma \dots}}_{2s}(s; w)\sigma_{k_\varsigma}^{l_\varsigma}(s; w) = \underbrace{\Omega^{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w) \underbrace{\Gamma^{l_\varsigma}_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w)$$

$$= \sigma(\frac{1}{2}; w)_{A_\varsigma} A'_\varsigma \underbrace{\Gamma^{l_\varsigma}_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w) + \sigma(\frac{1}{2}; w)_{B_\varsigma} B'_\varsigma \underbrace{\Gamma^{l_\varsigma}_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w) + \sigma(\frac{1}{2}; w)_{C_\varsigma} C'_\varsigma \underbrace{\Gamma^{l_\varsigma}_{A'_\varsigma B'_\varsigma C'_\varsigma \dots}}_{2s}(s; w) + \dots$$

$$= \sigma(\frac{1}{2}; w)_{A_\varsigma} A'_\varsigma \underbrace{\Gamma^{l_\varsigma}_{B_\varsigma C_\varsigma \dots A'_\varsigma}}_{2s}(s; w) + \sigma(\frac{1}{2}; w)_{B_\varsigma} A'_\varsigma \underbrace{\Gamma^{l_\varsigma}_{A_\varsigma C_\varsigma \dots A'_\varsigma}}_{2s}(s; w) + \sigma(\frac{1}{2}; w)_{C_\varsigma} A'_\varsigma \underbrace{\Gamma^{l_\varsigma}_{A_\varsigma B_\varsigma \dots A'_\varsigma}}_{2s}(s; w) + \dots$$



$$\begin{aligned}
&= \frac{1}{(2s-1)!} \sigma\left(\frac{1}{2}; w\right)_{A_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{(B_\zeta C_\zeta \dots) A'_\zeta}}_{2s} (s; w) + \frac{1}{(2s-1)!} \sigma\left(\frac{1}{2}; w\right)_{B_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{(A_\zeta C_\zeta \dots) A'_\zeta}}_{2s} (s; w) + \frac{1}{(2s-1)!} \sigma\left(\frac{1}{2}; w\right)_{C_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{(A_\zeta B_\zeta \dots) A'_\zeta}}_{2s} (s; w) \cdots \\
&\Leftrightarrow \frac{1}{(2s-1)!} \sigma\left(\frac{1}{2}; w\right)_{(A_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{B_\zeta C_\zeta \dots A'_\zeta}}_{2s} (s; w) \quad \square
\end{aligned}$$

**推论2.14.2.**  $\sigma\left(\frac{1}{2}; w\right) \times \sigma\left(\frac{1}{2}; w\right) = i\sigma\left(\frac{1}{2}; w\right) [\Rightarrow] \Omega(s; w) \times \Omega(s; w) = i\Omega(s; w) [\Rightarrow] \sigma(s; w) \times \sigma(s; w) = i\sigma(s; w)$

**证明:**  $\sigma\left(\frac{1}{2}; w\right) \times \sigma\left(\frac{1}{2}; w\right) = i\sigma\left(\frac{1}{2}; w\right)$

$$\Rightarrow \Omega(s; w) \times \Omega(s; w)$$

$$= [\sigma\left(\frac{1}{2}; w\right) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes \sigma\left(\frac{1}{2}; w\right) \otimes I_{(w+1)^{2s-2}} + \cdots + I_{(w+1)^{2s-1}} \otimes \sigma\left(\frac{1}{2}; w\right)]$$

$$\times [\sigma\left(\frac{1}{2}; w\right) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes \sigma\left(\frac{1}{2}; w\right) \otimes I_{(w+1)^{2s-2}} + \cdots + I_{(w+1)^{2s-1}} \otimes \sigma\left(\frac{1}{2}; w\right)]$$

$$= [\sigma\left(\frac{1}{2}; w\right) \times \sigma\left(\frac{1}{2}; w\right)] \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes [\sigma\left(\frac{1}{2}; w\right) \times \sigma\left(\frac{1}{2}; w\right)] \otimes I_{(w+1)^{2s-2}} + \cdots + I_{(w+1)^{2s-1}} \otimes [\sigma\left(\frac{1}{2}; w\right) \times \sigma\left(\frac{1}{2}; w\right)]$$

$$= i\sigma\left(\frac{1}{2}; w\right) \otimes I_{(w+1)^{2s-1}} + iI_{w+1} \otimes \sigma\left(\frac{1}{2}; w\right) \otimes I_{(w+1)^{2s-2}} + \cdots + iI_{(w+1)^{2s-1}} \otimes \sigma\left(\frac{1}{2}; w\right)$$

$$= i\Omega(s; w) \quad \square$$

**证明:**  $\Omega(s; w) \times \Omega(s; w) = i\Omega(s; w)$

$$\Rightarrow \bar{\Gamma}(s; w) \Omega(s; w) \times \Omega(s; w) \Gamma(s; w) = i\bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w) \Omega(s; w) \times \Gamma(s; w) \bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w) = i\bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w) \times \bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w) = i\bar{\Gamma}(s; w) \Omega(s; w) \Gamma(s; w)$$

$$\Leftrightarrow \sigma(s; w) \times \sigma(s; w) = i\sigma(s; w) \quad \square$$

## 2.15 关于常数矩阵 $\Omega_{ab}(s, \zeta; w)$ , $S_{ab}(s, \zeta; w)$ 的重要推论及洛伦兹群表示

**推论2.15.1.**  $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) S_{abk_\zeta}^{l_\zeta}(s; w) = \frac{1}{(2s-1)!} S_{ab}\left(\frac{1}{2}; w\right)_{(A_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{B_\zeta C_\zeta \dots A'_\zeta}}_{2s} (s; w)$

**证明:**  $\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}(s; w) S_{abk_\zeta}^{l_\zeta}(s; w) = \Omega_{ab} \underbrace{A'_\zeta B'_\zeta C'_\zeta \dots}_{2s} (s; w) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w)$

$$= S_{ab}\left(\frac{1}{2}; w\right)_{A_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{B'_\zeta C'_\zeta \dots}}_{2s} (s; w) + S_{ab}\left(\frac{1}{2}; w\right)_{B_\zeta} \underbrace{B'_\zeta \Gamma^{l_\zeta}_{A'_\zeta C'_\zeta \dots}}_{2s} (s; w) + S_{ab}\left(\frac{1}{2}; w\right)_{C_\zeta} \underbrace{C'_\zeta \Gamma^{l_\zeta}_{A'_\zeta B'_\zeta \dots}}_{2s} (s; w) + \cdots$$

$$= S_{ab}\left(\frac{1}{2}; w\right)_{A_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{B_\zeta C_\zeta \dots A'_\zeta}}_{2s} (s; w) + S_{ab}\left(\frac{1}{2}; w\right)_{B_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{A_\zeta C_\zeta \dots A'_\zeta}}_{2s} (s; w) + S_{ab}\left(\frac{1}{2}; w\right)_{C_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{A_\zeta B_\zeta \dots A'_\zeta}}_{2s} (s; w) + \cdots$$

$$= \frac{1}{(2s-1)!} S_{ab}\left(\frac{1}{2}; w\right)_{A_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{(B_\zeta C_\zeta \dots) A'_\zeta}}_{2s} (s; w) + \frac{1}{(2s-1)!} S_{ab}\left(\frac{1}{2}; w\right)_{B_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{(A_\zeta C_\zeta \dots) A'_\zeta}}_{2s} (s; w) + \cdots$$

$$\Leftrightarrow \frac{1}{(2s-1)!} S_{ab}\left(\frac{1}{2}; w\right)_{(A_\zeta} \underbrace{A'_\zeta \Gamma^{l_\zeta}_{B_\zeta C_\zeta \dots A'_\zeta}}_{2s} (s; w) \quad \square$$

**推论2.15.2.**  $i[S_{ab}\left(\frac{1}{2}, \zeta; w\right), S_{cd}\left(\frac{1}{2}, \zeta; w\right)] = \delta_{ad} S_{bc}\left(\frac{1}{2}, \zeta; w\right) - \delta_{ac} S_{bd}\left(\frac{1}{2}, \zeta; w\right) + \delta_{bc} S_{ad}\left(\frac{1}{2}, \zeta; w\right) - \delta_{bd} S_{ac}\left(\frac{1}{2}, \zeta; w\right)$

$$[\Rightarrow] i[\Omega_{ab}(s, \zeta; w), \Omega_{cd}(s, \zeta; w)] = \delta_{ad} \Omega_{bc}(s, \zeta; w) - \delta_{ac} \Omega_{bd}(s, \zeta; w) + \delta_{bc} \Omega_{ad}(s, \zeta; w) - \delta_{bd} \Omega_{ac}(s, \zeta; w)$$

$$[\Rightarrow] i[S_{ab}(s, \zeta; w), S_{cd}(s, \zeta; w)] = \delta_{ad} S_{bc}(s, \zeta; w) - \delta_{ac} S_{bd}(s, \zeta; w) + \delta_{bc} S_{ad}(s, \zeta; w) - \delta_{bd} S_{ac}(s, \zeta; w)$$

**证明:**  $i[\Omega_{ab}(s, \zeta; w), \Omega_{cd}(s, \zeta; w)]$

$$= i[S_{ab}\left(\frac{1}{2}, \zeta; w\right) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes S_{ab}\left(\frac{1}{2}, \zeta; w\right) \otimes I_{(w+1)^{2s-2}} + \cdots + I_{(w+1)^{2s-1}} \otimes S_{ab}\left(\frac{1}{2}, \zeta; w\right)]$$

$$, S_{cd}\left(\frac{1}{2}, \zeta; w\right) \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes S_{cd}\left(\frac{1}{2}, \zeta; w\right) \otimes I_{(w+1)^{2s-2}} + \cdots + I_{(w+1)^{2s-1}} \otimes S_{cd}\left(\frac{1}{2}, \zeta; w\right)]$$

$$= i\{[S_{ab}\left(\frac{1}{2}, \zeta; w\right), S_{cd}\left(\frac{1}{2}, \zeta; w\right)] \otimes I_{(w+1)^{2s-1}} + I_{w+1} \otimes [S_{ab}\left(\frac{1}{2}, \zeta; w\right), S_{cd}\left(\frac{1}{2}, \zeta; w\right)] \otimes I_{(w+1)^{2s-2}}$$

$$+ \cdots + I_{(w+1)^{2s-1}} \otimes [S_{ab}\left(\frac{1}{2}, \zeta; w\right), S_{cd}\left(\frac{1}{2}, \zeta; w\right)]\}$$

$$= [\delta_{ad} S_{bc}\left(\frac{1}{2}, \zeta; w\right) - \delta_{ac} S_{bd}\left(\frac{1}{2}, \zeta; w\right) + \delta_{bc} S_{ad}\left(\frac{1}{2}, \zeta; w\right) - \delta_{bd} S_{ac}\left(\frac{1}{2}, \zeta; w\right)] \otimes I_{(w+1)^{2s-1}}$$

$$+ I_{w+1} \otimes [\delta_{ad} S_{bc}\left(\frac{1}{2}, \zeta; w\right) - \delta_{ac} S_{bd}\left(\frac{1}{2}, \zeta; w\right) + \delta_{bc} S_{ad}\left(\frac{1}{2}, \zeta; w\right) - \delta_{bd} S_{ac}\left(\frac{1}{2}, \zeta; w\right)] \otimes I_{(w+1)^{2s-2}}$$

$$+ \cdots + I_{(w+1)^{2s-1}} \otimes [\delta_{ad} S_{bc}\left(\frac{1}{2}, \zeta; w\right) - \delta_{ac} S_{bd}\left(\frac{1}{2}, \zeta; w\right) + \delta_{bc} S_{ad}\left(\frac{1}{2}, \zeta; w\right) - \delta_{bd} S_{ac}\left(\frac{1}{2}, \zeta; w\right)]$$

$$= \delta_{ad} \Omega_{bc}(s, \zeta; w) - \delta_{ac} \Omega_{bd}(s, \zeta; w) + \delta_{bc} \Omega_{ad}(s, \zeta; w) - \delta_{bd} \Omega_{ac}(s, \zeta; w) \quad \square$$

$$\begin{aligned}
& \text{证明: } i[\Omega_{ab}(s, \varsigma; w), \Omega_{cd}(s, \varsigma; w)] = \delta_{ad}\Omega_{bc}(s, \varsigma; w) - \delta_{ac}\Omega_{bd}(s, \varsigma; w) + \delta_{bc}\Omega_{ad}(s, \varsigma; w) - \delta_{bd}\Omega_{ac}(s, \varsigma; w) \\
& \Rightarrow \bar{\Gamma}(s; w)i[\Omega_{ab}(s, \varsigma; w), \Omega_{cd}(s, \varsigma; w)]\Gamma(s; w) \\
& = \bar{\Gamma}(s; w)[\delta_{ad}\Omega_{bc}(s, \varsigma; w) - \delta_{ac}\Omega_{bd}(s, \varsigma; w) + \delta_{bc}\Omega_{ad}(s, \varsigma; w) - \delta_{bd}\Omega_{ac}(s, \varsigma; w)]\Gamma(s; w) \\
& \Leftrightarrow \bar{\Gamma}(s; w)i[\Omega_{ab}(s, \varsigma; w)\Omega_{cd}(s, \varsigma; w) - \Omega_{cd}(s, \varsigma; w)\Omega_{ab}(s, \varsigma; w)]\Gamma(s; w) \\
& = \bar{\Gamma}(s; w)[\delta_{ad}\Omega_{bc}(s, \varsigma; w) - \delta_{ac}\Omega_{bd}(s, \varsigma; w) + \delta_{bc}\Omega_{ad}(s, \varsigma; w) - \delta_{bd}\Omega_{ac}(s, \varsigma; w)]\Gamma(s; w) \\
& \Leftrightarrow \bar{\Gamma}(s; w)i[\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{cd}(s, \varsigma; w) - \Omega_{cd}(s, \varsigma; w)\Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)]\Gamma(s; w) \\
& = \bar{\Gamma}(s; w)[\delta_{ad}\Omega_{bc}(s, \varsigma; w) - \delta_{ac}\Omega_{bd}(s, \varsigma; w) + \delta_{bc}\Omega_{ad}(s, \varsigma; w) - \delta_{bd}\Omega_{ac}(s, \varsigma; w)]\Gamma(s; w) \\
& \Leftrightarrow i[\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{cd}(s, \varsigma; w)\Gamma(s; w) - \bar{\Gamma}(s; w)\Omega_{cd}(s, \varsigma; w)\Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)] \\
& = \bar{\Gamma}(s; w)[\delta_{ad}\Omega_{bc}(s, \varsigma; w) - \delta_{ac}\Omega_{bd}(s, \varsigma; w) + \delta_{bc}\Omega_{ad}(s, \varsigma; w) - \delta_{bd}\Omega_{ac}(s, \varsigma; w)]\Gamma(s; w) \\
& \Leftrightarrow i[S_{ab}(s, \varsigma; w), S_{cd}(s, \varsigma; w)] = \delta_{ad}S_{bc}(s, \varsigma; w) - \delta_{ac}S_{bd}(s, \varsigma; w) + \delta_{bc}S_{ad}(s, \varsigma; w) - \delta_{bd}S_{ac}(s, \varsigma; w) \quad \square
\end{aligned}$$

## 2.16 常数矩阵 $\Omega_{ab}(s, \varsigma; w)$ , $S_{ab}(s, \varsigma; w)$ 与 $\Omega(s; w)$ , $\sigma(s; w)$ 之间关系的讨论

$$\text{性质2.16.1. } S_{ab}(\frac{1}{2}, \varsigma) = \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(\frac{1}{2}) [\Rightarrow] \Omega_{ab}(s, \varsigma) = \sigma_{\varsigma ab}^{\alpha\varsigma} \Omega_{\alpha\varsigma}(s) [\Rightarrow] S_{ab}(s, \varsigma) = \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s)$$

第一种可能的关系(四维时空自旋型):

$$\text{猜想2.16.1. } S_{ab}(\frac{1}{2}, \varsigma; w) = \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(\frac{1}{2}; w) [\Rightarrow] \Omega_{ab}(s, \varsigma; w) = \sigma_{\varsigma ab}^{\alpha\varsigma} \Omega_{\alpha\varsigma}(s; w) [\Rightarrow] S_{ab}(s, \varsigma; w) = \sigma_{\varsigma ab}^{\alpha\varsigma} \sigma_{\alpha\varsigma}(s; w)$$

第二种可能的关系(任意时空自旋型, 四维时空自旋型是其特例。):

$$\begin{aligned}
\text{猜想2.16.2. } S_{ab}(\frac{1}{2}, \varsigma; w) &= \begin{bmatrix} -i[\sigma_{\alpha\varsigma}(\frac{1}{2}; w), \sigma_{\beta\varsigma}(\frac{1}{2}; w)] & -\varsigma\sigma_{\alpha\varsigma}(\frac{1}{2}; w) \\ \varsigma\sigma_{\beta\varsigma}(\frac{1}{2}; w) & 0 \end{bmatrix} \\
[\Rightarrow] \Omega_{ab}(s, \varsigma; w) &= \begin{bmatrix} -i[\Omega_{\alpha\varsigma}(s; w), \Omega_{\beta\varsigma}(s; w)] & -\varsigma\Omega_{\alpha\varsigma}(s; w) \\ \varsigma\Omega_{\beta\varsigma}(s; w) & 0 \end{bmatrix} \\
[\Rightarrow] S_{ab}(s, \varsigma; w) &= \begin{bmatrix} -i[\sigma_{\alpha\varsigma}(s; w), \sigma_{\beta\varsigma}(s; w)] & -\varsigma\sigma_{\alpha\varsigma}(s; w) \\ \varsigma\sigma_{\beta\varsigma}(s; w) & 0 \end{bmatrix}
\end{aligned}$$

第三种可能的关系(高维时空型Dirac型):

$$\begin{aligned}
\text{猜想2.16.3. } S_{ab}(\frac{1}{2}, \varsigma; w) &= -\frac{i}{4}[\gamma_a(\frac{1}{2}, \varsigma; w), \gamma_b(\frac{1}{2}, \varsigma; w)] \\
[\Rightarrow] \Omega_{ab}(s, \varsigma; w) &= -\frac{i}{4}[\Omega_a(s, \varsigma; w), \Omega_b(s, \varsigma; w)] [\Rightarrow] S_{ab}(s, \varsigma; w) = -\frac{i}{4}[\gamma_a(s, \varsigma; w), \gamma_b(s, \varsigma; w)]
\end{aligned}$$

第四种可能的关系(无关联型): 相对独立。

## 2.17 推论: 常数矩阵 $\Gamma(s; w)$ , $\bar{\Gamma}(s; w)$ 的几个恒等式

$$\begin{aligned}
\text{性质2.17.1. } & \begin{cases} \bar{\Gamma}(s; w)\Omega(s; w)\Gamma(s; w) = \sigma(s; w), [\Gamma(s; w)\bar{\Gamma}(s; w), \Omega(s; w)] = 0 \\ \Gamma(s; w)\sigma(s; w)\bar{\Gamma}(s; w) = \Omega(s; w)\Gamma(s; w)\bar{\Gamma}(s; w) = \Gamma(s; w)\bar{\Gamma}(s; w)\Omega(s; w) \end{cases} \\
\text{性质2.17.2. } & \begin{cases} \bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = S_{ab}(s, \varsigma; w), [\Gamma(s; w)\bar{\Gamma}(s; w), \Omega_{ab}(s, \varsigma; w)] = 0 \\ \Gamma(s; w)S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w) = \Omega_{ab}(s, \varsigma; w)\Gamma(s; w)\bar{\Gamma}(s; w) = \Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w) \end{cases} \\
\text{性质2.17.3. } & \begin{cases} \bar{\Gamma}(s; w)[\vartheta \cdot \Omega(s; w)]^n \Gamma(s; w) = [\vartheta \cdot \sigma(s; w)]^n, [\Gamma(s; w)\bar{\Gamma}(s; w), [\vartheta \cdot \Omega(s; w)]^n] = 0 \\ \Gamma(s; w)[\vartheta \cdot \sigma(s; w)]^n \bar{\Gamma}(s; w) = [\vartheta \cdot \Omega(s; w)]^n \Gamma(s; w)\bar{\Gamma}(s; w) = \Gamma(s; w)\bar{\Gamma}(s; w)[\vartheta \cdot \Omega(s; w)]^n \end{cases} \\
\text{性质2.17.4. } & \begin{cases} \bar{\Gamma}(s; w)[\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n \Gamma(s; w) = [\vartheta^{ab}S_{ab}(s, \varsigma; w)]^n, [\Gamma(s; w)\bar{\Gamma}(s; w), [\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n] = 0 \\ \Gamma(s; w)[\vartheta^{ab}S_{ab}(s, \varsigma; w)]^n \bar{\Gamma}(s; w) = [\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n \Gamma(s; w)\bar{\Gamma}(s; w) = \Gamma(s; w)\bar{\Gamma}(s; w)[\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n \end{cases} \\
\text{推论2.17.1. } & \begin{cases} \bar{\Gamma}(s; w)e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)}, [\Gamma(s; w)\bar{\Gamma}(s; w), e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}] = 0 \\ \Gamma(s; w)e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)}\bar{\Gamma}(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)\bar{\Gamma}(s; w) = \Gamma(s; w)\bar{\Gamma}(s; w)e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} \end{cases}
\end{aligned}$$

## 2.18 矩阵 $\Gamma(s; w), \bar{\Gamma}(s; w)$ 的常数不变张量性质

**定理2.18.1.**  $\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)}$ ,  $\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}$

**证明:**  $\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) = \Gamma(s; w)S_{ab}(s, \varsigma; w)$

$$\Leftrightarrow 0 = \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)\Gamma(s; w) - \Gamma(s; w)\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)$$

$$\Leftrightarrow \Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)} \quad \square$$

**定理2.18.2.**  $\bar{\Gamma}(s; w) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)}\bar{\Gamma}(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}$ ,  $\bar{\Gamma}(s; w) = e^{\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}\bar{\Gamma}(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}$

**证明:**  $\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w) = S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$

$$\Leftrightarrow 0 = -\bar{\Gamma}(s; w)\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w) + \frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)\bar{\Gamma}(s; w)$$

$$\Leftrightarrow \bar{\Gamma}(s; w) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s, \varsigma; w)}\bar{\Gamma}(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} \quad \square$$

**定理2.18.3.**  $\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}$

$$\Leftrightarrow \Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)\Gamma(s; w)}$$

**证明:**  $\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}$

$$\Leftrightarrow S^+(s; w)S(s; w)\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S^+(s; w)S(s; w)\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}$$

$$\Leftrightarrow S^+(s; w)\Gamma(s; w)\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S^+(s; w)\Gamma(s; w)\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}$$

$$\Leftrightarrow \Gamma(s; w) = [S(s; w)e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S^+(s; w)]\Gamma(s; w)[\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)]e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}[\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)]^+$$

$$\Leftrightarrow \Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)\bar{\Gamma}(s; w)S^+(s; w)\Gamma(s; w)}$$

$$\Leftrightarrow \Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)\Gamma(s; w)}$$

$$\Leftrightarrow \Gamma(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}\Gamma(s; w)e^{-\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)\Gamma(s; w)} \quad \square$$

**推论2.18.1.**

$$e^{\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)\Gamma(s; w)} = [\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)]e^{\frac{i}{2}\vartheta^{ab}\bar{\Gamma}(s; w)\Omega_{ab}(s, \varsigma; w)\Gamma(s; w)}[\bar{\Gamma}(s; w)S(s; w)\Gamma(s; w)]^+$$

**定理2.18.4.**  $S(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)}$

**证明:**  $\Omega_{ab}(s, \varsigma; w)S(s; w) = \Omega_{ab}(s, \varsigma; w)S(s; w), \forall \varsigma = \pm 1$

$$\Leftrightarrow S(s; w)S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w) = \Omega_{ab}(s, \varsigma; w)S(s; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow 0 = -S(s; w)\frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w) + \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)S(s; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = S(s; w)[1 - \frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)] + \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)S(s; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = [1 + \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]S(s; w)[1 - \frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)], \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)}, \forall \vartheta^{ab}, \varsigma = \pm 1 \quad \square$$

**定理2.18.5.**  $S(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}$

**证明:**  $S(s; w)\Omega_{ab}(s, \varsigma; w) = S(s; w)\Omega_{ab}(s, \varsigma; w), \forall \varsigma = \pm 1$

$$\Leftrightarrow S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)S(s; w) = S(s; w)\Omega_{ab}(s, \varsigma; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow 0 = \frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)S(s; w) - S(s; w)\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = [1 + \frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)]S(s; w) - S(s; w)\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w), \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = [1 + \frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)]S(s; w)[1 - \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)], \forall \vartheta^{ab}, \varsigma = \pm 1$$

$$\Leftrightarrow S(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}, \forall \vartheta^{ab}, \varsigma = \pm 1 \quad \square$$

**推论2.18.2.**

$$\begin{cases} S(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)} \\ S(s; w) = e^{\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)}S(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} \\ S^+(s; w) = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}S^+(s; w)e^{-\frac{i}{2}\vartheta^{ab}S(s; w)\Omega_{ab}(s, \varsigma; w)S^+(s; w)} \\ S^+(s; w) = e^{\frac{i}{2}\vartheta^{ab}S^+(s; w)\Omega_{ab}(s, \varsigma; w)S(s; w)}S^+(s; w)e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} \end{cases}$$

## 2.19 常数矩阵 $I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)$ , $I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)$ 的置换性质

推论2.19.1.  $\Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]$

证明:  $\Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]$

$$= \Omega(s; w)I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)$$

$$= [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}}] + I_{w+1} \otimes [\Gamma(s - \frac{1}{2}; w)\sigma(s - \frac{1}{2}; w)]$$

$$= [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] \quad \square$$

推论2.19.2.  $[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w) = [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]$

证明:  $[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w)$

$$= [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w)$$

$$= [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}}][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] + I_{w+1} \otimes [\sigma(s - \frac{1}{2}; w)\bar{\Gamma}(s - \frac{1}{2}; w)]$$

$$= [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \quad \square$$

## 2.20 推论: 常数矩阵 $\Gamma(s - \frac{1}{2}; w)$ , $\bar{\Gamma}(s - \frac{1}{2}; w)$ 的几个恒等式

性质2.20.1.

$$\begin{cases} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] \\ [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \\ = \Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w) \\ [[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)], \Omega(s; w)] = 0 \end{cases}$$

性质2.20.2.

$$\begin{cases} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega_{ab}(s, \varsigma; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \\ [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \\ = \Omega_{ab}(s, \varsigma; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega_{ab}(s, \varsigma; w) \\ [[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)], \Omega_{ab}(s, \varsigma; w)] = 0 \end{cases}$$

性质2.20.3.

$$\begin{cases} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vartheta \cdot \Omega(s; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = \{\vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]\}^n \\ [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]\{\vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]\}^n [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \\ = [\vartheta \cdot \Omega(s; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vartheta \cdot \Omega(s; w)]^n \\ [[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)], [\vartheta \cdot \Omega(s; w)]^n] = 0 \end{cases}$$

性质2.20.4.

$$\begin{cases} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = \{\vartheta^{ab}[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]\}^n \\ [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]\{\vartheta^{ab}[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]\}^n [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \\ = [\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n \\ [[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)], [\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)]^n] = 0 \end{cases}$$

推论2.20.1.

$$\begin{cases} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)] = e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]} \\ [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]e^{\frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]} [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \\ = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)} \\ [[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)], e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s, \varsigma; w)}] = 0 \end{cases}$$

## 推论2.20.2.

$$\begin{cases} I_{(w+1)^{2s-1}}\Gamma(s-\frac{1}{2};w) = \Gamma(s-\frac{1}{2};w)I_{C_{w+1}^{2s-1}}, \bar{\Gamma}(s-\frac{1}{2};w)I_{(w+1)^{2s-1}} = I_{C_{w+1}^{2s-1}}\bar{\Gamma}(s-\frac{1}{2};w) \\ [I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)]\sigma(\frac{1}{2};w) \otimes I_{(w+1)^{2s-1}}[I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)] = \sigma(\frac{1}{2};w) \otimes I_{C_{w+1}^{2s-1}} \\ [\sigma(\frac{1}{2};w) \otimes I_{(w+1)^{2s-1}}[I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)]] = [I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)][\sigma(\frac{1}{2};w) \otimes I_{C_{w+1}^{2s-1}}] \\ [I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)][\sigma(\frac{1}{2};w) \otimes I_{(w+1)^{2s-1}}] = [\sigma(\frac{1}{2};w) \otimes I_{C_{w+1}^{2s-1}}][I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)] \end{cases}$$

性质2.20.5.  $(\sigma \otimes I_{(w+1)^{2s-1}}, -i\zeta)_a [I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)]N(s;w) = [I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)]Z_a(s,\zeta;w)$

2.21 矩阵  $I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)$ ,  $I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)$  的常数不变张量性质

定理2.21.1.  $[I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)] = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\zeta;w)}[I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)]e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\zeta;w)} \otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\zeta;w)}$

证明:  $\Omega_{ab}(s,\zeta;w)[I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)] = [I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)][S_{ab}(\frac{1}{2},\zeta;w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s-\frac{1}{2},\zeta;w)]$   
 $\Leftrightarrow 0 = \frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\zeta;w)[I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)]$   
 $- \frac{i}{2}\vartheta^{ab}[I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)][S_{ab}(\frac{1}{2},\zeta;w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s-\frac{1}{2},\zeta;w)]$   
 $\Leftrightarrow [I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)] = e^{\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\zeta;w)}[I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)]e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\zeta;w)} \otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\zeta;w)} \quad \square$

定理2.21.2.  $[I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)] = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\zeta;w)} \otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\zeta;w)}[I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)]e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\zeta;w)}$

证明:  $[I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)]\Omega_{ab}(s,\zeta;w) = [S_{ab}(\frac{1}{2},\zeta;w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s-\frac{1}{2},\zeta;w)][I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)]$   
 $\Leftrightarrow 0 = -\frac{i}{2}\vartheta^{ab}[I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)]\Omega_{ab}(s,\zeta;w)$   
 $+ \frac{i}{2}\vartheta^{ab}[S_{ab}(\frac{1}{2},\zeta;w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s-\frac{1}{2},\zeta;w)][I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)]$   
 $\Leftrightarrow [I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)] = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2},\zeta;w)} \otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s-\frac{1}{2},\zeta;w)}[I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)]e^{-\frac{i}{2}\vartheta^{ab}\Omega_{ab}(s,\zeta;w)} \quad \square$

## 2.22 自洽性质验证

性质2.22.1.  $\psi' = S\psi \Rightarrow \hat{\psi}' = \Gamma(s;w)\psi' = \Gamma(s;w)S\psi = \bar{\Gamma}(s;w)S\bar{\Gamma}(s;w)\hat{\psi}$

性质2.22.2.  $\hat{\psi}' = (S \otimes \cdots \otimes S)\hat{\psi} \Rightarrow \psi' = \bar{\Gamma}(s;w)\hat{\psi}' = \bar{\Gamma}(s;w)(S \otimes \cdots \otimes S)\hat{\psi} = \bar{\Gamma}(s;w)(S \otimes \cdots \otimes S)\Gamma(s;w)\psi$

性质2.22.3.  $\psi' = S\psi, \hat{\psi}' = \Gamma(s;w)S\bar{\Gamma}(s;w)\hat{\psi}, \hat{\psi} = \Gamma(s;w)\psi \Rightarrow \hat{\psi}' = \Gamma(s;w)\psi'$

3 完美常数不变张量  $N_{A_\zeta l_\zeta}^{k_\zeta}(s;w)$ ,  $N_{k_\zeta}^{A_\zeta l_\zeta}(s;w)$ 3.1 常数不变张量  $N_{A_\zeta l_\zeta}^{k_\zeta}(s;w)$ ,  $N_{k_\zeta}^{A_\zeta l_\zeta}(s;w)$  的引入

定义3.1.1.  $N_{A_\zeta l_\zeta}^{k_\zeta}(s;w) := \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \cdots}^{k_\zeta}(s;w)}_{2s} \Gamma_{l_\zeta}^{\overbrace{B_\zeta C_\zeta \cdots}^{2s-1}}(s-\frac{1}{2};w)$ ,  $N_{k_\zeta}^{A_\zeta l_\zeta}(s;w) := \Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \cdots}^{2s}}(s;w) \Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta}(s-\frac{1}{2};w)$

推论3.1.1.  $N(s;w) = [I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)]\Gamma(s;w)$ ,  $\bar{N}(s;w) = \bar{\Gamma}(s;w)[I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)]$

性质3.1.1.  $\Gamma_{\underbrace{A_\zeta B_\zeta C_\zeta \cdots}_{2s}}^{k_\zeta}(s;w) = N_{A_\zeta l_\zeta}^{k_\zeta}(s;w) \Gamma_{\underbrace{B_\zeta C_\zeta \cdots}_{2s-1}}^{l_\zeta}(s-\frac{1}{2};w)$ ,  $\Gamma_{k_\zeta}^{\overbrace{A_\zeta B_\zeta C_\zeta \cdots}^{2s}}(s;w) = N_{k_\zeta}^{A_\zeta l_\zeta}(s;w) \Gamma_{l_\zeta}^{\overbrace{B_\zeta C_\zeta \cdots}^{2s-1}}(s-\frac{1}{2};w)$

推论3.1.2.  $\Gamma(s;w) = [I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)]N(s;w)$ ,  $\bar{\Gamma}(s;w) = \bar{N}(s;w)[I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)]$ ,

性质3.1.2.  $\Gamma(s;w) = [I_{w+1} \otimes \Gamma(s-\frac{1}{2};w)][I_{w+1} \otimes \bar{\Gamma}(s-\frac{1}{2};w)]\Gamma(s;w)$

3.2 常数矩阵  $N_{A_\zeta}(s;w)$ ,  $N^{A_\zeta}(s;w)$ ;  $\bar{N}_{A_\zeta}(s;w)$ ,  $\bar{N}^{A_\zeta}(s;w)$ ;  $N(s;w)$ ,  $\bar{N}(s;w)$  的引入

定义3.2.1.

$$\begin{cases} N_{A_\zeta}(s;w) \prec N_{A_\zeta l_\zeta}^{k_\zeta}(s;w), N^{A_\zeta}(s;w) \prec N_{k_\zeta}^{A_\zeta l_\zeta}(s;w) |_{I_{C_{w+1}^{2s}} \times I_{C_{w+1}^{2s-1}}} \\ \bar{N}_{A_\zeta}(s;w) := N_{A_\zeta}^+(s;w) \succ N_{A_\zeta l_\zeta}^{k_\zeta}(s;w), \bar{N}^{A_\zeta}(s;w) := N^{+A_\zeta}(s;w) \succ N_{k_\zeta}^{A_\zeta l_\zeta}(s;w) |_{I_{C_{w+1}^{2s-1}} \times I_{C_{w+1}^{2s}}} \\ N(s;w) \prec N_{A_\zeta \otimes l_\zeta}^{k_\zeta}(s;w) |_{(w+1)I_{C_{w+1}^{2s-1}} \times I_{C_{w+1}^{2s}}}, \bar{N}(s;w) = N^+(s;w) \prec N_{k_\zeta}^{A_\zeta \otimes l_\zeta}(s;w) |_{I_{C_{w+1}^{2s}} \times (w+1)I_{C_{w+1}^{2s-1}}} \end{cases}$$

推论3.2.1.  $[N_{A_\zeta}(s; w)] = C_{w+1}^{2s} \times C_{w+1}^{2s-1}$ ,  $[\bar{N}_{A_\zeta}(s; w)] = C_{w+1}^{2s-1} \times C_{w+1}^{2s}$

推论3.2.2.  $[N(s; w)] = (w+1)C_{w+1}^{2s-1} \times C_{w+1}^{2s}$ ,  $[\bar{N}(s; w)] = C_{w+1}^{2s} \times (w+1)C_{w+1}^{2s-1}$

### 3.3 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的基本性质

相等性:

性质3.3.1.

$$\begin{cases} N_{A'_\zeta l'_\zeta}^{k'_\zeta}(s; w) \simeq N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) \simeq N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) \simeq N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; w) \\ [N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)]^* \simeq N_{A'_\zeta l'_\zeta}^{k'_\zeta}(s; w), [N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)]^* \simeq N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; w) \end{cases}$$

推论3.3.1.

$$\begin{cases} N_{A_\zeta}(s; w) \simeq N^{A_\zeta}(s; w) \simeq N_{A'_\zeta}(s; w) \simeq N^{A'_\zeta}(s; w); \bar{N}_{A_\zeta}(s; w) \simeq \bar{N}^{A_\zeta}(s; w) \simeq \bar{N}_{A'_\zeta}(s; w) \simeq \bar{N}^{A'_\zeta}(s; w) \\ N_{A_\zeta}(s; w) = N_{A_\zeta}^*(s; w), \bar{N}_{A_\zeta}(s; w) = \bar{N}_{A_\zeta}^*(s; w); N(s; w) = N^*(s; w), \bar{N}(s; w) = \bar{N}^*(s; w) \end{cases}$$

### 3.4 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的正交性质

正交性:

$$\text{引理3.4.1. } N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; w) = \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}^{2s}(s; w)$$

证明:  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k'_\zeta}^{A'_\zeta l'_\zeta}(s; w)$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}_{2s-1}(s - \frac{1}{2}; w) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}^{2s}(s; w) \underbrace{\Gamma_{B'_\zeta C'_\zeta \dots}^{l'_\zeta}}_{2s-1}(s - \frac{1}{2}; w)$$

$$= \frac{1}{(2s-1)!} \delta_{[B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots] \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}^{2s}(s; w)$$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}^{2s}(s; w) \overbrace{\delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \dots}^{2s-1}$$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \overbrace{\Gamma_{k'_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}^{2s}(s; w)$$

□

定理3.4.1.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{m_\zeta}^{A_\zeta l_\zeta}(s; w) = \delta_{m_\zeta}^{k_\zeta} [\Leftrightarrow] N^{A_\zeta}(s; w) \bar{N}_{A_\zeta}(s; w) = I_{C_{w+1}^{2s}} [\Leftrightarrow] \bar{N}(s; w) N(s; w) = I_{C_{w+1}^{2s}} \\ N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{A_\zeta m_\zeta}(s; w) = \frac{w+(2s)\%2}{2s} \delta_{l_\zeta}^{m_\zeta} [\Leftrightarrow] \bar{N}_{A_\zeta}(s; w) N^{A_\zeta}(s; w) = \frac{w+(2s)\%2}{2s} \delta_{l_\zeta}^{m_\zeta} \\ N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{B_\zeta l_\zeta}(s; w) = \frac{1}{2s} [C_{w+1}^{2s-1} - \frac{(2s-1)\%2}{2s-1} C_{w+1}^{2s-2}] \delta_{A_\zeta}^{B_\zeta} [\Leftrightarrow] \text{tr}[\bar{N}_{A_\zeta}(s; w) N^{B_\zeta}(s; w)] = \frac{1}{2s} [C_{w+1}^{2s-1} - \frac{(2s-1)\%2}{2s-1} C_{w+1}^{2s-2}] \delta_{A_\zeta}^{B_\zeta} \end{cases}$$

证明:  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{m_\zeta}^{A_\zeta l_\zeta}(s; w)$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \overbrace{\Gamma_{m_\zeta}^{A_\zeta B_\zeta C_\zeta \dots}}^{2s}(s; w)$$

$$= \delta_{m_\zeta}^{k_\zeta}$$

□

证明:  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{A_\zeta m_\zeta}(s; w)$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}_{2s-1}(s - \frac{1}{2}; w) \overbrace{\Gamma_{k_\zeta}^{A_\zeta B'_\zeta C'_\zeta \dots}}^{2s}(s; w) \underbrace{\Gamma_{B'_\zeta C'_\zeta \dots}^{m_\zeta}}_{2s-1}(s - \frac{1}{2}; w)$$

$$= \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s}(s; w) \overbrace{\Gamma_{k_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots}}^{2s}(s; w) \delta_{A'_\zeta}^{A_\zeta} \underbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{B'_\zeta C'_\zeta \dots}^{m_\zeta}}_{2s-1}(s - \frac{1}{2}; w)$$

$$\begin{aligned}
&= \frac{1}{(2s)!} \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} \delta_{A'_\zeta}^{A_\zeta} \Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}; w) \\
&= \frac{1}{(2s)!} \delta_{A_\zeta}^{A'_\zeta} \Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots} (s - \frac{1}{2}; w) \Gamma_{B_\zeta C_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}; w) \\
&= \frac{1}{(2s)!} [\delta_{A_\zeta}^{A'_\zeta} \Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots} (s - \frac{1}{2}; w) \Gamma_{B_\zeta C_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}; w) - (2s-1) \% 2 \Gamma_{l_\zeta}^{A_\zeta C_\zeta \dots} (s - \frac{1}{2}; w) \Gamma_{A_\zeta C_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}; w)] \\
&= \frac{1}{(2s)!} [(2s-1)! \delta_{A_\zeta}^{A'_\zeta} \Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots} (s - \frac{1}{2}; w) \Gamma_{B_\zeta C_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}; w) - (2s-1) \% 2 (2s-1)! \Gamma_{l_\zeta}^{A_\zeta C_\zeta \dots} (s - \frac{1}{2}; w) \Gamma_{A_\zeta C_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}; w)] \\
&= \frac{1}{(2s)!} [(2s-1)! \delta_{A_\zeta}^{A'_\zeta} - (2s-1) \% 2 (2s-1)!] \delta_{l_\zeta}^{m_\zeta} \\
&= \frac{w+1 - (2s-1) \% 2}{2s} \delta_{l_\zeta}^{m_\zeta} = \frac{w+(2s) \% 2}{2s} \delta_{l_\zeta}^{m_\zeta} \quad \square
\end{aligned}$$

证明:  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{A'_\zeta l_\zeta}(s; w)$

$$\begin{aligned}
&= \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; w) \Gamma_{k_\zeta}^{A'_\zeta B_\zeta C_\zeta \dots} (s; w) \\
&= \frac{1}{(2s)!} \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} \\
&= \frac{1}{(2s)!} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} - \overbrace{\delta_{A_\zeta}^{B'_\zeta} \delta_{B_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} + \overbrace{\delta_{A_\zeta}^{C'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{A'_\zeta} \dots}^{2s} - \dots] \\
&= \frac{1}{(2s)!} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} - \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s} + \overbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{A_\zeta}^{A'_\zeta} \dots}^{2s} - \dots] \\
&= \frac{1}{(2s)!} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1} - (2s-1) \% 2 \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1}] \\
&= \frac{1}{(2s)!} [\delta_{A_\zeta}^{A'_\zeta} (2s-1)! \Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots} (s - \frac{1}{2}; w) \Gamma_{B_\zeta C_\zeta \dots}^{l_\zeta} (s - \frac{1}{2}; w) - (2s-1) \% 2 \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1}] \\
&= \frac{1}{(2s)!} [\delta_{A_\zeta}^{A'_\zeta} (2s-1)! \delta_{l_\zeta}^{l_\zeta} (s - \frac{1}{2}; w) - (2s-1) \% 2 \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1}] \\
&= \frac{1}{(2s)!} [\delta_{A_\zeta}^{A'_\zeta} (2s-1)! C_{w+1}^{2s-1} - (2s-1) \% 2 \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1}] \\
&= \delta_{A_\zeta}^{A'_\zeta} \frac{C_{w+1}^{2s-1}}{2s} - \frac{(2s-1) \% 2}{2s} \frac{1}{(2s-1)!} \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}^{2s-1} \\
&= \delta_{A_\zeta}^{A'_\zeta} \frac{C_{w+1}^{2s-1}}{2s} - \frac{(2s-1) \% 2}{2s} [\delta_{A_\zeta}^{A'_\zeta} \frac{C_{w+1}^{2s-2}}{2s-1} - \frac{(2s-2) \% 2}{2s-1} \frac{1}{(2s-2)!} \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{D_\zeta}^{D'_\zeta} \dots}^{2s-2}] \\
&= [\frac{C_{w+1}^{2s-1}}{2s} - \frac{(2s-1) \% 2}{2s} \frac{C_{w+1}^{2s-2}}{2s-1}] \delta_{A_\zeta}^{A'_\zeta} \\
&= \frac{1}{2s} [C_{w+1}^{2s-1} - \frac{(2s-1) \% 2}{2s-1} C_{w+1}^{2s-2}] \delta_{A_\zeta}^{A'_\zeta} \quad \square
\end{aligned}$$

性质3.4.1. 
$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{B_\zeta m_\zeta}(s; w) = \frac{1}{2s} [\delta_{A_\zeta}^{B_\zeta} \delta_{l_\zeta}^{m_\zeta} - (2s-1) \% 2 N_{A_\zeta n_\zeta}^{m_\zeta}(s - \frac{1}{2}; w) N_{l_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2}; w)] \\ \bar{N}_{A_\zeta}(s; w) N^{B_\zeta}(s; w) = \frac{1}{2s} [\delta_{A_\zeta}^{B_\zeta} I_{C_{w+1}^{2s-1}} - (2s-1) \% 2 N^{B_\zeta}(s - \frac{1}{2}; w) \bar{N}_{A_\zeta}(s - \frac{1}{2}; w)] \end{cases}$$

证明:  $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) N_{k_\zeta}^{A'_\zeta m_\zeta}(s; w)$

$$= \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; w) \Gamma_{l_\zeta}^{B_\zeta C_\zeta \dots} (s - \frac{1}{2}; w) \Gamma_{k_\zeta}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w) \Gamma_{B'_\zeta C'_\zeta \dots}^{m_\zeta} (s - \frac{1}{2}; w)$$

$$\begin{aligned}
 &= \frac{1}{(2s)!} \overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots} \overbrace{\Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots}}^{2s-1} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) \\
 &= \frac{1}{(2s)!} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} - \overbrace{\delta_{A_\zeta}^{B'_\zeta} \delta_{B_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} + \overbrace{\delta_{A_\zeta}^{C'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{A'_\zeta} \cdots}^{2s} - \cdots] \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) \\
 &= \frac{1}{(2s)!} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} - \overbrace{\delta_{A_\zeta}^{B'_\zeta} \delta_{B_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} + \overbrace{\delta_{A_\zeta}^{C'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{A'_\zeta} \cdots}^{2s} - \cdots] \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) \\
 &= \frac{1}{2s} (\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} - \overbrace{\delta_{A_\zeta}^{B'_\zeta} \delta_{B_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} + \overbrace{\delta_{A_\zeta}^{C'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{A'_\zeta} \cdots}^{2s} - \cdots) \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) \\
 &= \frac{1}{2s} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s} - (2s-1) \overbrace{2 \delta_{A_\zeta}^{B'_\zeta} \delta_{B_\zeta}^{A'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \cdots}^{2s}] \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) \\
 &= \frac{1}{2s} [\overbrace{\delta_{A_\zeta}^{A'_\zeta} \Gamma_{l_\zeta}^{B_\zeta C_\zeta \cdots}}^{2s-1} (s - \frac{1}{2}; w) \Gamma_{B'_\zeta C'_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w) - (2s-1) \overbrace{2 \Gamma_{l_\zeta}^{A'_\zeta C'_\zeta \cdots}}^{2s-1} (s - \frac{1}{2}; w) \Gamma_{A_\zeta C_\zeta \cdots}^{m_\zeta} (s - \frac{1}{2}; w)] \\
 &= \frac{1}{2s} [\delta_{A_\zeta}^{A'_\zeta} \delta_{l_\zeta}^{m_\zeta} - (2s-1) \overbrace{2 N_{A_\zeta n_\zeta}^{m_\zeta}}^{2s-1} (s - \frac{1}{2}; w) N_{l_\zeta}^{A'_\zeta n_\zeta} (s - \frac{1}{2}; w)]
 \end{aligned}$$

□

### 3.5 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的升降指标 (存在 $\varepsilon_{A_\zeta B_\zeta}$ 为前提条件)

升降指标:

性质3.5.1.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) = \varepsilon^{k_\zeta m_\zeta}(s; w) \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; w) N_{m_\zeta}^{B_\zeta n_\zeta}(s; w) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) = \varepsilon_{k_\zeta m_\zeta}(s; w) \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; w) N_{B_\zeta n_\zeta}^{m_\zeta}(s; w) \end{cases}$$

证明:  $\varepsilon^{k_\zeta m_\zeta}(s; w) \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; w) N_{m_\zeta}^{B_\zeta n_\zeta}(s; w)$

$$\begin{aligned}
 &= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} (s; w) \varepsilon_{A'_\zeta E'_\zeta} \varepsilon_{B'_\zeta F'_\zeta} \varepsilon_{C'_\zeta G'_\zeta} \cdots \Gamma_{E'_\zeta F'_\zeta G'_\zeta \cdots}^{m_\zeta} (s; w) \varepsilon_{A_\zeta B_\zeta} \\
 &\Gamma_{l_\zeta}^{B''_\zeta C''_\zeta \cdots} (s - \frac{1}{2}; w) \varepsilon_{B''_\zeta F''_\zeta} \varepsilon_{C''_\zeta G''_\zeta} \cdots \Gamma_{n_\zeta}^{F''_\zeta G''_\zeta \cdots} (s - \frac{1}{2}; w) N_{m_\zeta}^{B_\zeta n_\zeta}(s; w) \\
 &= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} (s; w) \varepsilon_{A'_\zeta E'_\zeta} \varepsilon_{B'_\zeta F'_\zeta} \varepsilon_{C'_\zeta G'_\zeta} \cdots \Gamma_{E'_\zeta F'_\zeta G'_\zeta \cdots}^{m_\zeta} (s; w) \varepsilon_{A_\zeta B_\zeta} \Gamma_{l_\zeta}^{B''_\zeta C''_\zeta \cdots} (s - \frac{1}{2}; w) \varepsilon_{B''_\zeta F''_\zeta} \varepsilon_{C''_\zeta G''_\zeta} \cdots \Gamma_{m_\zeta}^{B_\zeta F''_\zeta G''_\zeta \cdots} (s; w) \\
 &= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} (s; w) \varepsilon_{A'_\zeta E'_\zeta} \varepsilon_{B'_\zeta F'_\zeta} \varepsilon_{C'_\zeta G'_\zeta} \cdots \frac{1}{(2s)!} \underbrace{\delta_{[E'_\zeta} \delta_{F'_\zeta}^{F''_\zeta} \delta_{G'_\zeta}^{G''_\zeta} \cdots]}_{2s} \varepsilon_{A_\zeta B_\zeta} \underbrace{\varepsilon_{B''_\zeta F''_\zeta} \varepsilon_{C''_\zeta G''_\zeta} \cdots}_{2s-1} \Gamma_{l_\zeta}^{B''_\zeta C''_\zeta \cdots} (s - \frac{1}{2}; w) \\
 &= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} (s; w) \varepsilon_{A'_\zeta B_\zeta} \varepsilon_{B'_\zeta F''_\zeta} \varepsilon_{C'_\zeta G''_\zeta} \cdots \varepsilon_{A_\zeta B_\zeta} \varepsilon_{B''_\zeta F''_\zeta} \varepsilon_{C''_\zeta G''_\zeta} \cdots \Gamma_{l_\zeta}^{B''_\zeta C''_\zeta \cdots} (s - \frac{1}{2}; w) \\
 &= \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} (s; w) \underbrace{\delta_{A'_\zeta}^{A'_\zeta} \delta_{B'_\zeta}^{B'_\zeta} \delta_{C'_\zeta}^{C'_\zeta} \cdots}_{2s} \Gamma_{l_\zeta}^{B''_\zeta C''_\zeta \cdots} (s - \frac{1}{2}; w) \\
 &= N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)
 \end{aligned}$$

□

推论3.5.1.



$$\begin{cases} N_{A_\zeta}(s; w)\varepsilon(s - \frac{1}{2}; w) = \varepsilon_{A_\zeta B_\zeta}\varepsilon(s; w)N^{B_\zeta}(s; w), \varepsilon(s - \frac{1}{2}; w)\bar{N}_{A_\zeta}(s; w) = \bar{N}^{B_\zeta}(s; w)\varepsilon_{B_\zeta A_\zeta}\varepsilon(s; w) \\ N^{A_\zeta}(s; w)\varepsilon(s - \frac{1}{2}; w) = \varepsilon^{A_\zeta B_\zeta}\varepsilon(s; w)N_{B_\zeta}(s; w), \varepsilon(s - \frac{1}{2}; w)\bar{N}^{A_\zeta}(s; w) = \bar{N}_{B_\zeta}(s; w)\varepsilon^{B_\zeta A_\zeta}\varepsilon(s; w) \\ N(s; w)\varepsilon(s; w) = [\varepsilon(\frac{1}{2}; w) \otimes \varepsilon(s - \frac{1}{2}; w)]N(s; w), \varepsilon(s; w)\bar{N}(s; w) = \bar{N}(s; w)[\varepsilon(\frac{1}{2}; w) \otimes \varepsilon(s - \frac{1}{2}; w)] \end{cases}$$

$$\text{证明: } \Gamma(s; w)\varepsilon(s; w) = \underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s} \Gamma(s; w)$$

$$\Leftrightarrow [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Gamma(s; w)\varepsilon(s; w) = [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s} \Gamma(s; w)$$

$$\Leftrightarrow N(s; w)\varepsilon(s; w) = [I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s} \Gamma(s; w)$$

$$\Leftrightarrow N(s; w)\varepsilon(s; w) = \{\varepsilon(\frac{1}{2}; w) \otimes [\bar{\Gamma}(s - \frac{1}{2}; w) \underbrace{\varepsilon(\frac{1}{2}; w) \otimes \cdots \otimes \varepsilon(\frac{1}{2}; w)}_{2s-1}]\} \Gamma(s; w)$$

$$\Leftrightarrow N(s; w)\varepsilon(s; w) = [\varepsilon(\frac{1}{2}; w) \otimes \varepsilon(s - \frac{1}{2}; w)][I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Gamma(s; w)$$

$$\Leftrightarrow N(s; w)\varepsilon(s; w) = [\varepsilon(\frac{1}{2}; w) \otimes \varepsilon(s - \frac{1}{2}; w)]N(s; w) \quad \square$$

### Penrose标准升降规则:

#### 性质3.5.2.

$$\begin{cases} N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) = (-1)^{2s} [\zeta^{2s} \varepsilon^{k_\zeta m_\zeta}(s; w)] (-\zeta \varepsilon_{A_\zeta B_\zeta}) [(-\zeta)^{2s-1} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; w)] N_{m_\zeta}^{B_\zeta n_\zeta}(s; w) \\ N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) = (-1)^{2s} [(-\zeta)^{2s} \varepsilon_{k_\zeta m_\zeta}(s; w)] (\zeta \varepsilon^{A_\zeta B_\zeta}) [\zeta^{2s-1} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; w)] N_{B_\zeta n_\zeta}^{m_\zeta}(s; w) \end{cases}$$

### 3.6 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的自旋矩阵变换

#### 性质3.6.1.

$$\begin{cases} N_{k_\zeta}^{A_\zeta m_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) N_{B_\zeta m_\zeta}^{l_\zeta}(s; w) = \frac{1}{2s} \sigma^{\alpha_\zeta}_{k_\zeta l_\zeta}(s; w) [\Leftrightarrow] N^{A_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) = \frac{1}{2s} \sigma^{\alpha_\zeta}(s; w) \\ [\Leftrightarrow] \bar{N}(s; w) \sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} N(s; w) = \frac{1}{2s} \sigma(s; w) \end{cases}$$

$$\text{证明: } N_{k_\zeta}^{A_\zeta m_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) N_{A'_\zeta m_\zeta}^{l_\zeta}(s; w)$$

$$= \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}}_{2s}(s; w) \underbrace{\Gamma_{B_\zeta C_\zeta \cdots}^{m_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{l_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w)$$

$$= \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}}_{2s}(s; w) \frac{1}{(2s-1)!} \delta_{[B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{l_\zeta}}_{2s}(s; w)$$

$$= \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}}_{2s}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{l_\zeta}}_{2s}(s; w)$$

$$= \frac{1}{2s} \sigma^{\alpha_\zeta}_{m_\zeta l_\zeta}(s - \frac{1}{2}; w) \quad \square$$

$$\text{性质3.6.2. } N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) N_{B_\zeta m_\zeta}^{k_\zeta}(s; w) = -\frac{(2s-1)! 2}{2s(2s-1)} \sigma^{\alpha_\zeta}_{m_\zeta l_\zeta}(s - \frac{1}{2}; w)$$

$$[\Leftrightarrow] \bar{N}_{B_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta B_\zeta}(\frac{1}{2}; w) N^{A_\zeta}(s; w) = -\frac{(2s-1)! 2}{2s(2s-1)} \sigma^{\alpha_\zeta}(s - \frac{1}{2}; w)$$

$$\text{证明: } N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) N_{A'_\zeta m_\zeta}^{k_\zeta}(s; w)$$

$$= \underbrace{\Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots}}_{2s}(s; w) \underbrace{\Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta}}_{2s}(s; w) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w)$$

$$= \frac{1}{(2s)!} \delta_{[A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w)$$

$$= \frac{1}{(2s)!} [\delta_{A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] - \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots + \delta_{C'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{A'_\zeta}^{C_\zeta} \cdots - \cdots] \sigma^{\alpha_\zeta}_{A_\zeta A'_\zeta}(\frac{1}{2}; w) \underbrace{\Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta}}_{2s-1}(s - \frac{1}{2}; w) \underbrace{\Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots}}_{2s-1}(s - \frac{1}{2}; w)$$

$$\begin{aligned}
 &= \frac{1}{(2s)!} [\delta_{A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] - (2s-1) \% 2 \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] \sigma^{\alpha_\zeta} A'_\zeta \left(\frac{1}{2}; w\right) \Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta} \left(s - \frac{1}{2}; w\right) \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right) \\
 &= \frac{-(2s-1) \% 2}{(2s)!} \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] \sigma^{\alpha_\zeta} A'_\zeta \left(\frac{1}{2}; w\right) \Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta} \left(s - \frac{1}{2}; w\right) \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right) \\
 &= \frac{-(2s-1) \% 2}{(2s)!} \sigma^{\alpha_\zeta} B'_\zeta A'_\zeta \left(\frac{1}{2}; w\right) \Gamma_{A'_\zeta C'_\zeta \cdots}^{l_\zeta} \left(s - \frac{1}{2}; w\right) \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right) \\
 &= \frac{-(2s-1) \% 2}{2s} \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right) \sigma^{\alpha_\zeta} B'_\zeta A'_\zeta \left(\frac{1}{2}; w\right) \Gamma_{A'_\zeta C'_\zeta \cdots}^{l_\zeta} \left(s - \frac{1}{2}; w\right) \\
 &= -\frac{(2s-1) \% 2}{2s(2s-1)} \sigma^{\alpha_\zeta} m_\zeta l_\zeta \left(s - \frac{1}{2}; w\right) \quad \square
 \end{aligned}$$

性质3.6.3.

$$\begin{cases} N_{k_\zeta}^{A_\zeta m_\zeta}(s; w) S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) N_{B_\zeta m_\zeta}^{l_\zeta}(s; w) = \frac{1}{2s} S_{ab k_\zeta} l_\zeta(s; w) [\Leftrightarrow] N^{A_\zeta}(s; w) S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) \bar{N}_{B_\zeta}(s; w) = \frac{1}{2s} S_{ab}(s, \varsigma; w) \\ [\Leftrightarrow] \bar{N}(s; w) S_{ab} \left(\frac{1}{2}, \varsigma; w\right) \otimes I_{C_{w+1}^{2s-1}} N(s; w) = \frac{1}{2s} S_{ab}(s, \varsigma; w) \end{cases}$$

证明:  $N_{k_\zeta}^{A_\zeta m_\zeta}(s; w) S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) N_{A'_\zeta m_\zeta}^{l_\zeta}(s; w)$

$$\begin{aligned}
 &= \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots} \left(s; w\right) \Gamma_{B_\zeta C_\zeta \cdots}^{m_\zeta} \left(s - \frac{1}{2}; w\right) S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{l_\zeta} \left(s; w\right) \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right) \\
 &= \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots} \left(s; w\right) \frac{1}{(2s-1)!} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{l_\zeta} \left(s; w\right) \\
 &= \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots} \left(s; w\right) S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{l_\zeta} \left(s; w\right) \\
 &= \frac{1}{2s} \sigma^{\alpha_\zeta} m_\zeta l_\zeta \left(s - \frac{1}{2}; w\right) \quad \square
 \end{aligned}$$

性质3.6.4.  $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) N_{B_\zeta m_\zeta}^{k_\zeta}(s; w) = -\frac{(2s-1) \% 2}{2s(2s-1)} S_{ab m_\zeta} l_\zeta \left(s - \frac{1}{2}; w\right)$   
 $[\Leftrightarrow] \bar{N}_{B_\zeta}(s; w) S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) N^{A_\zeta}(s; w) = -\frac{(2s-1) \% 2}{2s(2s-1)} S_{ab} \left(s - \frac{1}{2}, \varsigma; w\right)$

证明:  $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) N_{A'_\zeta m_\zeta}^{k_\zeta}(s; w)$

$$\begin{aligned}
 &= \Gamma_{k_\zeta}^{A_\zeta B_\zeta C_\zeta \cdots} \left(s; w\right) \Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta} \left(s - \frac{1}{2}; w\right) S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \cdots}^{k_\zeta} \left(s; w\right) \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right) \\
 &= \frac{1}{(2s)!} \delta_{A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) \Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta} \left(s - \frac{1}{2}; w\right) \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right) \\
 &= \frac{1}{(2s)!} [\delta_{A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] - \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] + \delta_{C'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{A'_\zeta}^{C_\zeta} \cdots] - \cdots] S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) \Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta} \left(s - \frac{1}{2}; w\right) \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right) \\
 &= \frac{1}{(2s)!} [\delta_{A'_\zeta}^{A_\zeta} \delta_{B'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] - (2s-1) \% 2 \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) \Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta} \left(s - \frac{1}{2}; w\right) \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right) \\
 &= \frac{-(2s-1) \% 2}{(2s)!} \delta_{B'_\zeta}^{A_\zeta} \delta_{A'_\zeta}^{B_\zeta} \delta_{C'_\zeta}^{C_\zeta} \cdots] S_{ab A_\zeta B_\zeta} \left(\frac{1}{2}; w\right) \Gamma_{B_\zeta C_\zeta \cdots}^{l_\zeta} \left(s - \frac{1}{2}; w\right) \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right) \\
 &= \frac{-(2s-1) \% 2}{(2s)!} \sigma^{\alpha_\zeta} B'_\zeta A'_\zeta \left(\frac{1}{2}; w\right) \Gamma_{A'_\zeta C'_\zeta \cdots}^{l_\zeta} \left(s - \frac{1}{2}; w\right) \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \cdots} \left(s - \frac{1}{2}; w\right)
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{(2s-1)!2}{2s} \Gamma_{m_\zeta}^{B'_\zeta C'_\zeta \dots} (s - \frac{1}{2}; w) \sigma^{\alpha_\zeta} B'_\zeta A'_\zeta (\frac{1}{2}; w) \Gamma_{A'_\zeta C'_\zeta \dots}^{l_\zeta} (s - \frac{1}{2}; w) \\
&= -\frac{(2s-1)!2}{2s(2s-1)} \sigma^{\alpha_\zeta} m_\zeta l_\zeta (s - \frac{1}{2}; w)
\end{aligned}$$

□

### 3.7 常数不变张量 $N_{A_\zeta l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_\zeta l_\zeta}(s; w)$ 的置换性质

定理3.7.1.

$$\begin{cases}
\sigma^{\alpha_\zeta} A_\zeta B_\zeta (\frac{1}{2}; w) N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + \sigma^{\alpha_\zeta} l_\zeta m_\zeta (s - \frac{1}{2}; w) N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w) \sigma^{\alpha_\zeta} j_\zeta k_\zeta (s; w) \\
N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) \sigma^{\alpha_\zeta} A_\zeta B_\zeta (\frac{1}{2}; w) + N_{k_\zeta}^{B_\zeta m_\zeta}(s; w) \sigma^{\alpha_\zeta} m_\zeta l_\zeta (s - \frac{1}{2}; w) = \sigma^{\alpha_\zeta} k_\zeta j_\zeta (s; w) N_{j_\zeta}^{B_\zeta l_\zeta}(s; w) \\
\sigma^{\alpha_\zeta} A_\zeta B_\zeta (\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) + \sigma^{\alpha_\zeta} (s - \frac{1}{2}; w) \bar{N}_{A_\zeta}(s; w) = \bar{N}_{A_\zeta}(s; w) \sigma^{\alpha_\zeta}(s; w) \\
N^{A_\zeta}(s; w) \sigma^{\alpha_\zeta} A_\zeta B_\zeta (\frac{1}{2}; w) + N^{B_\zeta}(s; w) \sigma^{\alpha_\zeta} (s - \frac{1}{2}; w) = \sigma^{\alpha_\zeta}(s; w) N^{B_\zeta}(s; w) \\
[\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma^{\alpha_\zeta}(s - \frac{1}{2}; w)] N(s; w) = N(s; w) \sigma^{\alpha_\zeta}(s; w) \\
\bar{N}(s; w) [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma^{\alpha_\zeta}(s - \frac{1}{2}; w)] = \sigma^{\alpha_\zeta}(s; w) \bar{N}(s; w)
\end{cases}$$

$$\begin{aligned}
\text{证明: } & \Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta} (s; w) = \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; w) \sigma_{k_\zeta}^{l_\zeta} (s; w) \\
& \Rightarrow \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} \Omega_{A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta} (s; w) = \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; w) \sigma_{k_\zeta}^{l_\zeta} (s; w) \\
& \Leftrightarrow \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} [\sigma_{A_\zeta} A'_\zeta (\frac{1}{2}; w) \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s-1} + \delta_{A_\zeta}^{A'_\zeta} \Omega_{B_\zeta C_\zeta \dots}^{B'_\zeta C'_\zeta \dots} (s - \frac{1}{2}; w)] \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta} (s; w) = \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; w) \sigma_{k_\zeta}^{l_\zeta} (s; w) \\
& \Leftrightarrow [\sigma_{A_\zeta} A'_\zeta (\frac{1}{2}; w) \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} + \delta_{A_\zeta}^{A'_\zeta} \sigma_{j_\zeta} n_\zeta (s - \frac{1}{2}; w) \Gamma_{n_\zeta}^{B_\zeta C_\zeta \dots} (s; w)] \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta} (s; w) = \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; w) \sigma_{k_\zeta}^{l_\zeta} (s; w) \\
& \Leftrightarrow \sigma_{A_\zeta} A'_\zeta (\frac{1}{2}; w) N_{A_\zeta j_\zeta}^{l_\zeta}(s; w) + \sigma_{j_\zeta} n_\zeta (s - \frac{1}{2}; w) N_{A_\zeta n_\zeta}^{l_\zeta}(s; w) = N_{A_\zeta j_\zeta}^{k_\zeta}(s; w) \sigma_{k_\zeta}^{l_\zeta}(s; w) \\
& \Leftrightarrow \sigma^{\alpha_\zeta} A_\zeta B_\zeta (\frac{1}{2}; w) N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + \sigma^{\alpha_\zeta} l_\zeta m_\zeta (s - \frac{1}{2}; w) N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w) \sigma^{\alpha_\zeta} j_\zeta k_\zeta (s; w)
\end{aligned}$$

□

定理3.7.2.

$$\begin{cases}
S_{ab A_\zeta} B_\zeta (\frac{1}{2}; w) N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + S_{ab l_\zeta} m_\zeta (s - \frac{1}{2}; w) N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w) S_{ab j_\zeta} k_\zeta (s; w) \\
N_{k_\zeta}^{A_\zeta l_\zeta}(s; w) S_{ab A_\zeta} B_\zeta (\frac{1}{2}; w) + N_{k_\zeta}^{B_\zeta m_\zeta}(s; w) S_{ab m_\zeta} l_\zeta (s - \frac{1}{2}; w) = S_{ab k_\zeta} j_\zeta (s; w) N_{j_\zeta}^{B_\zeta l_\zeta}(s; w) \\
S_{ab A_\zeta} B_\zeta (\frac{1}{2}; w) \bar{N}_{B_\zeta}(s; w) + S_{ab} (s - \frac{1}{2}, \zeta; w) \bar{N}_{A_\zeta}(s; w) = \bar{N}_{A_\zeta}(s; w) S_{ab}(s, \zeta; w) \\
N^{A_\zeta}(s; w) S_{ab A_\zeta} B_\zeta (\frac{1}{2}; w) + N^{B_\zeta}(s; w) S_{ab} (s - \frac{1}{2}, \zeta; w) = S_{ab}(s, \zeta; w) N^{B_\zeta}(s; w) \\
[\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab} (s - \frac{1}{2}, \zeta; w)] N(s; w) = N(s; w) S_{ab}(s, \zeta; w) \\
\bar{N}(s; w) [S_{ab} (\frac{1}{2}, \zeta; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab} (s - \frac{1}{2}, \zeta; w)] = S_{ab}(s, \zeta; w) \bar{N}(s; w)
\end{cases}$$

$$\begin{aligned}
\text{证明: } & \Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta} (s; w) = \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; w) S_{ab k_\zeta}^{l_\zeta} (s; w) \\
& \Rightarrow \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} \Omega_{ab A_\zeta B_\zeta C_\zeta \dots}^{A'_\zeta B'_\zeta C'_\zeta \dots} (s; w) \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta} (s; w) = \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; w) S_{ab k_\zeta}^{l_\zeta} (s; w) \\
& \Leftrightarrow \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} [S_{ab A_\zeta} A'_\zeta (\frac{1}{2}; w) \underbrace{\delta_{B_\zeta}^{B'_\zeta} \delta_{C_\zeta}^{C'_\zeta} \dots}_{2s-1} + \delta_{A_\zeta}^{A'_\zeta} \Omega_{ab B_\zeta C_\zeta \dots}^{B'_\zeta C'_\zeta \dots} (s - \frac{1}{2}; w)] \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta} (s; w) = \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} (s; w) S_{ab k_\zeta}^{l_\zeta} (s; w)
\end{aligned}$$

$$\begin{aligned} &\Leftrightarrow [S_{abA_\zeta} A'_\zeta(\frac{1}{2}; w) \Gamma_{j_\zeta}^{B'_\zeta C'_\zeta \dots} \overset{2s-1}{\dots} + \delta_{A'_\zeta}^{A_\zeta} \sigma_{j_\zeta}^{n_\zeta}(s - \frac{1}{2}; w) \Gamma_{n_\zeta}^{B'_\zeta C'_\zeta \dots} \overset{2s}{\dots}(s; w)] \Gamma_{A'_\zeta B'_\zeta C'_\zeta \dots}^{l_\zeta}(s; w) = \Gamma_{j_\zeta}^{B_\zeta C_\zeta \dots} \overset{2s-1}{\dots} \Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta} \overset{2s}{\dots}(s; w) S_{abk_\zeta}^{l_\zeta}(s; w) \\ &\Leftrightarrow S_{abA_\zeta} A'_\zeta(\frac{1}{2}; w) N_{A'_\zeta j_\zeta}^{l_\zeta}(s; w) + S_{abj_\zeta}^{n_\zeta}(s - \frac{1}{2}; w) N_{A_\zeta n_\zeta}^{l_\zeta}(s; w) = N_{A_\zeta j_\zeta}^{k_\zeta}(s; w) S_{abk_\zeta}^{l_\zeta}(s; w) \\ &\Leftrightarrow S_{abA_\zeta} B'_\zeta(\frac{1}{2}; w) N_{B'_\zeta l_\zeta}^{k_\zeta}(s; w) + S_{abl_\zeta}^{m_\zeta}(s - \frac{1}{2}; w) N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w) S_{abj_\zeta}^{k_\zeta}(s; w) \quad \square \end{aligned}$$

### 3.8 常数不变张量 $N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w)$ , $N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; w)$ 的引入及其性质

$$\text{定义3.8.1. } \begin{cases} N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w) := \Gamma_{A_{\zeta 1} \dots A_{\zeta 2s}}^{k_\zeta}(s; w) \Gamma_{l_\zeta}^{A_{\zeta n+1} \dots A_{\zeta 2s}}(s - \frac{n}{2}; w) \\ N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; w) := \Gamma_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta 2s}}(s; w) \Gamma_{A_{\zeta n+1} \dots A_{\zeta 2s}}^{l_\zeta}(s - \frac{n}{2}; w) \end{cases}$$

相等性:

$$\text{性质3.8.1. } N_{A'_{\zeta 1} \dots A'_{\zeta n} l'_\zeta}^{k'_\zeta}(s; w) \simeq N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w) \simeq N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; w) \simeq N_{k'_\zeta}^{A'_{\zeta 1} \dots A'_{\zeta n} l'_\zeta}(s; w)$$

$$\text{性质3.8.2. } [N_{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w)]^* \simeq N_{A'_{\zeta 1} \dots A'_{\zeta n} l'_\zeta}^{k'_\zeta}(s; w), [N_{k_\zeta}^{A_{\zeta 1} \dots A_{\zeta n} l_\zeta}(s; w)]^* \simeq N_{k'_\zeta}^{A'_{\zeta 1} \dots A'_{\zeta n} l'_\zeta}(s; w)$$

展开性:

性质3.8.3.

$$\begin{cases} N_{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta n} l_\zeta}^{k_\zeta}(s; w) = N_{A_{\zeta 1} l_{\zeta 1}}^{k_\zeta}(s; w) N_{A_{\zeta 2} l_{\zeta 2}}^{l_{\zeta 1}}(s - \frac{1}{2}; w) \dots N_{A_{\zeta n} l_{\zeta n}}^{l_{\zeta n-1}}(s - \frac{n-1}{2}; w) \\ N_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta n} l_\zeta}(s; w) = N_{k_\zeta}^{A_{\zeta 1} l_{\zeta 1}}(s; w) N_{l_{\zeta 1}}^{A_{\zeta 2} l_{\zeta 2}}(s - \frac{1}{2}; w) \dots N_{l_{\zeta n-1}}^{A_{\zeta n} l_\zeta}(s - \frac{n-1}{2}; w) \end{cases}$$

性质3.8.4.

$$\begin{cases} \Gamma_{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta 2s}}^{k_\zeta}(s; w) = N_{A_{\zeta 1} l_{\zeta 1}}^{k_\zeta}(s; w) N_{A_{\zeta 2} l_{\zeta 2}}^{l_{\zeta 1}}(s - \frac{1}{2}; w) \dots N_{A_{\zeta 2s} l_{\zeta 2s}}^{l_{\zeta 2s-1}}(\frac{1}{2}; w) \\ \Gamma_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta 2s}}(s; w) = N_{k_\zeta}^{A_{\zeta 1} l_{\zeta 1}}(s; w) N_{l_{\zeta 1}}^{A_{\zeta 2} l_{\zeta 2}}(s - \frac{1}{2}; w) \dots N_{l_{\zeta 2s-1}}^{A_{\zeta 2s} l_{\zeta 2s}}(\frac{1}{2}; w) \\ \Gamma_{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta 2s}}^{k_\zeta}(s; w) \succ \Gamma_{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta 2s}}(s; w) = N_{A_{\zeta 1}}(s; w) N_{A_{\zeta 2}}(s - \frac{1}{2}; w) \dots N_{A_{\zeta 2s}}(\frac{1}{2}; w) \\ \Gamma_{k_\zeta}^{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta 2s}}(s; w) \succ \Gamma_{A_{\zeta 1} A_{\zeta 2} \dots A_{\zeta 2s}}(s; w) = N^{A_{\zeta 1}}(s; w) N^{A_{\zeta 2}}(s - \frac{1}{2}; w) \dots N^{A_{\zeta 2s}}(\frac{1}{2}; w) \\ \bar{\Gamma}(s; w) = \bar{N}(s; w) [I_{w+1} \otimes \bar{N}(s - \frac{1}{2}; w)] \cdot [I_{(w+1)^{2s-2}} \otimes \bar{N}(1)] [I_{(w+1)^{2s-1}} \otimes \bar{N}(\frac{1}{2}; w)] \\ \Gamma(s; w) = [I_{(w+1)^{2s-1}} \otimes N(\frac{1}{2}; w)] [I_{(w+1)^{2s-2}} \otimes N(1)] \cdot [I_{w+1} \otimes N(s - \frac{1}{2}; w)] N(s; w) \end{cases}$$

### 3.9 推论1: 常数矩阵 $N(s; w)$ , $\bar{N}(s; w)$ 的几个恒等式

性质3.9.1.

$$\begin{cases} \bar{N}(s; w) [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] N(s; w) = \sigma(s; w) \\ N(s; w) \sigma(s; w) \bar{N}(s; w) = [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] N(s; w) \bar{N}(s; w) \\ N(s; w) \sigma(s; w) \bar{N}(s; w) = N(s; w) \bar{N}(s; w) \{\vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]\}^n \\ [N(s; w) \bar{N}(s; w), \sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] = 0 \end{cases}$$

性质3.9.2.

$$\begin{cases} \bar{N}(s; w) [S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] N(s; w) = S_{ab}(s, \zeta; w) \\ N(s; w) S_{ab}(s, \zeta; w) \bar{N}(s; w) = [S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] N(s; w) \bar{N}(s; w) \\ N(s; w) S_{ab}(s, \zeta; w) \bar{N}(s; w) = N(s; w) \bar{N}(s; w) [S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] \\ [N(s; w) \bar{N}(s; w), S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] = 0 \end{cases}$$

性质3.9.3.

$$\begin{cases} \bar{N}(s; w) \{\vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]\}^n N(s; w) = [\vartheta \cdot \sigma(s; w)]^n \\ N(s; w) [\vartheta \cdot \sigma(s; w)]^n \bar{N}(s; w) = \{\vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]\}^n N(s; w) \bar{N}(s; w) \\ N(s; w) [\vartheta \cdot \sigma(s; w)]^n \bar{N}(s; w) = N(s; w) \bar{N}(s; w) \{\vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]\}^n \\ [N(s; w) \bar{N}(s; w), \{\vartheta \cdot [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]\}^n] = 0 \end{cases}$$

## 性质3.9.4.

$$\begin{cases} \bar{N}(s; w) \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \}^n N(s; w) = [\vartheta^{ab} S_{ab}(s, \varsigma; w)]^n \\ N(s; w) [\vartheta^{ab} S_{ab}(s, \varsigma; w)]^n \bar{N}(s; w) = \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \}^n N(s; w) \bar{N}(s; w) \\ N(s; w) [\vartheta^{ab} S_{ab}(s, \varsigma; w)]^n \bar{N}(s; w) = N(s; w) \bar{N}(s; w) \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \}^n \\ [N(s; w) \bar{N}(s; w), \{ \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] \}^n] = 0 \end{cases}$$

## 推论3.9.1.

$$\begin{cases} \bar{N}(s; w) e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]} N(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \\ N(s; w) e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \bar{N}(s; w) = e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]} N(s; w) \bar{N}(s; w) \\ N(s; w) e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \bar{N}(s; w) = N(s; w) \bar{N}(s; w) e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]} \\ [N(s; w) \bar{N}(s; w), e^{\frac{i}{2} \vartheta^{ab} [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]}] = 0 \end{cases}$$

3.10 推论2: 常数矩阵  $N(s; w)$ ,  $\bar{N}(s; w)$  的另外几个恒等式

## 推论3.10.1.

$$\begin{cases} \bar{N}(s; w) \sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} N(s; w) = \frac{1}{2s} \sigma(s; w) \\ \bar{N}(s; w) I_{w+1} \otimes \sigma(s - \frac{1}{2}; w) N(s; w) = (1 - \frac{1}{2s}) \sigma(s; w) \\ N^{A_c}(s; w) \sigma(s - \frac{1}{2}; w) \bar{N}_{A_c}(s; w) = (1 - \frac{1}{2s}) \sigma(s; w) \\ \bar{N}_{A_c}(s; w) \sigma(s; w) N^{A_c}(s; w) = [\frac{w}{2s} + \frac{(2s)\%2}{2s-1} - \frac{1}{2s(2s-1)}] \sigma(s - \frac{1}{2}; w) \end{cases}$$

## 推论3.10.2.

$$\begin{cases} \bar{N}(s; w) S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} N(s; w) = \frac{1}{2s} S_{ab}(s, \varsigma; w) \\ \bar{N}(s; w) I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w) N(s; w) = (1 - \frac{1}{2s}) S_{ab}(s, \varsigma; w) \\ N^{A_c}(s; w) S_{ab}(s - \frac{1}{2}, \varsigma; w) \bar{N}_{A_c}(s; w) = (1 - \frac{1}{2s}) S_{ab}(s, \varsigma; w) \\ \bar{N}_{A_c}(s; w) S_{ab}(s, \varsigma; w) N^{A_c}(s; w) = [\frac{w}{2s} + \frac{(2s)\%2}{2s-1} - \frac{1}{2s(2s-1)}] S_{ab}(s - \frac{1}{2}, \varsigma; w) \end{cases}$$

## 推论3.10.3.

$$\begin{cases} \bar{N}(1) [\sigma(\frac{1}{2}; w) \otimes I_2 + I_{w+1} \otimes \sigma(\frac{1}{2}; w)] N(1) = \sigma(1) \\ \bar{N}(\frac{3}{2}) \{ \sigma(\frac{1}{2}; w) \otimes I_3 + I_{w+1} \otimes \{ \bar{N}(1) [\sigma(\frac{1}{2}; w) \otimes I_2 + I_{w+1} \otimes \sigma(\frac{1}{2}; w)] N(1) \} \} N(\frac{3}{2}) = \sigma(\frac{3}{2}) \\ \bar{N}(s; w) \cdot \bar{N}(\frac{3}{2}) \{ \sigma(\frac{1}{2}; w) \otimes I_3 + I_{w+1} \otimes \{ \bar{N}(1) [\sigma(\frac{1}{2}; w) \otimes I_2 + I_{w+1} \otimes \sigma(\frac{1}{2}; w)] N(1) \} \} N(\frac{3}{2}) \cdot N(s; w) = \sigma(s; w) \end{cases}$$

3.11 矩阵  $N(s; w)$ ,  $\bar{N}(s; w)$  的常数不变张量性质

$$\text{定理3.11.1. } N(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s - \frac{1}{2}, \varsigma; w)} N(s; w) e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)}$$

$$\text{证明: } [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] N(s; w) = N(s; w) S_{ab}(s, \varsigma; w)$$

$$\Leftrightarrow 0 = [\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + \frac{i}{2} \vartheta^{ab} I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] N(s; w) - \frac{i}{2} \vartheta^{ab} N(s; w) S_{ab}(s, \varsigma; w)$$

$$\Leftrightarrow N(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s - \frac{1}{2}, \varsigma; w)} N(s; w) e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \quad \square$$

$$\text{定理3.11.2. } \bar{N}(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \bar{N}(s; w) e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(s - \frac{1}{2}, \varsigma; w)}$$

$$\text{证明: } \bar{N}(s; w) [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] = S_{ab}(s, \varsigma; w) \bar{N}(s; w)$$

$$\Leftrightarrow 0 = \frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w) \bar{N}(s; w) - \bar{N}(s; w) [\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + \frac{i}{2} \vartheta^{ab} I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]$$

$$\Leftrightarrow \bar{N}(s; w) = e^{\frac{i}{2} \vartheta^{ab} S_{ab}(s, \varsigma; w)} \bar{N}(s; w) e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2} \vartheta^{ab} S_{ab}(s - \frac{1}{2}, \varsigma; w)} \quad \square$$

### 3.12 两个定理的另外一种证明

**定理3.12.1.**  $\Omega(s)\Gamma(s) = \Gamma(s)\sigma(s), \bar{\Gamma}(s)\Omega(s) = \sigma(s)\bar{\Gamma}(s)$

证明: 采用数学归纳法

1: 当  $s = \frac{1}{2}$  时,  $\sigma(\frac{1}{2})\Gamma(\frac{1}{2}) = \Gamma(\frac{1}{2})\sigma(\frac{1}{2})$  成立。

2: 假设  $s = k$  时,  $\Omega(k)\Gamma(k) = \Gamma(k)\sigma(k)$  成立。

3: 当  $s = k + \frac{1}{2}$  时

$$\begin{aligned} & \Omega(k + \frac{1}{2})\Gamma(k + \frac{1}{2}) \\ &= [\sigma(\frac{1}{2}) \otimes I_{2^{2k}} + I \otimes \Omega(k)][I \otimes \Gamma(k)]N(k + \frac{1}{2}) \\ &= \{[I \otimes \Gamma(k)][\sigma(\frac{1}{2}) \otimes I_{2(k+1)}] + I \otimes [\Gamma(k)\sigma(k)]\}N(k + \frac{1}{2}) \\ &= [I \otimes \Gamma(k)][\sigma(\frac{1}{2}) \otimes I_{2(k+1)} + I \otimes \sigma(k)]N(k + \frac{1}{2}) \\ &= [I \otimes \Gamma(k)]N(k + \frac{1}{2})\sigma(k + \frac{1}{2}) \\ &= \Gamma(k + \frac{1}{2})\sigma(k + \frac{1}{2}) \end{aligned}$$

所以命题成立, 证毕。 □

**定理3.12.2.**  $\bar{\Gamma}(s)\Omega(s) = \sigma(s)\bar{\Gamma}(s)$

证明: 采用数学归纳法

1: 当  $s = \frac{1}{2}$  时,  $\bar{\Gamma}(\frac{1}{2})\sigma(\frac{1}{2}) = \sigma(\frac{1}{2})\bar{\Gamma}(\frac{1}{2})$  成立。

2: 假设  $s = k$  时,  $\bar{\Gamma}(k)\Omega(k) = \sigma(k)\bar{\Gamma}(k)$  成立。

3: 当  $s = k + \frac{1}{2}$  时

$$\begin{aligned} & \bar{\Gamma}(k + \frac{1}{2})\Omega(k + \frac{1}{2}) \\ &= \bar{N}(k + \frac{1}{2})[I \otimes \bar{\Gamma}(k)]\{\sigma(\frac{1}{2}) \otimes I_{2^{2k}} + I \otimes \Omega(k)\} \\ &= \bar{N}(k + \frac{1}{2})\{[\sigma(\frac{1}{2}) \otimes I_{2(k+1)}][I \otimes \bar{\Gamma}(k)] + [I \otimes \sigma(k)][I \otimes \bar{\Gamma}(k)]\} \\ &= \bar{N}(k + \frac{1}{2})[\sigma(\frac{1}{2}) \otimes I_{2(k+1)} + I \otimes \sigma(k)][I \otimes \bar{\Gamma}(k)] \\ &= \sigma(k + \frac{1}{2})\bar{N}(k + \frac{1}{2})[I \otimes \bar{\Gamma}(k)] \\ &= \sigma(k + \frac{1}{2})\bar{\Gamma}(k + \frac{1}{2}) \end{aligned}$$

所以命题成立, 证毕。 □

## 4 完美常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ (存在 $\varepsilon_{A_\zeta B_\zeta}$ 为前提)

只有满足  $\varepsilon_{A_\zeta B_\zeta} = \varepsilon_{B_\zeta A_\zeta}$  对称条件时, 本章节内容才全部成立, 否则只有部分成立。

### 4.1 完美常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的引入

**定义4.1.1.**  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) := \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon^{A_\zeta B_\zeta} N_{B_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) := \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{A_\zeta B_\zeta} N_{l_\zeta}^{B_\zeta m_\zeta}(s - \frac{1}{2}; w)$

**性质4.1.1.**  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) \simeq X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$

### 4.2 常数矩阵 $X(s; w), \bar{X}(s; w)$ 的引入

**定义4.2.1.** 
$$\begin{cases} X^{A_\zeta}(s; w) \prec X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta}(s; w) \prec X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) \\ \bar{X}_{A_\zeta}(s; w) \prec X_{A_\zeta l_\zeta}^{m_\zeta}(s; w), \bar{X}^{A_\zeta}(s; w) \prec X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) \\ X(s; w) \prec X_{A_\zeta \otimes l_\zeta}^{m_\zeta}(s; w), \bar{X}(s; w) \prec X_{m_\zeta}^{A_\zeta \otimes l_\zeta}(s; w) = X^+(s; w) \end{cases}$$

### 4.3 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w), X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的升降指标

**性质4.3.1.**

$$\begin{cases} X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) = \varepsilon^{A_\zeta B_\zeta} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; w) \varepsilon_{m_\zeta r_\zeta}(s - 1; w) X_{B_\zeta n_\zeta}^{r_\zeta}(s - \frac{1}{2}; w) \\ X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) = \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - \frac{1}{2}; w) \varepsilon^{m_\zeta r_\zeta}(s - 1; w) X_{r_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2}; w) \end{cases}$$

$$\begin{aligned}
& \text{证明: } N_{A_\zeta l_\zeta}^{k_\zeta}(s - \frac{1}{2}; w) = \varepsilon^{k_\zeta m_\zeta}(s - \frac{1}{2}; w) \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - 1; w) N_{m_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2}; w) \\
& \Leftrightarrow \varepsilon^{C_\zeta A_\zeta} N_{A_\zeta l_\zeta}^{k_\zeta}(s - \frac{1}{2}; w) = \varepsilon^{C_\zeta A_\zeta} \varepsilon^{k_\zeta m_\zeta}(s - \frac{1}{2}; w) \varepsilon_{A_\zeta B_\zeta} \varepsilon_{l_\zeta n_\zeta}(s - 1; w) N_{m_\zeta}^{B_\zeta n_\zeta}(s - \frac{1}{2}; w) \\
& \Leftrightarrow X_{l_\zeta}^{C_\zeta k_\zeta}(s; w) = \varepsilon^{C_\zeta A_\zeta} \varepsilon^{k_\zeta m_\zeta}(s - \frac{1}{2}; w) \varepsilon_{l_\zeta n_\zeta}(s - 1; w) X_{A_\zeta m_\zeta}^{n_\zeta}(s - \frac{1}{2}; w) \\
& \Leftrightarrow X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) = \varepsilon^{A_\zeta B_\zeta} \varepsilon^{l_\zeta n_\zeta}(s - \frac{1}{2}; w) \varepsilon_{m_\zeta r_\zeta}(s - 1; w) X_{B_\zeta n_\zeta}^{r_\zeta}(s - \frac{1}{2}; w)
\end{aligned}$$

□

#### 4.4 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)$ , $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的正交性

$$\text{性质4.4.1. } X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) = \delta_{m_\zeta}^{n_\zeta} [\Leftrightarrow] X^{A_\zeta}(s; w) \bar{X}_{A_\zeta}(s; w) = I_{C_{w+1}^{2s-2}} [\Leftrightarrow] \bar{X}(s; w) X(s; w) = I_{C_{w+1}^{2s-2}}$$

$$\begin{aligned}
& \text{证明: } X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) \\
& = \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{A_\zeta D_\zeta} N_{l_\zeta}^{D_\zeta n_\zeta}(s - \frac{1}{2}; w) \\
& = \frac{2s-1}{\sqrt{w+(2s-1)\%2}} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) \delta_{D_\zeta}^{C_\zeta} N_{l_\zeta}^{D_\zeta n_\zeta}(s - \frac{1}{2}; w) \\
& = \frac{2s-1}{w+(2s-1)\%2} \frac{w+(2s-1)\%2}{2s-1} \delta_{m_\zeta}^{n_\zeta} \\
& = \delta_{m_\zeta}^{n_\zeta}
\end{aligned}$$

□

$$\text{性质4.4.2. } X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) = 0$$

$$[\Leftrightarrow] X^{A_\zeta}(s; w) \bar{N}_{A_\zeta}(s; w) = 0, N_{A_\zeta}(s; w) \bar{X}^{A_\zeta}(s; w) = 0 [\Leftrightarrow] \bar{X}(s; w) N(s; w) = 0, \bar{N}(s; w) X(s; w) = 0$$

$$\begin{aligned}
& \text{证明: } X_{m_\zeta}^{A_\zeta l_\zeta}(s; w) N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) \\
& = \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) N_{A_\zeta l_\zeta}^{k_\zeta}(s; w) \\
& = \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon^{A_\zeta C_\zeta} \Gamma_{C_\zeta C_\zeta' D_\zeta'' \dots}^{l_\zeta}(s - \frac{1}{2}; w) \Gamma_{m_\zeta}^{C_\zeta'' D_\zeta'' \dots}(s - 1; w) \Gamma_{A_\zeta B_\zeta' C_\zeta' D_\zeta' \dots}^{k_\zeta}(s; w) \Gamma_{l_\zeta}^{B_\zeta' C_\zeta' D_\zeta' \dots}(s - \frac{1}{2}; w) \\
& = \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon^{A_\zeta C_\zeta} \frac{1}{(2s-1)!} \delta_{[C_\zeta}^{B_\zeta'} \delta_{C_\zeta''}^{C_\zeta'} \delta_{D_\zeta''}^{D_\zeta'} \dots] \Gamma_{m_\zeta}^{C_\zeta'' D_\zeta'' \dots}(s - 1; w) \Gamma_{A_\zeta B_\zeta' C_\zeta' D_\zeta' \dots}^{k_\zeta}(s; w) \\
& = \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon^{A_\zeta B_\zeta} \Gamma_{A_\zeta B_\zeta C_\zeta' D_\zeta'' \dots}^{k_\zeta}(s; w) \Gamma_{m_\zeta}^{C_\zeta'' D_\zeta'' \dots}(s - 1; w) \\
& = 0
\end{aligned}$$

□

$$\text{性质4.4.3. } X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) X_{m_\zeta}^{A_\zeta k_\zeta}(s; w) = \frac{2s-1}{w+(2s-1)\%2} \delta_{l_\zeta}^{k_\zeta} [\Leftrightarrow] \bar{X}_{A_\zeta}(s; w) X^{A_\zeta}(s; w) = \frac{2s-1}{w+(2s-1)\%2} I_{C_{w+1}^{2s-1}}$$

$$\begin{aligned}
& \text{证明: } X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) X_{m_\zeta}^{A_\zeta k_\zeta}(s; w) \\
& = \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{A_\zeta B_\zeta} N_{l_\zeta}^{B_\zeta m_\zeta}(s - \frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon^{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{k_\zeta}(s - \frac{1}{2}; w) \\
& = \frac{2s-1}{w+(2s-1)\%2} N_{l_\zeta}^{B_\zeta m_\zeta}(s - \frac{1}{2}; w) N_{B_\zeta m_\zeta}^{k_\zeta}(s - \frac{1}{2}; w) \\
& = \frac{2s-1}{w+(2s-1)\%2} \delta_{l_\zeta}^{k_\zeta}
\end{aligned}$$

□

$$\text{性质4.4.4. } X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) X_{m_\zeta}^{B_\zeta l_\zeta}(s; w) = \frac{C_{w+1}^{2s-2} - \frac{(2s-2)\%2}{2s-2} C_{w+1}^{2s-3}}{w+(2s-1)\%2} \delta_{A_\zeta}^{B_\zeta} [\Leftrightarrow] \text{tr}[\bar{X}_{A_\zeta}(s; w) X^{B_\zeta}(s; w)] = \frac{C_{w+1}^{2s-2} - \frac{(2s-2)\%2}{2s-2} C_{w+1}^{2s-3}}{w+(2s-1)\%2} \delta_{A_\zeta}^{B_\zeta}$$

$$\begin{aligned}
& \text{证明: } X_{A_\zeta l_\zeta}^{m_\zeta}(s; w) X_{m_\zeta}^{B_\zeta l_\zeta}(s; w) \\
& = \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{A_\zeta C_\zeta} N_{l_\zeta}^{C_\zeta m_\zeta}(s - \frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon^{B_\zeta D_\zeta} N_{D_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) \\
& = \frac{2s-1}{w+(2s-1)\%2} \varepsilon_{A_\zeta C_\zeta} \varepsilon^{B_\zeta D_\zeta} N_{l_\zeta}^{C_\zeta m_\zeta}(s - \frac{1}{2}; w) N_{D_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w) \\
& = \frac{2s-1}{w+(2s-1)\%2} \varepsilon_{A_\zeta C_\zeta} \varepsilon^{B_\zeta D_\zeta} \frac{1}{2s-1} [C_{w+1}^{2s-2} - \frac{(2s-2)\%2}{2s-2} C_{w+1}^{2s-3}] \delta_{D_\zeta}^{C_\zeta} \\
& = \frac{C_{w+1}^{2s-2} - \frac{(2s-2)\%2}{2s-2} C_{w+1}^{2s-3}}{w+(2s-1)\%2} \delta_{A_\zeta}^{B_\zeta}
\end{aligned}$$

□

$$\text{推论4.4.1. } \bar{N}(s; w) N(s; w) = I_{C_{w+1}^{2s-1}}, \bar{X}(s; w) X(s; w) = I_{C_{w+1}^{2s-2}}, \bar{N}(s; w) X(s; w) = 0, \bar{X}(s; w) N(s; w) = 0$$

#### 4.5 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)$ , $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的自旋变换

推论4.5.1.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)\sigma_{A_\zeta}^{A_\zeta} \sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)X_{B_\zeta l_\zeta}^{m_\zeta}(s; w) = \frac{1}{w+(2s-1)\%2} \frac{(2s-2)\%2}{2s-2} \sigma_{m_\zeta}^{A_\zeta} \sigma_{m_\zeta}^{n_\zeta}(s-1; w)$

$$[\Leftrightarrow] X_{A_\zeta}^{A_\zeta}(s; w)\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\bar{X}_{B_\zeta}(s; w) = \frac{1}{w+(2s-1)\%2} \frac{(2s-2)\%2}{2s-2} \sigma(s-1; w)$$

$$[\Leftrightarrow] \bar{X}(s; w)\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} X(s; w) = \frac{1}{w+(2s-1)\%2} \frac{(2s-2)\%2}{2s-2} \sigma(s-1; w)$$

证明:  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)\sigma_{A_\zeta}^{A_\zeta} \sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)X_{B_\zeta l_\zeta}^{m_\zeta}(s; w)$   
 $= \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)\sigma_{A_\zeta}^{A_\zeta} \sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{B_\zeta D_\zeta} N_{l_\zeta}^{D_\zeta n_\zeta}(s-\frac{1}{2}; w)$   
 $= -\frac{2s-1}{w+(2s-1)\%2} N_{C_\zeta m_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)\sigma_{D_\zeta}^{A_\zeta} \sigma_{D_\zeta}^{C_\zeta}(\frac{1}{2}; w)N_{l_\zeta}^{D_\zeta n_\zeta}(s-\frac{1}{2}; w)$   
 $= \frac{1}{w+(2s-1)\%2} \frac{(2s-2)\%2}{2s-2} \sigma_{m_\zeta}^{A_\zeta} \sigma_{m_\zeta}^{n_\zeta}(s-1; w)$  □

推论4.5.2.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)\sigma_{A_\zeta}^{A_\zeta} \sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)X_{B_\zeta k_\zeta}^{m_\zeta}(s; w) = -\frac{1}{w+(2s-1)\%2} \sigma_{k_\zeta}^{A_\zeta} \sigma_{k_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)$

$$[\Leftrightarrow] \bar{X}^{A_\zeta}(s; w)\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)X_{B_\zeta}(s; w) = -\frac{1}{w+(2s-1)\%2} \sigma_{k_\zeta}^{A_\zeta}(s-\frac{1}{2}; w)$$

证明:  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)\sigma_{A_\zeta}^{A_\zeta} \sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)X_{B_\zeta k_\zeta}^{m_\zeta}(s; w)$   
 $= \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)\sigma_{A_\zeta}^{A_\zeta} \sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{B_\zeta D_\zeta} N_{k_\zeta}^{D_\zeta m_\zeta}(s-\frac{1}{2}; w)$   
 $= -\frac{2s-1}{w+(2s-1)\%2} N_{C_\zeta m_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)\sigma_{D_\zeta}^{A_\zeta} \sigma_{D_\zeta}^{C_\zeta}(\frac{1}{2}; w)N_{k_\zeta}^{D_\zeta m_\zeta}(s-\frac{1}{2}; w)$   
 $= -\frac{1}{w+(2s-1)\%2} \sigma_{k_\zeta}^{A_\zeta} \sigma_{k_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)$  □

推论4.5.3.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w)X_{B_\zeta l_\zeta}^{n_\zeta}(s; w) = \frac{1}{w+(2s-1)\%2} \frac{(2s-2)\%2}{2s-2} S_{abm_\zeta}^{n_\zeta}(s-1; w)$

$$[\Leftrightarrow] X_{A_\zeta}^{A_\zeta}(s; w)S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{2s-2+w}^{2s-2}} \bar{X}_{A_\zeta}(s; w) = \frac{1}{w+(2s-1)\%2} \frac{(2s-2)\%2}{2s-2} S_{ab}(s-1, \zeta; w)$$

$$[\Leftrightarrow] \bar{X}(s; w)S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{w+1}^{2s-1}} X(s; w) = \frac{1}{w+(2s-1)\%2} \frac{(2s-2)\%2}{2s-2} S_{ab}(s-1, \zeta; w)$$

证明:  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w)X_{B_\zeta l_\zeta}^{n_\zeta}(s; w)$   
 $= \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{B_\zeta D_\zeta} N_{l_\zeta}^{D_\zeta n_\zeta}(s-\frac{1}{2}; w)$   
 $= -\frac{2s-1}{w+(2s-1)\%2} N_{C_\zeta m_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)S_{abD_\zeta}^{C_\zeta}(\frac{1}{2}; w)N_{l_\zeta}^{D_\zeta n_\zeta}(s-\frac{1}{2}; w)$   
 $= \frac{1}{w+(2s-1)\%2} \frac{(2s-2)\%2}{2s-2} S_{abm_\zeta}^{n_\zeta}(s-1; w)$  □

推论4.5.4.  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w)X_{B_\zeta k_\zeta}^{m_\zeta}(s; w) = -\frac{1}{2s} S_{abk_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)$

$$[\Leftrightarrow] \bar{X}^{A_\zeta}(s; w)S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w)X_{B_\zeta}(s; w) = -\frac{1}{w+(2s-1)\%2} S_{ab}(s-\frac{1}{2}, \zeta; w)$$

证明:  $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w)X_{B_\zeta k_\zeta}^{m_\zeta}(s; w)$   
 $= \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{A_\zeta C_\zeta} N_{C_\zeta m_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w) \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}} \varepsilon_{B_\zeta D_\zeta} N_{k_\zeta}^{D_\zeta m_\zeta}(s-\frac{1}{2}; w)$   
 $= -\frac{2s-1}{w+(2s-1)\%2} N_{C_\zeta m_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)S_{abD_\zeta}^{C_\zeta}(\frac{1}{2}; w)N_{k_\zeta}^{D_\zeta m_\zeta}(s-\frac{1}{2}; w)$   
 $= -\frac{1}{w+(2s-1)\%2} S_{abk_\zeta}^{l_\zeta}(s-\frac{1}{2}; w)$  □

#### 4.6 常数不变张量 $X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)$ , $X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)$ 的置换性质

引理4.6.1.  $\varepsilon_{A_\zeta C_\zeta} \sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\varepsilon_{E_\zeta B_\zeta} = \sigma_{E_\zeta}^{C_\zeta}(\frac{1}{2}; w)$

只有以上引理成立, 以下结论才成立。

定理4.6.1.

$$\begin{cases} X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)[\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{l_\zeta}^{k_\zeta}(s-\frac{1}{2}; w)] = \sigma_{m_\zeta}^{n_\zeta}(s-1; w)X_{n_\zeta}^{B_\zeta k_\zeta}(s; w) \\ [\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{l_\zeta}^{k_\zeta}(s-\frac{1}{2}; w)]X_{B_\zeta k_\zeta}^{n_\zeta}(s; w) = X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)\sigma_{m_\zeta}^{n_\zeta}(s-1; w) \\ X_{A_\zeta}^{A_\zeta}(s; w)[\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta}\sigma(s-\frac{1}{2}; w)] = \sigma(s-1; w)X^{B_\zeta}(s; w) \\ [\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta}\sigma(s-\frac{1}{2}; w)]\bar{X}_{B_\zeta}(s; w) = \bar{X}_{A_\zeta}(s; w)\sigma(s-1; w) \\ \bar{X}(s; w)[\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s-\frac{1}{2}; w)] = \sigma(s-1; w)\bar{X}(s; w) \\ [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s-\frac{1}{2}; w)]X(s; w) = X(s; w)\sigma(s-1; w) \end{cases}$$



证明:  $X_{m_\zeta}^{A_\zeta k_\zeta}(s; w)[\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{k_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - 1; w)X_{n_\zeta}^{B_\zeta l_\zeta}(s; w)$   
 $\Leftrightarrow \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}}\varepsilon^{A_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)[\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{k_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - 1; w)\frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}}\varepsilon^{B_\zeta D_\zeta}N_{D_\zeta n_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)$   
 $\Leftrightarrow \varepsilon^{A_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)[\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{k_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - 1; w)\varepsilon^{B_\zeta D_\zeta}N_{D_\zeta n_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)$   
 $\Leftrightarrow \varepsilon^{A_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{k_\zeta}(s; w)[\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{k_\zeta}^{l_\zeta}(s; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - \frac{1}{2}; w)\varepsilon^{B_\zeta D_\zeta}N_{D_\zeta n_\zeta}^{l_\zeta}(s; w)$   
 $\Leftrightarrow \varepsilon_{E_\zeta B_\zeta}\varepsilon^{A_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{k_\zeta}(s; w)[\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{k_\zeta}^{l_\zeta}(s; w)] = \varepsilon_{E_\zeta B_\zeta}\sigma_{m_\zeta}^{n_\zeta}(s - \frac{1}{2}; w)\varepsilon^{B_\zeta D_\zeta}N_{D_\zeta n_\zeta}^{l_\zeta}(s; w)$   
 $\Leftrightarrow N_{C_\zeta m_\zeta}^{k_\zeta}(s; w)[\sigma(\frac{1}{2}; w)_{E_\zeta C_\zeta}\delta_{k_\zeta}^{l_\zeta} - \delta_{E_\zeta C_\zeta}\sigma_{k_\zeta}^{l_\zeta}(s; w)] = -\sigma_{m_\zeta}^{n_\zeta}(s - \frac{1}{2}; w)N_{E_\zeta n_\zeta}^{l_\zeta}(s; w)$   
 $\Leftrightarrow [\sigma(\frac{1}{2}; w)_{E_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{l_\zeta}(s; w) + \sigma_{m_\zeta}^{n_\zeta}(s - \frac{1}{2}; w)N_{E_\zeta n_\zeta}^{l_\zeta}(s; w)] = N_{E_\zeta m_\zeta}^{k_\zeta}(s; w)\sigma_{k_\zeta}^{l_\zeta}(s; w)$   
 $\Leftrightarrow \sigma_{A_\zeta}^{\alpha_\zeta}N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + \sigma_{A_\zeta}^{\alpha_\zeta}N_{B_\zeta l_\zeta}^{m_\zeta}(s - \frac{1}{2}; w)N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w)\sigma_{j_\zeta}^{\alpha_\zeta}N_{A_\zeta m_\zeta}^{k_\zeta}(s; w)$  □

定理4.6.2.

$$\begin{cases} X_{m_\zeta}^{A_\zeta l_\zeta}(s; w)[S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)] = S_{abm_\zeta}^{n_\zeta}(s - 1; w)X_{n_\zeta}^{B_\zeta \otimes k_\zeta}(s; w) \\ [S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)]X_{B_\zeta k_\zeta}^{n_\zeta}(s; w) = X_{A_\zeta l_\zeta}^{m_\zeta}(s; w)S_{abm_\zeta}^{n_\zeta}(s - 1; w) \\ X_{A_\zeta}(s; w)[S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta}S_{ab}(s - \frac{1}{2}, \zeta; w)] = S_{ab}(s - 1, \zeta; w)X^{B_\zeta}(s; w) \\ [S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta}S_{ab}(s - \frac{1}{2}, \zeta; w)]\bar{X}_{B_\zeta}(s; w) = \bar{X}_{A_\zeta}(s; w)S_{ab}(s - 1, \zeta; w) \\ \bar{X}(s; w)[S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)] = S_{ab}(s - 1, \zeta; w)\bar{X}(s; w) \\ [S_{ab}(\frac{1}{2}, \zeta; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \zeta; w)]X(s; w) = X(s; w)S_{ab}(s - 1, \zeta; w) \end{cases}$$

证明:  $X_{m_\zeta}^{A_\zeta k_\zeta}(s; w)[S_{abA_\zeta}^{B_\zeta}\delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abk_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)] = S_{abm_\zeta}^{n_\zeta}(s - 1; w)X_{n_\zeta}^{B_\zeta l_\zeta}(s; w)$   
 $\Leftrightarrow \frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}}\varepsilon^{A_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)[S_{abA_\zeta}^{B_\zeta}\delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abk_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)]$   
 $= S_{abm_\zeta}^{n_\zeta}(s - 1; w)\frac{\sqrt{2s-1}}{\sqrt{w+(2s-1)\%2}}\varepsilon^{B_\zeta D_\zeta}N_{D_\zeta n_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)$   
 $\Leftrightarrow \varepsilon^{A_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)[S_{abA_\zeta}^{B_\zeta}\delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abk_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)] = S_{abm_\zeta}^{n_\zeta}(s - 1; w)\varepsilon^{B_\zeta D_\zeta}N_{D_\zeta n_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)$   
 $\Leftrightarrow \varepsilon^{A_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{k_\zeta}(s; w)[S_{abA_\zeta}^{B_\zeta}\delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abk_\zeta}^{l_\zeta}(s; w)] = S_{abm_\zeta}^{n_\zeta}(s - \frac{1}{2}; w)\varepsilon^{B_\zeta D_\zeta}N_{D_\zeta n_\zeta}^{l_\zeta}(s; w)$   
 $\Leftrightarrow \varepsilon_{E_\zeta B_\zeta}\varepsilon^{A_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{k_\zeta}(s; w)[S_{abA_\zeta}^{B_\zeta}\delta_{k_\zeta}^{l_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abk_\zeta}^{l_\zeta}(s; w)] = \varepsilon_{E_\zeta B_\zeta}S_{abm_\zeta}^{n_\zeta}(s - \frac{1}{2}; w)\varepsilon^{B_\zeta D_\zeta}N_{D_\zeta n_\zeta}^{l_\zeta}(s; w)$   
 $\Leftrightarrow N_{C_\zeta m_\zeta}^{k_\zeta}(s; w)[S_{abE_\zeta}^{C_\zeta}\delta_{k_\zeta}^{l_\zeta} - \delta_{E_\zeta C_\zeta}S_{abk_\zeta}^{l_\zeta}(s; w)] = -S_{abm_\zeta}^{n_\zeta}(s - \frac{1}{2}; w)N_{E_\zeta n_\zeta}^{l_\zeta}(s; w)$   
 $\Leftrightarrow [S_{abE_\zeta}^{C_\zeta}N_{C_\zeta m_\zeta}^{l_\zeta}(s; w) + S_{abm_\zeta}^{n_\zeta}(s - \frac{1}{2}; w)N_{E_\zeta n_\zeta}^{l_\zeta}(s; w)] = N_{E_\zeta m_\zeta}^{k_\zeta}(s; w)S_{abk_\zeta}^{l_\zeta}(s; w)$   
 $\Leftrightarrow S_{abA_\zeta}^{B_\zeta}N_{B_\zeta l_\zeta}^{k_\zeta}(s; w) + S_{abl_\zeta}^{m_\zeta}(s - \frac{1}{2}; w)N_{A_\zeta m_\zeta}^{k_\zeta}(s; w) = N_{A_\zeta l_\zeta}^{j_\zeta}(s; w)S_{abj_\zeta}^{k_\zeta}(s; w)$  □

推论4.6.1.

$$\begin{cases} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)\varepsilon^{C_\zeta A_\zeta}[\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{l_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)] = \sigma_{m_\zeta}^{n_\zeta}(s - 1; w)N_{D_\zeta n_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)\varepsilon^{D_\zeta B_\zeta} \\ [\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}\sigma_{l_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)]\varepsilon_{B_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{n_\zeta}(s - \frac{1}{2}; w) = \varepsilon_{A_\zeta D_\zeta}N_{l_\zeta}^{D_\zeta m_\zeta}(s - \frac{1}{2}; w)\sigma_{m_\zeta}^{n_\zeta}(s - 1; w) \\ N_{C_\zeta}(s - \frac{1}{2}; w)\varepsilon^{C_\zeta A_\zeta}[\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta}\sigma(s - \frac{1}{2}; w)] = \sigma(s - 1; w)N_{D_\zeta}(s - \frac{1}{2}; w)\varepsilon^{D_\zeta B_\zeta} \\ [\sigma_{A_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta}\sigma(s - \frac{1}{2}; w)]\varepsilon_{B_\zeta C_\zeta}N^{C_\zeta}(s - \frac{1}{2}; w) = \varepsilon_{A_\zeta D_\zeta}N^{D_\zeta}(s - \frac{1}{2}; w)\sigma(s - 1; w) \end{cases}$$

推论4.6.2.

$$\begin{cases} N_{C_\zeta m_\zeta}^{l_\zeta}(s - \frac{1}{2}; w)\varepsilon^{C_\zeta A_\zeta}[S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)] = S_{abm_\zeta}^{n_\zeta}(s - 1; w)N_{D_\zeta n_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)\varepsilon^{D_\zeta B_\zeta} \\ [S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w)\delta_{l_\zeta}^{k_\zeta} + \delta_{A_\zeta}^{B_\zeta}S_{abl_\zeta}^{k_\zeta}(s - \frac{1}{2}; w)]\varepsilon_{B_\zeta C_\zeta}N_{C_\zeta m_\zeta}^{n_\zeta}(s - \frac{1}{2}; w) = \varepsilon_{A_\zeta D_\zeta}N_{l_\zeta}^{D_\zeta m_\zeta}(s - \frac{1}{2}; w)S_{abm_\zeta}^{n_\zeta}(s - 1; w) \\ N_{C_\zeta}(s - \frac{1}{2}; w)\varepsilon^{C_\zeta A_\zeta}[S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta}S_{ab}(s - \frac{1}{2}, \zeta; w)] = S_{ab}(s - 1, \zeta; w)N_{D_\zeta}(s - \frac{1}{2}; w)\varepsilon^{D_\zeta B_\zeta} \\ [S_{abA_\zeta}^{B_\zeta}(\frac{1}{2}; w) + \delta_{A_\zeta}^{B_\zeta}S_{ab}(s - \frac{1}{2}, \zeta; w)]\varepsilon_{B_\zeta C_\zeta}N^{C_\zeta}(s - \frac{1}{2}; w) = \varepsilon_{A_\zeta D_\zeta}N^{D_\zeta}(s - \frac{1}{2}; w)S_{ab}(s - 1, \zeta; w) \end{cases}$$

#### 4.7 推论：关于常数矩阵 $X(s; w)$ , $\bar{X}(s; w)$ 的重要性质

推论4.7.1.

$$\begin{cases} \bar{X}(s; w)[\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]X(s; w) = \sigma(s - 1; w) \\ X(s; w)\sigma(s - 1; w)\bar{X}(s; w) = [\sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)]X(s; w)\bar{X}(s; w) \\ [X(s; w)\bar{X}(s; w), \sigma(\frac{1}{2}; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)] = 0 \end{cases}$$

推论4.7.2.

$$\begin{cases} \bar{X}(s; w)[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]X(s; w) = S_{ab}(s - 1, \varsigma; w) \\ X(s; w)S_{ab}(s, \varsigma - 1, \varsigma; w)\bar{X}(s; w) = [S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]X(s; w)\bar{X}(s; w) \\ [X(s; w)\bar{X}(s; w), S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] = 0 \end{cases}$$

推论4.7.3.  $X^{A_\varsigma}(s; w)\sigma(s - \frac{1}{2}; w)\bar{X}^{A_\varsigma}(s; w) = 1 + \frac{1}{w+(2s-1)\%2}\sigma(s - 1; w)$   
 $[\Leftrightarrow]\bar{X}(s; w)I_{w+1} \otimes \sigma(s - \frac{1}{2}; w)X(s; w) = 1 + \frac{1}{w+(2s-1)\%2}\sigma(s - 1; w)$

推论4.7.4.  $X^{A_\varsigma}(s; w)I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)\bar{X}^{A_\varsigma}(s; w) = 1 + \frac{1}{w+(2s-1)\%2}S_{ab}(s - 1, \varsigma; w)$   
 $[\Leftrightarrow]\bar{X}(s; w)I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)X(s; w) = 1 + \frac{1}{w+(2s-1)\%2}S_{ab}(s - 1, \varsigma; w)$

#### 4.8 矩阵 $X(s; w)$ , $\bar{X}(s; w)$ 的常数不变张量性质

定理4.8.1.  $X(s; w) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s - \frac{1}{2}, \varsigma; w)} X(s; w) e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s - 1, \varsigma; w)}$

证明:  $[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]X(s; w) = X(s; w)S_{ab}(s - 1, \varsigma; w)$   
 $\Leftrightarrow 0 = [\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + \frac{i}{2}\vartheta^{ab}I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]X(s; w) - \frac{i}{2}\vartheta^{ab}X(s; w)S_{ab}(s - 1, \varsigma; w)$   
 $\Leftrightarrow X(s; w) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s - \frac{1}{2}, \varsigma; w)} X(s; w) e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s - 1, \varsigma; w)} \quad \square$

定理4.8.2.  $\bar{X}(s; w) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s - 1, \varsigma; w)} \bar{X}(s; w) e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s - \frac{1}{2}, \varsigma; w)}$

证明:  $\bar{X}(s; w)[S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)] = S_{ab}(s - 1, \varsigma; w)\bar{X}(s; w)$   
 $\Leftrightarrow 0 = \frac{i}{2}\vartheta^{ab}S_{ab}(s - 1, \varsigma; w)\bar{X}(s; w) - \bar{X}(s; w)[\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2}, \varsigma; w) \otimes I_{C_{w+1}^{2s-1}} + \frac{i}{2}\vartheta^{ab}I_{w+1} \otimes S_{ab}(s - \frac{1}{2}, \varsigma; w)]$   
 $\Leftrightarrow \bar{X}(s; w) = e^{\frac{i}{2}\vartheta^{ab}S_{ab}(s - 1, \varsigma; w)} \bar{X}(s; w) e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(\frac{1}{2}, \varsigma; w)} \otimes e^{-\frac{i}{2}\vartheta^{ab}S_{ab}(s - \frac{1}{2}, \varsigma; w)} \quad \square$

#### 4.9 常数矩阵 $\Omega(s; w)$ , $\sigma(s - 1; w)$ 的置换性质

推论4.9.1.  $\begin{cases} \Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]X(s; w) = [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]X(s; w)\sigma(s - 1; w) \\ \bar{X}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w) = \sigma(s - 1; w)\bar{X}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)] \end{cases}$

推论4.9.2.  $\begin{cases} \sigma(s; w) = \bar{N}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]N(s; w) \\ \sigma(s - 1; w) = \bar{X}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]\Omega(s; w)[I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]X(s; w) \end{cases}$

推论4.9.3.  $\begin{cases} [\vec{\vartheta} \cdot \sigma(s; w)]^n = \bar{N}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vec{\vartheta} \cdot \Omega(s; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]N(s; w) \\ [\vec{\vartheta} \cdot \sigma(s - 1; w)]^n = \bar{X}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)][\vec{\vartheta} \cdot \Omega(s; w)]^n [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]X(s; w) \end{cases}$

推论4.9.4.  $\begin{cases} e^{\vec{\vartheta} \cdot \sigma(s; w)} = \bar{N}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]e^{\vec{\vartheta} \cdot \Omega(s; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]N(s; w) \\ e^{\vec{\vartheta} \cdot \sigma(s - 1; w)} = \bar{X}(s; w)[I_{w+1} \otimes \bar{\Gamma}(s - \frac{1}{2}; w)]e^{\vec{\vartheta} \cdot \Omega(s; w)} [I_{w+1} \otimes \Gamma(s - \frac{1}{2}; w)]X(s; w) \end{cases}$

#### 4.10 常数矩阵 $\Omega(s - l; w)$ , $[\vec{\vartheta} \cdot \Omega(s - l; w)]^n$ , $e^{\vec{\vartheta} \cdot \Omega(s - l; w)}$ 的同构性表示

推论4.10.1.  $\Omega(s; w) = \Omega(s - 1; w) \otimes I_{(w+1)^2} + I_{(w+1)^{2s-2}} \otimes \Omega(1; w)$

推论4.10.2.

$$\begin{cases} \Omega(s; w)I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\} = I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\}\Omega(s-1; w) \\ I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\}\Omega(s; w) = \Omega(s-1; w)I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\} \end{cases}$$

推论4.10.3.

$$\begin{cases} \Omega(s-1; w) = I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\}\Omega(s; w)I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\} \\ [\vec{\vartheta} \cdot \Omega(s-1; w)]^n = I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\}[\vec{\vartheta} \cdot \Omega(s; w)]^n I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\} \\ e^{\vec{\vartheta} \cdot \Omega(s-1; w)} = I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\}e^{\vec{\vartheta} \cdot \Omega(s; w)} I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\} \end{cases}$$

定义4.10.1.

$$\begin{cases} T(s; w) := I_{(w+1)^{2s-2}} \otimes \{[I_{w+1} \otimes \Gamma(\frac{1}{2}; w)]X(1; w)\} \\ \bar{T}(s; w) := I_{(w+1)^{2s-2}} \otimes \{\bar{X}(1; w)[I_{w+1} \otimes \bar{\Gamma}(\frac{1}{2}; w)]\} = T^+(s; w) \end{cases}$$

推论4.10.4.

$$\begin{cases} \Omega(s-l; w) = \bar{T}(s-l+1; w) \cdots \bar{T}(s-1; w)\bar{T}(s; w)\Omega(s; w)T(s; w)T(s-1; w) \cdots T(s-l+1; w) \\ [\vec{\vartheta} \cdot \Omega(s-l; w)]^n = \bar{T}(s-l+1; w) \cdots \bar{T}(s-1; w)\bar{T}(s; w)[\vec{\vartheta} \cdot \Omega(s; w)]^n T(s; w)T(s-1; w) \cdots T(s-l+1; w) \\ e^{\vec{\vartheta} \cdot \Omega(s-l; w)} = \bar{T}(s-l+1; w) \cdots \bar{T}(s-1; w)\bar{T}(s; w)e^{\vec{\vartheta} \cdot \Omega(s; w)} T(s; w)T(s-1; w) \cdots T(s-l+1; w) \end{cases}$$

推论4.10.5.

$$\begin{cases} \sigma(s-l; w) = \bar{\Gamma}(s-l; w)\bar{T}(s-l+1; w) \cdots \bar{T}(s; w)\Omega(s; w)T(s; w) \cdots T(s-l+1; w)\Gamma(s-l; w) \\ [\vec{\vartheta} \cdot \sigma(s-l; w)]^n = \bar{\Gamma}(s-l; w)\bar{T}(s-l+1; w) \cdots \bar{T}(s; w)[\vec{\vartheta} \cdot \Omega(s; w)]^n T(s; w) \cdots T(s-l+1; w)\Gamma(s-l; w) \\ e^{\vec{\vartheta} \cdot \sigma(s-l; w)} = \bar{\Gamma}(s-l; w)\bar{T}(s-l+1; w) \cdots \bar{T}(s; w)e^{\vec{\vartheta} \cdot \Omega(s; w)} T(s; w) \cdots T(s-l+1; w)\Gamma(s-l; w) \end{cases}$$

## 5 反对称常数不变张量 $\Gamma_{ab}^k(1; 3)$ 的应用-电磁张量的高旋量描述

### 5.1 基础

$$\text{电磁张量: } F_{ab} = \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix}, \text{对偶张量: } *F_{ab} = \begin{bmatrix} 0 & -iE_z & iE_y & B_x \\ iE_z & 0 & -iE_x & B_y \\ -iE_y & iE_x & 0 & B_z \\ -B_x & -B_y & -B_z & 0 \end{bmatrix} \quad (41.1)$$

推论5.1.1.

$$\begin{aligned} \sqrt{2}\Gamma_0^{ab} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = iR_z, \sqrt{2}\Gamma_1^{ab} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = -iR_y, \sqrt{2}\Gamma_2^{ab} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = -iL_x \\ \sqrt{2}\Gamma_3^{ab} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = iR_x, \sqrt{2}\Gamma_4^{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = -iL_y, \sqrt{2}\Gamma_5^{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = -iL_z \end{aligned}$$

推论5.1.2.  $F_0 = \sqrt{2}B_z, F_1 = -\sqrt{2}B_y, F_2 = -i\sqrt{2}E_x, F_3 = \sqrt{2}B_x, F_4 = -i\sqrt{2}E_y, F_5 = -i\sqrt{2}E_z$

$$\begin{bmatrix} \sqrt{2}\vec{E} \\ \sqrt{2}\vec{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = SF, S := \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, S^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}, SS^+ = S^+S = I_6$$

定理5.1.1.  $tr(R_\alpha R_\beta R_\gamma) = tr(L_\alpha L_\beta R_\gamma) = i\varepsilon_{\alpha\beta\gamma}$

证明:

$$\begin{aligned} tr(R_x R_y R_z) &= tr(R_z R_x R_y) = tr(R_y R_z R_x) = i \\ tr(R_y R_x R_z) &= tr(R_z R_y R_x) = tr(R_x R_z R_y) = -i \\ tr(R_x^2 R_x) &= tr(R_x^2 R_y) = tr(R_x^2 R_z) = 0 \end{aligned}$$

$$\begin{aligned}
tr(R_y^2 R_x) &= tr(R_y^2 R_y) = tr(R_y^2 R_z) = 0 \\
tr(R_z^2 R_x) &= tr(R_z^2 R_y) = tr(R_z^2 R_z) = 0 \\
tr(R_\alpha R_\beta L_x) &= tr(R_\alpha R_\beta L_y) = tr(R_\alpha R_\beta L_z) = 0 \\
tr(L_\alpha L_\beta L_x) &= tr(L_\alpha L_\beta L_y) = tr(L_\alpha L_\beta L_z) = 0 \\
tr(L_x L_y R_x) &= tr(L_x L_y R_y) = 0, tr(L_x L_y R_z) = i \\
tr(L_y L_z R_x) &= i, tr(L_y L_z R_y) = tr(L_y L_z R_z) = 0 \\
tr(L_z L_x R_x) &= 0, tr(L_z L_x R_y) = i, tr(L_z L_x R_z) = 0 \\
tr(L_y L_x R_x) &= tr(L_y L_x R_y) = 0, tr(L_y L_x R_z) = -i \\
tr(L_z L_y R_x) &= -i, tr(L_z L_y R_y) = tr(L_z L_y R_z) = 0 \\
tr(L_x L_z R_x) &= 0, tr(L_x L_z R_y) = -i, tr(L_x L_z R_z) = 0
\end{aligned}$$

□

推论5.1.3.  $tr(\gamma_\alpha \gamma_\beta \gamma_\gamma) = i\varepsilon_{\alpha\beta\gamma}$

推论5.1.4.  $R_k^l(1; 3) = 2\Gamma_k^{ab} R_a^{a'} \Gamma_{a'b}^l = -tr(\sqrt{2}\Gamma_k R \sqrt{2}\Gamma^l) \simeq -R_l^k(1; 3)$

## 5.2 $SR^x(1; 3)S^+$ 的计算

推论5.2.1.

$$\begin{aligned}
R_x^0{}^1(1; 3) &= -tr(\sqrt{2}\Gamma_0 R^x \sqrt{2}\Gamma^1) = -tr(R_z R^x R_y) = -i \\
R_x^0{}^2(1; 3) &= -tr(\sqrt{2}\Gamma_0 R^x \sqrt{2}\Gamma^2) = -tr(R_z R^x L_x) = 0 \\
R_x^0{}^3(1; 3) &= -tr(\sqrt{2}\Gamma_0 R^x \sqrt{2}\Gamma^3) = tr(R_z R^x R_x) = 0 \\
R_x^0{}^4(1; 3) &= -tr(\sqrt{2}\Gamma_0 R^x \sqrt{2}\Gamma^4) = -tr(R_z R^x L_y) = 0 \\
R_x^0{}^5(1; 3) &= -tr(\sqrt{2}\Gamma_0 R^x \sqrt{2}\Gamma^5) = -tr(R_z R^x L_z) = 0
\end{aligned}$$

推论5.2.2.

$$\begin{aligned}
R_x^1{}^2(1; 3) &= -tr(\sqrt{2}\Gamma_1 R^x \sqrt{2}\Gamma^2) = tr(R_y R^x L_x) = 0 \\
R_x^1{}^3(1; 3) &= -tr(\sqrt{2}\Gamma_1 R^x \sqrt{2}\Gamma^3) = -tr(R_y R^x R_x) = 0 \\
R_x^1{}^4(1; 3) &= -tr(\sqrt{2}\Gamma_1 R^x \sqrt{2}\Gamma^4) = tr(R_y R^x L_y) = 0 \\
R_x^1{}^5(1; 3) &= -tr(\sqrt{2}\Gamma_1 R^x \sqrt{2}\Gamma^5) = tr(R_y R^x L_z) = 0
\end{aligned}$$

推论5.2.3.

$$\begin{aligned}
R_x^2{}^3(1; 3) &= -tr(\sqrt{2}\Gamma_2 R^x \sqrt{2}\Gamma^3) = -tr(L_x R^x R_x) = 0 \\
R_x^2{}^4(1; 3) &= -tr(\sqrt{2}\Gamma_2 R^x \sqrt{2}\Gamma^4) = tr(L_x R^x L_y) = 0 \\
R_x^2{}^5(1; 3) &= -tr(\sqrt{2}\Gamma_2 R^x \sqrt{2}\Gamma^5) = tr(L_x R^x L_z) = 0
\end{aligned}$$

推论5.2.4.

$$\begin{aligned}
R_x^3{}^4(1; 3) &= -tr(\sqrt{2}\Gamma_3 R^x \sqrt{2}\Gamma^4) = -tr(R_x R^x L_y) = 0 \\
R_x^3{}^5(1; 3) &= -tr(\sqrt{2}\Gamma_3 R^x \sqrt{2}\Gamma^5) = -tr(R_x R^x L_z) = 0
\end{aligned}$$

推论5.2.5.

$$R_x^4{}^5(1; 3) = -tr(\sqrt{2}\Gamma_4 R^x \sqrt{2}\Gamma^5) = tr(L_y R^x L_z) = -i$$

推论5.2.6.

$$R^x(1; 3) = \begin{bmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \end{bmatrix}$$

推论5.2.7.

$$SR^x(1; 3)S^+ = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SR^x(1;3)S^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SR^x(1;3)S^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & i & 0 \end{bmatrix} = \gamma^x \otimes I$$

### 5.3 $SR^y(1;3)S^+$ 的计算

推论5.3.1.

$$R^y_0^1(1;3) = -tr(\sqrt{2}\Gamma_0 R^y \sqrt{2}\Gamma^1) = -tr(R_z R^y R_y) = 0$$

$$R^y_0^2(1;3) = -tr(\sqrt{2}\Gamma_0 R^y \sqrt{2}\Gamma^2) = -tr(R_z R^y L_x) = 0$$

$$R^y_0^3(1;3) = -tr(\sqrt{2}\Gamma_0 R^y \sqrt{2}\Gamma^3) = tr(R_z R^y R_x) = -i$$

$$R^y_0^4(1;3) = -tr(\sqrt{2}\Gamma_0 R^y \sqrt{2}\Gamma^4) = -tr(R_z R^y L_y) = 0$$

$$R^y_0^5(1;3) = -tr(\sqrt{2}\Gamma_0 R^y \sqrt{2}\Gamma^5) = -tr(R_z R^y L_z) = 0$$

推论5.3.2.

$$R^y_1^2(1;3) = -tr(\sqrt{2}\Gamma_1 R^y \sqrt{2}\Gamma^2) = tr(R_y R^y L_x) = 0$$

$$R^y_1^3(1;3) = -tr(\sqrt{2}\Gamma_1 R^y \sqrt{2}\Gamma^3) = -tr(R_y R^y R_x) = 0$$

$$R^y_1^4(1;3) = -tr(\sqrt{2}\Gamma_1 R^y \sqrt{2}\Gamma^4) = tr(R_y R^y L_y) = 0$$

$$R^y_1^5(1;3) = -tr(\sqrt{2}\Gamma_1 R^y \sqrt{2}\Gamma^5) = tr(R_y R^y L_z) = 0$$

推论5.3.3.

$$R^y_2^3(1;3) = -tr(\sqrt{2}\Gamma_2 R^y \sqrt{2}\Gamma^3) = -tr(L_x R^y R_x) = 0$$

$$R^y_2^4(1;3) = -tr(\sqrt{2}\Gamma_2 R^y \sqrt{2}\Gamma^4) = tr(L_x R^y L_y) = 0$$

$$R^y_2^5(1;3) = -tr(\sqrt{2}\Gamma_2 R^y \sqrt{2}\Gamma^5) = tr(L_x R^y L_z) = i$$

推论5.3.4.

$$R^y_3^4(1;3) = -tr(\sqrt{2}\Gamma_3 R^y \sqrt{2}\Gamma^4) = -tr(R_x R^y L_y) = 0$$

$$R^y_3^5(1;3) = -tr(\sqrt{2}\Gamma_3 R^y \sqrt{2}\Gamma^5) = -tr(R_x R^y L_z) = 0$$

推论5.3.5.

$$R^y_4^5(1;3) = -tr(\sqrt{2}\Gamma_4 R^y \sqrt{2}\Gamma^5) = tr(L_y R^y L_z) = 0$$

推论5.3.6.

$$R^y(1;3) = \begin{bmatrix} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

推论5.3.7.

$$SR^y(1;3)S^+ = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SR^y(1;3)S^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SR^y(1;3)S^+ = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \end{bmatrix} = \gamma^y \otimes I$$

### 5.4 $SR^z(1;3)S^+$ 的计算

推论5.4.1.

$$R^z_0^1(1;3) = -tr(\sqrt{2}\Gamma_0 R^z \sqrt{2}\Gamma^1) = -tr(R_z R^z R_y) = 0$$

$$R^z_0{}^2(1;3) = -tr(\sqrt{2}\Gamma_0 R^z \sqrt{2}\Gamma^2) = -tr(R_z R^z L_x) = 0$$

$$R^z_0{}^3(1;3) = -tr(\sqrt{2}\Gamma_0 R^z \sqrt{2}\Gamma^3) = tr(R_z R^z R_x) = 0$$

$$R^z_0{}^4(1;3) = -tr(\sqrt{2}\Gamma_0 R^z \sqrt{2}\Gamma^4) = -tr(R_z R^z L_y) = 0$$

$$R^z_0{}^5(1;3) = -tr(\sqrt{2}\Gamma_0 R^z \sqrt{2}\Gamma^5) = -tr(R_z R^z L_z) = 0$$

推论5.4.2.

$$R^z_1{}^2(1;3) = -tr(\sqrt{2}\Gamma_1 R^z \sqrt{2}\Gamma^2) = tr(R_y R^z L_x) = 0$$

$$R^z_1{}^3(1;3) = -tr(\sqrt{2}\Gamma_1 R^z \sqrt{2}\Gamma^3) = -tr(R_y R^z R_x) = -i$$

$$R^z_1{}^4(1;3) = -tr(\sqrt{2}\Gamma_1 R^z \sqrt{2}\Gamma^4) = tr(R_y R^z L_y) = 0$$

$$R^z_1{}^5(1;3) = -tr(\sqrt{2}\Gamma_1 R^z \sqrt{2}\Gamma^5) = tr(R_y R^z L_z) = 0$$

推论5.4.3.

$$R^z_2{}^3(1;3) = -tr(\sqrt{2}\Gamma_2 R^z \sqrt{2}\Gamma^3) = -tr(L_x R^z R_x) = 0$$

$$R^z_2{}^4(1;3) = -tr(\sqrt{2}\Gamma_2 R^z \sqrt{2}\Gamma^4) = tr(L_x R^z L_y) = -i$$

$$R^z_2{}^5(1;3) = -tr(\sqrt{2}\Gamma_2 R^z \sqrt{2}\Gamma^5) = tr(L_x R^z L_z) = 0$$

推论5.4.4.

$$R^z_3{}^4(1;3) = -tr(\sqrt{2}\Gamma_3 R^z \sqrt{2}\Gamma^4) = -tr(R_x R^z L_y) = 0$$

$$R^z_3{}^5(1;3) = -tr(\sqrt{2}\Gamma_3 R^z \sqrt{2}\Gamma^5) = -tr(R_x R^z L_z) = 0$$

推论5.4.5.

$$R^z_4{}^5(1;3) = -tr(\sqrt{2}\Gamma_4 R^z \sqrt{2}\Gamma^5) = tr(L_y R^z L_z) = 0$$

推论5.4.6.

$$R^z(1;3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

推论5.4.7.

$$SR^z(1;3)S^+ = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SR^z(1;3)S^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SR^z(1;3)S^+ = \begin{bmatrix} 0 & -i & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \gamma^z \otimes I$$

## 5.5 $SL^x(1;3)S^+$ 的计算

推论5.5.1.

$$L^x_0{}^1(1;3) = -tr(\sqrt{2}\Gamma_0 L^x \sqrt{2}\Gamma^1) = -tr(R_z L^x R_y) = 0$$

$$L^x_0{}^2(1;3) = -tr(\sqrt{2}\Gamma_0 L^x \sqrt{2}\Gamma^2) = -tr(R_z L^x L_x) = 0$$

$$L^x_0{}^3(1;3) = -tr(\sqrt{2}\Gamma_0 L^x \sqrt{2}\Gamma^3) = tr(R_z L^x R_x) = 0$$

$$L^x_0{}^4(1;3) = -tr(\sqrt{2}\Gamma_0 L^x \sqrt{2}\Gamma^4) = -tr(R_z L^x L_y) = -i$$

$$L^x_0{}^5(1;3) = -tr(\sqrt{2}\Gamma_0 L^x \sqrt{2}\Gamma^5) = -tr(R_z L^x L_z) = 0$$

推论5.5.2.

$$L^x_1{}^2(1;3) = -tr(\sqrt{2}\Gamma_1 L^x \sqrt{2}\Gamma^2) = tr(R_y L^x L_x) = 0$$

$$L^x_1{}^3(1;3) = -tr(\sqrt{2}\Gamma_1 L^x \sqrt{2}\Gamma^3) = -tr(R_y L^x R_x) = 0$$

$$L^x_1{}^4(1;3) = -tr(\sqrt{2}\Gamma_1 L^x \sqrt{2}\Gamma^4) = tr(R_y L^x L_y) = 0$$

$$L^x_1{}^5(1;3) = -tr(\sqrt{2}\Gamma_1 L^x \sqrt{2}\Gamma^5) = tr(R_y L^x L_z) = -i$$

推论5.5.3.

$$L^x_2{}^3(1;3) = -tr(\sqrt{2}\Gamma_2 L^x \sqrt{2}\Gamma^3) = -tr(L_x L^x R_x) = 0$$

$$L^x_2{}^4(1;3) = -tr(\sqrt{2}\Gamma_2 L^x \sqrt{2}\Gamma^4) = tr(L_x L^x L_y) = 0$$

$$L^x_2{}^5(1;3) = -tr(\sqrt{2}\Gamma_2 L^x \sqrt{2}\Gamma^5) = tr(L_x L^x L_z) = 0$$

推论5.5.4.

$$L^x_3{}^4(1;3) = -tr(\sqrt{2}\Gamma_3 L^x \sqrt{2}\Gamma^4) = -tr(R_x L^x L_y) = 0$$

$$L^x_3{}^5(1;3) = -tr(\sqrt{2}\Gamma_3 L^x \sqrt{2}\Gamma^5) = -tr(R_x L^x L_z) = 0$$

推论5.5.5.

$$L^x_4{}^5(1;3) = -tr(\sqrt{2}\Gamma_4 L^x \sqrt{2}\Gamma^5) = tr(L_y L^x L_z) = 0$$

推论5.5.6.

$$L^x(1;3) = \begin{bmatrix} 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \end{bmatrix}$$

推论5.5.7.

$$SL^x(1;3)S^+ = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SL^x(1;3)S^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & -i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SL^x(1;3)S^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} = \gamma^x \otimes \sigma_y$$

## 5.6 $SL^y(1;3)S^+$ 的计算

推论5.6.1.

$$L^y_0{}^1(1;3) = -tr(\sqrt{2}\Gamma_0 L^y \sqrt{2}\Gamma^1) = -tr(R_z L^y R_y) = 0$$

$$L^y_0{}^2(1;3) = -tr(\sqrt{2}\Gamma_0 L^y \sqrt{2}\Gamma^2) = -tr(R_z L^y L_x) = i$$

$$L^y_0{}^3(1;3) = -tr(\sqrt{2}\Gamma_0 L^y \sqrt{2}\Gamma^3) = tr(R_z L^y R_x) = 0$$

$$L^y_0{}^4(1;3) = -tr(\sqrt{2}\Gamma_0 L^y \sqrt{2}\Gamma^4) = -tr(R_z L^y L_y) = 0$$

$$L^y_0{}^5(1;3) = -tr(\sqrt{2}\Gamma_0 L^y \sqrt{2}\Gamma^5) = -tr(R_z L^y L_z) = 0$$

推论5.6.2.

$$L^y_1{}^2(1;3) = -tr(\sqrt{2}\Gamma_1 L^y \sqrt{2}\Gamma^2) = tr(R_y L^y L_x) = 0$$

$$L^y_1{}^3(1;3) = -tr(\sqrt{2}\Gamma_1 L^y \sqrt{2}\Gamma^3) = -tr(R_y L^y R_x) = 0$$

$$L^y_1{}^4(1;3) = -tr(\sqrt{2}\Gamma_1 L^y \sqrt{2}\Gamma^4) = tr(R_y L^y L_y) = 0$$

$$L^y_1{}^5(1;3) = -tr(\sqrt{2}\Gamma_1 L^y \sqrt{2}\Gamma^5) = tr(R_y L^y L_z) = 0$$

推论5.6.3.

$$L^y_2{}^3(1;3) = -tr(\sqrt{2}\Gamma_2 L^y \sqrt{2}\Gamma^3) = -tr(L_x L^y R_x) = 0$$

$$L^y_2{}^4(1;3) = -tr(\sqrt{2}\Gamma_2 L^y \sqrt{2}\Gamma^4) = tr(L_x L^y L_y) = 0$$

$$L^y_2{}^5(1;3) = -tr(\sqrt{2}\Gamma_2 L^y \sqrt{2}\Gamma^5) = tr(L_x L^y L_z) = 0$$

推论5.6.4.

$$L^y_3{}^4(1;3) = -tr(\sqrt{2}\Gamma_3 L^y \sqrt{2}\Gamma^4) = -tr(R_x L^y L_y) = 0$$

$$L^y_3{}^5(1;3) = -tr(\sqrt{2}\Gamma_3 L^y \sqrt{2}\Gamma^5) = -tr(R_x L^y L_z) = -i$$

推论5.6.5.

$$L^y_4{}^5(1;3) = -tr(\sqrt{2}\Gamma_4 L^y \sqrt{2}\Gamma^5) = tr(L_y L^y L_z) = 0$$

推论5.6.6.

$$L^y(1;3) = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \end{bmatrix}$$

推论5.6.7.

$$SL^y(1;3)S^+ = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SL^y(1;3)S^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SL^y(1;3)S^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \gamma^y \otimes \sigma_y$$

## 5.7 $SL^z(1;3)S^+$ 的计算

推论5.7.1.

$$L^z_0^1(1;3) = -tr(\sqrt{2}\Gamma_0 L^z \sqrt{2}\Gamma^1) = -tr(R_z L^z R_y) = 0$$

$$L^z_0^2(1;3) = -tr(\sqrt{2}\Gamma_0 L^z \sqrt{2}\Gamma^2) = -tr(R_z L^z L_x) = 0$$

$$L^z_0^3(1;3) = -tr(\sqrt{2}\Gamma_0 L^z \sqrt{2}\Gamma^3) = tr(R_z L^z R_x) = 0$$

$$L^z_0^4(1;3) = -tr(\sqrt{2}\Gamma_0 L^z \sqrt{2}\Gamma^4) = -tr(R_z L^z L_y) = 0$$

$$L^z_0^5(1;3) = -tr(\sqrt{2}\Gamma_0 L^z \sqrt{2}\Gamma^5) = -tr(R_z L^z L_z) = 0$$

推论5.7.2.

$$L^z_1^2(1;3) = -tr(\sqrt{2}\Gamma_1 L^z \sqrt{2}\Gamma^2) = tr(R_y L^z L_x) = i$$

$$L^z_1^3(1;3) = -tr(\sqrt{2}\Gamma_1 L^z \sqrt{2}\Gamma^3) = -tr(R_y L^z R_x) = 0$$

$$L^z_1^4(1;3) = -tr(\sqrt{2}\Gamma_1 L^z \sqrt{2}\Gamma^4) = tr(R_y L^z L_y) = 0$$

$$L^z_1^5(1;3) = -tr(\sqrt{2}\Gamma_1 L^z \sqrt{2}\Gamma^5) = tr(R_y L^z L_z) = 0$$

推论5.7.3.

$$L^z_2^3(1;3) = -tr(\sqrt{2}\Gamma_2 L^z \sqrt{2}\Gamma^3) = -tr(L_x L^z R_x) = 0$$

$$L^z_2^4(1;3) = -tr(\sqrt{2}\Gamma_2 L^z \sqrt{2}\Gamma^4) = tr(L_x L^z L_y) = 0$$

$$L^z_2^5(1;3) = -tr(\sqrt{2}\Gamma_2 L^z \sqrt{2}\Gamma^5) = tr(L_x L^z L_z) = 0$$

推论5.7.4.

$$L^z_3^4(1;3) = -tr(\sqrt{2}\Gamma_3 L^z \sqrt{2}\Gamma^4) = -tr(R_x L^z L_y) = i$$

$$L^z_3^5(1;3) = -tr(\sqrt{2}\Gamma_3 L^z \sqrt{2}\Gamma^5) = -tr(R_x L^z L_z) = 0$$

推论5.7.5.

$$L^z_4^5(1;3) = -tr(\sqrt{2}\Gamma_4 L^z \sqrt{2}\Gamma^5) = tr(L_y L^z L_z) = 0$$

推论5.7.6.

$$L^z(1;3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

推论5.7.7.

$$SL^z(1;3)S^+ = \begin{bmatrix} 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$

$$SL^z(1;3)S^+ = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \end{bmatrix}$$



$$SL^z(1;3)S^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \gamma^z \otimes \sigma_y$$

## 5.8 $SR(1;3)S^+$ 的计算结果

推论5.8.1.  $SR(1;3)S^+ = \gamma \otimes I, SL(1;3)S^+ = \gamma \otimes \sigma_y$

$$\Rightarrow S\bar{\Gamma}(1;3)(e^{i\omega \cdot R + \epsilon \cdot L} \otimes e^{i\omega \cdot R + \epsilon \cdot L})\Gamma(1;3)S^+ = S e^{i\omega \cdot R(1;3) + \epsilon \cdot L(1;3)} S^+ = e^{i\omega \cdot \gamma \otimes I + \epsilon \cdot \gamma \otimes \sigma_y}$$

$$\text{定理5.8.1. } \begin{bmatrix} \bar{E} \\ \bar{B} \end{bmatrix} \sim e^{i\omega \cdot \gamma \otimes I + \epsilon \cdot \gamma \otimes \sigma_y} = \frac{1}{2} \begin{bmatrix} [e^{(i\omega - \epsilon) \cdot \gamma} + e^{(i\omega + \epsilon) \cdot \gamma}] & i[e^{(i\omega - \epsilon) \cdot \gamma} - e^{(i\omega + \epsilon) \cdot \gamma}] \\ -i[e^{(i\omega - \epsilon) \cdot \gamma} - e^{(i\omega + \epsilon) \cdot \gamma}] & [e^{(i\omega - \epsilon) \cdot \gamma} + e^{(i\omega + \epsilon) \cdot \gamma}] \end{bmatrix}$$

$$\begin{aligned} \text{证明: } e^{i\omega \cdot \gamma \otimes I + \epsilon \cdot \gamma \otimes \sigma_y} &= \sum_{n=0}^{\infty} \frac{1}{n!} (i\omega \cdot \gamma \otimes I + \epsilon \cdot \gamma \otimes \sigma_y)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n C_n^k (i\omega \cdot \gamma \otimes I)^{n-k} (\epsilon \cdot \gamma \otimes \sigma_y)^k \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \sum_{k=0}^{[n/2]} C_n^{2k} (i\omega \cdot \gamma \otimes I)^{n-2k} (\epsilon \cdot \gamma \otimes \sigma_y)^{2k} + \sum_{k=0}^{[(n-1)/2]} C_n^{2k+1} (i\omega \cdot \gamma \otimes I)^{n-2k-1} (\epsilon \cdot \gamma \otimes \sigma_y)^{2k+1} \right] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \left[ \sum_{k=0}^{[n/2]} C_n^{2k} (i\omega \cdot \gamma)^{n-2k} (\epsilon \cdot \gamma)^{2k} \right] \otimes I + \left[ \sum_{k=0}^{[(n-1)/2]} C_n^{2k+1} (i\omega \cdot \gamma)^{n-2k-1} (\epsilon \cdot \gamma)^{2k+1} \right] \otimes \sigma_y \right\} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} \sum_{k=0}^{[n/2]} C_n^{2k} (i\omega \cdot \gamma)^{n-2k} (\epsilon \cdot \gamma)^{2k} & -i \sum_{k=0}^{[(n-1)/2]} C_n^{2k+1} (i\omega \cdot \gamma)^{n-2k-1} (\epsilon \cdot \gamma)^{2k+1} \\ i \sum_{k=0}^{[(n-1)/2]} C_n^{2k+1} (i\omega \cdot \gamma)^{n-2k-1} (\epsilon \cdot \gamma)^{2k+1} & \sum_{k=0}^{[n/2]} C_n^{2k} (i\omega \cdot \gamma)^{n-2k} (\epsilon \cdot \gamma)^{2k} \end{bmatrix} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} [(i\omega + \epsilon) \cdot \gamma]^n + [(i\omega - \epsilon) \cdot \gamma]^n & -i[(i\omega + \epsilon) \cdot \gamma]^n + i[(i\omega - \epsilon) \cdot \gamma]^n \\ i[(i\omega + \epsilon) \cdot \gamma]^n - i[(i\omega - \epsilon) \cdot \gamma]^n & [(i\omega + \epsilon) \cdot \gamma]^n + [(i\omega - \epsilon) \cdot \gamma]^n \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} [e^{(i\omega - \epsilon) \cdot \gamma} + e^{(i\omega + \epsilon) \cdot \gamma}] & i[e^{(i\omega - \epsilon) \cdot \gamma} - e^{(i\omega + \epsilon) \cdot \gamma}] \\ -i[e^{(i\omega - \epsilon) \cdot \gamma} - e^{(i\omega + \epsilon) \cdot \gamma}] & [e^{(i\omega - \epsilon) \cdot \gamma} + e^{(i\omega + \epsilon) \cdot \gamma}] \end{bmatrix} \quad \square \end{aligned}$$

## 5.9 反对称常数不变张量 $\Gamma_{abc}^k(\frac{3}{2};3)$ 及与 $\varepsilon_{abc}^k$ 的关系

引理5.9.1.  $\frac{1}{2} \varepsilon_k^{abc} (i\omega \cdot R + \epsilon \cdot L)_a^{a'} \delta_b^{b'} \delta_c^{c'} \varepsilon^{k'}_{a'b'c'} = (i\omega \cdot R + \epsilon \cdot L)_k^{k'}$

$$\begin{aligned} \text{证明: } \frac{1}{2} \varepsilon_k^{abc} (i\omega \cdot R + \epsilon \cdot L)_a^{a'} \delta_b^{b'} \delta_c^{c'} \varepsilon^{k'}_{a'b'c'} &= \frac{1}{2} \varepsilon_k^{abc} \varepsilon^{k'}_{a'bc} (i\omega \cdot R + \epsilon \cdot L)_a^{a'} \\ &= (\delta_k^{k'} \delta_a^{a'} - \delta_{ka'} \delta^{k'a}) (i\omega \cdot R + \epsilon \cdot L)_a^{a'} \\ &= -(i\omega \cdot R + \epsilon \cdot L)_k^{k'} \\ &= (i\omega \cdot R + \epsilon \cdot L)_k^{k'} \quad \square \end{aligned}$$

推论5.9.1.  $3\Gamma_k^{abc}(\frac{3}{2};3)(i\omega \cdot R + \epsilon \cdot L)_a^{a'} \delta_b^{b'} \delta_c^{c'} \Gamma_{a'b'c'}^{k'}(\frac{3}{2};3) = [(\sigma_y \otimes \sigma_x)(i\omega \cdot R + \epsilon \cdot L)(\sigma_y \otimes \sigma_x)]_k^{k'}$

$$\begin{aligned} \text{证明: } \sqrt{3!} \Gamma_{012}^0(\frac{3}{2};3) &= 1, \sqrt{3!} \Gamma_{013}^1(\frac{3}{2};3) = 1, \sqrt{3!} \Gamma_{023}^2(\frac{3}{2};3) = 1, \sqrt{3!} \Gamma_{123}^3(\frac{3}{2};3) = 1 \\ \Leftrightarrow \tilde{\Gamma}_{abc}^l(\frac{3}{2};3) &:= (i\sigma_y \otimes \sigma_x)^k l \Gamma_{abc}^l(\frac{3}{2};3) \\ \Leftrightarrow \sqrt{3!} \tilde{\Gamma}_{012}^3(\frac{3}{2};3) &= -1, \sqrt{3!} \tilde{\Gamma}_{013}^2(\frac{3}{2};3) = 1, \sqrt{3!} \tilde{\Gamma}_{023}^1(\frac{3}{2};3) = -1, \sqrt{3!} \tilde{\Gamma}_{123}^0(\frac{3}{2};3) = 1 \\ \Leftrightarrow \sqrt{3!} \tilde{\Gamma}_{abc}^k(\frac{3}{2};3) &= \varepsilon_{abc}^k \Leftrightarrow \Gamma_{abc}^k(\frac{3}{2};3) = (-i\sigma_y \otimes \sigma_x)_k^l \varepsilon^{labc} \\ \Rightarrow 3\Gamma_k^{abc}(\frac{3}{2};3)(i\omega \cdot R + \epsilon \cdot L)_a^{a'} \delta_b^{b'} \delta_c^{c'} \Gamma_{a'b'c'}^{k'}(\frac{3}{2};3) &= [(\sigma_y \otimes \sigma_x)(i\omega \cdot R + \epsilon \cdot L)(\sigma_y \otimes \sigma_x)]_k^{k'} \quad \square \end{aligned}$$

## 5.10 反对称常数不变张量 $\Gamma_{abcd}^k(2;3) = \varepsilon_{abcd}$

引理5.10.1.  $\varepsilon_{abcd} (i\omega \cdot R + \epsilon \cdot L)_a^{a'} \delta_b^{b'} \delta_c^{c'} \delta_d^{d'} \varepsilon_{a'b'c'd'} = 0$

推论5.10.1.  $4\Gamma_k^{abcd}(2;3)(i\omega \cdot R + \epsilon \cdot L)_a^{a'} \delta_b^{b'} \delta_c^{c'} \delta_d^{d'} \Gamma_{a'b'c'd'}^{k'}(2;3) = 0$

$$\begin{aligned} \text{证明: } \sqrt{4!} \Gamma_{0123}^0(2;3) &= 1 \Leftrightarrow \sqrt{4!} \Gamma_{abc}^k(2;3) = \varepsilon_{abcd} \\ \Rightarrow 4\Gamma_k^{abcd}(2;3)(i\omega \cdot R + \epsilon \cdot L)_a^{a'} \delta_b^{b'} \delta_c^{c'} \delta_d^{d'} \Gamma_{a'b'c'd'}^{k'}(2;3) &= 0 \quad \square \end{aligned}$$

## 6 反对称常数不变对偶张量

### 6.1 反对称常数不变对偶张量 $*\Gamma_{ab\dots}^k(s; 3)$ 的引入

$$\text{定义6.1.1. } *\Gamma_{\underbrace{ab\dots}_{n-2s}}^k(s; 3) := \varepsilon_{\underbrace{ab\dots}_{n-2s}} \overbrace{cd\dots}^{2s} \Gamma_{\underbrace{cd\dots}_{2s}}^k(s; 3)$$

### 6.2 常数不变张量 $\Gamma_{\alpha_s\beta_s\dots}^k(n; 3)$ 的引入

$$\text{定义6.2.1. } \Gamma_{\underbrace{\alpha_s\beta_s\dots}_n}^k(n; 3) := \underbrace{\sigma_{\zeta\alpha_s}^{ab} \sigma_{\zeta\beta_s}^{cd}}_n \Gamma_{\underbrace{abcd\dots}_{2n}}^k(n; 3)$$

# 第四十二章 有质量粒子的高旋量场量子化方案

自我评述：本章运用Dirac型完美常数不变张量，将四维时空中的Bargmann-Wigner方程等价改造成高旋量场形式。并从高旋量场角度重新描述有质量粒子，形式上与Dirac粒子更接近，可以类比讨论，更具统一形式。并对照之前章节的内容，直接写下对应的等价高旋量场表述结果。这为高自旋有质量粒子的量子化提供了另一个视角，并结合原来的方案就可以灵活切换，从多个角度审视检验各种物理结果。事实上就是无质量粒子自旋方程量子化方案的对应，这样对有质量粒子和无质量粒子都有两套量子化方案，即高旋量场方案和分量场方案。形式上更有统一性，更有美感，同时也体现了Dirac型完美常数不变张量方法的有用之处，找到了用武之地。

## 1 Bargmann-Wigner方程<sup>[17]</sup>自旋基和平面波解的高旋量场表述

### 1.1 有质量粒子旋量与无质量粒子旋量的对比

无质量粒子波函数= $C_{2s+1}^1$ -旋量，有质量粒子波函数= $C_{2s+3}^3$ -旋量。

中微子波函数=2分量旋量，电子波函数=4分量旋量。

光子波函数=3分量旋量，有质量矢量粒子波函数=10分量旋量。

引力微子波函数=4分量旋量，有质量引力微子波函数=15分量旋量。

引力子波函数=5分量旋量，有质量引力子波函数=35分量旋量。

### 1.2 Bargmann-Wigner方程高旋量场自旋基的定义及推论

定义1.2.1.

$$\begin{cases} U_{k_\zeta}(\vec{p}, h; s) := \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(s; 3) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)}_{2s} [\Leftrightarrow] \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)}_{2s} = \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \underbrace{U_{k_\zeta}(\vec{p}, h; s)}_{2s} \\ V_{k_\zeta}(\vec{p}, h; s) := \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(s; 3) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)}_{2s} [\Leftrightarrow] \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)}_{2s} = \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \underbrace{V_{k_\zeta}(\vec{p}, h; s)}_{2s} \end{cases}$$

推论1.2.1.

$$\begin{cases} U_{k_\zeta}(\vec{p}, h; s) = \sqrt{C_{2s}^{s-h}} \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(s; 3) \underbrace{u_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) u_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \dots u_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) u_{\tau_\zeta}(\vec{p}, -\frac{1}{2})}_{s+h} \\ V_{k_\zeta}(\vec{p}, h; s) = \sqrt{C_{2s}^{s-h}} \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(s; 3) \underbrace{v_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) v_{\mu_\zeta}(\vec{p}, \frac{1}{2}) \dots v_{\sigma_\zeta}(\vec{p}, -\frac{1}{2}) v_{\tau_\zeta}(\vec{p}, -\frac{1}{2})}_{s+h} \end{cases}$$

### 1.3 Bargmann-Wigner方程平面波解的高旋量场表述

定理1.3.1.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0$

$$\psi_{k_\zeta}(x; s) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=s}^{-s} E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} [a(\vec{p}, h) U_{k_\zeta}(\vec{p}, h; s) e^{ip \cdot x} + b^+(\vec{p}, h) V_{k_\zeta}(\vec{p}, h; s) e^{-ip \cdot x}] d^3 \vec{p}$$

$$U_{k_\zeta}(\vec{p}, h; s) := \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(s; 3) \underbrace{U_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)}_{2s}, V_{k_\zeta}(\vec{p}, h; s) := \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(s; 3) \underbrace{V_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(\vec{p}, h)}_{2s}$$

推论1.3.1.

$$\begin{cases} a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m}{E}}^{2s} U^{+k_\zeta}(\vec{p}, h; s) \psi_{k_\zeta}(x; s) e^{-ip \cdot x} d^3 \vec{r} \\ b^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m}{E}}^{2s} V^{+k_\zeta}(\vec{p}, h; s) \psi_{k_\zeta}(x; s) e^{ip \cdot x} d^3 \vec{r} \end{cases}$$

推论1.3.2.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0 [\Leftrightarrow] (\gamma^a \partial_a + m)_{\kappa_\zeta} \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0$

$$\text{证明: } (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0$$

$$\Leftrightarrow (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) \Gamma_{\mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{l_\zeta}(s - \frac{1}{2}; 3) \psi_{k_\zeta}(x; s) = 0$$

$$\Leftrightarrow (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta N_{\lambda_\zeta l_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0 \quad \square$$

$$\text{推论1.3.3. } (\gamma^a \otimes I_{**} \partial_a + m) \Gamma(s; 3) \psi(x; s) = 0 \Leftrightarrow (\gamma^a \otimes I_* \partial_a + m) N(s; 3) \psi(x; s) = 0$$

#### 1.4 Bargmann-Wigner高旋量场的自旋方程

$$\text{推论1.4.1. } (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0 \Leftrightarrow (\gamma^a \otimes I_{**} \partial_a + m) \Gamma(s; 3) \psi(x; s) = 0$$

$$\Rightarrow [s \partial_a + i S_{ab}(s; 3) \partial^b] \psi(x; s) = -\frac{1}{2} m \gamma_a(s; 3) \psi(x; s)$$

$$\text{证明: } (\gamma_a \partial^a + m)_{\kappa_\zeta} \lambda_\zeta \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0$$

$$\Rightarrow \gamma_a \eta_\zeta^{\kappa_\zeta} (\gamma_b \partial^b + m)_{\kappa_\zeta} \lambda_\zeta \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0$$

$$\Leftrightarrow [(\frac{1}{2}\{\gamma_a, \gamma_b\} + \frac{1}{2}[\gamma_a, \gamma_b])_{\eta_\zeta} \lambda_\zeta + m \gamma_a \eta_\zeta^{\kappa_\zeta}] \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0$$

$$\Leftrightarrow [(\delta_{ab} \partial^b + 2i S_{ab} \partial^b)_{\eta_\zeta} \lambda_\zeta + m \gamma_a \eta_\zeta^{\kappa_\zeta}] \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0, S_{ab} = -\frac{i}{4} [\gamma_a, \gamma_b]$$

$$\Rightarrow \Gamma_{j_\zeta}^{\eta_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(s; 3) [(\delta_{ab} \partial^b + 2i S_{ab} \partial^b)_{\eta_\zeta} \lambda_\zeta + m \gamma_a \eta_\zeta^{\kappa_\zeta}] \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0$$

$$\Leftrightarrow [\delta_{ab} \partial^b + \frac{i}{s} S_{ab}(s; 3) \partial^b + \frac{1}{2s} m \gamma_a(s; 3)]_{j_\zeta} \lambda_\zeta \psi_{k_\zeta}(x; s) = 0$$

$$\Leftrightarrow [s \delta_{ab} + i S_{ab}(s; 3)] \partial^b \psi(x; s) = -\frac{1}{2} m \gamma_a(s; 3) \psi(x; s)$$

$$\Leftrightarrow [s \partial_a + i S_{ab}(s; 3) \partial^b] \psi(x; s) = -\frac{1}{2} m \gamma_a(s; 3) \psi(x; s) \quad \square$$

$$\text{推论1.4.2. } (\gamma^a \otimes I_{**} \partial_a + m) \Gamma(s; 3) \psi(x; s) = 0 \Rightarrow [s \partial_a + i S_{ab}(s; 3) \partial^b] \psi(x; s) = -\frac{1}{2} m \gamma_a(s; 3) \psi(x; s)$$

$$\text{推论1.4.3. } [s \partial_a + i S_{ab}(s; 3) \partial^b] \psi(x; s) = -\frac{1}{2} m \gamma_a(s; 3) \psi(x; s)$$

$$\Leftrightarrow [s \partial_a + i \Omega_{ab}(s; 3) \partial^b] \Gamma(s; 3) \psi(x; s) = -\frac{1}{2} m \Omega_a(s; 3) \Gamma(s; 3) \psi(x; s)$$

$$\Leftrightarrow \{s \partial_a + i [S_{ab}(\frac{1}{2}; 3) \otimes I_* + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)] \partial^b\} N(s; 3) \psi(x; s) = -\frac{1}{2} m [\gamma^a \otimes I_* + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)] N(s; 3) \psi(x; s)$$

$$\text{证明: } [s \partial_a + i S_{ab}(s; 3) \partial^b] \psi(x; s) = -\frac{1}{2} m \gamma_a(s; 3) \psi(x; s)$$

$$\Leftrightarrow \Gamma(s; 3) [s \partial_a + i S_{ab}(s; 3) \partial^b] \psi(x; s) = -\frac{1}{2} m \Gamma(s; 3) \gamma_a(s; 3) \psi(x; s)$$

$$\Leftrightarrow [s \partial_a + i \Omega_{ab}(s; 3) \partial^b] \Gamma(s; 3) \psi(x; s) = -\frac{1}{2} m \Omega_a(s; 3) \Gamma(s; 3) \psi(x; s) \quad \square$$

$$\text{证明: } [s \partial_a + i S_{ab}(s; 3) \partial^b] \psi(x; s) = -\frac{1}{2} m \gamma_a(s; 3) \psi(x; s)$$

$$\Leftrightarrow N(s; 3) [s \partial_a + i S_{ab}(s; 3) \partial^b] \psi(x; s) = -\frac{1}{2} m N(s; 3) \gamma_a(s; 3) \psi(x; s)$$

$$\Leftrightarrow \{s \partial_a + i [S_{ab}(\frac{1}{2}; 3) \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes S_{ab}(s - \frac{1}{2}; 3)] \partial^b\} N(s; 3) \psi(x; s)$$

$$= -\frac{1}{2} m [\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)] N(s; 3) \psi(x; s) \quad \square$$

#### 1.5 Bargmann-Wigner高旋量场的类Dirac方程

$$\text{推论1.5.1. } (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0 \Rightarrow [\gamma^a(s; 3) \partial_a + 2sm] \psi(x; s) = 0$$

$$\text{证明: } (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0$$

$$\Rightarrow \Gamma_{j_\zeta}^{\kappa_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}(s; 3) (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \Gamma_{\lambda_\zeta \mu_\zeta \dots \sigma_\zeta \tau_\zeta}^{k_\zeta}(s; 3) \psi_{k_\zeta}(x; s) = 0$$

$$\Leftrightarrow [\frac{1}{2s} \gamma^a(s; 3) \partial_a + m]_{j_\zeta} \lambda_\zeta \psi_{k_\zeta}(x; s) = 0$$

$$\Leftrightarrow [\frac{1}{2s}\gamma^a(s; 3)\partial_a + m]\psi(x; s) = 0$$

$$\Leftrightarrow [\gamma^a(s; 3)\partial_a + 2sm]\psi(x; s) = 0 \quad \square$$

$$\text{推论1.5.2. } [\gamma^a(s; 3)\partial_a + 2sm]\psi(x; s) = 0 \Leftrightarrow [\Omega^a(s; 3)\partial_a + 2sm]\Gamma(s; 3)\psi(x; s) = 0$$

$$\Leftrightarrow \{[\gamma^a \otimes I_* + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)]\partial_a + 2sm\}N(s; 3)\psi(x; s) = 0$$

$$\Leftrightarrow [I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)\partial_a + (2s - 1)m]N(s; 3)\psi(x; s) = 0$$

$$\text{证明: } [\gamma^a(s; 3)\partial_a + 2sm]\psi(x; s) = 0$$

$$\Leftrightarrow \Gamma(s; 3)[\gamma^a(s; 3)\partial_a + 2sm]\psi(x; s) = 0$$

$$\Leftrightarrow [\Omega^a(s; 3)\partial_a + 2sm]\Gamma(s; 3)\psi(x; s) = 0 \quad \square$$

$$\text{证明: } [\gamma^a(s; 3)\partial_a + 2sm]\psi(x; s) = 0$$

$$\Leftrightarrow N(s; 3)[\gamma^a(s; 3)\partial_a + 2sm]\psi(x; s) = 0$$

$$\Leftrightarrow \{[\gamma^a \otimes I_{C_{2s-1+3}^{2s-1}} + I_4 \otimes \gamma^a(s - \frac{1}{2}; 3)]\partial_a + 2sm\}N(s; 3)\psi(x; s) = 0 \quad \square$$

## 1.6 Bargmann-Wigner方程高旋量场自旋基的矩阵表述

定义1.6.1.

$$\begin{cases} U(\vec{p}, h; s) := \bar{\Gamma}(s; 3) \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{p}, h) [\Leftrightarrow] \underbrace{U_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{p}, h) = \Gamma(s; 3)U(\vec{p}, h; s) \\ V(\vec{p}, h; s) := \bar{\Gamma}(s; 3) \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{p}, h) [\Leftrightarrow] \underbrace{V_{\lambda_\zeta \otimes \mu_\zeta \otimes \dots \otimes \sigma_\zeta \otimes \tau_\zeta}}_{2s}(\vec{p}, h) = \Gamma(s; 3)U(\vec{p}, h; s) \end{cases}$$

推论1.6.1.

$$\begin{cases} U(\vec{p}, h; s) = \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s; 3) \underbrace{u(\vec{p}, \frac{1}{2}) \otimes u(\vec{p}, \frac{1}{2}) \otimes \dots \otimes u(\vec{p}, -\frac{1}{2}) \otimes u(\vec{p}, -\frac{1}{2})}_{s+h} \\ V(\vec{p}, h; s) = \sqrt{C_{2s}^{s-h}} \bar{\Gamma}(s; 3) \underbrace{v(\vec{p}, \frac{1}{2}) \otimes v(\vec{p}, \frac{1}{2}) \otimes \dots \otimes v(\vec{p}, -\frac{1}{2}) \otimes v(\vec{p}, -\frac{1}{2})}_{s+h} \end{cases}$$

定义1.6.2.

$$\begin{cases} \bar{U}(\vec{p}, h; s) := \bar{\Gamma}(s; 3) \underbrace{\gamma_4 \otimes \gamma_4 \dots \gamma_4}_{2s} \Gamma(s; 3)U(\vec{p}, h; s) \\ \bar{V}(\vec{p}, h; s) := \bar{\Gamma}(s; 3) \underbrace{\gamma_4 \otimes \gamma_4 \dots \gamma_4}_{2s} \Gamma(s; 3)V(\vec{p}, h; s) \end{cases}$$

## 1.7 Bargmann-Wigner方程平面波解的高自旋矩阵表述

定理1.7.1.  $(\gamma^a \otimes I_* \partial_a + m)\Gamma(s; 3)\psi(x; s) = 0, * = 4^{2s-1}$

$$\psi(x; s) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=s}^{-s} E^{s-\frac{1}{2}} \sqrt{\frac{m}{E}}^{2s} [a(\vec{p}, h)U(\vec{p}, h; s)e^{ip \cdot x} + b^+(\vec{p}, h)V(\vec{p}, h; s)e^{-ip \cdot x}] d^3 \vec{p}$$

$$\begin{cases} a(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m}{E}}^{2s} U^+(\vec{p}, h; s)\psi(x; s)e^{-ip \cdot x} d^3 \vec{r} \\ b^+(\vec{p}, h) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} E^{-(s-\frac{1}{2})} \sqrt{\frac{m}{E}}^{2s} V^+(\vec{p}, h; s)\psi(x; s)e^{ip \cdot x} d^3 \vec{r} \end{cases}$$

## 1.8 Bargmann-Wigner方程的两种自旋基之间的关系

推论1.8.1.  $u(\vec{p}, h) = -\varsigma \gamma_5 v(\vec{p}, h), v(\vec{p}, h) = -\varsigma \gamma_5 u(\vec{p}, h), h = -\frac{1}{2}, \frac{1}{2}$

$$\text{推论1.8.2. } \begin{cases} U(\vec{p}, h; s) = (-\varsigma)^{2s} \bar{\Gamma}(s; 3) \underbrace{\gamma_5 \otimes \gamma_5 \dots \gamma_5}_{2s} \Gamma(s; 3)V(\vec{p}, h; s) \\ V(\vec{p}, h; s) = (-\varsigma)^{2s} \bar{\Gamma}(s; 3) \underbrace{\gamma_5 \otimes \gamma_5 \dots \gamma_5}_{2s} \Gamma(s; 3)U(\vec{p}, h; s) \end{cases}$$

$$\text{推论1.8.3. } \begin{cases} U^*(\vec{p}, h; s) = (-1)^{s+h} \varsigma^{2s} \bar{\Gamma}(s; 3) \underbrace{\sigma_y \otimes \sigma_y \dots \sigma_y}_{4s} \Gamma(s; 3)V(\vec{p}, -h; s) \\ V^*(\vec{p}, h; s) = (-1)^{s-h} \varsigma^{2s} \bar{\Gamma}(s; 3) \underbrace{\sigma_y \otimes \sigma_y \dots \sigma_y}_{4s} \Gamma(s; 3)U(\vec{p}, -h; s) \end{cases}$$

$$\text{推论1.8.4.} \begin{cases} U(\vec{p}, -h; s) = (-1)^{s-h} \zeta^{2s} \bar{\Gamma}(s; 3) \overbrace{\sigma_y \otimes \sigma_y \cdots}^{4s} \Gamma(s; 3) V^*(\vec{p}, h; s) \\ V(\vec{p}, -h; s) = (-1)^{s+h} \zeta^{2s} \bar{\Gamma}(s; 3) \overbrace{\sigma_y \otimes \sigma_y \cdots}^{4s} \Gamma(s; 3) U^*(\vec{p}, h; s) \end{cases}$$

$$\text{推论1.8.5.} \begin{cases} U^*(\vec{p}, h; s) = (-1)^{s-h} \zeta^{2s} \bar{\Gamma}(s; 3) \overbrace{(\gamma_2 \gamma_5) \otimes (\gamma_2 \gamma_5) \cdots}^{2s} \Gamma(s; 3) U(\vec{p}, -h; s) \\ V^*(\vec{p}, h; s) = (-1)^{s+h} \zeta^{2s} \bar{\Gamma}(s; 3) \overbrace{(\gamma_2 \gamma_5) \otimes (\gamma_2 \gamma_5) \cdots}^{2s} \Gamma(s; 3) V(\vec{p}, -h; s) \end{cases}$$

### 1.9 实表象下Bargmann-Wigner方程的两种自旋基之间的关系

$$\text{推论1.9.1.} \quad u_s(\vec{p}, h) = -\zeta \gamma_{s5} v_s(\vec{p}, h), \quad v_s(\vec{p}, h) = -\zeta \gamma_{s5} u_s(\vec{p}, h), \quad h = -\frac{1}{2}, \frac{1}{2}$$

$$\text{推论1.9.2.} \begin{cases} U_s(\vec{p}, h; s) = (-\zeta)^{2s} \bar{\Gamma}(s; 3) \overbrace{\gamma_{s5} \otimes \gamma_{s5} \cdots}^{2s} \Gamma(s; 3) V_s(\vec{p}, h; s) \\ V_s(\vec{p}, h; s) = (-\zeta)^{2s} \bar{\Gamma}(s; 3) \overbrace{\gamma_{s5} \otimes \gamma_{s5} \cdots}^{2s} \Gamma(s; 3) U_s(\vec{p}, h; s) \end{cases}$$

$$\text{推论1.9.3.} \begin{cases} U_s^*(\vec{p}, h; s) = (-1)^{s-h} \zeta^{2s} V_s(\vec{p}, -h; s) & U_s(\vec{p}, -h; s) = (-1)^{s+h} \zeta^{2s} V_s^*(\vec{p}, h; s) \\ V_s^*(\vec{p}, h; s) = (-1)^{s+h} \zeta^{2s} U_s(\vec{p}, -h; s) & V_s(\vec{p}, -h; s) = (-1)^{s-h} \zeta^{2s} U_s^*(\vec{p}, h; s) \end{cases}$$

$$\text{推论1.9.4.} \begin{cases} U_s^*(\vec{p}, h; s) = (-1)^{s+h} \bar{\Gamma}(s; 3) \overbrace{\gamma_{s5} \otimes \gamma_{s5} \cdots}^{2s} \Gamma(s; 3) U_s(\vec{p}, -h; s) \\ V_s^*(\vec{p}, h; s) = (-1)^{s-h} \bar{\Gamma}(s; 3) \overbrace{\gamma_{s5} \otimes \gamma_{s5} \cdots}^{2s} \Gamma(s; 3) V_s(\vec{p}, -h; s) \end{cases}$$

### 1.10 Bargmann-Wigner方程自旋基的正交性质

$$\text{推论1.10.1.} \begin{cases} \bar{U}(\vec{p}, h; s) U(\vec{p}, h'; s) = \delta_{hh'}, \quad \bar{U}(\vec{p}, h; s) V(\vec{p}, h'; s) = 0 \\ \bar{V}(\vec{p}, h; s) V(\vec{p}, h'; s) = \delta_{hh'}, \quad \bar{V}(\vec{p}, h; s) U(\vec{p}, h'; s) = 0 \end{cases}$$

$$\text{推论1.10.2.} \begin{cases} U^+(\vec{p}, h; s) U(\vec{p}, h'; s) = \left(\frac{E}{m}\right)^{2s} \delta_{hh'}, \quad U^+(\vec{p}, h; s) V(-\vec{p}, h'; s) = 0 \\ V^+(\vec{p}, h; s) V(\vec{p}, h'; s) = \left(\frac{E}{m}\right)^{2s} \delta_{hh'}, \quad V^+(\vec{p}, h; s) U(-\vec{p}, h'; s) = 0 \end{cases}$$

### 1.11 Dirac方程自旋基的广义二项式定理及其推论

定理1.11.1.

$$\begin{aligned} & \sum_{h=s}^{-s} C_{2s}^{s-h} \underbrace{u_{\{\lambda_\zeta(\vec{p}, \frac{1}{2}) u_{\mu_\zeta(\vec{p}, \frac{1}{2})} \cdots u_{\sigma_\zeta(\vec{p}, -\frac{1}{2}) u_{\tau_\zeta(\vec{p}, -\frac{1}{2})}\}}_{s+h} u_{\{\lambda'_\zeta(\vec{p}, \frac{1}{2}) u_{\mu'_\zeta(\vec{p}, \frac{1}{2})} \cdots u_{\sigma'_\zeta(\vec{p}, -\frac{1}{2}) u_{\tau'_\zeta(\vec{p}, -\frac{1}{2})}\}}_{s+h}} \\ &= \sum_{h=1/2}^{-1/2} \underbrace{u_{\{\lambda_\zeta(\vec{p}, h) u_{\lambda'_\zeta(\vec{p}, h)} \sum_{h=1/2}^{-1/2} u_{\mu_\zeta(\vec{p}, h) u_{\mu'_\zeta(\vec{p}, h)} \cdots \sum_{h=1/2}^{-1/2} u_{\sigma_\zeta(\vec{p}, h) u_{\sigma'_\zeta(\vec{p}, h)} \sum_{h=1/2}^{-1/2} u_{\tau_\zeta(\vec{p}, h) u_{\tau'_\zeta(\vec{p}, h)}\}}_{2s}} \\ &\Leftrightarrow \sum_{h=s}^{-s} C_{2s}^{s-h} \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}(s; 3) \underbrace{u_{\lambda_\zeta(\vec{p}, \frac{1}{2}) \cdots u_{\tau_\zeta(\vec{p}, -\frac{1}{2})}}_{s+h} u_{\{\lambda'_\zeta(\vec{p}, \frac{1}{2}) \cdots u_{\tau'_\zeta(\vec{p}, -\frac{1}{2})}\}}_{s+h}} \Gamma_{k'_\zeta}^{\lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}(s; 3) \\ &= \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \cdots \sigma_\zeta \tau_\zeta}(s; 3) \sum_{h=1/2}^{-1/2} \underbrace{u_{\lambda_\zeta(\vec{p}, h) u_{\lambda'_\zeta(\vec{p}, h)} \cdots \sum_{h=1/2}^{-1/2} u_{\tau_\zeta(\vec{p}, h) u_{\tau'_\zeta(\vec{p}, h)}\}}_{2s} \Gamma_{k'_\zeta}^{\lambda'_\zeta \mu'_\zeta \cdots \sigma'_\zeta \tau'_\zeta}(s; 3) \\ &\Leftrightarrow \sum_{h=s}^{-s} U(\vec{p}, h; s) U^+(\vec{p}, h; s) = \bar{\Gamma}(s; 3) \sum_{h=1/2}^{-1/2} \underbrace{u(\vec{p}, h) u^+(\vec{p}, h) \otimes \sum_{h=1/2}^{-1/2} u(\vec{p}, h) u^+(\vec{p}, h) \otimes \cdots \sum_{h=1/2}^{-1/2} u(\vec{p}, h) u^+(\vec{p}, h)}_{2s} \Gamma(s; 3) \end{aligned}$$

## 1.12 Bargmann-Wigner方程准投影算子的定义和性质

$$\text{定义1.12.1. } \begin{cases} \Lambda_+(\vec{p}; s) := \sum_{h=s}^{-s} U(\vec{p}, h; s) U^+(\vec{p}, h; s) \\ \Lambda_-(\vec{p}; s) := \sum_{h=s}^{-s} V(\vec{p}, h; s) V^+(\vec{p}, h; s) \end{cases}$$

$$\text{推论1.12.1. } \Lambda_{\pm}(\vec{p}; s) = \bar{\Gamma}(s; 3) \underbrace{\Lambda_{\pm}(\vec{p}, \frac{1}{2}) \otimes \Lambda_{\pm}(\vec{p}, \frac{1}{2}) \otimes \cdots \otimes \Lambda_{\pm}(\vec{p}, \frac{1}{2})}_{2s} \Gamma(s; 3)$$

$$\text{推论1.12.2. } \Lambda_{\pm}(\vec{p}; s) = \frac{1}{(2m)^{2s}} \bar{\Gamma}(s; 3) \underbrace{[(\pm m - i\gamma^a p_a) \gamma^4] \otimes [(\pm m - i\gamma^b p_b) \gamma^4] \otimes \cdots}_{2s} \Gamma(s; 3)$$

推论1.12.3.

$$\begin{cases} \sum_{h=s}^{-s} (-1)^{s-h} U(\vec{p}, h; s) V^T(\vec{p}, -h; s) = \zeta^{2s} \underbrace{[\Lambda_+(\vec{p}, \frac{1}{2}) \bar{C} \gamma_4] \otimes [\Lambda_+(\vec{p}, \frac{1}{2}) \bar{C} \gamma_4] \otimes \cdots \otimes [\Lambda_+(\vec{p}, \frac{1}{2}) \bar{C} \gamma_4]}_{2s} \\ \sum_{h=s}^{-s} (-1)^{s+h} V(\vec{p}, h; s) U^T(\vec{p}, -h; s) = \zeta^{2s} \underbrace{[\Lambda_-(\vec{p}, \frac{1}{2}) \bar{C} \gamma_4] \otimes [\Lambda_-(\vec{p}, \frac{1}{2}) \bar{C} \gamma_4] \otimes \cdots \otimes [\Lambda_-(\vec{p}, \frac{1}{2}) \bar{C} \gamma_4]}_{2s} \end{cases}$$

推论1.12.4.

$$\begin{cases} \sum_{h=s}^{-s} (-1)^{s-h} U(\vec{p}, h; s) V^T(\vec{p}, -h; s) = \frac{\zeta^{2s}}{(2m)^{2s}} \underbrace{[(m - i\gamma^a p_a) C] \otimes [(m - i\gamma^a p_a) C] \otimes \cdots \otimes [(m - i\gamma^a p_a) C]}_{2s} \\ \sum_{h=s}^{-s} (-1)^{s+h} V(\vec{p}, h; s) U^T(\vec{p}, -h; s) = \frac{\zeta^{2s}}{(2m)^{2s}} \underbrace{[(-m - i\gamma^a p_a) C] \otimes [(-m - i\gamma^a p_a) C] \otimes \cdots \otimes [(-m - i\gamma^a p_a) C]}_{2s} \end{cases}$$

## 2 Bargmann-Wigner方程对易规则的高旋量场表述

### 2.1 Bargmann-Wigner方程的协变对易规则

定理2.1.1.  $[a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [b(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [rest]_{-2s+1} = 0$

$$\Leftrightarrow \begin{cases} [\psi_{k_\zeta}(x; s), \psi_{k'_\zeta}^+(x'; s)]_{-2s+1} = \frac{i}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \underbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots}_{2s} \Gamma(s; 3)\}_{k_\zeta k'_\zeta} \Delta(x - x') \\ [rest]_{-2s+1} = 0 \\ [\psi_{k_\zeta}(x; s), \psi_{k'_\zeta}^+(x'; s)]_{-2s+1} = \frac{i}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \underbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots}_{2s} \Gamma(s; 3)\}_{k_\zeta k'_\zeta} \Delta(x - x') \\ [\psi_{k_\zeta}^{(+)}(x; s), \psi_{k'_\zeta}^{(+)}(x'; s)]_{-2s+1} = \frac{i}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \underbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots}_{2s} \Gamma(s; 3)\}_{k_\zeta k'_\zeta} \Delta^{(+)}(x - x') \\ [\psi_{k_\zeta}^{(-)}(x; s), \psi_{k'_\zeta}^{(-)}(x'; s)]_{-2s+1} = \frac{i}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \underbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots}_{2s} \Gamma(s; 3)\}_{k_\zeta k'_\zeta} \Delta^{(-)}(x - x') \\ [rest]_{-2s+1} = 0 \end{cases}$$

推论2.1.1.  $[\psi_{k_\zeta}(x; s), \psi_{k'_\zeta}^+(x'; s)]_{-2s+1} = 2im^{2s} \Lambda_+(-i\partial; s)$

### 2.2 Bargmann-Wigner方程玻色子对易规则两种等价描述

$$\text{定义2.2.1. } \underbrace{\mathbb{X}_{k_\zeta}^{ab \cdots}}_n(x; n) := \frac{1}{(2n)!} \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \cdots}(n; 3) \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x) \mathbb{X}_{\eta_\zeta \xi_\zeta}^b(x) \cdots}_n = \Gamma_{k_\zeta}^{\lambda_\zeta \mu_\zeta \eta_\zeta \xi_\zeta \cdots}(n; 3) \underbrace{\mathbb{X}_{\lambda_\zeta \mu_\zeta}^a(x) \mathbb{X}_{\eta_\zeta \xi_\zeta}^b(x) \cdots}_n$$

$$[\Leftrightarrow] \underbrace{\mathbb{X}_{k_\zeta}^{ab \cdots}}_n(x; n) = \bar{\Gamma}(n; 3) \underbrace{\mathbb{X}_{\lambda_\zeta \otimes \mu_\zeta}^a(x) \otimes \mathbb{X}_{\eta_\zeta \otimes \xi_\zeta}^b(x) \otimes \cdots}_n, \underbrace{\mathbb{X}^{+a'b' \cdots}}_n(x'; n) = \underbrace{[\mathbb{X}_{\lambda'_\zeta \otimes \mu'_\zeta}^{a'}(x) \otimes \mathbb{X}_{\eta'_\zeta \otimes \xi'_\zeta}^{b'}(x) \otimes \cdots]}_n^+ \Gamma(n; 3)$$

引理2.2.1.  $\mathbb{X}^a(p; 1)(\eta_{aa'} + \frac{p_a p_{a'}}{m^2}) \mathbb{X}^{+a'}(p; 1) = 2\bar{\Gamma}(1; 3) [(m - i\gamma^a p_a) \gamma^4] \otimes [(m - i\gamma^b p_b) \gamma^4] \Gamma(1; 3)$

引理2.2.2.  $\mathbb{X}^a(x; 1)(\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \mathbb{X}^{+a'}(x'; 1) \Delta(x - x') = 2\bar{\Gamma}(1; 3) [(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \Gamma(1; 3) \Delta(x - x')$

定理2.2.1.  $[a(\vec{p}, h), a^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [b(\vec{p}, h), b^+(\vec{p}', h')]_{-2s+1} = \delta_{hh'} \delta^3(\vec{p} - \vec{p}'), [rest]_{-2s+1} = 0$

$$\Leftrightarrow [\psi_{k_\zeta}(x; n), \psi_{k'_\zeta}^+(x'; n)] = \frac{i}{2^{2n-1}} \overbrace{\{\bar{\Gamma}(s; 3) [(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)\}}^{2n} \Delta(x - x')$$

$$\Leftrightarrow [\psi_{k_\zeta}(x; n), \psi_{k'_\zeta}^+(x'; n)] = \frac{i}{2^{2n-1}} \overbrace{\{\bar{\Gamma}(n; 3) \mathbb{X}^{ab \cdots}(x; n) \mathbb{X}^{+a'b' \cdots}(x'; n) \Gamma(n; 3) (\eta_{aa'} - \frac{\partial_a \partial_{a'}}{m^2}) \cdots\}}^n \Delta(x - x')$$

### 2.3 Bargmann-Wigner方程的对易函数、因果函数和费曼传播子

引理2.3.1.  $\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}$

$$= \sum_{n=0}^{2s} C_{2s}^n \bar{\Gamma}(s; 3) \overbrace{[-(\gamma^a \partial_a) \gamma^4] \otimes [-(\gamma^b \partial_b) \gamma^4] \otimes \cdots \otimes (m \gamma^4) \otimes (m \gamma^4) \otimes \cdots \Gamma(s; 3)}^{n} \overbrace{\Gamma(s; 3)}^{2s-n}$$

$$= \sum_{n=0}^{2s} (-1)^n m^{2s-n} C_{2s}^n \bar{\Gamma}(s; 3) \overbrace{(\gamma^a \gamma^4) \otimes (\gamma^b \gamma^4) \otimes \cdots \otimes (\gamma^4) \otimes (\gamma^4) \otimes \cdots \Gamma(s; 3)}^n \overbrace{\partial_a \partial_b \cdots}^n$$

引理2.3.2.  $[\theta(t), \bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}]_{-2s+1}$

$$= \sum_{n=0}^{2s} \sum_{l=0}^{n-1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \bar{\Gamma}(s; 3) \overbrace{(\gamma^i \gamma^4) \otimes (\gamma^j \gamma^4) \cdots \otimes I_4 \otimes I_4 \cdots \otimes (\gamma^4) \otimes (\gamma^4) \cdots \Gamma(s; 3)}^{l} \overbrace{[\partial_t^{n-1-l} \delta(t)]}_{2s-n} \overbrace{\partial_i \partial_j \cdots}^l$$

推论2.3.1.

$$\left\{ \begin{aligned} \Delta_{k_\zeta k'_\zeta}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{(+)}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta^{(+)}(x) \\ \Delta_{k_\zeta k'_\zeta}^{(-)}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta^{(-)}(x) \\ \Delta_{k_\zeta k'_\zeta}^{(l)}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta^{(l)}(x) \end{aligned} \right.$$

推论2.3.2.

$$\left\{ \begin{aligned} \Delta_{k_\zeta k'_\zeta}^{(c)}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta^{(c)}(x) + \frac{1}{2^{2s-1}} \\ &\quad \sum_{n=0}^{2s} \sum_{l=0}^{n-1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \bar{\Gamma}(s; 3) \overbrace{(\gamma^i \gamma^4) \otimes (\gamma^j \gamma^4) \cdots \otimes I_4 \otimes I_4 \cdots \otimes (\gamma^4) \otimes (\gamma^4) \cdots \Gamma(s; 3)}^{l} \overbrace{[\partial_t^{n-1-l} \delta(t)]}_{2s-n} \overbrace{\partial_i \partial_j \cdots}^l \Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{(F)}(x; s) = i \Delta_{k_\zeta k'_\zeta}^{(c)}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta_F(x) + \frac{i}{2^{2s-1}} \\ &\quad \sum_{n=0}^{2s} \sum_{l=0}^{n-1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \bar{\Gamma}(s; 3) \overbrace{(\gamma^i \gamma^4) \otimes (\gamma^j \gamma^4) \cdots \otimes I_4 \otimes I_4 \cdots \otimes (\gamma^4) \otimes (\gamma^4) \cdots \Gamma(s; 3)}^{l} \overbrace{[\partial_t^{n-1-l} \delta(t)]}_{2s-n} \overbrace{\partial_i \partial_j \cdots}^l \Delta(x) \end{aligned} \right.$$

推论2.3.3.

$$\left\{ \begin{aligned} \Delta_{k_\zeta k'_\zeta}^{(ret)}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta^{(ret)}(x) + \frac{1}{2^{2s-1}} \\ &\quad \sum_{n=0}^{2s} \sum_{l=0}^{n-1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \bar{\Gamma}(s; 3) \overbrace{(\gamma^i \gamma^4) \otimes (\gamma^j \gamma^4) \cdots \otimes I_4 \otimes I_4 \cdots \otimes (\gamma^4) \otimes (\gamma^4) \cdots \Gamma(s; 3)}^{l} \overbrace{[\partial_t^{n-1-l} \delta(t)]}_{2s-n} \overbrace{\partial_i \partial_j \cdots}^l \Delta(x) \\ \Delta_{k_\zeta k'_\zeta}^{(adv)}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta^{(adv)}(x) + \frac{1}{2^{2s-1}} \\ &\quad \sum_{n=0}^{2s} \sum_{l=0}^{n-1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \bar{\Gamma}(s; 3) \overbrace{(\gamma^i \gamma^4) \otimes (\gamma^j \gamma^4) \cdots \otimes I_4 \otimes I_4 \cdots \otimes (\gamma^4) \otimes (\gamma^4) \cdots \Gamma(s; 3)}^{l} \overbrace{[\partial_t^{n-1-l} \delta(t)]}_{2s-n} \overbrace{\partial_i \partial_j \cdots}^l \Delta(x) \end{aligned} \right.$$

引理2.3.3.  $\Delta(x) \partial_t^n \delta(t) = \sum_{l=0}^{[(n-1)/2]} C_n^{2l+1} (\nabla^2 - m^2)^l \partial_t^{n-2l-1} \delta^4(x)$



$$\text{推论2.3.4. } \Delta(x) \partial_t^{n-1-l} \delta(t) = \sum_{r=0}^{[(n-l-2)/2]} C_{n-1-l}^{2r+1} (\nabla^2 - m^2)^r \partial_t^{n-l-2-2r} \delta^4(x)$$

推论2.3.5.

$$\left\{ \begin{aligned} \Delta_{k_\zeta k'_\zeta}^{(c)}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta^{(c)}(x) \\ &+ \frac{1}{2^{2s-1}} \sum_{n=0}^{2s} \sum_{l=0}^{n-2} \sum_{r=0}^{[(n-l-2)/2]} C_{n-l-1}^{2r+1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \bar{\Gamma}(s; 3) \overbrace{(\gamma^i \gamma^4) \otimes (\gamma^j \gamma^4) \cdots}^l \otimes \overbrace{I_4 \otimes I_4 \cdots}^{n-l} \otimes \overbrace{(\gamma^4) \otimes (\gamma^4) \cdots}^{2s-n} \Gamma(s; 3) \\ &\underbrace{\partial_i \partial_j \cdots}_{l} (\nabla^2 - m^2)^r \partial_t^{n-l-2-2r} \delta^4(x) \\ \Delta_{k_\zeta k'_\zeta}^{(F)}(x; s) &= i \Delta_{k_\zeta k'_\zeta}^{(c)}(x; s) := \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta_F(x) \\ &+ \frac{i}{2^{2s-1}} \sum_{n=0}^{2s} \sum_{l=0}^{n-2} \sum_{r=0}^{[(n-l-2)/2]} C_{n-l-1}^{2r+1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \bar{\Gamma}(s; 3) \overbrace{(\gamma^i \gamma^4) \otimes (\gamma^j \gamma^4) \cdots}^l \otimes \overbrace{I_4 \otimes I_4 \cdots}^{n-l} \otimes \overbrace{(\gamma^4) \otimes (\gamma^4) \cdots}^{2s-n} \Gamma(s; 3) \\ &\underbrace{\partial_i \partial_j \cdots}_{l} (\nabla^2 - m^2)^r \partial_t^{n-l-2-2r} \delta^4(x) \\ \Delta_{k_\zeta k'_\zeta}^{(F)}(p; s) &= \frac{-i}{2^{2s-1}} \bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s} + \dots \end{aligned} \right.$$

推论2.3.6.

$$\left\{ \begin{aligned} \Delta_{k_\zeta k'_\zeta}^{(ret)}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta^{(ret)}(x) \\ &+ \frac{1}{2^{2s-1}} \sum_{n=0}^{2s} \sum_{l=0}^{n-2} \sum_{r=0}^{[(n-l-2)/2]} C_{n-l-1}^{2r+1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \bar{\Gamma}(s; 3) \overbrace{(\gamma^i \gamma^4) \otimes (\gamma^j \gamma^4) \cdots}^l \otimes \overbrace{I_4 \otimes I_4 \cdots}^{n-l} \otimes \overbrace{(\gamma^4) \otimes (\gamma^4) \cdots}^{2s-n} \Gamma(s; 3) \\ &\underbrace{\partial_i \partial_j \cdots}_{l} (\nabla^2 - m^2)^r \partial_t^{n-l-2-2r} \delta^4(x) \\ \Delta_{k_\zeta k'_\zeta}^{(adv)}(x; s) &:= \frac{1}{2^{2s-1}} \{\bar{\Gamma}(s; 3) \overbrace{[(m - \gamma^a \partial_a) \gamma^4] \otimes [(m - \gamma^b \partial_b) \gamma^4] \otimes \cdots \Gamma(s; 3)}^{2s}\}_{k_\zeta k'_\zeta} \Delta^{(adv)}(x) \\ &+ \frac{1}{2^{2s-1}} \sum_{n=0}^{2s} \sum_{l=0}^{n-2} \sum_{r=0}^{[(n-l-2)/2]} C_{n-l-1}^{2r+1} \frac{i^{n+l} m^{2s-n} (2s)!}{l!(n-l)!(2s-n)!} \bar{\Gamma}(s; 3) \overbrace{(\gamma^i \gamma^4) \otimes (\gamma^j \gamma^4) \cdots}^l \otimes \overbrace{I_4 \otimes I_4 \cdots}^{n-l} \otimes \overbrace{(\gamma^4) \otimes (\gamma^4) \cdots}^{2s-n} \Gamma(s; 3) \\ &\underbrace{\partial_i \partial_j \cdots}_{l} (\nabla^2 - m^2)^r \partial_t^{n-l-2-2r} \delta^4(x) \end{aligned} \right.$$

## 2.4 Bargmann-Wigner方程的等时对易规则

引理2.4.1.

$$\Delta_{k_\zeta k'_\zeta}(x; s)|_{t=0} = \frac{-i}{2^{2s-1}} \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \bar{\Gamma}(s; 3) \overbrace{[(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla) \otimes (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla) \cdots \otimes I_4] \Gamma(s; 3)}^{2s-2l-1} (m^2 - \nabla^2)^l \delta^3(\vec{r})$$

定理2.4.1.  $[\psi_{k_\zeta}(\vec{r}, t), \psi_{k'_\zeta}^\dagger(\vec{r}', t)]_{-2s+1}$ 

$$= \frac{1}{2^{2s-1}} \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} \bar{\Gamma}(s; 3) \overbrace{[(m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla) \otimes (m\gamma^4 + \gamma^4 \vec{\gamma} \cdot \nabla) \cdots \otimes I_4] \Gamma(s; 3)}^{2s-2l-1} (m^2 - \nabla^2)^l \delta^3(\vec{r} - \vec{r}')$$

## 2.5 Bargmann-Wigner方程的有关引理

$$\begin{aligned} \text{引理2.5.1. } & \bar{\Gamma}(s; 3) \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E] \otimes [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E] \otimes \cdots \Gamma(s; 3)}^{2s} \\ &= \sum_{l=0}^{2s} C_{2s}^l E^l \bar{\Gamma}(s; 3) \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) \otimes (m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) \cdots}^{2s-l} \otimes \overbrace{I_4 \otimes I_4 \cdots}^l \Gamma(s; 3) \end{aligned}$$

$$\begin{aligned} \text{引理2.5.2. } & \bar{\Gamma}(s; 3) \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E] \otimes [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E] \otimes \cdots \Gamma(s; 3)}^{2s} \\ &= \sum_{l=0}^{2s} (-1)^l C_{2s}^l E^l \bar{\Gamma}(s; 3) \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) \otimes (m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) \cdots}^{2s-l} \otimes \overbrace{I_4 \otimes I_4 \cdots}^l \Gamma(s; 3) \end{aligned}$$

$$\begin{aligned}
& \text{引理2.5.3. } \bar{\Gamma}(s; 3) \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E] \otimes [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E] \otimes \cdots \Gamma(s; 3)}^{2s} \\
& + \bar{\Gamma}(s; 3) \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E] \otimes [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E] \otimes \cdots \Gamma(s; 3)}^{2s} \\
& = 2 \sum_{l=0}^{[s]} C_{2s}^{2l} E^{2l} \bar{\Gamma}(s; 3) \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) \otimes (m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) \cdots \otimes I_4 \otimes I_4 \cdots}^{2s-2l} \overbrace{\Gamma(s; 3)}^{2l}
\end{aligned}$$

$$\begin{aligned}
& \text{引理2.5.4. } \bar{\Gamma}(s; 3) \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E] \otimes [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) + E] \otimes \cdots \Gamma(s; 3)}^{2s} \\
& - \bar{\Gamma}(s; 3) \overbrace{[(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E] \otimes [(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) - E] \otimes \cdots \Gamma(s; 3)}^{2s} \\
& = 2 \sum_{l=0}^{[s-\frac{1}{2}]} C_{2s}^{2l+1} E^{2l+1} \bar{\Gamma}(s; 3) \overbrace{(m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) \otimes (m\gamma^4 + i\gamma^4 \vec{\gamma} \cdot \vec{p}) \cdots \otimes I_4 \otimes I_4 \cdots}^{2s-2l-1} \overbrace{\Gamma(s; 3)}^{2l+1}
\end{aligned}$$

## 2.6 Bargmann-Wigner方程能量动量算符的提取

$$\text{定理2.6.1. } H(s) = \int \sum_{h=s}^{-s} E [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} = \int \psi^+(\vec{r}, t; s) \frac{(i\partial_t)^{2s}}{(m^2 - \nabla^2)^{2s-1}} \psi(\vec{r}, t; s) d^3 \vec{r}$$

$$\text{推论2.6.1. } \begin{cases} H(n) = \int \psi^+(\vec{r}, t; n) \frac{1}{(m^2 - \nabla^2)^{n-1}} \psi(\vec{r}, t; n) d^3 \vec{r} \\ H(n + \frac{1}{2}) = \int \psi^+(\vec{r}, t; n + \frac{1}{2}) \frac{i\partial_t}{(m^2 - \nabla^2)^n} \psi(\vec{r}, t; n + \frac{1}{2}) d^3 \vec{r} \end{cases}$$

$$\text{定理2.6.2. } P(s) = \int \sum_{h=s}^{-s} \vec{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} = \int \psi^+(\vec{r}, t; s) \frac{-i\nabla(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi(\vec{r}, t; s) d^3 \vec{r}$$

$$\text{推论2.6.2. } \begin{cases} P(n) = \int \psi^+(\vec{r}, t; n) \frac{(-i\nabla)(i\partial_t)^n}{(m^2 - \nabla^2)^n} \psi(\vec{r}, t; n) d^3 \vec{r} \\ P(n + \frac{1}{2}) = \int \psi^+(\vec{r}, t; n + \frac{1}{2}) \frac{-i\nabla}{(m^2 - \nabla^2)^n} \psi(\vec{r}, t; n + \frac{1}{2}) d^3 \vec{r} \end{cases}$$

## 2.7 Bargmann-Wigner方程的各种物理算符

定理2.7.1.

$$\begin{cases} P_u(s) = \int \psi^+(\vec{r}, t; s) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi(\vec{r}, t; s) d^3 \vec{r} = \int \sum_h p_u [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} \\ Q(s) = \int \psi^+(\vec{r}, t; s) \frac{(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi(\vec{r}, t; s) d^3 \vec{r} = \int \sum_h [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s-1} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} \\ N(s) = \int \psi^+(\vec{r}, t; s) \frac{(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi(\vec{r}, t; s) d^3 \vec{r} = \int \sum_h [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} \\ \vec{S}(s) = \int \psi^+(\vec{r}, t; s) \frac{\hat{\nabla}(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi(\vec{r}, t; s) d^3 \vec{r} = \int \sum_h \hat{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} \\ \vec{M}(s) = \int \psi^+(\vec{r}, t; s) \frac{\hat{\nabla}(i\partial_t)^{2s}}{(\sqrt{m^2 - \nabla^2})^{4s-1}} \psi(\vec{r}, t; s) d^3 \vec{r} = \int \sum_h \hat{p} [a^+(\vec{p}, h) a(\vec{p}, h) + (-1)^{2s-1} b(\vec{p}, h) b^+(\vec{p}, h)] d^3 \vec{p} \end{cases}$$

## 2.8 Bargmann-Wigner方程的量子方程

$$\text{定理2.8.1. } \begin{cases} (\gamma^a \otimes I_* \partial_a + m) \Gamma(s; 3) \psi(\vec{r}, t; s) = 0, * = 4^{2s-1} \\ P_u(s) = \int \psi^+(\vec{r}, t; s) \frac{-i\partial_u(i\partial_t)^{2s-1}}{(m^2 - \nabla^2)^{2s-1}} \psi(\vec{r}, t; s) d^3 \vec{r} \end{cases} \Rightarrow -i\partial_u \psi(\vec{r}, t; s) = [\psi(\vec{r}, t; s), P_u]$$

# 第四十三章 基本粒子内部分量作用

自我评述：在本章中我大胆猜测提出一种新型的相互作用：粒子内部相互作用，是否正确有待实践验证。

## 1 基本粒子内部分量假设

基本粒子内部分量假设：假设基本粒子存在内部分量，每个分量对应一个几何点。标量粒子存在一个内部分量，对应一个几何点；中微子<sup>[5]</sup>存在两个内部分量，对应两个几何点；光子<sup>[7,8]</sup>存在三个内部分量，对应三个几何点；引力微子<sup>[20]</sup>存在四个内部分量，对应四个几何点；引力子<sup>[11-14]</sup>存在五个内部分量，对应五个几何点；电子<sup>[4]</sup>存在四个内部分量，对应四个几何点。

内部分量作用本质上就是量子纠缠，作用形式与传统相互作用不同，不是吸引和排斥，而是相互内部分量关联，是一种新的相互作用。但相互作用的传递速度仍为光速，而不是瞬时传递的；几个夸克形成重子就是完全内部分量作用；平时的平面波叠加就是无内部分量作用。

## 2 粒子复合理论

### 2.1 两个中微子合成一个光子的物理机制

起初有2个独立的中微子 $\chi$ 和 $\varphi$ ，然后发生一种目前还不知道的新相互作用，即内部分量间的相互作用，该作用使第一个中微子 $\chi$ 的第2分量在任何参考系下恒等于第二个中微子 $\varphi$ 的第1分量，即 $\chi_2 \equiv \varphi_1$ ，也就是内部分量的两个几何点重合，发生这个作用后，将无法维持之前独立的协变性，因为在各自独立的洛仑兹变换到其它参考系时， $\chi_2 \equiv \varphi_1$ 不能成立。所以导致的结果是使2个粒子共同形成一个新的协变性，产生新的自旋，从而变为新的粒子：光子。过程如下：

$$\begin{cases} (\sigma, -i\zeta)^a \partial_a \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = 0, \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\sigma(\frac{1}{2})} \\ (\sigma, -i\zeta)^a \partial_a \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = 0, \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\sigma(\frac{1}{2})} \end{cases} \Leftrightarrow (\sigma \otimes I, -i\zeta)^a \partial_a \begin{bmatrix} \chi_1 \\ \chi_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} = 0, \begin{bmatrix} \chi_1 \\ \chi_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\sigma(\frac{1}{2})} \otimes I \quad (43.1)$$

两个内部分量几何点重合： $\chi_2 \equiv \varphi_1$ ，则有意义的方程变为

$$\rightarrow (\sigma \otimes I, -i\zeta)^a \partial_a \begin{bmatrix} \chi_1 \\ \chi_2 \equiv \varphi_1 \\ \varphi_1 \equiv \chi_2 \\ \varphi_2 \end{bmatrix} = 0, \begin{bmatrix} \chi_1 \\ \chi_2 \equiv \varphi_1 \\ \varphi_1 \equiv \chi_2 \\ \varphi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\sigma(\frac{1}{2})}_{\text{原}} \otimes e^{(i\omega+\zeta\epsilon)\cdot\sigma(\frac{1}{2})}_{\text{新}} \quad (43.2)$$

$$\Leftrightarrow [\partial_a + S_{ab}(1, \zeta)\partial^b] \begin{bmatrix} \chi_1 \\ \chi_2 \equiv \varphi_1 \\ \varphi_2 \end{bmatrix} = 0, \begin{bmatrix} \chi_1 \\ \chi_2 \equiv \varphi_1 \\ \varphi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\tau(1)} \quad (43.3)$$

### 2.2 2s个中微子合成一个自旋-s粒子的物理机制

起初有2s个独立的中微子 ${}^i\varphi, i = 1, 2, \dots, 2s$ ，然后发生一种目前还不知道的新相互作用，即内部分量间的相互作用，该作用使第i个中微子 ${}^i\varphi$ 的第2分量在任何参考系下恒等于第i+1个中微子 ${}^{i+1}\varphi$ 的第1分量，即 ${}^i\varphi_2 \equiv {}^{i+1}\varphi_1$ ，也就是内部分量的两个相邻几何点重合，发生这个作用后，将无法维持之前独立的协变性，因为在各自独立的洛仑兹变换到其它参考系时， ${}^i\varphi_2 \equiv {}^{i+1}\varphi_1$ 不能成立。所以导致的结果是使2s个粒子共同形成一个新

的协变性，产生新的自旋，从而变为新的粒子：自旋-s粒子。过程如下：

$$\left\{ \begin{array}{l} (\sigma, -i\zeta)^a \partial_a \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\sigma(\frac{1}{2})} \\ (\sigma, -i\zeta)^a \partial_a \begin{bmatrix} {}^2\varphi_1 \\ {}^2\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^2\varphi_1 \\ {}^2\varphi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\sigma(\frac{1}{2})} \\ \dots\dots\dots \\ (\sigma, -i\zeta)^a \partial_a \begin{bmatrix} {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\sigma(\frac{1}{2})} \end{array} \right. \quad (43.4)$$

$$\Leftrightarrow (\sigma \otimes I_{2s}, -i\zeta)^a \partial_a \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \\ {}^2\varphi_1 \\ {}^2\varphi_2 \\ \dots \\ {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \\ {}^2\varphi_1 \\ {}^2\varphi_2 \\ \dots \\ {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\sigma(\frac{1}{2})} \otimes I_{2s} \quad (43.5)$$

两个相邻粒子内部分量几何点重合： ${}^i\varphi_2 \equiv {}^{i+1}\varphi_1$ ，则有意义的方程变为：

$$\rightarrow (\sigma \otimes I_{2s}, -i\zeta)^a \partial_a \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \equiv {}^2\varphi_1 \\ {}^2\varphi_1 \equiv {}^1\varphi_2 \\ {}^2\varphi_2 \equiv {}^3\varphi_1 \\ \dots \\ {}^{2s-1}\varphi_2 \equiv {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \equiv {}^2\varphi_1 \\ {}^2\varphi_1 \equiv {}^1\varphi_2 \\ {}^2\varphi_2 \equiv {}^3\varphi_1 \\ \dots \\ {}^{2s-1}\varphi_2 \equiv {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\sigma(\frac{1}{2})_{原}} \otimes e^{(i\omega+\zeta\epsilon)\cdot\tau(s-\frac{1}{2})_{新}} \quad (43.6)$$

$$\Leftrightarrow [s\partial_a + S_{ab}(s, \zeta)\partial^b] \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \equiv {}^2\varphi_1 \\ {}^2\varphi_2 \equiv {}^3\varphi_1 \\ \dots \\ {}^{2s-2}\varphi_2 \equiv {}^{2s-1}\varphi_1 \\ {}^{2s-1}\varphi_2 \equiv {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} = 0, \begin{bmatrix} {}^1\varphi_1 \\ {}^1\varphi_2 \equiv {}^2\varphi_1 \\ {}^2\varphi_2 \equiv {}^3\varphi_1 \\ \dots \\ {}^{2s-2}\varphi_2 \equiv {}^{2s-1}\varphi_1 \\ {}^{2s-1}\varphi_2 \equiv {}^{2s}\varphi_1 \\ {}^{2s}\varphi_2 \end{bmatrix} \sim e^{(i\omega+\zeta\epsilon)\cdot\tau(s)} \quad (43.7)$$

以上的数学过程可以按多种方式进行正确理解，即若干个中微子先分成几组，每一组合成为一个新粒子，然后这些新粒子，再分组，每一组又合成为一个新粒子，这个过程可以循环往复，直到合成为单个自旋-s粒子为止，因此这个合成过程可以有好多组组合方式实现。所以两个中微子可以合成为一个光子；三个中微子可以合成为一个引力微子，一个中微子和一个光子可以合成为一个引力微子；四个中微子可以合成为一个引力子，两个中微子和一个光子可以合成为一个引力子，两个光子可以合成为一个引力子，一个中微子和一个引力微子可以合成为一个引力子等等。

### 3 常数张量与新相互作用

#### 3.1 新的相互作用

定义3.1.1.  $S_I = G \int dx^4 \psi_{k_\zeta}(s) \underbrace{\Gamma_{A_\zeta B_\zeta C_\zeta \dots}^{k_\zeta}}_{2s} \overbrace{\psi_1^{A_\zeta} \psi_2^{B_\zeta} \psi_3^{C_\zeta} \dots}^{2s} + \{\}^*$

### 3.2 引力子合成相互作用

$$\text{定义3.2.1. } S_{I1111} = G \int dx^4 \psi_{k_\zeta} (2) \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta}^{k_\zeta} (2) \psi_1^{A_\zeta} \psi_2^{B_\zeta} \psi_3^{C_\zeta} \psi_4^{D_\zeta} + \{\}^*$$

$$\text{定义3.2.2. } S_{I211} = G \int dx^4 \psi_{k_\zeta} (2) \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta}^{k_\zeta} (2) \psi_1^{A_\zeta B_\zeta} \psi_2^{C_\zeta} \psi_3^{D_\zeta} + \{\}^*$$

$$\text{定义3.2.3. } S_{I22} = G \int dx^4 \psi_{k_\zeta} (2) \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta}^{k_\zeta} (2) \psi_1^{A_\zeta B_\zeta} \psi_2^{C_\zeta D_\zeta} + \{\}^*$$

$$\text{定义3.2.4. } S_{I31} = G \int dx^4 \psi_{k_\zeta} (2) \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta}^{k_\zeta} (2) \psi_1^{A_\zeta B_\zeta C_\zeta} \psi_2^{D_\zeta} + \{\}^*$$

### 3.3 引力微子合成相互作用

$$\text{定义3.3.1. } S_{I111} = G \int dx^4 \psi_{k_\zeta} \left(\frac{3}{2}\right) \Gamma_{A_\zeta B_\zeta C_\zeta}^{k_\zeta} \left(\frac{3}{2}\right) \psi_1^{A_\zeta} \psi_2^{B_\zeta} \psi_3^{C_\zeta} + \{\}^*$$

$$\text{定义3.3.2. } S_{I21} = G \int dx^4 \psi_{k_\zeta} \left(\frac{3}{2}\right) \Gamma_{A_\zeta B_\zeta C_\zeta D_\zeta}^{k_\zeta} \left(\frac{3}{2}\right) \psi_1^{A_\zeta B_\zeta} \psi_2^{C_\zeta} + \{\}^*$$

### 3.4 光子合成相互作用

$$\text{定义3.4.1. } S_{I11} = G \int dx^4 \psi_{k_\zeta} (1) \Gamma_{A_\zeta B_\zeta}^{k_\zeta} (1) \psi_1^{A_\zeta} \psi_2^{B_\zeta} + \{\}^*$$

$$\text{定义3.4.2. } S_{I11} = G \int dx^4 \psi_{\alpha_\zeta} \sigma_{A_\zeta B_\zeta}^{\alpha_\zeta} \psi_1^{A_\zeta} \psi_2^{B_\zeta} + \{\}^*$$

$$\text{定义3.4.3. } S_{I11} = G \int dx^4 \psi_{\alpha_\zeta} \sigma_{k_\zeta l_\zeta}^{\alpha_\zeta} (s) \psi_1^{k_\zeta} (s) \psi_2^{l_\zeta} (s) + \{\}^*$$

### 3.5 新的类电磁相互作用

$$\text{定理3.5.1. } S = \int dx^4 \left\{ -\frac{\varepsilon}{4} F^{ab} F_{ab} - \nu_e^+ (\sigma, -i)^a \partial_a \nu_e - \nu_\mu^+ (\sigma, -i)^a \partial_a \nu_\mu - \nu_\tau^+ (\sigma, -i)^a \partial_a \nu_\tau \right\} \\ + \frac{1}{2} G \int dx^4 \left\{ F_{ab} \sigma_{+\alpha}^{ab} \sigma_{AB}^\alpha [(m_e - m_\mu) \nu_e^A \nu_\mu^B + (m_\mu - m_\tau) \nu_\mu^A \nu_\tau^B + (m_\tau - m_e) \nu_\tau^A \nu_e^B] \right\}$$

$$\text{定理3.5.2. } S_I = \frac{1}{2} G \int dx^4 \left\{ F_{ab} \sigma_{+\alpha}^{ab} \sigma_{AB}^\alpha [\alpha_{e\mu} \nu_e^A \nu_\mu^B + \alpha_{\mu\tau} \nu_\mu^A \nu_\tau^B + \alpha_{\tau e} \nu_\tau^A \nu_e^B] \right\}$$

$$\text{定理3.5.3. } S_I = \frac{1}{2} G \int dx^4 \left\{ F_{ab} \sigma_{+\alpha}^{ab} \sigma_{AB}^\alpha (\nu_e^A \nu_\mu^B + \nu_\mu^A \nu_\tau^B + \nu_\tau^A \nu_e^B) \right\}$$

定理3.5.4.

$$\nu_e \rightarrow \gamma + \bar{\nu}_\mu \rightarrow \nu_\tau : \alpha_{e\mu} \alpha_{\mu\tau}$$

$$\nu_e \rightarrow \gamma + \bar{\nu}_\tau \rightarrow \nu_\mu : \alpha_{\mu\tau} \alpha_{\tau e}$$

$$\nu_e \rightarrow \gamma + \bar{\nu}_\mu \rightarrow \nu_e : \alpha_{e\mu} \alpha_{e\mu}$$

$$\nu_e \rightarrow \gamma + \bar{\nu}_\tau \rightarrow \nu_e : \alpha_{\tau e} \alpha_{\tau e}$$

$$\nu_\mu \rightarrow \gamma + \bar{\nu}_\tau \rightarrow \nu_e$$

$$\Psi = \vec{E} + i\vec{B} = \vec{E} + i\nabla \times \vec{A}$$

$$\Psi_i = E_i + i\varepsilon_i^{jk} \partial_j A_k$$

$$[\Psi_i(x), \Psi_j(x')] = i\varepsilon_i^{kl} \partial_{x_k} [A_l(x), E_j(x')] + i\varepsilon_j^{kl} \partial_{x'_k} [E_i(x), A_l(x')]$$

$$[\Psi_i(x), \Psi_j(x')] = -\varepsilon_{ij}^k (\partial_{x_k} + \partial_{x'_k}) \delta^3(x - x') = 0$$

定理3.5.5.

$$\Psi = \vec{E} + i\vec{B} = \vec{E} + i\nabla \times \vec{A}$$

$$\Psi_i = E_i + i\varepsilon_i^{jk} \partial_j A_k$$

$$[\Psi_i(x), \Psi_j^+(x')] = i\varepsilon_i^{kl} \partial_{x_k} [A_l(x), E_j(x')] - i\varepsilon_j^{kl} \partial_{x'_k} [E_i(x), A_l(x')]$$

$$[\Psi_i(x), \Psi_j^+(x')] = -\varepsilon_{ij}^k (\partial_{x_k} - \partial_{x'_k}) \delta^3(x - x') = -2\varepsilon_{ij}^k \partial_{(x_k - x'_k)} \delta^3(x - x')$$

## 4 粒子内部分量作用

### 4.1 光子对内部分量作用

$$\text{猜想4.1.1. } \varepsilon_{ab}(\vec{p}_1, \vec{p}_2; 0; 1) = \frac{1}{\sqrt{2}} [\varepsilon_a(\vec{p}_1, 1) \varepsilon_b(\vec{p}_2, -1) - \varepsilon_a(\vec{p}_1, -1) \varepsilon_b(\vec{p}_2, 1)]$$

$$\text{猜想4.1.2. } \varepsilon_{ab}(\vec{p}, 0; 1) = \frac{1}{\sqrt{2}} [\varepsilon_a(\vec{p}, 1) \varepsilon_b(-\vec{p}, 1) - \varepsilon_a(-\vec{p}, 1) \varepsilon_b(\vec{p}, 1)]$$

# 第四十四章 自旋纠缠态分析初步

自我评述：在本章中我讨论了CG系数、自旋耦合和量子纠缠之间的本质联系，并基于表象变换技术对自旋纠缠态提供了一种美妙的数学描述。

## 1 电子方程的单模平面波解

### 1.1 双电子合成： $\frac{1}{2} \oplus \frac{1}{2} = (1, 1)$

$$\text{推论1.1.1.1.} \quad \begin{cases} (\gamma^a \partial_{1a} + m)_{\kappa_\zeta} \lambda_\zeta [U_{\lambda_\zeta}(\vec{p}_1, \frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, \frac{1}{2}) e^{ip_2 x_2}] = 0 \\ (\gamma^a \partial_{2a} + m)_{\kappa_\zeta} \mu_\zeta [U_{\lambda_\zeta}(\vec{p}_1, \frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, \frac{1}{2}) e^{ip_2 x_2}] = 0 \end{cases}$$

$$\text{推论1.1.2.} \quad \begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_2 x}] = 0 \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \mu_\zeta [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_2 x}] = 0 \end{cases}$$

$$\text{推论1.1.3.} \quad \begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta [U_{\lambda_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] = 0 \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \mu_\zeta [U_{\lambda_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] = 0 \end{cases}$$

$$\text{推论1.1.4.} \quad (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta [U_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) U_{\mu_\zeta}(\vec{p}, \frac{1}{2}) e^{ipx}] = 0$$

### 1.2 双电子合成： $\frac{1}{2} \oplus \frac{1}{2} = (1, -1)$

$$\text{推论1.2.1.} \quad \begin{cases} (\gamma^a \partial_{1a} + m)_{\kappa_\zeta} \lambda_\zeta [U_{\lambda_\zeta}(\vec{p}_1, -\frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, -\frac{1}{2}) e^{ip_2 x_2}] = 0 \\ (\gamma^a \partial_{2a} + m)_{\kappa_\zeta} \mu_\zeta [U_{\lambda_\zeta}(\vec{p}_1, -\frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, -\frac{1}{2}) e^{ip_2 x_2}] = 0 \end{cases}$$

$$\text{推论1.2.2.} \quad \begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_2 x}] = 0 \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \mu_\zeta [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_2 x}] = 0 \end{cases}$$

$$\text{推论1.2.3.} \quad \begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta [U_{\lambda_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] = 0 \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \mu_\zeta [U_{\lambda_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] = 0 \end{cases}$$

$$\text{推论1.2.4.} \quad (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta [U_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) U_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) e^{ipx}] = 0$$

### 1.3 双电子合成： $\frac{1}{2} \oplus \frac{1}{2} = (1, 0)$

推论1.3.1.

$$\begin{cases} (\gamma^a \partial_{1a} + m)_{\kappa_\zeta} \lambda_\zeta \{ [U_{\lambda_\zeta}(\vec{p}_1, \frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, -\frac{1}{2}) e^{ip_2 x_2}] + [U_{\lambda_\zeta}(\vec{p}_1, -\frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, \frac{1}{2}) e^{ip_2 x_2}] \} = 0 \\ (\gamma^a \partial_{2a} + m)_{\kappa_\zeta} \mu_\zeta \{ [U_{\lambda_\zeta}(\vec{p}_1, \frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, -\frac{1}{2}) e^{ip_2 x_2}] + [U_{\lambda_\zeta}(\vec{p}_1, -\frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, \frac{1}{2}) e^{ip_2 x_2}] \} = 0 \end{cases}$$

推论1.3.2.

$$\begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \{ [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_2 x}] + [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_2 x}] \} = 0 \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \mu_\zeta \{ [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_2 x}] + [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_2 x}] \} = 0 \end{cases}$$

推论1.3.3.

$$\begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \{ [U_{\lambda_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] + [U_{\lambda_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] \} = 0 \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \mu_\zeta \{ [U_{\lambda_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] + [U_{\lambda_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] \} = 0 \end{cases}$$

$$\text{推论1.3.4.} \quad (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \{ [U_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) U_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) e^{ipx}] + [U_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) U_{\mu_\zeta}(\vec{p}, \frac{1}{2}) e^{ipx}] \} = 0$$

## 1.4 双电子合成: $\frac{1}{2} \oplus \frac{1}{2} = (0, 0)$

推论1.4.1.

$$\begin{cases} (\gamma^a \partial_{1a} + m)_{\kappa_\zeta} \lambda_\zeta \{ [U_{\lambda_\zeta}(\vec{p}_1, \frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, -\frac{1}{2}) e^{ip_2 x_2}] - [U_{\lambda_\zeta}(\vec{p}_1, -\frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, \frac{1}{2}) e^{ip_2 x_2}] \} = 0 \\ (\gamma^a \partial_{2a} + m)_{\kappa_\zeta} \mu_\zeta \{ [U_{\lambda_\zeta}(\vec{p}_1, \frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, -\frac{1}{2}) e^{ip_2 x_2}] - [U_{\lambda_\zeta}(\vec{p}_1, -\frac{1}{2}) e^{ip_1 x_1}] [U_{\mu_\zeta}(\vec{p}_2, \frac{1}{2}) e^{ip_2 x_2}] \} = 0 \end{cases}$$

推论1.4.2.

$$\begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \{ [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_2 x}] - [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_2 x}] \} = 0 \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \mu_\zeta \{ [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_2 x}] - [U_{\lambda_\zeta}(\vec{p}_1 + \vec{p}_2, -\frac{1}{2}) e^{ip_1 x}] [U_{\mu_\zeta}(\vec{p}_1 + \vec{p}_2, \frac{1}{2}) e^{ip_2 x}] \} = 0 \end{cases}$$

推论1.4.3.

$$\begin{cases} (\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \{ [U_{\lambda_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] - [U_{\lambda_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] \} = 0 \\ (\gamma^a \partial_a + m)_{\kappa_\zeta} \mu_\zeta \{ [U_{\lambda_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] - [U_{\lambda_\zeta}(2\vec{p}, -\frac{1}{2}) e^{ipx}] [U_{\mu_\zeta}(2\vec{p}, \frac{1}{2}) e^{ipx}] \} = 0 \end{cases}$$

推论1.4.4.  $(\gamma^a \partial_a + m)_{\kappa_\zeta} \lambda_\zeta \{ [U_{\lambda_\zeta}(\vec{p}, \frac{1}{2}) U_{\mu_\zeta}(\vec{p}, -\frac{1}{2}) e^{ipx}] - [U_{\lambda_\zeta}(\vec{p}, -\frac{1}{2}) U_{\mu_\zeta}(\vec{p}, \frac{1}{2}) e^{ipx}] \} = 0$

## 2 光子方程的单模平面波解

### 2.1 双光子合成: $1 \oplus 1 = (2, 2)$

定理2.1.1.  $[\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \alpha_\zeta \psi_{\alpha_\zeta \beta_\zeta} = 0, S_{ab}(\gamma, \varsigma) = i\sigma_{\zeta ab}^{\alpha_\zeta} \gamma_{\alpha_\zeta}, \lambda_{\alpha_\zeta}(\hat{p}, h)$

推论2.1.1.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \alpha_\zeta [\lambda_{\alpha_\zeta}(\hat{p}_1, 1) e^{ip_1 x_1}] [\lambda_{\beta_\zeta}(\hat{p}_2, 1) e^{ip_2 x_2}] = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \beta_\zeta [\lambda_{\alpha_\zeta}(\hat{p}_1, 1) e^{ip_1 x_1}] [\lambda_{\beta_\zeta}(\hat{p}_2, 1) e^{ip_2 x_2}] = 0 \end{cases}$$

推论2.1.2.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \alpha_\zeta [\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, 1) e^{ip_1 x}] [\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, 1) e^{ip_2 x}] = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \beta_\zeta [\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, 1) e^{ip_1 x}] [\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, 1) e^{ip_2 x}] = 0 \end{cases}$$

推论2.1.3.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \alpha_\zeta [\lambda_{\alpha_\zeta}(2\vec{p}, 1) e^{ipx}] [\lambda_{\beta_\zeta}(2\vec{p}, 1) e^{ipx}] = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \beta_\zeta [\lambda_{\alpha_\zeta}(2\vec{p}, 1) e^{ipx}] [\lambda_{\beta_\zeta}(2\vec{p}, 1) e^{ipx}] = 0 \end{cases}$$

推论2.1.4.  $[\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \alpha_\zeta [\lambda_{\alpha_\zeta}(\hat{p}, 1) \lambda_{\beta_\zeta}(\hat{p}, 1) e^{ipx}] = 0$

### 2.2 双光子合成: $1 \oplus 1 = (2, -2)$

推论2.2.1.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \alpha_\zeta [\lambda_{\alpha_\zeta}(\hat{p}_1, -1) e^{ip_1 x_1}] [\lambda_{\beta_\zeta}(\hat{p}_2, -1) e^{ip_2 x_2}] = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \beta_\zeta [\lambda_{\alpha_\zeta}(\hat{p}_1, -1) e^{ip_1 x_1}] [\lambda_{\beta_\zeta}(\hat{p}_2, -1) e^{ip_2 x_2}] = 0 \end{cases}$$

推论2.2.2.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \alpha_\zeta [\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, -1) e^{ip_1 x}] [\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, -1) e^{ip_2 x}] = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \beta_\zeta [\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, -1) e^{ip_1 x}] [\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, -1) e^{ip_2 x}] = 0 \end{cases}$$

推论2.2.3.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \alpha_\zeta [\lambda_{\alpha_\zeta}(2\vec{p}, -1) e^{ipx}] [\lambda_{\beta_\zeta}(2\vec{p}, -1) e^{ipx}] = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \beta_\zeta [\lambda_{\alpha_\zeta}(2\vec{p}, -1) e^{ipx}] [\lambda_{\beta_\zeta}(2\vec{p}, -1) e^{ipx}] = 0 \end{cases}$$

推论2.2.4.  $[\partial_a + iS_{ab}(\gamma, \varsigma) \partial^b]_{\kappa_\zeta} \alpha_\zeta [\lambda_{\alpha_\zeta}(\hat{p}, -1) \lambda_{\beta_\zeta}(\hat{p}, -1) e^{ipx}] = 0$

### 2.3 双光子合成: $1 \oplus 1 = (2, 0)$

推论2.3.1.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\alpha_\zeta} \{[\lambda_{\alpha_\zeta}(\hat{p}_1, 1)e^{ip_1x_1}][\lambda_{\beta_\zeta}(\hat{p}_2, -1)e^{ip_2x_2}] + [\lambda_{\alpha_\zeta}(\hat{p}_1, -1)e^{ip_1x_1}][\lambda_{\beta_\zeta}(\hat{p}_2, 1)e^{ip_2x_2}]\} = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\beta_\zeta} \{[\lambda_{\alpha_\zeta}(\hat{p}_1, 1)e^{ip_1x_1}][\lambda_{\beta_\zeta}(\hat{p}_2, -1)e^{ip_2x_2}] + [\lambda_{\alpha_\zeta}(\hat{p}_1, -1)e^{ip_1x_1}][\lambda_{\beta_\zeta}(\hat{p}_2, 1)e^{ip_2x_2}]\} = 0 \end{cases}$$

推论2.3.2.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\alpha_\zeta} \{[\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_2x}] + [\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_1x}][\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}]\} = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\beta_\zeta} \{[\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_2x}] + [\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_1x}][\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}]\} = 0 \end{cases}$$

推论2.3.3.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\alpha_\zeta} \{[\lambda_{\alpha_\zeta}(2\vec{p}, 1)e^{ipx}][\lambda_{\beta_\zeta}(2\vec{p}, -1)e^{ipx}] + [\lambda_{\alpha_\zeta}(2\vec{p}, -1)e^{ipx}][\lambda_{\beta_\zeta}(2\vec{p}, 1)e^{ipx}]\} = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\beta_\zeta} \{[\lambda_{\alpha_\zeta}(2\vec{p}, 1)e^{ipx}][\lambda_{\beta_\zeta}(2\vec{p}, -1)e^{ipx}] + [\lambda_{\alpha_\zeta}(2\vec{p}, -1)e^{ipx}][\lambda_{\beta_\zeta}(2\vec{p}, 1)e^{ipx}]\} = 0 \end{cases}$$

推论2.3.4.  $[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\alpha_\zeta} \{[\lambda_{\alpha_\zeta}(\hat{p}, 1)\lambda_{\beta_\zeta}(\hat{p}, -1)e^{ipx}] + [\lambda_{\alpha_\zeta}(\hat{p}, -1)\lambda_{\beta_\zeta}(\hat{p}, 1)e^{ipx}]\} = 0$

### 2.4 双光子合成: $1 \oplus 1 = (0, 0)$

推论2.4.1.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\alpha_\zeta} \{[\lambda_{\alpha_\zeta}(\hat{p}_1, 1)e^{ip_1x_1}][\lambda_{\beta_\zeta}(\hat{p}_2, -1)e^{ip_2x_2}] - [\lambda_{\alpha_\zeta}(\hat{p}_1, -1)e^{ip_1x_1}][\lambda_{\beta_\zeta}(\hat{p}_2, 1)e^{ip_2x_2}]\} = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\beta_\zeta} \{[\lambda_{\alpha_\zeta}(\hat{p}_1, 1)e^{ip_1x_1}][\lambda_{\beta_\zeta}(\hat{p}_2, -1)e^{ip_2x_2}] - [\lambda_{\alpha_\zeta}(\hat{p}_1, -1)e^{ip_1x_1}][\lambda_{\beta_\zeta}(\hat{p}_2, 1)e^{ip_2x_2}]\} = 0 \end{cases}$$

推论2.4.2.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\alpha_\zeta} \{[\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_2x}] - [\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_1x}][\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}]\} = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\beta_\zeta} \{[\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_2x}] - [\lambda_{\alpha_\zeta}(\vec{p}_1 + \vec{p}_2, -1)e^{ip_1x}][\lambda_{\beta_\zeta}(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}]\} = 0 \end{cases}$$

推论2.4.3.

$$\begin{cases} [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\alpha_\zeta} \{[\lambda_{\alpha_\zeta}(2\vec{p}, 1)e^{ipx}][\lambda_{\beta_\zeta}(2\vec{p}, -1)e^{ipx}] - [\lambda_{\alpha_\zeta}(2\vec{p}, -1)e^{ipx}][\lambda_{\beta_\zeta}(2\vec{p}, 1)e^{ipx}]\} = 0 \\ [\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\beta_\zeta} \{[\lambda_{\alpha_\zeta}(2\vec{p}, 1)e^{ipx}][\lambda_{\beta_\zeta}(2\vec{p}, -1)e^{ipx}] - [\lambda_{\alpha_\zeta}(2\vec{p}, -1)e^{ipx}][\lambda_{\beta_\zeta}(2\vec{p}, 1)e^{ipx}]\} = 0 \end{cases}$$

推论2.4.4.  $[\partial_a + iS_{ab}(\gamma, \varsigma)\partial^b]_{\kappa_\zeta}^{\alpha_\zeta} \{[\lambda_{\alpha_\zeta}(\hat{p}, 1)\lambda_{\beta_\zeta}(\hat{p}, -1)e^{ipx}] - [\lambda_{\alpha_\zeta}(\hat{p}, -1)\lambda_{\beta_\zeta}(\hat{p}, 1)e^{ipx}]\} = 0$

## 3 矢量场方程的单模平面波解

### 3.1 双光子合成: $1 \oplus 1 = (2, 2)$

推论3.1.1.

$$\begin{cases} (-\partial^{1c}\partial_{1c} + m^2)[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}] = 0, \partial^{1a}[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}] = 0 \\ (-\partial^{2c}\partial_{2c} + m^2)[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}] = 0, \partial^{2b}[\varepsilon_a(\vec{p}_1, 1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, 1)e^{ip_2x_2}] = 0 \end{cases}$$

推论3.1.2.

$$\begin{cases} (-\partial^c\partial_c + m^2)[\varepsilon_a(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}] = 0, \partial^a[\varepsilon_a(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}] = 0 \\ (-\partial^c\partial_c + m^2)[\varepsilon_a(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}] = 0, \partial^b[\varepsilon_a(\vec{p}_1 + \vec{p}_2, 1)e^{ip_1x}][\varepsilon_b(\vec{p}_1 + \vec{p}_2, 1)e^{ip_2x}] = 0 \end{cases}$$

推论3.1.3.

$$\begin{cases} (-\partial^c\partial_c + m^2)[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}] = 0, \partial^a[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}] = 0 \\ (-\partial^c\partial_c + m^2)[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}] = 0, \partial^b[\varepsilon_a(2\vec{p}, 1)e^{ipx}][\varepsilon_b(2\vec{p}, 1)e^{ipx}] = 0 \end{cases}$$

推论3.1.4.  $(-\partial^c\partial_c + m^2)[\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1)e^{ipx}] = 0, \partial^a[\varepsilon_a(\vec{p}, 1)\varepsilon_b(\vec{p}, 1)e^{ipx}] = 0$

### 3.2 双光子合成: $1 \oplus 1 = (2, -2)$

推论3.2.1.

$$\begin{cases} (-\partial^{1c}\partial_{1c} + m^2)[\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] = 0, \partial^{1a}[\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] = 0 \\ (-\partial^{2c}\partial_{2c} + m^2)[\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] = 0, \partial^{2b}[\varepsilon_a(\vec{p}_1, -1)e^{ip_1x_1}][\varepsilon_b(\vec{p}_2, -1)e^{ip_2x_2}] = 0 \end{cases}$$





推论3.4.3.

$$\begin{cases} (-\partial^c \partial_c + m^2) \{ [\varepsilon_a(2\vec{p}, 1)e^{ipx}] [\varepsilon_b(2\vec{p}, -1)e^{ipx}] - [\varepsilon_a(2\vec{p}, -1)e^{ipx}] [\varepsilon_b(2\vec{p}, 1)e^{ipx}] \} = 0 \\ \partial^a \{ [\varepsilon_a(2\vec{p}, 1)e^{ipx}] [\varepsilon_b(2\vec{p}, -1)e^{ipx}] - [\varepsilon_a(2\vec{p}, -1)e^{ipx}] [\varepsilon_b(2\vec{p}, 1)e^{ipx}] \} = 0 \\ (-\partial^c \partial_c + m^2) \{ [\varepsilon_a(2\vec{p}, 1)e^{ipx}] [\varepsilon_b(2\vec{p}, -1)e^{ipx}] - [\varepsilon_a(2\vec{p}, -1)e^{ipx}] [\varepsilon_b(2\vec{p}, 1)e^{ipx}] \} = 0 \\ \partial^b \{ [\varepsilon_a(2\vec{p}, 1)e^{ipx}] [\varepsilon_b(2\vec{p}, -1)e^{ipx}] - [\varepsilon_a(2\vec{p}, -1)e^{ipx}] [\varepsilon_b(2\vec{p}, 1)e^{ipx}] \} = 0 \end{cases}$$

推论3.4.4.

$$\begin{cases} (-\partial^c \partial_c + m^2) \{ [\varepsilon_a(\vec{p}, 1)e^{ipx}] [\varepsilon_b(\vec{p}, -1)e^{ipx}] - [\varepsilon_a(\vec{p}, -1)e^{ipx}] [\varepsilon_b(\vec{p}, 1)e^{ipx}] \} = 0 \\ \partial^a \{ [\varepsilon_a(\vec{p}, 1)e^{ipx}] [\varepsilon_b(\vec{p}, -1)e^{ipx}] - [\varepsilon_a(\vec{p}, -1)e^{ipx}] [\varepsilon_b(\vec{p}, 1)e^{ipx}] \} = 0 \end{cases}$$

## 4 粒子自旋方程的单模平面波解

### 4.1 s-自旋方程及其单模平面波解

定理4.1.1.  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x) = 0$

$$\text{推论4.1.1. } \begin{cases} \psi(\vec{r}, t) := \frac{1}{(2\pi)^{3/2}} \int_{\vec{p} \neq 0} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma)e^{ip \cdot x} + a_2^+(\vec{p}, -s\varsigma)e^{-ip \cdot x}] d^3\vec{p} \\ |\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -s\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -s\varsigma) \psi(\vec{r}, t) e^{-ip \cdot x} d^3\vec{r} = \frac{1}{(2\pi)^{3/2}} \int \frac{i}{|\vec{p}|} \lambda^+(\hat{p}, -s\varsigma) \dot{\psi}(\vec{r}, t) e^{-ip \cdot x} d^3\vec{r} \\ |\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}, -s\varsigma) = \frac{1}{(2\pi)^{3/2}} \int \lambda^+(\hat{p}, -s\varsigma) \psi(\vec{r}, t) e^{ip \cdot x} d^3\vec{r} = \frac{1}{(2\pi)^{3/2}} \int \frac{-i}{|\vec{p}|} \lambda^+(\hat{p}, -s\varsigma) \dot{\psi}(\vec{r}, t) e^{ip \cdot x} d^3\vec{r} \end{cases}$$

$$\text{推论4.1.2. } \begin{cases} \psi_+(\vec{p}, -s\varsigma; x) = \frac{1}{(2\pi)^{3/2}} |\vec{p}|^{(s-\frac{1}{2})} a_1(\vec{p}, -s\varsigma) \lambda(\hat{p}, -s\varsigma) e^{ip \cdot x} = A_+(\vec{p}, -s\varsigma) \lambda(\hat{p}, -s\varsigma) e^{ip \cdot x} \\ \psi_-(\vec{p}, -s\varsigma; x) = \frac{1}{(2\pi)^{3/2}} |\vec{p}|^{(s-\frac{1}{2})} a_2^+(\vec{p}, -s\varsigma) \lambda(\hat{p}, -s\varsigma) e^{-ip \cdot x} = A_-(\vec{p}, -s\varsigma) \lambda(\hat{p}, -s\varsigma) e^{-ip \cdot x} \end{cases}$$

推论4.1.3.  $[s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \Leftrightarrow [sp_a + iS_{ab}(s, \varsigma)p^b]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0$

### 4.2 单模自旋方程的多种等价形式

定理4.2.1.

$$[s\hat{p}_a + iS_{ab}(s, \varsigma)\hat{p}^b]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \quad \Leftrightarrow \quad \sigma(s) \cdot \hat{p}\psi_{\pm}(\vec{p}, -s\varsigma; x) = \mp s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x), \quad O(s) \cdot \hat{p}\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0$$

$$[s\hat{p} - i\sigma(s) \times \hat{p} \pm \varsigma\sigma(s)]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \quad \Leftrightarrow \quad \begin{cases} \sigma(s) \cdot \hat{p}\psi_{\pm}(\vec{p}, -s\varsigma; x) = \mp s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x) \\ \{s^2\hat{p} - i\sigma(s) \times \hat{p} - \sigma(s)[\sigma(s) \cdot \hat{p}]\}\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \end{cases}$$

$$\{s\hat{p} - [\sigma(s) \cdot \hat{p}, \sigma(s)] \pm \varsigma\sigma(s)\}\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \Leftrightarrow \begin{cases} \sigma(s) \cdot \hat{p}\psi_{\pm}(\vec{p}, -s\varsigma; x) = \mp s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x) \\ \{s^2\hat{p} + (s-1)\sigma(s)[\sigma(s) \cdot \hat{p}] - s[\sigma(s) \cdot \hat{p}]\sigma(s)\}\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \end{cases}$$

$$\text{推论4.2.1. } \begin{cases} \sigma(1) \cdot \hat{p}\psi_{\pm}(\vec{p}, -\varsigma; x) = \mp \varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ [\sigma(1) \cdot \hat{p}]\sigma(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \hat{p}\psi_{\pm}(\vec{p}, -\varsigma; x) \end{cases}$$

### 4.3 自旋方程的沿z轴运动的单模平面波解

自旋方程的沿z轴正向运动的单模平面波解 $\hat{p}_a = (0, 0, 1, 1)$ :

定理4.3.1.

$$[s\hat{p}_a + iS_{ab}(s, \varsigma)\hat{p}^b]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \quad \Leftrightarrow \quad \sigma_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \mp s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x), \quad O_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0$$

$$[s\hat{p} - i\sigma(s) \times \hat{p} \pm \varsigma\sigma(s)]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \quad \Leftrightarrow \quad \begin{cases} \sigma_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \mp s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x) \\ \{s^2\hat{p} - i\sigma(s) \times \hat{p} - \sigma(s)\sigma_z(s)\}\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \end{cases}$$

$$\begin{aligned} & \{\hat{s}\hat{p} - [\sigma_z(s), \sigma(s)] \pm \varsigma\sigma(s)\}\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \Leftrightarrow \begin{cases} \sigma_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \mp s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x) \\ \{s^2\hat{p} + (s-1)\sigma(s)\sigma_z(s) - s\sigma_z(s)\sigma(s)\}\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \end{cases} \end{aligned}$$

$$\text{推论4.3.1. } [s\hat{p}_a + iS_{ab}(s, \varsigma)\hat{p}^b]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \Leftrightarrow \begin{cases} \sigma_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \mp s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x) \\ [\sigma_x(s) \mp i\varsigma\sigma_y(s)]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \end{cases}$$

$$\text{推论4.3.2. } \begin{cases} \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \mp \varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ \sigma_z(1)\sigma(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \hat{p}\psi_{\pm}(\vec{p}, -\varsigma; x) \end{cases}$$

自旋方程的沿z轴负向运动的平面波解 $\hat{p}_a = (0, 0, -1, 1)$ :

定理4.3.2.

$$[s\hat{p}_a + iS_{ab}(s, \varsigma)\hat{p}^b]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \Leftrightarrow \sigma_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \pm s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x), O_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0$$

$$[\hat{p} - i\sigma(s) \times \hat{p} \pm \varsigma\sigma(s)]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \Leftrightarrow \begin{cases} \sigma_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \pm s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x) \\ \{s^2\hat{p} - is\sigma(s) \times \hat{p} + \sigma(s)\sigma_z(s)\}\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \end{cases}$$

$$\{\hat{p} + [\sigma_z(s), \sigma(s)] \pm \varsigma\sigma(s)\}\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \Leftrightarrow \begin{cases} \sigma_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \pm s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x) \\ \{s^2\hat{p} - (s-1)\sigma(s)\sigma_z(s) + s\sigma_z(s)\sigma(s)\}\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \end{cases}$$

$$\text{推论4.3.3. } [s\hat{p}_a + iS_{ab}(s, \varsigma)\hat{p}^b]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \Leftrightarrow \begin{cases} \sigma_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \pm s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x) \\ [\sigma_x(s) \pm i\varsigma\sigma_y(s)]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \end{cases}$$

$$\text{推论4.3.4. } \begin{cases} \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \pm \varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ \sigma_z(1)\sigma(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = -\hat{p}\psi_{\pm}(\vec{p}, -\varsigma; x) \end{cases}$$

#### 4.4 1-自旋方程的沿z轴运动的单模平面波解

自旋方程的沿z轴正向运动的单模平面波解 $\hat{p}_a = (0, 0, 1, 1)$ :

定理4.4.1.

$$[\hat{p}_a + iS_{ab}(1, \varsigma)\hat{p}^b]\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \Leftrightarrow \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \mp \varsigma\psi_{\pm}(\vec{p}, -\varsigma; x), O_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = 0$$

$$[\hat{p} - i\sigma(1) \times \hat{p} \pm \varsigma\sigma(1)]\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \Leftrightarrow \begin{cases} \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \mp \varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ \{\hat{p} - i\sigma(1) \times \hat{p} - \sigma(1)\sigma_z(1)\}\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \end{cases}$$

$$\{\hat{p} - [\sigma_z(1), \sigma(1)] \pm \varsigma\sigma(1)\}\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \Leftrightarrow \begin{cases} \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \mp \varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ \{\hat{p} - \sigma_z(1)\sigma(1)\}\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \end{cases}$$

$$\text{推论4.4.1. } [\hat{p}_a + iS_{ab}(1, \varsigma)\hat{p}^b]\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \Leftrightarrow \begin{cases} \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \mp \varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ [\sigma_x(1) \mp i\varsigma\sigma_y(1)]\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \end{cases}$$

$$\text{推论4.4.2. } \begin{cases} \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \mp \varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ \sigma_z(1)\sigma(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \hat{p}\psi_{\pm}(\vec{p}, -\varsigma; x) \end{cases}$$

自旋方程的沿z轴负向运动的平面波解 $\hat{p}_a = (0, 0, -1, 1)$ :

定理4.4.2.

$$[\hat{p}_a + iS_{ab}(1, \varsigma)\hat{p}^b]\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \quad [\Leftrightarrow] \quad \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \pm\varsigma\psi_{\pm}(\vec{p}, -\varsigma; x), O_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = 0$$

$$[\hat{p} - i\sigma(1) \times \hat{p} \pm \varsigma\sigma(1)]\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \quad [\Leftrightarrow] \quad \begin{cases} \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \pm\varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ \{\hat{p} - i\sigma(1) \times \hat{p} + \sigma(1)\sigma_z(1)\}\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \end{cases}$$

$$\{\hat{p} + [\sigma_z(1), \sigma(1)] \pm \varsigma\sigma(1)\}\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \quad [\Leftrightarrow] \quad \begin{cases} \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \pm\varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ \{s^2\hat{p} + \sigma_z(1)\sigma(1)\}\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \end{cases}$$

$$\text{推论4.4.3. } [\hat{p}_a + iS_{ab}(1, \varsigma)\hat{p}^b]\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \quad [\Leftrightarrow] \quad \begin{cases} \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \pm\varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ [\sigma_x(1) \pm i\varsigma\sigma_y(1)]\psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \end{cases}$$

$$\text{推论4.4.4. } \begin{cases} \sigma_z(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = \pm\varsigma\psi_{\pm}(\vec{p}, -\varsigma; x) \\ \sigma_z(1)\sigma(1)\psi_{\pm}(\vec{p}, -\varsigma; x) = -\hat{p}\psi_{\pm}(\vec{p}, -\varsigma; x) \end{cases}$$

#### 4.5 单模平面波解的自旋矢量

$$\text{推论4.5.1. } \begin{cases} [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \\ \hat{P}_a := -i\partial_a, \hat{W}_a(s, \varsigma) := \varsigma S_{ab}(s, \varsigma)\partial^b \end{cases} \Rightarrow \begin{cases} \hat{P}_a\psi_{\pm}(\vec{p}, -s\varsigma; x) = \pm p_a\psi_{\pm}(\vec{p}, -s\varsigma; x) \\ \hat{W}_a(s, \varsigma)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \mp s\varsigma p_a\psi_{\pm}(\vec{p}, -s\varsigma; x) \end{cases}$$

$$\text{推论4.5.2. } W_a(s, \varsigma)\psi_{\pm}(\vec{p}, -s\varsigma; x) = -s\varsigma\vec{p}_a\psi_{\pm}(\vec{p}, -s\varsigma; x), W_a(s, \varsigma) = -i * S_{ab}(s, \varsigma)p^b = i\varsigma S_{ab}(s, \varsigma)p^b$$

#### 4.6 正z轴单模平面波解的完整量子态

$$\text{推论4.6.1. } \begin{cases} \sigma^2(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = s(s+1)\psi_{\pm}(\vec{p}, -s\varsigma; x), \sigma(s) \times \sigma(s) = i\sigma(s), \sigma^2(s) = s(s+1) \\ \sigma_z(s)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \mp s\varsigma\psi_{\pm}(\vec{p}, -s\varsigma; x), [\sigma_x(s) \mp i\varsigma\sigma_y(s)]\psi_{\pm}(\vec{p}, -s\varsigma; x) = 0 \\ \hat{P}_a\psi_{\pm}(\vec{p}, -s\varsigma; x) = \pm p_a\psi_{\pm}(\vec{p}, -s\varsigma; x), \hat{W}_a(s, \varsigma)\psi_{\pm}(\vec{p}, -s\varsigma; x) = \mp s\varsigma p_a\psi_{\pm}(\vec{p}, -s\varsigma; x) \end{cases}$$

#### 4.7 正z轴单模光子的完整量子态

$$\text{推论4.7.1. } \begin{cases} \gamma^2\Psi_{\pm}(\vec{p}, -\varsigma; x) = 2\Psi_{\pm}(\vec{p}, -\varsigma; x), \gamma \times \gamma = i\gamma, \gamma^2 = 2 \\ \gamma_z\Psi_{\pm}(\vec{p}, -\varsigma; x) = \mp\varsigma\Psi_{\pm}(\vec{p}, -\varsigma; x), [\gamma_x \mp i\varsigma\gamma_y]\Psi_{\pm}(\vec{p}, -\varsigma; x) = 0 \\ \hat{P}_a\Psi_{\pm}(\vec{p}, -\varsigma; x) = \pm p_a\Psi_{\pm}(\vec{p}, -\varsigma; x), \hat{W}_a(1, \varsigma)\Psi_{\pm}(\vec{p}, -\varsigma; x) = \mp\varsigma p_a\Psi_{\pm}(\vec{p}, -\varsigma; x) \end{cases}$$

#### 4.8 s-自旋方程的双模同向同步平面波解

$$\text{定义4.8.1. } \begin{cases} [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x) = 0, \vec{p} \neq 0 \\ \psi(x) := \frac{1}{(2\pi)^{3/2}} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma)e^{ip \cdot x} + a_2^+(\vec{p}, -s\varsigma)e^{-ip \cdot x}] \end{cases}$$

#### 4.9 s-自旋方程的双模异向同步正平面波解

$$\text{定义4.9.1. } \begin{cases} [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x) = 0, \vec{p} \neq 0 \\ \psi(x) := \frac{1}{(2\pi)^{3/2}} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -s\varsigma) [a_1(\vec{p}, -s\varsigma)e^{i\vec{p} \cdot \vec{r}} + a_2(-\vec{p}, -s\varsigma)e^{-i\vec{p} \cdot \vec{r}}] e^{-iEt} \end{cases}$$

#### 4.10 s-自旋方程的双模异向同步负平面波解

$$\text{定义4.10.1. } \begin{cases} [s\partial_a + iS_{ab}(s, \varsigma)\partial^b]\psi(x) = 0, \vec{p} \neq 0 \\ \psi(x) := \frac{1}{(2\pi)^{3/2}} |\vec{p}|^{(s-\frac{1}{2})} \lambda(\hat{p}, -s\varsigma) [a_1^+(\vec{p}, -s\varsigma)e^{-i\vec{p} \cdot \vec{r}} + a_2^+(-\vec{p}, -s\varsigma)e^{i\vec{p} \cdot \vec{r}}] e^{iEt} \end{cases}$$

### 4.11 s-自旋方程的双模异向同步纠缠平面波解猜测

猜想4.11.1.

$$\begin{cases} [\sigma(s) \otimes I + I \otimes \sigma(s)] \partial_t \psi(\vec{r}, \vec{r}'; t) = s_C (\nabla - \nabla') \psi(\vec{r}, \vec{r}'; t) - i s_C \{ [\sigma(s) \otimes I] \times \nabla - [I \otimes \sigma(s)] \times \nabla' \} \psi(\vec{r}, \vec{r}'; t) \\ \psi(\vec{r}, \vec{r}'; t) = a(\vec{p}) \lambda(\hat{p}, -s_C) \otimes I e^{i(\vec{p} \cdot \vec{r} - Et)} + b(\vec{p}) I \otimes \lambda(\hat{p}, s_C) e^{i(-\vec{p} \cdot \vec{r}' - Et)} + c(\vec{p}) \lambda(\hat{p}, -s_C) \otimes \lambda(\hat{p}, s_C) e^{i\vec{p} \cdot (\vec{r} - \vec{r}') - iEt} \end{cases}$$

## 5 几个粒子的自旋耦合例子

### 5.1 两个电子自旋的对角耦合表象

推论5.1.1. 对角耦合表象

$$\begin{cases} [\sigma(1) \oplus \sigma(0)]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, [\sigma(1) \oplus \sigma(0)]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ [\sigma(1) \oplus \sigma(0)]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(1) \oplus \sigma(0)]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ [\sigma(1) \oplus \sigma(0)]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(1) \oplus \sigma(0)]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ [\sigma(1) \oplus \sigma(0)]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(1) \oplus \sigma(0)]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

### 5.2 两个电子自旋的分离非耦合表象

推论5.2.1. 分离非耦合表象

$$\begin{cases} [\sigma(\frac{1}{2}) \otimes I]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, [I \otimes \sigma(\frac{1}{2})]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, [I \otimes \sigma(\frac{1}{2})]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ [\sigma(\frac{1}{2}) \otimes I]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [I \otimes \sigma(\frac{1}{2})]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [I \otimes \sigma(\frac{1}{2})]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ [\sigma(\frac{1}{2}) \otimes I]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [I \otimes \sigma(\frac{1}{2})]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [I \otimes \sigma(\frac{1}{2})]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ [\sigma(\frac{1}{2}) \otimes I]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [I \otimes \sigma(\frac{1}{2})]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [I \otimes \sigma(\frac{1}{2})]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

### 5.3 两个电子自旋的分离耦合表象

$$\text{推论5.3.1. } \hat{S}(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{1} & \sqrt{1} & 0 \end{bmatrix}, \hat{S}^+(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & -\sqrt{1} \\ 0 & \sqrt{1} & 0 & \sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$\text{推论5.3.2. } \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & \sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{1} & 0 & -\sqrt{1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & \sqrt{1} & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ 0 & -\sqrt{1} & \sqrt{1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\text{推论5.3.3. } \left\{ \left| \frac{1}{2} \right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \left| -\frac{1}{2} \right\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \left\{ |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, |-1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{定理5.3.1. } S(1) [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})] S^+(1) = \sigma(1) \oplus \sigma(0)$$

推论5.3.4. 分离耦合表象

$$\begin{cases} \frac{[\sigma(\frac{1}{2}) \otimes I]^2}{[I \otimes \sigma(\frac{1}{2})]^2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \frac{[\sigma(\frac{1}{2}) \otimes I]^2}{[I \otimes \sigma(\frac{1}{2})]^2} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]^2 \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]_z \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = 0 \\ \frac{[\sigma(\frac{1}{2}) \otimes I]^2}{[I \otimes \sigma(\frac{1}{2})]^2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]_z \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ \frac{[\sigma(\frac{1}{2}) \otimes I]^2}{[I \otimes \sigma(\frac{1}{2})]^2} \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]^2 \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = 0, [\sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})]_z \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = 0 \end{cases}$$

$$\text{推论5.3.5. } \hat{S}^+(1) = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & -\sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow |s, m\rangle = \begin{cases} \hat{S}^+(1) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |1, 1\rangle = |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle; \\ \hat{S}^+(1) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |1, 0\rangle = \frac{1}{\sqrt{2}}(|\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle + |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle); \\ \hat{S}^+(1) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |1, -1\rangle = |\frac{-1}{2}\rangle \otimes |\frac{-1}{2}\rangle; \\ \hat{S}^+(1) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |0, 0\rangle = \frac{1}{\sqrt{2}}(-|\frac{-1}{2}\rangle \otimes |\frac{1}{2}\rangle + |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle); \end{cases}$$

$$\text{推论5.3.6. } \begin{bmatrix} |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \\ |0,0\rangle \end{bmatrix} = [\hat{S}^+(1) \otimes I_4] \begin{bmatrix} |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \end{bmatrix} \otimes \begin{bmatrix} |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \end{bmatrix} = [I_4 \otimes \hat{S}(1)] \begin{bmatrix} |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \end{bmatrix} \otimes \begin{bmatrix} |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \end{bmatrix}$$

$$\text{推论5.3.7. } \begin{bmatrix} |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \\ |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & \sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{1} & 0 & -\sqrt{1} \end{bmatrix} \begin{bmatrix} |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \\ |0,0\rangle \end{bmatrix}, \begin{bmatrix} |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \\ |0,0\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{1} & 0 & \sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{1} & 0 & -\sqrt{1} \end{bmatrix} \begin{bmatrix} |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \\ |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \end{bmatrix}$$

$$\text{推论5.3.8. } \begin{bmatrix} |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \\ |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{1} & 0 & \sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{1} & 0 & \sqrt{1} \end{bmatrix} \begin{bmatrix} |1,1\rangle \\ |0,0\rangle \\ |1,-1\rangle \\ |1,0\rangle \end{bmatrix}, \begin{bmatrix} |1,1\rangle \\ |0,0\rangle \\ |1,-1\rangle \\ |1,0\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{1} & 0 & \sqrt{1} \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{1} & 0 & \sqrt{1} \end{bmatrix} \begin{bmatrix} |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \\ |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \end{bmatrix}$$

$$\text{推论5.3.9. } \begin{bmatrix} |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \\ |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{1} & \sqrt{1} \\ 0 & 0 & \sqrt{1} & -\sqrt{1} \end{bmatrix} \begin{bmatrix} |1,1\rangle \\ |1,-1\rangle \\ |1,0\rangle \\ |0,0\rangle \end{bmatrix}, \begin{bmatrix} |1,1\rangle \\ |1,-1\rangle \\ |1,0\rangle \\ |0,0\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{1} & \sqrt{1} \\ 0 & 0 & \sqrt{1} & -\sqrt{1} \end{bmatrix} \begin{bmatrix} |\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \\ |\frac{1}{2}\rangle \otimes |\frac{-1}{2}\rangle \\ |-\frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle \end{bmatrix}$$

### 5.4 两个有质量光子的自旋耦合表象

定理5.4.1.  $S(1 \otimes 1)[\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]S^+(1 \otimes 1) = \sigma(2) \oplus \sigma(1) \oplus \sigma(0)$

推论5.4.1.

$$S(1 \otimes 1) = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & \sqrt{4} & 0 & \sqrt{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 \end{bmatrix}, S^+(1 \otimes 1) = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 & 0 & -\sqrt{3} & 0 & \sqrt{2} \\ 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{4} & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{1} & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \end{bmatrix}$$

推论5.4.2.

$$\left\{ \begin{array}{l} |2, 2\rangle = |1\rangle \otimes |1\rangle; \\ |2, 1\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle) + |0\rangle \otimes |1\rangle; \\ |2, 0\rangle = \frac{1}{\sqrt{6}}(|1\rangle \otimes | -1\rangle + 2|0\rangle \otimes |0\rangle + | -1\rangle \otimes |1\rangle); \\ |2, -1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes | -1\rangle) + | -1\rangle \otimes |0\rangle; \\ |2, -2\rangle = | -1\rangle \otimes | -1\rangle; \\ |2; 1, 1\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle) - |0\rangle \otimes |1\rangle; \\ |2; 1, 0\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes | -1\rangle) - | -1\rangle \otimes |1\rangle; \\ |2; 1, -1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes | -1\rangle) - | -1\rangle \otimes |0\rangle; \\ |2; 0, 0\rangle = \frac{1}{\sqrt{3}}(|1\rangle \otimes | -1\rangle) - |0\rangle \otimes |0\rangle + | -1\rangle \otimes |1\rangle; \end{array} \right. \left\{ \begin{array}{l} |1\rangle \otimes |1\rangle = |2, 2\rangle; \\ |0\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}(|2, 1\rangle - |1, 1\rangle); \\ | -1\rangle \otimes |1\rangle = \frac{1}{\sqrt{6}}(|2, 0\rangle - \sqrt{3}|1, 0\rangle + \sqrt{2}|0, 0\rangle); \\ |1\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|2, 1\rangle + |1, 1\rangle); \\ |0\rangle \otimes |0\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|2, 0\rangle - |0, 0\rangle); \\ | -1\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|2, -1\rangle - |1, -1\rangle); \\ |1\rangle \otimes | -1\rangle = \frac{1}{\sqrt{6}}(|2, 0\rangle + \sqrt{3}|1, 0\rangle + \sqrt{2}|0, 0\rangle); \\ |0\rangle \otimes | -1\rangle = \frac{1}{\sqrt{2}}(|2, -1\rangle + |1, -1\rangle); \\ | -1\rangle \otimes | -1\rangle = |2, -2\rangle; \end{array} \right.$$

推论5.4.3.

$$\left\{ \begin{array}{l} |2, 2\rangle = |1\rangle \otimes |1\rangle; \\ |2, 1\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle) + |0\rangle \otimes |1\rangle; \\ |2, 0\rangle = \frac{1}{\sqrt{6}}(|1\rangle \otimes |-1\rangle + 2|0\rangle \otimes |0\rangle + |-1\rangle \otimes |1\rangle); \\ |2, -1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |-1\rangle) + |-1\rangle \otimes |0\rangle; \\ |2, -2\rangle = |-1\rangle \otimes |-1\rangle; \\ |2; 1, 1\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle) - |0\rangle \otimes |1\rangle; \\ |2; 1, 0\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |-1\rangle) - |-1\rangle \otimes |1\rangle; \\ |2; 1, -1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |-1\rangle) - |-1\rangle \otimes |0\rangle; \\ |2; 0, 0\rangle = \frac{1}{\sqrt{3}}(|1\rangle \otimes |-1\rangle) - |0\rangle \otimes |0\rangle + |-1\rangle \otimes |1\rangle; \end{array} \right. \quad \left\{ \begin{array}{l} |1\rangle \otimes |1\rangle = |2, 2\rangle; \\ |1\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|2, 1\rangle + |1, 1\rangle); \\ |1\rangle \otimes |-1\rangle = \frac{1}{\sqrt{6}}(|2, 0\rangle + \sqrt{3}|1, 0\rangle + \sqrt{2}|0, 0\rangle); \\ |0\rangle \otimes |-1\rangle = \frac{1}{\sqrt{2}}(|2, -1\rangle + |1, -1\rangle); \\ |-1\rangle \otimes |-1\rangle = |2, -2\rangle; \\ |0\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}(|2, 1\rangle - |1, 1\rangle); \\ |0\rangle \otimes |0\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|2, 0\rangle - |0, 0\rangle); \\ |-1\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|2, -1\rangle - |1, -1\rangle); \\ |-1\rangle \otimes |1\rangle = \frac{1}{\sqrt{6}}(|2, 0\rangle - \sqrt{3}|1, 0\rangle + \sqrt{2}|0, 0\rangle); \end{array} \right.$$

推论5.4.4.

$$\left\{ \begin{array}{l} |2, 2\rangle = |1\rangle \otimes |1\rangle; \\ |2, -2\rangle = |-1\rangle \otimes |-1\rangle; \\ |2, 1\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle) + |0\rangle \otimes |1\rangle; \\ |1, 1\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle) - |0\rangle \otimes |1\rangle; \\ |1, -1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |-1\rangle) - |-1\rangle \otimes |0\rangle; \\ |2, -1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |-1\rangle) + |-1\rangle \otimes |0\rangle; \\ |2, 0\rangle = \frac{1}{\sqrt{6}}(|1\rangle \otimes |-1\rangle + 2|0\rangle \otimes |0\rangle + |-1\rangle \otimes |1\rangle); \\ |1, 0\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |-1\rangle) - |-1\rangle \otimes |1\rangle; \\ |0, 0\rangle = \frac{1}{\sqrt{3}}(|1\rangle \otimes |-1\rangle) - |0\rangle \otimes |0\rangle + |-1\rangle \otimes |1\rangle; \end{array} \right. \quad \left\{ \begin{array}{l} |1\rangle \otimes |1\rangle = |2, 2\rangle; \\ |-1\rangle \otimes |-1\rangle = |2, -2\rangle; \\ |1\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|2, 1\rangle + |1, 1\rangle); \\ |0\rangle \otimes |1\rangle = \frac{1}{\sqrt{2}}(|2, 1\rangle - |1, 1\rangle); \\ |-1\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}(|2, -1\rangle - |1, -1\rangle); \\ |0\rangle \otimes |-1\rangle = \frac{1}{\sqrt{2}}(|2, -1\rangle + |1, -1\rangle); \\ |1\rangle \otimes |-1\rangle = \frac{1}{\sqrt{6}}(|2, 0\rangle + \sqrt{3}|1, 0\rangle + \sqrt{2}|0, 0\rangle); \\ |0\rangle \otimes |0\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|2, 0\rangle - |0, 0\rangle); \\ |-1\rangle \otimes |1\rangle = \frac{1}{\sqrt{6}}(|2, 0\rangle - \sqrt{3}|1, 0\rangle + \sqrt{2}|0, 0\rangle); \end{array} \right.$$

推论5.4.5.

$$\left\{ \begin{array}{l} \begin{bmatrix} |2, 2\rangle \\ |2, -2\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} |1\rangle \otimes |1\rangle \\ |-1\rangle \otimes |-1\rangle \end{bmatrix}, \begin{bmatrix} |1\rangle \otimes |1\rangle \\ |-1\rangle \otimes |-1\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} |2, 2\rangle \\ |2, -2\rangle \end{bmatrix} \\ \begin{bmatrix} |2, 1\rangle \\ |1, 1\rangle \\ |1, -1\rangle \\ |2, -1\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} |1\rangle \otimes |0\rangle \\ |0\rangle \otimes |1\rangle \\ |-1\rangle \otimes |0\rangle \\ |0\rangle \otimes |-1\rangle \end{bmatrix}, \begin{bmatrix} |1\rangle \otimes |0\rangle \\ |0\rangle \otimes |1\rangle \\ |-1\rangle \otimes |0\rangle \\ |0\rangle \otimes |-1\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} |2, 1\rangle \\ |1, 1\rangle \\ |1, -1\rangle \\ |2, -1\rangle \end{bmatrix} \\ \begin{bmatrix} |2, 0\rangle \\ |1, 0\rangle \\ |0, 0\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} |1\rangle \otimes |-1\rangle \\ |0\rangle \otimes |0\rangle \\ |-1\rangle \otimes |1\rangle \end{bmatrix}, \begin{bmatrix} |1\rangle \otimes |-1\rangle \\ |0\rangle \otimes |0\rangle \\ |-1\rangle \otimes |1\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} |2, 0\rangle \\ |1, 0\rangle \\ |0, 0\rangle \end{bmatrix} \end{array} \right.$$

推论5.4.6.

$$S^+(1 \otimes 1) \Rightarrow |s, m\rangle = \left\{ \begin{array}{l} |2, 2\rangle = |1\rangle \otimes |1\rangle; \\ |2, 1\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle); \\ |2, 0\rangle = \frac{1}{\sqrt{6}}(|-1\rangle \otimes |1\rangle + 2|0\rangle \otimes |0\rangle + |1\rangle \otimes |-1\rangle); \\ |2, -1\rangle = \frac{1}{\sqrt{2}}(|-1\rangle \otimes |0\rangle + |0\rangle \otimes |-1\rangle); \\ |2, -2\rangle = |-1\rangle \otimes |-1\rangle; \\ |2; 1, 1\rangle = \frac{1}{\sqrt{2}}(-|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle); \\ |2; 1, 0\rangle = \frac{1}{\sqrt{2}}(-|-1\rangle \otimes |1\rangle + |1\rangle \otimes |-1\rangle); \\ |2; 1, -1\rangle = \frac{1}{\sqrt{2}}(-|-1\rangle \otimes |0\rangle + |0\rangle \otimes |-1\rangle); \\ |2; 0, 0\rangle = \frac{1}{\sqrt{3}}(|-1\rangle \otimes |1\rangle - |0\rangle \otimes |0\rangle + |1\rangle \otimes |-1\rangle); \end{array} \right.$$

推论5.4.7. 
$$\begin{bmatrix} |2,2\rangle \\ |2,1\rangle \\ |2,0\rangle \\ |2,-1\rangle \\ |2,-2\rangle \\ |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \\ |0,0\rangle \end{bmatrix} = [S^+(1 \otimes 1) \otimes I_9] \begin{bmatrix} |1\rangle \\ |0\rangle \\ |-1\rangle \end{bmatrix} \otimes \begin{bmatrix} |1\rangle \\ |0\rangle \\ |-1\rangle \end{bmatrix} = [I_9 \otimes S(1 \otimes 1)] \begin{bmatrix} |1\rangle \\ |0\rangle \\ |-1\rangle \end{bmatrix} \otimes \begin{bmatrix} |1\rangle \\ |0\rangle \\ |-1\rangle \end{bmatrix}$$

### 5.5 两个无质量光子的自旋耦合

推论5.5.1. 对角耦合表象

$$\begin{aligned} [\sigma(2) \oplus \sigma(1) \oplus \sigma(0)]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} &= 6 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(2) \oplus \sigma(1) \oplus \sigma(0)]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ [\sigma(2) \oplus \sigma(1) \oplus \sigma(0)]^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} &= 6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(2) \oplus \sigma(1) \oplus \sigma(0)]_z \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ [\sigma(2) \oplus \sigma(1) \oplus \sigma(0)]^2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} &= 6 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(2) \oplus \sigma(1) \oplus \sigma(0)]_z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \\ [\sigma(2) \oplus \sigma(1) \oplus \sigma(0)]^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} &= 2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [\sigma(2) \oplus \sigma(1) \oplus \sigma(0)]_z \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \\ [\sigma(2) \oplus \sigma(1) \oplus \sigma(0)]^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} &= 0, [\sigma(2) \oplus \sigma(1) \oplus \sigma(0)]_z \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \end{aligned}$$

推论5.5.2. 非对角耦合表象

$$\begin{aligned} [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]^2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} &= 6 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]_z \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} &= 6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]_z \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]^2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} &= 6 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]_z \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \\ [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]^2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} &= 2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]_z \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0 \\ [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} &= 0, [\sigma(1) \otimes I_3 + I_3 \otimes \sigma(1)]_z \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \end{aligned}$$



$$\text{推论5.5.3. } S^+(2) \Rightarrow |s, m\rangle = \begin{cases} |2, 2\rangle = |1\rangle \otimes |1\rangle; \\ \frac{\sqrt{1}}{\sqrt{3}}|2, 0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|2; 0, 0\rangle = \frac{1}{\sqrt{2}}(|-1\rangle \otimes |1\rangle + |1\rangle \otimes |-1\rangle); \\ |2, -2\rangle = |-1\rangle \otimes |-1\rangle; \\ |2; 1, 0\rangle = \frac{1}{\sqrt{2}}(-|-1\rangle \otimes |1\rangle + |1\rangle \otimes |-1\rangle); \end{cases}$$

$$\text{推论5.5.4. } \begin{cases} |1\rangle \otimes |1\rangle = |2, 2\rangle; \\ |-1\rangle \otimes |1\rangle = \frac{1}{\sqrt{6}}|2, 0\rangle - \frac{1}{\sqrt{2}}|2; 1, 0\rangle + \frac{1}{\sqrt{3}}|2; 0, 0\rangle; \\ |1\rangle \otimes |-1\rangle = \frac{1}{\sqrt{6}}|2, 0\rangle + \frac{1}{\sqrt{2}}|2; 1, 0\rangle + \frac{1}{\sqrt{3}}|2; 0, 0\rangle; \\ |-1\rangle \otimes |-1\rangle = |2, -2\rangle; \end{cases}$$

$$\text{推论5.5.5. } \begin{cases} |2, 2\rangle = |1\rangle \otimes |1\rangle e^{i\vec{p}\cdot(\vec{r}_1 - \vec{r}_2)}; \\ \frac{\sqrt{1}}{\sqrt{3}}|2, 0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|2; 0, 0\rangle = \frac{1}{\sqrt{2}}(|-1\rangle \otimes |1\rangle e^{-2i\zeta pt} + |1\rangle \otimes |-1\rangle e^{2i\zeta pt}) e^{i\vec{p}\cdot(\vec{r}_1 - \vec{r}_2)}; \\ |2, -2\rangle = |-1\rangle \otimes |-1\rangle e^{i\vec{p}\cdot(\vec{r}_1 - \vec{r}_2)}; \\ |2; 1, 0\rangle = \frac{1}{\sqrt{2}}(-|-1\rangle \otimes |1\rangle e^{-2i\zeta pt} + |1\rangle \otimes |-1\rangle e^{2i\zeta pt}) e^{i\vec{p}\cdot(\vec{r}_1 - \vec{r}_2)}; \end{cases}$$

$$= [I_9 \otimes \hat{S}(1)] \begin{bmatrix} |1\rangle e^{i(\vec{p}\cdot\vec{r}_1 + \zeta pt)} \\ |-1\rangle e^{i(\vec{p}\cdot\vec{r}_1 - \zeta pt)} \end{bmatrix} \otimes \begin{bmatrix} |1\rangle e^{-i(\vec{p}\cdot\vec{r}_2 + \zeta pt)} \\ |-1\rangle e^{-i(\vec{p}\cdot\vec{r}_2 - \zeta pt)} \end{bmatrix}$$

推论5.5.6.

$$\begin{cases} |2, 2\rangle = |1\rangle \otimes |1\rangle; \\ i|2; 1, 0\rangle \sin(2\zeta pt) + (\frac{\sqrt{1}}{\sqrt{3}}|2, 0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|2; 0, 0\rangle) \cos(2\zeta pt) = \frac{1}{\sqrt{2}}(|-1\rangle \otimes |1\rangle e^{-2i\zeta pt} + |1\rangle \otimes |-1\rangle e^{2i\zeta pt}); \\ |2, -2\rangle = |-1\rangle \otimes |-1\rangle; \\ |2; 1, 0\rangle \cos(2\zeta pt) + i(\frac{\sqrt{1}}{\sqrt{3}}|2, 0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|2; 0, 0\rangle) \sin(2\zeta pt) = \frac{1}{\sqrt{2}}(-|-1\rangle \otimes |1\rangle e^{-2i\zeta pt} + |1\rangle \otimes |-1\rangle e^{2i\zeta pt}); \end{cases}$$

$$= [I_9 \otimes \hat{S}(1)] \begin{bmatrix} |1\rangle e^{i\zeta pt} \\ |-1\rangle e^{-i\zeta pt} \end{bmatrix} \otimes \begin{bmatrix} |1\rangle e^{-i\zeta pt} \\ |-1\rangle e^{i\zeta pt} \end{bmatrix}$$

$$\text{推论5.5.7. } \begin{bmatrix} |2, 2\rangle \\ \frac{\sqrt{1}}{\sqrt{3}}|2, 0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|0, 0\rangle \\ |2, -2\rangle \\ |1, 0\rangle \end{bmatrix} = [I_9 \otimes \hat{S}(1)] \begin{bmatrix} |1\rangle \\ |-1\rangle \end{bmatrix} \otimes \begin{bmatrix} |1\rangle \\ |-1\rangle \end{bmatrix}, \begin{bmatrix} |1, 1\rangle \\ |1, 0\rangle \\ |1, -1\rangle \\ |0, 0\rangle \end{bmatrix} = [I_4 \otimes \hat{S}(1)] \begin{bmatrix} |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \end{bmatrix} \otimes \begin{bmatrix} |\frac{1}{2}\rangle \\ |-\frac{1}{2}\rangle \end{bmatrix}$$

$$\text{推论5.5.8. } \begin{bmatrix} |2, 2\rangle \\ \frac{\sqrt{1}}{\sqrt{3}}|2, 0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|0, 0\rangle \\ |2, -2\rangle \\ |1, 0\rangle \end{bmatrix} = [\hat{S}^+(1) \otimes I_9] \begin{bmatrix} |1\rangle \otimes |1\rangle \\ \frac{\sqrt{1}}{\sqrt{3}}|-1\rangle \otimes |1\rangle + \frac{\sqrt{2}}{\sqrt{3}}|-1\rangle \otimes |-1\rangle \\ |0\rangle \otimes |0\rangle \\ |1\rangle \otimes |-1\rangle \end{bmatrix} = [I_9 \otimes \hat{S}(1)] \begin{bmatrix} |1\rangle \\ |-1\rangle \end{bmatrix} \otimes \begin{bmatrix} |1\rangle \\ |-1\rangle \end{bmatrix}$$

## 5.6 三个电子的自旋耦合

$$\text{推论5.6.1. } \hat{S}^+(\frac{3}{2}) = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & -\sqrt{4} & \sqrt{1} & 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{1} & 0 & -\sqrt{1} & \sqrt{4} & 0 \\ 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 \end{bmatrix}, \hat{S}^+(\frac{3}{2}) = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & -\sqrt{4} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{1} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{1} & 0 & -\sqrt{3} \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{1} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & -\sqrt{1} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{4} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 & 0 & 0 & 0 \end{bmatrix}$$

推论5.6.2.  $\hat{S}^+(\frac{3}{2}) \Rightarrow$

$$|s, m\rangle = \begin{cases} \left\{ \begin{aligned} \left| \frac{3}{2}, \frac{3}{2} \right\rangle &= \left| \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle; \\ \left| \frac{3}{2}, \frac{1}{2} \right\rangle &= \frac{1}{\sqrt{3}} \left( \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle + \left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle + \left| \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \right); \\ \left| \frac{3}{2}, -\frac{1}{2} \right\rangle &= \frac{1}{\sqrt{3}} \left( \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle + \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle + \left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \right); \\ \left| -\frac{3}{2}, -\frac{3}{2} \right\rangle &= \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle; \\ \left| \frac{3}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle_1 &= \frac{1}{\sqrt{6}} \left( -2 \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle + \left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle + \left| \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \right); \\ \left| \frac{3}{2}; \frac{1}{2}, -\frac{1}{2} \right\rangle_1 &= \frac{1}{\sqrt{6}} \left( -\left| \frac{-1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle - \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle + 2 \left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \right); \\ \left| \frac{3}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle_2 &= \frac{1}{\sqrt{2}} \left( -\left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle + \left| \frac{1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \right); \\ \left| \frac{3}{2}; \frac{1}{2}, -\frac{1}{2} \right\rangle_2 &= \frac{1}{\sqrt{2}} \left( -\left| \frac{-1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle + \left| \frac{-1}{2} \right\rangle \otimes \left| \frac{1}{2} \right\rangle \otimes \left| \frac{-1}{2} \right\rangle \right); \end{aligned} \right.$$

推论5.6.3. 
$$\begin{bmatrix} \left| \frac{3}{2}, \frac{3}{2} \right\rangle \\ \left| \frac{3}{2}, \frac{1}{2} \right\rangle \\ \left| \frac{3}{2}, -\frac{1}{2} \right\rangle \\ \left| \frac{3}{2}, -\frac{3}{2} \right\rangle \\ \left| \frac{1}{2}, \frac{1}{2} \right\rangle_1 \\ \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_1 \\ \left| \frac{1}{2}, \frac{1}{2} \right\rangle_2 \\ \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_2 \end{bmatrix} = [\hat{S}^+(\frac{3}{2}) \otimes I_8] \begin{bmatrix} \left| \frac{1}{2} \right\rangle \\ \left| \frac{-1}{2} \right\rangle \end{bmatrix} \otimes \begin{bmatrix} \left| \frac{1}{2} \right\rangle \\ \left| \frac{-1}{2} \right\rangle \end{bmatrix} \otimes \begin{bmatrix} \left| \frac{1}{2} \right\rangle \\ \left| \frac{-1}{2} \right\rangle \end{bmatrix} = [I_8 \otimes \hat{S}(\frac{3}{2})] \begin{bmatrix} \left| \frac{1}{2} \right\rangle \\ \left| \frac{-1}{2} \right\rangle \end{bmatrix} \otimes \begin{bmatrix} \left| \frac{1}{2} \right\rangle \\ \left| \frac{-1}{2} \right\rangle \end{bmatrix} \otimes \begin{bmatrix} \left| \frac{1}{2} \right\rangle \\ \left| \frac{-1}{2} \right\rangle \end{bmatrix}$$

## 6 经典物理+概率论解释光子纠缠的可能性(只是试探性的)

### 6.1 经典电磁学对自旋极化的描述

定义6.1.1.  $|\circ\rangle := \vec{E}_\circ = A[\cos(p \cdot x + \theta)\hat{e}_x + \sin(p \cdot x + \theta)\hat{e}_y], |\ominus\rangle := \vec{E}_\ominus = A[\cos(p \cdot x + \theta)\hat{e}_x - \sin(p \cdot x + \theta)\hat{e}_y]$

定义6.1.2.  $|\rightarrow\rangle := \vec{E}_\rightarrow = \sqrt{2}A\cos(p \cdot x + \theta)\hat{e}_x, |\uparrow\rangle := \vec{E}_\uparrow = \sqrt{2}A\sin(p \cdot x + \theta)\hat{e}_y$

证明:  $I_\circ = \frac{1}{2} \int_V (\vec{E}_\circ^2 + \vec{B}_\circ^2) d^3\vec{r} = \int_V \vec{E}_\circ^2 d^3\vec{r} = \frac{1}{T} \int_0^T \vec{E}_\circ^2 dt = A^2 = h\nu$  □

证明:  $I_\ominus = \frac{1}{2} \int_V (\vec{E}_\ominus^2 + \vec{B}_\ominus^2) d^3\vec{r} = \int_V \vec{E}_\ominus^2 d^3\vec{r} = \frac{1}{T} \int_0^T \vec{E}_\ominus^2 dt = A^2 = h\nu$  □

证明:  $I_\rightarrow = \frac{1}{2} \int_V (\vec{E}_\rightarrow^2 + \vec{B}_\rightarrow^2) d^3\vec{r} = \int_V \vec{E}_\rightarrow^2 d^3\vec{r} = \frac{1}{T} \int_0^T \vec{E}_\rightarrow^2 dt = A^2 = h\nu$  □

证明:  $I_\uparrow = \frac{1}{2} \int_V (\vec{E}_\uparrow^2 + \vec{B}_\uparrow^2) d^3\vec{r} = \int_V \vec{E}_\uparrow^2 d^3\vec{r} = \frac{1}{T} \int_0^T \vec{E}_\uparrow^2 dt = A^2 = h\nu$  □

推论6.1.1. 
$$\begin{cases} |\circ\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle + |\uparrow\rangle\}, |\ominus\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle - |\uparrow\rangle\} \\ |\rightarrow\rangle = \frac{1}{\sqrt{2}}\{|\circ\rangle + |\ominus\rangle\}, |\uparrow\rangle = \frac{1}{\sqrt{2}}\{|\circ\rangle - |\ominus\rangle\} \end{cases}$$

### 6.2 经典电磁学对线性极化的描述

定义6.2.1.  $|\rightarrow\rangle := \vec{E}_\rightarrow = \sqrt{2}A\cos(p \cdot x + \theta)\hat{e}_x, |\uparrow\rangle := \vec{E}_\uparrow = \sqrt{2}A\cos(p \cdot x + \theta)\hat{e}_y$

推论6.2.1. 
$$\begin{cases} |\nearrow\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle + |\uparrow\rangle\}, |\searrow\rangle = \frac{1}{\sqrt{2}}\{|\rightarrow\rangle - |\uparrow\rangle\} \\ |\rightarrow\rangle = \frac{1}{\sqrt{2}}\{|\nearrow\rangle + |\searrow\rangle\}, |\uparrow\rangle = \frac{1}{\sqrt{2}}\{|\nearrow\rangle - |\searrow\rangle\} \end{cases}$$

推论6.2.2. 
$$\left\{ P(\rightarrow) = 1 \quad \rightarrow \otimes \rightarrow \begin{cases} P(\nearrow) = \frac{1}{2} \\ P(\searrow) = 0 \end{cases} \quad \rightarrow \ominus \rightarrow \begin{cases} P(\rightarrow) = \frac{1}{4} \\ P(\uparrow) = 0 \end{cases} \quad \rightarrow \ominus \rightarrow \begin{cases} P(\rightarrow) = \frac{1}{4} \\ P(\uparrow) = 0 \end{cases} \right.$$

推论6.2.3. 
$$\left\{ P(\rightarrow) = 1 \quad \rightarrow \otimes \rightarrow \begin{cases} P(\nearrow) = \frac{1}{2} \\ P(\searrow) = \frac{1}{2} \end{cases} \quad \rightarrow \oplus \rightarrow \begin{cases} P(\uparrow) = \frac{1}{2} \\ P(\rightarrow) = \frac{1}{2} \end{cases} \quad \rightarrow \otimes \rightarrow \begin{cases} P(\nearrow) = \frac{1}{2} \\ P(\searrow) = \frac{1}{2} \end{cases} \right.$$

## 7 多粒子自旋体系

### 7.1 两电子紧耦合体系的总自旋

推论7.1.1.  $\vec{s} = \sigma(\frac{1}{2}) \otimes I + I \otimes \sigma(\frac{1}{2})$

### 7.2 两电子松耦合体系的总自旋

推论7.2.1.  $s_{ij} = \sigma_i(\frac{1}{2}) \otimes \sigma_j(\frac{1}{2})$

### 7.3 多电子紧耦合体系的总自旋

推论7.3.1.  $\vec{s} = \Omega(s)$

### 7.4 多电子松耦合体系的总自旋

推论7.4.1.  $s_{i_1 i_2 \dots i_{2s}} = \sigma_{i_1}(\frac{1}{2}) \otimes \sigma_{i_2}(\frac{1}{2}) \otimes \dots \otimes \sigma_{i_{2s}}(\frac{1}{2})$

## 8 多粒子纠缠方程的猜想

### 8.1 四阶矩阵的物理意义一

定理8.1.1.  $X_{ab}(\vec{r}, t) = \overbrace{\frac{1}{2!}[X_{\{ab\}}(\vec{r}, t) - \frac{1}{2}\delta_{ab}trX(\vec{r}, t)]}^{\text{自旋-2粒子}} + \overbrace{\frac{1}{2!}X_{[ab]}(\vec{r}, t)}^{\text{自旋-1粒子}} + \overbrace{\frac{1}{4}\delta_{ab}trX(\vec{r}, t)}^{\text{自旋-0粒子}}$

定理8.1.2.  $(\partial^c \partial_c - m^2) \overbrace{\frac{1}{2!}[X_{\{ab\}}(\vec{r}, t) - \frac{1}{2}\delta_{ab}trX(\vec{r}, t)]}^{\text{自旋-2粒子}} = 0, \partial^a \overbrace{\frac{1}{2!}[X_{\{ab\}}(\vec{r}, t) - \frac{1}{2}\delta_{ab}trX(\vec{r}, t)]}^{\text{自旋-2粒子}} = 0$

$$\overbrace{\frac{1}{2!}[X_{\{ab\}}(\vec{r}, t) - \frac{1}{2}\delta_{ab}trX(\vec{r}, t)]}^{\text{自旋-2粒子}} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \frac{1}{\sqrt{2^2 E}} \varepsilon_{ab}(\vec{p}, h; 2) [a(\vec{p}, h; 2)e^{ip \cdot x} + b^+(\vec{p}, h; 2)e^{-ip \cdot x}] d^3 \vec{p}$$

定理8.1.3.  $(\partial^c \partial_c - m^2) \overbrace{\frac{1}{2!}X_{[ab]}(\vec{r}, t)}^{\text{自旋-1粒子}} = 0, \partial^a \overbrace{\frac{1}{2!}X_{[ab]}(\vec{r}, t)}^{\text{自旋-1粒子}} = 0$

$$\overbrace{\frac{1}{2!}X_{[ab]}(\vec{r}, t)}^{\text{自旋-1粒子}} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \varepsilon_{ab}(\vec{p}, h; 1) [a(\vec{p}, h; 1)e^{ip \cdot x} + b^+(\vec{p}, h; 1)e^{-ip \cdot x}] d^3 \vec{p}$$

定理8.1.4.  $(\partial^c \partial_c - m^2) \overbrace{\frac{1}{4}\delta_{ab}trX(\vec{r}, t)}^{\text{自旋-0粒子}} = 0$

$$\overbrace{\frac{1}{4}\delta_{ab}trX(\vec{r}, t)}^{\text{自旋-0粒子}} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2E}} [\varepsilon_{ab}(\vec{p}, 0; 0) + \frac{p_a p_b}{m^2}] [a(\vec{p}, 0; 0)e^{ip \cdot x} + b^+(\vec{p}, 0; 0)e^{-ip \cdot x}] d^3 \vec{p}$$

### 8.2 四阶矩阵的物理意义二

定理8.2.1.  $X_{\lambda_s \mu_s}(\vec{r}, t) = \overbrace{\frac{1}{2!}X_{\{\lambda_s \mu_s\}}(\vec{r}, t)}^{\text{自旋-1粒子}} + \overbrace{\frac{1}{2!}X_{[\lambda_s \mu_s]}(\vec{r}, t)}^{\text{自旋-0粒子}}$

定理8.2.2.  $(\gamma^a \partial_a + m)_{\kappa_s} \lambda_s \overbrace{\frac{1}{2!}X_{\{\lambda_s \mu_s\}}(\vec{r}, t)}^{\text{自旋-1粒子}} = 0$

$$\overbrace{\frac{1}{2!}X_{\{\lambda_s \mu_s\}}(\vec{r}, t)}^{\text{自旋-1粒子}} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \sqrt{\frac{m^2}{E}} [a(\vec{p}, h; 1)U_{\lambda_s \mu_s}(\vec{p}, h; 1)e^{ip \cdot x} + b^+(\vec{p}, h; 1)V_{\lambda_s \mu_s}(\vec{p}, h; 1)e^{-ip \cdot x}] d^3 \vec{p}$$

定理8.2.3.  $(\gamma^a \partial_a + m)_{\kappa_s} \lambda_s \overbrace{\frac{1}{2!}X_{[\lambda_s \mu_s]}(\vec{r}, t)}^{\text{自旋-0粒子}} = 0$

$$\overbrace{\frac{1}{2!}X_{[\lambda_s \mu_s]}(\vec{r}, t)}^{\text{自旋-0粒子}} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2E}} [a(\vec{p}, 0; 0)U_{\lambda_s \mu_s}(\vec{p}, 0; 0)e^{ip \cdot x} + b^+(\vec{p}, 0; 0)V_{\lambda_s \mu_s}(\vec{p}, 0; 0)e^{-ip \cdot x}] d^3 \vec{p}$$

### 8.3 有质量双光子纠缠方程

定理8.3.1.  $(\partial^c \partial_c - m^2)X_{ab}(\vec{r}, t) = 0, \partial^a X_{ab}(\vec{r}, t) = 0, \partial^b X_{ab}(\vec{r}, t) = 0$

$$\begin{cases} X_{ab}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \frac{1}{\sqrt{2^2 E}} \varepsilon_{ab}(\vec{p}, h; 2) [a(\vec{p}, h; 2) e^{ip \cdot x} + b^+(\vec{p}, h; 2) e^{-ip \cdot x}] d^3 \vec{p} \\ + \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \varepsilon_{ab}(\vec{p}, h; 1) [a(\vec{p}, h; 1) e^{ip \cdot x} + b^+(\vec{p}, h; 1) e^{-ip \cdot x}] d^3 \vec{p} \\ + \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2E}} \varepsilon_{ab}(\vec{p}, 0; 0) [a(\vec{p}, 0; 0) e^{ip \cdot x} + b^+(\vec{p}, 0; 0) e^{-ip \cdot x}] d^3 \vec{p} \end{cases}$$

定理8.3.2.

$$\begin{cases} \hat{J}^2(\vec{p}, 2; R, L) \varepsilon_{a \otimes b}(\vec{p}, h; 2) = 2(2+1) \varepsilon_{a \otimes b}(\vec{p}, h; 2), \hat{J}_z(\vec{p}, 2; R, L) \varepsilon_{a \otimes b}(\vec{p}, h; 2) = h \varepsilon_{a \otimes b}(\vec{p}, h; 2), 2 \leq h \leq 2 \\ \hat{J}^2(\vec{p}, 2; R, L) \varepsilon_{a \otimes b}(\vec{p}, h; 1) = 1(1+1) \varepsilon_{a \otimes b}(\vec{p}, h; 1), \hat{J}_z(\vec{p}, 2; R, L) \varepsilon_{a \otimes b}(\vec{p}, h; 1) = h \varepsilon_{a \otimes b}(\vec{p}, h; 1), 1 \leq h \leq 1 \\ \hat{J}^2(\vec{p}, 2; R, L) \delta_{a \otimes b}(\vec{p}, 0; 0) = 0(0+1) \delta_{a \otimes b}(\vec{p}, 0; 0), \hat{J}_z(\vec{p}, 2; R, L) \delta_{a \otimes b}(\vec{p}, 0; 0) = 0 \delta_{a \otimes b}(\vec{p}, 0; 0) \\ \hat{J}^2(\vec{p}, 2; R, L) \varepsilon_{a \otimes b}(\vec{p}, 0; 0) = 0(0+1) \varepsilon_{a \otimes b}(\vec{p}, 0; 0), \hat{J}_z(\vec{p}, 2; R, L) \varepsilon_{a \otimes b}(\vec{p}, 0; 0) = 0 \varepsilon_{a \otimes b}(\vec{p}, 0; 0) \\ \hat{J}^2(\vec{p}, 2; R, L) \frac{p_a \otimes p_b}{m^2} = 0(0+1) \frac{p_a \otimes p_b}{m^2}, \hat{J}_z(\vec{p}, 2; R, L) \frac{p_a \otimes p_b}{m^2} = 0 \frac{p_a \otimes p_b}{m^2}, \varepsilon_{ab}(\vec{p}, 0; 0) + \frac{p_a p_b}{m^2} = -\delta_{ab} \end{cases}$$

### 8.4 双电子纠缠方程

定理8.4.1.  $(\gamma^a \partial_a + m)_{\kappa_s} X_{\lambda_s \mu_s}(\vec{r}, t) = 0, X_{\lambda_s \mu_s}(\vec{r}, t)$

$$\begin{aligned} &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \sqrt{\frac{m^2}{E}} [a(\vec{p}, h; 1) U_{\lambda_s \mu_s}(\vec{p}, h; 1) e^{ip \cdot x} + b^+(\vec{p}, h; 1) V_{\lambda_s \mu_s}(\vec{p}, h; 1) e^{-ip \cdot x}] d^3 \vec{p} \\ &+ \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2E}} [a(\vec{p}, 0; 0) U_{\lambda_s \mu_s}(\vec{p}, 0; 0) e^{ip \cdot x} + b^+(\vec{p}, 0; 0) V_{\lambda_s \mu_s}(\vec{p}, 0; 0) e^{-ip \cdot x}] d^3 \vec{p} \end{aligned}$$

定理8.4.2.

$$\begin{cases} \hat{J}^2(\vec{p}, 1; \gamma_a) U_{\lambda_s \otimes \mu_s}(\vec{p}, h; 1) = 1(1+1) U_{\lambda_s \otimes \mu_s}(\vec{p}, h; 1), \hat{J}_z(\vec{p}, 1; \gamma_a) U_{\lambda_s \otimes \mu_s}(\vec{p}, h; 1) = h U_{\lambda_s \otimes \mu_s}(\vec{p}, h; 1), -1 \leq h \leq 1 \\ \hat{J}^2(\vec{p}, 1; \gamma_a) U_{\lambda_s \otimes \mu_s}(\vec{p}, 0; 0) = 0(0+1) U_{\lambda_s \otimes \mu_s}(\vec{p}, 0; 0), \hat{J}_z(\vec{p}, 1; \gamma_a) U_{\lambda_s \otimes \mu_s}(\vec{p}, 0; 0) = 0 U_{\lambda_s \otimes \mu_s}(\vec{p}, 0; 0) \\ \hat{J}^2(\vec{p}, 1; \gamma_a) V_{\lambda_s \otimes \mu_s}(\vec{p}, h; 1) = 1(1+1) V_{\lambda_s \otimes \mu_s}(\vec{p}, h; 1), \hat{J}_z(\vec{p}, 1; \gamma_a) V_{\lambda_s \otimes \mu_s}(\vec{p}, h; 1) = h V_{\lambda_s \otimes \mu_s}(\vec{p}, h; 1), -1 \leq h \leq 1 \\ \hat{J}^2(\vec{p}, 1; \gamma_a) V_{\lambda_s \otimes \mu_s}(\vec{p}, 0; 0) = 0(0+1) V_{\lambda_s \otimes \mu_s}(\vec{p}, 0; 0), \hat{J}_z(\vec{p}, 1; \gamma_a) V_{\lambda_s \otimes \mu_s}(\vec{p}, 0; 0) = 0 V_{\lambda_s \otimes \mu_s}(\vec{p}, 0; 0) \end{cases}$$

### 8.5 二阶矩阵的物理意义

$$\text{定理8.5.1. } X_{AB}(\vec{r}, t) = \overbrace{\frac{1}{2!} X_{\{AB\}}(\vec{r}, t)}^{\text{自旋-1粒子}} + \overbrace{\frac{1}{2!} X_{[AB]}(\vec{r}, t)}^{\text{自旋-0粒子}}$$

$$\text{定理8.5.2. } (\partial^c \partial_c - m^2) \overbrace{\frac{1}{2!} X_{\{AB\}}(\vec{r}, t)}^{\text{自旋-1粒子}} = 0$$

$$\overbrace{\frac{1}{2!} X_{\{AB\}}(\vec{r}, t)}^{\text{自旋-1粒子}} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \sqrt{|\vec{p}|} \lambda_{AB}(\hat{p}, h; 1) [a(\vec{p}, h; 1) e^{ip \cdot x} + b^+(\vec{p}, h; 1) e^{-ip \cdot x}] d^3 \vec{p}$$

$$\text{定理8.5.3. } (\partial^c \partial_c - m^2) \overbrace{\frac{1}{2!} X_{[AB]}(\vec{r}, t)}^{\text{自旋-0粒子}} = 0$$

$$\overbrace{\frac{1}{2!} X_{[AB]}(\vec{r}, t)}^{\text{自旋-0粒子}} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2E}} \lambda_{AB}(\hat{p}, 0; 0) [a(\vec{p}, 0; 0) e^{ip \cdot x} + b^+(\vec{p}, 0; 0) e^{-ip \cdot x}] d^3 \vec{p}$$

### 8.6 双中微子纠缠方程

定理8.6.1.  $(\partial^c \partial_c - m^2)X_{AB}(\vec{r}, t) = 0, X_{AB}(\vec{r}, t)$

$$\begin{aligned} &= \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \sqrt{|\vec{p}|} \lambda_{AB}(\hat{p}, h; 1) [a(\vec{p}, h; 1) e^{ip \cdot x} + b^+(\vec{p}, h; 1) e^{-ip \cdot x}] d^3 \vec{p} \\ &+ \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2E}} \lambda_{AB}(\hat{p}, 0; 0) [a(\vec{p}, 0; 0) e^{ip \cdot x} + b^+(\vec{p}, 0; 0) e^{-ip \cdot x}] d^3 \vec{p} \end{aligned}$$

定理8.6.2.

$$\begin{cases} \hat{J}^2(\vec{p}, 1; \Omega(1))\lambda_{A\otimes B}(\vec{p}, h; 1) = 1(1+1)\lambda_{A\otimes B}(\vec{p}, h; 1), \hat{J}_z(\vec{p}, 1; \Omega(1))\lambda_{A\otimes B}(\vec{p}, h; 1) = h\lambda_{A\otimes B}(\vec{p}, h; 1), -1 \leq h \leq 1 \\ \hat{J}^2(\vec{p}, 1; \Omega(1))\lambda_{A\otimes B}(\vec{p}, 0; 0) = 0(0+1)\lambda_{A\otimes B}(\vec{p}, 0; 0), \hat{J}_z(\vec{p}, 1; \Omega(1))\lambda_{A\otimes B}(\vec{p}, 0; 0) = 0\lambda_{A\otimes B}(\vec{p}, 0; 0) \end{cases}$$

### 8.7 三阶矩阵的物理意义

$$\text{定理8.7.1. } X_{\alpha\beta}(\vec{r}, t) = \overbrace{\frac{1}{2!}[X_{\{\alpha\beta\}}(\vec{r}, t) - \frac{1}{2}\delta_{\alpha\beta}trX(\vec{r}, t)]}^{\text{自旋-2粒子}} + \overbrace{\frac{1}{2!}X_{[\alpha\beta]}(\vec{r}, t)}^{\text{自旋-1粒子}} + \overbrace{\frac{1}{4}\delta_{\alpha\beta}trX(\vec{r}, t)}^{\text{自旋-0粒子}}$$

$$\text{定理8.7.2. } (\partial^c\partial_c - m^2)\overbrace{\frac{1}{2!}[X_{\{\alpha\beta\}}(\vec{r}, t) - \frac{1}{2}\delta_{\alpha\beta}trX(\vec{r}, t)]}^{\text{自旋-2粒子}} = 0$$

$$\overbrace{\frac{1}{2!}[X_{\{\alpha\beta\}}(\vec{r}, t) - \frac{1}{2}\delta_{\alpha\beta}trX(\vec{r}, t)]}^{\text{自旋-2粒子}} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \frac{1}{\sqrt{2^2E}} \varepsilon_{\alpha\beta}(\vec{p}, h; 2)[a(\vec{p}, h; 2)e^{ip\cdot x} + b^+(\vec{p}, h; 2)e^{-ip\cdot x}]d^3\vec{p}$$

$$\text{定理8.7.3. } (\partial^c\partial_c - m^2)\overbrace{\frac{1}{2!}X_{[\alpha\beta]}(\vec{r}, t)}^{\text{自旋-1粒子}} = 0$$

$$\overbrace{\frac{1}{2!}X_{[\alpha\beta]}(\vec{r}, t)}^{\text{自旋-1粒子}} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \varepsilon_{\alpha\beta}(\vec{p}, h; 1)[a(\vec{p}, h; 1)e^{ip\cdot x} + b^+(\vec{p}, h; 1)e^{-ip\cdot x}]d^3\vec{p}$$

$$\text{定理8.7.4. } (\partial^c\partial_c - m^2)\overbrace{\frac{1}{4}\delta_{\alpha\beta}trX(\vec{r}, t)}^{\text{自旋-0粒子}} = 0$$

$$\overbrace{\frac{1}{4}\delta_{\alpha\beta}trX(\vec{r}, t)}^{\text{自旋-0粒子}} = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2E}} \varepsilon_{\alpha\beta}(\vec{p}, 0; 0)[a(\vec{p}, 0; 0)e^{ip\cdot x} + b^+(\vec{p}, 0; 0)e^{-ip\cdot x}]d^3\vec{p}$$

### 8.8 三阶矩阵的有质量双纠缠光子方程

定理8.8.1.  $(\partial^c\partial_c - m^2)X_{\alpha\beta}(\vec{r}, t) = 0$

$$\begin{cases} X_{\alpha\beta}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \frac{1}{\sqrt{2^2E}} \varepsilon_{\alpha\beta}(\vec{p}, h; 2)[a(\vec{p}, h; 2)e^{ip\cdot x} + b^+(\vec{p}, h; 2)e^{-ip\cdot x}]d^3\vec{p} \\ + \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2E}} \varepsilon_{\alpha\beta}(\vec{p}, h; 1)[a(\vec{p}, h; 1)e^{ip\cdot x} + b^+(\vec{p}, h; 1)e^{-ip\cdot x}]d^3\vec{p} \\ + \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2E}} \varepsilon_{\alpha\beta}(\vec{p}, 0; 0)[a(\vec{p}, 0; 0)e^{ip\cdot x} + b^+(\vec{p}, 0; 0)e^{-ip\cdot x}]d^3\vec{p} \end{cases}$$

推论8.8.1.  $\varepsilon_{\alpha\beta}(\vec{p}, h; 2) = \sum_{h'=1}^{-1} \frac{\sqrt{C_{2+h}^{1+h'}C_{2-h}^{1-h'}}}{\sqrt{C_4^2}} \varepsilon_{\alpha}(\vec{p}, h-h')\varepsilon_{\beta}(\vec{p}, h') =$

$$\begin{cases} \varepsilon_{\alpha\beta}(\vec{p}, 2; 2) = \varepsilon_{\alpha}(\vec{p}, 1)\varepsilon_{\beta}(\vec{p}, 1) \\ \varepsilon_{\alpha\beta}(\vec{p}, 1; 2) = \frac{1}{\sqrt{2}}[\varepsilon_{\alpha}(\vec{p}, 1; 2)\varepsilon_{\beta}(\vec{p}, 0) + \varepsilon_{\alpha}(\vec{p}, 0)\varepsilon_{\beta}(\vec{p}, 1)] \\ \varepsilon_{\alpha\beta}(\vec{p}, 0; 2) = \frac{1}{\sqrt{6}}[\varepsilon_{\alpha}(\vec{p}, 1)\varepsilon_{\beta}(\vec{p}, -1) + \varepsilon_{\alpha}(\vec{p}, -1)\varepsilon_{\beta}(\vec{p}, 1) + 2\varepsilon_{\alpha}(\vec{p}, 0)\varepsilon_{\beta}(\vec{p}, 0)] \\ \varepsilon_{\alpha\beta}(\vec{p}, -1; 2) = \frac{1}{\sqrt{2}}[\varepsilon_{\alpha}(\vec{p}, -1)\varepsilon_{\beta}(\vec{p}, 0) + \varepsilon_{\alpha}(\vec{p}, 0)\varepsilon_{\beta}(\vec{p}, -1)] \\ \varepsilon_{\alpha\beta}(\vec{p}, -2; 2) = \varepsilon_{\alpha}(\vec{p}, -1)\varepsilon_{\beta}(\vec{p}, -1) \end{cases}$$

推论8.8.2.  $\varepsilon_{\alpha\beta}(\vec{p}, h; 1) =$

$$\begin{cases} \varepsilon_{\alpha\beta}(\vec{p}, 1; 1) = \frac{1}{\sqrt{2}}[\varepsilon_{\alpha}(\vec{p}, 1)\varepsilon_{\beta}(\vec{p}, 0) - \varepsilon_{\alpha}(\vec{p}, 0)\varepsilon_{\beta}(\vec{p}, 1)] \\ \varepsilon_{\alpha\beta}(\vec{p}, 0; 1) = \frac{1}{\sqrt{2}}[\varepsilon_{\alpha}(\vec{p}, 1)\varepsilon_{\beta}(\vec{p}, -1) - \varepsilon_{\alpha}(\vec{p}, -1)\varepsilon_{\beta}(\vec{p}, 1)] \\ \varepsilon_{\alpha\beta}(\vec{p}, -1; 1) = \frac{1}{\sqrt{2}}[\varepsilon_{\alpha}(\vec{p}, -1)\varepsilon_{\beta}(\vec{p}, 0) - \varepsilon_{\alpha}(\vec{p}, 0)\varepsilon_{\beta}(\vec{p}, -1)] \end{cases}$$

推论8.8.3.  $\varepsilon_{\alpha\beta}(\vec{p}, 0; 0) = \frac{1}{\sqrt{3}}[\varepsilon_{\alpha}(\vec{p}, 1)\varepsilon_{\beta}(\vec{p}, -1) + \varepsilon_{\alpha}(\vec{p}, -1)\varepsilon_{\beta}(\vec{p}, 1) - \varepsilon_{\alpha}(\vec{p}, 0)\varepsilon_{\beta}(\vec{p}, 0)] = \frac{1}{\sqrt{3}}\delta_{\alpha\beta}$

定理8.8.2.

$$\begin{cases} \hat{J}^2(\vec{p}, 2; \gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p}, h; 2) = 2(2+1)\varepsilon_{\alpha\otimes\beta}(\vec{p}, h; 2), \hat{J}_z(\vec{p}, 2; \gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p}, h; 2) = h\varepsilon_{\alpha\otimes\beta}(\vec{p}, h; 2), 2 \leq h \leq 2 \\ \hat{J}^2(\vec{p}, 2; \gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p}, h; 1) = 1(1+1)\varepsilon_{\alpha\otimes\beta}(\vec{p}, h; 1), \hat{J}_z(\vec{p}, 2; \gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p}, h; 1) = h\varepsilon_{\alpha\otimes\beta}(\vec{p}, h; 1), 1 \leq h \leq 1 \\ \hat{J}^2(\vec{p}, 2; \gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p}, 0; 0) = 0(0+1)\varepsilon_{\alpha\otimes\beta}(\vec{p}, 0; 0), \hat{J}_z(\vec{p}, 2; \gamma)\varepsilon_{\alpha\otimes\beta}(\vec{p}, 0; 0) = 0\varepsilon_{\alpha\otimes\beta}(\vec{p}, 0; 0) \end{cases}$$

## 8.9 四阶矩阵的双引力微子纠缠方程

定理8.9.1.  $(\partial^c\partial_c - m^2)X_{kl}(\vec{r}, t) = 0$

$$\begin{cases} X_{kl}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=3}^{-3} \frac{1}{\sqrt{2^3 E}} \lambda_{kl}(\vec{p}, h; 3) [a(\vec{p}, h; 3)e^{ip \cdot x} + b^+(\vec{p}, h; 3)e^{-ip \cdot x}] d^3\vec{p} \\ + \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2}^{-2} \frac{1}{\sqrt{2^2 E}} \lambda_{kl}(\vec{p}, h; 2) [a(\vec{p}, h; 2)e^{ip \cdot x} + b^+(\vec{p}, h; 2)e^{-ip \cdot x}] d^3\vec{p} \\ + \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=1}^{-1} \frac{1}{\sqrt{2^1 E}} \lambda_{kl}(\vec{p}, h; 1) [a(\vec{p}, h; 1)e^{ip \cdot x} + b^+(\vec{p}, h; 1)e^{-ip \cdot x}] d^3\vec{p} \\ + \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2^0 E}} \lambda_{kl}(\vec{p}, 0; 0) [a(\vec{p}, 0; 0)e^{ip \cdot x} + b^+(\vec{p}, 0; 0)e^{-ip \cdot x}] d^3\vec{p} \end{cases}$$

## 8.10 $2s+1$ 阶矩阵的双 $s$ -自旋粒子纠缠方程

定理8.10.1.  $(\partial^c\partial_c - m^2)X_{kk'}(\vec{r}, t) = 0, (2s+1)^2 = (4s+1) + (4s-1) + \dots + 3 + 1$

$$\begin{cases} X_{kk'}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \sum_{n=1}^{2s} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2n+1}^{-(2n+1)} \frac{1}{\sqrt{2^n E}} \lambda_{kk'}(\vec{p}, h; n) [a(\vec{p}, h; n)e^{ip \cdot x} + b^+(\vec{p}, h; n)e^{-ip \cdot x}] d^3\vec{p} \\ + \frac{1}{(2\pi)^{3/2}} \int_{\vec{p}=-\infty}^{+\infty} \frac{1}{\sqrt{2^0 E}} \lambda_{kk'}(\vec{p}, 0; 0) [a(\vec{p}, 0; 0)e^{ip \cdot x} + b^+(\vec{p}, 0; 0)e^{-ip \cdot x}] d^3\vec{p} \end{cases}$$

## 8.11 $(2s_1+1) \times (2s_2+1)$ 矩阵的 $s_1 \otimes s_2$ 纠缠方程猜想

定理8.11.1.  $(\partial^c\partial_c - m^2)X_{kk'}(\vec{r}, t) = 0, (s_1+s_2+1)^2 = [2(s_1+s_2)+1] + [2(s_1+s_2)-1] + \dots + [2|s_1-s_2|+1]$

$$X_{kk'}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \sum_{s=2|s_1-s_2|+1}^{2(s_1+s_2)+1} \int_{\vec{p}=-\infty}^{+\infty} \sum_{h=2s+1}^{-(2s+1)} \frac{m^{s-[s]}}{\sqrt{2^{[s]} E}} \lambda_{kk'}(\vec{p}, h; s) [a(\vec{p}, h; s)e^{ip \cdot x} + b^+(\vec{p}, h; s)e^{-ip \cdot x}] d^3\vec{p}$$

# 第四十五章 相对论性玻色弦平面波解

## 1 数学准备

### 1.1 $\delta(x)$ 函数的性质

性质1.1.1.  $\int_{x=-\infty}^{+\infty} x\delta'(x)dx = -\int_{x=-\infty}^{+\infty} \delta(x)dx$

性质1.1.2.  $x\delta'(x) = -\delta(x), x^n\delta^{(n)}(x) = (-1)^n n!\delta(x), x\delta^{(n)}(x) = -n\delta^{(n-1)}(x)$

性质1.1.3.  $x\delta(x) = 0, x^2\delta'(x) = 0, x^n\delta^{(n-1)}(x) = 0$

性质1.1.4.  $\int_{k=0-}^{0+} \delta(k)e^{ikx}dx = 1, \int_{k=0-}^{0+} \delta'(k)e^{ikx}dx = -ix, \int_{k=0-}^{0+} \delta^{(n)}(k)e^{ikx}dx = (-ix)^n$

### 1.2 代数方程的 $\delta(x)$ 函数解

性质1.2.1.  $xf(x) = 0 \Leftrightarrow f(x) = c\delta(x)$

性质1.2.2.  $\psi(k, E)(E^2 - k^2) = 0 \Leftrightarrow \psi(k, E) = a(k, E)\delta(E^2 - k^2)$

性质1.2.3.  $\psi(k, E)(E^2 - k^2) = 0 \Leftrightarrow \psi(k, E) = c_{00}\delta(E)\delta(k) + c_{01}\delta(E)\delta'(k) + c_{10}\delta'(E)\delta(k) + c_{11}\delta'(E)\delta'(k)$

性质1.2.4.  $\psi(k, E)(E^2 - k^2) = 0 \Leftrightarrow \psi(k, E) = \sum_{n=0}^{\infty} [c_{1n}\delta^{(n)}(E+k)\delta(E-k) + c_{2n}\delta^{(n)}(E-k)\delta(E+k)]$

性质1.2.5.  $\psi(0, E)(E^2) = 0 \Leftrightarrow \psi(0, E) = c\delta(E^2)$

## 2 不同边界条件玻色弦方程的平面波解 [44, 46]

### 2.1 玻色闭弦方程波函数展开(自由闭弦)

定理2.1.1.  $X^u(\tau, \sigma) = \int_{k=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} X^u(k, E)e^{i(k\sigma-E\tau)}dkdE, X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi)$

$\Leftrightarrow X^u(\tau, \sigma) = \int_{E=-\infty}^{+\infty} \sum_{n=-\infty}^{\infty} \phi^u(n, E)e^{i(n\sigma-E\tau)}dE$

证明:  $X^u(\tau, \sigma) = \int_{k=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} X^u(k, E)e^{i(k\sigma-E\tau)}dkdE, X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi)$

$\Leftrightarrow \int_{k=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} X^u(k, E)(e^{i2\pi k} - 1)e^{i(k\sigma-E\tau)}dkdE = 0$

$\Leftrightarrow X^u(k, E)(e^{i2\pi k} - 1) = 0$

$\Leftrightarrow X^u(k, E) = \phi^u(k, E)\delta(e^{i2\pi k} - 1)$

$\Leftrightarrow X^u(k, E) = \sum_{n=-\infty}^{\infty} \phi^u(n, E)\delta(k - n)$

$\Leftrightarrow X^u(\tau, \sigma) = \int_{k=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \sum_{n=-\infty}^{\infty} \phi^u(n, E)\delta(k - n)e^{i(k\sigma-E\tau)}dkdE$

$\Leftrightarrow X^u(\tau, \sigma) = \int_{E=-\infty}^{+\infty} \sum_{n=-\infty}^{\infty} \phi^u(n, E)e^{i(n\sigma-E\tau)}dE$

□

评述: 傅立叶展开和傅立叶变换可以看作恒等式。

定理2.1.2.  $\partial_+\partial_-X^u(\tau, \sigma) = 0, X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi)$

$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}]$

证明:  $\partial_+ \partial_- X^u(\tau, \sigma) = 0, X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi)$

$$\Leftrightarrow \int_{E=-\infty}^{+\infty} \sum_{n=-\infty}^{\infty} \phi^u(n, E)(E^2 - n^2)e^{i(n\sigma - E\tau)} dE = 0$$

$$\Leftrightarrow \phi^u(n, E)(E^2 - n^2) = 0$$

$$\Leftrightarrow \phi^u(n, E) = \begin{cases} a^u(n, E)\delta(E^2 - n^2), \delta(E^2 - n^2) = \frac{1}{2|n|}[\delta(E - n) + \delta(E + n)], n \neq 0 \\ a^u(0, 0)c\delta(E) - a^u(0, 0)\delta'(E), n = 0 \end{cases}$$

$$\Leftrightarrow X^u(\tau, \sigma) = a^u(0, 0)c + ia^u(0, 0)\tau + \sum_{n=-\infty}^{\infty} \frac{1}{2|n|} [a^u(n, n)e^{in(\sigma - \tau)} + a^u(n, -n)e^{in(\sigma + \tau)}]$$

$$\Leftrightarrow X^u(\tau, \sigma) = a^u(0, 0)c + ia^u(0, 0)\tau + \sum_{n=-\infty}^{\infty} \frac{1}{2|n|} [a^u(n, n)e^{-in(\tau - \sigma)} + a^u(-n, n)e^{-in(\sigma + \tau)}]$$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}]$$

$$x^u = a^u(0, 0)c, p^u = i2\pi T a^u(0, 0), \alpha_n^u = -i\sqrt{\pi T} \frac{n}{|n|} a^u(n, n), \bar{\alpha}_n^u = -i\sqrt{\pi T} \frac{n}{|n|} a^u(-n, n) \quad \square$$

定理2.1.3.  $\partial_+ \partial_- X^u(\tau, \sigma) = 0, X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi), X^u(\tau, \sigma) = X^{*u}(\tau, \sigma)$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], \alpha_{-n}^u = \alpha_n^{*u}, \bar{\alpha}_{-n}^u = \bar{\alpha}_n^{*u}$$

评述: 玻色闭弦方程波函数展开是整个玻色弦理论的基础。

## 2.2 玻色N开弦方程波函数展开(对称闭弦)

定理2.2.1.  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], X'^u(\tau, \sigma)|_{\sigma=0, \pi} = 0$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], \alpha_n^u = \bar{\alpha}_n^u$$

证明:  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], X'^u(\tau, \sigma)|_{\sigma=0, \pi} = 0$

$$\Rightarrow \frac{-1}{\sqrt{4\pi T}} \sum_{n \neq 0} [\alpha_n^u e^{-in(\tau - \sigma)} - \bar{\alpha}_n^u e^{-in(\tau + \sigma)}]|_{\sigma=0, \pi} = 0$$

$$\Leftrightarrow \sum_{n \neq 0} [\alpha_n^u e^{-in\tau} - \bar{\alpha}_n^u e^{-in\tau}] = 0, \sum_{n \neq 0} [\alpha_n^u e^{in\pi} e^{-in\tau} - \bar{\alpha}_n^u e^{-in\pi} e^{-in\tau}] = 0,$$

$$\Leftrightarrow \alpha_n^u = \bar{\alpha}_n^u$$

$$\Rightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], \alpha_n^u = \bar{\alpha}_n^u \quad \square$$

定理2.2.2.  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], X^u(\tau, \sigma) = X^u(\tau, -\sigma)$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^u e^{-in\tau} \cos n\sigma, \alpha_n^u = \bar{\alpha}_n^u$$

证明:  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], X^u(\tau, \sigma) = X^u(\tau, -\sigma)$

$$\Rightarrow x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}] = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau + \sigma)} + \bar{\alpha}_n^u e^{-in(\tau - \sigma)}]$$

$$\Leftrightarrow \alpha_n^u = \bar{\alpha}_n^u$$

$$\Rightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], \alpha_n^u = \bar{\alpha}_n^u \quad \square$$

推论2.2.1.  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], X'^u(\tau, \sigma)|_{\sigma=0, \pi} = 0$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], X^u(\tau, \sigma) = X^u(\tau, -\sigma)$$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], \alpha_n^u = \bar{\alpha}_n^u$$

评述: N条件和对称条件等价, 也同时意味着两个分支合二为一, 三者相互等价。

推论2.2.2.  $\partial_+ \partial_- X^u(\tau, \sigma) = 0, X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi), X'^u(\tau, \sigma)|_{\sigma=0, \pi} = 0$

$$\Leftrightarrow \partial_+ \partial_- X^u(\tau, \sigma) = 0, X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi), X^u(\tau, \sigma) = X^u(\tau, -\sigma)$$



### 2.3 玻色D开弦方程波函数展开(反对称闭弦)

**定理2.3.1.**  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], \dot{X}^u(\tau, \sigma)|_{\sigma=0, \pi} = 0$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], p^u = 0, \alpha_n^u = -\bar{\alpha}_n^u$$

**证明:**  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], \dot{X}^u(\tau, \sigma)|_{\sigma=0, \pi} = 0$

$$\Rightarrow \frac{p^u}{2\pi T} + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}]|_{\sigma=0, \pi} = 0$$

$$\Leftrightarrow \frac{p^u}{2\pi T} + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} [\alpha_n^u e^{-in\tau} + \bar{\alpha}_n^u e^{-in\tau}] = 0, \frac{p^u}{2\pi T} + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} [\alpha_n^u e^{in\pi} e^{-in\tau} + \bar{\alpha}_n^u e^{-in\pi} e^{-in\tau}] = 0,$$

$$\Leftrightarrow p^u = 0, \alpha_n^u = -\bar{\alpha}_n^u$$

$$\Rightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], p^u = 0, \alpha_n^u = -\bar{\alpha}_n^u \quad \square$$

**定理2.3.2.**  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], X^u(\tau, \sigma) - x^u = -[X^u(\tau, -\sigma) - x^u]$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], p^u = 0, \alpha_n^u = -\bar{\alpha}_n^u$$

**证明:**  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], X^u(\tau, \sigma) - x^u = -[X^u(\tau, -\sigma) - x^u]$

$$\Rightarrow \frac{p^u}{\pi T} + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} (\alpha_n^u + \bar{\alpha}_n^u) [e^{-in(\tau-\sigma)} + e^{-in(\tau+\sigma)}] = 0$$

$$\Leftrightarrow p^u = 0, \alpha_n^u = -\bar{\alpha}_n^u$$

$$\Rightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], p^u = 0, \alpha_n^u = -\bar{\alpha}_n^u \quad \square$$

**推论2.3.1.**  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], \dot{X}^u(\tau, \sigma)|_{\sigma=0, \pi} = 0$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], X^u(\tau, \sigma) - x^u = -[X^u(\tau, -\sigma) - x^u]$$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], p^u = 0, \alpha_n^u = -\bar{\alpha}_n^u$$

**评述:** D条件和反对称条件等价, 也同时意味着两个分支合二为一, 三者相互等价。

**推论2.3.2.**  $\partial_+ \partial_- X^u(\tau, \sigma) = 0, X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi), \dot{X}^u(\tau, \sigma)|_{\sigma=0, \pi} = 0$

$$\Leftrightarrow \partial_+ \partial_- X^u(\tau, \sigma) = 0, X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi), X^u(\tau, \sigma) - x^u = -[X^u(\tau, -\sigma) - x^u]$$

### 2.4 玻色混合条件开弦方程波函数展开

**定理2.4.1.**

$$X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], [X'^u(\tau, \sigma) \cos\theta + \dot{X}^u(\tau, \sigma) \sin\theta]|_{\sigma=0, \pi} = 0$$

$$\Leftrightarrow X^u(\tau, \sigma, \theta) = x^u + \frac{p^u}{2\pi T}\tau \delta_{\sin\theta, 0} + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^u [e^{-in(\tau-\sigma)} + \frac{1-tg\theta}{1+tg\theta} e^{-in(\tau+\sigma)}]$$

**证明:**

$$X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], [X'^u(\tau, \sigma) \cos\theta + \dot{X}^u(\tau, \sigma) \sin\theta]|_{\sigma=0, \pi} = 0$$

$$\Rightarrow \frac{p^u}{2\pi T} \sin\theta + \frac{-1}{\sqrt{4\pi T}} \cos\theta \sum_{n \neq 0} [\alpha_n^u e^{-in(\tau-\sigma)} - \bar{\alpha}_n^u e^{-in(\tau+\sigma)}] + \frac{1}{\sqrt{4\pi T}} \sin\theta \sum_{n \neq 0} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}]|_{\sigma=0, \pi} = 0$$

$$\Rightarrow \frac{p^u}{2\pi T} \sin\theta + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} [(-\cos\theta + \sin\theta) \alpha_n^u e^{-in\tau} + (\cos\theta + \sin\theta) \bar{\alpha}_n^u e^{-in\tau}] = 0$$

$$, \frac{p^u}{2\pi T} \sin\theta + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} (-1)^n [(-\cos\theta + \sin\theta) \alpha_n^u e^{-in\tau} + (\cos\theta + \sin\theta) \bar{\alpha}_n^u e^{-in\tau}] = 0$$

$$\Leftrightarrow p^u \sin\theta = 0, (\cos\theta - \sin\theta) \alpha_n^u = (\cos\theta + \sin\theta) \bar{\alpha}_n^u$$

$$\Rightarrow X^u(\tau, \sigma, \theta) = x^u + \frac{p^u}{2\pi T}\tau \delta_{\sin\theta, 0} + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^u [e^{-in(\tau-\sigma)} + \frac{1-tg\theta}{1+tg\theta} e^{-in(\tau+\sigma)}] \quad \square$$

**定理2.4.2.**  $X^u(\tau, \sigma, \theta) = x^u + \frac{p^u}{2\pi T}\tau \delta_{\sin\theta, 0} + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^u [e^{-in(\tau-\sigma)} + \frac{1-tg\theta}{1+tg\theta} e^{-in(\tau+\sigma)}]$

$$\Rightarrow [X^u(\tau, \sigma, \theta) - x^u](\cos\theta + \sin\theta) = [X^u(\tau, -\sigma, -\theta) - x^u](\cos\theta - \sin\theta)$$

**评述:** 纯左移动解:  $X^u(\tau, \sigma, \frac{\pi}{4})$ , 纯右移动解:  $X^u(\tau, \sigma, -\frac{\pi}{4})$ , N弦解:  $X^u(\tau, \sigma, 0)$ , D弦解:  $X^u(\tau, \sigma, \pm\frac{\pi}{2})$

## 2.5 p-膜

**定理2.5.1.**  $X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}], \dot{X}^u(\tau, \sigma)|_{\sigma=0, \pi} = 0$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u - \frac{1}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^u e^{-in\tau} \sin n\sigma, p^u = 0, \alpha_n^u = -\bar{\alpha}_n^u, u = 1, 2, \dots, p$$

$X^l(\tau, \sigma) = x^l + \frac{p^l}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^l e^{-in(\tau-\sigma)} + \bar{\alpha}_n^l e^{-in(\tau+\sigma)}], X'^l(\tau, \sigma)|_{\sigma=0, \pi} = 0$

$$\Leftrightarrow X^l(\tau, \sigma) = x^l + \frac{p^l}{2\pi T}\tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^l e^{-in\tau} \cos n\sigma, \alpha_n^l = \bar{\alpha}_n^l, l = p+1, p+2, \dots, D$$

评述：p-膜的本质就是有的维度满足N条件，有的维度满足D条件。

## 2.6 一般边界条件的解

评述：更一般的p-膜是有的维度满足N条件，有的维度满足D条件，有的维度满足周期条件。

**定理2.6.1.**  $\partial_+ \partial_- X^u(\tau, \sigma) = 0, \dot{X}^I(\tau, \sigma)|_{\sigma=0, \pi} = 0, X'^J(\tau, \sigma)|_{\sigma=0, \pi} = 0, X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi)$

$$\Leftrightarrow X^I(\tau, \sigma) = x^I - \frac{p^I}{2\pi T}\sigma - \frac{1}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^I e^{-in\tau} \sin n\sigma, \alpha_n^I = -\bar{\alpha}_n^I, I = 1, 2, \dots, p$$

$X^J(\tau, \sigma) = x^J + \frac{p^J}{2\pi T}\tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^J e^{-in\tau} \cos n\sigma, \alpha_n^J = \bar{\alpha}_n^J, J = p+1, p+2, \dots, p+l$

$X^K(\tau, \sigma) = x^K + \frac{p^K}{2\pi T}\tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^K e^{-in(\tau-\sigma)} + \bar{\alpha}_n^K e^{-in(\tau+\sigma)}], K = p+l+1, \dots, D$

**推论2.6.1.**

$$\dot{X}^I(\tau, \sigma) = \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \alpha_n^I e^{-in\tau} \sin n\sigma, p^I = 0, \alpha_n^I = -\bar{\alpha}_n^I, I = 1, 2, \dots, p$$

$$\dot{X}^J(\tau, \sigma) = \frac{p^J}{2\pi T} + \frac{1}{\sqrt{\pi T}} \sum_{n \neq 0} \alpha_n^J e^{-in\tau} \cos n\sigma, \alpha_n^J = \bar{\alpha}_n^J, J = p+1, p+2, \dots, p+l$$

$$\dot{X}^K(\tau, \sigma) = \frac{p^K}{2\pi T} + \frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} [\alpha_n^K e^{-in(\tau-\sigma)} + \bar{\alpha}_n^K e^{-in(\tau+\sigma)}], K = p+l+1, p+l+2, \dots, D$$

**推论2.6.2.**

$$X'^I(\tau, \sigma) = -\frac{p^K}{2\pi T} - \frac{1}{\sqrt{\pi T}} \sum_{n \neq 0} \alpha_n^I e^{-in\tau} \cos n\sigma, p^I = 0, \alpha_n^I = -\bar{\alpha}_n^I, I = 1, 2, \dots, p$$

$$X'^J(\tau, \sigma) = -\frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \alpha_n^J e^{-in\tau} \sin n\sigma, \alpha_n^J = \bar{\alpha}_n^J, J = p+1, p+2, \dots, p+l$$

$$X'^K(\tau, \sigma) = -\frac{1}{\sqrt{4\pi T}} \sum_{n \neq 0} [\alpha_n^K e^{-in(\tau-\sigma)} - \bar{\alpha}_n^K e^{-in(\tau+\sigma)}], K = p+l+1, p+l+2, \dots, D$$

**推论2.6.3.**

$$\dot{X}^I(\tau, \sigma) + X'^I(\tau, \sigma) = \frac{1}{\sqrt{\pi T}} \sum_n \bar{\alpha}_n^I e^{-in(\tau+\sigma)}, \bar{\alpha}_n^I = -\alpha_n^I, I = 1, 2, \dots, p$$

$$\dot{X}^I(\tau, \sigma) - X'^I(\tau, \sigma) = \frac{1}{\sqrt{\pi T}} \sum_n \alpha_n^I e^{-in(\tau-\sigma)}, \alpha_0^I = \frac{p^I}{2\sqrt{\pi T}}, I = 1, 2, \dots, p$$

**推论2.6.4.**

$$\dot{X}^J(\tau, \sigma) + X'^J(\tau, \sigma) = \frac{1}{\sqrt{\pi T}} \sum_n \bar{\alpha}_n^J e^{-in(\tau+\sigma)}, \bar{\alpha}_n^J = \alpha_n^J, J = p+1, p+2, \dots, p+l$$

$$\dot{X}^J(\tau, \sigma) - X'^J(\tau, \sigma) = \frac{1}{\sqrt{\pi T}} \sum_n \alpha_n^J e^{-in(\tau-\sigma)}, \alpha_0^J = \frac{p^J}{2\sqrt{\pi T}}, J = p+1, p+2, \dots, p+l$$

**推论2.6.5.**

$$\dot{X}^K(\tau, \sigma) + X'^K(\tau, \sigma) = \frac{1}{\sqrt{\pi T}} \sum_n \bar{\alpha}_n^K e^{-in(\tau+\sigma)}, \bar{\alpha}_0^K = \frac{p^K}{2\sqrt{\pi T}}, K = p+l+1, \dots, D$$

$$\dot{X}^K(\tau, \sigma) - X'^K(\tau, \sigma) = \frac{1}{\sqrt{\pi T}} \sum_n \alpha_n^K e^{-in(\tau-\sigma)}, \alpha_0^K = \frac{p^K}{2\sqrt{\pi T}}, K = p+l+1, \dots, D$$

**推论2.6.6.**  $[\dot{X}^u(\tau, \sigma) + X'^u(\tau, \sigma)][\dot{X}_u(\tau, \sigma) + X'_u(\tau, \sigma)] = \frac{1}{\pi T} \sum_u \sum_n \bar{\alpha}_n^u e^{-in(\tau+\sigma)} \sum_m \bar{\alpha}_m^u e^{-im(\tau+\sigma)}$

$$= \frac{1}{\pi T} \sum_m \sum_u \sum_n \bar{\alpha}_n^u \bar{\alpha}_{m-n}^u e^{-im(\tau+\sigma)} = \frac{1}{\pi T} \sum_m \bar{L}_m e^{-im(\tau+\sigma)}, \bar{L}_m = \sum_u \sum_n \bar{\alpha}_n^u \bar{\alpha}_{m-n}^u$$

**推论2.6.7.**  $[\dot{X}^u(\tau, \sigma) - X'^u(\tau, \sigma)][\dot{X}_u(\tau, \sigma) - X'_u(\tau, \sigma)] = \frac{1}{\pi T} \sum_u \sum_n \alpha_n^u e^{-in(\tau-\sigma)} \sum_m \alpha_m^u e^{-im(\tau-\sigma)}$

$$= \frac{1}{\pi T} \sum_m \sum_u \sum_n \alpha_n^u \alpha_{m-n}^u e^{-im(\tau-\sigma)} = \frac{1}{\pi T} \sum_m L_m e^{-im(\tau-\sigma)}, L_m = \sum_u \sum_n \alpha_n^u \alpha_{m-n}^u$$

## 2.7 玻色开弦方程的严格简单解法

定理2.7.1.  $\partial_+ \partial_- X(\tau, \sigma) = 0 \Leftrightarrow X(\tau, \sigma) = f(\tau + \sigma) + g(\tau - \sigma)$

证明:  $\partial_+ \partial_- X(\tau, \sigma) = 0 \Leftrightarrow \partial_- X(\tau, \sigma) = h(\tau - \sigma) \Leftrightarrow X(\tau, \sigma) = \int h(\tau - \sigma) d(\tau - \sigma) + f(\tau + \sigma)$   
 $\Leftrightarrow X(\tau, \sigma) = f(\tau + \sigma) + g(\tau - \sigma)$  □

推论2.7.1.  $\partial_+ \partial_- X(\tau, \sigma) = 0, \partial_\sigma X(\tau, \sigma)|_{\sigma=0, \pi} = 0 \Leftrightarrow X(\tau, \sigma) = f(\tau + \sigma) + f(\tau - \sigma), f'(x - \pi) = f'(x + \pi)$

推论2.7.2.  $\partial_+ \partial_- X(\tau, \sigma) = 0, \partial_\tau X(\tau, \sigma)|_{\sigma=0, \pi} = 0 \Leftrightarrow X(\tau, \sigma) = f(\tau + \sigma) - f(\tau - \sigma), f'(x - \pi) = f'(x + \pi)$

## 3 玻色弦方程的平面波解完备性疑惑

### 3.1 对比研究一玻色闭弦方程波函数第二种展开方式

定理3.1.1.  $\partial_+ \partial_- X^u(\tau, \sigma) = 0 \Leftrightarrow \phi^u(\tau, \sigma) = \int_{k=-\infty}^{+\infty} \frac{1}{2\omega} [a^u(k, \omega) e^{i(k\sigma - \omega\tau)} + a^u(-k, -\omega) e^{-i(k\sigma - \omega\tau)}] dk$

证明:  $\partial_+ \partial_- X^u(\tau, \sigma) = 0 \Leftrightarrow \int_{k=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} X^u(k, E) (E^2 - k^2) e^{i(k\sigma - E\tau)} dk dE = 0$

$\Leftrightarrow X^u(k, E) (E^2 - k^2) = 0$

$\Leftrightarrow \begin{cases} X^u(k, E)|_{(k, E)=0} = C^u \delta(k) \delta(E) + C_E^u \dot{\delta}(E) \delta(k) + C_k^u \delta(E) \delta'(k) + C_{Ek}^u \dot{\delta}(E) \delta'(k) \\ X^u(k, E)|_{(k, E) \neq 0} = a^u(k, E) \delta(E^2 - k^2) \end{cases}$

$\Leftrightarrow X^u(\tau, \sigma) = \int_{k=0_-}^{0_+} \int_{E=0_-}^{0_+} [C^u \delta(\tau) \delta(\sigma) + C_\tau^u \dot{\delta}(\tau) \delta(\sigma) + C_\sigma^u \delta(\tau) \delta'(\sigma)] e^{i(k\sigma - E\tau)} dk dE$

$+ \int_{k=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} a^u(k, E) \delta(E^2 - k^2) e^{i(k\sigma - E\tau)} dk dE|_{(k, E) \neq 0}$

$\Leftrightarrow X^u(\tau, \sigma) = (C^u + iC_\tau^u \tau - iC_\sigma^u \sigma + C_{\tau\sigma}^u \tau \sigma)$

$+ \int_{k=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \frac{1}{2\omega} a^u(k, E) [\delta(E - \omega) + \delta(E + \omega)] e^{i(k\sigma - E\tau)} dk dE|_{(k, E) \neq 0, \omega = |k|}$

$\Leftrightarrow X^u(\tau, \sigma) = (C^u + iC_\tau^u \tau - iC_\sigma^u \sigma + C_{\tau\sigma}^u \tau \sigma) + \int_{k=-\infty}^{+\infty} \frac{1}{2\omega} [a^u(k, \omega) e^{i(k\sigma - \omega\tau)} + a^u(k, -\omega) e^{i(k\sigma + \omega\tau)}] dk|_{k \neq 0}$

$\Leftrightarrow X^u(\tau, \sigma) = (C^u + iC_\tau^u \tau - iC_\sigma^u \sigma + C_{\tau\sigma}^u \tau \sigma) + \int_{k=-\infty}^{+\infty} \frac{1}{2\omega} [a^u(k, \omega) e^{i(k\sigma - \omega\tau)} + a^u(-k, -\omega) e^{-i(k\sigma - \omega\tau)}] dk|_{k \neq 0}$  □

定理3.1.2.  $\partial_+ \partial_- X^u(\tau, \sigma) = 0, X^u(\tau, \sigma) = X^{*u}(\tau, \sigma)$

$\Leftrightarrow X^u(\tau, \sigma) = (c^u + c_\tau^u \tau + c_\sigma^u \sigma + c_{\tau\sigma}^u \tau \sigma) + \int_{k=-\infty}^{+\infty} \frac{1}{2\omega} [a^u(k, \omega) e^{i(k\sigma - \omega\tau)} + a^{*u}(k, \omega) e^{-i(k\sigma - \omega\tau)}] dk|_{k \neq 0}$

定理3.1.3.  $\partial_+ \partial_- X^u(\tau, \sigma) = 0, X^u(\tau, \sigma) = X^{*u}(\tau, \sigma), X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi)$

$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T} \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau - \sigma)} + \bar{\alpha}_n^u e^{-in(\tau + \sigma)}], \alpha_{-n}^u = \alpha_n^{*u}, \bar{\alpha}_{-n}^u = \bar{\alpha}_n^{*u}$

证明:  $\partial_+ \partial_- X^u(\tau, \sigma) = 0, X^u(\tau, \sigma) = X^{*u}(\tau, \sigma), X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi)$

$\Leftrightarrow X^u(\tau, \sigma) = (c^u + c_\tau^u \tau + c_\sigma^u \sigma + c_{\tau\sigma}^u \tau \sigma) + \int_{k=-\infty}^{+\infty} \frac{1}{2\omega} [a^u(k, \omega) e^{i(k\sigma - \omega\tau)} + a^{*u}(k, \omega) e^{-i(k\sigma - \omega\tau)}] dk|_{k \neq 0}$

$X^u(\tau, \sigma) = X^u(\tau, \sigma + 2\pi)$

$\Rightarrow a^u(k, \omega) (e^{i2\pi k} - 1) = 0, a^{*u}(k, \omega) (e^{-i2\pi k} - 1) = 0, c_\sigma^u = 0, c_{\tau\sigma}^u = 0$

$\Leftrightarrow a^u(k, \omega) = \alpha^u(k) \delta(e^{i2\pi k} - 1), c_\sigma^u = 0, c_{\tau\sigma}^u = 0$

$\Rightarrow X^u(\tau, \sigma) = c^u + c_\tau^u \tau + \int_{k=-\infty}^{+\infty} \frac{1}{2\omega} [\alpha^u(k) \delta(e^{i2\pi k} - 1) e^{i(k\sigma - \omega\tau)} + \alpha^{*u}(k) \delta(e^{-i2\pi k} - 1) e^{-i(k\sigma - \omega\tau)}] dk|_{k \neq 0}$

$\Leftrightarrow X^u(\tau, \sigma) = c^u + c_\tau^u \tau + \int_{k=-\infty}^{+\infty} \frac{1}{2\omega} [\alpha^u(k) \sum_{n=-\infty}^{\infty} \delta(k - n) e^{i(k\sigma - \omega\tau)} + \alpha^{*u}(k) \sum_{n=-\infty}^{\infty} \delta(k - n) e^{-i(k\sigma - \omega\tau)}] dk|_{k \neq 0}$

$\Leftrightarrow X^u(\tau, \sigma) = c^u + c_\tau^u \tau + \sum_{n \neq 0} \frac{1}{2|n|} [\alpha^u(n) e^{i(n\sigma - |n|\tau)} + \alpha^{*u}(n) e^{-i(n\sigma - |n|\tau)}]$

$$\Leftrightarrow X^u(\tau, \sigma) = c^u + c_\tau^u \tau + \sum_{n=1}^{\infty} \frac{1}{2n} [\alpha^u(n) e^{in(\sigma-\tau)} + \alpha^{*u}(n) e^{-in(\sigma-\tau)}]$$

$$+ \sum_{n=-\infty}^{-1} -\frac{1}{2n} [\alpha^u(n) e^{in(\tau+\sigma)} + \alpha^{*u}(n) e^{-in(\tau+\sigma)}]$$

$$\Leftrightarrow X^u(\tau, \sigma) = c^u + c_\tau^u \tau + \sum_{n \neq 0} \frac{1}{2n} \alpha_n^u e^{-in(\tau-\sigma)} + \sum_{n \neq 0} \frac{1}{2n} \bar{\alpha}_n^u e^{-in(\tau+\sigma)}$$

$$\alpha_n^u = -\alpha_{-n}^{*u} = \begin{cases} \alpha^u(n), n > 0 \\ -\alpha^{*u}(-n), n < 0 \end{cases}, \bar{\alpha}_n^u = -\bar{\alpha}_{-n}^{*u} = \begin{cases} \alpha^u(-n), n > 0 \\ -\alpha^{*u}(n), n < 0 \end{cases}$$

$$\Leftrightarrow X^u(\tau, \sigma) = x^u + \frac{p^u}{2\pi T} \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} [\alpha_n^u e^{-in(\tau-\sigma)} + \bar{\alpha}_n^u e^{-in(\tau+\sigma)}]$$

$$c_\tau^u = \frac{p^u}{2\pi T}, \alpha_n^u = -i\sqrt{\pi T} \alpha_n^u, \bar{\alpha}_n^u = -i\sqrt{\pi T} \bar{\alpha}_n^u, \alpha_{-n}^u = \alpha_n^{*u}, \bar{\alpha}_{-n}^u = \bar{\alpha}_n^{*u}$$

□

评述：改变波函数的限制条件顺序，导致需要不同的数学技巧，但结论应是一样的。

### 3.2 对比研究一玻色闭弦方程波函数第三种展开方式

$$\text{定理3.2.1. } \partial_+ \partial_- X^u(\tau, \sigma) = 0 \Leftrightarrow X^u(\tau, \sigma) = \int_{k=-\infty}^{+\infty} \frac{1}{2\omega} [a^u(k, \omega) e^{i(k\sigma - \omega\tau)} + a^u(-k, -\omega) e^{-i(k\sigma - \omega\tau)}] dk$$

$$\text{证明: } \partial_+ \partial_- X^u(\tau, \sigma) = 0 \Leftrightarrow \int_{E=-\infty}^{+\infty} X^u(E, \sigma) e^{-iE\tau} dE = 0$$

$$\Leftrightarrow (\partial_\sigma^2 + E^2) X^u(E, \sigma) = 0$$

$$\Leftrightarrow X^u(E, \sigma) = c_{00}^u \delta(E) + c_{01}^u \delta'(E) + c_{10}^u \sigma \delta(E) + c_{11}^u \sigma \delta'(E) + a^u(E) e^{iE\sigma} + b^u(E) e^{-iE\sigma}$$

$$\Leftrightarrow X^u(\tau, \sigma) = c_{00} + c_{01} \tau + c_{10} \sigma + c_{11} \sigma \tau + \int_{E=-\infty}^{+\infty} [a^u(E) e^{-iE(\tau-\sigma)} + b^u(E) e^{-iE(\tau+\sigma)}] dE$$

□

评述：改变波函数的限制条件顺序，导致需要不同的数学技巧，但结论仍是一样的。

### 3.3 性质

$$\text{定理3.3.1. } \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} [a(\vec{k}, t) e^{i\vec{k}\cdot\vec{r}} + a^*(\vec{k}, t) e^{-i\vec{k}\cdot\vec{r}}] d^3 \vec{k}$$

$$\nabla \phi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} i\vec{k} [a(\vec{k}, t) e^{i\vec{k}\cdot\vec{r}} - a^*(\vec{k}, t) e^{-i\vec{k}\cdot\vec{r}}] d^3 \vec{k}$$

$$\dot{\phi}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{\vec{k}=-\infty}^{+\infty} [\dot{a}(\vec{k}, t) e^{i\vec{k}\cdot\vec{r}} + \dot{a}^*(\vec{k}, t) e^{-i\vec{k}\cdot\vec{r}}] d^3 \vec{k}$$

$$\text{定理3.3.2. } \phi(\vec{r}, t)^2 = \int_{\vec{k}=-\infty}^{+\infty} [a(\vec{k}, t) a(-\vec{k}, t) + a^*(\vec{k}, t) a^*(-\vec{k}, t) + 2a(\vec{k}, t) a^*(\vec{k}, t)] d^3 \vec{k}$$

$$\nabla \phi(\vec{r}, t) \cdot \nabla \phi(\vec{r}, t) = \int_{\vec{k}=-\infty}^{+\infty} \vec{k}^2 [a(\vec{k}, t) a(-\vec{k}, t) + a^*(\vec{k}, t) a^*(-\vec{k}, t) + 2a(\vec{k}, t) a^*(\vec{k}, t)] d^3 \vec{k}$$

$$\dot{\phi}(\vec{r}, t)^2 = \int_{\vec{k}=-\infty}^{+\infty} [\dot{a}(\vec{k}, t) \dot{a}(-\vec{k}, t) + \dot{a}^*(\vec{k}, t) \dot{a}^*(-\vec{k}, t) + 2\dot{a}(\vec{k}, t) \dot{a}^*(\vec{k}, t)] d^3 \vec{k}$$

#### 定理3.3.3.

$$\int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \phi(\vec{r}, t)^2 d^3 \vec{r} dt = \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} a(\vec{k}, E) a(-\vec{k}, -E) d^3 \vec{k} dE$$

$$\int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \nabla \phi(\vec{r}, t) \cdot \nabla \phi(\vec{r}, t) d^3 \vec{r} dt = \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \vec{k}^2 a(\vec{k}, E) a(-\vec{k}, -E) d^3 \vec{k} dE$$

$$\int_{\vec{r}=-\infty}^{+\infty} \int_{t=-\infty}^{+\infty} \dot{\phi}(\vec{r}, t)^2 d^3 \vec{r} dt = \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} E^2 a(\vec{k}, E) a(-\vec{k}, -E) d^3 \vec{k} dE$$

$$\phi(\vec{r}, t) = \frac{1}{(2\pi)^2} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} a(\vec{k}, E) e^{i(\vec{k}\cdot\vec{r} - Et)} d^3 \vec{k} dE$$

$$S = \frac{1}{2} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} (E^2 - \vec{k}^2 - m^2) a(\vec{k}, E) a(-\vec{k}, -E) d^3 \vec{k} dE \Rightarrow (E^2 - \vec{k}^2 - m^2) a(\vec{k}, E) = 0$$

**定理3.3.4.**

$$\begin{aligned}
\int_{\vec{r}=-\infty}^{+\infty} \phi(\vec{r}, t)^2 d^3\vec{r} &= \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \int_{E'=-\infty}^{+\infty} a(\vec{k}, E) a(-\vec{k}, E') e^{-i(E+E')t} d^3\vec{k} dE dE' \\
\int_{\vec{r}=-\infty}^{+\infty} \nabla\phi(\vec{r}, t) \cdot \nabla\phi(\vec{r}, t) d^3\vec{r} &= \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \int_{E'=-\infty}^{+\infty} \vec{k}^2 a(\vec{k}, E) a(-\vec{k}, E') e^{-i(E+E')t} d^3\vec{k} dE dE' \\
\int_{\vec{r}=-\infty}^{+\infty} \dot{\phi}(\vec{r}, t)^2 d^3\vec{r} &= \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \int_{E'=-\infty}^{+\infty} -EE' a(\vec{k}, E) a(-\vec{k}, E') e^{-i(E+E')t} d^3\vec{k} dE \\
\phi(\vec{r}, t) &= \frac{1}{(2\pi)^2} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} a(\vec{k}, E) e^{i(\vec{k}\cdot\vec{r}-Et)} d^3\vec{k} dE \\
H &= \frac{1}{2} \int_{\vec{k}=-\infty}^{+\infty} \int_{E=-\infty}^{+\infty} \int_{E'=-\infty}^{+\infty} (\vec{k}^2 + m^2 - EE') a(\vec{k}, E) a(-\vec{k}, E') e^{-i(E+E')t} d^3\vec{k} dE dE' \\
&= \frac{1}{4} \int_{\vec{k}=-\infty}^{+\infty} [a(\vec{k}, \omega_k) a(-\vec{k}, -\omega_k) + a(-\vec{k}, -\omega_k) a(\vec{k}, \omega_k)] d^3\vec{k}
\end{aligned}$$

**3.4 玻色闭弦方程的推广**

$$\text{定理3.4.1. } \partial_+ \partial_- X^\alpha(\tau, \sigma) = 0 \Leftrightarrow X^\alpha(\tau, \sigma) = \int_{k=-\infty}^{+\infty} \frac{1}{2\omega} [a^\alpha(k, \omega) e^{i(k\sigma - \omega\tau)} + a^\alpha(-k, -\omega) e^{-i(k\sigma - \omega\tau)}] dk$$

# 第四十六章 简单超对称理论初步

## 1 二维中的电磁场

### 1.0.1 二维中的光锥坐标和导数

定义1.0.1.  $z \equiv \tau + \sigma, \tilde{z} \equiv \tau - \sigma, \tau = \frac{1}{2}(z + \tilde{z}), \sigma = \frac{1}{2}(z - \tilde{z}), z_\zeta := \tau + \zeta\sigma, \bar{z}_\zeta := \tau - \zeta\sigma$

定义1.0.2.  $\begin{bmatrix} z \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} 1 & -i \\ -1 & -i \end{bmatrix} \begin{bmatrix} \sigma \\ i\tau \end{bmatrix}, \begin{bmatrix} \sigma \\ i\tau \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \begin{bmatrix} z \\ \tilde{z} \end{bmatrix}$

推论1.0.1.  $\begin{cases} dz = d\tau + d\sigma, d\tilde{z} = d\tau - d\sigma \\ \partial_z = \frac{1}{2}(\partial_\tau + \partial_\sigma), \partial_{\tilde{z}} = \frac{1}{2}(\partial_\tau - \partial_\sigma) \end{cases} \begin{cases} d\tau = \frac{1}{2}(dz + d\tilde{z}), d\sigma = \frac{1}{2}(dz - d\tilde{z}) \\ \partial_\tau = \partial_z + \partial_{\tilde{z}}, \partial_\sigma = \partial_z - \partial_{\tilde{z}} \end{cases}$

推论1.0.2.  $dz \wedge d\tilde{z} = 2d\sigma \wedge d\tau$

定义1.0.3.  $P_z \equiv -i\partial_z, P_{\tilde{z}} \equiv -i\partial_{\tilde{z}}, P_\tau \equiv i\partial_\tau, P_\sigma \equiv -i\partial_\sigma$

推论1.0.3.  $P_z = -\frac{1}{2}(P_\tau - P_\sigma), P_{\tilde{z}} = -\frac{1}{2}(P_\tau + P_\sigma), -P_\tau = P_z + P_{\tilde{z}}, P_\sigma = P_z - P_{\tilde{z}}$

推论1.0.4.  $e^{i(P_\sigma\sigma - P_\tau\tau)} = e^{i(P_z z + P_{\tilde{z}}\tilde{z})}$

### 1.1 二维中的电磁场

定义1.1.1. 场势:  $A_a = (A_\sigma, A_\pi) = (A_\sigma, iA_\tau)$ , 场强:  $F_{ab} = \partial_a A_b - \partial_b A_a$ , 电场:  $E = -iF_{\sigma\pi}$

定义1.1.2. 光锥场势:  $\partial_z? = (\partial_\tau + \partial_\sigma), \partial_{\tilde{z}}? = (\partial_\tau - \partial_\sigma), A_z \equiv A_\tau + A_\sigma, A_{\tilde{z}} \equiv A_\tau - A_\sigma, A'_z = e^{-\epsilon} A_z, A'_{\tilde{z}} = e^\epsilon A_{\tilde{z}}$

定义1.1.3. 光锥场源:  $J_z \equiv J_\tau + J_\sigma, J_{\tilde{z}} \equiv J_\tau - J_\sigma, J'_z = e^{-\epsilon} J_z, J'_{\tilde{z}} = e^\epsilon J_{\tilde{z}}$

推论1.1.1.  $F_{\sigma\pi} = \partial_\sigma A_\pi - \partial_\pi A_\sigma = i(\partial_\sigma A_\tau + \partial_\tau A_\sigma) = i(\partial_z A_z - \partial_{\tilde{z}} A_{\tilde{z}}) = iE$

推论1.1.2.  $E = \partial_z A_z - \partial_{\tilde{z}} A_{\tilde{z}}, F_{ab} = i\epsilon_{ab} E$

所以在二维中电场 $E$ 是标量, 且没有磁场, 场强 $F_{ab}$ 既可以看作张量也可以看作标量。

推论1.1.3. 电磁场方程:  $\partial^a F_{ab} = -J_b \Leftrightarrow \begin{cases} \partial_\tau E = J_\sigma \\ \partial_\sigma E = -J_\tau \end{cases} \Leftrightarrow \begin{cases} \partial_z E = -\frac{1}{2}J_{\tilde{z}} \\ \partial_{\tilde{z}} E = \frac{1}{2}J_z \end{cases}$

推论1.1.4. 洛伦兹条件:  $\partial^a A_a = 0 \Leftrightarrow \partial_z A_z + \partial_{\tilde{z}} A_{\tilde{z}} = 0$

推论1.1.5.  $\begin{cases} \partial^a F_{ab} = -J_b \\ \partial^a A_a = 0 \end{cases} \Rightarrow \begin{cases} \partial_z^2 A_z = -\frac{1}{4}J_{\tilde{z}}, \partial_z \partial_{\tilde{z}} A_{\tilde{z}} = \frac{1}{4}J_{\tilde{z}} \\ \partial_{\tilde{z}}^2 A_{\tilde{z}} = -\frac{1}{4}J_z, \partial_z \partial_{\tilde{z}} A_z = \frac{1}{4}J_z \end{cases}$

推论1.1.6.  $\begin{cases} \partial^a F_{ab} = 0 \\ \partial^a A_a = 0 \end{cases} \Leftrightarrow \begin{cases} \partial_z A_z = \frac{1}{2}E_0 \\ \partial_{\tilde{z}} A_{\tilde{z}} = -\frac{1}{2}E_0 \end{cases} \Leftrightarrow \begin{cases} A_z = f(\tilde{z}) + \frac{1}{2}E_0 z \\ A_{\tilde{z}} = g(z) - \frac{1}{2}E_0 \tilde{z} \end{cases}$

推论1.1.7.  $S = \int dz d\tilde{z} \{-\frac{1}{4}F_{ab}F^{ab}\} = \int dz d\tilde{z} \{\frac{1}{2}E^2\} = \int dz d\tilde{z} \{\frac{1}{2}(\partial_z A_z - \partial_{\tilde{z}} A_{\tilde{z}})^2\}$

推论1.1.8. 规范变换:  $\delta A_a = \partial_a \theta \Leftrightarrow \begin{cases} \delta A_z = 2\partial_z \theta \\ \delta A_{\tilde{z}} = 2\partial_{\tilde{z}} \theta \end{cases} \Rightarrow \delta S = 0$

## 1.2 二维中的矢量旋量超对称 [47]

$$\text{定理1.2.1. } S = \int dzd\bar{z} \{ (\partial_z A_z)^2 + \bar{\varphi} \partial_z \varphi \}, \begin{cases} \delta A_z = \bar{\epsilon}(\bar{z}) \bar{\varphi} \\ \delta \bar{\varphi} = -\bar{\epsilon}(\bar{z}) \partial_z A_z \end{cases} \Rightarrow \delta S = 0$$

$$\text{推论1.2.1. } \begin{cases} [\delta_{\bar{\epsilon}_1(\bar{z})}, \delta_{\bar{\epsilon}_2(\bar{z})}] A_z = 2\bar{\epsilon}_1(\bar{z}) \bar{\epsilon}_2(\bar{z}) \partial_z A_z \\ [\delta_{\bar{\epsilon}_1(\bar{z})}, \delta_{\bar{\epsilon}_2(\bar{z})}] \bar{\varphi} = 2\bar{\epsilon}_1(\bar{z}) \bar{\epsilon}_2(\bar{z}) \partial_z \bar{\varphi} \end{cases}$$

$$\text{定理1.2.2. } S = \int dzd\bar{z} \{ (\partial_{\bar{z}} A_{\bar{z}})^2 + \varphi \partial_{\bar{z}} \varphi \}, \begin{cases} \delta A_{\bar{z}} = \epsilon(z) \varphi \\ \delta \varphi = -\epsilon(z) \partial_{\bar{z}} A_{\bar{z}} \end{cases} \Rightarrow \delta S = 0$$

$$\text{推论1.2.2. } \begin{cases} [\delta_{\epsilon_1(z)}, \delta_{\epsilon_2(z)}] A_{\bar{z}} = 2\epsilon_1(z) \epsilon_2(z) \partial_{\bar{z}} A_{\bar{z}} \\ [\delta_{\epsilon_1(z)}, \delta_{\epsilon_2(z)}] \varphi = 2\epsilon_1(z) \epsilon_2(z) \partial_{\bar{z}} \varphi \end{cases}$$

$$\text{定理1.2.3. } S = \int dzd\bar{z} \{ (\partial_z A_z)^2 + (\partial_{\bar{z}} A_{\bar{z}})^2 + \bar{\varphi} \partial_z \bar{\varphi} + \varphi \partial_{\bar{z}} \varphi \}, \begin{cases} \delta A_z = \bar{\epsilon}(\bar{z}) \bar{\varphi}, \delta A_{\bar{z}} = \epsilon(z) \varphi \\ \delta \varphi = -\epsilon(z) \partial_{\bar{z}} A_{\bar{z}}, \delta \bar{\varphi} = -\bar{\epsilon}(\bar{z}) \partial_z A_z \end{cases} \Rightarrow \delta S = 0$$

$$\text{定理1.2.4. } S = \int dzd\bar{z} \{ (\partial_z A_z - \partial_{\bar{z}} A_{\bar{z}})^2 + \bar{\varphi} \partial_z \bar{\varphi} + \varphi \partial_{\bar{z}} \varphi \}, \begin{cases} \delta A_z = \bar{\epsilon}(\bar{z}) \bar{\varphi}, \delta A_{\bar{z}} = \epsilon(z) \varphi \\ \delta \varphi = -\epsilon(z) (\partial_{\bar{z}} A_{\bar{z}} - \partial_z A_z) \\ \delta \bar{\varphi} = -\bar{\epsilon}(\bar{z}) (\partial_z A_z - \partial_{\bar{z}} A_{\bar{z}}) \end{cases} \Rightarrow \delta S = 0$$

## 1.3 二维中的Majorana-Weyl任意子方程及其相应作用量

$$\text{推论1.3.1. } [s\partial_a + iS_{ab}(s)\partial^b]\psi(s) = 0, \psi'(s) = e^{\frac{i}{2}\theta^{ab}S_{ab}(s)}\psi(s) = e^{-s\epsilon}\psi(s), iS_{ab}(s) = \begin{bmatrix} 0 & is \\ -is & 0 \end{bmatrix}$$

$$\Leftrightarrow \partial_z \psi(s) = 0$$

$$\text{推论1.3.2. } S = \int dzd\bar{z} \psi(1-s)\partial_z \psi(s)$$

$$\text{推论1.3.3. } [-s\partial_a + iS_{ab}(s)\partial^b]\psi(s) = 0, \psi'(s) = e^{-s\epsilon}\psi(s), iS_{ab}(s) = \begin{bmatrix} 0 & is \\ -is & 0 \end{bmatrix} \Leftrightarrow \partial_{\bar{z}} \psi(s) = 0$$

$$\text{推论1.3.4. } S = \int dzd\bar{z} \psi(-1-s)\partial_{\bar{z}} \psi(s)$$

$$\text{推论1.3.5. } [s\partial_a + iS_{ab}(-s)\partial^b]\psi(-s) = 0, \psi'(s) = e^{s\epsilon}\psi(-s), iS_{ab}(-s) = \begin{bmatrix} 0 & -is \\ is & 0 \end{bmatrix} \Leftrightarrow \partial_{\bar{z}} \psi(-s) = 0$$

$$\text{推论1.3.6. } S = \int dzd\bar{z} \psi(-1+s)\partial_{\bar{z}} \psi(-s)$$

$$\text{推论1.3.7. } [-s\partial_a + iS_{ab}(-s)\partial^b]\psi(-s) = 0, \psi'(s) = e^{s\epsilon}\psi(-s), iS_{ab}(-s) = \begin{bmatrix} 0 & -is \\ is & 0 \end{bmatrix} \Leftrightarrow \partial_z \psi(-s) = 0$$

$$\text{推论1.3.8. } S = \int dzd\bar{z} \psi(1+s)\partial_z \psi(-s)$$

## 1.4 二维中的Majorana-Weyl任意子作用量?

$$\text{推论1.4.1. } S = \sum_s \int dzd\bar{z} [\psi(s+1)\partial_z \psi(-s) + \psi(s-1)\partial_{\bar{z}} \psi(-s)]$$

## 1.5 二维中Dirac方程的经典构造

二维中的无质量Dirac方程:

$$\text{推论1.5.1. } S = \int i\bar{\psi}\gamma^a \partial_a \psi d\sigma d\tau, (\gamma^a, \gamma^3) = (\sigma_x, \sigma_y, \sigma_z)$$

$$S = \int i\psi^T \sigma_y (\sigma_x \partial_\sigma + \sigma_y \partial_{i\tau}) \psi d\sigma d\tau = \int \psi^T (\sigma_z \partial_\sigma + \partial_\tau) \psi d\sigma d\tau$$

$$= \int [\psi_1(\partial_\tau + \partial_\sigma)\psi_1 + \psi_2(\partial_\tau - \partial_\sigma)\psi_2] d\sigma d\tau = \int (\psi_1 \partial_z \psi_1 + \psi_2 \partial_{\bar{z}} \psi_2) dzd\bar{z}$$

二维中的中微子方程：

$$\text{推论1.5.2. } S = \int i\bar{\psi} \frac{1+\gamma^3}{2} \gamma^a \partial_a \psi d\sigma d\tau, (\gamma^a, \gamma^3) = (\sigma_x, \sigma_y, \sigma_z)$$

$$S = \int \psi_1 (\partial_\tau + \partial_\sigma) \psi_1 d\sigma d\tau = \int \psi_1 \partial_z \psi_1 dz d\bar{z}$$

二维中的反中微子方程：

$$\text{推论1.5.3. } S = \int i\bar{\psi} \frac{1-\gamma^3}{2} \gamma^a \partial_a \psi d\sigma d\tau, (\gamma^a, \gamma^3) = (\sigma_x, \sigma_y, \sigma_z)$$

$$S = \int \psi_2 (\partial_\tau - \partial_\sigma) \psi_2 d\sigma d\tau = \int \psi_2 \partial_{\bar{z}} \psi_2 dz d\bar{z}$$

## 2 左超对称弦 [44, 46, 47]

### 2.1 左超对称弦作用量及运动方程

$$\text{定理2.1.1. } S = \int (\partial_z X^u \partial_{\bar{z}} X_u + \varphi^u \partial_{\bar{z}} \varphi_u) dz d\bar{z}, \begin{cases} \delta X^u = \epsilon(z) \varphi^u \\ \delta \varphi^u = -\epsilon(z) \partial_z X^u \equiv -\frac{1}{2} \epsilon(z) (1, i)^a \partial_a X^u \end{cases} \Rightarrow \delta S = 0$$

$$\text{证明: } \delta S = \delta \int (\partial_z X^u \partial_{\bar{z}} X_u + \varphi^u \partial_{\bar{z}} \varphi_u) dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int (\partial_z \delta X^u \partial_{\bar{z}} X_u + \partial_z X^u \partial_{\bar{z}} \delta X_u + \delta \varphi^u \partial_{\bar{z}} \varphi_u + \varphi^u \partial_{\bar{z}} \delta \varphi_u) dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int \{ \partial_z [\epsilon(z) \varphi^u] \partial_{\bar{z}} X_u + \partial_z X^u \partial_{\bar{z}} [\epsilon(z) \varphi_u] + [-\epsilon(z) \partial_z X^u] \partial_{\bar{z}} \varphi_u + \varphi^u \partial_{\bar{z}} [-\epsilon(z) \partial_z X_u] \} dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int \{ \partial_z [\epsilon(z) \varphi^u] \partial_{\bar{z}} X_u + \partial_z X^u \epsilon(z) \partial_{\bar{z}} \varphi_u + [-\epsilon(z) \partial_z X^u] \partial_{\bar{z}} \varphi_u + \varphi^u [-\epsilon(z) \partial_z \partial_{\bar{z}} X_u] \} dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int \{ \partial_z [\epsilon(z) \varphi^u] \partial_{\bar{z}} X_u + \epsilon(z) \varphi^u \partial_z \partial_{\bar{z}} X_u \} dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int \partial_z [\epsilon(z) \varphi^u \partial_{\bar{z}} X_u] dz d\bar{z}$$

$$\Rightarrow \delta S = 0$$

□

超对称变换的封闭性

$$\text{推论2.1.1. } S = \int (\partial_z X^u \partial_{\bar{z}} X_u + \varphi^u \partial_{\bar{z}} \varphi_u) dz d\bar{z} \equiv \int (\frac{1}{2} \partial^a X^u \partial_a X_u + \varphi^u (-1, i)^a \partial_a \varphi_u) d\tau d\sigma$$

$$\begin{cases} [\delta_{\epsilon_1(z)}, \delta_{\epsilon_2(z)}] X^u = 2\epsilon_1(z) \epsilon_2(z) \partial_z X^u \equiv \frac{1}{2} [\epsilon_1(z) (1, i)^a \epsilon_2(z) - \epsilon_2(z) (1, i)^a \epsilon_1(z)] \partial_a X^u \\ [\delta_{\epsilon_1(z)}, \delta_{\epsilon_2(z)}] \varphi^u = 2\epsilon_1(z) \epsilon_2(z) \partial_z \varphi^u + \partial_z [\epsilon_1(z) \epsilon_2(z)] \varphi^u \\ \equiv \frac{1}{2} [\epsilon_1(z) (1, i)^a \epsilon_2(z) - \epsilon_2(z) (1, i)^a \epsilon_1(z)] \partial_a \varphi^u + \partial_z [\epsilon_1(z) \epsilon_2(z)] \varphi^u \end{cases}$$

$$\text{推论2.1.2. } S = \int (\partial_z X^u \partial_{\bar{z}} X_u + \varphi^u \partial_{\bar{z}} \varphi_u) dz d\bar{z} \equiv \int (\frac{1}{2} \partial^a X^u \partial_a X_u + \varphi^u (-1, i)^a \partial_a \varphi_u) d\tau d\sigma$$

$$\begin{cases} [\delta_{\epsilon_1}, \delta_{\epsilon_2}] X^u = 2\epsilon_1 \epsilon_2 \partial_z X^u \equiv \frac{1}{2} [\epsilon_1 (1, i)^a \epsilon_2 - \epsilon_2 (1, i)^a \epsilon_1] \partial_a X^u \\ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \varphi^u = 2\epsilon_1 \epsilon_2 \partial_z \varphi^u \equiv \frac{1}{2} [\epsilon_1 (1, i)^a \epsilon_2 - \epsilon_2 (1, i)^a \epsilon_1] \partial_a \varphi^u \end{cases}$$

$$\text{推论2.1.3. } [\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 2\epsilon_1 \epsilon_2 \partial_z \equiv \frac{1}{2} [\epsilon_1 (1, i)^a \epsilon_2 - \epsilon_2 (1, i)^a \epsilon_1] \partial_a$$

$$\text{定义2.1.1. } \delta_\epsilon Y = [\epsilon Q, Y], \partial_a Y = i[P_a, Y]$$

$$\text{推论2.1.4. } [\delta_{\epsilon_1}, \delta_{\epsilon_2}] Y = [\epsilon_2 Q, \delta_{\epsilon_1} Y] - [\epsilon_1 Q, \delta_{\epsilon_2} Y] = [\epsilon_2 Q, [\epsilon_1 Q, Y]] - [\epsilon_1 Q, [\epsilon_2 Q, Y]] = [[\epsilon_2 Q, \epsilon_1 Q], Y]$$

$$\text{推论2.1.5. } [[\epsilon_2 Q, \epsilon_1 Q], Y] = 2\epsilon_1 \epsilon_2 i[P_z, Y] \Rightarrow Q^2 = iP_z \Rightarrow [Q, P_z] = 0$$

### 2.2 左超对称弦作用量的局域玻色变换

$$\text{定理2.2.1. } S = \int \partial_z X^u \partial_{\bar{z}} X_u dz d\bar{z}, \delta X^u = v(z) \partial_z X^u \Rightarrow \delta S = 0$$

$$\text{证明: } \delta S = \delta \int \partial_z X^u \partial_{\bar{z}} X_u dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int (\partial_z \delta X^u \partial_{\bar{z}} X_u + \partial_z X^u \partial_{\bar{z}} \delta X_u) dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int \{ \partial_z [v(z) \partial_z X^u] \partial_{\bar{z}} X_u + \partial_z X^u \partial_{\bar{z}} [v(z) \partial_z X_u] \} dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int \{ \partial_z [v(z) \partial_z X^u] \partial_{\bar{z}} X_u + v(z) \partial_z X^u \partial_z \partial_{\bar{z}} X_u \} dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int \partial_z [v(z) \partial_z X^u \partial_{\bar{z}} X_u] dz d\bar{z}$$

$$\Rightarrow \delta S = 0$$

□



定理2.2.2.  $S = \int \varphi^u \partial_z \varphi_u dz d\bar{z}, \delta \varphi^u = v(z) \partial_z \varphi^u \Rightarrow \delta S = 0$

证明:  $\delta S = \delta \int \varphi^u \partial_z \varphi_u dz d\bar{z}$

$$\Leftrightarrow \delta S = \int \delta \varphi^u \partial_z \varphi_u + \varphi^u \partial_z \delta \varphi_u dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int [v(z) \partial_z \varphi^u] \partial_z \varphi_u + \varphi^u \partial_z [v(z) \partial_z \varphi_u] dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int v(z) \partial_z [\varphi^u \partial_z \varphi_u] dz d\bar{z}$$

□

### 2.3 左超对称弦作用量的全局玻色变换

推论2.3.1.  $S = \int \partial_z X^u \partial_{\bar{z}} X_u dz d\bar{z}, \delta X^u = v \partial_z X^u \Rightarrow \delta S = 0$

推论2.3.2.  $S = \int \varphi^u \partial_z \varphi_u dz d\bar{z}, \delta \varphi^u = v \partial_z \varphi^u \Rightarrow \delta S = 0$

证明:  $\delta S = \delta \int \varphi^u \partial_z \varphi_u dz d\bar{z}$

$$\Leftrightarrow \delta S = \int \delta \varphi^u \partial_z \varphi_u + \varphi^u \partial_z \delta \varphi_u dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int [v \partial_z \varphi^u] \partial_z \varphi_u + \varphi^u \partial_z [v \partial_z \varphi_u] dz d\bar{z}$$

$$\Leftrightarrow \delta S = \int \partial_z [v \varphi^u \partial_z \varphi_u] dz d\bar{z}$$

$$\Rightarrow \delta S = 0$$

□

### 2.4 左超对称弦作用量的全局变换的封闭性

定义2.4.1.  $\delta_v Y = i[v P_z, Y] = v \partial_z$

推论2.4.1.  $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] Y = 2i\epsilon_1 \epsilon_2 [P_z, Y]$

推论2.4.2.  $[\delta_{v_1}, \delta_{v_2}] Y = [[iv_2 P_z, iv_1 P_z], Y] = 0$

推论2.4.3.  $[\delta_{v_1}, \delta_{\epsilon_2}] Y = [[\epsilon_2 Q, iv_1 P_z], Y] = 0$

推论2.4.4.  $[Q, Q] = 2iP_z, [Q, P_z] = 0, [P_z, P_z] = 0$

## 3 右超对称弦 [44, 46, 47]

### 3.1 右超对称弦作用量及运动方程

定理3.1.1.  $S = \int (\partial_z X^u \partial_{\bar{z}} X_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u) dz d\bar{z}, \begin{cases} \delta X^u = \bar{\epsilon}(\bar{z}) \bar{\varphi}^u \\ \delta \bar{\varphi}^u = -\bar{\epsilon}(\bar{z}) \partial_z X^u \equiv -\frac{1}{2} \bar{\epsilon}(\bar{z}) (-1, i)^a \partial_a X^u \end{cases} \Rightarrow \delta S = 0$

超对称变换的封闭性

推论3.1.1.  $S = \int (\partial_z X^u \partial_{\bar{z}} X_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u) dz d\bar{z} \equiv \int (\frac{1}{2} \partial^a X^u \partial_a X_u + \bar{\varphi}^u (1, i)^a \partial_a \bar{\varphi}_u) d\tau d\sigma$

$$\begin{cases} [\delta_{\bar{\epsilon}_1(\bar{z})}, \delta_{\bar{\epsilon}_2(\bar{z})}] X^u = 2\bar{\epsilon}_1(\bar{z}) \bar{\epsilon}_2(\bar{z}) \partial_z X^u \\ [\delta_{\bar{\epsilon}_1(\bar{z})}, \delta_{\bar{\epsilon}_2(\bar{z})}] \bar{\varphi}^u = 2\bar{\epsilon}_1(\bar{z}) \bar{\epsilon}_2(\bar{z}) \partial_z \bar{\varphi}^u + \partial_z [\bar{\epsilon}_1(\bar{z}) \bar{\epsilon}_2(\bar{z})] \bar{\varphi}^u \end{cases}$$

推论3.1.2.  $S = \int (\partial_z X^u \partial_{\bar{z}} X_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u) dz d\bar{z} \equiv \int (\frac{1}{2} \partial^a X^u \partial_a X_u + \bar{\varphi}^u (1, i)^a \partial_a \bar{\varphi}_u) d\tau d\sigma$

$$\begin{cases} [\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}] X^u = 2\bar{\epsilon}_1 \bar{\epsilon}_2 \partial_z X^u \\ [\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}] \bar{\varphi}^u = 2\bar{\epsilon}_1 \bar{\epsilon}_2 \partial_z \bar{\varphi}^u \end{cases}$$

推论3.1.3.  $[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}] = 2\bar{\epsilon}_1 \bar{\epsilon}_2 \partial_z \equiv \frac{1}{2} [\bar{\epsilon}_1 (-1, i)^a \bar{\epsilon}_2 - \bar{\epsilon}_2 (-1, i)^a \bar{\epsilon}_1] \partial_a$

定义3.1.1.  $\delta_{\bar{\epsilon}} Y = [\bar{\epsilon} \bar{Q}, Y], \partial_a Y = i[P_a, Y]$

推论3.1.4.  $[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}] Y = [\bar{\epsilon}_2 \bar{Q}, \delta_{\bar{\epsilon}_1} Y] - [\bar{\epsilon}_1 \bar{Q}, \delta_{\bar{\epsilon}_2} Y] = [\bar{\epsilon}_2 \bar{Q} +, [\bar{\epsilon}_1 \bar{Q}, Y]] - [\bar{\epsilon}_1 \bar{Q}, [\bar{\epsilon}_2 \bar{Q}, Y]] = [[\bar{\epsilon}_2 \bar{Q}, \bar{\epsilon}_1 \bar{Q}], Y]$

推论3.1.5.  $[[\bar{\epsilon}_2 \bar{Q}, \bar{\epsilon}_1 \bar{Q}], Y] = 2\bar{\epsilon}_1 \bar{\epsilon}_2 i [P_z, Y] \Rightarrow \bar{Q}^2 = iP_z \Rightarrow [\bar{Q}, P_z] = 0$

### 3.2 右超对称弦作用量的局域玻色变换

推论3.2.1.  $S = \int \partial_z X^u \partial_{\bar{z}} X_u dz d\bar{z}, \delta X^u = \bar{v}(\bar{z}) \partial_{\bar{z}} X^u \Rightarrow \delta S = 0$

推论3.2.2.  $S = \int \bar{\varphi}^u \partial_z \bar{\varphi}_u dz d\bar{z}, \delta \bar{\varphi}^u = \bar{v}(\bar{z}) \partial_{\bar{z}} \bar{\varphi}^u \Rightarrow \delta S = \int \bar{v}(\bar{z}) \partial_{\bar{z}} [\bar{\varphi}^u \partial_z \bar{\varphi}_u] dz d\bar{z}$

### 3.3 右超对称弦作用量的全局玻色变换

推论3.3.1.  $S = \int \partial_z X^u \partial_{\bar{z}} X_u dz d\bar{z}, \delta X^u = \bar{v} \partial_{\bar{z}} X^u \Rightarrow \delta S = 0$

推论3.3.2.  $S = \int \bar{\varphi}^u \partial_z \bar{\varphi}_u dz d\bar{z}, \delta \bar{\varphi}^u = \bar{v} \partial_{\bar{z}} \bar{\varphi}^u \Rightarrow \delta S = 0$

### 3.4 右超对称弦作用量的全局变换的封闭性

定义3.4.1.  $\delta_v Y = i[\bar{v} P_{\bar{z}}, Y] = \bar{v} \partial_{\bar{z}}$

推论3.4.1.  $[\delta_{\bar{\epsilon}_1}, \delta_{\bar{\epsilon}_2}] Y = 2i\bar{\epsilon}_1 \bar{\epsilon}_2 [P_{\bar{z}}, Y]$

推论3.4.2.  $[\delta_{\bar{v}_1}, \delta_{\bar{v}_2}] Y = [[i\bar{v}_2 P_{\bar{z}}, i\bar{v}_1 P_{\bar{z}}], Y] = 0$

推论3.4.3.  $[\delta_{\bar{v}_1}, \delta_{\bar{\epsilon}_2}] Y = [[\bar{\epsilon}_2 \bar{Q}, i\bar{v}_1 P_{\bar{z}}], Y] = 0$

推论3.4.4.  $[\bar{Q}, \bar{Q}] = 2iP_{\bar{z}}, [\bar{Q}, P_{\bar{z}}] = 0, [P_{\bar{z}}, P_{\bar{z}}] = 0$

## 4 左右超对称弦 [44, 46, 47]

### 4.1 质壳左右超对称弦作用量及运动方程

推论4.1.1.  $S_B = \frac{1}{2\pi\alpha'} \int \partial_z X^u \partial_{\bar{z}} X_u dz d\bar{z}, S_F = \frac{1}{4\pi} \int (\bar{\varphi}^u \partial_z \bar{\varphi}_u + \varphi^u \partial_{\bar{z}} \varphi_u) dz d\bar{z}$

定理4.1.1.  $S = \int dz d\bar{z} (\partial_z X^u \partial_{\bar{z}} X_u + \varphi^u \partial_{\bar{z}} \varphi_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u)$

$$\begin{cases} \delta X^u = \epsilon(z) \varphi^u + \bar{\epsilon}(\bar{z}) \bar{\varphi}^u \\ \delta \varphi^u = -\epsilon(z) \partial_z X^u \\ \delta \bar{\varphi}^u = -\bar{\epsilon}(\bar{z}) \partial_{\bar{z}} X^u \end{cases} \Leftrightarrow \delta \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \end{bmatrix} = \begin{bmatrix} 0 & \epsilon(z) & \bar{\epsilon}(\bar{z}) \\ -\epsilon(z) \partial_z & 0 & 0 \\ -\bar{\epsilon}(\bar{z}) \partial_{\bar{z}} & 0 & 0 \end{bmatrix} \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \end{bmatrix} \Rightarrow \delta S = 0$$

证明:  $\delta S = \delta \int dz d\bar{z} (\partial_z X^u \partial_{\bar{z}} X_u + \varphi^u \partial_{\bar{z}} \varphi_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u)$

$$\Leftrightarrow \delta S = \int dz d\bar{z} (\partial_z \delta X^u \partial_{\bar{z}} X_u + \partial_z X^u \partial_{\bar{z}} \delta X_u + \delta \varphi^u \partial_{\bar{z}} \varphi_u + \varphi^u \partial_{\bar{z}} \delta \varphi_u + \delta \bar{\varphi}^u \partial_z \bar{\varphi}_u + \bar{\varphi}^u \partial_z \delta \bar{\varphi}_u)$$

$$\Leftrightarrow \delta S = \int dz d\bar{z} \{ \partial_z [\epsilon(z) \varphi^u] \partial_{\bar{z}} X_u + \bar{\epsilon}(\bar{z}) \partial_{\bar{z}} X_u \partial_z \bar{\varphi}^u + \epsilon(z) \partial_z X^u \partial_{\bar{z}} \varphi^u + \partial_{\bar{z}} [\bar{\epsilon}(\bar{z}) \bar{\varphi}^u] \partial_z X^u$$

$$- \epsilon(z) \partial_z X^u \partial_{\bar{z}} \varphi_u + \epsilon(z) \varphi^u \partial_z \partial_{\bar{z}} X^u - \bar{\epsilon}(\bar{z}) \partial_{\bar{z}} X^u \partial_z \bar{\varphi}_u + \bar{\epsilon}(\bar{z}) \bar{\varphi}^u \partial_z \partial_z X^u \}$$

$$\Leftrightarrow \delta S = \int dz d\bar{z} \{ \partial_z [\epsilon(z) \varphi^u \partial_{\bar{z}} X_u] + \partial_{\bar{z}} [\bar{\epsilon}(\bar{z}) \bar{\varphi}^u \partial_z X^u] \}$$

$$\Rightarrow \delta S = 0$$

□

以上虽然是超对称变换，但只满足质壳封闭性。

### 4.2 非质壳左右超对称弦作用量及运动方程

定理4.2.1.  $S = \int dz d\bar{z} (\partial_z X^u \partial_{\bar{z}} X_u + \varphi^u \partial_{\bar{z}} \varphi_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u + F^u F_u)$

$$\begin{cases} \delta X^u = \epsilon(z) \varphi^u + \bar{\epsilon}(\bar{z}) \bar{\varphi}^u \\ \delta \varphi^u = -\epsilon(z) \partial_z X^u - \bar{\epsilon}(\bar{z}) F^u \\ \delta \bar{\varphi}^u = -\bar{\epsilon}(\bar{z}) \partial_{\bar{z}} X^u - \epsilon(z) F^u \\ \delta F^u = \bar{\epsilon}(\bar{z}) \partial_{\bar{z}} \varphi^u + \epsilon(z) \partial_z \bar{\varphi}^u \end{cases} \Leftrightarrow \delta \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix} = \begin{bmatrix} 0 & \epsilon(z) & \bar{\epsilon}(\bar{z}) & 0 \\ -\epsilon(z) \partial_z & 0 & 0 & -\bar{\epsilon}(\bar{z}) \\ -\bar{\epsilon}(\bar{z}) \partial_{\bar{z}} & 0 & 0 & -\epsilon(z) \\ 0 & \bar{\epsilon}(\bar{z}) \partial_{\bar{z}} & \epsilon(z) \partial_z & 0 \end{bmatrix} \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix} \Rightarrow \delta S = 0$$

证明:  $\delta S = \delta \int dz d\bar{z} (\partial_z X^u \partial_{\bar{z}} X_u + \varphi^u \partial_{\bar{z}} \varphi_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u + F^u F_u)$

$$\Leftrightarrow \delta S = \int dz d\bar{z} (\partial_z \delta X^u \partial_{\bar{z}} X_u + \partial_z X^u \partial_{\bar{z}} \delta X_u + \delta \varphi^u \partial_{\bar{z}} \varphi_u + \varphi^u \partial_{\bar{z}} \delta \varphi_u + \delta \bar{\varphi}^u \partial_z \bar{\varphi}_u + \bar{\varphi}^u \partial_z \delta \bar{\varphi}_u + \frac{1}{2} \delta F^u F_u + \frac{1}{2} F^u \delta F_u)$$

$$\Leftrightarrow \delta S = \int dz d\bar{z} \{ \partial_z [\epsilon(z) \varphi^u] \partial_{\bar{z}} X_u + \bar{\epsilon}(\bar{z}) \partial_{\bar{z}} X_u \partial_z \bar{\varphi}^u + \epsilon(z) \partial_z X^u \partial_{\bar{z}} \varphi^u + \partial_{\bar{z}} [\bar{\epsilon}(\bar{z}) \bar{\varphi}^u] \partial_z X^u$$

$$\begin{aligned}
& -\epsilon(z)\partial_z X^u \partial_{\bar{z}} \varphi_u - \bar{\epsilon}(\bar{z})F^u \partial_{\bar{z}} \varphi_u + \epsilon(z)\varphi^u \partial_z \partial_{\bar{z}} X^u + \partial_{\bar{z}}[\bar{\epsilon}(\bar{z})F_u]\varphi^u \\
& - \bar{\epsilon}(\bar{z})\partial_{\bar{z}} X^u \partial_z \bar{\varphi}_u - \epsilon(z)F^u \partial_z \bar{\varphi}_u + \bar{\epsilon}(\bar{z})\bar{\varphi}^u \partial_z \partial_z X^u + \partial_z[\epsilon(z)F_u]\bar{\varphi}^u \\
& + 2[\bar{\epsilon}(\bar{z})F_u]\partial_{\bar{z}} \varphi^u + 2[\epsilon(z)F_u]\partial_z \bar{\varphi}^u \\
& \Leftrightarrow \delta S = \int dz d\bar{z} \{ \partial_z [\epsilon(z)(\varphi^u \partial_z X_u + \bar{\varphi}^u F_u)] + \partial_{\bar{z}} [\bar{\epsilon}(\bar{z})(\bar{\varphi}^u \partial_z X^u + \varphi^u F_u)] \} \\
& \Rightarrow \delta S = 0
\end{aligned}$$

□

以上虽然是超对称变换，但不满足封闭性。下面的既是超对称变换，也满足非质壳封闭性。

$$\begin{aligned}
& \text{定理4.2.2. } S = \int dz d\bar{z} (\partial_z X^u \partial_{\bar{z}} X_u + \varphi^u \partial_{\bar{z}} \varphi_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u + F^u F_u) \\
& \begin{cases} \delta X^u = \epsilon(z)\varphi^u + \bar{\epsilon}(\bar{z})\bar{\varphi}^u \\ \delta \varphi^u = -\epsilon(z)\partial_z X^u - \bar{\epsilon}(\bar{z})F^u \\ \delta \bar{\varphi}^u = -\bar{\epsilon}(\bar{z})\partial_{\bar{z}} X^u + \epsilon(z)F^u \\ \delta F^u = \bar{\epsilon}(\bar{z})\partial_{\bar{z}} \varphi^u - \epsilon(z)\partial_z \bar{\varphi}^u \end{cases} \Leftrightarrow \delta \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix} = \begin{bmatrix} 0 & \epsilon(z) & \bar{\epsilon}(\bar{z}) & 0 \\ -\epsilon(z)\partial_z & 0 & 0 & -\bar{\epsilon}(\bar{z}) \\ -\bar{\epsilon}(\bar{z})\partial_{\bar{z}} & 0 & 0 & \epsilon(z) \\ 0 & \bar{\epsilon}(\bar{z})\partial_{\bar{z}} & -\epsilon(z)\partial_z & 0 \end{bmatrix} \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix} \Rightarrow \delta S = 0
\end{aligned}$$

$$\begin{aligned}
& \text{证明: } \delta S = \delta \int dz d\bar{z} (\partial_z X^u \partial_{\bar{z}} X_u + \varphi^u \partial_{\bar{z}} \varphi_u + \bar{\varphi}^u \partial_z \bar{\varphi}_u + F^u F_u) \\
& \Leftrightarrow \delta S = \int dz d\bar{z} (\partial_z \delta X^u \partial_{\bar{z}} X_u + \partial_z X^u \partial_{\bar{z}} \delta X_u + \delta \varphi^u \partial_{\bar{z}} \varphi_u + \varphi^u \partial_{\bar{z}} \delta \varphi_u + \delta \bar{\varphi}^u \partial_z \bar{\varphi}_u + \bar{\varphi}^u \partial_z \delta \bar{\varphi}_u + \frac{1}{2} \delta F^u F_u + \frac{1}{2} F^u \delta F_u) \\
& \Leftrightarrow \delta S = \int dz d\bar{z} \{ \partial_z [\epsilon(z)\varphi^u] \partial_{\bar{z}} X_u + \bar{\epsilon}(\bar{z})\partial_{\bar{z}} X_u \partial_z \bar{\varphi}^u + \epsilon(z)\partial_z X^u \partial_{\bar{z}} \varphi^u + \partial_{\bar{z}} [\bar{\epsilon}(\bar{z})\bar{\varphi}^u] \partial_z X^u \\
& - \epsilon(z)\partial_z X^u \partial_{\bar{z}} \varphi_u - \bar{\epsilon}(\bar{z})F^u \partial_{\bar{z}} \varphi_u + \epsilon(z)\varphi^u \partial_z \partial_{\bar{z}} X^u + \partial_{\bar{z}} [\bar{\epsilon}(\bar{z})F_u]\varphi^u \\
& - \bar{\epsilon}(\bar{z})\partial_{\bar{z}} X^u \partial_z \bar{\varphi}_u + \epsilon(z)F^u \partial_z \bar{\varphi}_u + \bar{\epsilon}(\bar{z})\bar{\varphi}^u \partial_z \partial_z X^u - \partial_z [\epsilon(z)F_u]\bar{\varphi}^u \\
& + 2[\bar{\epsilon}(\bar{z})F_u]\partial_{\bar{z}} \varphi^u - 2[\epsilon(z)F_u]\partial_z \bar{\varphi}^u \} \\
& \Leftrightarrow \delta S = \int dz d\bar{z} \{ \partial_z [\epsilon(z)(\varphi^u \partial_z X_u - \bar{\varphi}^u F_u)] + \partial_{\bar{z}} [\bar{\epsilon}(\bar{z})(\bar{\varphi}^u \partial_z X^u + \varphi^u F_u)] \} \\
& \Rightarrow \delta S = 0
\end{aligned}$$

□

$$\text{定义4.2.1. } \delta_\epsilon \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix} = \begin{bmatrix} 0 & \epsilon & \bar{\epsilon} & 0 \\ -\epsilon\partial_z & 0 & 0 & -\bar{\epsilon} \\ -\bar{\epsilon}\partial_{\bar{z}} & 0 & 0 & \epsilon \\ 0 & \bar{\epsilon}\partial_{\bar{z}} & -\epsilon\partial_z & 0 \end{bmatrix} \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix}$$

$$\text{定理4.2.3. } [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix} = ([\epsilon_1, \epsilon_2]\partial_z + [\bar{\epsilon}_1, \bar{\epsilon}_2]\partial_{\bar{z}}) \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix}$$

$$\text{证明: } [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix} = - \left[ \begin{bmatrix} 0 & \epsilon_1 & \bar{\epsilon}_1 & 0 \\ -\epsilon_1\partial_z & 0 & 0 & -\bar{\epsilon}_1 \\ -\bar{\epsilon}_1\partial_{\bar{z}} & 0 & 0 & \epsilon_1 \\ 0 & \bar{\epsilon}_1\partial_{\bar{z}} & -\epsilon_1\partial_z & 0 \end{bmatrix}, \begin{bmatrix} 0 & \epsilon_2 & \bar{\epsilon}_2 & 0 \\ -\epsilon_2\partial_z & 0 & 0 & -\bar{\epsilon}_2 \\ -\bar{\epsilon}_2\partial_{\bar{z}} & 0 & 0 & \epsilon_2 \\ 0 & \bar{\epsilon}_2\partial_{\bar{z}} & -\epsilon_2\partial_z & 0 \end{bmatrix} \right] \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix}$$

$$\Leftrightarrow [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix} = ([\epsilon_1, \epsilon_2]\partial_z + [\bar{\epsilon}_1, \bar{\epsilon}_2]\partial_{\bar{z}}) \begin{bmatrix} X^u \\ \varphi^u \\ \bar{\varphi}^u \\ F^u \end{bmatrix}$$

□

### 4.3 左右超对称弦的超对称代数

$$\text{定义4.3.1. } \delta_\epsilon Y = [\epsilon Q + \bar{\epsilon} \bar{Q}, Y], \partial_a Y = i[P_a, Y]$$

$$\begin{aligned}
& \text{推论4.3.1. } [\delta_{\epsilon_1}, \delta_{\epsilon_2}] Y = [\epsilon_2 Q + \bar{\epsilon}_2 \bar{Q}, \delta_{\epsilon_1} Y] - [\epsilon_1 Q + \bar{\epsilon}_1 \bar{Q}, \delta_{\epsilon_2} Y] \\
& = [\epsilon_2 Q + \bar{\epsilon}_2 \bar{Q}, [\epsilon_1 Q + \bar{\epsilon}_1 \bar{Q}, Y]] - [\epsilon_1 Q + \bar{\epsilon}_1 \bar{Q}, [\epsilon_2 Q + \bar{\epsilon}_2 \bar{Q}, Y]] \\
& = [[\epsilon_2 Q + \bar{\epsilon}_2 \bar{Q}, \epsilon_1 Q + \bar{\epsilon}_1 \bar{Q}], Y]
\end{aligned}$$

推论4.3.2.  $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]Y = \{[\epsilon_1, \epsilon_2]\partial_z + [\bar{\epsilon}_1, \bar{\epsilon}_2]\partial_{\bar{z}}\}Y$

$\Leftrightarrow [[\epsilon_2 Q + \bar{\epsilon}_2 \bar{Q}, \epsilon_1 Q + \bar{\epsilon}_1 \bar{Q}], Y] = i[[\epsilon_1, \epsilon_2]P_{\bar{z}} + [\bar{\epsilon}_1, \bar{\epsilon}_2]P_z, Y]$

$\Leftrightarrow [[\epsilon_2 Q + \bar{\epsilon}_2 \bar{Q}, \epsilon_1 Q + \bar{\epsilon}_1 \bar{Q}], Y] = i[2\epsilon_1 \epsilon_2 P_{\bar{z}} + 2\bar{\epsilon}_1 \bar{\epsilon}_2 P_z, Y]$

$\Leftrightarrow [\epsilon_2 Q + \bar{\epsilon}_2 \bar{Q}, \epsilon_1 Q + \bar{\epsilon}_1 \bar{Q}] = i(2\epsilon_1 \epsilon_2 P_{\bar{z}} + 2\bar{\epsilon}_1 \bar{\epsilon}_2 P_z)$

$\Leftrightarrow \{Q, \bar{Q}\} = 0, \{Q, Q\} = 2iP_{\bar{z}}, \{\bar{Q}, \bar{Q}\} = 2iP_z$

$\Leftrightarrow \{Q, \bar{Q}\} = 0, \{Q, Q\} = 2iP_{\bar{z}}, \{\bar{Q}, \bar{Q}\} = 2iP_z, [Q, P_{\bar{z}}] = 0, [Q, P_z] = 0, [\bar{Q}, P_{\bar{z}}] = 0, [\bar{Q}, P_z] = 0, [P_z, P_{\bar{z}}] = 0$

#### 4.4 左右超对称弦的超空间表示

推论4.4.1.  $D_\theta = \partial_\theta + \theta\partial_z, D_{\bar{\theta}} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}$

推论4.4.2.  $\hat{X}(z, \bar{z}, \theta, \bar{\theta}) = X + \theta\varphi + \bar{\theta}\bar{\varphi} + \theta\bar{\theta}F$

推论4.4.3.  $S = \int D_\theta \hat{X} D_{\bar{\theta}} \hat{X} d\theta d\bar{\theta} dz d\bar{z}$

### 5 局域超对称弦作用量 [44, 46, 47]

#### 5.1 世界面的超对称作用量

定理5.1.1.  $S = -\frac{1}{2} \int d\sigma d\tau e \{g^{ab} \partial_a X^u \partial_b X_u + i\bar{\psi}^u \gamma^a \partial_a \psi_u + 2i\bar{\chi}_a \gamma^b \gamma^a \psi^u [\partial_b X_u + \frac{i}{2} \bar{\psi}_u \chi_b]\}$

$$\begin{cases} \delta X^u = i\bar{\eta} \psi^u; \\ \delta e_\beta^b = -2i\bar{\eta} \gamma^b \psi_\beta; \\ \delta \psi^u = \gamma^a (\partial_a X^u + i\bar{\psi}^u \chi_a) \eta; \\ \delta \chi_\beta = -D_\beta \eta = -\partial_\beta \eta + \frac{i}{2} \omega_\beta \gamma^x \gamma^y \eta \end{cases}$$

#### 5.2 时空中的超对称作用量

定理5.2.1.  $S = -\frac{1}{2\pi} \int d\sigma d\tau e \{g^{ab} (\partial_a X^u - i \sum_A \bar{\theta}^A \gamma^u \partial_a \theta^A) (\partial_b X^u - i \sum_A \bar{\theta}^A \gamma^u \partial_b \theta^A)\}$

$+ \frac{1}{\pi} \int \{-idX^u \wedge (\bar{\theta}^1 \gamma_u d\theta^1 - \bar{\theta}^2 \gamma_u d\theta^2) + \bar{\theta}^1 \gamma^u d\theta^1 \wedge \bar{\theta}^2 \gamma_u d\theta^2\}$

$$\begin{cases} \delta \theta^A = \varepsilon^A \\ \delta X^u = \frac{i}{2} \sum_A (\bar{\varepsilon}^A \gamma^u \theta^A - \bar{\theta}^A \gamma^u \varepsilon^A) \equiv i \sum_A \bar{\varepsilon}^A \gamma^u \theta^A \end{cases}$$

#### 5.3 超对称理论的自旋表述

猜想5.3.1.

$$\begin{cases} [\frac{1}{2} \partial_a + iS_{ab}(\frac{1}{2}, \varsigma) \partial^b] \psi = 0 \\ [\partial_a + iS_{ab}(1, \varsigma) \partial^b] \hat{Q}^+ \psi = 0 \\ [\frac{3}{2} \partial_a + iS_{ab}(\frac{3}{2}, \varsigma) \partial^b] \hat{Q}^{+2} \psi = 0 \\ [2\partial_a + iS_{ab}(2, \varsigma) \partial^b] \hat{Q}^{+3} \psi = 0 \\ \dots \\ [s\partial_a + iS_{ab}(s, \varsigma) \partial^b] \hat{Q}^{+(2s-1)} \psi = 0 \end{cases}$$

是否可以找到满足以上条件的 $\hat{Q}$ 算子？但显然不是通常的超对称算子。

## 参考文献

- [1] Roger Penrose and Wolfgang Rindler. *Spinors and Space-Time, Volume 1: Two-Spinor Calculus and Relativistic Fields*. Cambridge University Press, 1984. 1, 4, 9, 10, 11, 12, 13, 26, 37, 38, 8, 14, 122, 138, 139, 143, 144, 146, 154, 156, 157, 162, 166, 172, 174, 183, 185, 192, 201, 519, 784, 806, 810, 811, 825, 827
- [2] Roger Penrose and Wolfgang Rindler. *Spinors and Space-Time, Volume 2: Spinor and Twistor Methods in Space-Time Geometry*. Cambridge University Press, 1986. 1, 4, 9, 10, 11, 12, 13, 26, 37, 38, 8, 14, 122, 138, 139, 143, 144, 146, 154, 156, 157, 162, 166, 172, 174, 183, 185, 187, 192, 193, 201, 519, 784, 806, 810, 811, 825, 827
- [3] R. Penrose. Twistor algebra. *Journal of Mathematical Physics*, 8(2):345–366, 1967. 1, 13, 187, 193
- [4] P.A.M. Dirac. *The Principles of Quantum Mechanics*. Oxford:Clarendon Press, 1930. 1, 12, 13, 14, 29, 36, 37, 38, 40, 136, 183, 196, 197, 208, 528, 534, 541, 604, 767, 781, 816, 824, 854, 950
- [5] H. Weyl. Quantenmechanik und gruppentheorie. *Zeitschrift F ü r Physik*, 46(1-2):1–46, 1927. 1, 12, 37, 38, 134, 136, 183, 541, 781, 824, 950
- [6] C.N. Yang and R.L. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Physical Review*, 96(1):191–195, 1954. 1, 11, 12, 149, 183
- [7] 余恒清, 全泽松. 《电磁波理论解题指南》. 电子科技大学出版社, 1995年. 1, 10, 11, 12, 137, 140, 143, 146, 163, 167, 171, 175, 950
- [8] 李子平. 《经典和量子约束系统及其对称性质》. 北京工业大学出版社, 1993年. 1, 12, 14, 11, 180, 223, 950
- [9] J.D.比约肯, S.D.德雷尔. 《相对论量子力学》. 科学出版社, 1984年. 1, 13, 196, 197
- [10] 胡瑶光. 《规范场论》. 华东师范大学出版社, 1984年. 1
- [11] A.爱因斯坦. 《相对论的意义》. 科学出版社, 1979年. 1, 11, 149, 157, 168, 169, 950
- [12] S.温伯格. 《引力论和宇宙论》. 科学出版社, 1980年. 1, 11, 14, 149, 157, 169, 223, 950
- [13] 刘辽. 《广义相对论》. 高等教育出版社, 1987年. 1, 11, 149, 157, 169, 950
- [14] S.W.霍金, G.F.R.埃利斯. 《时空的大尺度结构》. 湖南科技出版社, 2006年. 1, 11, 149, 157, 165, 169, 950
- [15] Alan H. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Physical Review D*, 23(2):347–356, 1981. 1
- [16] Edward Witten. M-theory. *American Mathematical Society Providence Ri*, 1, 1998. 1

- [17] V. Bargmann and E.P. Wigner. Group theoretical discussion of relativistic wave equations. *Proceedings of the National Academy of Sciences of the United States of America*, 34(5):211–23, 1948. 1, 13, 14, 27, 28, 29, 30, 34, 35, 37, 38, 45, 195, 198, 201, 204, 205, 543, 561, 570, 585, 604, 605, 614, 615, 621, 726, 727, 733, 746, 747, 749, 751, 752, 773, 796, 798, 799, 803, 806, 811, 812, 813, 814, 816, 818, 819, 820, 821, 823, 942
- [18] A. Salam, R. Delbourgo, and J. Strathdee. The covariant theory of strong interaction symmetries. *Proceedings of the Royal Society A*, 284(1396):146–158, 1965. 1, 14, 203
- [19] 黄时中. 高自旋粒子的波动方程、波函数、投影算符和传播子研究. PhD thesis, 中国科学技术大学, 2002年. 1, 5, 14, 28, 29, 34, 35, 38, 22, 23, 204, 205, 570, 584, 585, 614, 615, 726, 746, 751, 798, 799, 803, 811, 812, 813, 814, 818, 819, 820, 823
- [20] William Rarita and Julian Schwinger. On a theory of particles with half-integral spin. *Physical Review*, 60(1):61, 1941. 1, 12, 14, 28, 29, 35, 177, 205, 207, 585, 615, 746, 950
- [21] P.A.M. Dirac. Relativistic wave equations. *Proceedings of the Royal Society A*, 155:447–459, 1936. 1
- [22] M. Fierz and W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proceedings of the Royal Society A*, 173(953):211–232, 1939. 1, 14, 28, 29, 34, 35, 204, 207, 570, 614, 709, 726, 751
- [23] J.D. Jackson. 《经典电动力学》. 人民教育出版社, 1980年. 1, 15, 16, 22, 24, 225, 229, 232, 246, 247, 402, 454, 465
- [24] 王永久, 唐智明. 《引力理论与引力效应》. 湖南科学技术出版社, 1990年. 1, 15, 225, 229
- [25] 张宗燧. 《电动力学及狭义相对论(第二版)》. 北京大学出版社, 2004年. 1, 15, 16, 22, 24, 225, 229, 249, 402, 454, 465
- [26] 周邦融. 《量子场论》. 高等教育出版社, 2007年. 2, 8, 9, 21, 22, 23, 35, 118, 119, 120, 121, 367, 372, 373, 385, 389, 413, 420, 427, 727
- [27] S. Weinberg. 《*The Quantum Theory of Fields Vol I: Foundations*》. Cambridge University Press, 1995年. 2, 8, 9, 21, 22, 23, 118, 119, 120, 121, 367, 385, 389, 413, 420, 427
- [28] 施水荣. 《各种粒子的量子方程》. 百度文库, 2006年. 2
- [29] 施水荣. 《粒子方程的重新表述-第一版》. 豆丁文库, 2015年. 2
- [30] S.R. Shi. *A New Formalism of Arbitrary Spin Particle Equations*. <http://vixra.org/abs/1610.0277>, 2016. 2
- [31] 施水荣. 《各种自旋粒子方程的重新表述与推广》. 豆丁文库, 2016年. 2
- [32] 施水荣. 《各种自旋粒子方程的重新表述与推广第二版》. 豆丁文库, 2018年. 2
- [33] 施水荣. 《各种自旋粒子方程的重新表述与推广第三版》. 豆丁文库, 2019年. 2
- [34] [德]W.顾莱纳, [美]繆勒. 《量子力学:对称性》. 北京大学出版社, 2002年. 4, 33, 1, 697, 700, 706
- [35] 北京大学物理系编写组. 《量子统计物理学》. 北京大学出版社, 1987年. 8, 116

- [36] A. Ashtekar. New variables for classical and quantum gravity. *Physical Review Letters*, 57(18):2244–2247, 1986. 11, 150, 151
- [37] E.P. Wigner. On unitary representations of the inhomogeneous lorentz group. *Ann.math*, 40(1):149, 1939. 16, 248
- [38] 何宝鹏, 熊钰庆编著. 《量子场论导论》. 华南理工大学出版社, 1991年. 16, 250
- [39] H.H.波戈留波夫, Д.В.希尔科夫. 《量子场论导引》. 科学出版社, 1957年. 20, 22, 23, 35, 357, 385, 388, 389, 413, 420, 427, 727
- [40] D.卢里著, 董明德等译. 《粒子和场》. 科学出版社, 1981年. 21, 22, 23, 35, 378, 385, 389, 413, 420, 427, 727
- [41] 杨泽森. 《高等量子力学》. 北京大学出版社, 2007年. 33, 697, 700
- [42] 刘希明. 《高等量子力学》. 山东科学技术出版社, 2002年. 33, 10, 697, 700
- [43] 张启仁. 《量子力学》. 科学出版社, 2002年. 33, 697, 700
- [44] J. Polchinski. 《String theory》. Cambridge University Press, 1998年. 38, 48, 825, 970, 979, 980, 981, 983
- [45] E. Kiritsis, 吕海港翻译. 《超弦理论导论》. 2002年. 39, 831
- [46] B. Zwiebach. 《A First Course in String Theory》. Cambridge University Press, 2004年. 48, 970, 979, 980, 981, 983
- [47] 李新洲, 徐建军. 《超对称物理导论》. 复旦大学出版社, 1992年. 48, 978, 979, 980, 981, 983
- [48] Ryoyu Utiyama. Invariant theoretical interpretation of interaction. *Physical Review*, 101(5):1597–1607, 1956. 158
- [49] T.W.B. Kibble. Lorentz invariance and the gravitational field. *Journal of Mathematical Physics*, 2(2):212–221, 1961. 158
- [50] 郭汉英, 吴詠时, 张元仲. 引力规范理论的一种方案. 科学通报, 18(2):72–74, 1973年. 158
- [51] 邹振隆, 黄珊, 张元仲, 李根道, 安琪, 陈时, 吴詠时, 张历宁, 何祚庥, 郭汉英. 关于引力规范理论的研究. 中国科学, 22(04):366–372, 1979年. 158
- [52] R.E. Behrends and C. Fronsdal. *Physical Review*, 106(345), 1957. 584, 710, 711
- [53] C. Fronsdal. *Nuovo Cimento Suppl.*, 9(416), 1958. 584, 710, 711