ON THE GEOMETRY OF AXES OF COMPLEX CIRCLES OF PARTITION - PART 1

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ABSTRACT. In this paper we continue the development of the circles of partition by introducing a certain geometry of the axes of complex circles of partition. We use this geometry to verify the condition in the squeeze principle in special cases with regards to the orientation of the axes of complex circles of partition.

1. **Introduction**

In our seminal paper [1], we introduced and developed the method of circles of partition. This method is underpinned by a combinatorial structure that encodes certain additive properties of the subsets of the integers and invariably equipped with a certain geometric structure that allows to view the elements as points in the plane whose weights are just elements of the underlying subset. We call this combinatorial structure the circles of partition and is refereed to as the set of points

$$\mathcal{C}(n,\mathbb{M}) = \{[x] \mid x, n-x \in \mathbb{M}\}.$$

Each point in this set - except the center point - must have a uniquely distinct point that are join by a line which we refer to as an axis of the CoP. We denote an axis of a CoP with $\mathbb{L}_{[x],[y]}$ and an axis contained in the CoP as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}) \text{ which means } [x],[y] \in \mathcal{C}(n,\mathbb{M}) \text{ with } x+y=n.$$

The method of circles of partition and their associated structures have been well advanced in [2], where the corresponding points have complex numbers as their weights and a line (axis) joining co-axis points. The following structure was considered as a complex circle of partition

$$\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) = \{[z] \mid z, n-z \in \mathbb{C}_M, \Im(z)^2 = \Re(z)\,(n-\Re(z))\}$$

where

$$\mathbb{C}_{\mathbb{M}} := \{z = x+iy \mid x \in \mathbb{M}, y \in \mathbb{R}\} \subset \mathbb{C}$$

with $\mathbb{M} \subseteq \mathbb{N}$. We abbreviate this complex additive structure as cCoP. The condition $\Im(z)^2 = \Re(z)(n-\Re(z)$ is referred to as the **circle condition** and it pretty much guarantees that all points on the cCoP lie on a circle in the complex. This circle is the embedding circle of the cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$, denoted as $\mathfrak{C}_n$. The embedding circles of cCoPs have the property that they reside fully inside those embedding

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circle with a relatively larger generators, except the origin as a common point [2]. For each axis with the following assignment

$$\mathbb{L}_{[z_1],[z_1]} \hat{\in} \mathcal{C}(n, \mathbb{C}_{\mathbb{M}}) \text{ which means } [z_1],[z_2] \in \mathcal{C}(n, \mathbb{C}_{\mathbb{M}}) \text{ with } z_1 + z_2 = n.$$

The structure of the complex circles of partition is much more versatile and has extra structures that are not readily available in the structures of circles of partition. Most notably, for each axis $\mathbb{L}_{[z],[n-z]}$ of a cCoP there exists

$$\mathbb{L}_{\overline{[z]},\overline{[n-z]}}$$

a *conjugate axis*, where $\overline{[z]}, \overline{[n-z]}$. The space occupied by the embedding circles of partition and correspondingly outside the embedding circle had turned out to be very interesting, since this notion can be passed down to studying a certain ordering principle of the points of two interacting axes of distinct cCoPs. Much more striking is the fact which comes with ease by virtue of the circle condition that

$$|\mathbb{L}_{[z_1],[z_2]}| = n$$

for any axis $\mathbb{L}_{[z_1],[z_2]} \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}}) = \{[z] \mid z, n-z \in \mathbb{C}_M, \Im(z)^2 = \Re(z)\,(n - \Re(z))\}$. The squeeze principle [3] can be considered as a black box for studying the binary Goldbach conjecture. A slightly different version of this principle appears in [2]. For the sake of the reader, we provide a brief recap of this elegant principle as below

Lemma 1.1 (The squeeze principle)**.** *Let $\mathbb{B} \subset \mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n+t, \mathbb{C}_{\mathbb{M}})$ with $t \geq 4$ be non–empty cCoPs with integers n, t, s of the same parity. If there exist an axis $\mathbb{L}_{[v_1],[w_1]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ with $w_1 \in \mathbb{C}_{\mathbb{B}}$ and an axis $\mathbb{L}_{[v_2],[w_2]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_{\mathbb{M}})$ with $v_2 \in \mathbb{C}_{\mathbb{B}}$ such that*

$$\Re(v_1) < \Re(v_2) \text{ and } \Re(w_1) < \Re(w_2) \tag{1.1}$$

then there exists an axis $\mathbb{L}_{[v_2],[w_1]} \hat{\in} \mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{B}})$ with $0 < s < t$. Hence also $\mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{M}})$ is not empty.

Proof. From the existence of an axis $\mathbb{L}_{[v_1],[w_1]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ follows $\Re(w_1) = n - \Re(v_1)$. With the requirement (1.1) we get

$$\Re(w_1) > n - \Re(v_2). \tag{1.2}$$

On the other hand from the existence of an axis $\mathbb{L}_{[v_2],[w_2]} \hat{\in} \mathcal{C}^o(n+t, \mathbb{C}_{\mathbb{M}})$ follows $\Re(w_2) = n + t - \Re(v_2)$ and with the requirement (1.1) and the result (1.2) we get

$$\begin{aligned} n - \Re(v_2) < \Re(w_1) &< n + t - \Re(v_2) \mid +\Re(v_2) \\ n < \Re(w_1) + \Re(v_2) &< n + t \\ n < n + s &< n + t. \end{aligned}$$

By virtue of the requirements $w_1, v_2 \in \mathbb{C}_{\mathbb{B}}$ and $n + s = \Re(w_1) + \Re(v_2)$ there is an axis $\mathbb{L}_{[v_2],[w_1]} \hat{\in} \mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{B}})$ and hence holds $\mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{B}}) \neq \emptyset$. And from $\mathbb{B} \subset \mathbb{M}$ follows immediately $\mathbb{C}_{\mathbb{B}} \subset \mathbb{C}_{\mathbb{M}}$ and therefore holds also $\mathcal{C}^o(n+s, \mathbb{C}_{\mathbb{M}}) \neq \emptyset$. This completes the proof. $\square$

The Lemma 1.1 referred to as the squeeze principle may be regarded as a fundamental tool set for investigating the viability of dividing integers of a particular parity, utilizing constituent elements originating from a specific subset of the integers. The mechanism operates by discerning a pair of cCoPs that are both non-vacuous and share a common base set. Subsequently, supplementary cCoPs that are non-vacuous and have generators restrained within the interstice of these two generators are identified. This principle may be applied in a resourceful manner to investigate the overarching matter of the practicality of divvying up numbers such that each addend is a member of the identical subset of positive integers. The squeeze principle tends to provide an impetus to investigate the conditions for which it holds, by investigating using a particularly innate geometry. This quest is motivated and driven by the following deep questions:

Question 1. *How the are the notion of interiors and exteriors with respect to cCoPs facilitate proving the squeeze principle?*

Question 2. *Which role do the imaginary weights of members of cCoPs play?*

Question 3. *Are the embedding circles of cCoPs the key for proving the BGC?*

2. Orientations of axes of Complex circles of partition and related Geometries

In this section we introduce and study the geometry of the axis of cCoPs. We launch the following languages as a precursor to our studies. In this section we consider only axes of distinct cCoPs $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ such that $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ with $\Re(z_1) \neq \Re(w_1)$ and $\Re(z_2) \neq \Re(w_2)$.

Definition 2.1. Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be a non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$. We say it is an axis of positive orientation if the gradient is positive. On the other hand if the gradient is negative, then we say it is an axis of the cCoP with a negative orientation. We say the axis of the cCoP is horizontal if it is parallel to the real axis of the complex plane. We denote the gradient of the axis $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ with

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \frac{\Im(z_2) - \Im(z_1)}{\Re(z_2) - \Re(z_1)}.$$

Definition 2.2. Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$. We say the axes are of **homogeneous** orientation if they point to the same direction. We denote this relation with $\mathbb{L}_{[z_1],[z_2]} \,||\, \mathbb{L}_{[w_1],[w_2]}$. If they point to different directions, then we say the axes are of **mixed** orientation. We denote the axes of distinct orientation that are perpendicular with the relation $\mathbb{L}_{[z_1],[z_2]} \perp \mathbb{L}_{[w_1],[w_2]}$. If they point to different directions and do not intersect, then we say the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ are **skewed**.

Proposition 2.3. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z],[\overline{z}]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w],[\overline{w}]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$. Then*

$$\mathbb{L}_{[z],[\overline{z}]} \,||\, \mathbb{L}_{[w],[\overline{w}]}.$$

Proof. The claim follows since $\mathbb{L}_{[z],[\overline{z}]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w],[\overline{w}]} \hat{\in} \mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ are the degenerate axes of their corresponding cCoPs and each degenerate axis must be parallel to the imaginary axis. It follows by transitivity that the axes must be parallel to each other. □

Lemma 2.4. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ such that the axes are of positive orientation. If $\mathbb{L}_{[z_1],[z_2]} \,||\, \mathbb{L}_{[w_1],[w_2]}$, then $\Re(z_1) < \Re(w_1)$ and $\Re(z_2) < \Re(w_2)$.*

Proof. We note that the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ passes through the point $(\frac{n}{2},0)$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ also passes through the point $(\frac{m}{2},0)$ with $m > n$. It follows that $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})$ so that we can write

$$\frac{\Im(z_2)}{\Re(z_2) - \frac{n}{2}} = \frac{\Im(w_2)}{\Re(w_2) - \frac{m}{2}}$$

since $\mathbb{L}_{[z_1],[z_2]} \,||\, \mathbb{L}_{[w_1],[w_2]}$. Since $m > n$, it follows that $\mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]$. Since the axes are of positive orientation with $\mathbb{L}_{[z_1],[z_2]} \,||\, \mathbb{L}_{[w_1],[w_2]}$ then it implies that $\Im(z_2) < \Im(w_2)$ so that $\Re(z_2) - \frac{n}{2} < \Re(w_2) - \frac{m}{2} \iff \Re(z_2) < \Re(w_2)$. Let us join the point w_2 to the point z_2 by a straight line, then it is easy to see that the gradient of this line is given by

$$\frac{\Im(w_2) - \Im(z_2)}{\Re(w_2) - \Re(z_2)} > 0.$$

Similarly, let us join the point z_1 to the point z_2 by a straight line. We compute the gradient of this line as

$$\frac{\Im(z_1) - \Im(w_1)}{\Re(z_1) - \Re(w_1)} < 0.$$

Let us suppose that

$$\frac{\Im(z_1) - \Im(w_1)}{\Re(z_1) - \Re(w_1)} > 0$$

then $\Im(z_1) - \Im(w_1) > 0$ since $\mathbb{L}_{[z_1],[z_2]} \,||\, \mathbb{L}_{[w_1],[w_2]}$ with $m > n$ so that $\Re(z_1) > \Re(w_1)$. It follows that

$$\Re(w_1) = \Re(\overline{w_1}) < \Re(z_1) = \Re(\overline{z_1})$$

with

$$|w_1 - (\frac{m}{2},0)| = |\overline{w_1} - (\frac{m}{2},0)| > |z_1 - (\frac{m}{2},0)| = |\overline{z_1} - (\frac{m}{2},0)|.$$

Since

$$|\Im(w_1)| = |\Im(\overline{w_1})| > |\Im(z_1)| = |\Im(\overline{z_1})|$$

and the points $[w_1], [\overline{w_1}]$ are opposite points on the embedding circle $\mathfrak{C}_m$ and similarly the points $[z_1], [\overline{z_1}]$ on $\mathfrak{C}_n$ with $m > n$, it follows that $\mathfrak{C}_m$ and $\mathfrak{C}_n$ cannot have a common point at the origin. This violates the Big Bang theorem. □

We obtain a similar result of the natural ordering principle of the real part of axes of cCoPs in the case where the axes are all of negative orientation.

Lemma 2.5. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ such that the axes are of negative orientation. If $\mathbb{L}_{[z_1],[z_2]} \,||\, \mathbb{L}_{[w_1],[w_2]}$, then $\Re(z_1) < \Re(w_1)$ and $\Re(z_2) < \Re(w_2)$.*

Proof. We note that the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ passes through the point $(\frac{n}{2},0)$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ also passes through the point $(\frac{m}{2},0)$ with $m>n$. It follows that $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})$ so that we can write

$$\frac{\Im(z_2)}{\Re(z_2)-\frac{n}{2}} = \frac{\Im(w_2)}{\Re(w_2)-\frac{m}{2}}$$

since $\mathbb{L}_{[z_1],[z_2]} \,||\, \mathbb{L}_{[w_1],[w_2]}$. Since $m>n$, it follows that $\mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]$. Since the axes are of negative orientation with $\mathbb{L}_{[z_1],[z_2]} \,||\, \mathbb{L}_{[w_1],[w_2]}$ then it implies that $\Im(z_1) = -\Im(z_2) < -\Im(w_2) = \Im(w_1)$ so that we have

$$\frac{-\Im(z_1)}{\Re(z_2)-\frac{n}{2}} = \frac{-\Im(w_1)}{\Re(w_2)-\frac{m}{2}} \iff \frac{\Im(z_1)}{\Re(z_2)-\frac{n}{2}} = \frac{\Im(w_1)}{\Re(w_2)-\frac{m}{2}}.$$

Since $\Im(w_1) > \Im(z_1)$, it follows that $\Re(z_2)-\frac{n}{2} < \Re(w_2)-\frac{m}{2} \iff \Re(z_2) < \Re(w_2)$ for $m>n$. Let us join the point w_2 to the point z_2 by a straight line, then it is easy to see that the gradient of this line is given by

$$\frac{\Im(w_2)-\Im(z_2)}{\Re(w_2)-\Re(z_2)} < 0$$

since $\Im(z_2) < \Im(w_2)$. Similarly, let us join the point z_1 to the point z_2 by a straight line. We compute the gradient of this line as

$$\frac{\Im(z_1)-\Im(w_1)}{\Re(z_1)-\Re(w_1)} > 0.$$

Let us suppose that

$$\frac{\Im(z_1)-\Im(w_1)}{\Re(z_1)-\Re(w_1)} > 0$$

then $\Im(z_1)-\Im(w_1) < 0$ since $\mathbb{L}_{[z_1],[z_2]} \,||\, \mathbb{L}_{[w_1],[w_2]}$ with $m>n$ so that $\Re(z_1) > \Re(w_1)$. It follows that

$$\Re(w_1) = \Re(\overline{w_1}) < \Re(z_1) = \Re(\overline{z_1})$$

with

$$|w_1 - (\frac{m}{2},0)| = |\overline{w_1} - (\frac{m}{2},0)| > |z_1 - (\frac{m}{2},0)| = |\overline{z_1} - (\frac{m}{2},0)|.$$

Since

$$|\Im(w_1)| = |\Im(\overline{w_1})| > |\Im(z_1)| = |\Im(\overline{z_1})|$$

and the points $[w_1],[\overline{w_1}]$ are opposite points on the embedding circle $\mathfrak{C}_m$ and similarly the points $[z_1],[\overline{z_1}]$ on $\mathfrak{C}_n$ with $m>n$, it follows that $\mathfrak{C}_m$ and $\mathfrak{C}_n$ cannot have a common point at the origin. This violates the Big Bang theorem. $\square$

We prove an important fact concerning the relationship between an axis of a cCoP and other axes of cCoPs with higher generators. It basically purports that the slope of cCoPs with higher generators must be relatively steeper so long as these axis intersect. We launch formally the following fact in the lemma below.

Lemma 2.6. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ for $m>n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the diameter that passes through the origin, then*

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}).$$

Proof. Suppose the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ intersect at the point $v \in \mathbb{C}$. We note that the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ passes through the point $(\frac{n}{2}, 0)$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ also passes through the point $(\frac{m}{2}, 0)$ so that we can compute the gradient of the axes. We obtain

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) = \frac{\Im(v)}{\Re(v) - \frac{m}{2}}$$

and

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

Let us suppose that $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) \leq \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$, then it follows that

$$\frac{\Im(v)}{\Re(v) - \frac{m}{2}} \leq \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

Since the axes intersect at a point above the diameter that passes through the origin, it must be that $\Im(v) > 0$ so that we obtain

$$\frac{1}{\Re(v) - \frac{m}{2}} \leq \frac{1}{\Re(v) - \frac{n}{2}} \iff \Re(v) - \frac{n}{2} \leq \Re(v) - \frac{m}{2} \iff m < n$$

which violates the inequality $m > n$. □

We obtain an analogous result in the case the axes intersect below the diameter that passes through the origin.

Lemma 2.7. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the diameter that passes through the origin, then*

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}).$$

Proof. Suppose the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$ intersect at the point $v \in \mathbb{C}$. We note that the axes $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ passes through the point $(\frac{n}{2}, 0)$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ also passes through the point $(\frac{m}{2}, 0)$ so that we can compute the gradient of the axes. We obtain

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) = \frac{\Im(v)}{\Re(v) - \frac{m}{2}}$$

and

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

Let us suppose that $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) \geq \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$, then it follows that

$$\frac{\Im(v)}{\Re(v) - \frac{m}{2}} \geq \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

Since the axes intersect at a point below the diameter that passes through the origin, it must be that $\Im(v) < 0$ so that we obtain

$$\frac{1}{\Re(v) - \frac{m}{2}} \leq \frac{1}{\Re(v) - \frac{n}{2}} \iff \Re(v) - \frac{n}{2} \leq \Re(v) - \frac{m}{2} \iff m < n$$

which violates the inequality $m > n$. □

We obtain a certain characterization of the gradient of axes of two interacting cCoPs. This is an immediate consequence of Lemma 2.6. It will also serve in many ways as a guiding principle for our further investigations.

Theorem 2.8. *Let $\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]}\hat{\in}\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]}\hat{\in}\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ for $m>n$ with $\Re(z_1)<\Re(z_2)$ and $\Re(w_1)<\Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the diameter that passes through the origin with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})<0$, then*

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})<0.$$

Proof. Let us assume to the contrary that

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})\geq 0$$

then $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})>0$, since $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})\neq 0$. The axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the diameter that passes through the origin so that

$$0>\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})>\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})>0$$

by virtue of Lemma 2.6, which is absurd. □

Theorem 2.9. *Let $\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]}\hat{\in}\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]}\hat{\in}\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ for $m>n$ with $\Re(z_1)<\Re(z_2)$ and $\Re(w_1)<\Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the diameter that passes through the origin with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})>0$, then*

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})>0.$$

Proof. Let us assume to the contrary that

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})\leq 0$$

then $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})<0$, since $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})\neq 0$. The axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the diameter that passes through the origin so that

$$0<\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})<\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})<0$$

by virtue of Lemma 2.7, which is absurd. □

We obtain variants of Theorem 2.8 and Theorem 2.9 in the sequel.

Theorem 2.10. *Let $\mathbb{M}\subseteq\mathbb{N}$ and $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]}\hat{\in}\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]}\hat{\in}\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ for $m>n$ with $\Re(z_1)<\Re(z_2)$ and $\Re(w_1)<\Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the diameter that passes through the origin with $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})>0$, then*

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})>0.$$

Proof. The axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the diameter that passes through the origin so that

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})>\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})>0$$

by virtue of Lemma 2.6, and the claim follows. □

Theorem 2.11. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the diameter that passes through the origin with $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$, then*

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < 0.$$

Proof. The axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the diameter that passes through the origin so that

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$$

by virtue of Lemma 2.7, and the claim follows. □

It is worthwhile noting that we have only confirmed the natural ordering principle of the real parts of the upper axes points of cCoPs in the case the corresponding axes of distinct cCoPs are parallel. We would like this behaviour to be propagated among the remaining configuration of the axes of cCoPs that we have not yet exhaust. It is possible that certain imagined configuration may not hold in this geometry. In the following sequel, we will examine this naturally exhibiting principle in the cases where any two axes of distinct non-empty cCoPs are skewed. We launch the following result.

Lemma 2.12. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ are skewed with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$ then $\Re(z_1) < \Re(w_1)$ and $\Re(z_2) < \Re(w_2)$.*

Proof. Let axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ be skewed with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$. Let us join z_2 to w_2 by a straight line. Then by the embedding $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ with the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ passing through the point $(\frac{n}{2}, 0)$ and $(\frac{m}{2}, 0)$, respectively with $m > n$, it follows that $\Im(w_2) > \Im(z_2)$. Let us suppose that the gradient of this line

$$\frac{\Im(w_2) - \Im(z_2)}{\Re(w_2) - \Re(z_2)} < 0.$$

Then it follows that $\Re(w_2) < \Re(z_2)$ so that the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ must intersect at a point since $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$ and $\mathbb{L}_{[w_1],[w_2]}$ passes through the point $(\frac{m}{2}, 0)$ with $m > n$, contradicting the requirement that the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ are skewed . Thus we must have $\Re(w_2) > \Re(z_2)$. Similarly let us join the point z_1 to the point w_1 by a straight line. Then by the embedding $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ with the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ passing through the point $(\frac{n}{2}, 0)$ and $(\frac{m}{2}, 0)$, respectively with $m > n$ and $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > 0$, it follows that $\Im(w_1) < \Im(z_1) < 0$. Let us suppose that the gradient of this line

$$\frac{\Im(w_1) - \Im(z_1)}{\Re(w_1) - \Re(z_1)} > 0$$

then it implies that $\Re(z_1) > \Re(w_1)$. It follows that

$$\Re(w_1) = \Re(\overline{w_1}) < \Re(z_1) = \Re(\overline{z_1})$$

with

$$|w_1 - (\frac{m}{2}, 0)| = |\overline{w_1} - (\frac{m}{2}, 0)| > |z_1 - (\frac{m}{2}, 0)| = |\overline{z_1} - (\frac{m}{2}, 0)|.$$

Since

$$|\Im(w_1)| = |\Im(\overline{w_1})| > |\Im(z_1)| = |\Im(\overline{z_1})|$$

and the points $[w_1], [\overline{w_1}]$ are opposite points on the embedding circle $\mathfrak{C}_m$ and similarly the points $[z_1], [\overline{z_1}]$ on $\mathfrak{C}_n$ with $m > n$, it follows that $\mathfrak{C}_m$ and $\mathfrak{C}_n$ cannot have a common point at the origin. This violates the Big Bang theorem. □

We obtain an analogous result in the case all the axis are of negative orientation and slopes down negatively.

Lemma 2.13. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ are skewed with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$ then $\Re(z_1) < \Re(w_1)$ and $\Re(z_2) < \Re(w_2)$.*

Proof. Let the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ be skewed with $0 > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) > \mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})$. Let us join z_1 to w_1 by a straight line. Then by the embedding $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ with the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ passing through the point $(\frac{n}{2}, 0)$ and $(\frac{m}{2}, 0)$, respectively with $m > n$, it follows that $\Im(w_1) < \Im(z_1) < 0$. Let us suppose that the gradient of this line

$$\frac{\Im(w_1) - \Im(z_1)}{\Re(w_1) - \Re(z_1)} < 0$$

then it follows that $\Re(w_1) < \Re(z_1)$. It follows that

$$\Re(w_1) = \Re(\overline{w_1}) < \Re(z_1) = \Re(\overline{z_1})$$

with

$$|w_1 - (\frac{m}{2}, 0)| = |\overline{w_1} - (\frac{m}{2}, 0)| > |z_1 - (\frac{m}{2}, 0)| = |\overline{z_1} - (\frac{m}{2}, 0)|.$$

Since

$$|\Im(w_1)| = |\Im(\overline{w_1})| > |\Im(z_1)| = |\Im(\overline{z_1})|$$

and the points $[w_1], [\overline{w_1}]$ are opposite points on the embedding circle $\mathfrak{C}_m$ and similarly the points $[z_1], [\overline{z_1}]$ on $\mathfrak{C}_n$ with $m > n$, it follows that $\mathfrak{C}_m$ and $\mathfrak{C}_n$ cannot have a common point at the origin. This violates the Big Bang theorem. Similarly, let us join z_2 to w_2 by a straight line. Then by the embedding $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ with the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ passing through the point $(\frac{n}{2}, 0)$ and $(\frac{m}{2}, 0)$, respectively with $m > n$, it follows that $\Im(w_2) < \Im(z_2) < 0$. Let us suppose that the gradient of this line

$$\frac{\Im(w_2) - \Im(z_2)}{\Re(w_2) - \Re(z_2)} > 0.$$

Then it follows that $\Re(w_2) < \Re(z_2)$ so that the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ must intersect at a point since $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$ and $\mathbb{L}_{[w_1],[w_2]}$ passes through the point $(\frac{m}{2}, 0)$ with $m > n$, contradicting the requirement that the axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ are skewed . Thus we must have $\Re(w_2) > \Re(z_2)$. □

We examine the remaining skew case of interacting axes of distinct cCoPs in the scenario where they have gradient of opposite signs.

Lemma 2.14. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ are skewed with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > 0$ and $\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) < 0$ such that*

$$|\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})| > |\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})|$$

then $\Re(z_1) < \Re(w_1)$ and $\Re(z_2) < \Re(w_2)$.

Proof. Under the requirement $|\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]})| > |\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})|$ with the embedding

$$\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$$

it implies that $\Im(z_1) > \Im(w_1)$ and $\Im(z_2) < \Im(w_2)$. Let us join the point z_1 to the point w_1 by a straight line and suppose for the gradient of this line

$$\frac{\Im(z_1) - \Im(w)}{\Re(z_1) - \Re(w_1)} > 0.$$

then it follows that $\Re(w_1) < \Re(z_1)$. It follows that

$$\Re(w_1) = \Re(\overline{w_1}) < \Re(z_1) = \Re(\overline{z_1})$$

with

$$|w_1 - (\frac{m}{2}, 0)| = |\overline{w_1} - (\frac{m}{2}, 0)| > |z_1 - (\frac{m}{2}, 0)| = |\overline{z_1} - (\frac{m}{2}, 0)|.$$

Since

$$|\Im(w_1)| = |\Im(\overline{w_1})| > |\Im(z_1)| = |\Im(\overline{z_1})|$$

and the points $[w_1], [\overline{w_1}]$ are opposite points on the embedding circle $\mathfrak{C}_m$ and similarly the points $[z_1], [\overline{z_1}]$ on $\mathfrak{C}_n$ with $m > n$, it follows that $\mathfrak{C}_m$ and $\mathfrak{C}_n$ cannot have a common point at the origin. This violates the Big Bang theorem. Similarly, let us join z_2 to w_2 by a straight line and suppose of the gradient of this line

$$\frac{\Im(z_2) - \Im(w_2)}{\Re(z_2) - \Re(w_2)} < 0.$$

Then it implies that $\Re(z_2) > \Re(w_2)$ since $\Im(z_2) - \Im(w_2) < 0$. Since $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > 0$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ must pass through the point $(\frac{m}{2}, 0)$ with $m > n$, it follows that both axes $\mathbb{L}_{[z_1],[z_2]}$ and $\mathbb{L}_{[w_1],[w_2]}$ must intersect at a point. This violates the requirement that the axes are skewed. □

Thus far we have almost exhausted this naturally inherent ordering behaviour of the real part of the axes points of interacting cCoPs with distinct generators in the cases where the axes are parallel and skewed. We are still left with investigating this property for interacting intersecting axes of distinct cCoPs intersecting. It is worth noting that one may likely run into a deadlock replicating the same argument for the cases where the two axes of distinct cCoP intersects. We now examine the corresponding converse of Lemma 2.6 and 2.7.

Lemma 2.15. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$, then the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ cannot intersect at a point above the diameter passing through the origin.*

Proof. Suppose the axes $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point above the diameter passing through the origin with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) < \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$. Now let $v \in \mathbb{C}$ be their point of intersection, then $\Im(v) > 0$ and we obtain

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) = \frac{\Im(v)}{\Re(v) - \frac{m}{2}}$$

and

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

It follows that

$$\frac{\Im(v)}{\Re(v) - \frac{m}{2}} < \frac{\Im(v)}{\Re(v) - \frac{n}{2}} \iff \Re(v) - \frac{n}{2} < \Re(v) - \frac{m}{2} \iff m < n$$

which violates the inequality $m > n$. □

Lemma 2.16. *Let $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ be non–empty cCoPs with $\mathbb{L}_{[z_1],[z_2]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and $\mathbb{L}_{[w_1],[w_2]} \hat{\in} \mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ for $m > n$ with $\Re(z_1) < \Re(z_2)$ and $\Re(w_1) < \Re(w_2)$. If $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$, then the axis $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ cannot intersect at a point below the diameter passing through the origin.*

Proof. Suppose the axes $\mathbb{L}_{[z_1],[z_2]}$ and the axis $\mathbb{L}_{[w_1],[w_2]}$ intersect at a point below the diameter passing through the origin with $\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) > \mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]})$. Now let $v \in \mathbb{C}$ be their point of intersection, then $\Im(v) < 0$ and we obtain

$$\mathrm{Grad}(\mathbb{L}_{[w_1],[w_2]}) = \frac{\Im(v)}{\Re(v) - \frac{m}{2}}$$

and

$$\mathrm{Grad}(\mathbb{L}_{[z_1],[z_2]}) = \frac{\Im(v)}{\Re(v) - \frac{n}{2}}.$$

It follows that

$$\frac{\Im(v)}{\Re(v) - \frac{m}{2}} > \frac{\Im(v)}{\Re(v) - \frac{n}{2}} \iff \Re(v) - \frac{n}{2} < \Re(v) - \frac{m}{2} \iff m < n$$

which violates the inequality $m > n$. □

It is worth noting that by piecing Lemma 2.6 with Lemma 2.16 one obtains an equivalent statement. The same equivalence also hold by piecing together Lemma 2.7 and Lemma 2.15. These equivalence in their own right could serve as benchmark for proving or disproving such configurations in the geometry. It turns out that the arguments and the method employed in this paper does not help to analyze the cases where arbitrary axes of distinct cCoPs intersect. Consequently, we will analyze these cases in a separate paper using a different method.

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