On Clifford-valued Actions, Generalized Dirac Equation and Quantization of Branes

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Abstract

We explore the construction of a generalized Dirac equation via the introduction of the notion of Clifford-valued actions, and which was inspired by the work of [1], [2] on the De Donder-Weyl theory formulation of field theory. Crucial in this construction is the evaluation of the *exponentials* of *multivectors* associated with Clifford (hypercomplex) analysis. Exact *matrix* solutions (instead of spinors) of the generalized Dirac equation in $D = 2, 3$ spacetime dimensions were found. This formalism can be extended to curved spacetime backgrounds like it happens with the Schroedinger-Dirac equation. We conclude by proposing a wave-functional equation governing the quantum dynamics of branes living in C-spaces (Clifford spaces), and which is based on the De Donder-Weyl Hamiltonian formulation of field theory.

Keywords : Quantum Mechanics; Clifford Algebras; Dirac Equation; De Donder-Weyl theory.

1 Dirac Equation, Exponentials of Multivectors

The $4D$ Dirac equation, in units of $\hbar = c = 1$, is

$$(i\gamma^\mu \partial_\mu \ - m\ )\psi \ = \ 0, \quad \mu = 0, 1, 2, 3 \tag{1}$$

where ψ is a Dirac spinor, a column matrix with 4 complex entries. It is well known among the experts that Dirac spinors are left/right ideal elements of the complex Clifford $Cl(4, C)$ algebra in $4D$. Such left/right ideals of the Clifford algebra can be represented by 4×4 complex matrices with one, and only one, non-vanishing column/row, while the remaining three columns/rows are set to zero.

Inspired by the work of [1], [2] we shall begin to generalize the Dirac equation (1), firstly, by replacing the spinor ψ with a matrix Ψ defined as $\Psi \equiv R\, e^{-iS_\mu \gamma^\mu} = R\, e^{-iS(\frac{S_\mu}{S})\gamma^\mu}$, where S^μ is a four-vector extension of the Hamilton-Jacobi function $\mathcal{S}$ and whose norm is $S \equiv \sqrt{S_\mu S^\mu}$. Performing a power series Taylor expansion and taking into account $\gamma^\mu \gamma^\nu = \eta^{\mu\nu} \mathbf{1} + \gamma^{[\mu\nu]}$, one arrives at

$$R\, e^{-iS_\mu \gamma^\mu} \;=\; R\, e^{-iS(\frac{S_\mu}{S})\gamma^\mu} \;=\; R\, cos(S)\; \mathbf{1} \;-i\; \gamma^\mu\; (\frac{RS_\mu}{S})\; sin(S) \qquad (2)$$

One could extend the above definition of Ψ by writing $\Psi = Re^{-iS_M \Gamma^M}$ where the Γ^M's span the $2^4 = 16$ matrices $\mathbf{1}, \gamma^\mu, \gamma^{[\mu\nu]}, \gamma^{[\mu\nu\sigma]}, \gamma^{[\mu\nu\sigma\tau]}$ associated with the 16-dim Clifford algebra $Cl(4, C)$. However, one cannot any longer write the Clifford-valued quantity Ψ in the same functional form displayed by eq-(2)

$$\Psi \;=\; R\, e^{-iS_M \Gamma^M} \;\neq\; R\, cos[\sqrt{S_M S^M}]\; \mathbf{1} \;-i\; \Gamma^N\; (\frac{RS_N}{\sqrt{S_M S^M}})\; sin[\sqrt{S_M S^M}] \quad (3)$$

Since Ψ in (3) is a 4×4 matrix it can be expanded in a Clifford basis as $\Psi = \Psi_M \Gamma^M$, but now the expressions for the coefficients Ψ_M's are very complicated functions of R, and the polyvector-valued entries S_M associated with the Clifford-valued $\mathbf{S} = S_M \Gamma^M$ extension of the Hamilton-Jacobi function $\mathcal{S}$.

The exponentials of generalized multivectors associated with real Clifford algebras in $3D$ have been found explicitly by [4] (see also [5]). Given the multivector in $3D$ $\mathbf{A} = a_0 + a_i e_i + a_{ij} e_{ij} + a_{123} I$, where I is a pseudo-scalar, its exponential $exp(\mathbf{A}) = \mathbf{B}$ is another multivector $\mathbf{B} = b_0 + b_i e_i + b_{ij} e_{ij} + b_{123} I$ whose components (coefficients) $b_0, b_i, b_{ij}, b_{123}$ are explicitly given in terms of $a_0, a_i, a_{ij}, a_{123}$. For instance, in the $Cl(0, 3)$ algebra case, the coefficients found by [4] are given by

$$b_0 \;=\; \frac{1}{2}\; e^{a_0}\; \left(\; e^{a_{123}}\; cos(a_+) \;+\; e^{-a_{123}}\; cos(a_-)\; \right) \qquad (4a)$$

$$b_{123} \;=\; \frac{1}{2}\; e^{a_0}\; \left(\; e^{a_{123}}\; cos(a_+) \;-\; e^{-a_{123}}\; cos(a_-)\; \right) \qquad (4b)$$

$$b_1 \;=\; \frac{1}{2}\; e^{a_0}\; \left(\; e^{a_{123}}\; (a_1 - a_{23})\; \frac{sin(a_+)}{a_+} \;+\; e^{-a_{123}}\; (a_1 + a_{23})\; \frac{sin(a_-)}{a_-} \right) \quad (4c)$$

$$b_2 \;=\; \frac{1}{2}\; e^{a_0}\; \left(\; e^{a_{123}}\; (a_2 + a_{13})\; \frac{sin(a_+)}{a_+} \;+\; e^{-a_{123}}\; (a_2 - a_{13})\; \frac{sin(a_-)}{a_-} \right) \quad (4d)$$

$$b_3 \;=\; \frac{1}{2}\; e^{a_0}\; \left(\; e^{a_{123}}\; (a_3 - a_{12})\; \frac{sin(a_+)}{a_+} \;+\; e^{-a_{123}}\; (a_3 + a_{12})\; \frac{sin(a_-)}{a_-} \right) \quad (4e)$$

$$b_{12} \;=\; \frac{1}{2}\; e^{a_0}\; \left(\; - e^{a_{123}}\; (a_3 - a_{12})\; \frac{sin(a_+)}{a_+} \;+\; e^{-a_{123}}\; (a_3 + a_{12})\; \frac{sin(a_-)}{a_-} \right) \tag{4f}$$

$$b_{13} \;=\; \frac{1}{2}\; e^{a_0}\; \left(\; e^{a_{123}}\; (a_2 + a_{13})\; \frac{sin(a_+)}{a_+} \;-\; e^{-a_{123}}\; (a_2 - a_{13})\; \frac{sin(a_-)}{a_-} \right) \tag{4g}$$

$$b_{23} \;=\; \frac{1}{2}\; e^{a_0}\; \left(\; - e^{a_{123}}\; (a_1 - a_{23})\; \frac{sin(a_+)}{a_+} \;+\; e^{-a_{123}}\; (a_1 + a_{23})\; \frac{sin(a_-)}{a_-} \right) \tag{4h}$$

where

$$a_+ \;=\; \sqrt{(a_3 - a_{12})^2 \;+\; (a_2 + a_{13})^2 \;+\; (a_1 - a_{23})^2} \tag{5a}$$

$$a_- \;=\; \sqrt{(a_3 + a_{12})^2 \;+\; (a_2 - a_{13})^2 \;+\; (a_1 + a_{23})^2} \tag{5b}$$

To find the explicit components of the exponential of a multivector associated with a Clifford algebra in $4D$ is a more difficult task.

Let us then generalize the Dirac equation (1) by writing the following equation

$$(i\gamma^\mu \partial_\mu)^{\alpha\beta}\; \Psi^\sigma_\beta \;- m\; \Psi^{\alpha\sigma} \;=\; 0, \quad \mu = 0, 1, 2, 3 \tag{6}$$

where $\Psi^{\alpha\sigma}$ is no longer a Dirac spinor but a 4×4 complex matrix given by $\Psi = Re^{-iS_\mu \gamma^\mu}$ and whose Taylor expansion is provided by eq-(2). Inserting the expression for Ψ provided by eq-(2) into eq-(6) yields the following 3 equations after matching the terms multiplying the unit matrix and the $\gamma^\mu, \gamma^{[\mu\nu]}$ matrices

$$\partial_\mu (\; \frac{RS^\mu}{S}\; sin(S)\;) \;=\; m\; R\; cos(S) \tag{7}$$

$$\partial_\mu (\; R\; cos(S)\;) \;=\; -\; m\; \frac{RS_\mu}{S}\; sin(S) \tag{8}$$

$$\partial_\mu (\; \frac{RS_\nu}{S}\; sin(S)\;) \;-\; \partial_\nu (\; \frac{RS_\mu}{S}\; sin(S)\;) \;=\; 0 \tag{9}$$

Eq-(9) is trivially satisfied since from eq-(8) one learns that $\frac{RS_\mu}{S} sin(S)$ is a total derivative given by $\frac{1}{m}\partial_\mu(R\, cos(S))$. Thus one arrives at $\frac{1}{m}\partial_{[\mu}\partial_{\nu]}(R\, cos(S)) = 0$. After some straightforward algebra by treating the sine and cosine as independent functions, one learns from eqs-(7,8) that

$$S^\mu \partial_\mu S \;=\; mS; \;\; \partial_\mu R \;=\; 0; \;\; \partial_\mu (\frac{S^\mu}{S}) \;=\; \frac{S\partial_\mu S^\mu \;-\; S^\mu \partial_\mu S}{S^2} \;=\; 0 \tag{10}$$

And, finally, from eq-(10) one arrives at a *covariant* Hamilton-Jacobi equation

$$\partial_\mu S^\mu \;-\; m \;=\; 0 \tag{11}$$

Eq-(11) can be interpreted as being the "square-root" of the relativistic Hamilton-Jacobi equation

$$(\partial_\mu \mathcal{S})^2 \;-\; m^2 \;=\; 0 \;\leftrightarrow\; p_\mu\, p^\mu \;-\; m^2 \;=\; 0, \;\; p_\mu = \partial_\mu \mathcal{S} \tag{12}$$

where $\mathcal{S}$ is the relativistic (scalar) action associated with the massive spin-$\frac{1}{2}$ particle. Note that $\mathcal{S} \neq S = \sqrt{S_\mu S^\mu}$.

The Dirac equation (1) is commonly referred as the "square-root" of the Klein-Gordon equation. This is just a simple consequence of $(i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu + m) = -(\partial_\mu \partial^\mu + m^2) = 0$. Therefore, one has found another realization of the "square-root" procedure of eq-(12) given by the covariant Hamilton-Jacobi equation (11), after recurring to a vector-valued extension of the Hamilton-Jacobi function given by S^μ, and to a generalization of the Dirac equation displayed in eq-(6) where now $\Psi^{\alpha\sigma}$ is a 4×4 complex matrix given by $\Psi = Re^{-iS_\mu \gamma^\mu}$, and not a (column) spinor.

One should emphasize that eq-(6) is *not* the same as the Dirac-Hestenes equation (DHE) [7] describing a Dirac-Hestenes spinor field (DHSF), and whose relation with the relativistic de Broglie-Bohm theory was studied by [8], in order to show that the classical relativistic Hamilton-Jacobi equation is equivalent to a DHE satisfied by a particular class of DHSF. This was required in order to obtain the correct relativistic quantum potential when the Dirac theory is interpreted as a de Broglie-Bohm theory.

As a reminder, the DHE is obtained from the Dirac equation (1) simply by replacing $i\psi \to \Psi_{4\times 4}\gamma_{21}$ and $m\psi \to \Psi_{4\times 4}\gamma_0$, with $\Psi_{4\times 4}$ a suitable complex 4×4 matrix and whose entries are given in terms of the 4 complex components of the Dirac spinor ψ. For more details see [8].

We are going to solve the generalized Dirac equation (6) in $D = 2 + 1$ dimensions via the substitution $\Psi = Re^{-iS_\mu \gamma^\mu}$ when $R =$ constant. A trivial solution of the equations (10) for S_μ, when $\eta_{\mu\nu} = diag(1, -1, -1)$, is of the form

$$S_\mu = m\ (t,\ 0,\ 0),\ S = (S_\mu S^\mu) = mt \ \Rightarrow\ \partial_\mu S^\mu\ - m\ =\ m - m\ =\ 0,$$

$$S\ \partial_t S\ =\ m\ S_t\ =\ m^2 t \tag{13}$$

and it yields the following solution of the generalized Dirac equation (6) in $D = 2 + 1$ for the 2×2 matrix Ψ given by

$$\Psi\ =\ R\ \left(\ cos(mt)\ \mathbf{1}\ - i\ \gamma^t\ sin(mt)\ \right) \tag{14}$$

with R *constant*. A representation of the 2×2 matrix γ^t can be chosen to have $1, -1$ along the diagonals and zero entries off the diagonal. One can verify explicitly that Ψ given by eq-(14) solves eq-(6).

Under Lorentz transformations one has

$$S^2\ =\ S_\mu S^\mu \to S'^2\ =\ S'_\mu S'^\mu\ =\ S_\mu S^\mu\ =\ S^2,\ \ \gamma^\mu\ S_\mu \to \Lambda\ \gamma^\mu\ S_\mu\ \Lambda^{-1}, \tag{15}$$

$$\Psi \to \Lambda\ \Psi\ \Lambda^{-1},\ \ \gamma^\mu \partial_\mu \to \Lambda\ \gamma^\mu \partial_\mu\ \Lambda^{-1} \tag{16}$$

and one can verify that the generalized Dirac equation is covariant under the above Lorenz transformations

$$i\gamma^\mu \partial_\mu \Psi \ - m \ \Psi \ = \ 0 \ \to \Lambda \ (\ i\gamma^\mu \partial_\mu \Psi \ - m \ \Psi \ ) \ \Lambda^{-1} \ = \ 0 \tag{17}$$

where Λ is a 2×2 matrix encoding the Lorenz transformations. mt is *not* a Lorentz scalar, mt is just a component of S_μ that happens to coincide with S in the rest frame of the particle. Under Lorentz transformations $S_\mu = m(t,0,0) \to S'_\mu = [S'_{t'}(t',x',y'); S'_{x'}(t',x',y'); S'_{y'}(t',x',y')]$, such that $S^2 = m^2 t^2 = S'^2 = (S'_{t'})^2 - (S'_{x'})^2 - (S'_{y'})^2$.

Given the particular solution (14), a Lorentz transformation will generate a family of solutions of the form

$$\Psi' \ = \ R \ \left(\ cos(S') \ \mathbf{1} \ - i \ \frac{\gamma'^\mu S'_\mu}{S'} \ sin(S') \ \right) \ =$$

$$\Lambda \ \Psi \ \Lambda^{-1} \ = \ R \ \Lambda \ \big(\ cos(mt) \ \mathbf{1} \ - i \ \gamma^t \ sin(mt) \ \big) \ \Lambda^{-1} \tag{18}$$

with $S' = S = mt$. There are many other solutions $different$ from (14) and belonging to $different$ orbits of the Lorentz group. For instance, given $S^2 = (S_t)^2 - (S_x)^2 - (S_y)^2 \neq m^2 t^2$, the equations $S\partial_\mu S = mS_\mu$ in the most general case yield the following differential equations

$$S_t \partial_t S_t \ - \ S_x \partial_t S_x \ - \ S_y \partial_t S_y \ = \ m \ S_t \tag{19a}$$

$$S_t \partial_x S_t \ - \ S_x \partial_x S_x \ - \ S_y \partial_x S_y \ = \ m \ S_x \tag{19b}$$

$$S_t \partial_y S_t \ - \ S_x \partial_y S_x \ - \ S_y \partial_y S_y \ = \ m \ S_y \tag{19c}$$

and whose nontrivial solutions $S_\mu = (S_t(t,x,y), S_x(t,x,y), S_y(t,x,y))$ are such that $S^2 \neq m^2 t^2$, and are very different from the trivial solution $S_\mu = (mt, 0, 0)$. A Lorentz transformation of the nontrivial solutions will generate another family of solutions belonging to a different Lorentz orbit from the trivial solution.

Let us find now solutions to eqs-(10) in $D = 1+1$. From eqs-(10) one learns

$$\partial_\mu S \ = \ m \ \frac{S_\mu}{S}; \ \ \partial_\mu (\frac{S^\mu}{S}) \ = \ 0 \ \Rightarrow \ \partial^\mu \partial_\mu S \ = \ 0 \tag{20}$$

The solutions for S have the usual (right-moving, left-moving) wave-like form

$$S \ = \ f(x-t) \ + \ h(x+t) \ \equiv \ f(u) \ + \ h(v), \ \ u \equiv x - t, \ \ v \equiv x + t \tag{21}$$

for arbitrary functions f, h. Consequently, given $S_\mu = (S_t, S_x)$, one has

$$S^2 \ = \ [f(u) + h(u)]^2 \ = \ (S_t)^2 \ - (S_x)^2, \tag{22a}$$

$$\partial_t S \ = \ - f'_u \ + \ h'_v \ \ = \ m \ \frac{S_t}{f(u)+h(v)}; \ \ \partial_x S \ = \ f'_u \ + \ h'_v \ = \ m \ \frac{S_x}{f(u)+h(v)} \tag{22b}$$

Eliminating S_t, S_x from eqs-(22b) by recurring to eq-(22a) yields

$$m^2 \ = \ (-f'_u + h'_v)^2 \ - \ (f'_u + h'_v)^2 \ = \ - \ 4 \ f'_u \ h'_v \tag{23}$$

A trivial solution to eq-(23) is

$$ f(u) \ = - \ \frac{mu}{2}, \ \ h(v) \ = \ \frac{mv}{2} \ \Rightarrow f = -\frac{m}{2}(x-t), \ \ h \ = \ \frac{m}{2}(x+t) \ \Rightarrow $$

$$ S \ = \ f \ + \ h \ = \ mt, \ \ S_t \ = \ mt, \ \ S_x \ = \ 0 \qquad (24a) $$

and once again one recovers the same functional form found in eq-(13) in the rest frame of the particle. There are many other solutions to (23). Since f'_u is solely a function of u, and h'_v is solely a function of v, one learns from eq-(23) that $f'_u = c_1$ (constant) and $h'_v = c_2$ (constant) leading to $f(u) = c_1 u + d_1$, $h(v) = c_2 v + d_2$, with d_1, d_2 arbitrary constants and c_1, c_2 obeying $c_1 c_2 = -\frac{m^2}{4}$. Hence, one finds that the most general solution to eqs-(10) in $D = 1+1$ are

$$ S \ = \ f + h \ = \ c_1(x-t) + c_2(x+t) + (d_1+d_2) \ = \ x\,(c_1+c_2) + t\,(c_2-c_1) + d_1+d_2 \qquad (24b) $$

leading to the most general solutions

$$ S_t \ = \ \frac{c_2 - c_1}{m} \ [\ x\ (c_1+c_2) \ + \ t\ (c_2-c_1) \ + \ d_1 + d_2 \ ] $$

$$ S_x \ = \ \frac{c_2 + c_1}{m} \ [\ x\ (c_1+c_2) \ + \ t\ (c_2-c_1) \ + \ d_1 + d_2 \ ] \qquad (24c) $$

with $c_1 c_2 = -\frac{m^2}{4}$. One can also verify from eqs-(24c) that $\partial_\mu S^\mu - m = \partial_t S_t - \partial_x S_x - m = 0$. And, finally, by inserting the solutions found in eqs-(24b,24c) directly into eq-(2) (with R constant) one has then found *exact* solutions to the generalized Dirac equation (6) in $D = 1+1$ where Ψ is now a 2×2 matrix rather than a column spinor.

The above solutions of the generalized Dirac equation (6) in $D = 2+1$ and $D = 1+1$ were straightforward via the substitution $\Psi = Re^{-iS_\mu \gamma^\mu}$. However, this is *no* longer the case when one has the exponential of the full multivector-valued action

$$ \Psi \ = \ R\ exp\,(\ S_0 \mathbf{1} \ + \ S_\mu \gamma^\mu \ + \ S_{\mu\nu}\gamma^{\mu\nu} \ + \ S_{\mu\nu\rho}\gamma^{\mu\nu\rho}\ ) \qquad (25) $$

after reabsorbing the $-i$ factors into the components $S_0, S_\mu, S_{\mu\nu}, \cdots$ of the multivector-valued action

The exponential of the multivector associated with the $Cl(0,3)$ algebra, such that $e_1^2 = e_2^2 = e_3^2 = -1$, was explicitly displayed above in eqs-(4,5). The authors [4] also wrote down the exponentials of the multivectors associated with the remaining $Cl(3,0), Cl(2,1), Cl(1,2)$ real Clifford algebras $Cl(p,q), p+q = 3$ in $3D$. Below we shall discuss the plausible physical interpretation of the components $S_0, S_\mu, S_{\mu\nu}, \cdots$ of the multivector-valued action.

Inserting the substitution (25) into eq-(6), and recurring to similar results as in [4] which determine the functional relations among the components of $\Psi = \Psi_M \Gamma^M$ and $\mathbf{S} = S_M \Gamma^M$, leads to a very complicated system of coupled nonlinear differential equations for the multivector components $S_0, S_\mu, S_{\mu\nu}, S_{\mu\nu\rho}$. Nothing has been gained in this case by making the substitution (25) as compared to the trivial solutions found above.

2 The Generalized Dirac Equation in C-space (Clifford space)

Let us proceed in constructing the most general Dirac-like equation in C-space (Clifford-space).The analog of mass $\mathcal{M}$ in C-space is defined in terms of the on-shell condition of the polyparticle's polymomentum [10] as follows

$$p^2 + P_\mu P^\mu + P_{\mu\nu} P^{\mu\nu} + P_{\mu\nu\rho} P^{\mu\nu\rho} + P_{\mu\nu\rho\tau} P^{\mu\nu\rho\tau} = \mathcal{M}^2 \qquad (26)$$

Powers of a length parameter must be introduced in eq-(26) on dimensional grounds to match units. In [10] we introduced the Planck scale L_P which was set to 1 in units of $G = \hbar = c = 1$. A Taylor expansion of the exponential $\Psi(\mathbf{X}) = R(\mathbf{X}) e^{-iS_J(\mathbf{X})\Gamma^J} = \Psi_I \Gamma^I$ *determines* the explicit (and complicated) functional form of the expansion coefficients $\Psi_I(R, S_J)$ in terms of R, S_J as shown in eqs-(4,5). This requires using the explicit formulae for the Clifford geometric products [9] of the Γ's; i.e. $\Gamma^I \Gamma^J = f^{IJ}{}_K \Gamma^K$. These geometric products (expressed in terms of commutators and anti-commutators) [9] are very useful in finding solutions to the most general Dirac-like equation in C-space given by

$$i\, \Gamma^I \frac{\partial}{\partial X^I} \Psi - \mathcal{M}\Psi = 0. \; I = 1, 2, 3, \cdots, 2^D \qquad (27)$$

Upon decomposing Ψ in the form $\Psi(X^I) = \Psi_J(X^I)\Gamma^J$, inserting it into eq-(27), and recurring to the geometric products of the Clifford algebra generators, yields the following 2^D equations corresponding to the 2^D dimensions of the Clifford algebra in D-dim

$$i\, f^{IJ}{}_K \frac{\partial \Psi_J}{\partial X^I} = \mathcal{M}\, \Psi_K, \quad I, J, K = 1, 2, 3, \cdots, 2^D \qquad (28)$$

After solving the 2^D equations (28) for the 2^D functions $\Psi_J(X^I)$ of the multivector coordinates $\mathbf{X} = X^I \Gamma_I$, one can go back to the expressions relating the Ψ_J components in terms of the S_J components (as indicated by eqs-(4,5), for example), and *implicitly* establish the solutions for the multivector components $S_J(X^I)$ of the Clifford-valued action $\mathbf{S} = S_J \Gamma^J$.

Note that one has 2^D equations (28) for $1+2^D$ functions after adding $R(\mathbf{X})$ to the 2^D components $S_K(\mathbf{X})$ of the Clifford-valued action $\mathbf{S} = S_K \Gamma^K$. This issue can be easily resolved in the *complex* Clifford algebra case simply by *absorbing* R into a re-definition of the scalar part of the action S_0. One simply rewrites $Re^{-iS_0} = e^{lnR - iS_0} \equiv e^{-iS'_0}$, where S'_0 is now complex. Therefore, effectively one ends up with 2^D equations corresponding to 2^D unknowns $S'_0, S_\mu, S_{\mu\nu}, \cdots$.

Earlier on, we were able to derive in the simplest case that $\partial_\mu S^\mu - m = 0$ in eq-(11). The natural extension of this relation (in the most general case) in C-space is $\partial_I S^I - \mathcal{M} = 0$, where I are multivector-valued indices spanning the full of C-space. To *prove* the latter relation from eqs-(28), after expressing the Ψ_I components in terms of the S_I components, is a very difficult task due to the complexity of these relations, as displayed by eqs-(4,5), for example. This is beyond the scope of this work.

Bohm's introduction of his quantum potential $Q \sim \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}$ into the classical non-relativistic Hamilton-Jacobi equation, followed up by the use of the pilot-wave guide equation (continuity equation) $\partial_t \rho + \frac{1}{m} \partial_i (\rho \partial^i S) = 0$ leads to the Schroedinger equation via the substitution $\psi = \sqrt{\rho}\ e^{-iS}$. Is there an analogy of Bohm's procedure here ? Namely, given $\mathcal{W}_Q$, the analog of a putative Bohm's quantum potential (which is to be determined), after including it into the covariant Hamilton-Jacobi equation in C-space as follows

$$\partial_I S^I \ = \ \mathcal{M} \ + \ \mathcal{W}_Q \tag{29}$$

and adding the continuity equation[1]

$$\partial_I J^I \ = \ 0, \quad J^I \ \equiv \ Trace\ (\Psi^\dagger\ \Gamma^I\ \Psi), \quad \Gamma^I \ \equiv \ \mathbf{1}, \gamma^\mu, \gamma^{\mu_1 \mu_2}, \gamma^{\mu_1 \mu_2 \mu_3}, \cdots \tag{30}$$

does this lead to the generalized Dirac equation (27) after making the standard substitution $\Psi(\mathbf{X}) = R(\mathbf{X}) e^{-i S_I(\mathbf{X}) \Gamma^I}$?? In other words, can one find a judicious expression for $\mathcal{W}_Q$ in terms of Ψ and $\Psi^\dagger$ which attains this goal ??

This is a difficult problem for several reasons. Firstly, because one has to construct the "logarithm" of a multivector in order to express $\mathbf{S} = S_I \Gamma^I$ in terms of $\Psi = \Psi_I \Gamma^I$, which is the inverse operation from obtaining the exponential of a multivector. Secondly, the expansion $\mathbf{S} = S_I \Gamma^I$ is comprised of hermitian and anti-hermitian matrices, hence $\Psi^\dagger \Psi \neq\ R^2\ \mathbf{1}$, as a result of the Baker-Campbell-Hausdorff formula. Therefore, R **cannot** be expressed in terms of Ψ and $\Psi^\dagger$. Therefore, it is highly unlikely that one can recover the generalized Dirac equation (27) from eqs-(29,30). Note also the *difference* of eq-(30) involving the actual matrices Ψ to the case involving a Dirac spinor ψ_D

$$\partial_\mu J^\mu \ = \ 0, \quad J^\mu \ \equiv \ \bar{\psi}_D\ \gamma^\mu\ \psi_D, \quad \bar{\psi}_D \equiv \psi_D^\dagger \gamma^0 \tag{31}$$

Eq-(31) is obtained from the Dirac equation and its conjugate as shown in the standard textbooks.

It remains to find what is the physical significance of the components of the multivector-valued action $\mathbf{S} = S_M \Gamma^M$? Given a Clifford-valued multivector coordinate $\mathbf{X} = x\mathbf{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_{\mu\nu} + \cdots$ associated to a polyparticle in C-space [10], and the polyparticle's multivector-valued momentum $\mathbf{P} = p\mathbf{1} + p^\mu \gamma_\mu + p^{\mu\nu} \gamma_{\mu\nu} + \cdots$, the Clifford geometric product $\mathbf{PX}$ can be used to define a physical quantity which has the same characteristics as a multivector-valued action $\mathbf{S} =$

[1] The trace of $\Psi^\dagger \Psi$ is positive definite since the matrix $\Psi^\dagger$ is the hermitian adjoint of the matrix Ψ

$S_M \Gamma^M$. For instance, let us define $\mathbf{S} \equiv \mathbf{PX}$, and look at the product $p_\mu x_\nu \gamma^\mu \gamma^\nu = p_\mu x^\mu \mathbf{1} + \frac{1}{2}\gamma^{\mu\nu}(p_\mu x_\nu - p_\nu x_\mu)$. The scalar part $p_\mu x^\mu$ is the standard phase factor (with units of action) associated to a plane wave solution to the Klein-Gordon equation. Whereas the bivector piece includes the orbital angular momentum $(p_\mu x_\nu - p_\nu x_\mu)$ associated to the Lorentz generators.

The scalar part of the Clifford geometric product yields

$$S_0 \ = \ < \mathbf{P} \ \mathbf{X} > \ = \ px \ + \ p_\mu x^\mu \ + \ p_{\mu\nu} x^{\mu\nu} \ + \ \cdots \tag{32}$$

and furnishes the generalization of the plane wave phase factor to the full C-space. Whereas the higher-grade multivectors in $\mathbf{PX}$ will yield the other components $S_\mu, S_{\mu\nu}, S_{\mu\nu\sigma}, \cdots$ of $\mathbf{S}$ in terms of the other combinations of products.

The author [1] pointed out that similar to the Hamiltonian-Jacobi formulation of classical mechanics, an analog can be developed in the De-Donder-Weyl theory in terms of D Hamilton-Jacobi functions S^μ on the *field* configuration space and which satisfy the De-Donder-Weyl Hamilton-Jacobi equation $\partial_\mu S^\mu + H_{DW} = 0$, where $H_{DW} = p^\mu_a \partial_\mu \phi^a - L$ is the De-Donder-Weyl Hamiltonian which is a function of the fields ϕ^a, their polymomenta $p^\mu_a \equiv (\partial L / \partial(\partial_\mu \phi^a))$, and the space-time coordinates x^μ. We find that the DW Hamilton-Jacobi equation has the *same* form as $\partial_I S^I - \mathcal{M} = 0$ (up to a sign of S^I due to our choice of sign in the phase factor $\Psi = Re^{-iS_I \Gamma^I}$). More work is required in order to explore further analogies with the DW (De Donder-Weyl) formalism.

Our results have been restricted to flat spacetimes. They can be extended to curved spacetime backgrounds via the introduction of vierbeins e^a_μ obeying $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, and which allows to relate the Clifford basis generators in the tangent space γ_a to the Clifford basis generators in the curved spacetime $\gamma_\mu = e^a_\mu \gamma_a$. This procedure allows to construct the Schroedinger-Dirac equation [11] in curved spacetime backgrounds

$$(\ g^{\mu\nu} \ \nabla_\mu \ \nabla_\nu \ + \frac{R}{4} \ + m^2 \ ) \ \Psi_D \ = \ 0 \tag{33}$$

and which is obtained from "squaring" the covariant Dirac equation after recurring to the relation

$$\gamma^\mu \gamma^\nu \ \nabla_\mu \ \nabla_\nu \ = \ g^{\mu\nu} \ \nabla_\mu \ \nabla_\nu \ + \ \gamma^{\mu\nu} \ [\nabla_\mu, \nabla_\nu]$$

and where the scalar curvature term $\frac{R}{4}$ stems from the commutator $[\nabla_\mu, \nabla_\nu] = [\partial_\mu + \omega_\mu, \partial_\nu + \omega_\nu]$ of the covariant derivatives which are defined in terms of the Clifford-valued spin connection $\omega_\mu = \omega^{ab}_\mu \gamma_{ab}$.

The Schroedinger-Dirac equation was recently revisited by [12]. Despite that the Schroedinger-Dirac equation is *not* conformally invariant, there exists a generalization of the equation that is conformally invariant but which requires a different conformal transformation of the spinor than the one required by the Dirac equation. The new conformal factor acquired by the spinor is found to be a *matrix-valued* factor [12] obeying a differential equation that involves the Fock-Ivanenko line element.

We conclude by discussing an interesting physical application of this work in the study of branes in C-spaces [10]. These are characterized by maps from the multivector-valued world-manifold of the brane embedded into a target C-space background given by $X^M(\sigma^A)$. The multivector-valued index in X^M spans the dimension 2^D of the target space Clifford algebra $Cl(D)$. Whereas the multivector-valued index in σ^A spans the 2^d-dim of the world-manifold Clifford algebra $Cl(d)$. There is a natural wave-functional $\Psi[X^M(\sigma^A)]$ associated with the C-space brane field configurations. This wave-functional is similar to the string field $\Psi[X^\mu(\sigma^1,\sigma^2)]$ in open and closed strings, where X^μ are the embedding background target spacetime coordinates and σ^1,σ^2 are the string world-sheet coordinates.

This is where the DW formulation of field theory becomes important. In such DW formulation the spacetime variables x^μ also enter into picture, in addition to the fields ϕ^a and their polymomenta $p^\mu_a \equiv (\partial L/\partial(\partial_\mu \phi^a))$. Hence, the DW formalism requires to extend $\Psi[X^M(\sigma^A)]$ to $\Psi[X^M(\sigma^A),\sigma^A]$, such that the latter can be interpreted as a probability amplitude for the C-space brane in the quantum state Ψ to have a field configuration $X^M(\sigma^A)$ at the point σ^A. In other words, $\Psi[X^M(\sigma^A),\sigma^A] \equiv < X^M(\sigma^A),\sigma^A|\Psi >$.

Finally, following similar steps to the work by [1], [2] one can then write a Dirac-like equation of the form

$$i\Gamma^A \frac{\partial}{\partial \sigma^A} \Psi[X^M(\sigma^A),\sigma^A] \ = \ \hat{H}_{DW} \Psi[X^M(\sigma^A),\sigma^A], \quad A = 1,2,3,\cdots,2^d \qquad (34)$$

where $\hat{H}_{DW}$ is the operator corresponding to the DW Hamiltonian function and where we set κ to unity. κ is a constant which is required on dimensional grounds to match units [1],[2]. Ψ is Clifford-valued $\Psi = \Psi_0 \mathbf{1} + \Psi_a \gamma^a + \Psi_{ab}\gamma^{ab} + \cdots$ where the gammas Γ^A span the 2^d-dimensions of the world-manifold associated with the $Cl(d)$ algebra of the C-space brane.

In the case of $free$ (non-interacting) branes, eq-(34) is of the form

$$i\Gamma^A \frac{\partial}{\partial \sigma^A} \Psi[X^M(\sigma^A),\sigma^A] \ = \ -\frac{1}{2} \ \frac{\delta^2 \ \Psi[X^M(\sigma^A),\sigma^A]}{(\delta X^M(\sigma^A))^2}, \quad A = 1,2,3,\cdots,2^d \qquad (35)$$

Suffice to say that matters are not that simple due to the complexity of eq-(35), otherwise the quantization of branes would have been attained long ago. Introducing brane interactions will complicate matters since one would be required to introduce a term of the form $V[X^M(\sigma^A)]\Psi$ into eq-(35) where V is the potential.

Concluding, we have explored the construction of a generalized Dirac equation via the introduction of the notion of Clifford-valued actions, and which was inspired by the work of [1], [2] on the De Donder-Weyl theory formulation of field theory. Crucial in this construction is the evaluation of the *exponentials* of *multivectors* associated with Clifford (hypercomplex) analysis. Exact *matrix* solutions (instead of spinors) of the generalized Dirac equation in $D = 2,3$ spacetime dimensions were found. This formalism can be extended to curved spacetime backgrounds like it happens with the Schroedinger-Dirac equation. We finalized by proposing a wave-functional equation governing the quantum dynamics of branes living in C-spaces, and which was based on the De Donder-Weyl Hamiltonian formulation of field theory.

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