Notes on hydrodynamics and decoherence of ideal quantum gases

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I. MODEL

$$
\begin{aligned}
S[\psi^\dagger,\psi] &= \int_x \left[\frac{1}{2}\psi^\dagger i\hbar\partial_t\psi - \frac{1}{2}i\hbar(\partial_t\psi^\dagger)\psi - \frac{\hbar^2}{2m}\Delta\psi + (\mu - eA_0)\psi^\dagger\psi\right] \\
&\to \int_x \psi^\dagger\left[i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta + \mu - eu\right]\psi \qquad (1)
\end{aligned}
$$

Dimensions: $[\hbar] = ML^2T^{-1}$, $[\hbar c] = ML^3T^{-2}$, $[G^{-1}_{x,x'}] = [\hbar]T^{-1}/L^3T = L^{-3}T^{-2}$, $L^6T^3[\psi^\dagger G^{-1}\psi] = L^3T[\hbar]T^{-1}[\psi]^2 = [\hbar]$, $[\psi] = L^{-3/2}$

$$
\begin{aligned}
e^{\frac{i}{\hbar}W[\hat{a}]} &= \mathrm{Tr}\bar{T}[e^{\frac{i}{\hbar}\int_{t_i}^{t_f}dt\int_{\boldsymbol{x}}[H(x)+a^-(x)J^-(x)]}]T[e^{-\frac{i}{\hbar}\int_{t_i}^{t_f}dt\int_{\boldsymbol{x}}[H(x)-a^+(x)J^+(x)]}]\rho_i \\
&= \int D[\hat{\psi}]D[\hat{\psi}^\dagger]e^{\frac{i}{\hbar}\hat{\psi}^\dagger\cdot[\hat{G}^{-1}+\hat{\not{a}}]\cdot\hat{\psi}} \qquad (2)
\end{aligned}
$$

$$
\hat{\psi} = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \qquad \hat{a} = \begin{pmatrix}a^+\\ a^-\end{pmatrix}. \qquad (3)
$$

Density-current:

$$
\begin{aligned}
(\hat{\not{a}})^\sigma &= a^\sigma_\mu\mathcal{C}^{\sigma\sigma}_\mu = a^\sigma_0\mathcal{C}^{\sigma\sigma} + \boldsymbol{a}^\sigma\boldsymbol{\mathcal{C}}^{\sigma\sigma} \\
\psi^\dagger\cdot\mathcal{C}^{\sigma\sigma'}_{0\ x}\cdot\psi &= \psi^{\dagger\sigma}\cdot\mathcal{C}_{0\ x}\cdot\psi^{\sigma'} = \psi^{\dagger\sigma}_x\psi^{\sigma'}_x \\
(\mathcal{C}^{\sigma\sigma'}_{0\ x})^{\sigma_1\sigma_2} &= \mathcal{C}_{0\ x}\delta^{\sigma_1\sigma}\delta^{\sigma_2\sigma'} \\
(\mathcal{C}_{0\ x})_{yz} &= \delta_{y,x}\delta_{z,x} \\
\psi^\dagger\cdot\boldsymbol{\mathcal{C}}^{\sigma\sigma'}_x\cdot\psi &= \psi^{\dagger\sigma}\cdot\boldsymbol{\mathcal{C}}_{0\ x}\cdot\psi^{\sigma'} = -\frac{i\hbar}{2m}[\psi^{\dagger\sigma}\boldsymbol{\nabla}\psi^{\sigma'} - (\boldsymbol{\nabla}\psi^{\dagger\sigma})\psi^{\sigma'}]_x \\
(\boldsymbol{\mathcal{C}}^{\sigma\sigma'}_x)^{\sigma_1\sigma_2} &= \boldsymbol{\mathcal{C}}_{0\ x}\delta^{\sigma_1\sigma}\delta^{\sigma_2\sigma'} \\
(\boldsymbol{\mathcal{C}}_x)_{yz} &= -\frac{i\hbar}{2m}\delta_{y,x}\delta_{z,x}(\boldsymbol{\nabla}_z - \boldsymbol{\nabla}_y) \qquad (4)
\end{aligned}
$$

summation over repeated Greec indices. $\bar{r} = 1/\bar{k}$ characteristic length scale.

$$
\hat{G}^{-1} = \hat{G}^{-1}_0 + \hat{G}^{-1}_{\mathrm{BC}} \qquad (5)
$$

where

$$
\hat{G}^{-1} = \begin{pmatrix}G^{-1}_0 & 0\\ 0 & -G^{-1*}_0\end{pmatrix} + G^{-1}_{\mathrm{BC}}, \qquad (6)
$$

$$
G^{-1}_{0\ x',x} = \left(i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta + \mu\right)\delta_{x,x'} \qquad (7)
$$

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II. CURRENT GREEN FUNTION

$$W[\hat{a}] = i\xi\hbar \mathrm{Tr}\ln(\hat{G}^{-1} + \hat{\not{a}}) \approx \hat{a}\hat{w}^{(1)} - \frac{1}{2}\hat{a}\tilde{G}\hat{a} \tag{8}$$

$$\tilde{G} = \begin{pmatrix} -\tilde{G}^n & \tilde{G}^f \\ -\tilde{G}^f & \tilde{G}^n \end{pmatrix} + i\tilde{G}^i \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{9}$$

$\tilde{G}^n = \tilde{G}^{n\mathrm{tr}}$, $\tilde{G}^{f\mathrm{tr}} = -\tilde{G}^f$, $\tilde{G}^r = \tilde{G}^n + \tilde{G}^f$, $\tilde{G}^a = \tilde{G}^n - \tilde{G}^f = \tilde{G}^{a\mathrm{tr}}$, $\tilde{G}^i = \tilde{G}^{i\mathrm{tr}}$

$$a^{\pm} = \frac{a}{2}(1 \pm \kappa) \pm \bar{a} = \frac{a}{2} \pm \left(\bar{a} + \frac{\kappa}{2}a\right) \tag{10}$$

$$\begin{aligned}
\tilde{G}\hat{a} &= \begin{pmatrix} -\tilde{G}^n & \tilde{G}^f \\ -\tilde{G}^f & \tilde{G}^n \end{pmatrix} \begin{pmatrix} \frac{a}{2} + \bar{a} + \frac{\kappa}{2}a \\ \frac{a}{2} - \bar{a} - \frac{\kappa}{2}a \end{pmatrix} = \begin{pmatrix} -\tilde{G}^r\bar{a} - \frac{\kappa\tilde{G}^r + \tilde{G}^a}{2}a \\ -\tilde{G}^r\bar{a} - \frac{\kappa\tilde{G}^r - \tilde{G}^a}{2}a \end{pmatrix} \\
\hat{a}\tilde{G}\hat{a} &= -(\frac{a}{2} + \bar{a} + \frac{\kappa}{2}a, \frac{a}{2} - \bar{a} - \frac{\kappa}{2}a) \begin{pmatrix} \tilde{G}^r\bar{a} + \frac{\kappa\tilde{G}^r + \tilde{G}^a}{2}a \\ \tilde{G}^r\bar{a} + \frac{\kappa\tilde{G}^r - \tilde{G}^a}{2}a \end{pmatrix} \\
&= -\bar{a}\tilde{G}^r\bar{a} - \frac{1+\kappa}{2}a\frac{\kappa\tilde{G}^r + \tilde{G}^a}{2}a - \bar{a}\frac{\kappa\tilde{G}^r + \tilde{G}^a}{2}a - \frac{1+\kappa}{2}a\tilde{G}^r\bar{a} \\
&\quad + \bar{a}\tilde{G}^r\bar{a} + \frac{1-\kappa}{2}a\frac{-\kappa\tilde{G}^r + \tilde{G}^a}{2}a + \bar{a}\frac{\kappa\tilde{G}^r - \tilde{G}^a}{2}a - \frac{1-\kappa}{2}a\tilde{G}^r\bar{a} \\
&= -\kappa a\tilde{G}^n a - \bar{a}\tilde{G}^a a - a\tilde{G}^r\bar{a}
\end{aligned} \tag{11}$$

$$\Re W[a,\bar{a}] = \bar{a}w^{aux(1)} + aJ_{\mathrm{gr}} + \frac{1}{2}(\bar{a}, a) \begin{pmatrix} 0 & \tilde{G}^a \\ \tilde{G}^r & \kappa\tilde{G}^n \end{pmatrix} \begin{pmatrix} \bar{a} \\ a \end{pmatrix} \tag{12}$$

(earlier signs)

$$J_{\mathrm{gr}} = \frac{1+\kappa}{2}w^{(1)+} + \frac{1-\kappa}{2}w^{(1)-}, \qquad w^{aux(1)} = w^{(1)+} - w^{(1)-} \tag{13}$$

Unitarity: $W[0,\bar{a}] = 0$: $w^{aux(1)} = 0$

III. EFFECTIVE ACTION

A. Two fields

$$W[\hat{a}_0, \hat{\boldsymbol{a}}] = \Gamma[\hat{\rho}, \hat{\boldsymbol{J}}] + \hat{a}_0\hat{\rho} + \hat{\boldsymbol{a}}\hat{\boldsymbol{J}} \tag{14}$$

where

$$\hat{\rho} = \frac{\delta W}{\delta \hat{a}_0}, \quad \hat{\boldsymbol{J}} = \frac{\delta W}{\delta \hat{\boldsymbol{a}}} \tag{15}$$

Inverse Legendre transformation:

$$\frac{\delta\Gamma}{\delta\hat{\boldsymbol{J}}} = \frac{\delta W[\hat{a}]}{\delta\hat{a}} \cdot \frac{\delta\hat{a}}{\delta\hat{J}} - \frac{\delta a}{\delta J} \cdot J - a = -\hat{\boldsymbol{a}}, \qquad \frac{\delta\Gamma}{\delta\hat{\rho}} = -\hat{a}_0 \tag{16}$$

$$W[\hat{a}] = w^{(1)}\hat{a} + \frac{1}{2}\hat{a}w^{(2)}\hat{a} = \Gamma[\hat{J}] + \hat{a}\hat{J} = \Gamma[J^a, J] + \bar{a}J^a + aJ \tag{17}$$

where

$$\hat{J} = \frac{\delta W}{\delta \hat{a}} = w^{(1)} + w^{(2)} \cdot \hat{a} \tag{18}$$

Inversion:

$$\hat{a} = w^{(2)-1} \cdot (\hat{J} - w^{(1)}) \tag{19}$$

$$\begin{aligned}\Gamma[\hat{J}] &= w^{(1)}w^{(2)-1}(\hat{J} - w^{(1)}) + \frac{1}{2}(\hat{J} - w^{(1)})w^{(2)-1}w^{(2)}w^{(2)-1}(\hat{J} - w^{(1)}) - \hat{J}w^{(2)-1}(\hat{J} - w^{(1)}) \\ &= w^{(1)}w^{(2)-1}w^{(1)} + \hat{J}w^{(2)-1}w^{(1)} - \frac{1}{2}\hat{J}w^{(2)-1}\hat{J}\end{aligned} \tag{20}$$

Physical parametrization:

$$\Gamma[J, J^a] = \Re W[\hat{a}] - \bar{a}J^a - aJ, \qquad J = \frac{\delta W}{\delta a}, \qquad J^a = \frac{\delta W}{\delta \bar{a}} \tag{21}$$

$$\mathbb{1} = \begin{pmatrix} 0 & \tilde{G}^a \\ \tilde{G}^r & \kappa\tilde{G}^n \end{pmatrix} \cdot \begin{pmatrix} -\kappa\tilde{G}^{r-1}\tilde{G}^n\tilde{G}^{a-1} & \tilde{G}^{r-1} \\ \tilde{G}^{a-1} & 0 \end{pmatrix} \tag{22}$$

$$\begin{aligned}\Gamma[\hat{J}] &= (J^a, J) \cdot \begin{pmatrix} -\kappa\tilde{G}^{r-1}\tilde{G}^n\tilde{G}^{a-1} & \tilde{G}^{r-1} \\ \tilde{G}^{a-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ J_{\rm gr} \end{pmatrix} - \frac{1}{2}(J^a, J) \cdot \begin{pmatrix} -\kappa\tilde{G}^{r-1}\tilde{G}^n\tilde{G}^{a-1} & \tilde{G}^{r-1} \\ \tilde{G}^{a-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} J^a \\ J \end{pmatrix} \\ &= (J^a, J) \cdot \begin{pmatrix} \tilde{G}^{r-1}J_{\rm gr} \\ 0 \end{pmatrix} + \frac{1}{2}(J^a, J) \cdot \begin{pmatrix} \kappa\tilde{G}^{r-1}\tilde{G}^n\tilde{G}^{a-1} & -\tilde{G}^{r-1} \\ -\tilde{G}^{a-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} J^a \\ J \end{pmatrix} \\ &= J^a \cdot \tilde{G}^{r-1} \cdot J_{\rm gr} + \frac{1}{2}J^a \cdot \kappa\tilde{G}^{r-1}\tilde{G}^n\tilde{G}^{a-1} \cdot J^a - \frac{1}{2}J^a \cdot \tilde{G}^{r-1} \cdot J - \frac{1}{2}J \cdot \tilde{G}^{a-1} \cdot J^a\end{aligned} \tag{23}$$

Equations of motion:

$$-a = \frac{\delta\Gamma[J^a, J]}{\delta J}, \qquad -\bar{a} = \frac{\delta\Gamma[J^a, J]}{\delta J^a} \tag{24}$$

$$\begin{aligned}-\bar{a} &= \tilde{G}^{r-1}J_{\rm gr} + \kappa\tilde{G}^{r-1}\tilde{G}^n\tilde{G}^{a-1}J^a - \tilde{G}^{r-1}J \\ \tilde{G}^{r-1}J - \kappa\tilde{G}^{r-1}\tilde{G}^n\tilde{G}^{a-1}J^a &= \bar{a} + \tilde{G}^{r-1}J_{\rm gr}\end{aligned} \tag{25}$$

$$\tilde{G}^{a-1}J^a = a \tag{26}$$

and

$$\tilde{G}^{r-1}J = \tilde{G}^{r-1}J_{\rm gr} + \kappa\tilde{G}^{r-1} \cdot \tilde{G}^n \cdot a + \bar{a} \tag{27}$$

B. Physical field

$$\Re W[a,\bar a]=\frac{\kappa}{2}a\tilde G^n a+a(\tilde G^r\bar a+J_{\rm gr})=w^{(1)}a+\frac{1}{2}aw^{(2)}a=\Gamma[J]+aJ=\Gamma[J]+aJ \tag{28}$$

$$J=\frac{\delta W}{\delta a}=w^{(1)}+w^{(2)}a \tag{29}$$

Inversion: $a=w^{(2)-1}(J-w^{(1)})$

$$\begin{aligned}\Gamma[J]&=w^{(1)}w^{(2)-1}(J-w^{(1)})+\frac{1}{2}(J-w^{(1)})w^{(2)-1}w^{(2)}w^{(2)-1}(J-w^{(1)})-Jw^{(2)-1}(J-w^{(1)})\\&=w^{(1)}w^{(2)-1}w^{(1)}+Jw^{(2)-1}w^{(1)}-\frac{1}{2}Jw^{(2)-1}J\\&=\frac{1}{\kappa}J\tilde G^{n-1}(\tilde G^r\bar a+J_{\rm gr})-\frac{1}{2\kappa}J\tilde G^{n-1}J\end{aligned} \tag{30}$$

and

$$J=\tilde G^r\bar a+J_{\rm gr} \tag{31}$$

C. Bare action

$$\begin{aligned}e^{\frac{i}{\hbar}W[\hat a]}&=\int D[\hat\psi]D[\hat\psi^\dagger]e^{\frac{i}{\hbar}\hat\psi^\dagger\cdot[\hat G^{-1}+\hat{\not a}]\cdot\hat\psi}\\&=\int D[\hat J]e^{\frac{i}{\hbar}S_B[\hat J]+\frac{i}{\hbar}\hat J\hat a}=\int D[\hat J]e^{\frac{i}{2\hbar}\hat J\tilde G^{-1}\hat J+\frac{i}{\hbar}\hat J(\hat A+\hat a)}=e^{-\frac{i}{2\hbar}(\hat A+\hat a)\tilde G(\hat A+\hat a)}=e^{-\frac{i}{2\hbar}\hat a\tilde G\hat a-\frac{i}{\hbar}\hat a\tilde G\hat A-\frac{i}{2\hbar}\hat A\tilde G\hat A}\end{aligned} \tag{32}$$

$$W[\hat a]=\hat a\hat w^{(1)}-\frac{1}{2}\hat a\tilde G\hat a\quad\to\quad S_B[\hat J]=\frac{1}{2}\hat J\tilde G^{-1}\hat J-\hat J\tilde G^{-1}\hat w^{(1)} \tag{33}$$

IV. IDENTITIES

A. CTP symmetry

$$R^{\mu\nu\pm}_{\omega,\boldsymbol q}=R^{(\nu\mu)\mp}_{-\omega,-\boldsymbol q} \tag{34}$$

Check:

$$R^{\pm}_{\omega,\boldsymbol q}=\begin{pmatrix}R^{\pm(tt)}_{\omega,q^2}&\boldsymbol nR^{\pm(ts)}_{\omega,q^2}\\\boldsymbol nR^{\pm(st)}_{\omega,q^2}&TR^{\pm(T)}_{\omega,q^2}+LR^{\pm(L)}_{\omega,q^2}\end{pmatrix} \tag{35}$$

$$\begin{aligned}\hat R^{\mu\nu\pm}_{-\omega,\boldsymbol q}&=\frac{\hbar\eta_s^3}{\pi^5}\int d^3kd^3p\frac{n_{\boldsymbol k}(1-n_{\boldsymbol k+\boldsymbol q-\boldsymbol p})F^{\mu\nu\pm}(\boldsymbol k,\boldsymbol q-\boldsymbol p)}{(\boldsymbol p^2+\eta^2)^3\{[-\omega\pm(\omega_{\boldsymbol k}-\omega_{\boldsymbol k+\boldsymbol q-\boldsymbol p})]^2+\gamma^2\}}\\&=\frac{\hbar\eta_s^3}{\pi^5}\int d^3kd^3p\frac{n_{\boldsymbol k}(1-n_{\boldsymbol k+\boldsymbol q-\boldsymbol p})F^{\mu\nu\pm}(\boldsymbol k,\boldsymbol q-\boldsymbol p)}{(\boldsymbol p^2+\eta^2)^3\{[\omega\mp(\omega_{\boldsymbol k}-\omega_{\boldsymbol k+\boldsymbol q-\boldsymbol p})]^2+\gamma^2\}}\\&=\begin{pmatrix}R^{\mp(tt)}_{\omega,q^2}&-\boldsymbol nR^{\mp(ts)}_{\omega,q^2}\\-\boldsymbol nR^{\mp(st)}_{\omega,q^2}&TR^{\mp(T)}_{\omega,q^2}+LR^{\mp(L)}_{\omega,q^2}\end{pmatrix}\\&=\hat R^{(\mu\nu)\mp}_{\omega,-\boldsymbol q}\end{aligned} \tag{36}$$

B. Transposition

$$\begin{pmatrix} L_{\omega,\boldsymbol{q}} - iR^+_{\omega,\boldsymbol{q}} - iR^-_{\omega,\boldsymbol{q}} & -2iR^-_{\omega,\boldsymbol{q}} \\ -2iR^+_{\omega,\boldsymbol{q}} & -L_{-\omega,-\boldsymbol{q}} - iR^+_{\omega,\boldsymbol{q}} - iR^-_{\omega,\boldsymbol{q}} \end{pmatrix} = \begin{pmatrix} \tilde{L}_{-\omega,-\boldsymbol{q}} - i\tilde{R}^+_{-\omega,-\boldsymbol{q}} - i\tilde{R}^-_{-\omega,-\boldsymbol{q}} & -2i\tilde{R}^+_{-\omega,-\boldsymbol{q}} \\ -2i\tilde{R}^-_{-\omega,\boldsymbol{q}} & -\tilde{L}_{\omega,\boldsymbol{q}} - i\tilde{R}^+_{-\omega,-\boldsymbol{q}} - i\tilde{R}^-_{-\omega,-\boldsymbol{q}} \end{pmatrix} \tag{37}$$

Real part:

$$L^{\mu\nu}_{\omega,\boldsymbol{q}} = \begin{pmatrix} L^{tt}_{\omega,q^2} & \boldsymbol{n}L^{ts}_{\omega,q^2} \\ \boldsymbol{n}L^{st}_{\omega,q^2} & TL^T_{\omega,q^2} + LL^L_{\omega,q^2} \end{pmatrix} \tag{38}$$

$$\begin{aligned} L^{\mu\nu}_{\omega,\boldsymbol{q}} &= L^{(\nu\mu)}_{-\omega,-\boldsymbol{q}} \\ L^{ts}_{\omega,q^2} &= L^{st}_{\omega,q^2} = -L^{st}_{-\omega,q^2} \\ L^{tt}_{-\omega,q^2} &= L^{tt}_{\omega,q^2} \\ L^T_{-\omega,q^2} &= L^T_{\omega,q^2} \\ L^L_{\omega,q^2} &= L^L_{-\omega,q^2} \end{aligned} \tag{39}$$

Check:

$$\begin{aligned} L^{\mu\nu}_{-\omega,\boldsymbol{q}} &= 2\hbar(2\pi)^3\frac{4\eta_s^3}{\pi^2}\int_{\boldsymbol{k},\boldsymbol{p}}\frac{(-\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})(n_{\boldsymbol{k}}-n_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}-\boldsymbol{p})}{(\boldsymbol{p}^2+\eta^2)^3[(\omega-\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})^2+\gamma^2]} \\ &= 2\hbar(2\pi)^3\frac{4\eta_s^3}{\pi^2}\int_{\boldsymbol{k},\boldsymbol{p}}\frac{(-\omega-\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}})(-n_{\boldsymbol{k}}+n_{\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}})F^{\mu\nu}(\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p},\boldsymbol{q}-\boldsymbol{p})}{(\boldsymbol{p}^2+\eta^2)^3[(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}})^2+\gamma^2]} \\ &= 2\hbar(2\pi)^3\frac{4\eta_s^3}{\pi^2}\int_{\boldsymbol{k},\boldsymbol{p}}\frac{(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})(n_{\boldsymbol{k}}-n_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p},\boldsymbol{q}-\boldsymbol{p})}{(\boldsymbol{p}^2+\eta^2)^3[(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})^2+\gamma^2]} \end{aligned} \tag{40}$$

$$\begin{aligned} F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}-\boldsymbol{p}) &= \begin{pmatrix} 1 & -\bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \\ -\bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) & \bar{r}^2\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \end{pmatrix} \\ F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p},\boldsymbol{q}-\boldsymbol{p}) &= \begin{pmatrix} 1 & -\bar{r}\left(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \\ -\bar{r}\left(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) & \bar{r}^2\left(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right)\otimes\left(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \end{pmatrix} \\ &= \begin{pmatrix} 1 & \bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \\ \bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) & \bar{r}^2\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}-\boldsymbol{p}}{2}\right) \end{pmatrix} \end{aligned} \tag{41}$$

$$\begin{aligned} L^{ts}_{-\omega,\boldsymbol{q}} &= -\frac{\hbar^2\eta_s^3}{\pi^5 mq^2}\int d^3k d^3p\frac{(-\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})(n_{\boldsymbol{k}}-n_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})(\boldsymbol{q}\boldsymbol{k}+\frac{q^2-\boldsymbol{q}\boldsymbol{p}}{2})}{(\boldsymbol{p}^2+\eta^2)^3[(-\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})^2+\gamma^2]} \\ &= -\frac{\hbar^2\eta_s^3}{\pi^5 mq^2}\int d^3k d^3p\frac{(-\omega-\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})(-n_{\boldsymbol{k}}+n_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})[\boldsymbol{q}(-\boldsymbol{k}-\boldsymbol{q}+\boldsymbol{p})+\frac{q^2-\boldsymbol{q}\boldsymbol{p}}{2}]}{(\boldsymbol{p}^2+\eta^2)^3[(-\omega-\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})^2+\gamma^2]} \\ &= -\frac{\hbar^2\eta_s^3}{\pi^5 mq^2}\int d^3k d^3p\frac{(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})(n_{\boldsymbol{k}}-n_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})(-\boldsymbol{q}\boldsymbol{k}-\frac{q^2-\boldsymbol{q}\boldsymbol{p}}{2})}{(\boldsymbol{p}^2+\eta^2)^3[(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}-\boldsymbol{p}})^2+\gamma^2]} = -L^{ts}_{\omega,\boldsymbol{q}} \end{aligned} \tag{42}$$

Imaginary part: same as the CTP symmetry,

$$\begin{aligned} R^{\mu\nu\pm}_{-\omega,-\boldsymbol{q}} &= R^{(\nu\mu)\mp}_{\omega,\boldsymbol{q}} \\ R^{\pm(ts)}_{\omega,q^2} &= R^{\pm(st)}_{\omega,q^2} = -R^{\mp(st)}_{-\omega,q^2} \\ R^{\pm(tt)}_{-\omega,q^2} &= R^{\mp(tt)}_{\omega,q^2} \\ R^{\pm(T)}_{-\omega,q^2} &= R^{\mp(T)}_{\omega,q^2} \\ R^{\pm(L)}_{\omega,q^2} &= R^{\mp(L)}_{-\omega,q^2} \end{aligned} \tag{43}$$

C. Ward-identity

$$\tilde{G}_{\omega,\boldsymbol{q}} = \begin{pmatrix} \tilde{G}^{tt}_{\omega,q^2} & \boldsymbol{n}\tilde{G}^{ts}_{\omega,q^2} \\ \boldsymbol{n}\tilde{G}^{st}_{\omega,q^2} & T\tilde{G}^{T}_{\omega,q^2} + L\tilde{G}^{L}_{\omega,q^2} \end{pmatrix} \tag{44}$$

$$\partial_t \rho + \boldsymbol{\nabla} \boldsymbol{j} = 0 \tag{45}$$

$$0 = \tilde{G}^{\mu 0}\omega - \tilde{G}^{\mu j} q_j = \begin{pmatrix} \omega\tilde{G}^{tt}_{\omega,q^2} - q\tilde{G}^{ts}_{\omega,q^2} \\ \boldsymbol{n}(\omega\tilde{G}^{st}_{\omega,q^2} - q\tilde{G}^{L}_{\omega,q^2}) \end{pmatrix} \tag{46}$$

$$\tilde{G}^{ts}_{\omega,q^2} = \frac{\omega}{q}\tilde{G}^{tt}_{\omega,q^2}, \qquad \tilde{G}^{L}_{\omega,q^2} = \frac{\omega}{q}\tilde{G}^{st}_{\omega,q^2} = \frac{\omega^2}{q^2}\tilde{G}^{tt}_{\omega,q^2} \tag{47}$$

V. GREEN FUNCTION AND ITS INVERSE

A. Different Green functions

$$\begin{aligned} \hat{\tilde{G}}^{(\sigma\mu)(\sigma'\nu)}_{\omega,\boldsymbol{q}} &= \begin{pmatrix} \tilde{G}^{tt}_{\omega,q^2} & \boldsymbol{n}\frac{\omega}{q}\tilde{G}^{tt}_{\omega,q^2} \\ \boldsymbol{n}\frac{\omega}{q}\tilde{G}^{tt}_{\omega,q^2} & L\frac{\omega^2}{q^2}\tilde{G}^{tt}_{\omega,q^2} + T\tilde{G}^{T}_{\omega,q^2} \end{pmatrix}^{\sigma\sigma'} \\ &= \begin{pmatrix} L_{\omega,\boldsymbol{q}} - iR^{+}_{\omega,\boldsymbol{q}} - iR^{-}_{\omega,\boldsymbol{q}} & -2iR^{-}_{\omega,\boldsymbol{q}} \\ -2iR^{+}_{\omega,\boldsymbol{q}} & -L_{\omega,\boldsymbol{q}} - iR^{+}_{\omega,\boldsymbol{q}} - iR^{-}_{\omega,\boldsymbol{q}} \end{pmatrix}^{\mu\nu} \end{aligned} \tag{48}$$

$$\begin{aligned} \Re\tilde{G}_{xx'} &= \frac{1}{2}\int_q (\tilde{G}_q + \tilde{G}^{*}_{-q}) e^{-iq(x-x')} \\ &= \int_q \begin{pmatrix} L_{\omega,\boldsymbol{q}} & i(R_{\omega,\boldsymbol{q}} - R_{-\omega,-\boldsymbol{q}}) \\ -i(R_{\omega,\boldsymbol{q}} - R_{-\omega,-\boldsymbol{q}}) & -L_{\omega,\boldsymbol{q}} \end{pmatrix} e^{-iq(x-x')} \\ i\Im\tilde{G}_{xx'} &= \frac{1}{2}\int_q (\tilde{G}_q - \tilde{G}^{*}_{-q}) e^{-iq(x-x')} = \int_q (-iR_{\omega,\boldsymbol{q}} - iR_{-\omega,-\boldsymbol{q}}) e^{-iq(x-x')} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \Im\tilde{G}_{xx'} &= -\int_q (R_{\omega,\boldsymbol{q}} + R_{-\omega,-\boldsymbol{q}}) e^{-iq(x-x')} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ G^{r}_{xx'} &= \Re\tilde{G}^{++}_{xx'} - \Im\tilde{G}^{-+}_{xx'} \\ &= \frac{1}{2}\int_q (\tilde{G}^{++}_q e^{-iq(x-x')} + \tilde{G}^{++*}_q e^{iq(x-x')}) - (\tilde{G}^{-+}_q e^{-iq(x-x')} + \tilde{G}^{-+*}_q e^{iq(x-x')}) \\ &= \frac{1}{2}\int_{\omega,\boldsymbol{q}} [(L_{\omega,\boldsymbol{q}} - iR^{+}_{\omega,\boldsymbol{q}} - iR^{-}_{\omega,\boldsymbol{q}}) e^{-iq(x-x')} + (L_{\omega,\boldsymbol{q}} + iR^{+}_{\omega,\boldsymbol{q}} + iR^{-}_{\omega,\boldsymbol{q}}) e^{iq(x-x')} \\ &\quad +2iR^{+}_{\omega,\boldsymbol{q}} e^{-iq(x-x')} - 2iR^{+}_{\omega,\boldsymbol{q}} e^{iq(x-x')}] \\ &= \frac{1}{2}\int_{\omega,\boldsymbol{q}} (L_{\omega,\boldsymbol{q}} - iR^{+}_{\omega,\boldsymbol{q}} - iR^{-}_{\omega,\boldsymbol{q}} + L_{-\omega,-\boldsymbol{q}} + iR^{+}_{-\omega,-\boldsymbol{q}} + iR^{-}_{-\omega,-\boldsymbol{q}} + 2iR^{+}_{\omega,\boldsymbol{q}} - 2iR^{+}_{-\omega,-\boldsymbol{q}}] e^{-iq(x-x')} \\ &= \int_{\omega,\boldsymbol{q}} (L_{\omega,\boldsymbol{q}} + iR^{+}_{\omega,\boldsymbol{q}} - iR^{+}_{-\omega,-\boldsymbol{q}}] e^{-iq(x-x')} \end{aligned} \tag{49}$$

Removing the dimensions for the linear algebra:

$$\begin{aligned} \tilde{J} &= \begin{pmatrix} \rho \\ \frac{m}{\hbar k}\boldsymbol{j} \end{pmatrix} = gJ, \qquad \tilde{a} = g^{-1}a, \qquad g = \begin{pmatrix} 1 & 0 \\ 0 & \frac{m}{\hbar k} \end{pmatrix} \\ S &= aWa = \tilde{a}\tilde{W}\tilde{a} = ag^{-1}\tilde{W}g^{-1}a, \qquad W = g^{-1}\tilde{W}g^{-1}, \qquad W^{-1} = g\tilde{W}^{-1}g \end{aligned} \tag{50}$$

Null-space $\omega\rho - kj^L = 0$, $j^L = \frac{\omega}{k}\rho$, $J^{(0)} = (\frac{\omega}{k}, -1)\rho$, $\omega\tilde\rho - \frac{\hbar\bar kk}{m}\tilde j^L = 0$, $z\tilde\rho = \tilde j^L$:

$$\begin{aligned}\tilde a^\mu_{\omega,\boldsymbol{k}} &= \begin{pmatrix}\tilde u_{\omega,\boldsymbol{k}}\\ \tilde{\boldsymbol{a}}_{\omega,\boldsymbol{k}}\end{pmatrix} = \frac{\tilde v^{(0)}_{\omega,\boldsymbol{k}} z}{\sqrt{1+z^2}}\begin{pmatrix}1\\ -\frac{1}{z}\boldsymbol{n}\end{pmatrix} + \frac{\tilde v_{\omega,\boldsymbol{k}}}{\sqrt{1+z^2}}\begin{pmatrix}1\\ z\boldsymbol{n}\end{pmatrix} + \begin{pmatrix}0\\ \tilde{\boldsymbol{a}}^T_{\omega,\boldsymbol{k}}\end{pmatrix}\\ a^\mu_{\omega,\boldsymbol{k}} &= \frac{\tilde v^{(0)}_{\omega,\boldsymbol{k}} z}{\sqrt{1+z^2}}\begin{pmatrix}1\\ -\frac{m}{z\hbar\bar k}\boldsymbol{n}\end{pmatrix} + \frac{\tilde v_{\omega,\boldsymbol{k}}}{\sqrt{1+z^2}}\begin{pmatrix}1\\ z\frac{m}{\hbar\bar k}\boldsymbol{n}\end{pmatrix} + \begin{pmatrix}0\\ \tilde{\boldsymbol{a}}^T_{\omega,\boldsymbol{k}}\frac{m}{\hbar\bar k}\end{pmatrix}\end{aligned}\tag{51}$$

$\boldsymbol{k}\boldsymbol{j}^T_{\omega,\boldsymbol{k}} = \boldsymbol{k}\boldsymbol{a}^T_{\omega,\boldsymbol{k}} = 0$

$$\begin{aligned}\hat a\tilde G\hat a &= \left[\frac{\tilde v^\sigma_{-\omega,-\boldsymbol{k}}}{\sqrt{1+z^2}}\begin{pmatrix}1\\ z\frac{m}{\hbar\bar k}\boldsymbol{n}\end{pmatrix} + \begin{pmatrix}0\\ \tilde{\boldsymbol{a}}^{T\sigma}_{-\omega,-\boldsymbol{k}}\frac{m}{\hbar\bar k}\end{pmatrix}\right]\begin{pmatrix}\tilde G^{tt}_{\omega,q^2} & \boldsymbol{n}\frac{\omega}{q}\tilde G^{tt}_{\omega,q^2}\\ \boldsymbol{n}\frac{\omega}{q}\tilde G^{tt}_{\omega,q^2} & L\frac{\omega^2}{q^2}\tilde G^{tt}_{\omega,q^2} + T\tilde G^T_{\omega,q^2}\end{pmatrix}^{\sigma\sigma'}\left[\frac{\tilde v^{\sigma'}_{\omega,\boldsymbol{k}}}{\sqrt{1+z^2}}\begin{pmatrix}1\\ z\frac{m}{\hbar\bar k}\boldsymbol{n}\end{pmatrix} + \begin{pmatrix}0\\ \tilde{\boldsymbol{a}}^{T\sigma'}_{\omega,\boldsymbol{k}}\frac{m}{\hbar\bar k}\end{pmatrix}\right]\\ &= \left[\frac{\tilde v_{-\omega,-\boldsymbol{k}}}{\sqrt{1+z^2}}\begin{pmatrix}1\\ z\frac{m}{\hbar\bar k}\boldsymbol{n}\end{pmatrix} + \begin{pmatrix}0\\ \tilde{\boldsymbol{a}}^T_{-\omega,-\boldsymbol{k}}\frac{m}{\hbar\bar k}\end{pmatrix}\right]\left[\tilde G^{tt}_{\omega,q^2}\sqrt{1+z^2}\begin{pmatrix}1\\ \boldsymbol{n}\frac{\omega}{q}\end{pmatrix}\tilde v_{\omega,\boldsymbol{k}} + \begin{pmatrix}0\\ \tilde G^T_{\omega,q^2}\tilde{\boldsymbol{a}}^T_{\omega,\boldsymbol{k}}\frac{m}{\hbar\bar k}\end{pmatrix}\right]\\ &= \tilde v^\sigma_{-\omega,-\boldsymbol{k}}\tilde G^{tt\sigma\sigma'}_{\omega,q^2}(1+z^2)\tilde v^{\sigma'}_{\omega,\boldsymbol{k}} + \tilde{\boldsymbol{a}}^{T\sigma}_{-\omega,-\boldsymbol{k}}\frac{m^2}{\hbar^2\bar k^2}\tilde G^{T\sigma\sigma'}_{\omega,q^2}\tilde{\boldsymbol{a}}^{T\sigma'}_{\omega,\boldsymbol{k}}\end{aligned}\tag{52}$$

B. Inverse

CTP bloks:

$$\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}Ax+By\\ Cx+Dy\end{pmatrix}\tag{53}$$

y first:

$$\begin{aligned}y &= D^{-1}(b-Cx)\\ Ax &= a - By = a - BD^{-1}(b-Cx)\\ x &= \frac{1}{A-BD^{-1}C}(a-BD^{-1}b)\\ y &= D^{-1}\left[b - C\frac{1}{A-BD^{-1}C}(a-BD^{-1}b)\right]\\ \begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1} &= \begin{pmatrix}\frac{1}{A-BD^{-1}C} & -\frac{1}{A-BD^{-1}C}BD^{-1}\\ -D^{-1}C\frac{1}{A-BD^{-1}C} & D^{-1}+D^{-1}C\frac{1}{A-BD^{-1}C}BD^{-1}\end{pmatrix}\end{aligned}\tag{54}$$

x first:

$$\begin{aligned}x &= A^{-1}(a-By)\\ Dy &= b - Cx = b - CA^{-1}(a-By)\\ y &= \frac{1}{D-CA^{-1}B}(b-CA^{-1}a)\\ x &= A^{-1}\left[a - B\frac{1}{D-CA^{-1}B}(b-CA^{-1}a)\right]\\ \begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1} &= \begin{pmatrix}A^{-1}+A^{-1}B\frac{1}{D-CA^{-1}B}CA^{-1} & -A^{-1}B\frac{1}{D-CA^{-1}B}\\ -\frac{1}{D-CA^{-1}B}CA^{-1} & \frac{1}{D-CA^{-1}B}\end{pmatrix}\end{aligned}\tag{55}$$

$$
\begin{aligned}
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} L - iR^+ - iR^- & -2iR^- \\ -2iR^+ & -L - iR^+ - iR^- \end{pmatrix} = \begin{pmatrix} L - iS_+ & -2iR^- \\ -2iR^+ & -L - iS_+ \end{pmatrix}, \quad S_\pm = R^+ \pm R^- \\
\frac{1}{A - BD^{-1}C} &= \frac{1}{L - iR^+ - iR^- - 4R^- \frac{1}{L + iR^+ + iR^-} R^+} \\
&= \frac{L + iR^+ + iR^-}{L^2 + (R^+ + R^-)^2 - 4R^- R^+} = \frac{L + iR^+ + iR^-}{L^2 + (R^+ - R^-)^2} = \frac{L + iS_+}{L^2 + S_-^2} \\
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} &= \begin{pmatrix} \frac{L+iS_+}{L^2+S_-^2} & -2i \frac{L+iS_+}{L^2+S_-^2} R^- \frac{1}{L+iS_+} \\ -2i \frac{1}{L+iS_+} R^+ \frac{L+iS_+}{L^2+S_-^2} & -\frac{1}{L+iS_+} - 4 \frac{1}{L+iS_+} R^+ \frac{L+iS_+}{L^2+S_-^2} R^- \frac{1}{L+iS_+} \end{pmatrix} \\
&= \begin{pmatrix} \frac{L+iS_+}{L^2+S_-^2} & -\frac{2iR^-}{L^2+S_-^2} \\ -\frac{2iR^+}{L^2+S_-^2} & -\frac{1}{L+iS_+}(1 + \frac{4R^+R^-}{L^2+S_-^2}) \end{pmatrix} = \begin{pmatrix} \frac{L+iS_+}{L^2+S_-^2} & -\frac{2iR^-}{L^2+S_-^2} \\ -\frac{2iR^+}{L^2+S_-^2} & -\frac{1}{L+iS_+} \frac{L^2+S_+^2}{L^2+S_-^2} \end{pmatrix} \\
&= \frac{1}{L^2 + S_-^2} \begin{pmatrix} L + iS_+ & -2iR^- \\ -2iR^+ & -L + iS_+ \end{pmatrix} = \frac{1}{L^2 + S_-^2} \left[\begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix} + i \begin{pmatrix} S_+ & -2R^- \\ -2R^+ & S_+ \end{pmatrix} \right]
\end{aligned} \tag{56}
$$

$$
J^\mu_{\omega,\boldsymbol{k}} = \frac{n_{\omega,\boldsymbol{k}}}{\sqrt{1+z^2}} \begin{pmatrix} 1 \\ z \frac{\hbar \bar{k}}{m} \boldsymbol{n} \end{pmatrix} + \begin{pmatrix} 0 \\ \boldsymbol{j}^T_{\omega,\boldsymbol{k}} \end{pmatrix} \tag{57}
$$

$$
\begin{aligned}
\hat{a}\tilde{G}\hat{a} &= \tilde{v}^\sigma_{-\omega,-\boldsymbol{k}} \tilde{G}^{tt\sigma\sigma'}_{\omega,q^2} (1 + z^2) \tilde{v}^{\sigma'}_{\omega,\boldsymbol{k}} + \tilde{\boldsymbol{a}}^{T\sigma}_{-\omega,-\boldsymbol{k}} \frac{m^2}{\hbar^2 \bar{k}^2} \tilde{G}^{T\sigma\sigma'}_{\omega,q^2} \tilde{\boldsymbol{a}}^{T\sigma'}_{\omega,\boldsymbol{k}} \\
\hat{J}\tilde{G}^{-1}\hat{J} &= n^\sigma_{-\omega,-\boldsymbol{k}} \frac{\tilde{G}^{tt-1\sigma\sigma'}_{\omega,q^2}}{1+z^2} n^{\sigma'}_{\omega,\boldsymbol{k}} + \boldsymbol{j}^{T\sigma}_{-\omega,-\boldsymbol{k}} \tilde{G}^{T-1\sigma\sigma'}_{\omega,q^2} \boldsymbol{j}^{T\sigma'}_{\omega,\boldsymbol{k}}
\end{aligned} \tag{58}
$$

VI. EQUATIONS OF MOTION

$j^L = \boldsymbol{n}\boldsymbol{j}$, $\boldsymbol{j}^T = T\boldsymbol{j}$, $a^L = \boldsymbol{n}\boldsymbol{a}$, $\boldsymbol{a}^T = T\boldsymbol{a}$, $(\rho, \boldsymbol{j}) = J - J_{\rm gr}$

A. Fourier transforms

$$
\int_{t'\omega} i\omega e^{-i\omega(t-t')} f_{t'} = \int_{t'\omega} (\partial_{t'} e^{-i\omega(t-t')}) f_{t'} = -\partial_t f_t, \qquad \int_{x'q} iqe^{iq(x-x')} f_{x'} = \int_{x'q} (-\partial_{x'} e^{iq(x-x')}) f_{x'} = \partial_x f_x \tag{59}
$$

$$
\begin{aligned}
\int_{\boldsymbol{k}} e^{i\boldsymbol{y}\boldsymbol{k}} k &= \frac{1}{4\pi^2} \int dk k^3 dC e^{ikrC} = \frac{1}{4\pi^2 ir} \int dk k^2 (e^{ikr} - e^{-ikr}) = \frac{1}{4\pi^2 ir} \int dk k^2 e^{ikr} \mathrm{sign}(k) \\
&= -\frac{1}{4\pi^2 ir} \partial_r^2 \int dk e^{ikr} \mathrm{sign}(k) \\
\mathrm{sign}(k) &= i \int \frac{dz}{2\pi} \left(\frac{e^{-izk}}{z + i\epsilon} - \frac{e^{izk}}{z + i\epsilon} \right) = i \int \frac{dz}{2\pi} e^{-izk} \left(\frac{1}{z + i\epsilon} - \frac{1}{-z + i\epsilon} \right) = 2i \int \frac{dz}{2\pi} e^{-izk} \frac{z}{z^2 + \epsilon^2} \\
\int_{\boldsymbol{k}} e^{i\boldsymbol{y}\boldsymbol{k}} k &= -\frac{1}{2\pi^2 r} \partial_r^2 \int dk \frac{dz}{2\pi} e^{ik(r-z)} \frac{z}{z^2 + \epsilon^2} = -\frac{1}{2\pi^2 r} \partial_r^2 \frac{r}{r^2 + \epsilon^2} \\
\int_{\boldsymbol{k}} \frac{e^{i\boldsymbol{y}\boldsymbol{k}}}{k} &= \frac{1}{4\pi^2} \int dk dC e^{ikrC} = \frac{1}{4\pi^2 ir} \int dk (e^{ikr} - e^{-ikr}) = \frac{1}{4\pi^2 ir} \int dk e^{ikr} 2i \int \frac{dz}{2\pi} e^{-izk} \frac{z}{z^2 + \epsilon^2} \\
&= \frac{1}{2\pi^2 r} \frac{r}{r^2 + \epsilon^2} = \frac{1}{2\pi^2 (r^2 + \epsilon^2)} \\
\int_{\boldsymbol{k}} \frac{e^{i\boldsymbol{y}\boldsymbol{k}}}{k^2 + q^2} &= \frac{1}{4\pi^2} \int dk dC \frac{k^2 e^{ikrC}}{k^2 + q^2} = \frac{1}{4\pi^2 ir} \int \frac{dk k}{k^2 + q^2} (e^{ikr} - e^{-ikr}) = \frac{1}{4\pi^2 ir} \int \frac{dk k e^{ikr}}{(k + iq)(k - iq)} = \frac{e^{-qr}}{4\pi r}
\end{aligned} \tag{60}
$$

Back to real space: $F_{\boldsymbol{q}} = f_{\boldsymbol{q}} q$

$$F_{\boldsymbol{x}} = \int_{\boldsymbol{q}} e^{i\boldsymbol{x}\boldsymbol{q}} q f_{\boldsymbol{q}} = \int_{\boldsymbol{q}\boldsymbol{y}} e^{i(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{q}} q f_{\boldsymbol{y}} = -\frac{1}{2\pi^2}\int_{\boldsymbol{y}} f_{\boldsymbol{x}-y\boldsymbol{n}} \frac{1}{y}\partial_y^2 \frac{y}{y^2+\epsilon^2} = -\frac{1}{2\pi^2}\int d^2n dy \frac{y}{y^2+\epsilon^2}\partial_y^2 y f_{\boldsymbol{x}-y\boldsymbol{n}} \tag{61}$$

$$\partial_y f_{\boldsymbol{x}-y\boldsymbol{n}} = -\boldsymbol{n}\boldsymbol{\nabla} f_{\boldsymbol{x}-y\boldsymbol{n}}, \qquad \partial_y^2 f_{\boldsymbol{x}-y\boldsymbol{n}} = (\boldsymbol{n}\boldsymbol{\nabla})^2 f_{\boldsymbol{x}-y\boldsymbol{n}} \tag{62}$$

$$\begin{aligned} F_{\boldsymbol{x}} &= -\frac{1}{2\pi^2}\int d^2n dy \frac{y}{y^2+\epsilon^2}[y(\boldsymbol{n}\boldsymbol{\nabla})^2 f_{\boldsymbol{x}-y\boldsymbol{n}} - 2\boldsymbol{n}\boldsymbol{\nabla} f_{\boldsymbol{x}-y\boldsymbol{n}}] \\ &= -\frac{1}{2\pi^2}\int d^2n \frac{dy}{y}[y(\boldsymbol{n}\boldsymbol{\nabla})^2 f_{\boldsymbol{x}-y\boldsymbol{n}} - 2\boldsymbol{n}\boldsymbol{\nabla} f_{\boldsymbol{x}-y\boldsymbol{n}}] \\ &= -\frac{1}{2\pi^2}\int \frac{d^3y}{y^2}\left[(\boldsymbol{n}\boldsymbol{\nabla})^2 f_{\boldsymbol{x}-\boldsymbol{y}} - \frac{2}{y}\boldsymbol{n}\boldsymbol{\nabla} f_{\boldsymbol{x}-\boldsymbol{y}}\right] \\ &= -\frac{1}{2\pi^2}\int \frac{d^3y}{(\boldsymbol{x}-\boldsymbol{y})^2}\left[(\boldsymbol{n}\boldsymbol{\nabla})^2 f_{\boldsymbol{y}} - \frac{2}{|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{n}\boldsymbol{\nabla} f_{\boldsymbol{y}}\right] \\ &= \int_{\boldsymbol{y}} q_{\boldsymbol{x},\boldsymbol{y}} f_{\boldsymbol{y}} \end{aligned} \tag{63}$$

$$q_{\boldsymbol{x},\boldsymbol{y}} = -\frac{1}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4}[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y] \tag{64}$$

$f_{\boldsymbol{x}} = \delta_{\boldsymbol{x},0}$

$$\begin{aligned} \tilde{f}_{\boldsymbol{x}} &= -\frac{1}{2\pi^2}\int_{\boldsymbol{y}} \delta_{\boldsymbol{x},\boldsymbol{y}} \frac{1}{y}\partial_y^2 \frac{y}{y^2+\epsilon^2} = -\frac{1}{2\pi^2}\frac{1}{r}\partial_r^2 \frac{r}{r^2+\epsilon^2} = -\frac{1}{2\pi^2}\frac{1}{r}\partial_r\left[\frac{1}{r^2+\epsilon^2} - \frac{2r^2}{(r^2+\epsilon^2)^2}\right] \\ &= -\frac{1}{2\pi^2}\frac{1}{r}\left[-\frac{2r}{(r^2+\epsilon^2)^2} - \frac{4r}{(r^2+\epsilon^2)^2} + \frac{8r^3}{(r^2+\epsilon^2)^3}\right] = \frac{1}{2\pi^2}\frac{1}{r}\frac{6r(r^2+\epsilon^2)-8r^3}{(r^2+\epsilon^2)^3} \\ &= -\frac{r^2}{\pi^2(r^2+\epsilon^2)^3} \end{aligned} \tag{65}$$

$F_{\boldsymbol{q}} = f_{\boldsymbol{q}} q^{-1}$

$$F_{\boldsymbol{x}} = \int_{\boldsymbol{q}} e^{i\boldsymbol{x}\boldsymbol{q}} q^{-1} f_{\boldsymbol{q}} = \int_{\boldsymbol{q}\boldsymbol{y}} e^{i(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{q}} q^{-1} f_{\boldsymbol{y}} = \int_{\boldsymbol{y}} \frac{1}{2\pi^2[(\boldsymbol{x}-\boldsymbol{y})^2+\epsilon^2]} f_{\boldsymbol{y}} = \int_{\boldsymbol{y}} q^{-1}_{\boldsymbol{x}-\boldsymbol{y}} f_{\boldsymbol{y}} \tag{66}$$

$$q^{-1}_{\boldsymbol{x}} = \frac{1}{2\pi^2(\boldsymbol{x}^2+\epsilon^2)} \tag{67}$$

$F_\omega = i\pi \mathrm{sign}(\omega) f_\omega$

$$F_t = \int_{t'\omega} e^{-i\omega(t-t')} i\pi \mathrm{sign}(\omega) f_{t'} = \int_{t'\omega} e^{-i\omega(t-t')} i\pi 2i \int \frac{dz}{2\pi} e^{-iz\omega}\frac{z}{z^2+\epsilon^2} f_{t'} = -\int d\tau \frac{\tau}{\tau^2+\epsilon^2} f_{t+\tau} = \int d\tau \Sigma_\tau f_{t-\tau} \tag{68}$$

$$\Sigma_\tau = \frac{\tau}{\tau^2+\epsilon^2} = P\frac{1}{\tau} \tag{69}$$

B. Equations of motion in real space

$\boldsymbol{j} = \boldsymbol{j}_T + \boldsymbol{n} j_L$, $\boldsymbol{n} = \boldsymbol{q}/|\boldsymbol{q}|$

$$\begin{aligned}
a^0 &= \frac{4\pi^2}{n_s m k_F} \frac{n}{-2 + i\pi z + 2z^2 + \frac{Q^2}{6}} = -\frac{1}{2} \frac{4\pi^2}{n_s m k_F} \frac{n}{1 - \frac{1}{2} i\pi z - z^2 - \frac{Q^2}{12}} \\
&= -\frac{1}{2} \frac{4\pi^2}{n_s m k_F} \left[1 + \frac{1}{2} i\pi z + \left(1 - \frac{\pi^2}{4}\right) z^2 + \frac{Q^2}{12} \right] n = \frac{4\pi^2}{n_s m k_F} \underbrace{\left(-\frac{1}{2} - \frac{1}{4}\pi i z + \frac{\pi^2 - 8}{8} z^2 - \frac{Q^2}{24} \right)}_{a_0 + i a_z z + a_{zz} z^2 + a_{qq} Q^2} n \\
\boldsymbol{a}^T &= \frac{4\pi^2}{n_s m k_F} \frac{m^2}{\hbar^2 k_F^2} \frac{1}{-\frac{2}{3} + i\pi z + 2z^2 + \frac{1}{6} Q^2} \boldsymbol{j}^T = -\frac{3}{2} \frac{4\pi^2}{n_s m k_F} \frac{m^2}{\hbar^2 k_F^2} \frac{1}{1 - \frac{3}{2} i\pi z - 3z^2 - \frac{1}{4} Q^2} \boldsymbol{j}^T \\
&= -\frac{3}{2} \frac{4\pi^2}{n_s m k_F} \frac{m^2}{\hbar^2 k_F^2} \left(1 + \frac{3}{2} i\pi z + \left(3 - \frac{4\pi^2}{9}\right) z^2 + \frac{1}{4} Q^2 \right) \boldsymbol{j}^T \\
&= \frac{4\pi^2}{n_s m k_F} \frac{m^2}{\hbar^2 k_F^2} \underbrace{\left(-\frac{3}{2} - \frac{9\pi}{4} i z + \frac{4\pi^2 - 27}{6} z^2 - \frac{3}{8} Q^2 \right)}_{b_0 + i b_z z + b_{zz} z^2 + b_{qq} Q^2} \boldsymbol{j}^T
\end{aligned} \tag{70}$$

$$\begin{aligned}
a_0 &= -\frac{1}{2}, \quad a_z = -\frac{\pi}{4}, \quad a_{zz} = \frac{\pi^2 - 8}{8}, \quad a_{qq} = -\frac{1}{24} \\
b_0 &= -\frac{3}{2}, \quad b_z = -\frac{9\pi}{4}, \quad b_{zz} = \frac{4\pi^2 - 27}{6}, \quad b_{qq} = -\frac{3}{8}
\end{aligned} \tag{71}$$

$$\omega n = \boldsymbol{k}\boldsymbol{j} = q j_L \tag{72}$$

$$\begin{aligned}
\frac{n_s m k_F}{4\pi^2} a^0 &= \left(a_0 + a_z \frac{m}{\hbar \bar{k}} \frac{i\omega}{q} + a_{zz} \frac{m^2}{\hbar^2 \bar{k}^2} \frac{\omega^2}{q^2} + \frac{a_{qq}}{\bar{k}^2} q^2 \right) n \\
\frac{n_s m k_F}{4\pi^2} \frac{\hbar^2 k_F^2}{m^2} \boldsymbol{a}^T &= \left(b_0 + b_z \frac{m}{\hbar \bar{k}} \frac{i\omega}{q} + b_{zz} \frac{m^2}{\hbar^2 \bar{k}^2} \frac{\omega^2}{q^2} + \frac{b_{qq}}{\bar{k}^2} q^2 \right) \boldsymbol{j}^T
\end{aligned} \tag{73}$$

$\mathcal{O}(\omega)$:

$$\begin{aligned}
\frac{n_s m k_F}{4\pi^2} q a^0 &= \left(q a_0 + a_z \frac{m}{\hbar \bar{k}} i\omega + \frac{a_{qq}}{\bar{k}^2} q^3 \right) n \\
\frac{n_s m k_F}{4\pi^2} \frac{\hbar^2 k_F^2}{m^2} q \boldsymbol{a}^T &= \left(q b_0 + b_z \frac{m}{\hbar \bar{k}} i\omega + \frac{b_{qq}}{\bar{k}^2} q^3 \right) \boldsymbol{j}^T
\end{aligned} \tag{74}$$

$$\begin{aligned}
a_z \frac{m}{\hbar \bar{k}} i\omega n &= \frac{n_s m k_F}{4\pi^2} q a^0 - \left(q a_0 + \frac{a_{qq}}{\bar{k}^2} q^3 \right) n \\
b_z \frac{m}{\hbar \bar{k}} i\omega \boldsymbol{j}^T &= \frac{n_s m k_F}{4\pi^2} \frac{\hbar^2 k_F^2}{m^2} q \boldsymbol{a}^T - \left(q b_0 + \frac{b_{qq}}{\bar{k}^2} q^3 \right) \boldsymbol{j}^T
\end{aligned} \tag{75}$$

$$\begin{aligned}
b_z \frac{m}{\hbar\bar{k}} i\omega \boldsymbol{j} &= \frac{n_s m k_F}{4\pi^2}\frac{\hbar^2 k_F^2}{m^2} q\boldsymbol{a}^T - \left(q b_0 + \frac{b_{qq}}{\bar{k}^2} q^3\right)\left(1 - \frac{\boldsymbol{q}\otimes\boldsymbol{q}}{q^2}\right)\boldsymbol{j} + b_z \frac{m}{\hbar\bar{k}} i \frac{\omega^2}{\boldsymbol{q}^2}\boldsymbol{q} n \\
&\approx \frac{n_s m k_F}{4\pi^2}\frac{\hbar^2 k_F^2}{m^2} q\boldsymbol{a}^T - \left(q b_0 + \frac{b_{qq}}{\bar{k}^2} q^3\right)\left(\boldsymbol{j} - \frac{\boldsymbol{q}}{q} j_L\right) \\
&= \frac{n_s m k_F}{4\pi^2}\frac{\hbar^2 k_F^2}{m^2} q\boldsymbol{a}^T - \left(q b_0 + \frac{b_{qq}}{\bar{k}^2} q^3\right)\boldsymbol{j} + \boldsymbol{q}\left(b_0 + \frac{b_{qq}}{\bar{k}^2} q^2\right) j_L \\
&= \frac{n_s m k_F}{4\pi^2}\frac{\hbar^2 k_F^2}{m^2} q\boldsymbol{a}^T - \left(q b_0 + \frac{b_{qq}}{\bar{k}^2} q^3\right)\boldsymbol{j} + \boldsymbol{q}\left(b_0 + \frac{b_{qq}}{\bar{k}^2} q^2\right)\frac{\omega}{q} n \\
&= \frac{n_s m k_F}{4\pi^2}\frac{\hbar^2 k_F^2}{m^2} q\boldsymbol{a}^T - q\left(b_0 + \frac{b_{qq}}{\bar{k}^2} q^2\right)\boldsymbol{j} \\
&\quad - i\boldsymbol{q}\left(b_0 + \frac{b_{qq}}{\bar{k}^2} q^2\right)\frac{\hbar\bar{k}}{a_z m}\left[\frac{n_s m k_F}{4\pi^2} a^0 - \left(a_0 + \frac{a_{qq}}{\bar{k}^2} q^2\right) n\right]
\end{aligned} \tag{76}$$

$$\begin{aligned}
a_z \frac{m}{\hbar\bar{k}} \partial_t n_{\boldsymbol{x}} &= \frac{n_s m k_F}{4\pi^2}\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y] a^0_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} + \left(-a_0 + \frac{a_{qq}}{\bar{k}^2}\Delta\right)\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y] n_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} \\
b_z \frac{m}{\hbar\bar{k}} \partial_t \boldsymbol{j}_{\boldsymbol{x}} &= \frac{n_s m k_F}{4\pi^2}\frac{\hbar^2 k_F^2}{m^2}\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y] \boldsymbol{a}^T_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} + \left(-b_0 + \frac{b_{qq}}{\bar{k}^2}\Delta\right)\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y] \boldsymbol{j}^T_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} - b_z \frac{m}{\hbar\bar{k}}\boldsymbol{\nabla}\Phi_x
\end{aligned} \tag{77}$$

$$\begin{aligned}
\Phi &= \frac{\hbar\bar{k}}{b_z m}\left(b_0 + \frac{b_{qq}}{\bar{k}^2} q^2\right)\frac{\hbar\bar{k}}{a_z m}\left[\left(a_0 + \frac{a_{qq}}{\bar{k}^2} q^2\right) n - \frac{n_s m k_F}{4\pi^2} a^0\right] \\
&\approx -\frac{n_s \hbar^2 \bar{k}^3}{4\pi^2 a_z b_z m}\left(b_0 + \frac{b_{qq}}{\bar{k}^2} q^2\right) a^0 + \frac{\hbar^2\bar{k}^2}{a_z b_z m^2}\left(a_0 b_0 + \frac{a_0 b_{qq} + b_0 a_{qq}}{\bar{k}^2} q^2\right) n \\
&= -\frac{n_s \hbar^2 \bar{k}^3}{4\pi^2 a_z b_z m}\left(b_0 - \frac{b_{qq}}{\bar{k}^2}\Delta\right) a^0 + \frac{\hbar^2\bar{k}^2}{a_z b_z m^2}\left(a_0 b_0 - \frac{a_0 b_{qq} + b_0 a_{qq}}{\bar{k}^2}\Delta\right) n
\end{aligned} \tag{78}$$

$$\begin{aligned}
\partial_t n_{\boldsymbol{x}} &= \frac{n_s \hbar\bar{k}^2}{4\pi^2 a_z}\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y] a^0_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} + \frac{\hbar\bar{k}^2}{m}\left(-\frac{a_0}{a_z} + \frac{a_{qq}}{a_z\bar{k}^2}\Delta\right)\frac{1}{\bar{k}}\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y] n_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} \\
\partial_t \boldsymbol{j}^T_{\boldsymbol{x}} &= \frac{n_s \hbar\bar{k}^2}{4\pi^2 b_z}\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y] \boldsymbol{a}^T_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} + \frac{\hbar\bar{k}^2}{m}\left(-\frac{b_0}{b_z} + \frac{b_{qq}}{b_z\bar{k}^2}\Delta\right)\frac{1}{\bar{k}}\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y] \boldsymbol{j}^T_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} - \boldsymbol{\nabla}\Phi_x
\end{aligned} \tag{79}$$

$\mathcal{O}(\omega^2)$:

$$\begin{aligned}
\frac{n_s m k_F}{4\pi^2} q^2 a^0 &= \left(q^2 a_0 + a_z \frac{m}{\hbar\bar{k}} q i\omega + a_{zz}\frac{m^2}{\hbar^2\bar{k}^2}\omega^2 + \frac{a_{qq}}{\bar{k}^2} q^4\right) n \\
\frac{n_s m k_F}{4\pi^2}\frac{\hbar^2 k_F^2}{m^2} q^2 \boldsymbol{a}^T &= \left(q^2 b_0 + b_z \frac{m}{\hbar\bar{k}} q i\omega + b_{zz}\frac{m^2}{\hbar^2\bar{k}^2}\omega^2 + \frac{b_{qq}}{\bar{k}^2} q^4\right)\boldsymbol{j}^T
\end{aligned} \tag{80}$$

$$\begin{aligned}
a_{zz}\frac{m^2}{\hbar^2\bar{k}^2}\omega^2 n &= \frac{n_s m k_F}{4\pi^2} q^2 a^0 - \left(q^2 a_0 + a_z \frac{m}{\hbar\bar{k}} q i\omega + \frac{a_{qq}}{\bar{k}^2} q^4\right) n \\
b_{zz}\frac{m^2}{\hbar^2\bar{k}^2}\omega^2 \boldsymbol{j}^T &= \frac{n_s m k_F}{4\pi^2}\frac{\hbar^2 k_F^2}{m^2} q^2 \boldsymbol{a}^T - \left(q^2 b_0 + b_z \frac{m}{\hbar\bar{k}} q i\omega + \frac{b_{qq}}{\bar{k}^2} q^4\right)\boldsymbol{j}^T
\end{aligned} \tag{81}$$

For a given wave vector:

$$\begin{aligned}
n &= \frac{\tilde{a}^0}{\omega^2 + c i\omega + d} \\
\boldsymbol{j}^T &= \frac{\tilde{\boldsymbol{a}}^T}{\omega^2 + e i\omega + f}
\end{aligned} \tag{82}$$

with

$$c = \frac{a_z q}{a_{zz}}\frac{\hbar\bar{k}}{m}, \quad d = \frac{a_0 q^2 + \frac{a_{qq}}{\bar{k}^2}q^4}{a_{zz}\frac{m^2}{\hbar^2\bar{k}^2}} \tag{83}$$

Overdamped modes: $-c^2 < 4d$, $-e^2 < 4f$

$$\begin{aligned}
-\frac{a_z^2 q^2}{a_{zz}^2}\frac{\hbar^2\bar{k}^2}{m^2} &< 4\frac{a_0 q^2 + \frac{a_{qq}}{\bar{k}^2}q^4}{a_{zz}\frac{m^2}{\hbar^2\bar{k}^2}} \\
-\frac{a_z^2}{a_{zz}^2} &< 4\frac{a_0 + \frac{a_{qq}}{\bar{k}^2}q^2}{a_{zz}} \\
-\frac{a_z^2}{4a_{zz}^2} - \frac{a_0}{a_{zz}} &< \frac{a_{qq}}{a_{zz}}\frac{q^2}{\bar{k}^2} \\
\frac{q^2}{\bar{k}^2} &< -\frac{a_z^2}{4a_{zz}a_{qq}} - \frac{a_0}{a_{qq}} = \frac{8\cdot 24\pi^2}{16\cdot 4(\pi^2-8)} - \frac{24}{2} = 12\left[\frac{\pi^2}{4(\pi^2-8)} - 1\right] \sim 3.84 \\
\frac{q^2}{\bar{k}^2} &< -\frac{b_z^2}{4b_{zz}b_{qq}} - \frac{b_0}{b_{qq}} = \frac{6\cdot 8\cdot 81\pi^2}{16\cdot 4(4\pi^2-27)3} - \frac{3\cdot 8}{3\cdot 2} = 4\left[\frac{81\pi^2}{16(4\pi^2-27)} - 1\right] \sim 12.02
\end{aligned} \tag{84}$$

Near field Green function only: $\pi \to 0$:

$$\begin{aligned}
a_0 &= -\frac{1}{2}, \quad a_z = -\frac{\pi}{4} \to 0, \quad a_{zz} = \frac{\pi^2-8}{8} \to -1, \quad a_{qq} = -\frac{1}{24} \\
b_0 &= -\frac{3}{2}, \quad b_z = -\frac{9\pi}{4} \to 0, \quad b_{zz} = \frac{4\pi^2-27}{6} \to -\frac{9}{2}, \quad b_{qq} = -\frac{3}{8}
\end{aligned} \tag{85}$$

and

$$\omega^2 = -q^2\frac{\hbar^2\bar{k}^2}{m^2}\frac{a_0 + \frac{a_{qq}}{\bar{k}^2}q^2}{a_{zz}} = -q^2\frac{\hbar^2\bar{k}^2}{m^2}\left(\frac{1}{2} + \frac{q^2}{24\bar{k}^2}\right) \tag{86}$$

Static flow:

$$\begin{aligned}
n_{\boldsymbol{q}} &= \frac{n_s m k_F}{4\pi^2}\left[-1 + \left(\frac{1}{Q} - \frac{Q}{4}\right)\ln\left|\frac{1-\frac{Q}{2}}{1+\frac{Q}{2}}\right|\right]a^0 \\
&\approx -\frac{n_s m k_F}{4\pi^2}\frac{2}{3}e^{-\frac{Q^2}{3}}a^0
\end{aligned} \tag{87}$$

$a^0 = const$, $n_{\boldsymbol{x}} = n_x$, no dependence in y, z:

$$\begin{aligned}
n_x &= -\frac{2a_0 n_s m k_F}{24\pi^3}\int dq e^{iqx - \frac{q^2}{3k_F^2}} \\
&= -\frac{a_0 n_s m k_F}{12\pi^3}\sqrt{\frac{2\pi}{\frac{2}{3k_F^2}}}e^{-\frac{x^2}{2\frac{2}{3k_F^2}}} = -\frac{a_0 n_s m k_F^2}{12\pi^3}\sqrt{3\pi}e^{-\frac{3}{4}k_F^2 x^2}
\end{aligned} \tag{88}$$

Navier-Stokes equation:

$$nD_0 v_a = F_a + \partial_\ell\left[\eta\left(\partial_\ell v_a + \partial_a v_\ell - \frac{2}{3}\delta_{a,\ell}\vec{\partial}\vec{v}\right)\right] + \partial_a(\xi\vec{\partial}\vec{v}) \tag{89}$$

for $\partial\eta = \partial\xi = 0$

$$n[\partial_t\boldsymbol{v} + (\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v}] = -\boldsymbol{\nabla}p + \eta\Delta\boldsymbol{v} + \left(\xi + \frac{\eta}{3}\right)\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{v}) \tag{90}$$

VII. DECOHERENCE

$$\begin{aligned}
a^{\pm} &= \frac{a}{2}(1\pm\kappa)\pm\bar{a} = \frac{a}{2}\pm\left(\bar{a}+\frac{\kappa}{2}a\right)\\
aJ &= a\underbrace{\left(\frac{1+\kappa}{2}J^{+}+\frac{1-\kappa}{2}J^{-}\right)}_{J}+\bar{a}(\underbrace{J^{+}-J^{-}}_{J^{a}})\\
J &= \frac{1+\kappa}{2}J^{+}+\frac{1-\kappa}{2}J^{-} = \frac{1}{2}(J^{+}+J^{-})+\frac{\kappa}{2}(J^{+}-J^{-}), \qquad J^{a}=J^{+}-J^{-}\\
J^{-} &= J^{+}-J^{a}, \quad J = J^{+}-\frac{1}{2}J^{a}+\frac{\kappa}{2}J^{a}\\
J^{+} &= J+\frac{1-\kappa}{2}J^{a} = J+\frac{J^{a}}{2}-\frac{\kappa}{2}J^{a}, \quad J^{-} = J-\frac{1+\kappa}{2}J^{a} = J-\frac{J^{a}}{2}-\frac{\kappa}{2}J^{a}, \quad J^{\pm} = J-\frac{\kappa}{2}J^{a}\pm\frac{J^{a}}{2} \qquad (91)
\end{aligned}$$

$$\begin{aligned}
iS_B[\hat{J}] &= \frac{i}{2(L^2+S_-^2)}\left(u+\frac{v}{2}, u-\frac{v}{2}\right)\left[\begin{pmatrix} L & 0\\ 0 & -L\end{pmatrix}+i\begin{pmatrix} S_+ & -2R^-\\ -2R^+ & S_+\end{pmatrix}\right]\begin{pmatrix} u+\frac{v}{2}\\ u-\frac{v}{2}\end{pmatrix}\\
&= \frac{i}{2(L^2+S_-^2)}\left(u+\frac{v}{2}, u-\frac{v}{2}\right)\left[\begin{pmatrix} L(u+\frac{v}{2})\\ -L(u-\frac{v}{2})\end{pmatrix}+i\begin{pmatrix} uS_-+\frac{v}{2}S_++vR^-\\ -uS_--\frac{v}{2}S_+-vR^+\end{pmatrix}\right]\\
&= \frac{i}{2(L^2+S_-^2)}[uLv+vLu+i(-uS_-v+vS_-u+vS_+v)]\\
&= i\frac{uLv}{L^2+S_-^2}+\frac{uS_-v}{L^2+S_-^2}-\frac{vS_+v}{2(L^2+S_-^2)} \qquad (92)
\end{aligned}$$

$J\to J-\frac{\kappa}{2}J^a$:

$$\begin{aligned}
iS_B[\hat{J}] &= i\frac{(J-\frac{\kappa}{2}J^a)LJ^a}{L^2+S_-^2}+\frac{(J-\frac{\kappa}{2}J^a)S_-J^a}{L^2+S_-^2}-\frac{J^aS_+J^a}{2(L^2+S_-^2)}\\
&= \frac{1}{L^2+S_-^2}\left[i\left(JLJ^a-\frac{\kappa}{2}J^aLJ^a\right)+JS_-J^a-\frac{\kappa}{2}J^aS_-J^a-\frac{1}{2}J^aS_+J^a\right]\\
&= \frac{1}{L^2+S_-^2}\left[iJLJ^a+JS_-J^a-\frac{1}{2}J^aS_+J^a\right] \qquad (93)
\end{aligned}$$

and κ-dependence drops out.

A. Mode-by-mode

Generating functional:

$$\begin{aligned}
E &= i\frac{u^*(L-iS_-)v}{2(L^2+S_-^2)}+i\frac{v^*(L+iS_-)u}{2(L^2+S_-^2)}-\frac{v^*S_+v}{2(L^2+S_-^2)}+ix^*u+iu^*x+iy^*v+iv^*y\\
&= i\frac{(u^*+a^*)(L-iS_-)(v+b)}{2(L^2+S_-^2)}+i\frac{(v^*+b^*)(L+iS_-)(u+a)}{2(L^2+S_-^2)}-\frac{(v^*+b^*)S_+(v+b)}{2(L^2+S_-^2)}\\
&\quad +ix^*(u+a)+i(u^*+a^*)x+iy^*(v+b)+i(v^*+b^*)y\\
&= i\frac{u^*(L-iS_-)v}{2(L^2+S_-^2)}+i\frac{v^*(L+iS_-)u}{2(L^2+S_-^2)}-\frac{v^*S_+v}{2(L^2+S_-^2)}+i\left(x+\frac{b(L-iS_-)}{2(L^2+S_-^2)}\right)^*u\\
&\quad +iu^*\left(x+\frac{(L-iS_-)b}{2(L^2+S_-^2)}\right)+i\left(y+\frac{a(L+iS_-)-ibS_+}{2(L^2+S_-^2)}\right)^*v+iv^*\left(y+\frac{(L+iS_-)a+iS_+b}{2(L^2+S_-^2)}\right)\\
&\quad i\frac{a^*(L-iS_-)b}{2(L^2+S_-^2)}+i\frac{b^*(L+iS_-)a}{2(L^2+S_-^2)}-\frac{b^*S_+b}{2(L^2+S_-^2)} \qquad (94)
\end{aligned}$$

$$\begin{aligned}
b &= -\frac{2(L^2+S_-^2)}{L-iS_-}x = 2(L+iS_-)x \\
\frac{a(L+iS_-)}{2(L^2+S_-^2)} &= -\frac{ibS_+}{2(L^2+S_-^2)} - y \\
a &= -\frac{ibS_+}{L+iS_-} - y\frac{2(L^2+S_-^2)}{L+iS_-} = -\frac{ibS_+}{L+iS_-} - 2y(L-iS_-) = -2iS_+x - 2y(L-iS_-) \\
b^* &= -\frac{2(L^2+S_-^2)}{L+iS_-}x^* = 2(L-iS_-)x^* \\
\frac{a^*(L-iS_-)}{2(L^2+S_-^2)} &= -\frac{ib^*S_+}{2(L^2+S_-^2)} - y^* \\
a^* &= -\frac{ib^*S_+}{L-iS_-} - y^*\frac{2(L^2+S_-^2)}{L-iS_-} = -\frac{ib^*S_+}{L-iS_-} - 2y^*(L+iS_-) = -2iS_+x^* - 2y^*(L+iS_-)
\end{aligned} \tag{95}$$

$$\begin{aligned}
iS_B &= i\frac{u^*(L-iS_-)v + c.c.}{2(L^2+S_-^2)} - \frac{v^*S_+v}{2(L^2+S_-^2)} \\
&= i\frac{(u_1-iu_2)(L-iS_-)(v_1+iv_2) + c.c.}{2(L^2+S_-^2)} - \frac{(v_1-iv_2)S_+(v_1+iv_2)}{2(L^2+S_-^2)} \\
&= i\frac{u_1Lv_1 + u_2Lv_2 - u_2S_-v_1 + u_1S_-v_2}{L^2+S_-^2} - \frac{v_1S_+v_1 + v_2S_+v_2}{2(L^2+S_-^2)} \\
&= \frac{1}{2(L^2+S_-^2)}(u_1,u_2,v_1,v_2)\begin{pmatrix} 0 & 0 & iL & iS_- \\ 0 & 0 & -iS_- & iL \\ iL & iS_- & -S_+ & 0 \\ -iS_- & iL & 0 & -S_+ \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}
\end{aligned} \tag{96}$$

$$D = det\begin{pmatrix} 0 & 0 & iL & iS_- \\ 0 & 0 & -iS_- & iL \\ iL & iS_- & -S_+ & 0 \\ -iS_- & iL & 0 & -S_+ \end{pmatrix} = iLiL(-L^2-S_-^2) + iS_-iS_-(-L^2-S_-^2)) = (L^2+S_-^2)^2 \tag{97}$$

$$e^{iW} = \int dudu^*dvdv^* e^{i\frac{u^*(L-iS_-)v}{2(L^2+S_-^2)} + i\frac{v^*(L+iS_-)u}{2(L^2+S_-^2)} - \frac{v^*S_+v}{2(L^2+S_-^2)} + ix^*u + iu^*x + iy^*v + iv^*y} \tag{98}$$

$$\begin{aligned}
W &= -i\ln\frac{(2\pi)^2}{L^2+S_-^2} + \frac{a^*(L-iS_-)b}{2(L^2+S_-^2)} + \frac{b^*(L+iS_-)a}{2(L^2+S_-^2)} + i\frac{b^*S_+b}{2(L^2+S_-^2)} \\
&\approx \frac{1}{2(L^2+S_-^2)}\{[-2iS_+x^* - 2y^*(L+iS_-)](L-iS_-)2(L+iS_-)x \\
&\quad +2(L-iS_-)x^*(L+iS_-)[-2iS_+x - 2y(L-iS_-)] + i2(L-iS_-)x^*S_+2(L+iS_-)x\} \\
&= [-2iS_+x^* - 2y^*(L+iS_-)]x + x^*[-2iS_+x - 2y(L-iS_-)] + ix^*S_+2x \\
&= -2(L+iS_-)y^*x - 2(L-iS_-)x^*y - i2S_+xx^*
\end{aligned} \tag{99}$$

$$
\begin{aligned}
i\langle u\rangle &= [-2i(L-iS_-)y+2S_+x]e^{-2i(L+iS_-)y^*x-2i(L-iS_-)x^*y+2S_+xx^*} \\
-\langle u^*u\rangle &= \{[-2i(L-iS_-)y+2S_+x][-2i(L+iS_-)y^*+2S_+x^*]+2S_+\} \\
&= \{-4(L^2+S_-^2)y^*y+4S_+^2x^*x-4iS_+(L+iS_-)y^*x-4iS_+(L-iS_-)x^*y+2S_+\}\to 2S_+ \\
-i\langle u^*u^*u\rangle &= [(-2i(L+iS_-)y^*+2S_+x^*]\{[-2i(L-iS_-)y+2S_+x][-2i(L+iS_-)y^*+2S_+x^*]+2S_+\} \\
&\quad +2S_+[-2i(L+iS_-)y^*+2S_+x^*] \\
&= [-2i(L-iS_-)y+2S_+x][-2i(L+iS_-)y^*+2S_+x^*]^2+4S_+[(-2i(L+iS_-)y^*+2S_+x^*] \\
\langle (u^*u)^2\rangle &= [-2i(L-iS_-)y+2S_+x]\big([-2i(L-iS_-)y+2S_+x][-2i(L+iS_-)y^*+2S_+x^*]^2 \\
&\quad +4S_+[(-2i(L+iS_-)y^*+2S_+x^*]\big)+4S_+[-2i(L-iS_-)y+2S_+x][-2i(L+iS_-)y^*+2S_+x^*]+8S_+^2 \\
&= [-2i(L-iS_-)y+2S_+x]^2[-2i(L+iS_-)y^*+2S_+x^*]^2 \\
&\quad +8S_+[(-2i(L+iS_-)y^*+2S_+x^*][-2i(L-iS_-)y+2S_+x]+8S_+^2\to 8S_+^2 \\
i\langle v\rangle &= -2i(L+iS_-)xe^{-2i(L+iS_-)y^*x-2i(L-iS_-)x^*y+2S_+xx^*} \\
-\langle u^*v\rangle &= -2i(L+iS_-) \\
i\langle v^*\rangle &= -2i(L-iS_-)x^*e^{-2i(L+iS_-)y^*x-2i(L-iS_-)x^*y+2S_+xx^*} \\
-\langle v^*u\rangle &= -2i(L-iS_-)
\end{aligned}
\tag{100}
$$

Relative length:

$$
R=\frac{\langle (u^*u)^2\rangle}{\langle u^*v\rangle\langle v^*u\rangle}=\frac{2S_+^2}{L^2+S_-^2} \tag{101}
$$

Decoherence:

$$
D=\frac{\langle u^*u\rangle}{\langle u^*v\rangle\langle v^*u\rangle}=\frac{S_+}{2(L^2+S_-^2)} \tag{102}
$$

Classicality:

$$
C=\frac{S_+}{L} \tag{103}
$$

B. Whole configuration

$$
\begin{aligned}
iS_B[\hat{J}] &= i\frac{(J-\frac{\kappa}{2}J^a)LJ^a}{L^2+S_-^2}+\frac{(J-\frac{\kappa}{2}J^a)S_-J^a}{L^2+S_-^2}-\frac{J^aS_+J^a}{2(L^2+S_-^2)} \\
&= \frac{1}{L^2+S_-^2}\left[i\left(JLJ^a-\frac{\kappa}{2}J^aLJ^a\right)+JS_-J^a-\frac{\kappa}{2}J^aS_-J^a-\frac{1}{2}J^aS_+J^a\right] \\
&= \frac{1}{L^2+S_-^2}\left[iJLJ^a+JS_-J^a-\frac{1}{2}J^aS_+J^a\right]
\end{aligned}
\tag{104}
$$

$$
\begin{aligned}
Z[b,b^a] &= \int D[J]D[J^a]e^{\frac{i}{\hbar}S_B[\hat{J}]+\frac{i}{\hbar}bJ+\frac{i}{\hbar}b^aJ^a} \\
&= \frac{1}{L^2+S_-^2}\left[iJLJ^a+JS_-J^a-\frac{1}{2}J^aS_+J^a\right]
\end{aligned}
\tag{105}
$$

Appendix A: Density-current two-point functions

$$\begin{aligned}
\tilde{G}^{(00)(\sigma\sigma')}_{x,y} &= in_s\xi\hbar\int_{r,s} G^{\sigma'\sigma}_r e^{-ir(y-x)} G^{\sigma\sigma'}_s e^{-is(x-y)} \\
&= in_s\xi\hbar\int_{r,s} G^{\sigma'\sigma}_r G^{\sigma\sigma'}_s e^{-i(s-r)(x-y)}, \\
\tilde{G}^{(0j)(\sigma\sigma')}_{x,y} &= -\frac{n_s\xi\hbar^2}{2m}\int_{a,b,r,s} G^{\sigma'\sigma}_r e^{-ir(a-b)}\delta_{b,x} G^{\sigma\sigma'}_s e^{-is(b-a)}(\leftarrow \partial_{a_j}\delta_{a,y} - \delta_{a,y}\partial_{a_j}\rightarrow) \\
&= -i\frac{n_s\xi\hbar^2}{2m}\int_{a,b,r,s} G^{\sigma'\sigma}_r e^{-ir(a-b)}\delta_{b,x} G^{\sigma\sigma'}_s e^{-is(b-a)}\delta_{a,y}(s_j+r_j)\chi^{ts}_r\chi^{ts}_s \\
&= -i\frac{n_s\xi\hbar^2}{2m}\int_{r,s} G^{\sigma'\sigma}_r G^{\sigma\sigma'}_s e^{-i(s-r)(x-y)}(s_j+r_j), \\
\tilde{G}^{(jk)(\sigma\sigma')}_{x,y} &= -i\frac{n_s\xi\hbar^3}{4m^2}\int_{a,b,r,s} G^{\sigma'\sigma}_r e^{-ir(a-b)}(\leftarrow \partial_j\delta_{b,x} - \delta_{b,x}\partial_j\rightarrow) G^{\sigma\sigma'}_s e^{-is(b-a)}(\leftarrow \partial_k\delta_{a,y} - \delta_{a,y}\partial_k\rightarrow) \\
&= i\frac{n_s\xi\hbar^3}{4m^2}\int_{a,b,r,s} G^{\sigma'\sigma}_r e^{-ir(a-b)}\delta_{b,x}(r_j+s_j) G^{\sigma\sigma'}_s e^{-is(b-a)}\delta_{a,y}(s_k+r_k) \\
&= i\frac{n_s\xi\hbar^3}{4m^2}\int_{r,s} G^{\sigma'\sigma}_r G^{\sigma\sigma'}_s e^{-i(s-r)(x-y)}(r_j+s_j)(s_k+r_k),
\end{aligned} \tag{A1}$$

$$\tilde{G}_{x,y} = \int_p \tilde{G}_p e^{-ip(x-y)}, \tag{A2}$$

and we find

$$\begin{aligned}
\tilde{G}^{(00)(\sigma\sigma')}_p &= in_s\xi\hbar\int_r G^{\sigma'\sigma}_r G^{\sigma\sigma'}_{p+r} \\
\tilde{G}^{(0j)(\sigma\sigma')}_p &= -i\frac{n_s\xi\hbar^2}{m}\int_r G^{\sigma'\sigma}_r G^{\sigma\sigma'}_{p+r)}\left(r_j+\frac{p_j}{2}\right) \\
\tilde{G}^{(j0)(\sigma\sigma')}_p &= -i\frac{n_s\xi\hbar^2}{m}\int_r G^{\sigma'\sigma}_r G^{\sigma\sigma'}_{p+r)}\left(r_j+\frac{p_j}{2}\right) \\
\tilde{G}^{(jk)(\sigma\sigma')}_p &= i\frac{n_s\xi\hbar^3}{m^2}\int_r G^{\sigma'\sigma}_r G^{\sigma\sigma'}_{p+r}\left(r_j+\frac{p_j}{2}\right)\left(r_k+\frac{p_k}{2}\right)
\end{aligned} \tag{A3}$$

$$F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) = \begin{pmatrix} 1 & -\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \\ -\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \end{pmatrix} \tag{A4}$$

$$\begin{aligned}
F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q}) &= \begin{pmatrix} 1 & -\frac{\hbar}{m}\left(-\boldsymbol{k}-\boldsymbol{q}+\frac{\boldsymbol{q}}{2}\right) \\ -\frac{\hbar}{m}\left(-\boldsymbol{k}-\boldsymbol{q}+\frac{\boldsymbol{q}}{2}\right) & \bar{r}^2\left(-\boldsymbol{k}-\boldsymbol{q}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(-\boldsymbol{k}-\boldsymbol{q}+\frac{\boldsymbol{q}}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \\ \frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \end{pmatrix}
\end{aligned} \tag{A5}$$

$$\begin{aligned}
\tilde{G}^{(\mu\nu)\sigma\sigma'}_q &= in_s\xi\hbar\int_p G^{\sigma\sigma'}_{p+q} G^{\sigma'\sigma}_p e^{4i\eta p^0} \\
&= in_s\xi\hbar\int_k \begin{pmatrix} \frac{1}{k^0+q^0-\tilde{\omega}_{\boldsymbol{k}+\boldsymbol{q}}} - i2\pi\delta_\gamma(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})\xi n_{\boldsymbol{k}+\boldsymbol{q}} & -i2\pi\delta(k^0+q^0-\tilde{\omega}_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}+\boldsymbol{q}} \\ -i2\pi\delta(k^0+q^0-\tilde{\omega}_{\boldsymbol{k}+\boldsymbol{q}})(\xi+n_{\boldsymbol{k}+\boldsymbol{q}}) & \frac{1}{\tilde{\epsilon}^*_{\boldsymbol{k}+\boldsymbol{q}}-k^0-q^0} - i2\pi\delta_\gamma(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})\xi n_{\boldsymbol{k}+\boldsymbol{q}} \end{pmatrix}^{\sigma\sigma'} \\
&\times \begin{pmatrix} \frac{1}{k^0-\tilde{\omega}_{\boldsymbol{k}}} - i2\pi\delta_\gamma(k^0-\omega_{\boldsymbol{k}})\xi n_{\boldsymbol{k}} & -i2\pi\delta(k^0-\tilde{\omega}_{\boldsymbol{k}})n_{\boldsymbol{k}} \\ -i2\pi\delta(k^0-\tilde{\omega}_{\boldsymbol{k}})(\xi+n_{\boldsymbol{k}}) & \frac{1}{\tilde{\epsilon}^*_{\boldsymbol{k}}-k^0} - i2\pi\delta_\gamma(k^0-\omega_{\boldsymbol{k}})\xi n_{\boldsymbol{k}} \end{pmatrix}^{\sigma'\sigma} F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})
\end{aligned} \tag{A6}$$

$$\tilde{G}_q^{(\mu\nu)++} = in_s\xi\hbar\int_k \left(\frac{1}{k^0+q^0-\tilde{\omega}_{\boldsymbol{k}+\boldsymbol{q}}} - i2\pi\delta_\gamma(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})\xi n_{\boldsymbol{k}+\boldsymbol{q}}\right)\left(\frac{1}{k^0-\tilde{\omega}_{\boldsymbol{k}}} - i2\pi\delta_\gamma(k^0-\omega_{\boldsymbol{k}})\xi n_{\boldsymbol{k}}\right)F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})$$

$$\begin{aligned}
\tilde{G}_{vq}^{(\mu\nu)++} &= in_s\xi\hbar\int_k \frac{F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\gamma)(k^0-\omega_{\boldsymbol{k}}+i\gamma)} \\
&= in_s\xi\hbar\int_{\boldsymbol{k}} \frac{F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{-\omega_{\boldsymbol{k}}-q^0+\omega_{\boldsymbol{k}+\boldsymbol{q}}}\int_{k^0}\left[\frac{1}{k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\gamma} - \frac{1}{k^0-\omega_{\boldsymbol{k}}+i\gamma}\right] = 0 \\
\tilde{G}_{eq}^{(\mu\nu)++} &= in_s\xi\hbar\int_k \Bigg[-\frac{i2\pi\delta_\gamma(k^0-\omega_{\boldsymbol{k}})\xi n_{\boldsymbol{k}}}{k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\gamma} - \frac{i2\pi\delta_\gamma(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})\xi n_{\boldsymbol{k}+\boldsymbol{q}}}{k^0-\omega_{\boldsymbol{k}}+i\gamma} \\
&\quad -2\pi\delta_\gamma(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}+\boldsymbol{q}}2\pi\delta_\gamma(k^0-\omega_{\boldsymbol{k}})n_{\boldsymbol{k}}\Bigg]F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})
\end{aligned} \tag{A7}$$

+-:

$$\begin{aligned}
\tilde{G}_q^{+-} &= -in_s\xi\hbar\int_k 2\pi\delta(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\gamma)n_{\boldsymbol{k}+\boldsymbol{q}}2\pi\delta(k^0-\omega_{\boldsymbol{k}}+i\gamma)(\xi+n_{\boldsymbol{k}})F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\
&= -in_s\xi\hbar\int_{\boldsymbol{k}} 2\pi\delta(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}+q^0)n_{\boldsymbol{k}+\boldsymbol{q}}(\xi+n_{\boldsymbol{k}})F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\
&= -in_s\xi\hbar\int_{\boldsymbol{k}} 2\pi\delta(\omega_{\boldsymbol{k}-\boldsymbol{q}}-\omega_{\boldsymbol{k}}+q^0)n_{\boldsymbol{k}}(\xi+n_{\boldsymbol{k}-\boldsymbol{q}})F^{\mu\nu}(\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q}) \\
&= -in_s\xi\hbar\int_{\boldsymbol{k}} 2\pi\delta(\omega_{\boldsymbol{k}+\boldsymbol{q}}-\omega_{\boldsymbol{k}}+q^0)n_{\boldsymbol{k}}(\xi+n_{\boldsymbol{k}+\boldsymbol{q}})F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q}) \\
&= -2iR_q^{\mu\nu-} = -2iR_{-q}^{\mu\nu} \\
\tilde{G}_q^{-+} &= -in_s\xi\hbar\int_k 2\pi\delta(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\gamma)(\xi+n_{\boldsymbol{k}+\boldsymbol{q}})2\pi\delta(k^0-\omega_{\boldsymbol{k}}+i\gamma)n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\
&= -in_s\xi\hbar\int_{\boldsymbol{k}} 2\pi\delta(\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}}(\xi+n_{\boldsymbol{k}+\boldsymbol{q}})F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\
&= -2iR_q^{\mu\nu+}
\end{aligned} \tag{A8}$$

$$F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q}) = \begin{pmatrix} 1 & \mp\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \\ \mp\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\end{pmatrix} \tag{A9}$$

$$R_q^{\mu\nu} = n_s\pi\hbar\int_{\boldsymbol{k}}\delta(\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})n_{\boldsymbol{k}}(1+\xi n_{\boldsymbol{k}+\boldsymbol{q}})F^{+\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \tag{A10}$$

$R_q^{\mu\nu\pm} = R_{\pm q}^{\mu\nu}$ is independent of γ.
++ for $\gamma = 0$:

$$\tilde{G}_q^{(\mu\nu)++} = n_s\hbar\int_{\boldsymbol{k}}\left[\frac{n_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\epsilon} + \frac{n_{\boldsymbol{k}+\boldsymbol{q}}}{-q^0+\omega_{\boldsymbol{k}+\boldsymbol{q}}-\omega_{\boldsymbol{k}}+i\epsilon} - i\xi 2\pi\delta(\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}\right]F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \tag{A11}$$

$$\begin{aligned}
\Re\tilde{G}_q^{(\mu\nu)++} &= n_s\hbar P\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}}-n_{\boldsymbol{k}+\boldsymbol{q}}}{q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\
&= n_s\hbar P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}} - \frac{F^{\mu\nu}(\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q})}{q^0+\omega_{\boldsymbol{k}-\boldsymbol{q}}-\omega_{\boldsymbol{k}}}\right] \\
&= n_s\hbar P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}} + \frac{F^{\mu\nu}(\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q})}{-q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}-\boldsymbol{q}}}\right] \\
&= n_s\hbar P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}} + \frac{F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q})}{-q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}}\right]
\end{aligned} \tag{A12}$$

$$
\begin{aligned}
\Im\tilde{G}_q^{(\mu\nu)++} &= -n_s\pi\hbar\int_{\boldsymbol{k}}\delta(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})(n_{\boldsymbol{k}}+n_{\boldsymbol{k}+\boldsymbol{q}}+2\xi n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}})F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})\\
&= -n_s\pi\hbar\int_{\boldsymbol{k}}\delta(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})[n_{\boldsymbol{k}}(1+\xi n_{\boldsymbol{k}+\boldsymbol{q}})+n_{\boldsymbol{k}+\boldsymbol{q}}(1+\xi n_{\boldsymbol{k}})]F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})\\
&= -n_s\pi\hbar\int_{\boldsymbol{k}}[\delta(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})n_{\boldsymbol{k}}(1+\xi n_{\boldsymbol{k}+\boldsymbol{q}})F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})+\delta(q^0-\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}-\boldsymbol{q}})n_{\boldsymbol{k}}(1+\xi n_{\boldsymbol{k}-\boldsymbol{q}})F^{\mu\nu}(\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q})]\\
&= -n_s\pi\hbar\int_{\boldsymbol{k}}[\delta(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})n_{\boldsymbol{k}}(1+\xi n_{\boldsymbol{k}+\boldsymbol{q}})F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})+\delta(q^0-\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}}(1+\xi n_{\boldsymbol{k}+\boldsymbol{q}})F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q})]\\
&= -R_q^{\mu\nu+}-R_q^{\mu\nu-}
\end{aligned}
\tag{A13}
$$

++ for finite life-time, $\tau = 1/\gamma$:

$$
\begin{aligned}
\tilde{G}_{eq}^{(\mu\nu)++} &= n_s\hbar\int_k\Bigg[\underbrace{\frac{2\pi\delta_\gamma(k^0-\omega_{\boldsymbol{k}})n_{\boldsymbol{k}}}{k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\gamma}}_{A}+\underbrace{\frac{2\pi\delta_\gamma(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}+\boldsymbol{q}}}{k^0-\omega_{\boldsymbol{k}}+i\gamma}}_{B}\\
&\qquad\underbrace{-i\xi 2\pi\delta_\gamma(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}+\boldsymbol{q}}2\pi\delta_\gamma(k^0-\omega_{\boldsymbol{k}})n_{\boldsymbol{k}}}_{C}\Bigg]F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})
\end{aligned}
\tag{A14}
$$

$$
\begin{aligned}
A &= 2n_s\hbar\gamma\int_k\frac{n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{[(k^0-\omega_{\boldsymbol{k}})^2+\gamma^2](k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\gamma)}\\
&= 2n_s\hbar\gamma\int_k\frac{n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{(k^0-\omega_{\boldsymbol{k}}+i\gamma)(k^0-\omega_{\boldsymbol{k}}-i\gamma)(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\gamma)}\\
&= 2n_s i\hbar\gamma\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}}+i2\gamma)(\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i2\gamma)}\\
&= n_s\hbar\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}+2\gamma i}
\end{aligned}
\tag{A15}
$$

by closing the countour upward.

$$
\begin{aligned}
B &= 2n_s\hbar\gamma\int_k\frac{n_{\boldsymbol{k}+\boldsymbol{q}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{[(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})^2+\gamma^2](k^0-\omega_{\boldsymbol{k}}+i\gamma)}\\
&= 2n_s\hbar\gamma\int_k\frac{n_{\boldsymbol{k}+\boldsymbol{q}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\gamma)(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}-i\gamma)(k^0-\omega_{\boldsymbol{k}}+i\gamma)}\\
&= 2n_s\hbar i\gamma\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}+\boldsymbol{q}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{2i\gamma(-q^0+\omega_{\boldsymbol{k}+\boldsymbol{q}}-\omega_{\boldsymbol{k}}+i2\gamma)}\\
&= -n_s\hbar\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}+\boldsymbol{q}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}}-2\gamma i}
\end{aligned}
\tag{A16}
$$

$$
\begin{aligned}
C &= -n_s i\xi\hbar\int_k 2\pi\delta_\gamma(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}+\boldsymbol{q}}2\pi\delta_\gamma(k^0-\omega_{\boldsymbol{k}})n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})\\
&= -4n_s i\xi\gamma^2\hbar\int_k \frac{n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{[(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})^2+\gamma^2][(k^0-\omega_{\boldsymbol{k}})^2+\gamma^2]}\\
&= -4n_s i\xi\gamma^2\hbar\int_k \frac{n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\gamma)(k^0+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}-i\gamma)(k^0-\omega_{\boldsymbol{k}}+i\gamma)(k^0-\omega_{\boldsymbol{k}}-i\gamma)}\\
&= 4n_s\xi\gamma^2\hbar\int_{\boldsymbol{k}} \frac{n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{2i\gamma(-q^0+\omega_{\boldsymbol{k}+\boldsymbol{q}}-\omega_{\boldsymbol{k}}+i2\gamma)(-q^0+\omega_{\boldsymbol{k}+\boldsymbol{q}}-\omega_{\boldsymbol{k}})}\\
&\quad +4n_s\xi\gamma^2\hbar\int_{\boldsymbol{k}} \frac{n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{(\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i2\gamma)(\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})2i\gamma}\\
&= -2n_s i\xi\gamma\hbar\int_{\boldsymbol{k}} \frac{n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}}-i2\gamma)(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})} - c.c.\\
&= -4n_s i\xi\gamma\hbar\Re\int_{\boldsymbol{k}} \frac{n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}}-i2\gamma)(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})}
\end{aligned}
\tag{A17}
$$

$$
\begin{aligned}
\Re\tilde{G}^{(\gamma)\mu\nu++}_{eq} &= n_s\hbar\Re\int_{\boldsymbol{k}} F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})\left[\frac{n_{\boldsymbol{k}}}{q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i2\gamma}-\frac{n_{\boldsymbol{k}+\boldsymbol{q}}}{q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}}-i2\gamma}\right]\\
\Im\tilde{G}^{(\gamma)\mu\nu++}_{q} &= n_s\hbar\Im\int_{\boldsymbol{k}} F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})\left[\frac{n_{\boldsymbol{k}}}{q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i2\gamma}-\frac{n_{\boldsymbol{k}+\boldsymbol{q}}}{q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}}-i2\gamma}\right]\\
&\quad -4n_s\xi\gamma\hbar\Re\int_{\boldsymbol{k}} \frac{n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}}-i2\gamma)(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})}\\
&= -2n_s\gamma\hbar\int_{\boldsymbol{k}} \frac{(n_{\boldsymbol{k}}+n_{\boldsymbol{k}+\boldsymbol{q}})F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{(q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}})^2+4\gamma^2} - 4n_s\gamma\xi\hbar\int_{\boldsymbol{k}} \frac{n_{\boldsymbol{k}}n_{\boldsymbol{k}+\boldsymbol{q}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{[(q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}})^2+4\gamma^2]}\\
&= -n_s\pi\hbar\int_{\boldsymbol{k}} \delta_{2\gamma}(q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})(n_{\boldsymbol{k}}+n_{\boldsymbol{k}+\boldsymbol{q}}+2\xi n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}})F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})
\end{aligned}
\tag{A18}
$$

Appendix B: Zero temperature fermions

1. Real part

$$
\Re\tilde{G}^{(\mu\nu)++}_{\omega,\boldsymbol{q}} = L^{(\mu\nu)++}_{\omega,\boldsymbol{q}} = n_s\hbar P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu+}(\boldsymbol{k},\boldsymbol{q})}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}\boldsymbol{q}}{m}}+\frac{F^{\mu\nu-}(\boldsymbol{k},\boldsymbol{q})}{-\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}\boldsymbol{q}}{m}}\right]
\tag{B1}
$$

$z=\omega m/\hbar\bar{k}q$, $b_\pm=\frac{Q}{2}\pm z$, $\frac{1}{Q}(b_-^3+b_+^3)=\frac{Q^2}{4}+3z^2$

$$
\begin{aligned}
L^{tt}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar}{4\pi^2}P\int dkk^2dCn_k\frac{1}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar kqC}{m}}+(\omega\to-\omega)\\
&= -\frac{n_sm}{4\pi^2q}P\int_0^{k_F}dkk\ln\left|\frac{2m\omega-\hbar q^2-2\hbar kq}{2m\omega-\hbar q^2+2\hbar kq}\right|+(\omega\to-\omega)\\
&= \frac{n_smk_F}{4\pi^2}\left\{-1+\frac{k_F}{2q}\left[1-\left(z-\frac{q}{2k_F}\right)^2\right]\ln\left|\frac{1+z-\frac{q}{2k_F}}{1-z+\frac{q}{2k_F}}\right|-\frac{k_F}{2q}\left[1-\left(z+\frac{q}{2k_F}\right)^2\right]\ln\left|\frac{1+z+\frac{q}{2k_F}}{1-z-\frac{q}{2k_F}}\right|\right\}\\
&= \frac{n_smk_F}{4\pi^2}\sum_\tau\left[-\frac{1}{2}+\frac{1}{2Q}(1-b_\tau^2)\ln\left|\frac{1-b_\tau}{1+b_\tau}\right|\right]
\end{aligned}
\tag{B2}
$$

Gradient expansion:

$$\begin{aligned} L^{tt}_{\omega,\boldsymbol{q}} &= \frac{n_s m k_F}{4\pi^2}\sum_\tau\left[-\frac{1}{2}-\frac{1}{Q}(1-b_\tau^2)(b_\tau+\frac{1}{3}b_\tau^3+\mathcal{O}(b^5))\right] \\ &= -\frac{n_s m k_F}{4\pi^2}\sum_\tau\left[\frac{1}{2}+\frac{b_\tau}{Q}-\frac{2}{3Q}b_\tau^3\right]+\frac{1}{Q}\mathcal{O}(b^5) \\ &\approx -\frac{n_s m k_F}{4\pi^2}\left(2-\frac{Q^2}{6}-2z^2\right) \end{aligned} \tag{B3}$$

Imaginary energy: $z=iw$, $b=\frac{Q}{2}+iw$

$$\begin{aligned} L^{tt}_{i\omega,\boldsymbol{q}} &= \frac{n_s m k_F}{4\pi^2}\left[-1+\frac{1-\frac{Q^2}{4}+w^2}{Q}\ln\left|\frac{1+\frac{Q^2}{4}-Q+w^2}{1+\frac{Q^2}{4}+Q+w^2}\right|\right] \\ &\approx -\frac{n_s m k_F}{4\pi^2}\left(2-\frac{Q^2}{6}+2w^2\right) \end{aligned} \tag{B4}$$

$Q=k/k_F=1/R$

$$a=\bar{k}z-\frac{q}{2}=\frac{\omega m}{\hbar q}-\frac{q}{2}=\frac{q}{2}\left(\frac{\hbar\omega}{\frac{\hbar^2 q^2}{2m}}-1\right),\qquad b_\pm=\frac{q}{2\bar{k}}\pm z=\frac{q}{2\bar{k}}\left(1\pm\frac{\hbar\omega}{\frac{\hbar^2 q^2}{2m}}\right) \tag{B5}$$

Static case: $z=0$:

$$\begin{aligned} L^{tt}_{\boldsymbol{q}} &= \frac{n_s m k_F}{4\pi^2}\left[-1+\frac{1}{Q}\left(1-\frac{Q^2}{4}\right)\ln\left|\frac{1-\frac{Q}{2}}{1+\frac{Q}{2}}\right|\right] \\ &= \frac{n_s m k_F}{4\pi^2}\left[-1+\left(\frac{1}{Q}-\frac{Q}{4}\right)\ln\left|\frac{1-\frac{Q}{2}}{1+\frac{Q}{2}}\right|\right] \end{aligned} \tag{B6}$$

$$\begin{aligned} \Re\tilde{G}^{ss}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar^3}{m^2}P\int_{\boldsymbol{k}} n_k\frac{(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})\otimes(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}}+(\omega\to-\omega) \\ \mathrm{Tr}\Re\tilde{G}^{ss}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar^3}{m^2}P\int_{\boldsymbol{k}} n_k\frac{k^2+\frac{q^2}{4}+\boldsymbol{kq}}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}}+(\omega\to-\omega) \\ \boldsymbol{q}\Re\tilde{G}^{ss}_{\omega,\boldsymbol{q}}\boldsymbol{q} &= \frac{n_s\hbar^3}{m^2}P\int_{\boldsymbol{k}} n_k\frac{(\boldsymbol{qk})^2+\frac{q^4}{4}+q^2\boldsymbol{kq}}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}}+(\omega\to-\omega) \\ \Re\tilde{G}^{T}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar^3}{2m^2}P\int_{\boldsymbol{k}} n_k\frac{k^2+\frac{q^2}{4}+\boldsymbol{kq}-(\boldsymbol{nk})^2-\frac{q^2}{4}-\boldsymbol{kq}}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}}+(\omega\to-\omega) \\ &= \frac{n_s\hbar^3}{2m^2}P\int_{\boldsymbol{k}} n_k\frac{k^2-(\boldsymbol{nk})^2}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}}+(\omega\to-\omega) \end{aligned} \tag{B7}$$

because $T=\frac{1}{2}(\mathrm{Tr}-L)$

$$\begin{aligned} \int dc\frac{1}{a+bc} &= \frac{1}{b}\ln(a+bc) \\ \int dc\frac{c}{a+bc} &= \frac{1}{b}\int dc\left[\frac{a+bc}{a+bc}-\frac{a}{a+bc}\right]=\frac{c}{b}-\frac{a}{b^2}\ln(a+bc) \\ \int dc\frac{c^2}{a+bc} &= \frac{1}{b}\int dcc\left[1-\frac{a}{a+bc}\right]=\frac{c^2}{2b}-\frac{a}{b}\left(\frac{c}{b}-\frac{a}{b^2}\ln(a+bc)\right)=\frac{c^2}{2b}-\frac{ac}{b^2}+\frac{a^2}{b^3}\ln(a+bc) \\ \int dc\frac{1-c^2}{a+bc} &= -\frac{c^2}{2b}+\frac{ac}{b^2}+\frac{b^2-a^2}{b^3}\ln(a+bc) \end{aligned} \tag{B8}$$

$$\begin{aligned}
\Re\tilde{G}^T_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar^3}{8\pi^2m^2}P\int_0^\infty dkk^4n_k\int_{-1}^1 dC\frac{1-C^2}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar kqC}{m}}+(\omega\to-\omega)\\
&= \frac{n_s\hbar^3}{8\pi^2m^2}P\int_0^\infty dkk^4n_k\left[\frac{c^2}{2\frac{\hbar kq}{m}}+\frac{(\omega-\frac{\hbar q^2}{2m})c}{(\frac{\hbar kq}{m})^2}-\frac{(\frac{\hbar kq}{m})^2-(\omega-\frac{\hbar q^2}{2m})^2}{(\frac{\hbar kq}{m})^3}\ln(\omega-\frac{\hbar q^2}{2m}-\frac{\hbar kqC}{m})\right]_{-1}^1\\
&= \frac{n_s\hbar^3}{8\pi^2m^2}P\int_0^\infty dkk^4n_k\left[\frac{2\omega-\frac{\hbar q^2}{m}}{(\frac{\hbar kq}{m})^2}+\frac{(\frac{\hbar kq}{m})^2-(\omega-\frac{\hbar q^2}{2m})^2}{(\frac{\hbar kq}{m})^3}\ln\frac{\omega-\frac{\hbar q^2}{2m}+\frac{\hbar kq}{m}}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar kq}{m}}\right]+(\omega\to-\omega)\\
&= \frac{n_s\hbar^3}{8\pi^2m^2}P\int_0^\infty dkk^4n_k\left[\frac{2\omega m^2-m\hbar q^2}{\hbar^2k^2q^2}+\frac{m}{\hbar kq}\left(1-(\frac{m\omega}{\hbar kq}-\frac{q}{2k})^2\right)\ln\frac{\omega-\frac{\hbar q^2}{2m}+\frac{\hbar kq}{m}}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar kq}{m}}\right]+(\omega\to-\omega)\\
&= \frac{n_s\hbar(2\omega m-\hbar q^2)}{8\pi^2mq^2}\int_0^\infty dkk^2n_k+\frac{n_s\hbar^2}{8\pi^2mq}\int_0^\infty dkn_k\left(k^3-k(\frac{m\omega}{\hbar q}-\frac{q}{2})^2\right)\ln\frac{\frac{m\omega}{\hbar q}-\frac{q}{2}+k}{\frac{m\omega}{\hbar q}-\frac{q}{2}-k}+(\omega\to-\omega)
\end{aligned}\tag{B9}$$

$$\begin{aligned}
\int_0^{k_F}dkk\ln(a+bk) &= \frac{ak}{2b}-\frac{k^2}{4}+\frac{1}{2}\left(k^2-\frac{a^2}{b^2}\right)\ln(a+bk)\Big|_0^{k_F}\\
&= \frac{ak_F}{2b}-\frac{k_F^2}{4}+\frac{k_F^2}{2}\ln(a+bk_F)-\frac{a^2}{2b^2}\ln\frac{a+bk_F}{a}\\
\int_0^{k_F}dkk\ln\frac{a+bk}{a-bk} &= \frac{ak_F}{b}+\left(\frac{k_F^2}{2}-\frac{a^2}{2b^2}\right)\ln\frac{a+bk_F}{a-bk_F}\\
\int_0^{k_F}dkk^3ln(a+bk) &= -\frac{k^4}{16}+\frac{ak^3}{12b}-\frac{a^2k^2}{8b^2}+\frac{a^3k}{4b^3}+\frac{1}{4}\left(k^4-\frac{a^4}{b^4}\right)\ln(a+bk)\Big|_0^{k_F}\\
&= -\frac{k_F^4}{16}+\frac{ak_F^3}{12b}-\frac{a^2k_F^2}{8b^2}+\frac{a^3k_F}{4b^3}+\frac{1}{4}k_F^4\ln(a+bk_F)-\frac{1}{4}\frac{a^4}{b^4}\ln\frac{a+bk_F}{a}\\
\int_0^{k_F}dkk^3\ln\frac{a+bk}{a-bk} &= \frac{ak_F^3}{6b}+\frac{a^3k_F}{2b^3}+\frac{1}{4}\left(k_F^4-\frac{a^4}{b^4}\right)\ln\frac{a+bk_F}{a-bk_F}
\end{aligned}\tag{B10}$$

$$\begin{aligned}
\Re\tilde{G}^T_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar(2\omega m-\hbar q^2)k_F^3}{24\pi^2 mq^2}+\frac{n_s\hbar^2}{8\pi^2 mq}\Bigg[\frac{ak_F^3}{6b}+\frac{a^3k_F}{2b^3}-\frac{ak_F}{b}(\frac{m\omega}{\hbar q}-\frac{q}{2})^2 \\
&+\left(\frac{k_F^4}{4}-\frac{a^4}{4b^4}-(\frac{m\omega}{\hbar q}-\frac{q}{2})^2\left(\frac{k_F^2}{2}-\frac{a^2}{2b^2}\right)\right)\ln\frac{\frac{m\omega}{\hbar q}-\frac{q}{2}+k_F}{\frac{m\omega}{\hbar q}-\frac{q}{2}-k_F}\Bigg]+(\omega\to-\omega) \\
&= \frac{n_s\hbar^2k_Fq(2z-\frac{q}{k_F})k_F^3}{24\pi^2mq^2}+\frac{n_s\hbar^2}{8\pi^2mq}\Bigg[\frac{1}{6}(\frac{m\omega}{\hbar q}-\frac{q}{2})k_F^3+\frac{1}{2}(\frac{m\omega}{\hbar q}-\frac{q}{2})^3k_F-(\frac{m\omega}{\hbar q}-\frac{q}{2})k_F(\frac{m\omega}{\hbar q}-\frac{q}{2})^2 \\
&+\frac{1}{4}\left(k_F^4-(\frac{m\omega}{\hbar q}-\frac{q}{2})^4-2(\frac{m\omega}{\hbar q}-\frac{q}{2})^2\left(k_F^2-(\frac{m\omega}{\hbar q}-\frac{q}{2})^2\right)\right)\ln\frac{\frac{m\omega}{\hbar q}-\frac{q}{2}+k_F}{\frac{m\omega}{\hbar q}-\frac{q}{2}-k_F}\Bigg]+(\omega\to-\omega) \\
&= \frac{n_s\hbar^2k_F^4(z-\frac{q}{2k_F})}{12\pi^2mq}+\frac{n_s\hbar^2k_F^4}{8\pi^2mq}\Bigg[\frac{1}{6}(z-\frac{q}{2k_F})-\frac{1}{2}(z-\frac{q}{2k_F})^3 \\
&+\frac{1}{4}\left(1-(z-\frac{q}{2k_F})^4-2(z-\frac{q}{2k_F})^2\left(1-(z-\frac{q}{2k_F})^2\right)\right)\ln\frac{z-\frac{q}{2k_F}+1}{z-\frac{q}{2k_F}-1}\Bigg]+(\omega\to-\omega) \\
&= -\frac{n_s\hbar^2k_F^4b_-}{12\pi^2mq}+\frac{n_s\hbar^2k_F^4}{8\pi^2mq}\left[-\frac{b_-}{6}+\frac{b_-^3}{2}+\frac{1}{4}\left(1-b_-^4-2b_-^2\left(1-b_-^2\right)\right)\ln\frac{1-b_-}{1+b_-}\right]+(\omega\to-\omega) \\
&= -\frac{n_s\hbar^2k_F^4b_-}{12\pi^2mq}+\frac{n_s\hbar^2k_F^4}{8\pi^2mq}\left[-\frac{b_-}{6}+\frac{b_-^3}{2}+\frac{1}{4}\left(1-2b_-^2+b_-^4\right)\ln\frac{1-b_-}{1+b_-}\right]+(\omega\to-\omega) \\
&= -\frac{n_s\hbar^2k_F^4(b_-+b_+)}{12\pi^2mq}+\frac{n_s\hbar^2k_F^4}{16\pi^2mq}\left[-\frac{b_-+b_+}{3}+b_-^3+b_+^3+\frac{1}{2}(1-b_-^2)^2\ln\frac{1-b_-}{1+b_-}+\frac{1}{2}(1-b_+^2)^2\ln\frac{1-b_+}{1+b_+}\right] \\
&= \frac{n_smk_F}{4\pi^2Q}\frac{\hbar^2k_F^2}{m^2}\sum_{\tau=\pm}\left[-\frac{b_\tau}{3}+\frac{1}{4}\left(-\frac{b_\tau}{3}+b_\tau^3+\frac{1}{2}(1-b_\tau^2)^2\ln\left|\frac{1-b_\tau}{1+b_\tau}\right|\right)\right] \qquad \text{(B11)}
\end{aligned}$$

$b_\pm=\frac{Q}{2}\pm z$, $\frac{1}{Q}(b_-^3+b_+^3)=\frac{Q^2}{4}+3z^2$

$$\begin{aligned}
\Re\tilde{G}^T_{\omega,\boldsymbol{q}} &= \frac{n_smk_F}{4\pi^2Q}\frac{\hbar^2k_F^2}{m^2}\sum_\tau\left[-\frac{5b_\tau}{12}+\frac{b_\tau^3}{4}-\frac{(1-b_\tau^2)^2}{4}(b_\tau+\frac{1}{3}b_\tau^3)+\mathcal{O}(b^5)\right] \\
&\approx \frac{n_smk_F}{4\pi^2Q}\frac{\hbar^2k_F^2}{m^2}\sum_\tau\left[-\frac{5b_\tau}{12}+\frac{b_\tau^3}{4}-\frac{1+b_\tau^4-2b_\tau^2}{4}(b_\tau+\frac{1}{3}b_\tau^3)\right] \\
&\approx \frac{n_smk_F}{4\pi^2Q}\frac{\hbar^2k_F^2}{m^2}\sum_\tau\left[-\frac{5b_\tau}{12}+\frac{b_\tau^3}{4}-\frac{b_\tau-2b_\tau^3+\frac{1}{3}b_\tau^3}{4}\right] \\
&= \frac{n_smk_F}{4\pi^2Q}\frac{\hbar^2k_F^2}{m^2}\sum_\tau\left(-\frac{8b_\tau}{12}+\frac{8b_\tau^3}{12}\right) \\
&= \frac{n_smk_F}{4\pi^2}\frac{\hbar^2k_F^2}{m^2}\frac{2}{3}\left(-1+\frac{Q^2}{4}+3z^2\right) \qquad \text{(B12)}
\end{aligned}$$

Imaginary energy: $z=iw$, $b=\frac{Q}{2}+iw$

$$\begin{aligned}
\Re\tilde{G}^T_{i\omega,\boldsymbol{q}} &= \frac{n_smk_F}{4\pi^2Q}\frac{\hbar^2k_F^2}{m^2}\left[-\frac{2Q}{3}+\frac{1}{4}\left(-\frac{2Q}{3}+\frac{Q^3}{4}-3Qw^2+\left[\left(1-\frac{Q^2}{4}+w^2\right)^2-Q^2w^2\right]\ln\frac{1+\frac{Q^2}{4}-Q+w^2}{1+\frac{Q^2}{4}+Q+w^2}\right)\right] \\
&= \frac{n_smk_F}{4\pi^2}\frac{\hbar^2k_F^2}{4m^2}\left[-\frac{10}{3}+\frac{Q^2}{4}-3Qw^2+\left[\left(1-\frac{Q^2}{4}+w^2\right)^2-Q^2w^2\right]\ln\frac{1+\frac{Q^2}{4}-Q+w^2}{1+\frac{Q^2}{4}+Q+w^2}\right] \\
&\approx \frac{n_smk_F}{4\pi^2}\frac{\hbar^2k_F^2}{m^2}\frac{2}{3}\left(-1+\frac{Q^2}{4}-3z^2\right) \qquad \text{(B13)}
\end{aligned}$$

Clasical limit $Q \to 0$: $C^{tt} = 1$, $C^T = \frac{\hbar^2 k_F^2}{m^2}$, $c_0^{tt} = 1$, $c_0^T = -\frac{2}{3}$

$$\begin{aligned} b_+^{2n+1} + b_-^{2n+1} &= (\frac{1}{2}Q + z)^{2n+1} + (\frac{1}{2}Q - z)^{2n+1} \to (2n+1)Qz^{2n} \\ L^X_{\omega,\boldsymbol{q}} &= -\frac{n_s m k_F}{4\pi^2} C^X \sum_{n=0}^{\infty} c_n^X z^{2n} \end{aligned} \tag{B14}$$

2. Imaginary part

$$F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q}) = \begin{pmatrix} 1 & \mp\frac{\hbar}{m}\left(\boldsymbol{k} + \frac{\boldsymbol{q}}{2}\right) \\ \mp\frac{\hbar}{m}\left(\boldsymbol{k} + \frac{\boldsymbol{q}}{2}\right) & \frac{\hbar^2}{m^2}\left(\boldsymbol{k} + \frac{\boldsymbol{q}}{2}\right) \otimes \left(\boldsymbol{k} + \frac{\boldsymbol{q}}{2}\right) \end{pmatrix} \tag{B15}$$

$$\begin{aligned} R_q^{\mu\nu\pm} &= n_s \pi\hbar \int_{\boldsymbol{k}} \delta(\pm\omega - \omega_{\boldsymbol{k}+\boldsymbol{q}} + \omega_{\boldsymbol{k}}) n_{\boldsymbol{k}} (1 + \xi n_{\boldsymbol{k}+\boldsymbol{q}}) F^{\pm\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\ &= n_s \pi\hbar \int_{\boldsymbol{k}} \delta(\pm\omega - \epsilon_{\boldsymbol{k}+\boldsymbol{q}} + \epsilon_{\boldsymbol{k}}) n_{\boldsymbol{k}} (1 - n_{\boldsymbol{k}+\boldsymbol{q}}) F^{\pm\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\ &= \frac{n_s\hbar}{8\pi^2} \int_{k<k_F} d^3k \Theta(|\boldsymbol{k}+\boldsymbol{q}| - k_F) \delta\left(\pm\omega - \frac{\hbar \boldsymbol{q}\boldsymbol{k}}{m} - \frac{\hbar q^2}{2m}\right) F^{\pm\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\ &= \frac{n_s m}{8\pi^2 q} \int_{k<k_F} d^2k \Theta(|\boldsymbol{k}-\boldsymbol{q}| - k_F) F^{\pm\mu\nu}(-\boldsymbol{k},\boldsymbol{q})_{|k_3 = \mp\frac{m\omega}{\hbar q} + \frac{q}{2}} \end{aligned} \tag{B16}$$

$$\begin{aligned} (a):\ q > 2k_F \quad -k_F &< \mp\frac{\omega m}{\hbar q} + \frac{q}{2} < k_F \\ -1 &< \mp z + \frac{q}{2k_F} < 1 \\ (b):\ q < 2k_F \quad -k_F &< \mp\frac{\omega m}{\hbar q} + \frac{q}{2} < q - k_F \\ -1 &< \mp z + \frac{q}{2k_F} < \frac{q}{k_F} - 1 \\ (c):\ q < 2k_F \quad q - k_F &< \mp\frac{\omega m}{\hbar q} + \frac{q}{2} < \frac{q}{2} \\ \frac{q}{k_F} - 1 &< \mp z + \frac{q}{2k_F} < \frac{q}{2k_F} \end{aligned} \tag{B17}$$

$$R^{tt\pm}_{\omega,\boldsymbol{q}} = \frac{n_s m}{8\pi^2 q} \int_{k_1}^{k_2} 2\pi dk k = \frac{n_s m}{8\pi q}(q_2^2 - q_1^2) \tag{B18}$$

(a,b): $k_2 = \sqrt{k_F^2 - (\frac{q}{2} \mp \frac{m\omega}{\hbar q})^2}$, $k_1 = 0$, (c): $k_2 = \sqrt{k_F^2 - (\frac{q}{2} \mp \frac{m\omega}{\hbar q})^2}$, $k_1 = \sqrt{k_F^2 - (\frac{q}{2} \pm \frac{m\omega}{\hbar q})^2}$, $Q = q/k_F$

$$R^{tt+}_{\omega,\boldsymbol{q}} = \frac{n_s m k_F}{8\pi Q} \begin{cases} 1 - (\frac{Q}{2} - z)^2 & Q > 2, \quad \overbrace{-1 - \frac{Q}{2}}^{<-2} < -z < 1 - \frac{Q}{2} < 0 \\ 1 - (\frac{Q}{2} - z)^2 & Q < 2, \quad -2 < -1 - \frac{Q}{2} < -z < -1 + \frac{Q}{2} < 0 \\ 2zQ & Q < 2, \quad 0 < z < 1 - \frac{Q}{2} \end{cases} = R^{tt-}_{-\omega,\boldsymbol{q}} \tag{B19}$$

$$R^{tt+}_{\omega,\boldsymbol{q}} = \frac{n_s m k_F}{8\pi Q} \begin{cases} 1 - (\frac{Q}{2} - z)^2 & Q > 2, \quad -1 < \frac{Q}{2} - z < 1 \\ 1 - (\frac{Q}{2} - z)^2 & Q < 2, \quad -1 < \frac{Q}{2} - z < -1 + Q < 1 \\ 2zQ & Q < 2, \quad 0 < z < 1 - \frac{Q}{2} \end{cases} = R^{tt-}_{-\omega,\boldsymbol{q}} \tag{B20}$$

$$\begin{aligned}
\mathrm{Tr}(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})\otimes(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}) &= k^2+\frac{q^2}{4}+\boldsymbol{kq} \\
\boldsymbol{n}(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})\otimes(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})\boldsymbol{n} &= (\boldsymbol{nk})^2+\frac{q^2}{4}+\boldsymbol{kq} \\
(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})\otimes(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})^T &= k^2-(\boldsymbol{nk})^2=k^2-k_3^2
\end{aligned} \tag{B21}$$

$$\begin{aligned}
R^{T\pm}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar^2}{8\pi^2 qm}\int_{k_1}^{k_2}2\pi dk k^3=\frac{n_s\hbar^2}{16\pi qm}(q_2^4-k_1^4) \\
&= \frac{n_s\hbar^2k_F^4}{16\pi qm}\begin{cases}[1-(\frac{q}{2k_F}\mp z)^2]^2 & \frac{q}{k_F}>2, \quad -1-\frac{q}{2k_F}<\mp z<1-\frac{q}{2k_F} \\ [1-(\frac{q}{2k_F}\mp z)^2]^2 & \frac{q}{k_F}<2, \quad -1-\frac{q}{2k_F}<\mp z<-1+\frac{q}{2k_F} \\ \mp z\frac{q}{k_F}[\frac{q^2}{k_F^2}+4(z^2-1)] & \frac{q}{k_F}<2, \quad 0<\pm z<1-\frac{q}{2k_F}\end{cases}
\end{aligned} \tag{B22}$$

because

$$[1-(\frac{q}{2k_F}\pm z)^2]^2-[1-(\frac{q}{2k_F}\mp z)^2]^2=(\frac{q}{2k_F}\pm z)^4-(\frac{q}{2k_F}\mp z)^4-4\frac{q}{k_F}(\pm z)=\pm z\frac{q}{k_F}\left[\frac{q^2}{k_F^2}+4(z^2-1)\right] \tag{B23}$$

3. Flow with boundary

$$L^{(\mu\nu)++}_{\omega,\ell,\boldsymbol{q}_\perp} = \frac{n_s\hbar}{L}\sum_j P\int_{\boldsymbol{k}_\perp} n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu+}(\boldsymbol{k},\boldsymbol{q})}{\omega-\frac{\hbar(2\pi)^2}{2mL^2}\ell^2-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}_\perp\boldsymbol{q}_\perp}{m}-\frac{\hbar(2\pi)^2 j\ell}{L^2 m}}+\frac{F^{\mu\nu-}(\boldsymbol{k},\boldsymbol{q})}{-\omega-\frac{\hbar(2\pi)^2}{2mL^2}\ell^2-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}_\perp\boldsymbol{q}_\perp}{m}-\frac{\hbar(2\pi)^2 j\ell}{L^2 m}}\right] \tag{B24}$$

$z=\omega m/\hbar\bar{k}q$

$$\begin{aligned}
\int\frac{d^2k}{\boldsymbol{qk}+c} &= \frac{(\boldsymbol{qk}+c)\ln(\boldsymbol{qk}+c)-\frac{1}{2}\boldsymbol{qk}}{q_1q_2} \\
\int\frac{dk_1}{\boldsymbol{qk}+c} &= \frac{q_2\ln(\boldsymbol{qk}+c)+q_2-\frac{1}{2}q_2}{q_1q_2} \\
\frac{1}{\boldsymbol{qk}+c} &= \frac{q_1q_2}{q_1q_2(\boldsymbol{qk}+c)}
\end{aligned} \tag{B25}$$

$$\begin{aligned}
L^{tt}_{\omega,\ell,\boldsymbol{q}} &= \frac{n_s\hbar}{4\pi^2 L}\sum_j P\int d^2k n_{\boldsymbol{k}}\frac{1}{\omega-\frac{\hbar(2\pi)^2}{2mL^2}\ell^2-\frac{\hbar q^2}{2m}-\frac{\hbar(k_1q_1+k_2q_2)}{m}-\frac{\hbar(2\pi)^2 j\ell}{L^2m}}+(\omega\to-\omega) \\
&= \frac{n_s\hbar}{4\pi^2 L}\sum_j P\frac{\frac{\hbar\boldsymbol{kq}}{2m}+[\omega-\frac{\hbar(2\pi)^2}{2mL^2}(\ell^2+2j\ell)-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}]\ln[\omega-\frac{\hbar(2\pi)^2}{2mL^2}(\ell^2+2j\ell)-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}]}{\frac{\hbar^2q_1q_2}{m^2}}+(\omega\to-\omega) \\
&= \frac{n_s\hbar}{4\pi^2 L}\sum_j P\int dkkd\phi n_k\frac{1}{\omega-\frac{\hbar(2\pi)^2}{2mL^2}\ell^2-\frac{\hbar q^2}{2m}-\frac{\hbar kq\cos\phi}{m}-\frac{\hbar(2\pi)^2 j\ell}{L^2m}}+(\omega\to-\omega)
\end{aligned} \tag{B26}$$

Appendix C: Both statics, arbitrary temperature

1. Infinite lifetime, Lindhart function

$$\begin{aligned}
\Re\tilde{G}^{\mu\nu}_{\omega,\boldsymbol{q}} = L^{\mu\nu}_{q} &= n_s\hbar P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\frac{\begin{pmatrix} 1 & -\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \\ -\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\end{pmatrix}}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{q}\boldsymbol{k}}{m}} + (\omega\to-\omega) \\
&= \frac{n_s m}{q\bar{k}}P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\frac{\begin{pmatrix} 1 & -\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \\ -\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\end{pmatrix}}{z-\frac{q}{2\bar{k}}-\frac{\boldsymbol{n}\boldsymbol{k}}{\bar{k}}} + (z\to-z)
\end{aligned} \tag{C1}$$

$$\begin{aligned}
L^{tt}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar}{4\pi^2}P\int_0^\infty dk k^2 n_k\int_{-1}^{1}dC\frac{1}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar kqC}{m}} + (\omega\to-\omega) \\
&= -\frac{n_s m}{4\pi^2 q}P\int_0^\infty dk k n_k\ln\left(\omega-\frac{\hbar q^2}{2m}-\frac{\hbar kqC}{m}\right)_{-1}^{1} + (\omega\to-\omega) \\
&= -\frac{n_s m}{4\pi^2 q}P\int_0^\infty dk k n_k\ln\left|\frac{2m\omega-\hbar q^2-2\hbar kq}{2m\omega-\hbar q^2+2\hbar kq}\right| + (\omega\to-\omega) \\
&= -\frac{n_s m}{4\pi^2 q}P\int_0^\infty dk k n_k\ln\left|\frac{k-\frac{m\omega}{\hbar q}+\frac{q}{2}}{k+\frac{m\omega}{\hbar q}-\frac{q}{2}}\right| + (\omega\to-\omega) \\
&= -\frac{n_s m}{4\pi^2 q}P\int_0^\infty dk k n_k\ln\left|\frac{k-a}{k+a}\right| + (\omega\to-\omega) \\
&= \mathrm{sign}(a)\frac{n_s m}{4\pi^2 q}P\int_0^\infty dk k n_k\ln\frac{k+a}{|k-a|} + (\omega\to-\omega)
\end{aligned} \tag{C2}$$

$$\begin{aligned}
I &= \int_0^\infty dk f_k\ln\frac{k+a}{|k-a|} = \int_0^a dk f_k\ln\frac{k+a}{a-k} + \int_a^\infty dk f_k\ln\frac{k+a}{k-a} \\
&= \int_0^a dk f_k\ln\frac{1+\frac{k}{a}}{1-\frac{k}{a}} + \int_a^\infty dk f_k\ln\frac{1+\frac{a}{k}}{1+\frac{a}{k}} \\
&\approx \int_0^a dk f_k\left(2\frac{k}{a}-\frac{2k^3}{3a^3}\right) + \int_a^\infty dk f_k\left(2\frac{a}{k}-\frac{2a^3}{3k^3}\right)
\end{aligned} \tag{C3}$$

for $a>0$

$$\begin{aligned}
L^{tt}_{\omega,\boldsymbol{q}} &= \mathrm{sign}(a)\frac{n_s m}{2\pi^2 q}\left(\frac{1}{a}\int_0^a dk k^2 n_k + a\int_a^\infty dk n_k\right) + (\omega\to-\omega) \\
&\approx \mathrm{sign}(a)\frac{n_s m}{2\pi^2 q}\left(\frac{a^3}{3a} + a\int_0^\infty dk n_k - a^2\right) + (\omega\to-\omega) \\
&= |a|\frac{n_s m}{2\pi^2 q}\left(\int_0^\infty dk n_k - \frac{2a}{3}\right) + (\omega\to-\omega) \\
&\approx \left|\frac{\hbar\omega}{\frac{\hbar^2 q^2}{2m}}-1\right|\frac{n_s m}{4\pi^2}\int_0^\infty dk n_k + (\omega\to-\omega)
\end{aligned} \tag{C4}$$

$$L^{s}_{q} = \frac{n_s m}{q\bar{k}}P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\frac{\frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)}{z-\frac{q}{2\bar{k}}-\frac{\boldsymbol{n}\boldsymbol{k}}{\bar{k}}} + (z\to-z) \tag{C5}$$

$$\begin{aligned} L^{tt}_{\boldsymbol q} &= 2n_s\hbar P\int_{\boldsymbol k} n_{\boldsymbol k}\frac{1}{\omega_{\boldsymbol k}-\omega_{\boldsymbol k+\boldsymbol q}} \\ &= -4n_s m P\int_{\boldsymbol k}\frac{n_{\boldsymbol k}}{q^2+2\boldsymbol{kq}} \\ &= -n_s\frac{m}{\pi^2}P\int dkk^2dC\frac{n_k}{q^2+2kqC} \\ &= -n_s\frac{m}{\pi^2}\int dkk^2n_k\frac{\ln\frac{q^2+2kq}{|q^2-2kq|}}{2kq} \\ &= -\frac{n_sm}{2\pi^2q}\int dkkn_k\ln\frac{q+2k}{|q-2k|} \end{aligned} \tag{C6}$$

$$\begin{aligned} L^{tt}_{\boldsymbol q} &\approx -\frac{n_sm}{2\pi^2q}\int_0^{q/2}dkkn_k\left(\frac{4k}{q}-\frac{16k^3}{q^3}\right)-\frac{n_sm}{2\pi^2q}\int_{q/2}^{\infty}dkkn_k\left(\frac{q}{k}-\frac{q^3}{12k^3}\right) \\ &\approx -\frac{2n_sm}{\pi^2q^2}\int_0^{q/2}dkk^2n_k-\frac{n_sm}{2\pi^2}\int_{q/2}^{\infty}dkn_k \\ &\approx -\frac{n_smq}{12\pi^2}+\frac{n_smq}{4\pi^2}-\frac{n_sm}{2\pi^2}\int_0^{\infty}dkn_k=-\frac{n_sm}{2\pi^2}\int_0^{\infty}dkn_k+\frac{n_smq}{6\pi^2} \end{aligned} \tag{C7}$$

$\boldsymbol n=\boldsymbol q/q$

$$L^{ss}_{\boldsymbol q}=\frac{4n_s\hbar^2}{qm}P\int_{\boldsymbol k}n_{\boldsymbol k}\frac{\boldsymbol k\otimes\boldsymbol k+\frac{q}{2}\boldsymbol n\otimes\boldsymbol k+\frac{q}{2}\boldsymbol k\otimes\boldsymbol n+\frac{q^2}{4}\boldsymbol n\otimes\boldsymbol n}{q+2\boldsymbol{kn}}\approx\frac{4n_s\hbar^2}{qm}P\int_{\boldsymbol k}n_{\boldsymbol k}\frac{\boldsymbol k\otimes\boldsymbol k}{q+2\boldsymbol{kn}} \tag{C8}$$

$$\begin{aligned} trL^{ss1}_{\boldsymbol q} &= \frac{4n_s\hbar^2}{qm}P\int_{\boldsymbol k}n_{\boldsymbol k}\frac{k^2}{q+2\boldsymbol{kn}} \\ &= \frac{n_s\hbar^2}{qm\pi^2}P\int_0^{\infty}dkk^4n_k\int_{-1}^{1}dC\frac{1}{2kC+q} \\ &= \frac{n_s\hbar^2}{2qm\pi^2}P\int_0^{\infty}dkk^3n_k\ln\left|\frac{2k+q}{2k-q}\right| \\ &\approx \frac{n_s\hbar^2}{2m\pi^2}\int_0^{\infty}dkk^2n_k=\frac{n_s\hbar^2}{m}\int_{\boldsymbol k}n_k \end{aligned} \tag{C9}$$

2. Infinite lifetime, R

$$\begin{aligned} R^{\mu\nu\pm}_q &= n_s\pi\hbar\int_{\boldsymbol k}\delta(\pm\omega-\omega_{\boldsymbol k+\boldsymbol q}+\omega_{\boldsymbol k})n_{\boldsymbol k}(n_{\boldsymbol k+\boldsymbol q}-1)F^{\mu\nu\pm}(\boldsymbol k,\boldsymbol q) \\ &= n_s\pi\hbar\int_{\boldsymbol k}\frac{\delta\left(\pm\omega-\frac{\hbar\boldsymbol{qk}}{m}-\frac{\hbar q^2}{2m}\right)[1+\xi-e^{\beta(\frac{\hbar^2(\boldsymbol k+\boldsymbol q)^2}{2m}-\mu)}]}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)(e^{\beta(\frac{\hbar^2(\boldsymbol k+\boldsymbol q)^2}{2m}-\mu)}-\xi)}F^{\mu\nu\pm}(\boldsymbol k,\boldsymbol q) \\ &= \frac{n_s\pi m}{q}\int_{\boldsymbol k}\frac{\delta\left(-k_3-\frac{q}{2}\pm\frac{m\omega}{\hbar q}\right)[1+\xi-e^{\beta(\frac{\hbar^2(\boldsymbol k+\boldsymbol q)^2}{2m}-\mu)}]}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)(e^{\beta(\frac{\hbar^2(\boldsymbol k+\boldsymbol q)^2}{2m}-\mu}-\xi)}F^{\mu\nu\pm}(\boldsymbol k,\boldsymbol q) \\ &= \frac{n_sm}{8\pi^2q}\int d^2k\frac{F^{\mu\nu\pm}(\boldsymbol k,\boldsymbol q)[1+\xi-e^{\beta\{\frac{\hbar^2}{2m}[(\pm\frac{m\omega}{\hbar q}+\frac{q}{2})^2+k^2]-\mu)\}}]}{(e^{\beta\{\frac{\hbar^2}{2m}[(\pm\frac{m\omega}{\hbar q}-\frac{q}{2})^2+k^2]-\mu)\}}-\xi)(e^{\beta\{\frac{\hbar^2}{2m}[(\pm\frac{m\omega}{\hbar q}+\frac{q}{2})^2+k^2]-\mu)\}}-\xi)}_{|k_3=\pm\frac{m\omega}{\hbar q}-\frac{q}{2}} \end{aligned} \tag{C10}$$

$$
\begin{aligned}
\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)_{|\frac{\boldsymbol{q}}{q}\boldsymbol{k}=\pm\frac{m\omega}{\hbar q}-\frac{q}{2}} &= \boldsymbol{k}_\perp + \frac{\boldsymbol{q}}{q}\left(\pm\frac{m\omega}{\hbar q}-\frac{q}{2}\right)+\frac{\boldsymbol{q}}{2} \\
&= \boldsymbol{k}_\perp \pm \frac{m\omega\boldsymbol{q}}{\hbar q^2} \\
F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})_{|\frac{\boldsymbol{q}}{q}\boldsymbol{k}=\pm\frac{m\omega}{\hbar q}-\frac{q}{2}} &= \begin{pmatrix} 1 & \mp\bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \\ \mp\bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \bar{r}^2\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\end{pmatrix}_{|\frac{\boldsymbol{q}}{q}\boldsymbol{k}=\pm\frac{m\omega}{\hbar q}-\frac{q}{2}} \\
&= \begin{pmatrix} 1 & \mp\bar{r}\left(\boldsymbol{k}_\perp\pm\frac{m\omega\boldsymbol{q}}{\hbar q^2}\right) \\ \mp\bar{r}\left(\boldsymbol{k}_\perp\pm\frac{m\omega\boldsymbol{q}}{\hbar q^2}\right) & \bar{r}^2\left(\boldsymbol{k}_\perp\pm\frac{m\omega\boldsymbol{q}}{\hbar q^2}\right)\otimes\left(\boldsymbol{k}_\perp\pm\frac{m\omega\boldsymbol{q}}{\hbar q^2}\right)\end{pmatrix}
\end{aligned}
\tag{C11}
$$

$$
\begin{aligned}
R^{\mu\nu\pm}_{\boldsymbol{q}} &= \frac{n_s m}{8\pi^2 q}\int d^2k \frac{F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})_{|k_3=-\frac{q}{2}}[1+\xi-e^{\beta\{\frac{\hbar^2}{2m}(\frac{q^2}{4}+k^2)-\mu)\}}]}{(e^{\beta\{\frac{\hbar^2}{2m}[\frac{q^2}{4}+k^2]-\mu)\}}-\xi)^2} \\
&\approx \frac{n_s m}{8\pi^2 q}\int d^2k \frac{\begin{pmatrix}1 & 0\\ 0 & \frac{\hbar^2k^2}{2m^2}(\mathbb{1}-\boldsymbol{n}\otimes\boldsymbol{n})\end{pmatrix}[1+\xi-e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}]}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)^2}
\end{aligned}
\tag{C12}
$$

$$
\begin{aligned}
R^{tt\pm}_{\boldsymbol{q}} &= \frac{n_s m}{4\pi q}\int dkk\frac{1+\xi-e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)^2} \\
R^{T\pm}_{\boldsymbol{q}} &= \frac{n_s\hbar^2}{8m\pi q}\int dkk^3\frac{1+\xi-e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)^2}
\end{aligned}
\tag{C13}
$$

$$
\begin{aligned}
\partial_\omega R^{\mu\nu\pm}_{\boldsymbol{q}} &= \frac{n_s m}{8\pi^2 q}\int d^2k\frac{\pm\frac{m}{\hbar q}\partial_{k_3}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})_{|k_3=-\frac{q}{2}}[1+\xi-e^{\beta\{\frac{\hbar^2}{2m}(\frac{q^2}{4}+k^2)-\mu)\}}]}{(e^{\beta\{\frac{\hbar^2}{2m}[\frac{q^2}{4}+k^2]-\mu)\}}-\xi)^2} \\
&-\frac{n_s m}{8\pi^2 q}\int d^2k\frac{\pm\frac{m}{\hbar}\beta\frac{\hbar^2}{2m}e^{\beta\{\frac{\hbar^2}{2m}(\frac{q^2}{4}+k^2)-\mu)\}}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})_{|k_3=-\frac{q}{2}}}{(e^{\beta\{\frac{\hbar^2}{2m}[\frac{q^2}{4}+k^2]-\mu)\}}-\xi)^2} \\
&= \pm\frac{n_s m}{8\pi^2 q}\int d^2k\left[\frac{\frac{m}{\hbar q}\partial_{k_3}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})[1+\xi-e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}]}{(e^{\beta\{\frac{\hbar^2}{2m}[\frac{q^2}{4}+k^2]-\mu)\}}-\xi)^2}-\frac{\beta\hbar e^{\beta\{\frac{\hbar^2}{2m}(\frac{q^2}{4}+k^2)-\mu)\}}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})}{2(e^{\beta\{\frac{\hbar^2}{2m}[\frac{q^2}{4}+k^2]-\mu)\}}-\xi)^2}\right]_{|k_3=-\frac{q}{2}}
\end{aligned}
\tag{C14}
$$

$$
\partial_{k_3}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q}) = \begin{pmatrix} 0 & \mp\frac{\hbar}{m}\boldsymbol{n} \\ \mp\frac{\hbar}{m}\boldsymbol{n} & \frac{\hbar^2}{m^2}(\boldsymbol{k}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{k})\end{pmatrix}
\tag{C15}
$$

For $\mu\nu = tt$ or ss

$$
\begin{aligned}
\partial_\omega R^{\mu\nu\pm}_{\boldsymbol{q}} &\to \mp\frac{n_s m\hbar}{8\pi^2 qk_BT}\int d^2k\frac{e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{0})}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)^2} \\
&= \pm\frac{n_s m\xi\hbar}{8\pi^2 qk_BT}\int d^2k\frac{F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{0})}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)(e^{-\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)} \\
&= \pm\frac{n_s m\xi\hbar}{4\pi qk_BT}\int_0^\infty dkk\frac{\begin{pmatrix}1 & 0\\ 0 & \frac{\hbar^2k^2}{2m^2}(\mathbb{1}-\boldsymbol{n}\otimes\boldsymbol{n})\end{pmatrix}}{2-\xi[e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}+e^{-\beta(\frac{\hbar^2k^2}{2m}-\mu)}]}
\end{aligned}
\tag{C16}
$$

$$
\begin{aligned}
R^{\mu\nu\pm}_{\omega,\boldsymbol{q}} &= R^{\mu\nu\pm}_{\boldsymbol{q}} + R^{\mu\nu\pm}_1 \frac{\omega}{q} \\
R^{tt\pm}_1 &= \pm \frac{n_s m \xi \hbar}{4\pi k_B T} \int_0^\infty dk k \frac{1}{2 - \xi[e^{\beta(\frac{\hbar^2 k^2}{2m} - \mu)} + e^{-\beta(\frac{\hbar^2 k^2}{2m} - \mu)}]} \\
R^{T\pm}_1 &= \pm \frac{n_s \xi \hbar^2}{8\pi m k_B T} \int_0^\infty dk k^3 \frac{1}{2 - \xi[e^{\beta(\frac{\hbar^2 k^2}{2m} - \mu)} + e^{-\beta(\frac{\hbar^2 k^2}{2m} - \mu)}]}
\end{aligned}
\tag{C17}
$$

$$
\tilde{G}^{(X)}_{\omega,q^2} = \tilde{G}^{(X)}_0 \left[1 + \frac{\bar{k}}{q}\mathcal{O}\left(\frac{m\omega}{\hbar q^2}\right) + \mathcal{O}\left(\frac{q}{\bar{k}}\right)\right] + \tilde{G}^{(X)}_1 \frac{\omega}{q}\left[1 + \frac{\bar{k}}{q}\mathcal{O}\left(\frac{m\omega}{\hbar q^2}\right) + \mathcal{O}\left(\frac{q}{\bar{k}}\right)\right]
\tag{C18}
$$

$$
\begin{aligned}
\tilde{G}^{tt}_0 &= -\frac{n_s \xi m}{2\pi^2} \int_0^\infty dk n_k, \qquad \tilde{G}^{tt}_1 = \frac{n_s m \hbar}{2\pi k_B T} \int_0^\infty dk k \frac{1}{2 - \xi[e^{\beta(\frac{\hbar^2 k^2}{2m} - \mu)} + e^{-\beta(\frac{\hbar^2 k^2}{2m} - \mu)}]}, \\
\tilde{G}^{T}_0 &= -\xi m \bar{r}^2 n_0, \qquad \tilde{G}^{T}_1 = \frac{n_s m \hbar \bar{r}^2}{4\pi k_B T} \int_0^\infty dk k^3 \frac{1}{2 - \xi[e^{\beta(\frac{\hbar^2 k^2}{2m} - \mu)} + e^{-\beta(\frac{\hbar^2 k^2}{2m} - \mu)}]},
\end{aligned}
\tag{C19}
$$

3. Finite lifetime, Lindhart function

$$
\begin{aligned}
L^{(\gamma)\mu\nu}_{eq} &= 2\hbar\Re \int_{\boldsymbol{k}} F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \left[\frac{n_{\boldsymbol{k}}}{q^0 + \omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}+\boldsymbol{q}} + i2\gamma} - \frac{n_{\boldsymbol{k}+\boldsymbol{q}}}{q^0 - \omega_{\boldsymbol{k}+\boldsymbol{q}} + \omega_{\boldsymbol{k}} - i2\gamma}\right] \\
&= 2\hbar\Re \int_{\boldsymbol{k}} \left[\frac{n_{\boldsymbol{k}} F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{q^0 + \omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}+\boldsymbol{q}} + i2\gamma} + \frac{n_{\boldsymbol{k}} F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q})}{-q^0 - \omega_{\boldsymbol{q}+\boldsymbol{k}} + \omega_{\boldsymbol{k}} + i2\gamma}\right] \\
&= 2\hbar\Re \int_{\boldsymbol{k}} \frac{n_{\boldsymbol{k}} \begin{pmatrix} 1 & 0 \\ 0 & \bar{r}^2 \left(\boldsymbol{k} + \frac{\boldsymbol{q}}{2}\right) \otimes \left(\boldsymbol{k} + \frac{\boldsymbol{q}}{2}\right) \end{pmatrix}}{q^0 + \omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}+\boldsymbol{q}} + i2\gamma} + (\omega \to -\omega)
\end{aligned}
\tag{C20}
$$

for $\mu\nu = ss$ or tt.

$$
\begin{aligned}
L^{(\gamma)tt}_{e\omega,\boldsymbol{q}} &= 2\hbar\Re \int_{\boldsymbol{k}} \frac{n_{\boldsymbol{k}}}{\omega + \omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}+\boldsymbol{q}} + i2\gamma} + (\omega \to -\omega) \\
&= \frac{\hbar}{2\pi^2}\Re \int_0^\infty dk k^2 n_k \int_{-1}^1 \frac{dC}{\omega - \frac{\hbar q^2}{2m} - \frac{\hbar k q C}{m} + 2i\gamma} + (\omega \to -\omega) \\
&= \frac{m}{2\pi^2 q}\Re \int_0^\infty dk k n_k \ln \frac{2m\omega - \hbar q^2 - 2\hbar k q + 4im\gamma}{2m\omega - \hbar q^2 + 2\hbar k q + 4im\gamma} + (\omega \to -\omega) \\
&= \frac{m}{4\pi^2 q} \int_0^\infty dk k n_k \ln \frac{(2m\omega - \hbar q^2 - 2\hbar k q)^2 + 16m^2\gamma^2}{(2m\omega - \hbar q^2 + 2\hbar k q)^2 + 16m^2\gamma^2} + (\omega \to -\omega) \\
&= \frac{m}{4\pi^2 q} \int_0^\infty dk k n_k \ln \frac{(k - \frac{m\omega}{\hbar q} + \frac{q}{2})^2 + \frac{4m^2\gamma^2}{\hbar^2 q^2}}{(k + \frac{m\omega}{\hbar q} - \frac{q}{2})^2 + \frac{4m^2\gamma^2}{\hbar^2 q^2}} + (\omega \to -\omega)
\end{aligned}
\tag{C21}
$$

or

$$
\begin{aligned}
L^{(\gamma)tt}_{e\omega,\boldsymbol{q}} &= \frac{\hbar}{\gamma}\Im \int_{\boldsymbol{k}} \frac{n_{\boldsymbol{k}}}{1 + \frac{1}{2i\gamma}(\omega - \frac{\hbar q^2}{2m} - \frac{\hbar \boldsymbol{k}\boldsymbol{q}}{m})} + (\omega \to -\omega) \\
&\approx \frac{\hbar}{\gamma}\Im \int_{\boldsymbol{k}} n_{\boldsymbol{k}} \left[1 - \frac{1}{2i\gamma}(\omega - \frac{\hbar q^2}{2m} - \frac{\hbar \boldsymbol{k}\boldsymbol{q}}{m}) - \frac{1}{4\gamma^2}(\omega - \frac{\hbar q^2}{2m} - \frac{\hbar \boldsymbol{k}\boldsymbol{q}}{m})^2\right] + (\omega \to -\omega) \\
&= -\frac{\hbar^2 \boldsymbol{q}^2}{2m\gamma^2} n_0
\end{aligned}
\tag{C22}
$$

$$
\begin{aligned}
L^{(\gamma)ss}_{e\omega,\boldsymbol{q}} &= \frac{\bar{r}^2\hbar}{\gamma}\Im\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)}{1+\frac{1}{2i\gamma}(\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m})}+(\omega\to-\omega)\\
&\approx -\frac{\bar{r}^2\hbar}{\gamma^2}\int_{\boldsymbol{k}}n_{\boldsymbol{k}}\left[\left(\mathbb{1}\frac{k^2}{3}+\frac{\boldsymbol{q}\otimes\boldsymbol{q}}{4}\right)\frac{\hbar q^2}{2m}+\frac{1}{2}(\boldsymbol{k}\otimes\boldsymbol{q}+\boldsymbol{q}\otimes\boldsymbol{k})\frac{\hbar\boldsymbol{kq}}{m}\right]\\
&= -\frac{\bar{r}^2\hbar^2\boldsymbol{q}^2}{2m\gamma^2}\int_{\boldsymbol{k}}n_{\boldsymbol{k}}\left(\mathbb{1}\frac{k^2}{3}+\frac{\boldsymbol{q}\otimes\boldsymbol{q}}{4}\right)-\boldsymbol{q}\otimes\boldsymbol{q}\frac{\bar{r}^2\hbar^2}{3m\gamma^2}\int_{\boldsymbol{k}}n_{\boldsymbol{k}}k^2\\
&= -\frac{\bar{r}^2\hbar^2}{6m\gamma^2}n_0\overline{k^2}(\mathbb{1}\boldsymbol{q}^2+2\boldsymbol{q}\otimes\boldsymbol{q})\\
&= -\frac{\bar{r}^2\hbar^2}{6m\gamma^2}n_0\overline{k^2}\boldsymbol{q}^2(T+3L)
\end{aligned}
\tag{C23}
$$

4. Finite lifetime, K

$$
\begin{aligned}
K^{tt}_{\omega,\boldsymbol{q}} &= 2\hbar\Im\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}}}{\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i2\gamma}+(\omega\to-\omega)-8i\xi\gamma\hbar\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}}{(\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})^2+4\gamma^2}\\
&= -\frac{\hbar}{\gamma}\Re\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}}}{1+\frac{1}{2i\gamma}(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}})}+(\omega\to-\omega)-\frac{2i\xi\hbar}{\gamma}\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}}{1-\frac{1}{4\gamma^2}(\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})^2}\\
&\approx -\frac{\hbar}{\gamma}\Re\int_{\boldsymbol{k}}n_{\boldsymbol{k}}\left[1-\frac{1}{2i\gamma}(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}})-\frac{1}{4\gamma^2}(\omega+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}})^2\right]+(\omega\to-\omega)\\
&\quad -\frac{2i\xi\hbar}{\gamma}\int_{\boldsymbol{k}}n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}\left[1+\frac{1}{4\gamma^2}(\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})^2\right]\\
&= -\frac{\hbar}{\gamma}\Re\int_{\boldsymbol{k}}n_{\boldsymbol{k}}\left[1-\frac{1}{2i\gamma}(\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m})-\frac{1}{4\gamma^2}(\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m})^2\right]+(\omega\to-\omega)\\
&\quad -\frac{2i\xi\hbar}{\gamma}\int_{\boldsymbol{k}}n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}\left[1+\frac{1}{4\gamma^2}(\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m})^2\right]\\
&= -\frac{2\hbar}{\gamma}\int_{\boldsymbol{k}}n_{\boldsymbol{k}}\left[1-\frac{\omega^2}{4\gamma^2}-\frac{\hbar^2}{12m^2\gamma^2}\boldsymbol{k}^2\boldsymbol{q}^2\right]-\frac{2i\xi\hbar}{\gamma}\int_{\boldsymbol{k}}n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}\left[1+\frac{\omega^2}{4\gamma^2}+\frac{\hbar^2}{12m^2\gamma^2}\boldsymbol{k}^2\boldsymbol{q}^2\right]
\end{aligned}
\tag{C24}
$$

$K_q-K_{-q}\approx 0.$

Appendix D: Finte temperature fermions

$n=\frac{1}{e^{\beta(\epsilon-\mu)}+1}$

1. Real part

$$
\Re\tilde{G}^{(\mu\nu)++}_{\omega,\boldsymbol{q}}=L^{(\mu\nu)++}_{\omega,\boldsymbol{q}} = n_s\hbar P\int_{\boldsymbol{k}}n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu+}(\boldsymbol{k},\boldsymbol{q})}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}}+\frac{F^{\mu\nu-}(\boldsymbol{k},\boldsymbol{q})}{-\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}}\right]
\tag{D1}
$$

$$\begin{aligned}
L^{tt}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar}{4\pi^2}P\int dkk^2dC\frac{1}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}+1)(\omega-\frac{\hbar q^2}{2m}-\frac{\hbar kqC}{m})}+(\omega\to-\omega)\\
&= -\frac{n_sm}{4\pi^2q}P\int\frac{dkk}{e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}+1}\ln\left|\frac{2m\omega-\hbar q^2-2\hbar kq}{2m\omega-\hbar q^2+2\hbar kq}\right|+(\omega\to-\omega)\\
&= -\frac{n_sm}{8\pi^2q}P\int\frac{dk^2}{e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}+1}\ln\left|\frac{(\frac{m\omega}{\hbar kq}-\frac{q}{2k}-1)(-\frac{m\omega}{\hbar kq}-\frac{q}{2k}-1)}{(\frac{m\omega}{\hbar kq}-\frac{q}{2k}+1)(-\frac{m\omega}{\hbar kq}-\frac{q}{2k}+1)}\right| \qquad x=\frac{\beta\hbar^2k^2}{2m}\\
&= -\frac{n_sm}{8\pi^2q}P\int\frac{dx\frac{2m}{\beta\hbar^2}}{e^{x-\beta\mu}+1}\ln\left|\frac{(-\frac{m\omega}{q}\sqrt{\frac{\beta}{2mx}}+\frac{\hbar q}{2}\sqrt{\frac{\beta}{2mx}}+1)(\frac{m\omega}{q}\sqrt{\frac{\beta}{2mx}}+\frac{\hbar q}{2}\sqrt{\frac{\beta}{2mx}}+1)}{(\frac{m\omega}{q}\sqrt{\frac{\beta}{2mx}}-\frac{\hbar q}{2}\sqrt{\frac{\beta}{2mx}}+1)(\frac{m\omega}{q}\sqrt{\frac{\beta}{2mx}}+\frac{\hbar q}{2}\sqrt{\frac{\beta}{2mx}}-1)}\right| \qquad \hbar k=\sqrt{\frac{2mx}{\beta}}
\end{aligned} \tag{D2}$$

$\bar{k}=\sqrt{\frac{mk_BT}{2\pi\hbar^2}}$, $z=\omega m/\hbar\bar{k}q$, $Q=q/\bar{k}$

$$\begin{aligned}
\frac{m\omega}{q}\sqrt{\frac{\beta}{2mx}}+\frac{\hbar q}{2}\sqrt{\frac{\beta}{2mx}} &= \frac{1}{\sqrt{2x}}\left(\frac{m\omega}{q}\frac{1}{\sqrt{mk_BT}}+\frac{\hbar q}{2}\frac{1}{\sqrt{mk_BT}}\right)=\frac{1}{\sqrt{2x}}\left(\frac{m\omega}{\hbar\bar{k}q}\bar{k}+\frac{q}{2\bar{k}}\bar{k}\right)\sqrt{\frac{\hbar^2}{mk_BT}}=\frac{z+\frac{q}{2\bar{k}}}{\sqrt{4\pi x}}\\
\frac{n_sm}{8\pi^2q}\frac{2m}{\beta\hbar^2} &= \frac{n_sm^2k_BT}{4\pi^2q\hbar^2}=\frac{n_sm\bar{k}^2}{2\pi q}=\frac{n_sm\bar{k}}{2\pi Q}
\end{aligned} \tag{D3}$$

$$\begin{aligned}
L^{tt}_{\omega,\boldsymbol{q}} &= -\frac{n_sm\bar{k}}{2\pi Q}P\int\frac{dx}{e^{x-\beta\mu}+1}\ln\left|\frac{(\sqrt{4\pi x}-z+\frac{Q}{2})(\sqrt{4\pi x}+z+\frac{Q}{2})}{(\sqrt{4\pi x}+z-\frac{Q}{2})(\sqrt{4\pi x}-z-\frac{Q}{2})}\right|\\
&= -\frac{n_sm\bar{k}}{2\pi Q}\left[I^{tt}\left(z-\frac{Q}{2}\right)-I^{tt}\left(-z+\frac{Q}{2}\right)\right]+(z\to-z)
\end{aligned} \tag{D4}$$

where

$$I^{tt}(u)=P\int_0^\infty dx\frac{\ln|\sqrt{4\pi x}-u|}{e^{x-\beta\mu}+1} \tag{D5}$$

$$\begin{aligned}
\Re\tilde{G}^T_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar(2\omega m-\hbar q^2)}{8\pi^2mq^2}\int_0^\infty dkk^2n_k+\frac{n_s\hbar^2}{8\pi^2mq}\int_0^\infty dkn_k\left(k^3-k(\frac{m\omega}{\hbar q}-\frac{q}{2})^2\right)\ln\frac{\frac{m\omega}{\hbar q}-\frac{q}{2}+k}{\frac{m\omega}{\hbar q}-\frac{q}{2}-k}+(\omega\to-\omega)\\
&= \frac{n_s\hbar^2\bar{k}q(2z-Q)}{8\pi^2mq^2}\int_0^\infty\frac{dkk^2}{e^{\frac{\hbar^2k^2}{2mk_BT}-\beta\mu}+1}+\frac{n_s\hbar^2}{8\pi^2mq}\int_0^\infty\frac{dk\left(k^3-k\bar{k}^2(z-\frac{Q}{2})^2\right)}{e^{\frac{\hbar^2k^2}{2mk_BT}-\beta\mu}+1}\ln\frac{z-\frac{Q}{2}+\frac{k}{\bar{k}}}{z-\frac{Q}{2}-\frac{k}{\bar{k}}}+(z\to-z)\\
&= \frac{n_s\hbar^2\bar{k}^4q(2z-Q)}{8\pi^2mq^2}\int_0^\infty\frac{dyy^2}{e^{\frac{\hbar^2}{2mk_BT}\frac{mk_BT}{2\pi\hbar^2}y^2-\beta\mu}+1}+\frac{n_s\hbar^2\bar{k}^4}{8\pi^2mq}\int_0^\infty\frac{dx\left(y^3-y(z-\frac{Q}{2})^2\right)}{e^{\frac{\hbar^2}{2mk_BT}\frac{mk_BT}{2\pi\hbar^2}y^2-\beta\mu}+1}\ln\frac{z-\frac{Q}{2}+y}{z-\frac{Q}{2}-y}\\
&= \frac{n_s\hbar^2\bar{k}^4q(2z-Q)}{8\pi^2mq^2}\int_0^\infty\frac{dyy^2}{e^{\frac{1}{4\pi}y^2-\beta\mu}+1}+\frac{n_s\hbar^2\bar{k}^4}{8\pi^2mq}\int_0^\infty\frac{dy\left(y^3-y(z-\frac{Q}{2})^2\right)}{e^{\frac{1}{4\pi}y^2-\beta\mu}+1}\ln\frac{z-\frac{Q}{2}+y}{z-\frac{Q}{2}-y}+(z\to-z)
\end{aligned} \tag{D6}$$

$x=y^2$, $dx=2dyy$, $y=\sqrt{x}$, $dy=dx/2\sqrt{x}$

$$\begin{aligned}
\Re\tilde{G}^T_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar^2\bar{k}^4q(2z-Q)}{16\pi^2mq^2}\int_0^\infty\frac{dx\sqrt{x}}{e^{\frac{x}{4\pi}-\beta\mu}+1}+\frac{n_s\hbar^2\bar{k}^4}{16\pi^2mq}\int_0^\infty\frac{dx\left(x-(z-\frac{Q}{2})^2\right)}{e^{\frac{x}{4\pi}-\beta\mu}+1}\ln\frac{z-\frac{Q}{2}+\sqrt{x}}{z-\frac{Q}{2}-\sqrt{x}}+(z\to-z)\\
&= \frac{n_s\hbar^2(4\pi)^{3/2}\bar{k}^4q(z-\frac{Q}{2})}{8\pi^2mq^2}\int_0^\infty\frac{dx\sqrt{x}}{e^{x-\beta\mu}+1}+\frac{n_s\hbar^2\bar{k}^4}{4\pi mq}\int_0^\infty\frac{dx\left(4\pi x-(z-\frac{Q}{2})^2\right)}{e^{x-\beta\mu}+1}\ln\frac{z-\frac{Q}{2}+\sqrt{4\pi x}}{z-\frac{Q}{2}-\sqrt{4\pi x}}+(z\to-z)
\end{aligned} \tag{D7}$$

$$I_1^T = \int_0^\infty \frac{dx\sqrt{x}}{e^{x-\beta\mu}+1}$$
$$I_2^T(u) = \int_0^\infty \frac{dx 4\pi x}{e^{x-\beta\mu}+1}\ln|\sqrt{4\pi x}-u| \quad \text{(D8)}$$

$$\Re\tilde{G}^T_{\omega,\boldsymbol{q}} = \frac{n_s\hbar^2\bar{k}^3(z-\frac{Q}{2})}{\sqrt{\pi}mQ}I_1^T + \frac{n_s\hbar^2\bar{k}^3}{4\pi mQ}\left[I_2^T\left(z-\frac{Q}{2}\right)-\left(z-\frac{Q}{2}\right)^2 I^{tt}\left(z-\frac{Q}{2}\right)\right]+(z\to -z)$$
$$= \frac{n_s m\bar{k}}{2\pi Q}\frac{\hbar^2\bar{k}^2}{m^2}\left[2\sqrt{\pi}\left(z-\frac{Q}{2}\right)I_1^T+\frac{1}{2}I_2^T\left(z-\frac{Q}{2}\right)-\frac{1}{2}\left(z-\frac{Q}{2}\right)^2 I^{tt}\left(z-\frac{Q}{2}\right)\right]+(z\to -z) \quad \text{(D9)}$$

2. Imaginary part

$$F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q}) = \begin{pmatrix} 1 & \mp\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \\ \mp\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\end{pmatrix} \quad \text{(D10)}$$

$$R_q^{\mu\nu\pm} = n_s\pi\hbar\int_{\boldsymbol{k}}\delta(\pm\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})n_{\boldsymbol{k}}(1+\xi n_{\boldsymbol{k}+\boldsymbol{q}})F^{\pm\mu\nu}(\boldsymbol{k},\boldsymbol{q})$$
$$= n_s\pi\hbar\int_{\boldsymbol{k}}\delta(\pm\omega-\epsilon_{\boldsymbol{k}+\boldsymbol{q}}+\epsilon_{\boldsymbol{k}})n_{\boldsymbol{k}}(1-n_{\boldsymbol{k}+\boldsymbol{q}})F^{\pm\mu\nu}(\boldsymbol{k},\boldsymbol{q})$$
$$= \frac{n_s\hbar}{8\pi^2}\int d^3k n_{\boldsymbol{k}}(1-n_{\boldsymbol{k}+\boldsymbol{q}})\delta\left(\pm\omega-\frac{\hbar\boldsymbol{q}\boldsymbol{k}}{m}-\frac{\hbar q^2}{2m}\right)F^{\pm\mu\nu}(\boldsymbol{k},\boldsymbol{q})$$
$$= \frac{n_s m}{8\pi^2 q}\int d^2k n_{\boldsymbol{k}}(1-n_{\boldsymbol{k}+\boldsymbol{q}})F^{\pm\mu\nu}(\boldsymbol{k},\boldsymbol{q})_{|k_3=\pm\frac{m\omega}{\hbar q}-\frac{q}{2}=\bar{k}(\pm z-\frac{Q}{2})} \quad \text{(D11)}$$

$$R_q^{tt+} = \frac{n_s m}{8\pi^2 q}\int\frac{d^2k}{(e^{\frac{\hbar^2}{2mk_BT}[k^2+\bar{k}^2(z-\frac{Q}{2})^2]-\beta\mu}+1)(e^{-\frac{\hbar^2}{2mk_BT}[k^2+\bar{k}^2(z+\frac{Q}{2})^2]+\beta\mu}+1)}$$
$$= \frac{n_s m\bar{k}}{8\pi Q}\int\frac{dx}{(e^{\frac{\hbar^2\bar{k}^2}{2mk_BT}[x+(z-\frac{Q}{2})^2]-\beta\mu}+1)(e^{-\frac{\hbar^2\bar{k}^2}{2mk_BT}[x+(z+\frac{Q}{2})^2]+\beta\mu}+1)} \quad \text{(D12)}$$

$\frac{\hbar^2\bar{k}^2}{2mk_BT} = \frac{\hbar^2}{2mk_BT}\frac{mk_BT}{2\pi\hbar^2} = \frac{1}{4\pi}$

$$R_q^{tt+} = \frac{n_s m\bar{k}}{2Q}\int_0^\infty\frac{dx}{(e^{x+\frac{1}{4\pi}(z-\frac{Q}{2})^2-\beta\mu}+1)(e^{-x-\frac{1}{4\pi}(z+\frac{Q}{2})^2+\beta\mu}+1)} \quad \text{(D13)}$$

$T = \frac{1}{2}(\mathrm{Tr}-L)$

$$R_q^{T+} = \frac{n_s m}{16\pi^2 q}\frac{\hbar^2}{m^2}\int d^2k n_{\boldsymbol{k}}(1-n_{\boldsymbol{k}+\boldsymbol{q}})\left[\mathrm{Tr}(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})\otimes(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})-\boldsymbol{n}(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})\otimes(\boldsymbol{k}+\frac{\boldsymbol{q}}{2})\boldsymbol{n}\right]_{|k_3=\bar{k}(z-\frac{Q}{2})}$$
$$= \frac{n_s m}{16\pi^2 q}\frac{\hbar^2}{m^2}\int d^2k n_{\boldsymbol{k}}(1-n_{\boldsymbol{k}+\boldsymbol{q}})\left[k^2+\frac{q^2}{4}+\boldsymbol{k}\boldsymbol{q}-(\boldsymbol{n}\boldsymbol{k})^2-\frac{q^2}{4}-\boldsymbol{k}\boldsymbol{q}\right]_{|k_3=\bar{k}(z-\frac{Q}{2})}$$
$$= \frac{n_s m}{16\pi^2 q}\frac{\hbar^2}{m^2}\int\frac{d^2k k^2}{(e^{\frac{\hbar^2}{2mk_BT}[k^2+\bar{k}^2(z-\frac{Q}{2})^2]-\beta\mu}+1)(e^{-\frac{\hbar^2}{2mk_BT}[k^2+\bar{k}^2(z+\frac{Q}{2})^2]+\beta\mu}+1)}$$
$$= \frac{n_s m\bar{k}}{16\pi Q}\frac{\hbar^2\bar{k}^2}{m^2}\int\frac{dx x}{(e^{\frac{\hbar^2\bar{k}^2}{2mk_BT}[x+(z-\frac{Q}{2})^2]-\beta\mu}+1)(e^{-\frac{\hbar^2\bar{k}^2}{2mk_BT}[x+(z+\frac{Q}{2})^2]+\beta\mu}+1)}$$
$$= \frac{n_s\pi m\bar{k}}{Q}\frac{\hbar^2\bar{k}^2}{m^2}\int\frac{dx x}{(e^{x+\frac{1}{4\pi}(z-\frac{Q}{2})^2-\beta\mu}+1)(e^{-x-\frac{1}{4\pi}(z+\frac{Q}{2})^2+\beta\mu}+1)} \quad \text{(D14)}$$

Conversion:

$$n = n_s \int_{\boldsymbol{k}} \frac{1}{e^{\beta(\epsilon_{\boldsymbol{k}}-\mu)}+1} = \frac{n_s}{2\pi^2} \int \frac{dkk^2}{e^{\frac{\hbar^2 k^2}{2mk_BT}-\beta\mu}+1} = \frac{n_s}{4\pi^2} \int \frac{dy\sqrt{y}}{e^{\frac{\hbar^2 y}{2mk_BT}-\beta\mu}+1} = \frac{n_s}{4\pi^2}\left(\frac{2mk_BT}{\hbar^2}\right)^{3/2} \int \frac{dx\sqrt{x}}{e^{x-\beta\mu}+1}$$
$$= \frac{n_s}{4\pi^2}(4\pi\bar{k})^{3/2} \int \frac{dx\sqrt{x}}{e^{x-\beta\mu}+1} \tag{D15}$$

Appendix E: Finite quantization volume

$\int_{\boldsymbol{k}} = \frac{1}{V}\sum_{\boldsymbol{k}}$

$$\begin{aligned}\tilde{G}_q^{(\mu\nu)++} &= \frac{n_s\hbar}{V}\sum_{\boldsymbol{k}}\left[\frac{n_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\epsilon} + \frac{n_{\boldsymbol{k}+\boldsymbol{q}}}{-q^0+\omega_{\boldsymbol{k}+\boldsymbol{q}}-\omega_{\boldsymbol{k}}+i\epsilon} - i\xi 2\pi\delta(\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}\right]F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\ &= \frac{n_s\hbar}{V}\sum_{\boldsymbol{k}}\left[\frac{n_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\epsilon} + \frac{n_{\boldsymbol{k}+\boldsymbol{q}}}{-q^0+\omega_{\boldsymbol{k}+\boldsymbol{q}}-\omega_{\boldsymbol{k}}+i\epsilon} - i\xi 2\pi\delta(\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}\right]F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\ &= \tilde{G}_q^{1(\mu\nu)++} + \tilde{G}_q^{2(\mu\nu)++}\end{aligned} \tag{E1}$$

$$\begin{aligned}\tilde{G}_q^{1(\mu\nu)++} &= \frac{n_s\hbar}{V}\sum_{\boldsymbol{k}} n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\epsilon} - \frac{F^{\mu\nu}(\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q})}{q^0+\omega_{\boldsymbol{k}-\boldsymbol{q}}-\omega_{\boldsymbol{k}}-i\epsilon}\right] \\ &= \frac{n_s\hbar}{V}\sum_{\boldsymbol{k}} n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\epsilon} + \frac{F^{\mu\nu}(\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q})}{-q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}-\boldsymbol{q}}+i\epsilon}\right] \\ &= \frac{n_s\hbar}{V}\sum_{\boldsymbol{k}} n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})}{q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\epsilon} + \frac{F^{\mu\nu}(-\boldsymbol{k}-\boldsymbol{q},\boldsymbol{q})}{-q^0+\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}+i\epsilon}\right]\end{aligned} \tag{E2}$$

$\tilde{G}_{\pm\omega,\boldsymbol{q}}^{2(\mu\nu)++} = 0$ (non-vanishing denominators): Trivial thermodynamical limit when
$\tilde{G}_{\pm\omega,\boldsymbol{q}}^{2(\mu\nu)++} \neq 0$: $\omega_{\boldsymbol{k}'}+\sigma\omega-\omega_{\boldsymbol{k}'+\boldsymbol{q}} = \sigma\omega - \frac{\hbar q^2}{2m} - \frac{\hbar\boldsymbol{k}'\boldsymbol{q}}{m} = 0$, $\sigma = -\tilde{\sigma} = \pm 1$

$$\begin{aligned}\tilde{G}_{\omega,\boldsymbol{q}}^{1(\mu\nu)++} &= \frac{n_s\hbar}{V}\sum_{\boldsymbol{k}} n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu+}(\boldsymbol{k},\boldsymbol{q})}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}\boldsymbol{q}}{m}+i\epsilon} + \frac{F^{\mu\nu-}(\boldsymbol{k},\boldsymbol{q})}{-\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}\boldsymbol{q}}{m}+i\epsilon}\right] \\ &= \frac{n_s\hbar}{V}\sum_{\sigma}\sum_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}}F^{\mu\nu\sigma}(\boldsymbol{k},\boldsymbol{q})}{\sigma\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}\boldsymbol{q}}{m}+i\epsilon} \\ &= \frac{n_s\hbar}{V}\sum_{\sigma}\sum_{\boldsymbol{k}\neq\boldsymbol{k}'}\frac{n_{\boldsymbol{k}}F^{\mu\nu\sigma}(\boldsymbol{k},\boldsymbol{q})}{\sigma\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}\boldsymbol{q}}{m}+i\epsilon} + \frac{n_s\hbar}{V}n_{\boldsymbol{k}'}\left(\frac{F^{\mu\nu\sigma}(\boldsymbol{k}',\boldsymbol{q})}{i\epsilon} + \frac{F^{\mu\nu\tilde{\sigma}}(\boldsymbol{k}',\boldsymbol{q})}{-2\sigma\omega+i\epsilon}\right)\end{aligned} \tag{E3}$$

$$\begin{aligned}\Re\tilde{G}_q^{1(\mu\nu)++} &= \frac{n_s\hbar}{V}\sum_{\sigma}\sum_{\boldsymbol{k}\neq\boldsymbol{k}'}\frac{n_{\boldsymbol{k}}F^{\mu\nu\sigma}(\boldsymbol{k},\boldsymbol{q})}{\sigma\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}\boldsymbol{q}}{m}+i\epsilon} \\ \Im\tilde{G}_q^{1(\mu\nu)++} &= \frac{n_s\hbar}{V}n_{\boldsymbol{k}'}\left(\frac{F^{\mu\nu\sigma}(\boldsymbol{k}',\boldsymbol{q})}{i\epsilon} + \frac{F^{\mu\nu\tilde{\sigma}}(\boldsymbol{k}',\boldsymbol{q})}{-2\sigma\omega+i\epsilon}\right) - i\xi\frac{n_s\hbar}{V}\sum_{\boldsymbol{k}}2\pi\delta(\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\ &= \frac{n_s\hbar}{V}n_{\boldsymbol{k}'}\left(\frac{F^{\mu\nu\sigma}(\boldsymbol{k}',\boldsymbol{q})}{i\epsilon} + \frac{F^{\mu\nu\tilde{\sigma}}(\boldsymbol{k}',\boldsymbol{q})}{-2\sigma\omega+i\epsilon}\right) - i\delta_{1,\sigma}\xi\frac{n_s\hbar}{V}2\pi\delta(\omega_{\boldsymbol{k}}+q^0-\omega_{\boldsymbol{k}+\boldsymbol{q}})n_{\boldsymbol{k}+\boldsymbol{q}}n_{\boldsymbol{k}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})\end{aligned} \tag{E4}$$

$z = \omega m/\hbar\bar{k}q$, $b_\pm = \frac{Q}{2} \pm z$, $\frac{1}{Q}(b_-^3 + b_+^3) = \frac{Q^2}{4} + 3z^2$

$$\begin{aligned} L^{tt}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar}{4\pi^2} P \int dk k^2 dC n_k \frac{1}{\omega - \frac{\hbar q^2}{2m} - \frac{\hbar k q C}{m}} + (\omega \to -\omega) \\ &= -\frac{n_s m}{4\pi^2 q} P \int_0^{k_F} dk k \ln\left|\frac{2m\omega - \hbar q^2 - 2\hbar k q}{2m\omega - \hbar q^2 + 2\hbar k q}\right| + (\omega \to -\omega) \\ &= \frac{n_s m k_F}{4\pi^2}\left\{-1 + \frac{k_F}{2q}\left[1 - \left(z - \frac{q}{2k_F}\right)^2\right]\ln\left|\frac{1 + z - \frac{q}{2k_F}}{1 - z + \frac{q}{2k_F}}\right| - \frac{k_F}{2q}\left[1 - \left(z + \frac{q}{2k_F}\right)^2\right]\ln\left|\frac{1 + z + \frac{q}{2k_F}}{1 - z - \frac{q}{2k_F}}\right|\right\} \\ &= \frac{n_s m k_F}{4\pi^2}\sum_\tau\left[-\frac{1}{2} + \frac{1}{2Q}(1 - b_\tau^2)\ln\left|\frac{1 - b_\tau}{1 + b_\tau}\right|\right] \end{aligned} \tag{E5}$$

Gradient expansion:

$$\begin{aligned} L^{tt}_{\omega,\boldsymbol{q}} &= \frac{n_s m k_F}{4\pi^2}\sum_\tau\left[-\frac{1}{2} - \frac{1}{Q}(1 - b_\tau^2)(b_\tau + \frac{1}{3}b_\tau^3 + \mathcal{O}(b^5))\right] \\ &= -\frac{n_s m k_F}{4\pi^2}\sum_\tau\left[\frac{1}{2} + \frac{b_\tau}{Q} - \frac{2}{3Q}b_\tau^3\right] + \frac{1}{Q}\mathcal{O}(b^5) \\ &\approx -\frac{n_s m k_F}{4\pi^2}\left(2 - \frac{Q^2}{6} - 2z^2\right) \end{aligned} \tag{E6}$$

Imaginary energy: $z = iw$, $b = \frac{Q}{2} + iw$

$$\begin{aligned} L^{tt}_{i\omega,\boldsymbol{q}} &= \frac{n_s m k_F}{4\pi^2}\left[-1 + \frac{1 - \frac{Q^2}{4} + w^2}{Q}\ln\left|\frac{1 + \frac{Q^2}{4} - Q + w^2}{1 + \frac{Q^2}{4} + Q + w^2}\right|\right] \\ &\approx -\frac{n_s m k_F}{4\pi^2}\left(2 - \frac{Q^2}{6} + 2w^2\right) \end{aligned} \tag{E7}$$

$Q = k/k_F = 1/R$

$$a = \bar{k}z - \frac{q}{2} = \frac{\omega m}{\hbar q} - \frac{q}{2} = \frac{q}{2}\left(\frac{\hbar\omega}{\frac{\hbar^2 q^2}{2m}} - 1\right), \qquad b_\pm = \frac{q}{2\bar{k}} \pm z = \frac{q}{2\bar{k}}\left(1 \pm \frac{\hbar\omega}{\frac{\hbar^2 q^2}{2m}}\right) \tag{E8}$$

Static case: $z = 0$:

$$\begin{aligned} L^{tt}_{\boldsymbol{q}} &= \frac{n_s m k_F}{4\pi^2}\left[-1 + \frac{1}{Q}\left(1 - \frac{Q^2}{4}\right)\ln\left|\frac{1 - \frac{Q}{2}}{1 + \frac{Q}{2}}\right|\right] \\ &= \frac{n_s m k_F}{4\pi^2}\left[-1 + \left(\frac{1}{Q} - \frac{Q}{4}\right)\ln\left|\frac{1 - \frac{Q}{2}}{1 + \frac{Q}{2}}\right|\right] \end{aligned} \tag{E9}$$

$$\begin{aligned} \Re\tilde{G}^{ss}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar^3}{m^2} P \int_{\boldsymbol{k}} n_k \frac{(\boldsymbol{k} + \frac{\boldsymbol{q}}{2}) \otimes (\boldsymbol{k} + \frac{\boldsymbol{q}}{2})}{\omega - \frac{\hbar q^2}{2m} - \frac{\hbar \boldsymbol{k}\boldsymbol{q}}{m}} + (\omega \to -\omega) \\ \mathrm{Tr}\Re\tilde{G}^{ss}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar^3}{m^2} P \int_{\boldsymbol{k}} n_k \frac{k^2 + \frac{q^2}{4} + \boldsymbol{k}\boldsymbol{q}}{\omega - \frac{\hbar q^2}{2m} - \frac{\hbar \boldsymbol{k}\boldsymbol{q}}{m}} + (\omega \to -\omega) \\ \boldsymbol{q}\Re\tilde{G}^{ss}_{\omega,\boldsymbol{q}}\boldsymbol{q} &= \frac{n_s\hbar^3}{m^2} P \int_{\boldsymbol{k}} n_k \frac{(\boldsymbol{q}\boldsymbol{k})^2 + \frac{q^4}{4} + q^2\boldsymbol{k}\boldsymbol{q}}{\omega - \frac{\hbar q^2}{2m} - \frac{\hbar \boldsymbol{k}\boldsymbol{q}}{m}} + (\omega \to -\omega) \\ \Re\tilde{G}^{T}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar^3}{2m^2} P \int_{\boldsymbol{k}} n_k \frac{k^2 + \frac{q^2}{4} + \boldsymbol{k}\boldsymbol{q} - (\boldsymbol{n}\boldsymbol{k})^2 - \frac{q^2}{4} - \boldsymbol{k}\boldsymbol{q}}{\omega - \frac{\hbar q^2}{2m} - \frac{\hbar \boldsymbol{k}\boldsymbol{q}}{m}} + (\omega \to -\omega) \\ &= \frac{n_s\hbar^3}{2m^2} P \int_{\boldsymbol{k}} n_k \frac{k^2 - (\boldsymbol{n}\boldsymbol{k})^2}{\omega - \frac{\hbar q^2}{2m} - \frac{\hbar \boldsymbol{k}\boldsymbol{q}}{m}} + (\omega \to -\omega) \end{aligned} \tag{E10}$$

because $T = \frac{1}{2}(\mathrm{Tr} - L)$

$$\begin{aligned}
\int dc \frac{1}{a+bc} &= \frac{1}{b}\ln(a+bc) \\
\int dc \frac{c}{a+bc} &= \frac{1}{b}\int dc\left[\frac{a+bc}{a+bc} - \frac{a}{a+bc}\right] = \frac{c}{b} - \frac{a}{b^2}\ln(a+bc) \\
\int dc \frac{c^2}{a+bc} &= \frac{1}{b}\int dcc\left[1 - \frac{a}{a+bc}\right] = \frac{c^2}{2b} - \frac{a}{b}\left(\frac{c}{b} - \frac{a}{b^2}\ln(a+bc)\right) = \frac{c^2}{2b} - \frac{ac}{b^2} + \frac{a^2}{b^3}\ln(a+bc) \\
\int dc \frac{1-c^2}{a+bc} &= -\frac{c^2}{2b} + \frac{ac}{b^2} + \frac{b^2-a^2}{b^3}\ln(a+bc)
\end{aligned} \tag{E11}$$

$$\begin{aligned}
\Re\tilde{G}^T_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar^3}{8\pi^2m^2}P\int_0^\infty dkk^4n_k\int_{-1}^1 dC\frac{1-C^2}{\omega - \frac{\hbar q^2}{2m} - \frac{\hbar kqC}{m}} + (\omega\to-\omega) \\
&= \frac{n_s\hbar^3}{8\pi^2m^2}P\int_0^\infty dkk^4n_k\left[\frac{c^2}{2\frac{\hbar kq}{m}} + \frac{(\omega - \frac{\hbar q^2}{2m})c}{(\frac{\hbar kq}{m})^2} - \frac{(\frac{\hbar kq}{m})^2 - (\omega-\frac{\hbar q^2}{2m})^2}{(\frac{\hbar kq}{m})^3}\ln(\omega - \frac{\hbar q^2}{2m} - \frac{\hbar kqC}{m})\right]_{-1}^1 \\
&= \frac{n_s\hbar^3}{8\pi^2m^2}P\int_0^\infty dkk^4n_k\left[\frac{2\omega - \frac{\hbar q^2}{m}}{(\frac{\hbar kq}{m})^2} + \frac{(\frac{\hbar kq}{m})^2 - (\omega - \frac{\hbar q^2}{2m})^2}{(\frac{\hbar kq}{m})^3}\ln\frac{\omega - \frac{\hbar q^2}{2m} + \frac{\hbar kq}{m}}{\omega - \frac{\hbar q^2}{2m} - \frac{\hbar kq}{m}}\right] + (\omega\to-\omega) \\
&= \frac{n_s\hbar^3}{8\pi^2m^2}P\int_0^\infty dkk^4n_k\left[\frac{2\omega m^2 - m\hbar q^2}{\hbar^2k^2q^2} + \frac{m}{\hbar kq}\left(1 - (\frac{m\omega}{\hbar kq} - \frac{q}{2k})^2\right)\ln\frac{\omega - \frac{\hbar q^2}{2m} + \frac{\hbar kq}{m}}{\omega - \frac{\hbar q^2}{2m} - \frac{\hbar kq}{m}}\right] + (\omega\to-\omega) \\
&= \frac{n_s\hbar(2\omega m - \hbar q^2)}{8\pi^2mq^2}\int_0^\infty dkk^2n_k + \frac{n_s\hbar^2}{8\pi^2mq}\int_0^\infty dkn_k\left(k^3 - k(\frac{m\omega}{\hbar q} - \frac{q}{2})^2\right)\ln\frac{\frac{m\omega}{\hbar q} - \frac{q}{2} + k}{\frac{m\omega}{\hbar q} - \frac{q}{2} - k} + (\omega\to-\omega)
\end{aligned} \tag{E12}$$

$$\begin{aligned}
\int_0^{k_F} dkk\ln(a+bk) &= \frac{ak}{2b} - \frac{k^2}{4} + \frac{1}{2}\left(k^2 - \frac{a^2}{b^2}\right)\ln(a+bk)\Big|_0^{k_F} \\
&= \frac{ak_F}{2b} - \frac{k_F^2}{4} + \frac{k_F^2}{2}\ln(a+bk_F) - \frac{a^2}{2b^2}\ln\frac{a+bk_F}{a} \\
\int_0^{k_F} dkk\ln\frac{a+bk}{a-bk} &= \frac{ak_F}{b} + \left(\frac{k_F^2}{2} - \frac{a^2}{2b^2}\right)\ln\frac{a+bk_F}{a-bk_F} \\
\int_0^{k_F} dkk^3 ln(a+bk) &= -\frac{k^4}{16} + \frac{ak^3}{12b} - \frac{a^2k^2}{8b^2} + \frac{a^3k}{4b^3} + \frac{1}{4}\left(k^4 - \frac{a^4}{b^4}\right)\ln(a+bk)\Big|_0^{k_F} \\
&= -\frac{k_F^4}{16} + \frac{ak_F^3}{12b} - \frac{a^2k_F^2}{8b^2} + \frac{a^3k_F}{4b^3} + \frac{1}{4}k_F^4\ln(a+bk_F) - \frac{1}{4}\frac{a^4}{b^4}\ln\frac{a+bk_F}{a} \\
\int_0^{k_F} dkk^3\ln\frac{a+bk}{a-bk} &= \frac{ak_F^3}{6b} + \frac{a^3k_F}{2b^3} + \frac{1}{4}\left(k_F^4 - \frac{a^4}{b^4}\right)\ln\frac{a+bk_F}{a-bk_F}
\end{aligned} \tag{E13}$$

$$
\begin{aligned}
\Re\tilde{G}^T_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar(2\omega m-\hbar q^2)k_F^3}{24\pi^2 mq^2}+\frac{n_s\hbar^2}{8\pi^2 mq}\Bigg[\frac{ak_F^3}{6b}+\frac{a^3k_F}{2b^3}-\frac{ak_F}{b}(\frac{m\omega}{\hbar q}-\frac{q}{2})^2\\
&+\left(\frac{k_F^4}{4}-\frac{a^4}{4b^4}-(\frac{m\omega}{\hbar q}-\frac{q}{2})^2\left(\frac{k_F^2}{2}-\frac{a^2}{2b^2}\right)\right)\ln\frac{\frac{m\omega}{\hbar q}-\frac{q}{2}+k_F}{\frac{m\omega}{\hbar q}-\frac{q}{2}-k_F}\Bigg]+(\omega\to-\omega)\\
&= \frac{n_s\hbar^2 k_F q(2z-\frac{q}{k_F})k_F^3}{24\pi^2 mq^2}+\frac{n_s\hbar^2}{8\pi^2 mq}\Bigg[\frac{1}{6}(\frac{m\omega}{\hbar q}-\frac{q}{2})k_F^3+\frac{1}{2}(\frac{m\omega}{\hbar q}-\frac{q}{2})^3k_F-(\frac{m\omega}{\hbar q}-\frac{q}{2})k_F(\frac{m\omega}{\hbar q}-\frac{q}{2})^2\\
&+\frac{1}{4}\left(k_F^4-(\frac{m\omega}{\hbar q}-\frac{q}{2})^4-2(\frac{m\omega}{\hbar q}-\frac{q}{2})^2\left(k_F^2-(\frac{m\omega}{\hbar q}-\frac{q}{2})^2\right)\right)\ln\frac{\frac{m\omega}{\hbar q}-\frac{q}{2}+k_F}{\frac{m\omega}{\hbar q}-\frac{q}{2}-k_F}\Bigg]+(\omega\to-\omega)\\
&= \frac{n_s\hbar^2k_F^4(z-\frac{q}{2k_F})}{12\pi^2 mq}+\frac{n_s\hbar^2k_F^4}{8\pi^2 mq}\Bigg[\frac{1}{6}(z-\frac{q}{2k_F})-\frac{1}{2}(z-\frac{q}{2k_F})^3\\
&+\frac{1}{4}\left(1-(z-\frac{q}{2k_F})^4-2(z-\frac{q}{2k_F})^2\left(1-(z-\frac{q}{2k_F})^2\right)\right)\ln\frac{z-\frac{q}{2k_F}+1}{z-\frac{q}{2k_F}-1}\Bigg]+(\omega\to-\omega)\\
&= -\frac{n_s\hbar^2k_F^4b_-}{12\pi^2 mq}+\frac{n_s\hbar^2k_F^4}{8\pi^2 mq}\left[-\frac{b_-}{6}+\frac{b_-^3}{2}+\frac{1}{4}\left(1-b_-^4-2b_-^2\left(1-b_-^2\right)\right)\ln\frac{1-b_-}{1+b_-}\right]+(\omega\to-\omega)\\
&= -\frac{n_s\hbar^2k_F^4b_-}{12\pi^2 mq}+\frac{n_s\hbar^2k_F^4}{8\pi^2 mq}\left[-\frac{b_-}{6}+\frac{b_-^3}{2}+\frac{1}{4}\left(1-2b_-^2+b_-^4\right)\ln\frac{1-b_-}{1+b_-}\right]+(\omega\to-\omega)\\
&= -\frac{n_s\hbar^2k_F^4(b_-+b_+)}{12\pi^2 mq}+\frac{n_s\hbar^2k_F^4}{16\pi^2 mq}\left[-\frac{b_-+b_+}{3}+b_-^3+b_+^3+\frac{1}{2}(1-b_-^2)^2\ln\frac{1-b_-}{1+b_-}+\frac{1}{2}(1-b_+^2)^2\ln\frac{1-b_+}{1+b_+}\right]\\
&= \frac{n_s mk_F}{4\pi^2 Q}\frac{\hbar^2k_F^2}{m^2}\sum_{\tau=\pm}\left[-\frac{b_\tau}{3}+\frac{1}{4}\left(-\frac{b_\tau}{3}+b_\tau^3+\frac{1}{2}(1-b_\tau^2)^2\ln\left|\frac{1-b_\tau}{1+b_\tau}\right|\right)\right] \qquad \text{(E14)}
\end{aligned}
$$

$b_\pm=\frac{Q}{2}\pm z$, $\frac{1}{Q}(b_-^3+b_+^3)=\frac{Q^2}{4}+3z^2$

$$
\begin{aligned}
\Re\tilde{G}^T_{\omega,\boldsymbol{q}} &= \frac{n_s mk_F}{4\pi^2 Q}\frac{\hbar^2k_F^2}{m^2}\sum_\tau\left[-\frac{5b_\tau}{12}+\frac{b_\tau^3}{4}-\frac{(1-b_\tau^2)^2}{4}(b_\tau+\frac{1}{3}b_\tau^3)+\mathcal{O}(b^5)\right]\\
&\approx \frac{n_s mk_F}{4\pi^2 Q}\frac{\hbar^2k_F^2}{m^2}\sum_\tau\left[-\frac{5b_\tau}{12}+\frac{b_\tau^3}{4}-\frac{1+b_\tau^4-2b_\tau^2}{4}(b_\tau+\frac{1}{3}b_\tau^3)\right]\\
&\approx \frac{n_s mk_F}{4\pi^2 Q}\frac{\hbar^2k_F^2}{m^2}\sum_\tau\left[-\frac{5b_\tau}{12}+\frac{b_\tau^3}{4}-\frac{b_\tau-2b_\tau^3+\frac{1}{3}b_\tau^3}{4}\right]\\
&= \frac{n_s mk_F}{4\pi^2 Q}\frac{\hbar^2k_F^2}{m^2}\sum_\tau\left(-\frac{8b_\tau}{12}+\frac{8b_\tau^3}{12}\right)\\
&= \frac{n_s mk_F}{4\pi^2}\frac{\hbar^2k_F^2}{m^2}\frac{2}{3}\left(-1+\frac{Q^2}{4}+3z^2\right) \qquad \text{(E15)}
\end{aligned}
$$

Imaginary energy: $z=iw$, $b=\frac{Q}{2}+iw$

$$
\begin{aligned}
\Re\tilde{G}^T_{i\omega,\boldsymbol{q}} &= \frac{n_s mk_F}{4\pi^2 Q}\frac{\hbar^2k_F^2}{m^2}\left[-\frac{2Q}{3}+\frac{1}{4}\left(-\frac{2Q}{3}+\frac{Q^3}{4}-3Qw^2+\left[\left(1-\frac{Q^2}{4}+w^2\right)^2-Q^2w^2\right]\ln\frac{1+\frac{Q^2}{4}-Q+w^2}{1+\frac{Q^2}{4}+Q+w^2}\right)\right]\\
&= \frac{n_s mk_F}{4\pi^2}\frac{\hbar^2k_F^2}{4m^2}\left[-\frac{10}{3}+\frac{Q^2}{4}-3Qw^2+\left[\left(1-\frac{Q^2}{4}+w^2\right)^2-Q^2w^2\right]\ln\frac{1+\frac{Q^2}{4}-Q+w^2}{1+\frac{Q^2}{4}+Q+w^2}\right]\\
&\approx \frac{n_s mk_F}{4\pi^2}\frac{\hbar^2k_F^2}{m^2}\frac{2}{3}\left(-1+\frac{Q^2}{4}-3z^2\right) \qquad \text{(E16)}
\end{aligned}
$$

Clasical limit $Q \to 0$: $C^{tt} = 1$, $C^T = \frac{\hbar^2 k_F^2}{m^2}$, $c_0^{tt} = 1$, $c_0^T = -\frac{2}{3}$

$$\begin{aligned} b_+^{2n+1} + b_-^{2n+1} &= (\frac{1}{2}Q + z)^{2n+1} + (\frac{1}{2}Q - z)^{2n+1} \to (2n+1)Qz^{2n} \\ L_{\omega,\boldsymbol{q}}^X &= -\frac{n_s m k_F}{4\pi^2} C^X \sum_{n=0}^{\infty} c_n^X z^{2n} \end{aligned} \tag{E17}$$

1. Imaginary part

$$F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q}) = \begin{pmatrix} 1 & \mp\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \\ \mp\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) \end{pmatrix} \tag{E18}$$

$$\begin{aligned} R_q^{\mu\nu\pm} &= n_s\pi\hbar \int_{\boldsymbol{k}} \delta(\pm\omega - \omega_{\boldsymbol{k}+\boldsymbol{q}} + \omega_{\boldsymbol{k}}) n_{\boldsymbol{k}} (1 + \xi n_{\boldsymbol{k}+\boldsymbol{q}}) F^{\pm\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\ &= n_s\pi\hbar \int_{\boldsymbol{k}} \delta(\pm\omega - \epsilon_{\boldsymbol{k}+\boldsymbol{q}} + \epsilon_{\boldsymbol{k}}) n_{\boldsymbol{k}} (1 - n_{\boldsymbol{k}+\boldsymbol{q}}) F^{\pm\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\ &= \frac{n_s\hbar}{8\pi^2} \int_{k<k_F} d^3k \Theta(|\boldsymbol{k}+\boldsymbol{q}| - k_F) \delta\left(\pm\omega - \frac{\hbar \boldsymbol{q}\boldsymbol{k}}{m} - \frac{\hbar q^2}{2m}\right) F^{\pm\mu\nu}(\boldsymbol{k},\boldsymbol{q}) \\ &= \frac{n_s m}{8\pi^2 q} \int_{k<k_F} d^2k \Theta(|\boldsymbol{k}-\boldsymbol{q}| - k_F) F^{\pm\mu\nu}(-\boldsymbol{k},\boldsymbol{q})_{|k_3 = \mp\frac{m\omega}{\hbar q} + \frac{q}{2}} \end{aligned} \tag{E19}$$

$$\begin{aligned} (a): \ q > 2k_F \quad -k_F &< \mp\frac{\omega m}{\hbar q} + \frac{q}{2} < k_F \\ -1 &< \mp z + \frac{q}{2k_F} < 1 \\ (b): \ q < 2k_F \quad -k_F &< \mp\frac{\omega m}{\hbar q} + \frac{q}{2} < q - k_F \\ -1 &< \mp z + \frac{q}{2k_F} < \frac{q}{k_F} - 1 \\ (c): \ q < 2k_F \quad q - k_F &< \mp\frac{\omega m}{\hbar q} + \frac{q}{2} < \frac{q}{2} \\ \frac{q}{k_F} - 1 &< \mp z + \frac{q}{2k_F} < \frac{q}{2k_F} \end{aligned} \tag{E20}$$

$$R_{\omega,\boldsymbol{q}}^{tt\pm} = \frac{n_s m}{8\pi^2 q} \int_{k_1}^{k_2} 2\pi dk k = \frac{n_s m}{8\pi q}(q_2^2 - q_1^2) \tag{E21}$$

(a,b): $k_2 = \sqrt{k_F^2 - (\frac{q}{2} \mp \frac{m\omega}{\hbar q})^2}$, $k_1 = 0$, (c): $k_2 = \sqrt{k_F^2 - (\frac{q}{2} \mp \frac{m\omega}{\hbar q})^2}$, $k_1 = \sqrt{k_F^2 - (\frac{q}{2} \pm \frac{m\omega}{\hbar q})^2}$

$$R_{\omega,\boldsymbol{q}}^{tt\pm} = \frac{n_s m k_F^2}{8\pi q} \begin{cases} 1 - (\frac{q}{2k_F} \mp z)^2 & \frac{q}{k_F} > 2, \quad -1 - \frac{q}{2k_F} < \mp z < 1 - \frac{q}{2k_F} \\ 1 - (\frac{q}{2k_F} \mp z)^2 & \frac{q}{k_F} < 2, \quad -1 - \frac{q}{2k_F} < \mp z < -1 + \frac{q}{2k_F} \\ \pm 2z\frac{q}{k_F} & \frac{q}{k_F} < 2, \quad 0 < \pm z < 1 - \frac{q}{2k_F} \end{cases} \tag{E22}$$

$$\begin{aligned} \mathrm{Tr}(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}) \otimes (\boldsymbol{k}+\frac{\boldsymbol{q}}{2}) &= k^2 + \frac{q^2}{4} + \boldsymbol{k}\boldsymbol{q} \\ \boldsymbol{n}(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}) \otimes (\boldsymbol{k}+\frac{\boldsymbol{q}}{2})\boldsymbol{n} &= (\boldsymbol{n}\boldsymbol{k})^2 + \frac{q^2}{4} + \boldsymbol{k}\boldsymbol{q} \\ (\boldsymbol{k}+\frac{\boldsymbol{q}}{2}) \otimes (\boldsymbol{k}+\frac{\boldsymbol{q}}{2})^T &= k^2 - (\boldsymbol{n}\boldsymbol{k})^2 = k^2 - k_3^2 \end{aligned} \tag{E23}$$

$$R^{T\pm}_{\omega,\boldsymbol{q}} = \frac{n_s\hbar^2}{8\pi^2 qm}\int_{k_1}^{k_2} 2\pi dk k^3 = \frac{n_s\hbar^2}{16\pi qm}(q_2^4 - k_1^4)$$
$$= \frac{n_s\hbar^2 k_F^4}{16\pi qm}\begin{cases}[1-(\frac{q}{2k_F}\mp z)^2]^2 & \frac{q}{k_F}>2, \quad -1-\frac{q}{2k_F}<\mp z<1-\frac{q}{2k_F}\\ [1-(\frac{q}{2k_F}\mp z)^2]^2 & \frac{q}{k_F}<2, \quad -1-\frac{q}{2k_F}<\mp z<-1+\frac{q}{2k_F}\\ \mp z\frac{q}{k_F}[\frac{q^2}{k_F^2}+4(z^2-1)] & \frac{q}{k_F}<2, \quad 0<\pm z<1-\frac{q}{2k_F}\end{cases} \tag{E24}$$

because

$$[1-(\frac{q}{2k_F}\pm z)^2]^2 - [1-(\frac{q}{2k_F}\mp z)^2]^2 = (\frac{q}{2k_F}\pm z)^4 - (\frac{q}{2k_F}\mp z)^4 - 4\frac{q}{k_F}(\pm z) = \pm z\frac{q}{k_F}\left[\frac{q^2}{k_F^2}+4(z^2-1)\right] \tag{E25}$$

2. Flow with boundary

$$L^{(\mu\nu)++}_{\omega,\ell,\boldsymbol{q}_\perp} = \frac{n_s\hbar}{L}\sum_j P\int_{\boldsymbol{k}_\perp} n_{\boldsymbol{k}}\left[\frac{F^{\mu\nu+}(\boldsymbol{k},\boldsymbol{q})}{\omega-\frac{\hbar(2\pi)^2}{2mL^2}\ell^2-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}_\perp\boldsymbol{q}_\perp}{m}-\frac{\hbar(2\pi)^2 j\ell}{L^2 m}} + \frac{F^{\mu\nu-}(\boldsymbol{k},\boldsymbol{q})}{-\omega-\frac{\hbar(2\pi)^2}{2mL^2}\ell^2-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{k}_\perp\boldsymbol{q}_\perp}{m}-\frac{\hbar(2\pi)^2 j\ell}{L^2 m}}\right] \tag{E26}$$

$z = \omega m/\hbar\bar{k}q$

$$\int\frac{d^2k}{\boldsymbol{qk}+c} = \frac{(\boldsymbol{qk}+c)\ln(\boldsymbol{qk}+c)-\frac{1}{2}\boldsymbol{qk}}{q_1q_2}$$
$$\int\frac{dk_1}{\boldsymbol{qk}+c} = \frac{q_2\ln(\boldsymbol{qk}+c)+q_2-\frac{1}{2}q_2}{q_1q_2}$$
$$\frac{1}{\boldsymbol{qk}+c} = \frac{q_1q_2}{q_1q_2(\boldsymbol{qk}+c)} \tag{E27}$$

$$L^{tt}_{\omega,\ell,\boldsymbol{q}} = \frac{n_s\hbar}{4\pi^2 L}\sum_j P\int d^2k n_{\boldsymbol{k}}\frac{1}{\omega-\frac{\hbar(2\pi)^2}{2mL^2}\ell^2-\frac{\hbar q^2}{2m}-\frac{\hbar(k_1q_1+k_2q_2)}{m}-\frac{\hbar(2\pi)^2 j\ell}{L^2m}} + (\omega\to-\omega)$$
$$= \frac{n_s\hbar}{4\pi^2 L}\sum_j P\frac{\frac{\hbar\boldsymbol{kq}}{2m}+[\omega-\frac{\hbar(2\pi)^2}{2mL^2}(\ell^2+2j\ell)-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}]\ln[\omega-\frac{\hbar(2\pi)^2}{2mL^2}(\ell^2+2j\ell)-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{kq}}{m}]}{\frac{\hbar^2 q_1q_2}{m^2}} + (\omega\to-\omega)$$
$$= \frac{n_s\hbar}{4\pi^2 L}\sum_j P\int dkkd\phi n_k\frac{1}{\omega-\frac{\hbar(2\pi)^2}{2mL^2}\ell^2-\frac{\hbar q^2}{2m}-\frac{\hbar kq\cos\phi}{m}-\frac{\hbar(2\pi)^2 j\ell}{L^2m}} + (\omega\to-\omega) \tag{E28}$$

Appendix F: Both statics, arbitrary temperature

1. Infinite lifetime, Lindhart function

$$\Re\tilde{G}^{\mu\nu}_{\omega,\boldsymbol{q}} = L^{\mu\nu}_q = n_s\hbar P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\frac{\begin{pmatrix}1 & -\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\\ -\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\end{pmatrix}}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar\boldsymbol{qk}}{m}} + (\omega\to-\omega)$$
$$= \frac{n_s m}{q\bar{k}}P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\frac{\begin{pmatrix}1 & -\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\\ -\frac{\hbar}{m}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\end{pmatrix}}{z-\frac{q}{2k}-\frac{\boldsymbol{nk}}{k}} + (z\to-z) \tag{F1}$$

$$
\begin{aligned}
L^{tt}_{\omega,\boldsymbol{q}} &= \frac{n_s\hbar}{4\pi^2}P\int_0^\infty dkk^2 n_k \int_{-1}^1 dC \frac{1}{\omega-\frac{\hbar q^2}{2m}-\frac{\hbar kqC}{m}} + (\omega\to-\omega)\\
&= -\frac{n_s m}{4\pi^2 q}P\int_0^\infty dkkn_k \ln\left(\omega-\frac{\hbar q^2}{2m}-\frac{\hbar kqC}{m}\right)_{-1}^{1} + (\omega\to-\omega)\\
&= -\frac{n_s m}{4\pi^2 q}P\int_0^\infty dkkn_k \ln\left|\frac{2m\omega-\hbar q^2-2\hbar kq}{2m\omega-\hbar q^2+2\hbar kq}\right| + (\omega\to-\omega)\\
&= -\frac{n_s m}{4\pi^2 q}P\int_0^\infty dkkn_k \ln\left|\frac{k-\frac{m\omega}{\hbar q}+\frac{q}{2}}{k+\frac{m\omega}{\hbar q}-\frac{q}{2}}\right| + (\omega\to-\omega)\\
&= -\frac{n_s m}{4\pi^2 q}P\int_0^\infty dkkn_k \ln\left|\frac{k-a}{k+a}\right| + (\omega\to-\omega)\\
&= \mathrm{sign}(a)\frac{n_s m}{4\pi^2 q}P\int_0^\infty dkkn_k \ln\frac{k+a}{|k-a|} + (\omega\to-\omega)
\end{aligned}
\tag{F2}
$$

$$
\begin{aligned}
I &= \int_0^\infty dkf_k \ln\frac{k+a}{|k-a|} = \int_0^a dkf_k\ln\frac{k+a}{a-k} + \int_a^\infty dkf_k\ln\frac{k+a}{k-a}\\
&= \int_0^a dkf_k\ln\frac{1+\frac{k}{a}}{1-\frac{k}{a}} + \int_a^\infty dkf_k\ln\frac{1+\frac{a}{k}}{1+\frac{a}{k}}\\
&\approx \int_0^a dkf_k\left(2\frac{k}{a}-\frac{2k^3}{3a^3}\right) + \int_a^\infty dkf_k\left(2\frac{a}{k}-\frac{2a^3}{3k^3}\right)
\end{aligned}
\tag{F3}
$$

for $a>0$

$$
\begin{aligned}
L^{tt}_{\omega,\boldsymbol{q}} &= \mathrm{sign}(a)\frac{n_s m}{2\pi^2 q}\left(\frac{1}{a}\int_0^a dkk^2 n_k + a\int_a^\infty dkn_k\right) + (\omega\to-\omega)\\
&\approx \mathrm{sign}(a)\frac{n_s m}{2\pi^2 q}\left(\frac{a^3}{3a} + a\int_0^\infty dkn_k - a^2\right) + (\omega\to-\omega)\\
&= |a|\frac{n_s m}{2\pi^2 q}\left(\int_0^\infty dkn_k - \frac{2a}{3}\right) + (\omega\to-\omega)\\
&\approx \left|\frac{\hbar\omega}{\frac{\hbar^2 q^2}{2m}}-1\right|\frac{n_s m}{4\pi^2}\int_0^\infty dkn_k + (\omega\to-\omega)
\end{aligned}
\tag{F4}
$$

$$
L^s_q = \frac{n_s m}{q\bar{k}}P\int_{\boldsymbol{k}} n_{\boldsymbol{k}} \frac{\frac{\hbar^2}{m^2}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)}{z-\frac{q}{2\bar{k}}-\frac{\boldsymbol{nk}}{\bar{k}}} + (z\to-z)
\tag{F5}
$$

$$
\begin{aligned}
L^{tt}_{\boldsymbol{q}} &= 2n_s\hbar P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\frac{1}{\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}+\boldsymbol{q}}}\\
&= -4n_s mP\int_{\boldsymbol{k}}\frac{n_{\boldsymbol{k}}}{q^2+2\boldsymbol{kq}}\\
&= -n_s\frac{m}{\pi^2}P\int dkk^2 dC\frac{n_k}{q^2+2kqC}\\
&= -n_s\frac{m}{\pi^2}\int dkk^2 n_k\frac{\ln\frac{q^2+2kq}{|q^2-2kq|}}{2kq}\\
&= -\frac{n_s m}{2\pi^2 q}\int dkkn_k\ln\frac{q+2k}{|q-2k|}
\end{aligned}
\tag{F6}
$$

$$\begin{aligned}
L^{tt}_{\boldsymbol{q}} &\approx -\frac{n_s m}{2\pi^2 q}\int_0^{q/2} dk k n_k \left(\frac{4k}{q}-\frac{16k^3}{q^3}\right) - \frac{n_s m}{2\pi^2 q}\int_{q/2}^{\infty} dk k n_k\left(\frac{q}{k}-\frac{q^3}{12k^3}\right)\\
&\approx -\frac{2n_s m}{\pi^2 q^2}\int_0^{q/2} dk k^2 n_k - \frac{n_s m}{2\pi^2}\int_{q/2}^{\infty} dk n_k\\
&\approx -\frac{n_s m q}{12\pi^2}+\frac{n_s m q}{4\pi^2}-\frac{n_s m}{2\pi^2}\int_0^{\infty} dk n_k = -\frac{n_s m}{2\pi^2}\int_0^{\infty} dk n_k + \frac{n_s m q}{6\pi^2} \qquad \text{(F7)}
\end{aligned}$$

$\boldsymbol{n} = \boldsymbol{q}/q$

$$L^{ss}_{\boldsymbol{q}} = \frac{4n_s\hbar^2}{qm}P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\frac{\boldsymbol{k}\otimes\boldsymbol{k}+\frac{q}{2}\boldsymbol{n}\otimes\boldsymbol{k}+\frac{q}{2}\boldsymbol{k}\otimes\boldsymbol{n}+\frac{q^2}{4}\boldsymbol{n}\otimes\boldsymbol{n}}{q+2\boldsymbol{kn}} \approx \frac{4n_s\hbar^2}{qm}P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\frac{\boldsymbol{k}\otimes\boldsymbol{k}}{q+2\boldsymbol{kn}} \qquad \text{(F8)}$$

$$\begin{aligned}
trL^{ss1}_{\boldsymbol{q}} &= \frac{4n_s\hbar^2}{qm}P\int_{\boldsymbol{k}} n_{\boldsymbol{k}}\frac{k^2}{q+2\boldsymbol{kn}}\\
&= \frac{n_s\hbar^2}{qm\pi^2}P\int_0^{\infty} dk k^4 n_k\int_{-1}^{1} dC\frac{1}{2kC+q}\\
&= \frac{n_s\hbar^2}{2qm\pi^2}P\int_0^{\infty} dk k^3 n_k \ln\left|\frac{2k+q}{2k-q}\right|\\
&\approx \frac{n_s\hbar^2}{2m\pi^2}\int_0^{\infty} dk k^2 n_k = \frac{n_s\hbar^2}{m}\int_{\boldsymbol{k}} n_k \qquad \text{(F9)}
\end{aligned}$$

2. Infinite lifetime, R

$$\begin{aligned}
R^{\mu\nu\pm}_q &= n_s\pi\hbar\int_{\boldsymbol{k}}\delta(\pm\omega-\omega_{\boldsymbol{k}+\boldsymbol{q}}+\omega_{\boldsymbol{k}})n_{\boldsymbol{k}}(n_{\boldsymbol{k}+\boldsymbol{q}}-1)F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})\\
&= n_s\pi\hbar\int_{\boldsymbol{k}}\frac{\delta\left(\pm\omega-\frac{\hbar\boldsymbol{qk}}{m}-\frac{\hbar q^2}{2m}\right)[1+\xi-e^{\beta(\frac{\hbar^2(\boldsymbol{k}+\boldsymbol{q})^2}{2m}-\mu)}]}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)(e^{\beta(\frac{\hbar^2(\boldsymbol{k}+\boldsymbol{q})^2}{2m}-\mu)}-\xi)}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})\\
&= \frac{n_s\pi m}{q}\int_{\boldsymbol{k}}\frac{\delta\left(-k_3-\frac{q}{2}\pm\frac{m\omega}{\hbar q}\right)[1+\xi-e^{\beta(\frac{\hbar^2(\boldsymbol{k}+\boldsymbol{q})^2}{2m}-\mu)}]}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)(e^{\beta(\frac{\hbar^2(\boldsymbol{k}+\boldsymbol{q})^2}{2m}-\mu}-\xi)}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})\\
&= \frac{n_s m}{8\pi^2 q}\int d^2k\frac{F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})[1+\xi-e^{\beta\{\frac{\hbar^2}{2m}[(\pm\frac{m\omega}{\hbar q}+\frac{q}{2})^2+k^2]-\mu)\}}]}{(e^{\beta\{\frac{\hbar^2}{2m}[(\pm\frac{m\omega}{\hbar q}-\frac{q}{2})^2+k^2]-\mu)\}}-\xi)(e^{\beta\{\frac{\hbar^2}{2m}[(\pm\frac{m\omega}{\hbar q}+\frac{q}{2})^2+k^2]-\mu)\}}-\xi)}_{|k_3=\pm\frac{m\omega}{\hbar q}-\frac{q}{2}} \qquad \text{(F10)}
\end{aligned}$$

$$\begin{aligned}
\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)_{|\frac{\boldsymbol{q}}{q}\boldsymbol{k}=\pm\frac{m\omega}{\hbar q}-\frac{q}{2}} &= \boldsymbol{k}_\perp+\frac{\boldsymbol{q}}{q}\left(\pm\frac{m\omega}{\hbar q}-\frac{q}{2}\right)+\frac{\boldsymbol{q}}{2}\\
&= \boldsymbol{k}_\perp\pm\frac{m\omega\boldsymbol{q}}{\hbar q^2}\\
F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})_{|\frac{\boldsymbol{q}}{q}\boldsymbol{k}=\pm\frac{m\omega}{\hbar q}-\frac{q}{2}} &= \begin{pmatrix}1 & \mp\bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\\ \mp\bar{r}\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right) & \bar{r}^2\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2}\right)\end{pmatrix}_{|\frac{\boldsymbol{q}}{q}\boldsymbol{k}=\pm\frac{m\omega}{\hbar q}-\frac{q}{2}}\\
&= \begin{pmatrix}1 & \mp\bar{r}\left(\boldsymbol{k}_\perp\pm\frac{m\omega\boldsymbol{q}}{\hbar q^2}\right)\\ \mp\bar{r}\left(\boldsymbol{k}_\perp\pm\frac{m\omega\boldsymbol{q}}{\hbar q^2}\right) & \bar{r}^2\left(\boldsymbol{k}_\perp\pm\frac{m\omega\boldsymbol{q}}{\hbar q^2}\right)\otimes\left(\boldsymbol{k}_\perp\pm\frac{m\omega\boldsymbol{q}}{\hbar q^2}\right)\end{pmatrix} \qquad \text{(F11)}
\end{aligned}$$

$$R_{\boldsymbol{q}}^{\mu\nu\pm} = \frac{n_s m}{8\pi^2 q}\int d^2k \frac{F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})_{|k_3=-\frac{q}{2}}[1+\xi - e^{\beta\{\frac{\hbar^2}{2m}(\frac{q^2}{4}+k^2)-\mu)\}}]}{(e^{\beta\{\frac{\hbar^2}{2m}[\frac{q^2}{4}+k^2]-\mu)\}}-\xi)^2}$$
$$\approx \frac{n_s m}{8\pi^2 q}\int d^2k \frac{\begin{pmatrix}1 & 0\\ 0 & \frac{\hbar^2k^2}{2m^2}(\mathbb{1}-\boldsymbol{n}\otimes\boldsymbol{n})\end{pmatrix}[1+\xi-e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}]}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)^2} \tag{F12}$$

$$R_{\boldsymbol{q}}^{tt\pm} = \frac{n_s m}{4\pi q}\int dkk\frac{1+\xi-e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)^2}$$
$$R_{\boldsymbol{q}}^{T\pm} = \frac{n_s\hbar^2}{8m\pi q}\int dkk^3\frac{1+\xi-e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)^2} \tag{F13}$$

$$\partial_\omega R_{\boldsymbol{q}}^{\mu\nu\pm} = \frac{n_s m}{8\pi^2 q}\int d^2k\frac{\pm\frac{m}{\hbar q}\partial_{k_3}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})_{|k_3=-\frac{q}{2}}[1+\xi-e^{\beta\{\frac{\hbar^2}{2m}(\frac{q^2}{4}+k^2)-\mu)\}}]}{(e^{\beta\{\frac{\hbar^2}{2m}[\frac{q^2}{4}+k^2]-\mu)\}}-\xi)^2}$$
$$-\frac{n_s m}{8\pi^2 q}\int d^2k\frac{\pm\frac{m}{\hbar}\beta\frac{\hbar^2}{2m}e^{\beta\{\frac{\hbar^2}{2m}(\frac{q^2}{4}+k^2)-\mu)\}}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})_{|k_3=-\frac{q}{2}}}{(e^{\beta\{\frac{\hbar^2}{2m}[\frac{q^2}{4}+k^2]-\mu)\}}-\xi)^2}$$
$$= \pm\frac{n_s m}{8\pi^2 q}\int d^2k\left[\frac{\frac{m}{\hbar q}\partial_{k_3}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})[1+\xi-e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}]}{(e^{\beta\{\frac{\hbar^2}{2m}[\frac{q^2}{4}+k^2]-\mu)\}}-\xi)^2}-\frac{\beta\hbar e^{\beta\{\frac{\hbar^2}{2m}(\frac{q^2}{4}+k^2)-\mu)\}}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q})}{2(e^{\beta\{\frac{\hbar^2}{2m}[\frac{q^2}{4}+k^2]-\mu)\}}-\xi)^2}\right]_{|k_3=-\frac{q}{2}} \tag{F14}$$

$$\partial_{k_3}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{q}) = \begin{pmatrix}0 & \mp\frac{\hbar}{m}\boldsymbol{n}\\ \mp\frac{\hbar}{m}\boldsymbol{n} & \frac{\hbar^2}{m^2}(\boldsymbol{k}\otimes\boldsymbol{n}+\boldsymbol{n}\otimes\boldsymbol{k})\end{pmatrix} \tag{F15}$$

For $\mu\nu = tt$ or ss

$$\partial_\omega R_{\boldsymbol{q}}^{\mu\nu\pm} \to \mp\frac{n_s m\hbar}{8\pi^2 q k_B T}\int d^2k\frac{e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{0})}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)^2}$$
$$= \pm\frac{n_s m\xi\hbar}{8\pi^2 q k_B T}\int d^2k\frac{F^{\mu\nu\pm}(\boldsymbol{k},\boldsymbol{0})}{(e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)(e^{-\beta(\frac{\hbar^2k^2}{2m}-\mu)}-\xi)}$$
$$= \pm\frac{n_s m\xi\hbar}{4\pi q k_B T}\int_0^\infty dkk\frac{\begin{pmatrix}1 & 0\\ 0 & \frac{\hbar^2k^2}{2m^2}(\mathbb{1}-\boldsymbol{n}\otimes\boldsymbol{n})\end{pmatrix}}{2-\xi[e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}+e^{-\beta(\frac{\hbar^2k^2}{2m}-\mu)}]} \tag{F16}$$

$$R_{\omega,\boldsymbol{q}}^{\mu\nu\pm} = R_{\boldsymbol{q}}^{\mu\nu\pm}+R_1^{\mu\nu\pm}\frac{\omega}{q}$$
$$R_1^{tt\pm} = \pm\frac{n_s m\xi\hbar}{4\pi k_B T}\int_0^\infty dkk\frac{1}{2-\xi[e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}+e^{-\beta(\frac{\hbar^2k^2}{2m}-\mu)}]}$$
$$R_1^{T\pm} = \pm\frac{n_s\xi\hbar^2}{8\pi m k_B T}\int_0^\infty dkk^3\frac{1}{2-\xi[e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}+e^{-\beta(\frac{\hbar^2k^2}{2m}-\mu)}]} \tag{F17}$$

$$\tilde{G}_{\omega,q^2}^{(X)} = \tilde{G}_0^{(X)}\left[1+\frac{\bar{k}}{q}\mathcal{O}\left(\frac{m\omega}{\hbar q^2}\right)+\mathcal{O}\left(\frac{q}{\bar{k}}\right)\right]+\tilde{G}_1^{(X)}\frac{\omega}{q}\left[1+\frac{\bar{k}}{q}\mathcal{O}\left(\frac{m\omega}{\hbar q^2}\right)+\mathcal{O}\left(\frac{q}{\bar{k}}\right)\right] \tag{F18}$$

$$\tilde{G}_0^{tt} = -\frac{n_s\xi m}{2\pi^2}\int_0^\infty dkn_k, \qquad \tilde{G}_1^{tt} = \frac{n_s m\hbar}{2\pi k_B T}\int_0^\infty dkk\frac{1}{2-\xi[e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}+e^{-\beta(\frac{\hbar^2k^2}{2m}-\mu)}]},$$
$$\tilde{G}_0^{T} = -\xi m\bar{r}^2 n_0, \qquad \tilde{G}_1^{T} = \frac{n_s m\hbar\bar{r}^2}{4\pi k_B T}\int_0^\infty dkk^3\frac{1}{2-\xi[e^{\beta(\frac{\hbar^2k^2}{2m}-\mu)}+e^{-\beta(\frac{\hbar^2k^2}{2m}-\mu)}]}, \tag{F19}$$

3. Summary

$\tilde{G}^{\mu\nu++}_{\omega,\boldsymbol{q}} = L_{\omega,\boldsymbol{q}} - iR^{+}_{\omega,\boldsymbol{q}} - iR^{-}_{\omega,\boldsymbol{q}}$, $G^{\mu\nu r}_{q} = -L^{\mu\nu}_{q} + iR^{\mu\nu+}_{q} - iR^{\mu\nu+}_{-q}$
$T=0$: $z = \omega m/\hbar\bar{k}q$, $\bar{k} = k_F$, $Q = q/\bar{k}$, $b_\pm = \frac{Q}{2} \pm z$, $w = \frac{2m\omega}{\hbar q^2} = \frac{2z}{Q}$, $b_\pm = \frac{Q}{2}(1\pm w)$, $\mu = \hbar^2 k_F^2/2m$, $k_F = \sqrt{2m\mu}/\hbar$

$$
\begin{aligned}
L^{tt}_{\omega,\boldsymbol{q}} &= \frac{n_s m k_F}{4\pi^2}\sum_{\tau=\pm}\left(-\frac{1}{2} + \frac{R}{2}(1-b_\tau^2)\ln\left|\frac{1-b_\tau}{1+b_\tau}\right|\right) \to \frac{n_s m k_F}{4\pi^2}\left(-2 + \frac{Q^2}{6}\right)\\
L^{T}_{\omega,\boldsymbol{q}} &= \frac{n_s m k_F}{4\pi^2 Q}\frac{\hbar^2 k_F^2}{m^2}\sum_{\tau=\pm}\left(-\frac{5b_\tau}{12} + \frac{b_\tau^3}{4} + \frac{1}{8}(1-b_\tau^2)^2\ln\frac{1-b_\tau}{1+b_\tau}\right) \to \frac{n_s m k_F}{4\pi^2}\frac{\hbar^2 k_F^2}{m^2}\left(-\frac{2}{3} + \frac{13Q^2}{96}\right)\\
R^{tt\pm}_{\omega,\boldsymbol{q}} &= \frac{n_s m k_F}{8\pi Q}\begin{cases} 1-(\frac{Q}{2}\mp z)^2 & Q>2, \quad -1-\frac{Q}{2} < \mp z < 1-\frac{Q}{2}\\ 1-(\frac{Q}{2}\mp z)^2 & Q<2, \quad -1-\frac{Q}{2} < \mp z < -1+\frac{Q}{2}\\ \pm 2zQ & Q<2, \quad 0<\pm z<1-\frac{Q}{2}\end{cases} \to \pm\frac{n_s m k_F}{4\pi}\Theta(\pm z)z\\
R^{T\pm}_{\omega,\boldsymbol{q}} &= \frac{n_s m k_F}{16\pi Q}\frac{\hbar^2 k_F^2}{m^2}\begin{cases} [1-(\frac{Q}{2}\mp z)^2]^2 & Q>2, \ -1-\frac{Q}{2} < \mp z < 1-\frac{Q}{2}\\ [1-(\frac{Q}{2}\mp z)^2]^2 & Q<2, \ -1-\frac{Q}{2} < \mp z < -1+\frac{Q}{2}\\ \mp zQ[Q^2+4(z^2-1)] & Q<2, \ 0<\pm z<1-\frac{Q}{2}\end{cases} \to \pm\frac{n_s m k_F}{4\pi}\frac{\hbar^2 k_F^2}{m^2}\Theta(\pm z)z
\end{aligned}
\tag{F20}
$$

$T>0$: $\bar{k} = \sqrt{\frac{mk_BT}{2\pi\hbar^2}} = K_T$, $z = \omega m/\hbar\bar{k}q$

$$
\begin{aligned}
L^{tt}_{\omega,\boldsymbol{q}} &= -\frac{n_s m\bar{k}}{2\pi Q}\left[I^{tt}\left(z-\frac{Q}{2}\right) - I^{tt}\left(-z+\frac{Q}{2}\right)\right] + (z\to -z)\\
L^{T}_{\omega,\boldsymbol{q}} &= \frac{n_s m\bar{k}}{2\pi Q}\frac{\hbar^2\bar{k}^2}{m^2}\left[2\sqrt{\pi}\left(z-\frac{Q}{2}\right)I_1^T + \frac{1}{2}I_2^T\left(z-\frac{Q}{2}\right) - \frac{1}{2}\left(z-\frac{Q}{2}\right)^2 I^{tt}\left(z-\frac{Q}{2}\right)\right] + (z\to -z)\\
I^{tt}(u) &= P\int_0^\infty dx\frac{\ln|\sqrt{4\pi x}-u|}{e^{x-\beta\mu}+1}\\
I_1^T &= \int_0^\infty \frac{dx\sqrt{x}}{e^{x-\beta\mu}+1}\\
I_2^T(u) &= \int_0^\infty \frac{dx 4\pi x}{e^{x-\beta\mu}+1}\ln|\sqrt{4\pi x}-u|\\
R^{tt+}_{q} &= \frac{n_s m\bar{k}}{2Q}\int_0^\infty \frac{dx}{(e^{x+\frac{1}{4\pi}(z-\frac{Q}{2})^2-\beta\mu}+1)(e^{-x-\frac{1}{4\pi}(z+\frac{Q}{2})^2+\beta\mu}+1)}\\
R^{T+}_{q} &= \frac{n_s\pi m\bar{k}}{Q}\frac{\hbar^2\bar{k}^2}{m^2}\int \frac{dxx}{(e^{x+\frac{1}{4\pi}(z-\frac{Q}{2})^2-\beta\mu}+1)(e^{-x-\frac{1}{4\pi}(z+\frac{Q}{2})^2+\beta\mu}+1)}
\end{aligned}
\tag{F21}
$$

Appendix G: Classical case

$p_F = \hbar k_F = \mathcal{O}(\hbar^0)$, $k_F = \mathcal{O}(\hbar^{-1})$, $\hbar\omega_p = \frac{\boldsymbol{p}^2}{2m} = E(\boldsymbol{p})$, $n_E = \frac{1}{e^{\beta(E-\mu)}+1}$

$$
\begin{aligned}
\Re\tilde{G}^{(\mu\nu)++}_{\omega,\boldsymbol{q}} &= n_s\hbar P\int\frac{d^3k}{(2\pi)^3}\frac{n_{\boldsymbol{k}} - n_{\boldsymbol{k}+\boldsymbol{q}}}{\omega - \omega_{\boldsymbol{k}+\boldsymbol{q}} + \omega_{\boldsymbol{k}}}F^{\mu\nu}(\boldsymbol{k},\boldsymbol{q})\\
&= n_s\hbar^2 P\int\frac{d^3p}{(2\pi\hbar)^3}\frac{n_{\boldsymbol{p}} - n_{\boldsymbol{p}+\hbar\boldsymbol{q}}}{\hbar\omega - E(\boldsymbol{p}+\hbar\boldsymbol{q}) + E(\boldsymbol{p})}\begin{pmatrix} 1 & \mp\frac{1}{m}\left(\boldsymbol{p}+\frac{\hbar\boldsymbol{q}}{2}\right)\\ \mp\frac{1}{m}\left(\boldsymbol{p}+\frac{\hbar\boldsymbol{q}}{2}\right) & \frac{1}{m^2}\left(\boldsymbol{p}+\frac{\hbar\boldsymbol{q}}{2}\right)\otimes\left(\boldsymbol{p}+\frac{\hbar\boldsymbol{q}}{2}\right)\end{pmatrix}
\end{aligned}
\tag{G1}
$$

$E = \hbar\omega$, $\boldsymbol{k} = \hbar\boldsymbol{q}$, $\boldsymbol{p} = m\boldsymbol{v}$, $\hbar\to 0$:

$$
\Re\tilde{G}^{(\mu\nu)++}_{\omega,\boldsymbol{q}} \to -n_s\hbar^3 P\int\frac{d^3p}{(2\pi\hbar)^3}\frac{\partial n_E}{\partial E}\frac{\frac{\partial E}{\partial\boldsymbol{p}}\boldsymbol{q}}{\hbar(\omega - \boldsymbol{q}\frac{\partial E(\boldsymbol{p})}{\partial\boldsymbol{p}})}\begin{pmatrix} 1 & \mp\frac{1}{m}\boldsymbol{p}\\ \mp\frac{1}{m}\boldsymbol{p} & \frac{1}{m^2}\boldsymbol{p}\otimes\boldsymbol{p}\end{pmatrix}
\tag{G2}
$$

$T \to 0,\ \frac{\partial n_E}{\partial E} \to -\delta(E - E_F)$

$$\begin{aligned}
\Re\tilde{G}^{(\mu\nu)++}_{\omega,\boldsymbol{k}} &\to n_s\hbar^2 P\int \frac{d^3p}{(2\pi\hbar)^3}\delta(\mu - E(\boldsymbol{p}))\frac{\boldsymbol{vq}}{\omega - \boldsymbol{qv}}\begin{pmatrix} 1 & \mp\boldsymbol{v} \\ \mp\boldsymbol{v} & \boldsymbol{v}\otimes\boldsymbol{v}\end{pmatrix} \\
&= 2mn_s\hbar^2 P\int \frac{d^3p}{(2\pi\hbar)^3}\delta(2m\mu - \boldsymbol{p}^2)\frac{\boldsymbol{vq}}{\omega - \boldsymbol{qv}}\begin{pmatrix} 1 & \mp\boldsymbol{v} \\ \mp\boldsymbol{v} & \boldsymbol{v}\otimes\boldsymbol{v}\end{pmatrix} \\
&= \frac{2mn_s}{(2\pi)^3\hbar}P\int_{|\boldsymbol{p}|=\sqrt{2m\mu}} d^2p\frac{\boldsymbol{vq}}{\omega - \boldsymbol{qv}}\begin{pmatrix} 1 & \mp\boldsymbol{v} \\ \mp\boldsymbol{v} & \boldsymbol{v}\otimes\boldsymbol{v}\end{pmatrix} \\
&= \frac{2m^2n_s}{(2\pi)^3\hbar}P\int_{|\boldsymbol{v}|=\sqrt{\frac{2\mu}{m}}} d^2v\frac{\boldsymbol{vq}}{\omega - \boldsymbol{qv}}\begin{pmatrix} 1 & \mp\boldsymbol{v} \\ \mp\boldsymbol{v} & \boldsymbol{v}\otimes\boldsymbol{v}\end{pmatrix}
\end{aligned} \tag{G3}$$

Appendix H: Poiseuille flow

$$\begin{aligned}
\bar{a}^0_{\boldsymbol{x}} &= \frac{u_d}{(2\pi\ell^2)^{3/2}}e^{-\frac{x^2}{2\ell^2}} \\
\bar{\boldsymbol{a}}^T_{\boldsymbol{x}} &= \boldsymbol{z}\frac{u_c}{(2\pi\ell^2)^{3/2}}e^{-\frac{x^2}{2\ell^2}}
\end{aligned} \tag{H1}$$

$$\begin{aligned}
\bar{a}^0_{\boldsymbol{q}} &= \frac{u_d\ell^3(2\pi)^{3/2}}{(2\pi\ell^2)^{3/2}}e^{-\frac{q^2\ell^2}{2}} = u_d e^{-\frac{q^2\ell^2}{2}} \\
\bar{\boldsymbol{a}}^T_{\boldsymbol{q}} &= \boldsymbol{z}u_c e^{-\frac{q^2\ell^2}{2}}
\end{aligned} \tag{H2}$$

$$\begin{aligned}
n_{\boldsymbol{k}} &= \frac{n_smk_F}{4\pi^2}\left[-1 + \left(\frac{1}{Q} - \frac{Q}{4}\right)\ln\left|\frac{2-Q}{2+Q}\right|\right]a^0_{\boldsymbol{k}}, \\
&\approx -\frac{n_smk_F}{4\pi^2}2e^{-\frac{Q^2}{6}}\bar{a}^0 = -\frac{n_smk_F}{4\pi^2}2e^{-\frac{Q^2}{6}}u_d e^{-\frac{q^2\ell^2}{2}} = -2u_d\frac{n_smk_F}{4\pi^2}e^{-\frac{k^2}{2}(\frac{1}{3k_F^2}+\ell^2)} = -2u_d\frac{n_smk_F}{4\pi^2}e^{-\frac{k^2}{2}(\xi_F^2+\ell^2)}, \\
\boldsymbol{j}^T_{\boldsymbol{k}} &= \frac{n_smk_F}{4\pi^2}\frac{\hbar^2k_F^2}{m^2}\left[-\frac{5}{12} + \frac{Q^2}{16} + \frac{1}{Q}\left(1 - \frac{Q^2}{4}\right)^2\ln\left|\frac{2-Q}{2+Q}\right|\right]\bar{\boldsymbol{a}}^T_{\boldsymbol{k}} \\
&\approx -\boldsymbol{z}\frac{n_smk_F}{4\pi^2}\frac{\hbar^2k_F^2}{m^2}\frac{2}{3}e^{-\frac{Q^2}{3}}\bar{a}^T_0 = -\boldsymbol{z}\frac{2}{3}u_c\frac{n_smk_F}{4\pi^2}\frac{\hbar^2k_F^2}{m^2}e^{-\frac{k^2}{2}(\frac{1}{3k_F^2}+\ell^2)} = -\boldsymbol{z}\frac{2}{3}u_c\frac{n_smk_F}{4\pi^2}\frac{\hbar^2k_F^2}{m^2}e^{-\frac{k^2}{2}(\xi_F^2+\ell^2)}.
\end{aligned} \tag{H3}$$

$\xi_F = 1/\sqrt{3}k_F$

$$\begin{aligned}
n_x &= -\frac{n_smk_F}{4\pi^2}\frac{2u_d}{[2\pi(\xi_F^2+\ell^2)]^{3/2}}e^{-\frac{x^2}{2(\xi_F^2+\ell^2)}}, \\
\boldsymbol{j}^T_x &= -\boldsymbol{z}\frac{2}{3}\frac{n_smk_F}{4\pi^2}\frac{\hbar^2k_F^2}{m^2}\frac{u_c}{[2\pi(\xi_F^2+\ell^2)]^{3/2}}e^{-\frac{k^2}{2(\xi_F^2+\ell^2)}}.
\end{aligned} \tag{H4}$$

Appendix I: Spread of the wave-packet

$$\psi_{t,x} = \int\frac{dk}{2\pi}e^{-\frac{1}{2}k^2\ell^2 - i\frac{\hbar k^2t}{2m}+ikx} = \int\frac{dk}{2\pi}e^{-\frac{1}{2}k^2(\ell^2+i\frac{\hbar t}{2m})+ikx} = \frac{e^{-\frac{1}{2}x^2\frac{\ell^2-i\frac{\hbar t}{2m}}{\ell^4+\frac{\hbar^2t^2}{4m^2}}}}{\sqrt{2\pi(\ell^2+i\frac{\hbar t}{2m})}} \tag{I1}$$

$\ell^4(t) = \ell^4 + \frac{\hbar^2 t^2}{4m^2}$

$$
\begin{aligned}
\psi_{t,x,y} &= e^{-\frac{\ell^2 - i\frac{\hbar t}{2m}}{2\ell^4(t)}[(x-a)^2+(y+a)^2]} + \sigma(a \to -a) \\
|\psi_{t,x,y}|^2 &= e^{-\frac{\ell^2}{\ell^4(t)}[(x-a)^2+(y+a)^2]} + e^{-\frac{\ell^2}{\ell^4(t)}[(x+a)^2+(y-a)^2]} + 2\sigma\Re e^{-\frac{\ell^2 - i\frac{\hbar t}{2m}}{2\ell^4(t)}[(x-a)^2+(y+a)^2] - \frac{\ell^2 + i\frac{\hbar t}{2m}}{2\ell^4(t)}[(x+a)^2+(y-a)^2]} \\
n_{t,z} &= \int_x |\psi_{t,x,z}|^2 + \int_y |\psi_{t,z,y}|^2 = 2\int_x |\psi_{t,x,z}|^2 \\
&= 2\int_y \left[e^{-\frac{\ell^2}{\ell^4(t)}[(z-a)^2+(y+a)^2]} + e^{-\frac{\ell^2}{\ell^4(t)}[(z+a)^2+(y-a)^2]} + 2\sigma\Re e^{-\frac{\ell^2 - i\frac{\hbar t}{2m}}{2\ell^4(t)}[(z-a)^2+(y+a)^2] - \frac{\ell^2 + i\frac{\hbar t}{2m}}{2\ell^4(t)}[(z+a)^2+(y-a)^2]} \right] \\
&= 2\sqrt{\frac{2\pi\ell^4(t)}{2\ell^2}} \left[e^{-\frac{\ell^2}{\ell^4(t)}(z-a)^2} + e^{-\frac{\ell^2}{\ell^4(t)}(z+a)^2} \right] + X
\end{aligned} \tag{I2}
$$

$$
\begin{aligned}
X &= 4\sigma\Re \int_y e^{-\frac{\ell^2}{2\ell^4(t)}[(z-a)^2+(y+a)^2+(z+a)^2+(y-a)^2] + i\frac{\hbar t}{4m\ell^4(t)}[(z-a)^2+(y+a)^2-(z+a)^2-(y-a)^2]} \\
&= 4\sigma\Re \int_y e^{-\frac{\ell^2}{\ell^4(t)}(z^2+y^2+2a^2) + i\frac{\hbar t}{m\ell^4(t)}a(y-z)} \\
&= 4\sigma\Re \sqrt{\frac{2\pi\ell^4(t)}{2\ell^2}} e^{-\frac{\ell^2}{\ell^4(t)}(z^2+2a^2) - i\frac{\hbar t}{m\ell^4(t)}az - \frac{a^2\ell^4(t)\hbar^2 t^2}{4m^2\ell^2\ell^8(t)}}
\end{aligned} \tag{I3}
$$

$$
\begin{aligned}
n_{t,z} &= 2\sqrt{\frac{\pi\ell^4(t)}{\ell^2}} \left[e^{-\frac{\ell^2}{\ell^4(t)}(z-a)^2} + e^{-\frac{\ell^2}{\ell^4(t)}(z+a)^2} + 2\sigma e^{-\frac{\ell^2}{\ell^4(t)}(z^2+2a^2) - \frac{a^2\hbar^2 t^2}{4m^2\ell^2\ell^4(t)}} \right] \\
&= 2\sqrt{\frac{\pi\ell^4(t)}{\ell^2}} \left[e^{-\frac{\ell^2}{\ell^4(t)}(z-a)^2} + e^{-\frac{\ell^2}{\ell^4(t)}(z+a)^2} + 2\sigma e^{-\frac{\ell^2}{2\ell^4(t)}[(z-a)^2+(z+a)^2] - \frac{\ell^2}{\ell^4(t)}a^2 - \frac{a^2\hbar^2 t^2}{4m^2\ell^2\ell^4(t)}} \right] \\
&= 2\sqrt{\frac{\pi\ell^4(t)}{\ell^2}} \left[e^{-\frac{\ell^2}{\ell^4(t)}(z-a)^2} + e^{-\frac{\ell^2}{\ell^4(t)}(z+a)^2} + 2\sigma e^{-\frac{\ell^2}{2\ell^4(t)}[(z-a)^2+(z+a)^2] - \frac{\ell^2 a^2}{\ell^4(t)}\left(1+\frac{\hbar^2 t^2}{4m^2\ell^4}\right)} \right] \\
&= 2\sqrt{\frac{\pi\ell^4(t)}{\ell^2}} \left[e^{-\frac{\ell^2}{\ell^4(t)}(z-a)^2} + e^{-\frac{\ell^2}{\ell^4(t)}(z+a)^2} + 2\sigma e^{-\frac{\ell^2}{2\ell^4(t)}[(z-a)^2+(z+a)^2] - \frac{a^2}{\ell^2}} \right] \\
&= 2\sqrt{\frac{\pi\ell^4(t)}{\ell^2}} \left[\left(e^{-\frac{\ell^2}{2\ell^4(t)}(z-a)^2} + \sigma e^{-\frac{\ell^2}{2\ell^4(t)}(z+a)^2} \right)^2 - 2\sigma e^{-\frac{\ell^2}{2\ell^4(t)}[(z-a)^2+(z+a)^2]} \left(1 - e^{-\frac{a^2}{\ell^2}}\right) \right] \\
&= 2\sqrt{\frac{\pi\ell^4(t)}{\ell^2}} \left[\left(e^{-\frac{\ell^2}{2\ell^4(t)}(z-a)^2} + \sigma e^{-\frac{\ell^2}{2\ell^4(t)}(z+a)^2} \right)^2 - 2\sigma e^{-\frac{\ell^2}{\ell^4(t)}(z^2+a^2)} \left(1 - e^{-\frac{a^2}{\ell^2}}\right) \right]
\end{aligned} \tag{I4}
$$

Appendix J: Galilean boost-invariance

$\vec{x} \to \vec{x} + t\vec{u},\ \vec{v} \to \vec{v} + \vec{u},\ f(\vec{x},t) \to f(\vec{x} - t\vec{u}, t) = f(\vec{x},t) - t\vec{u}\vec{\partial} f(\vec{x},t),\ \partial_0 \to \partial_0 - \vec{u}\vec{\partial}$

$$D = \partial_0 + \vec{v}\vec{\partial} \to \partial_0 - \vec{u}\vec{\partial} + (\vec{v} + \vec{u})\vec{\partial} = D \tag{J1}$$

Therefore Galilean boost invariance excludes $\vec{j}$ without derivatives except surface terms ($\mathcal{O}(j)$) and introduces the convective time derivatives.

$$i\hbar\partial_0\psi(t, \boldsymbol{x} - t\boldsymbol{u}) = H\psi(t, \boldsymbol{x} - t\boldsymbol{u}) - i\hbar\boldsymbol{u}\boldsymbol{\nabla}\psi(t, \boldsymbol{x} - t\boldsymbol{u}) = (H + \boldsymbol{u}\boldsymbol{p})\psi(t, \boldsymbol{x} - t\boldsymbol{u}) \tag{J2}$$

Continuity equation reinforces the correct Galilean-boost transformation rule for the current: $(\rho, \boldsymbol{j}) \to (\rho', \boldsymbol{j}')$

$$0 = \partial_0\rho + \boldsymbol{\nabla}\boldsymbol{j} \to 0 = -\boldsymbol{u}\boldsymbol{\nabla}\rho + \boldsymbol{\nabla}(\boldsymbol{j}' - \boldsymbol{j}) \quad \to \quad \boldsymbol{j}' = \boldsymbol{j} + \rho\boldsymbol{u} \tag{J3}$$

$\boldsymbol{j} = \boldsymbol{k} + \boldsymbol{\nabla}\phi$, $\boldsymbol{\nabla}\boldsymbol{k} = 0$, $\phi = -\Delta^{-1}\partial_0\rho$
Galilei boost: $(\rho, \boldsymbol{k}) \to (\rho, \boldsymbol{k})$,

$$\begin{aligned} \phi = -\Delta^{-1}\partial_0\rho &\to -\Delta^{-1}(\partial_0 - \boldsymbol{u}\boldsymbol{\nabla})\rho \\ \boldsymbol{\nabla}\phi = -\Delta^{-1}\boldsymbol{\nabla}\partial_0\rho &\to -\Delta^{-1}(\boldsymbol{\nabla}\partial_0 - \boldsymbol{u}\Delta)\rho = \boldsymbol{\nabla}\phi + \boldsymbol{u}\rho \end{aligned} \tag{J4}$$

$\boldsymbol{j} \to \boldsymbol{j} + \rho\boldsymbol{u}$
Convective derivative:

$$D = \partial_0 + \frac{\boldsymbol{\nabla}\phi}{\rho}\boldsymbol{\nabla} = \partial_0 - \frac{\Delta^{-1}\boldsymbol{\nabla}\partial_0\rho}{\rho}\boldsymbol{\nabla} \tag{J5}$$

or

$$D = \partial_0 + \frac{\boldsymbol{j}}{\rho}\boldsymbol{\nabla} = \partial_0 + \frac{\boldsymbol{k} - \Delta^{-1}\boldsymbol{\nabla}\partial_0\rho}{\rho}\boldsymbol{\nabla} \tag{J6}$$

Gradient expansion in time

$$\begin{aligned} \Gamma[\rho, \boldsymbol{j}, \partial_0\rho, \partial_0\boldsymbol{j}] &= \Gamma[\rho, \boldsymbol{k} - \Delta^{-1}\boldsymbol{\nabla}\partial_0\rho, \partial_0\rho, \partial_0(\boldsymbol{k} - \Delta^{-1}\boldsymbol{\nabla}\partial_0\rho)] \\ &= \Gamma[\rho, \boldsymbol{k} - \Delta^{-1}\boldsymbol{\nabla}\partial_0\rho, \partial_0\rho, \partial_0\boldsymbol{k} - \Delta^{-1}\boldsymbol{\nabla}\partial_0^2\rho] \\ &= \Gamma[\rho, \boldsymbol{k} - \Delta^{-1}\boldsymbol{\nabla}D\rho, D\rho, D\boldsymbol{k} - \Delta^{-1}\boldsymbol{\nabla}D^2\rho] \end{aligned} \tag{J7}$$

Linearized equation of motion:

$$\begin{aligned} \partial_0\rho(x) &= \int_y A(x-y)\rho(y) + \int_y \boldsymbol{B}(x-y)\boldsymbol{k}(y) \\ &\to \frac{\Delta^{-1}\boldsymbol{\nabla}\partial_0\rho}{\rho}\boldsymbol{\nabla}\rho(x) + \int_y A(x-y)\rho(y) + \int_y \boldsymbol{B}(x-y)\boldsymbol{k}(y) \\ \partial_0\boldsymbol{k}(x) &= \int_y \boldsymbol{C}(x-y)\rho(y) + \int_y \boldsymbol{D}(x-y)\boldsymbol{k}(y) \\ &\to \frac{\Delta^{-1}\nabla_\ell\partial_0\rho}{\rho}\nabla_\ell\boldsymbol{k}(x) + \int_y \boldsymbol{C}(x-y)\rho(y) + \int_y \boldsymbol{D}(x-y)\boldsymbol{k}(y) \end{aligned} \tag{J8}$$

in particular, for $a = 0$

$$\begin{aligned} \partial_t n_x &= \frac{\Delta^{-1}\boldsymbol{\nabla}\partial_t n_x \boldsymbol{\nabla} n_x}{n_0 + n_x} + \frac{\hbar\bar{k}^2}{m}\left(-\frac{a_0}{a_z} + \frac{a_{qq}}{a_z\bar{k}^2}\Delta\right)\frac{1}{\bar{\bar{k}}}\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y]n_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} \\ &= -\frac{\boldsymbol{\nabla}n_x}{4\pi(n_0 + n_x)}\int_{\boldsymbol{y}} \frac{\boldsymbol{\nabla}\partial_t n_y}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{\hbar\bar{k}^2}{m}\left(-\frac{a_0}{a_z} + \frac{a_{qq}}{a_z\bar{k}^2}\Delta\right)\frac{1}{\bar{\bar{k}}}\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y]n_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} \\ \partial_t \boldsymbol{j}^T_{\boldsymbol{x}} &= \frac{\Delta^{-1}\nabla_\ell\partial_t n_x \nabla_\ell \boldsymbol{k}_x}{n_0 + n_x} + \frac{\hbar\bar{k}^2}{m}\left(-\frac{b_0}{b_z} + \frac{b_{qq}}{b_z\bar{k}^2}\Delta\right)\frac{1}{\bar{\bar{k}}}\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y]\boldsymbol{j}^T_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} \\ &= -\frac{\nabla_\ell\boldsymbol{k}_x}{4\pi(n_0 + n_x)}\int_{\boldsymbol{y}} \frac{\nabla_\ell\partial_t n_y}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{\hbar\bar{k}^2}{m}\left(-\frac{b_0}{b_z} + \frac{b_{qq}}{b_z\bar{k}^2}\Delta\right)\frac{1}{\bar{k}}\int_{\boldsymbol{y}} \frac{[(\boldsymbol{y}\boldsymbol{\nabla}_y)^2 - 2\boldsymbol{y}\boldsymbol{\nabla}_y]\boldsymbol{j}^T_{\boldsymbol{y}}}{2\pi^2(\boldsymbol{x}-\boldsymbol{y})^4} \end{aligned} \tag{J9}$$

"Reynolds number": T is the total time,

$$R_n = Tn_0\frac{\hbar\bar{k}}{m}\frac{1}{\bar{k}^2} = \frac{Tn_0\hbar}{m\bar{k}} \to \frac{T\bar{k}^2\hbar}{m} \tag{J10}$$

or measure of nonlinearity:

$$\frac{1}{R_n} = \frac{m}{T\bar{k}^2\hbar} \tag{J11}$$