Seiberg-Witten and the chemical potential

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I. $SU(N_c)$ THEORY WITH N_f FLAVOURS

Let us consider a supersymmetric $\mathcal{N} = 2$, $SU(N_c)$ gauge theory with one $\mathcal{N} = 2$ chiral multiplet and N_f massive hypermultiplets transforming under the fundamental representation of $SU(N_f)$. The $\mathcal{N} = 2$ chiral multiplet consists of $\mathcal{N} = 1$ vector W_α and chiral Φ multiplets. The super-potential in the $\mathcal{N} = 1$ language is

$$W = \sqrt{2}\tilde{Q}_i \Phi Q^i + \sum_i m_i \tilde{Q}_i Q^i. \tag{1}$$

The effective $\mathcal{N} = 2$ potential with $\mathcal{N} = 1$ multiplets (A_i, W_i) is

$$\mathcal{L}_{eff} = \mathrm{Im}\frac{1}{4\pi}\left[\int d^4\theta \partial_i \mathcal{F}(A)\bar{A}^i + \frac{1}{2}\int d^2\theta \partial_i \partial_j \mathcal{F}(A) W^i W^j\right], \tag{2}$$

where A is the $\mathcal{N} = 1$ chiral super-field. It is important to note that this pre-potential form of the effective Wilsonian lagrangian does not provide higher than second-derivative terms of fields in the Kahler potential. But it does give all non-renormalisable terms without derivatives. Given that we are most interested in the IR, we can probably argue using gradient expansion that we do not need those terms and that we can still find the desired effective potential to extract the ground state. If I remember correctly the 1PI-effective action doesn't provide any higher derivatives either, or does it?

Hanany and Oz [1] solve the problem of finding the SW curve for the $SU(N_c)$ theory with flavour. For curves without flavour multiplets, see also [2] (published 10 days after [1]). The curve for $N_f < N_c$, which is the regime in which we are presumably most interested is the following. They also find solutions for other regimes. The hyper-eliptic curve, including instanton contributions, is

$$y^2 = \mathcal{C}^2_{N_c}(x) - \Lambda^{2N_c - N_f}_{N_f} \prod_{i=1}^{N_f}(x + m_i) \tag{3}$$

$$\equiv \mathcal{C}^2_{N_c}(x) - \Lambda^{2N_c - N_f}_{N_f} G(x, m_i) \tag{4}$$

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where in our case all masses of flavours will be equal $m_i = m$ so that we have a conserved $U(1)$ baryon charge. This gives

$$y^2 = \mathcal{C}_{N_c}^2(x) - \Lambda_{N_f}^{2N_c - N_f}(x+m)^{N_f} \tag{5}$$

where

$$\mathcal{C}_{N_c} = x^{N_c} + \sum_{i=2}^{N_c} s_i x^{N_c - i} \tag{6}$$

and s_i can be found recursively from

$$k s_k + \sum_{i=1}^{k} s_{k-i} u_i = 0, \qquad\qquad k = \{1, 2, ...\} \tag{7}$$

with $s_0 = 1$ and $s_1 = u_1 = 0$ for the $SU(N_c)$ group.

Once we have the SW curve then we can find the pre-potential $\mathcal{F}$. Define a^i to be the vevs of the scalar components of A, where $i = 1, ..., N_c$ and $\sum_i A_i = 0$. Its dual is then

$$a_D^i = \frac{\partial \mathcal{F}(a)}{\partial a^i}. \tag{8}$$

They can be found from

$$a_D^i = \oint_{\alpha_i} \lambda, \qquad\qquad a^i = \oint_{\beta_i} \lambda, \tag{9}$$

where α_i, β_i are the basis of homology cycles on the curve. λ is the meromorphic one-form, which can be computed directly from y as

$$\lambda = \frac{x dx}{2\pi i y}\left(\frac{\mathcal{C}_{N_c}}{2G}\frac{\partial G}{\partial x} - \frac{\partial \mathcal{C}_{N_c}}{\partial x}\right) \tag{10}$$

$$= \frac{x dx}{2\pi i y}\left(\frac{N_f}{2(x+m)}\mathcal{C}_{N_c} - \frac{\partial \mathcal{C}_{N_c}}{\partial x}\right) \tag{11}$$

The genus of the hyper-eliptic curve is $g = N_c - 1$, which gives $2N_c - 2$ cycles α_i and β_i.

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[1] A. Hanany and Y. Oz, "On the quantum moduli space of vacua of N=2 supersymmetric SU(N(c)) gauge theories," Nucl. Phys. B **452** (1995) 283 [hep-th/9505075].

[2] A. Klemm, W. Lerche and S. Theisen, "Nonperturbative effective actions of N=2 supersymmetric gauge theories," Int. J. Mod. Phys. A **11** (1996) 1929 [hep-th/9505150].