Radiation in polarizable medium

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I. EFFECTIVE THEORY

$$e^{\frac{i}{\hbar}W[j]} = \int D[\psi]D[\bar{\psi}]D[A]e^{\frac{i}{\hbar}\bar{\psi}\cdot[G^{-1}-\frac{e}{\hbar c}\not{A}]\cdot\psi+\frac{i}{2\hbar}A\cdot D_0^{-1}\cdot A+\frac{i}{\hbar}(j-en)\cdot A} \tag{1}$$

where

$$G_0^{-1} = i\not{\partial} - \frac{mc}{\hbar} + i\epsilon, \qquad D_0^{-1} = \Box T + \xi\Box L + i\epsilon, \tag{2}$$

$$T^{ab} = g^{ab} - L^{ab}, \quad L^{ab} = \frac{\partial^a\partial^b}{\Box}, \tag{3}$$

$$\begin{aligned}
S_M[A] &= \int_x \left(-\frac{1}{4}(\partial_a A_b - \partial_b A_a)^2 - \frac{\xi}{2}(\partial^a A_a)^2\right) \\
&= \int_x \left(-\frac{1}{2}\partial_a A_b(\partial_a A_b - \partial_b A_a - \frac{\xi}{2}(\partial^a A_a)^2\right) \\
&= \int_x A_a \left(\frac{1}{2}g^{ab}\Box - \frac{1}{2}\partial_a\partial_b + \frac{\xi}{2}\partial^a\partial^b\right) A_b \\
&= \frac{1}{2}\int_x A_a \left[g^{ab}\Box - (1-\xi)\partial_a\partial_b\right] A_b \\
&= \frac{1}{2}A_a \cdot D_0^{-1ab} \cdot A_b \\
D_{0a,b}^{-1} &= g^{ab}\Box - (1-\xi)\partial_a\partial_b = \Box(T+\xi L)_{ab}
\end{aligned} \tag{4}$$

A. One-loop approximation for $W[j]$

$$\ln[G^{-1} - A] = \log G^{-1} - \sum_{n=1}^{\infty}\frac{1}{n}(G\cdot A)^n. \tag{5}$$

$$\begin{aligned}
e^{\frac{i}{\hbar}W[j]} &= \int D[A]e^{\mathrm{Tr}\ln[G^{-1}-\not{A}]+\frac{i}{2\hbar}A\cdot D_0^{-1}\cdot A+\frac{i}{\hbar}(j-en)\cdot A} \\
&= \int D[A]e^{\mathrm{Tr}\ln[G^{-1}+\not{A}]-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}(G\cdot A)^n+\frac{i}{2\hbar}A\cdot D_0^{-1}\cdot A+\frac{i}{\hbar}E\cdot(j-en)} \\
&= \int D[A]e^{\mathrm{Tr}\ln G^{-1}-e\mathrm{Tr}[G\cdot A]-\frac{e^2}{2}\mathrm{Tr}[G^{-1}\cdot A\cdot G\cdot A]+\mathcal{O}(A^3)+\frac{i}{2\hbar}A\cdot D_0^{-1}\cdot A+\frac{i}{\hbar}A\cdot(j-en)} \\
&= \int D[A]e^{\mathrm{Tr}\ln G^{-1}+\frac{i}{2\hbar}A\cdot D^{-1}\cdot A+\frac{i}{\hbar}A\cdot(j-\bar{j})+\mathcal{O}(A^3)} \\
&= \mathcal{N}e^{-\frac{i}{2\hbar}j\cdot D\cdot j+\mathcal{O}(j^3)}
\end{aligned} \tag{6}$$

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$$\bar{j}^\mu = n^\mu - i\hbar \mathrm{tr}[G^{-1}_{xx}\gamma^\mu] \to 0 \tag{7}$$

$$\tilde{G}^{\mu\nu}_{xy} = -i\hbar \mathrm{tr}[G_{yx}\gamma^\mu G_{xy}\gamma^\nu] \tag{8}$$

$$D^{-1} = D_0^{-1} - e^2 G_0 \tag{9}$$

B. Effective action

Generic case:

$$W[j] = w^{(1)}j + \frac{1}{2} j w^{(2)} j = \Gamma[A] + Aj \tag{10}$$

where

$$A = \frac{\delta W}{\delta j} = w^{(1)} + w^{(2)} j \tag{11}$$

Inverse Legendre transformation:

$$\frac{\delta \Gamma}{\delta A} = -j \tag{12}$$

Inversion:

$$\begin{aligned} \Gamma[A] &= w^{(1)}j + \frac{1}{2} j w^{(2)} j - jA \\ &= w^{(1)} w^{(2)-1}(A - w^{(1)}) + \frac{1}{2}(A - w^{(1)}) w^{(2)-1} w^{(2)} w^{(2)-1}(A - w^{(1)}) - (A - w^{(1)}) w^{(2)-1} A \\ &= w^{(1)} w^{(2)-1} A - \frac{1}{2} A w^{(2)-1} A \end{aligned} \tag{13}$$

In this case:

$$\Gamma[A] = \frac{1}{2} A \cdot D^{-1} \cdot A \tag{14}$$

One-loop gradient expansion expression: $|k^2| \ll M^2 = \frac{15\pi}{\alpha}\frac{m^2c^2}{\hbar^2} \approx 5 \cdot 10^{24}[cm^{-2}]$ (CGS units)

$$e^2 \tilde{G}_{xy} = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{(k^2)^2}{M^2}\left[1 + \mathcal{O}(\left(\frac{k^2}{m^2}\right)^2)\right] \tag{15}$$

Propagator:

$$D^{\mu\nu}_{0xy} = T^{\mu\nu} D_{0xy} + \frac{1}{\xi}\frac{L^{\mu\nu}}{\Box - i\epsilon} \tag{16}$$

$$D_{0xy} = -\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2 + i\epsilon} \tag{17}$$

Retarded Green functions:

$$\begin{aligned} D^r_k &\approx -\frac{1}{k^2 + \frac{(k^2)^2}{M^2} + e^{-5.7\cdot 10^9/T}[2.2 \cdot 10^{-4} T^{3/2} + \frac{3\cdot 10^{-19} T^{3/4}}{\boldsymbol{k}^2} + 1.4 \cdot 10^{20}\frac{T}{|\boldsymbol{k}|} \cdot e^{-\frac{1.2\cdot 10^{-12}}{\sqrt{T}}\cdot(|\boldsymbol{q}|+1.5\frac{k^0}{|\boldsymbol{k}|})^2}]} \\ &\approx -\frac{1}{k^2 + 2\cdot 10^{-25}(k^2)^2 + e^{-5.7\cdot 10^9/T}[2.2 \cdot 10^{-4} T^{3/2} + \frac{3\cdot 10^{-19} T^{3/4}}{\boldsymbol{k}^2} + 1.4 \cdot 10^{20}\frac{T}{|\boldsymbol{k}|} \cdot e^{-\frac{1.2\cdot 10^{-12}}{\sqrt{T}}\cdot(|\boldsymbol{q}|+1.5\frac{k^0}{|\boldsymbol{k}|})^2}]} \end{aligned} \tag{18}$$

where $\boldsymbol{k}$ and k^0 are given in cm^{-1} and sec^{-1}, respectively, T in Kelvin and $k^0 \to k^0 + i\epsilon$.

II. POLYNOMIAL SUPPRESSION

Suppression at high momenta:

$$
\begin{aligned}
D^{\mu\nu}_{xy} &= -\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2+\frac{(k^2)^2}{M^2}+i\epsilon} \\
&\to -\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2+\frac{(k^2)^2}{M^2}\frac{K^{2n}}{P^{(n)}(k^2)}+i\epsilon} \\
&= -\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{(k^2+i\epsilon)[1+\frac{k^2K^{2n}}{M^2P^{(n)}(k^2)}]} \\
&= -\frac{M^2}{K^{2n}}\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}P^{(n)}(k^2)}{(k^2+i\epsilon)[\frac{M^2}{K^{2n}}P^{(n)}(k^2)+k^2]}
\end{aligned}
\tag{19}
$$

In order not to generate new poles:

$$
\frac{M^2}{K^{2n}}P^{(n)}(k^2)+k^2=\prod_{j=1}^{n}(k^2-m_j^2) \tag{20}
$$

with $m_j^2 \geq 0$

$$
P^{(n)}(0)=K^{2n}=(-1)^n\prod_{j=1}^{n}m_j^2 \tag{21}
$$

$$
D^{\mu\nu}_{k} = -\frac{\prod_{j=1}^{n}(k^2-m_j^2)-k^2}{(k^2+i\epsilon)\prod_{j=1}^{n}(k^2-m_j^2+i\epsilon)} \tag{22}
$$

Need - in front of k0 (corrected from +).

$$
D^{r\mu\nu}_{x} = -\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}\int\frac{dk^0}{2\pi}e^{-ik^0ct}\left[\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)\frac{\prod_{j=1}^{n}(k^2-m_j^2)-k^2}{k^2\prod_{j=1}^{n}(k^2-m_j^2)}-\frac{1}{\xi}\frac{k^\mu k^\nu}{k^2}\frac{1}{k^2}\right]_{|k^0\to k^0+i\epsilon} \tag{23}
$$

Using SI units we have a four-vector $x=(ct,\boldsymbol{x})$, hence we have to have $k=(k^0,\boldsymbol{k})=(\omega/c,\boldsymbol{k})$. Let us divide the Green's function into three integrals

$$
D^{\mu\nu}_{r}(x)=D^{\mu\nu}_{r,1}(x)+D^{\mu\nu}_{r,2}(x)+D^{\mu\nu}_{r,3}(x), \tag{24}
$$

where $t>0$. We have

$$
\begin{aligned}
D^{\mu\nu}_{r,1} &= -\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \int \frac{dk^0}{2\pi} e^{-ik^0ct} g^{\mu\nu} \frac{\prod_{j=1}^n (k^2-m_j^2) - k^2}{k^2 \prod_{j=1}^n (k^2-m_j^2)}\Big|_{k^0\to k^0+i\epsilon} \\
&= -g^{\mu\nu} \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \int \frac{dk^0}{2\pi} e^{-ik^0ct} \\
&\quad \frac{\prod_{j=1}^n \left[(k^0 - \sqrt{\boldsymbol{k}^2+m_j^2})(k^0+\sqrt{\boldsymbol{k}^2+m_j^2})\right] - (k^0-|\boldsymbol{k}|)(k^0+|\boldsymbol{k}|)}{(k^0+i\epsilon-|\boldsymbol{k}|)(k^0+i\epsilon+|\boldsymbol{k}|)\prod_{j=1}^n \left[(k^0+i\epsilon-\sqrt{\boldsymbol{k}^2+m_j^2})(k^0+i\epsilon+\sqrt{\boldsymbol{k}^2+m_j^2})\right]} \\
&= -g^{\mu\nu} i \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \Bigg\{ \frac{e^{-i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|} - \frac{e^{i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|} - \sum_{j=1}^n \frac{e^{-i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^n \left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]} + \\
&\quad + \sum_{j=1}^n \frac{e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^n \left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]} \Bigg\} \\
&= -\frac{g^{\mu\nu} i}{2(2\pi)^3} \int d^3k e^{i\boldsymbol{kx}} \Bigg\{ \frac{e^{-i|\boldsymbol{k}|ct} - e^{i|\boldsymbol{k}|ct}}{|\boldsymbol{k}|} - \sum_{j=1}^n \frac{1}{\prod_{p=1}^{j-1}\prod_{q=j+1}^n \left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]} \\
&\quad \left[\frac{e^{-i\sqrt{\boldsymbol{k}^2+m_j^2}ct} - e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{\sqrt{\boldsymbol{k}^2+m_j^2}}\right] \Bigg\} \\
&= -\frac{g^{\mu\nu}}{2(2\pi)^2 r} \Bigg\{ \int_0^\infty d|\boldsymbol{k}| (e^{i|\boldsymbol{k}|r} - e^{-i|\boldsymbol{k}|r})(e^{-i|\boldsymbol{k}|ct} - e^{i|\boldsymbol{k}|ct}) - \\
&\quad - \sum_{j=1}^n \frac{1}{\prod_{p=1}^{j-1}\prod_{q=j+1}^n \left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]} \int_0^\infty d|\boldsymbol{k}||\boldsymbol{k}|(e^{i|\boldsymbol{k}|r} - e^{-i|\boldsymbol{k}|r}) \left[\frac{e^{-i\sqrt{\boldsymbol{k}^2+m_j^2}ct} - e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{\sqrt{\boldsymbol{k}^2+m_j^2}}\right] \Bigg\} \\
&= -\frac{\delta(ct-r)}{4\pi r} g^{\mu\nu} + \frac{g^{\mu\nu}}{8\pi r} \sum_{j=1}^n \frac{1}{\prod_{p=1}^{j-1}\prod_{q=j+1}^n \left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]} \int_{-\infty}^\infty \frac{d|\boldsymbol{k}|}{2\pi} |\boldsymbol{k}| (e^{i|\boldsymbol{k}|r} - e^{-i|\boldsymbol{k}|r}) \\
&\quad \left[\frac{e^{-i\sqrt{\boldsymbol{k}^2+m_j^2}ct} - e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{\sqrt{\boldsymbol{k}^2+m_j^2}}\right]
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
D^{\mu\nu}_{r,2} &= \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \int \frac{dk^0}{2\pi} e^{-ik^0ct} \frac{1}{\xi} \frac{k^\mu k^\nu}{k^4}\Big|_{k^0\to k^0+i\epsilon} \\
D^{00}_{r,2} &= \frac{1}{\xi} \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \int \frac{dk^0}{2\pi} e^{-ik^0ct} \frac{(k^0)^2}{(k^0+i\epsilon-|\boldsymbol{k}|)^2(k^0+i\epsilon+|\boldsymbol{k}|)^2} \\
&= \frac{i}{\xi} \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \left[\frac{e^{-i|\boldsymbol{k}|ct} - e^{i|\boldsymbol{k}|ct}}{4|\boldsymbol{k}|} - \frac{ict}{4}\left(e^{-i|\boldsymbol{k}|ct} + e^{i|\boldsymbol{k}|ct}\right)\right] \\
&= -\frac{\delta(ct-r)}{8\pi r\xi} - \frac{ict}{4(2\pi)^2 r\xi} \int_0^\infty dkk(e^{ikr} - e^{-ikr})(e^{-ikct} + e^{ikct}) \\
&= -\frac{\delta(ct-r)}{8\pi r\xi} - \frac{ict}{8(2\pi)^2 r\xi} \int_{-\infty}^\infty dkk(e^{ikr} - e^{-ikr})(e^{-ikct} + e^{ikct})
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
D^{0i}_{r,2} &= \frac{1}{\xi} \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} k^i \int \frac{dk^0}{2\pi} e^{-ik^0ct} \frac{k^0}{(k^0+i\epsilon-|\boldsymbol{k}|)^2(k^0+i\epsilon+|\boldsymbol{k}|)^2} \\
&= \frac{ct}{4\xi} \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} k^i \frac{e^{-i|\boldsymbol{k}|ct} - e^{i|\boldsymbol{k}|ct}}{|\boldsymbol{k}|}
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
D^{01}_{r,2} &= \frac{ct}{4\xi(2\pi)^3}\int d|\boldsymbol{k}||\boldsymbol{k}|^2 d\phi d\theta e^{i|\boldsymbol{k}|r\cos\theta}\sin^2\theta\cos\phi(e^{-i|\boldsymbol{k}|ct}-e^{i|\boldsymbol{k}|ct})=0\\
D^{02}_{r,2} &= \frac{ct}{4\xi(2\pi)^3}\int d|\boldsymbol{k}||\boldsymbol{k}|^2 d\phi d\theta e^{i|\boldsymbol{k}|r\cos\theta}\sin^2\theta\sin\phi(e^{-i|\boldsymbol{k}|ct}-e^{i|\boldsymbol{k}|ct})=0\\
D^{03}_{r,2} &= \frac{ct}{4\xi(2\pi)^3}\int d|\boldsymbol{k}||\boldsymbol{k}|^2 d\phi d\theta e^{i|\boldsymbol{k}|r\cos\theta}\sin\theta\cos\theta(e^{-i|\boldsymbol{k}|ct}-e^{i|\boldsymbol{k}|ct})=\\
&= \frac{ct}{4\xi(2\pi)^2r^2}\int_0^\infty d|\boldsymbol{k}|(e^{i|\boldsymbol{k}|r}-e^{-i|\boldsymbol{k}|r})(e^{-i|\boldsymbol{k}|ct}-e^{i|\boldsymbol{k}|ct})\\
&\quad -\frac{ict}{4\xi(2\pi)^2r}\int_0^\infty d|\boldsymbol{k}||\boldsymbol{k}|(e^{i|\boldsymbol{k}|r}+e^{-i|\boldsymbol{k}|r})(e^{-i|\boldsymbol{k}|ct}-e^{i|\boldsymbol{k}|ct})\\
&= \frac{ct\delta(ct-r)}{8\pi r^2\xi}-\frac{ict}{8(2\pi)^2r\xi}\int_{-\infty}^\infty d|\boldsymbol{k}||\boldsymbol{k}|(e^{i|\boldsymbol{k}|r}+e^{-i|\boldsymbol{k}|r})(e^{-i|\boldsymbol{k}|ct}-e^{i|\boldsymbol{k}|ct})
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
D^{ij}_{r,2} &= \frac{1}{\xi}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}k^ik^j\int\frac{dk^0}{2\pi}e^{-ik^0ct}\frac{1}{(k^0+i\epsilon-|\boldsymbol{k}|)^2(k^0+i\epsilon+|\boldsymbol{k}|)^2}\\
&= \frac{i}{4\xi}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}k^ik^j\frac{(e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct})}{|\boldsymbol{k}|^3}+\frac{ct}{4\xi}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}k^ik^j\frac{(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct})}{|\boldsymbol{k}|^2}
\end{aligned}
\tag{29}
$$

Because of the ϕ dependence of k^ik^j we know that $D^{12}_{r,2}=D^{13}_{r,2}=D^{23}_{r,2}=0$.

$$
\begin{aligned}
D^{11}_{r,2} &= \frac{i}{4\xi}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}k^1k^1\frac{(e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct})}{|\boldsymbol{k}|^3}+\frac{ct}{4\xi}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}k^1k^1\frac{(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct})}{|\boldsymbol{k}|^2}=\\
&= \frac{i}{4\xi(2\pi)^3}\int d|\boldsymbol{k}||\boldsymbol{k}|d\phi d\theta e^{i|\boldsymbol{k}|r\cos\theta}\sin^3\theta\cos^2\phi(e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct})\\
&\quad +\frac{ct}{4\xi(2\pi)^3}\int d|\boldsymbol{k}||\boldsymbol{k}|^2 d\phi d\theta e^{i|\boldsymbol{k}|r\cos\theta}\sin^3\theta\cos^2\phi(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct})=\\
&= \frac{1}{4\xi(2\pi)^2r^3}\int_0^\infty d|\boldsymbol{k}|\frac{(e^{i|\boldsymbol{k}|r}-e^{-i|\boldsymbol{k}|r})}{|\boldsymbol{k}|^2}(e^{i|\boldsymbol{k}|ct}\\
&\quad -e^{-i|\boldsymbol{k}|ct})-\frac{i}{4\xi(2\pi)^2r^2}\int_0^\infty d|\boldsymbol{k}|\frac{(e^{i|\boldsymbol{k}|r}+e^{-i|\boldsymbol{k}|r})}{|\boldsymbol{k}|}(e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct})\\
&\quad -\frac{ict}{4\xi(2\pi)^2r^3}\int_0^\infty d|\boldsymbol{k}|\frac{(e^{i|\boldsymbol{k}|r}-e^{-i|\boldsymbol{k}|r})}{|\boldsymbol{k}|}(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct})\\
&\quad -\frac{ct}{4\xi(2\pi)^2r^2}\int_0^\infty d|\boldsymbol{k}|(e^{i|\boldsymbol{k}|r}+e^{-i|\boldsymbol{k}|r})(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct})=
\end{aligned}
\tag{30}
$$

$$\begin{aligned}
D^{22}_{r,2} &= \frac{i}{4\xi}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}k^2k^2\frac{(e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct})}{|\boldsymbol{k}|^3}+\frac{ct}{4\xi}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}k^2k^2\frac{(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct})}{|\boldsymbol{k}|^2}= \\
&= \frac{i}{4\xi(2\pi)^3}\int d|\boldsymbol{k}||\boldsymbol{k}|d\phi d\theta e^{i|\boldsymbol{k}|r\cos\theta}\sin^3\theta\sin^2\phi(e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}) \\
&\quad+\frac{ct}{4\xi(2\pi)^3}\int d|\boldsymbol{k}||\boldsymbol{k}|^2d\phi d\theta e^{i|\boldsymbol{k}|r\cos\theta}\sin^3\theta\sin^2\phi(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct})= \\
&= \frac{1}{4\xi(2\pi)^2r^3}\int_0^\infty d|\boldsymbol{k}|\frac{(e^{i|\boldsymbol{k}|r}-e^{-i|\boldsymbol{k}|r})}{|\boldsymbol{k}|^2}(e^{i|\boldsymbol{k}|ct} \\
&\quad-e^{-i|\boldsymbol{k}|ct})-\frac{i}{4\xi(2\pi)^2r^2}\int_0^\infty d|\boldsymbol{k}|\frac{(e^{i|\boldsymbol{k}|r}+e^{-i|\boldsymbol{k}|r})}{|\boldsymbol{k}|}(e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}) \\
&\quad-\frac{ict}{4\xi(2\pi)^2r^3}\int_0^\infty d|\boldsymbol{k}|\frac{(e^{i|\boldsymbol{k}|r}-e^{-i|\boldsymbol{k}|r})}{|\boldsymbol{k}|}(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct}) \\
&\quad-\frac{ct}{4\xi(2\pi)^2r^2}\int_0^\infty d|\boldsymbol{k}|(e^{i|\boldsymbol{k}|r}+e^{-i|\boldsymbol{k}|r})(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct})=
\end{aligned} \tag{31}$$

$$\begin{aligned}
D^{33}_{r,2} &= \frac{i}{4\xi}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}k^3k^3\frac{(e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct})}{|\boldsymbol{k}|^3}+\frac{ct}{4\xi}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}k^3k^3\frac{(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct})}{|\boldsymbol{k}|^2}= \\
&= \frac{i}{4\xi(2\pi)^3}\int d|\boldsymbol{k}||\boldsymbol{k}|d\phi d\theta e^{i|\boldsymbol{k}|r\cos\theta}\sin\theta\cos^2\theta(e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}) \\
&\quad+\frac{ct}{4\xi(2\pi)^3}\int d|\boldsymbol{k}||\boldsymbol{k}|^2d\phi d\theta e^{i|\boldsymbol{k}|r\cos\theta}\sin\theta\cos^2\theta(e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct})= \\
&=
\end{aligned} \tag{32}$$

It will be useful to evaluate the derivative of the following product:

$$\begin{aligned}
&\partial_{k^0}\prod_{j=1}^n\left[(k^0-\sqrt{\boldsymbol{k}^2+m_j^2})(k^0+\sqrt{\boldsymbol{k}^2+m_j^2})\right]\Big|_{k^0\to|\boldsymbol{k}|}= \\
&= \partial_{k^0}\prod_{j=1}^n((k^0)^2-\boldsymbol{k}^2-m_j^2)\Big|_{k^0\to|\boldsymbol{k}|}= \\
&= 2|\boldsymbol{k}|(-1)^{n-1}[m_2^2m_3^2m_4^2...+m_1^2m_3^2m_4^2...+m_1^2m_2^2m_4^2...+...]= \\
&= 2|\boldsymbol{k}|(-1)^{n-1}\sum_{j=1}^n\prod_{p=1}^{j-1}\prod_{q=j+1}^n m_p^2m_q^2
\end{aligned} \tag{33}$$

$$D^{\mu\nu}_{r,3}=\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}\int\frac{dk^0}{2\pi}e^{-ik^0ct}\frac{k^\mu k^\nu}{k^2}\frac{\prod_{j=1}^n(k^2-m_j^2)-k^2}{k^2\prod_{j=1}^n(k^2-m_j^2)}\Big|_{k^0\to k^0+i\epsilon} \tag{34}$$

$$
\begin{aligned}
D^{00}_{r,3} = &\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} \int \frac{dk^0}{2\pi} e^{-ik^0ct} \frac{k^0k^0\left(\prod_{j=1}^{n}(k^2-m_j^2)-k^2\right)}{k^4\prod_{j=1}^{n}(k^2-m_j^2)}\Bigg|_{k^0\to k^0+i\epsilon} \\
= \; i &\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} \int \frac{dk^0}{2\pi} e^{-ik^0ct} \frac{(k^0)^2\left[\prod_{j=1}^{n}(k^0-\sqrt{\boldsymbol{k}^2+m_j^2})(k^0+\sqrt{\boldsymbol{k}^2+m_j^2})-(k^0-|\boldsymbol{k}|)(k^0+|\boldsymbol{k}|)\right]}{(k^0+i\epsilon-|\boldsymbol{k}|)^2(k^0+i\epsilon+|\boldsymbol{k}|)^2\prod_{j=1}^{n}\left[(k^0+i\epsilon-\sqrt{\boldsymbol{k}^2+m_j^2})(k^0+i\epsilon+\sqrt{\boldsymbol{k}^2+m_j^2})\right]} \\
= \; i &\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} \Bigg\{ \frac{|\boldsymbol{k}|}{2(-1)^n\prod_{j=1}^{n}m_j^2}[e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}] - \frac{1}{4|\boldsymbol{k}|}[e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}] - \frac{ict}{4}[e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct}] \\
&-\sum_{j=1}^{n} \frac{(\boldsymbol{k}^2+m_j^2)e^{-i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{2m_j^2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]} \\
&+\sum_{j=1}^{n} \frac{(\boldsymbol{k}^2+m_j^2)e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{2m_j^2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]}\Bigg\}
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
D^{ij}_{r,3} = &\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} \int \frac{dk^0}{2\pi} e^{-ik^0ct} \frac{k^ik^j\left(\prod_{j=1}^{n}(k^2-m_j^2)-k^2\right)}{k^4\prod_{j=1}^{n}(k^2-m_j^2)}\Bigg|_{k^0\to k^0+i\epsilon} \\
\not{i} &\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} k^ik^j \int \frac{dk^0}{2\pi} e^{-ik^0ct} \frac{\left[\prod_{j=1}^{n}(k^0-\sqrt{\boldsymbol{k}^2+m_j^2})(k^0+\sqrt{\boldsymbol{k}^2+m_j^2})-(k^0-|\boldsymbol{k}|)(k^0+|\boldsymbol{k}|)\right]}{(k^0+i\epsilon-|\boldsymbol{k}|)^2(k^0+i\epsilon+|\boldsymbol{k}|)^2\prod_{j=1}^{n}\left[(k^0+i\epsilon-\sqrt{\boldsymbol{k}^2+m_j^2})(k^0+i\epsilon+\sqrt{\boldsymbol{k}^2+m_j^2})\right]} \\
= \; i &\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} k^ik^j \Bigg\{ \frac{1}{2|\boldsymbol{k}|(-1)^n\prod_{j=1}^{n}m_j^2}[e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}] + \frac{1}{4|\boldsymbol{k}|^3}[e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}] - \frac{ict}{4|\boldsymbol{k}|^2}[e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct}] - \\
&-\sum_{j=1}^{n} \frac{e^{-i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{2m_j^2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]} \\
&+\sum_{j=1}^{n} \frac{e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{2m_j^2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]}\Bigg\}
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
D^{11}_{r,3} &= i \int \frac{d|\boldsymbol{k}| d\theta d\phi}{(2\pi)^3} e^{i|\boldsymbol{k}|r\cos\theta} \sin^3\theta \cos^2\phi \\
&\Bigg\{ \frac{|\boldsymbol{k}|^3}{2(-1)^n \prod_{j=1}^n m_j^2} [e^{i|\boldsymbol{k}|ct} - e^{-i|\boldsymbol{k}|ct}] + \frac{|\boldsymbol{k}|}{4} [e^{i|\boldsymbol{k}|ct} - e^{-i|\boldsymbol{k}|ct}] - \frac{ict|\boldsymbol{k}|^2}{4} [e^{i|\boldsymbol{k}|ct} + e^{-i|\boldsymbol{k}|ct}] - \\
&- \sum_{j=1}^n \frac{|\boldsymbol{k}|^4 e^{-i\sqrt{\boldsymbol{k}^2 + m_j^2}ct}}{2m_j^2 \sqrt{\boldsymbol{k}^2 + m_j^2} \prod_{p=1}^{j-1} \prod_{q=j+1}^n \left[(m_j^2 - m_p^2)(m_j^2 - m_q^2) \right]} + \\
&+ \sum_{j=1}^n \frac{|\boldsymbol{k}|^4 e^{i\sqrt{\boldsymbol{k}^2 + m_j^2}ct}}{2m_j^2 \sqrt{\boldsymbol{k}^2 + m_j^2} \prod_{p=1}^{j-1} \prod_{q=j+1}^n \left[(m_j^2 - m_p^2)(m_j^2 - m_q^2) \right]} \Bigg\} = \\
= \ & \frac{i}{2\pi^2 r^3} \int_0^\infty d|\boldsymbol{k}| [\sin(|\boldsymbol{k}|r) - |\boldsymbol{k}|r\cos(|\boldsymbol{k}|r)] \\
&\Bigg\{ \frac{1}{2(-1)^n \prod_{j=1}^n m_j^2} [e^{i|\boldsymbol{k}|ct} - e^{-i|\boldsymbol{k}|ct}] + \frac{1}{4|\boldsymbol{k}|^2} [e^{i|\boldsymbol{k}|ct} - e^{-i|\boldsymbol{k}|ct}] - \frac{ict}{4|\boldsymbol{k}|} [e^{i|\boldsymbol{k}|ct} + e^{-i|\boldsymbol{k}|ct}] - \\
&- \sum_{j=1}^n \frac{|\boldsymbol{k}| e^{-i\sqrt{\boldsymbol{k}^2 + m_j^2}ct}}{2m_j^2 \sqrt{\boldsymbol{k}^2 + m_j^2} \prod_{p=1}^{j-1} \prod_{q=j+1}^n \left[(m_j^2 - m_p^2)(m_j^2 - m_q^2) \right]} + \\
&+ \sum_{j=1}^n \frac{|\boldsymbol{k}| e^{i\sqrt{\boldsymbol{k}^2 + m_j^2}ct}}{2m_j^2 \sqrt{\boldsymbol{k}^2 + m_j^2} \prod_{p=1}^{j-1} \prod_{q=j+1}^n \left[(m_j^2 - m_p^2)(m_j^2 - m_q^2) \right]} \Bigg\} =
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
D^{22}_{r,3} &= \frac{i}{2\pi^2 r^3} \int_0^\infty d|\boldsymbol{k}| [\sin(|\boldsymbol{k}|r) - |\boldsymbol{k}|r\cos(|\boldsymbol{k}|r)] \\
&\Bigg\{ \frac{1}{2(-1)^n \prod_{j=1}^n m_j^2} [e^{i|\boldsymbol{k}|ct} - e^{-i|\boldsymbol{k}|ct}] + \frac{1}{4|\boldsymbol{k}|^2} [e^{i|\boldsymbol{k}|ct} - e^{-i|\boldsymbol{k}|ct}] - \frac{ict}{4|\boldsymbol{k}|} [e^{i|\boldsymbol{k}|ct} + e^{-i|\boldsymbol{k}|ct}] - \\
&- \sum_{j=1}^n \frac{|\boldsymbol{k}| e^{-i\sqrt{\boldsymbol{k}^2 + m_j^2}ct}}{2m_j^2 \sqrt{\boldsymbol{k}^2 + m_j^2} \prod_{p=1}^{j-1} \prod_{q=j+1}^n \left[(m_j^2 - m_p^2)(m_j^2 - m_q^2) \right]} + \\
&+ \sum_{j=1}^n \frac{|\boldsymbol{k}| e^{i\sqrt{\boldsymbol{k}^2 + m_j^2}ct}}{2m_j^2 \sqrt{\boldsymbol{k}^2 + m_j^2} \prod_{p=1}^{j-1} \prod_{q=j+1}^n \left[(m_j^2 - m_p^2)(m_j^2 - m_q^2) \right]} \Bigg\} =
\end{aligned}
\tag{38}
$$

Complete propagator:

$$
\begin{aligned}
D_r^{00} &= i\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}}\Bigg\{-\frac{e^{-i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|}+\frac{e^{i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|}+\sum_{j=1}^{n}\frac{e^{-i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]}-\\
&\quad -\sum_{j=1}^{n}\frac{e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]}-\\
&\quad +\frac{e^{-i|\boldsymbol{k}|ct}-e^{i|\boldsymbol{k}|ct}}{4|\boldsymbol{k}|\xi}-\frac{ict}{4\xi}\left(e^{-i|\boldsymbol{k}|ct}+e^{i|\boldsymbol{k}|ct}\right)+\\
&\quad +\frac{|\boldsymbol{k}|}{2(-1)^n\prod_{j=1}^{n}m_j^2}[e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}]-\frac{1}{4|\boldsymbol{k}|}[e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}]-\frac{ict}{4}[e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct}]-\\
&\quad -\sum_{j=1}^{n}\frac{(\boldsymbol{k}^2+m_j^2)e^{-i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{2m_j^2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]}+\\
&\quad +\sum_{j=1}^{n}\frac{(\boldsymbol{k}^2+m_j^2)e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}}{2m_j^2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]}\Bigg\}\\
&= i\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}}\Bigg\{\frac{e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}}{4|\boldsymbol{k}|}(1-\frac{1}{\xi})-\frac{ict}{4}[e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct}](1-\frac{1}{\xi})+\frac{|\boldsymbol{k}|}{2(-1)^n\prod_{j=1}^{n}m_j^2}[e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}]\\
&\quad +\sum_{j=1}^{n}\frac{(\boldsymbol{k}^2+m_j^2)\left[e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}-e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}\right]}{2m_j^2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]}\left(1-\frac{m_j^2}{\boldsymbol{k}^2+m_j^2}\right)\Bigg\}\\
&= i\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}}\Bigg\{\frac{e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}}{4|\boldsymbol{k}|}(1-\frac{1}{\xi})-\frac{ict}{4}[e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct}](1-\frac{1}{\xi})+\frac{|\boldsymbol{k}|}{2(-1)^n\prod_{j=1}^{n}m_j^2}[e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}]\\
&\quad +\sum_{j=1}^{n}\frac{\boldsymbol{k}^2\left[e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}-e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}\right]}{2m_j^2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]}\Bigg\}\\
&= -\frac{\delta(ct-r)}{8\pi r}(1-\frac{1}{\xi})-i\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}}\Bigg\{\frac{ict}{4}[e^{i|\boldsymbol{k}|ct}+e^{-i|\boldsymbol{k}|ct}](1-\frac{1}{\xi})-\frac{|\boldsymbol{k}|}{2(-1)^n\prod_{j=1}^{n}m_j^2}[e^{i|\boldsymbol{k}|ct}-e^{-i|\boldsymbol{k}|ct}]\\
&\quad -\sum_{j=1}^{n}\frac{\boldsymbol{k}^2\left[e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}-e^{i\sqrt{\boldsymbol{k}^2+m_j^2}ct}\right]}{2m_j^2\sqrt{\boldsymbol{k}^2+m_j^2}\prod_{p=1}^{j-1}\prod_{q=j+1}^{n}\left[(m_j^2-m_p^2)(m_j^2-m_q^2)\right]}\Bigg\}
\end{aligned}
\tag{39}
$$

III. GAUSSIAN SUPPRESSION

Try suppressing the contribution that comes to the propagator from gradient expansion with a gaussian. Gradient expansion is valid only for small $|k|$ which means that our expression only gives correction to the propagation near the lightcone.

$$
\begin{aligned}
D^{\mu\nu}_{xy} &= -\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2+\frac{(k^2)^2}{M^2}+i\epsilon} \\
&\rightarrow -\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2+\frac{(k^2)^2}{M^2}e^{-Ck^2}} \\
&= -\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2[1+\frac{k^2e^{-Ck^2}}{M^2}]} \\
&= -M^2\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}e^{Ck^2}}{k^2[M^2e^{Ck^2}+k^2]}
\end{aligned} \tag{40}
$$

Taylor expanding $\frac{M^2e^{Ck^2}}{k^2[M^2e^{Ck^2}+k^2]}$ to $O(k^6)$ and ignoring higher power contributions we get:

$$
\begin{aligned}
D^{\mu\nu}_{xy} = -\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\Big[&\frac{1}{k^2}-\frac{1}{M^2}+\left(\frac{1}{M^4}+\frac{C}{M^2}\right)k^2-\left(\frac{1}{M^6}+\frac{2C}{M^4}+\frac{C^2}{2M^2}\right)k^4 \\
&+\left(\frac{1}{M^8}+\frac{3C}{M^6}+\frac{2C^2}{M^4}+\frac{C^3}{6M^2}\right)k^6+...\Big]
\end{aligned} \tag{41}
$$

For small k C will have to be large in order to suppress the k^4 term so that the gaussian decays fast. Further reducing small terms gives: (note: not sure about definition of Fourier transform - or + sign... maybe metric is -+++ not +—)

$$
D^{\mu\nu}_{xy} = -\int \frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\left[\frac{1}{k^2}-\frac{1}{M^2}+\frac{C}{M^2}k^2-\frac{C^2}{2M^2}k^4+\frac{C^3}{6M^2}k^6+...\right] \tag{42}
$$

Now evaluate

$$
\begin{aligned}
D^{r\mu\nu}_{x\ 1} &= -g^{\mu\nu}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}\int\frac{dk^0}{2\pi}e^{-ik^0ct}\left[\frac{1}{k^2}-\frac{1}{M^2}+\frac{C}{M^2}k^2-\frac{C^2}{2M^2}k^4+\frac{C^3}{6M^2}k^6+...\right]\Bigg|_{|k^0\rightarrow k^0+i\epsilon} \\
&= -(or+)\frac{\delta(ct-r)}{4\pi r}g^{\mu\nu}+\frac{\delta^4(x)}{M^2}g^{\mu\nu}-\frac{g^{\mu\nu}C}{M^2}\int\frac{d^4k}{(2\pi)^4}e^{-(or+)ik_\alpha x^\alpha}\left[k^2-\frac{C}{2}k^4+\frac{C^2}{6}k^6+...\right]
\end{aligned} \tag{43}
$$

IV. NO SUPPRESSION

This time do not use function to suppress k, just assume that it is small.

$$
D^{\mu\nu}_{xy} = -g^{\mu\nu}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+\frac{(k^2)^2}{M^2}}\Bigg|_{k^0\rightarrow k^0+i\epsilon} \tag{44}
$$

Then this gives

$$
\begin{aligned}
D^{r\mu\nu}_{x\ 1} &= -g^{\mu\nu}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}\int\frac{dk^0}{2\pi}e^{-ik^0ct}\frac{M^2}{k^2(M^2+k^2)}\Bigg|_{k^0\rightarrow k^0+i\epsilon} \\
&= -g^{\mu\nu}\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}\int\frac{dk^0}{2\pi}e^{-ik^0ct}\frac{M^2}{(k^0-|\boldsymbol{k}|)(k^0+|\boldsymbol{k}|)((k^0)^2-|\boldsymbol{k}|^2+M^2)}\Bigg|_{k^0\rightarrow k^0+i\epsilon} \\
&= -g^{\mu\nu}M^2\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{kx}}\int\frac{dk^0}{2\pi}e^{-ik^0ct} \\
&\quad \frac{1}{(k^0+i\epsilon-|\boldsymbol{k}|)(k^0+i\epsilon+|\boldsymbol{k}|)(k^0+i\epsilon-\sqrt{|\boldsymbol{k}|^2-M^2})(k^0+i\epsilon+\sqrt{|\boldsymbol{k}|^2-M^2})}
\end{aligned} \tag{45}
$$

We have a restriction that $|k^2| << M^2 \to |(k^0)^2 - |\boldsymbol{k}|^2| << M^2$. So the sizes of temporal and spatial components of k must be of similar sizes. This means that we have two cases.

I) $|\boldsymbol{k}|^2 - M^2 > 0$

II) $|\boldsymbol{k}|^2 - M^2 < 0$

Case I) for $(t > 0)$:

$$\begin{aligned} D^{r\mu\nu}_{x\ 1} &= -g^{\mu\nu} i \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \left[\frac{e^{-i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|} - \frac{e^{i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|} + \frac{e^{i\sqrt{\boldsymbol{k}^2 - M^2}ct}}{2\sqrt{\boldsymbol{k}^2 - M^2}} - \frac{e^{-i\sqrt{\boldsymbol{k}^2 - M^2}ct}}{2\sqrt{\boldsymbol{k}^2 - M^2}} \right] \\ &= -g^{\mu\nu} \frac{\delta(ct - r)}{4\pi r} + \frac{g^{\mu\nu}}{8\pi r} \int_{-\infty}^{\infty} \frac{d|\boldsymbol{k}|}{2\pi} |\boldsymbol{k}| (e^{i|\boldsymbol{k}|r} - e^{-i|\boldsymbol{k}|r}) \left[\frac{e^{-i\sqrt{\boldsymbol{k}^2 - M^2}ct} - e^{i\sqrt{\boldsymbol{k}^2 - M^2}ct}}{\sqrt{\boldsymbol{k}^2 - M^2}} \right] \end{aligned} \quad (46)$$

Derivation of zitterbewegung as an effect of vacuum polarization.

Case II) for $(t > 0)$:

$$\begin{aligned} D^{r\mu\nu}_{x\ 1} &= -g^{\mu\nu} M^2 \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \int \frac{dk^0}{2\pi} e^{-ik^0 ct} \frac{1}{(k^0 + i\epsilon - |\boldsymbol{k}|)(k^0 + i\epsilon + |\boldsymbol{k}|)(k^0 + i\sqrt{M^2 - |\boldsymbol{k}|^2})(k^0 - i\sqrt{M^2 - |\boldsymbol{k}|^2})} \\ &= -g^{\mu\nu} i \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \left[\frac{e^{-i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|} - \frac{e^{i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|} + \frac{e^{-\sqrt{M^2 - |\boldsymbol{k}|^2}ct}}{2i\sqrt{M^2 - |\boldsymbol{k}|^2}} \right] \\ &= -g^{\mu\nu} \frac{\delta(ct - r)}{4\pi r} - g^{\mu\nu} \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \frac{e^{-\sqrt{M^2 - |\boldsymbol{k}|^2}ct}}{2\sqrt{M^2 - |\boldsymbol{k}|^2}} \\ &= -g^{\mu\nu} \frac{\delta(ct - r)}{4\pi r} - g^{\mu\nu} \frac{i}{4\pi^2 r} \int_0^{\infty} d|\boldsymbol{k}| |\boldsymbol{k}| (e^{-i|\boldsymbol{k}|r} - e^{i|\boldsymbol{k}|r}) \frac{e^{-\sqrt{M^2 - |\boldsymbol{k}|^2}ct}}{2\sqrt{M^2 - |\boldsymbol{k}|^2}} \\ &= -g^{\mu\nu} \frac{\delta(ct - r)}{4\pi r} + g^{\mu\nu} \frac{i}{8\pi r} \int_{-\infty}^{\infty} \frac{d|\boldsymbol{k}|}{2\pi} |\boldsymbol{k}| (e^{i|\boldsymbol{k}|r} - e^{-i|\boldsymbol{k}|r}) \frac{e^{-\sqrt{M^2 - |\boldsymbol{k}|^2}ct}}{\sqrt{M^2 - |\boldsymbol{k}|^2}} \end{aligned} \quad (47)$$

Case III)

$$\begin{aligned} D^{r\mu\nu}_{x\ 1} &= -g^{\mu\nu} M^2 \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \int \frac{dk^0}{2\pi} e^{-ik^0 ct} \frac{1}{(k^0 + i\epsilon - |\boldsymbol{k}|)(k^0 + i\epsilon + |\boldsymbol{k}|)(k^0 + i\epsilon)^2} \\ &= -g^{\mu\nu} M^2 i \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \left[\frac{e^{-i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|^3} - \frac{e^{i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|^3} + \frac{ict}{|\boldsymbol{k}|^2} \right] \\ &= g^{\mu\nu} \frac{M^2}{8\pi^2 r} \int_0^{\infty} dk (e^{-ikr} - e^{ikr}) \left[\frac{e^{-i|kct}}{k^2} - \frac{e^{ikct}}{k^2} + \frac{2ict}{k} \right] \\ &= -g^{\mu\nu} i \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} \left[\frac{e^{-i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|^3} - \frac{e^{i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|^3} + \frac{ict}{|\boldsymbol{k}|^2} \right] \end{aligned} \quad (48)$$

V. USEFUL

$$\int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{kx}} = \frac{1}{(2\pi)^3} \int e^{ikr\cos\theta} k^2 dk d(\cos\theta) d\phi = \frac{i}{4\pi^2 r} \int_0^{\infty} dkk(e^{-ikr} - e^{ikr}) \quad (49)$$

$$\frac{(n/2)!}{n!} = (\frac{1}{2})^{n/2} \frac{n!!}{n!} = (\frac{1}{2})^{n/2} \frac{1}{(n-1)!!} \; for \; even \; n \quad (50)$$

VI. HEAVYSIDE FUNCTION

First a little lemma:
Let us consider

$$F_x = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} \int \frac{dk^0}{2\pi} e^{-ik^0 ct} \frac{f_k}{\prod_j (k^0 - \omega_{j,\boldsymbol{k}})} \tag{51}$$

and

$$G_x = \Theta(t) F_x \tag{52}$$

where f_k is analytic. Then

$$G_x = \int \frac{d\omega}{2\pi i} \frac{e^{i\omega ct}}{\omega - i\epsilon} \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} \int \frac{dk^0}{2\pi} e^{-ik^0 ct} \frac{f_k}{\prod_j (k^0 - \omega_{j,\boldsymbol{k}})} \tag{53}$$

and

$$\begin{aligned} G_q &= \int d^4x e^{iqx} G_x \\ &= \int d^4x \int \frac{d\omega}{2\pi i} \frac{1}{\omega - i\epsilon} \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{x}(\boldsymbol{k}-\boldsymbol{q})} \int \frac{dk^0}{2\pi} e^{ict(\omega - k^0 + q^0)} \frac{f_k}{\prod_j (k^0 - \omega_{j,\boldsymbol{k}})} \\ &= \int \frac{d\omega}{2\pi i} \frac{f_{\omega + q^0 \boldsymbol{q}}}{(\omega - i\epsilon) \prod_j (\omega + q^0 - \omega_{j,\boldsymbol{q}})} \end{aligned} \tag{54}$$

We close the contour upward,

$$\begin{aligned} G_q &= F_q + \sum{}' \mathrm{Res} \frac{f_{\omega + q^0 \boldsymbol{q}}}{(\omega - i\epsilon) \prod_j (\omega + q^0 - \omega_{j,\boldsymbol{q}})} \\ &= F_q + \sum{}'_k \frac{f_{\omega_{k,\boldsymbol{q}},\boldsymbol{q}}}{(\omega_{k,\boldsymbol{q}} - q^0 - i\epsilon) \prod_{j \neq k} (\omega_{k,\boldsymbol{q}} - \omega_{j,\boldsymbol{q}})} \\ &= F_q - \sum{}'_k \frac{f_{\omega_{k,\boldsymbol{q}},\boldsymbol{q}}}{(q^0 - \omega_{k,\boldsymbol{q}} + i\epsilon) \prod_{j \neq k} (\omega_{k,\boldsymbol{q}} - \omega_{j,\boldsymbol{q}})} \end{aligned} \tag{55}$$

where $\sum'$ denotes the summation over poles with $\Im(\omega_{j,\boldsymbol{q}}) > 0$ and find

$$G_x = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\boldsymbol{x}} \int \frac{dk^0}{2\pi} e^{-ik^0 ct} \left[F_q - \sum{}'_k \frac{f_{\omega_{k,\boldsymbol{q}},\boldsymbol{q}}}{(q^0 - \omega_{k,\boldsymbol{q}}) \prod_{j \neq k} (\omega_{k,\boldsymbol{q}} - \omega_{j,\boldsymbol{q}})} \right] \tag{56}$$

The residuu of the function

$$G_q = \frac{f_q}{\prod_j (q^0 - \omega_{j,\boldsymbol{q}})} - \sum{}'_k \frac{f_{\omega_{k,\boldsymbol{q}},\boldsymbol{q}}}{(q^0 - \omega_{k,\boldsymbol{q}}) \prod_{j \neq k} (\omega_{k,\boldsymbol{q}} - \omega_{j,\boldsymbol{q}})} \tag{57}$$

is obtained from those of

$$F_q = \frac{f_q}{\prod_j (q^0 - \omega_{j,\boldsymbol{q}})} \tag{58}$$

by omitting the poles on the upper half plane. Thue the multiplication by the Heavyside funtion is equivalent by the removal of the residuu of the ”wrong” poles.

The removal of the "wrong" pole contribution cancels the contribution in $D_{t,\boldsymbol{x}}$ for $t<0$, $|ct|\ll|\boldsymbol{x}|$,

$$\begin{aligned}
D_x &\to -\Theta(t)\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+\frac{(k^2)^2}{M^2}}\Big|_{k^0\to k^0+i\epsilon}\\
&= -\Theta(t)\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}\int\frac{dk^0}{2\pi}e^{-ik^0ct}\frac{M^2}{k^2(M^2+k^2)}\Big|_{k^0\to k^0+i\epsilon}\\
&= -\Theta(t)\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}\int\frac{dk^0}{2\pi}e^{-ik^0ct}\frac{M^2}{(k^0-|\boldsymbol{k}|)(k^0+|\boldsymbol{k}|)((k^0)^2-|\boldsymbol{k}|^2+M^2)}\Big|_{k^0\to k^0+i\epsilon}\\
&= -\Theta(t)M^2\int\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}\int\frac{dk^0}{2\pi}e^{-ik^0ct}\\
&\quad\frac{1}{(k^0+i\epsilon-|\boldsymbol{k}|)(k^0+i\epsilon+|\boldsymbol{k}|)(k^0+i\epsilon-\sqrt{|\boldsymbol{k}|^2-M^2})(k^0+i\epsilon+\sqrt{|\boldsymbol{k}|^2-M^2})}
\end{aligned}\tag{59}$$

$$\begin{aligned}
D_x &= -\Theta(t)i\int_{|\boldsymbol{k}|>M}\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}\left[\frac{e^{-i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|}-\frac{e^{i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|}+\frac{e^{i\sqrt{\boldsymbol{k}^2-M^2}ct}}{2\sqrt{\boldsymbol{k}^2-M^2}}-\frac{e^{-i\sqrt{\boldsymbol{k}^2-M^2}ct}}{2\sqrt{\boldsymbol{k}^2-M^2}}\right]\\
&\quad -i\Theta(t)\int_{|\boldsymbol{k}|<M}\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}\left[\frac{e^{-i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|}-\frac{e^{i|\boldsymbol{k}|ct}}{2|\boldsymbol{k}|}+\frac{e^{-\sqrt{M^2-|\boldsymbol{k}|^2}ct}}{2i\sqrt{M^2-|\boldsymbol{k}|^2}}\right]\\
&= -\Theta(t)\frac{\delta(ct-r)}{4\pi r}-\Theta(t)i\int_{|\boldsymbol{k}|>M}\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}\frac{e^{i\sqrt{\boldsymbol{k}^2-M^2}ct}-e^{-i\sqrt{\boldsymbol{k}^2-M^2}ct}}{2\sqrt{\boldsymbol{k}^2-M^2}}\\
&\quad -i\Theta(t)\int_{\boldsymbol{k}<M}\frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\boldsymbol{x}}\frac{e^{-\sqrt{M^2-|\boldsymbol{k}|^2}ct}}{2i\sqrt{M^2-|\boldsymbol{k}|^2}}\\
&= -\Theta(t)\frac{\delta(ct-r)}{4\pi r}+\frac{\Theta(t)}{8\pi r}\int_{|\boldsymbol{k}|>M}\frac{d|\boldsymbol{k}|}{2\pi}|\boldsymbol{k}|(e^{i|\boldsymbol{k}|r}-e^{-i|\boldsymbol{k}|r})\frac{e^{-i\sqrt{\boldsymbol{k}^2-M^2}ct}-e^{i\sqrt{\boldsymbol{k}^2-M^2}ct}}{\sqrt{\boldsymbol{k}^2-M^2}}\\
&\quad +\frac{\Theta(t)i}{8\pi r}\int_{-M}^{M}\frac{d|\boldsymbol{k}|}{2\pi}|\boldsymbol{k}|(e^{i|\boldsymbol{k}|r}-e^{-i|\boldsymbol{k}|r})\frac{e^{-\sqrt{M^2-|\boldsymbol{k}|^2}ct}}{\sqrt{M^2-|\boldsymbol{k}|^2}}
\end{aligned}\tag{60}$$

Lorentz invariance: $D_{t,r}=D(c^2t^2-r^2)$,

$$\begin{aligned}
D_x &= \lim_{r\to0}\Bigg[\frac{\Theta(t)}{8\pi r}\int_{|\boldsymbol{k}|>M}\frac{d|\boldsymbol{k}|}{2\pi}|\boldsymbol{k}|(e^{i|\boldsymbol{k}|r}-e^{-i|\boldsymbol{k}|r})\frac{e^{-i\sqrt{\boldsymbol{k}^2-M^2}ct}-e^{i\sqrt{\boldsymbol{k}^2-M^2}ct}}{\sqrt{\boldsymbol{k}^2-M^2}}\\
&\quad +\frac{\Theta(t)i}{8\pi r}\int_{-M}^{M}\frac{d|\boldsymbol{k}|}{2\pi}|\boldsymbol{k}|(e^{i|\boldsymbol{k}|r}-e^{-i|\boldsymbol{k}|r})\frac{e^{-\sqrt{M^2-\boldsymbol{k}^2}ct}}{\sqrt{M^2-\boldsymbol{k}^2}}\Bigg]\\
&= i\frac{\Theta(t)}{4\pi}\int_{|\boldsymbol{k}|>M}\frac{d|\boldsymbol{k}|}{2\pi}\boldsymbol{k}^2\frac{e^{-i\sqrt{(\boldsymbol{k}^2-M^2)x^2}}-e^{i\sqrt{(\boldsymbol{k}^2-M^2)x^2}}}{\sqrt{\boldsymbol{k}^2-M^2}}+i\frac{\Theta(t)}{4\pi}\int_{-M}^{M}\frac{d|\boldsymbol{k}|}{2\pi}\boldsymbol{k}^2\frac{e^{-\sqrt{(M^2-\boldsymbol{k}^2)x^2}}}{\sqrt{M^2-\boldsymbol{k}^2}}
\end{aligned}\tag{61}$$

for $x^2>0$ and

$$D_x = \frac{\Theta(t)}{8\pi\sqrt{-x^2}}\int_{-M}^{M}\frac{d|\boldsymbol{k}|}{2\pi}|\boldsymbol{k}|\frac{e^{i|\boldsymbol{k}|\sqrt{-x^2}}-e^{-i|\boldsymbol{k}|\sqrt{-x^2}}}{\sqrt{M^2-\boldsymbol{k}^2}}\tag{62}$$

for $t\to0$ and $x^2<0$, ie.

$$\begin{aligned}
D_x &= i\frac{\Theta(x^2)\Theta(t)}{4\pi x^2}\int_{z>M\sqrt{x^2}}\frac{dz}{2\pi}z^2\frac{e^{-i\sqrt{z^2-M^2x^2}}-e^{i\sqrt{z^2-M^2x^2}}}{\sqrt{z^2-M^2x^2}}+i\frac{\Theta(x^2)\Theta(t)}{4\pi x^2}\int_{-M\sqrt{x^2}}^{M\sqrt{x^2}}\frac{dz}{2\pi}z^2\frac{e^{-\sqrt{M^2x^2-z^2}}}{\sqrt{M^2x^2-z^2}}\\
&\quad +\frac{\Theta(-x^2)\Theta(t)}{8\pi|x^2|}\int_{-M\sqrt{|x^2|}}^{M\sqrt{|x^2|}}\frac{dz}{2\pi}z\frac{e^{iz}-e^{-iz}}{\sqrt{M^2|x^2|-z^2}}
\end{aligned}\tag{63}$$

This self energy contribution is a regular function (and not distribution) because it is finite for $x^2 = 0$.

$$
\begin{aligned}
I(u) &= \int_{z>M\sqrt{x^2}} dz z^2 e^{iu\sqrt{z^2-M^2x^2}} \\
J(u) &= \int_{-M\sqrt{x^2}}^{M\sqrt{x^2}} dz z^2 e^{-u\sqrt{M^2x^2-z^2}} \\
K(u) &= \int_{-M\sqrt{|x^2|}}^{M\sqrt{|x^2|}} dz \frac{e^{iuz}}{\sqrt{M^2|x^2|-z^2}}
\end{aligned} \tag{64}
$$

$r \gg 1/M$:

$$
\begin{aligned}
D_x &= -\Theta(t)\frac{\delta(ct-r)}{4\pi r} + \frac{\Theta(t)i}{8\pi r}\int_{-M}^{M}\frac{d|\boldsymbol{k}|}{2\pi}|\boldsymbol{k}|(e^{i|\boldsymbol{k}|r} - e^{-i|\boldsymbol{k}|r})\frac{e^{-\sqrt{M^2-|\boldsymbol{k}|^2}ct}}{\sqrt{M^2-|\boldsymbol{k}|^2}} \\
&= -\Theta(t)\frac{\delta(ct-r)}{4\pi r} + \frac{\Theta(t)}{8\pi r}\partial_r\int_{-M}^{M}\frac{d|\boldsymbol{k}|}{2\pi}\frac{e^{i|\boldsymbol{k}|r-\sqrt{M^2-|\boldsymbol{k}|^2}ct}}{\sqrt{M^2-|\boldsymbol{k}|^2}} + (r \leftrightarrow -r) \\
&= -\Theta(t)\frac{\delta(ct-r)}{4\pi r} + \Theta(t)D'_x
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
\partial_{ct}D'_x &= -\frac{1}{16\pi^2 r}\partial_r\int_{-M}^{M} dk e^{ikr-\sqrt{M^2-k^2}ct} + (r \leftrightarrow -r) \\
&= -\frac{M}{16\pi^2 r}\partial_r\int_{-1}^{1} dz e^{M(izr-\sqrt{1-z^2}ct)} + (r \leftrightarrow -r)
\end{aligned} \tag{66}
$$

$R = Mr$, $T = ctM$,

$$
\begin{aligned}
\partial_T D'_{R,T} &= -\frac{M^2}{16\pi^2 R}\partial_R\int_{-1}^{1} dz e^{izR-\sqrt{1-z^2}T} + (R \leftrightarrow -R) \\
&= -\frac{M^2}{8\pi^2 R}\partial_R\int_{-1}^{1} dz \cos(zR)e^{-\sqrt{1-z^2}T}
\end{aligned} \tag{67}
$$

Notice that the integral $Q \equiv \int_{-1}^{1} dz\cos(zR)e^{-\sqrt{1-z^2}T}$ can be rewritten as a partial differential equation $\frac{\partial^2}{\partial T^2}Q + \frac{\partial^2}{\partial R^2}Q = Q \to \nabla^2 Q = Q$. But since we cannot be sure that Q is separable in R and T, the solution and boundary conditions might not be trivial. Another way to do this is to use power series expansion and expand $e^{-\sqrt{1-z^2}T}$.

$$
\begin{aligned}
\int_{-1}^{1} dz\cos(zR)e^{-\sqrt{1-z^2}T} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}T^n\int_{-1}^{1}\cos(zR)(1-z^2)^{n/2}dz = \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{\pi}}{n!}T^n\Gamma\left(1+\frac{n}{2}\right) Hypergeometric01FRegularized[\frac{3+n}{2}, -\frac{R^2}{4}] = \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{\pi}}{n!}T^n\Gamma\left(\frac{n}{2}+1\right)\left[\sum_{k=0}^{\infty}\left(-\frac{1}{4}\right)^k\frac{R^{2k}}{k!\Gamma\left(\frac{3+n}{2}+k\right)}\right] = \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{\pi}}{n!}T^n\Gamma\left(\frac{n}{2}+1\right)2^{\frac{1+n}{2}}R^{-\frac{1+n}{2}}J_{\frac{1+n}{2}}(R)
\end{aligned} \tag{68}
$$

where J_α is the "Bessel function of the first kind".

$$\begin{aligned}\int_{-1}^{1} dz\cos(zR)e^{-\sqrt{1-z^2}T} &= \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{\pi}}{n!}T^n\Gamma\left(\frac{n}{2}+1\right)2^{\frac{1+n}{2}}R^{-\frac{1+n}{2}}J_{\frac{1+n}{2}}(R) = \\ &= \sum_{n\ even\geq 0}^{\infty}\frac{\sqrt{\pi}(n/2)!}{n!}T^n 2^{\frac{1+n}{2}}R^{-\frac{1+n}{2}}J_{\frac{1+n}{2}}(R) - \\ &\quad - \sum_{n\ odd\geq 1}^{\infty}\frac{\pi}{(n-1)!!}T^n R^{-\frac{1+n}{2}}J_{\frac{1+n}{2}}(R) = \\ &= \sum_{n\ even\geq 0}^{\infty}\frac{\sqrt{\pi}(n/2)!}{n!}T^n 2^{\frac{1+n}{2}}R^{-\frac{1+n}{2}}J_{\frac{1+n}{2}}(R) - \\ &\quad - \sum_{n\ odd\geq 1}^{\infty}\frac{\pi}{(n-1)!!}T^n R^{-\frac{1+n}{2}}J_{\frac{1+n}{2}}(R) = \end{aligned} \tag{69}$$

$$\begin{aligned}\int_{-1}^{1} dz\cos(zR)e^{-\sqrt{1-z^2}T} &= \sum_{n\ even\geq 0}^{\infty}\frac{\sqrt{2\pi}}{(n-1)!!}T^n R^{-\frac{1+n}{2}}J_{\frac{1+n}{2}}(R) - \\ &\quad - \sum_{n\ odd\geq 1}^{\infty}\frac{\pi}{(n-1)!!}T^n R^{-\frac{1+n}{2}}J_{\frac{1+n}{2}}(R) = \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\boldsymbol{\bar{\pi}_2}(n)}{(n-1)!!}T^n R^{-\frac{1+n}{2}}J_{\frac{1+n}{2}}(R) \end{aligned} \tag{70}$$

where we have defined a function $\boldsymbol{\bar{\pi}_2}(j)$ that equals to $\sqrt{2\pi}$ if j is even; and π if j is odd. Also see its gamma function representation.

$$\begin{aligned}\boldsymbol{\bar{\pi}_2}(j) &\equiv \begin{cases}\sqrt{2\pi} & if\ j=0,2,4,...,2k \\ \pi & if\ j=1,3,...,2k+1\end{cases} \\ \boldsymbol{\bar{\pi}_2}(j) &= \frac{\sqrt{\pi}\Gamma(\frac{n}{2}+1)2^{\frac{n+1}{2}}(n-1)!!}{n!}\end{aligned} \tag{71}$$

Now the propagator is

$$\begin{aligned}\partial_T D'_{R,T} &= -\frac{M^2}{8\pi^2 R}\partial_R\int_{-1}^{1}dz\cos(zR)e^{-\sqrt{1-z^2}T} = \\ &= -\frac{M^2}{8\pi^2 R}\frac{\partial}{\partial R}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\boldsymbol{\bar{\pi}_2}(n)}{(n-1)!!}T^n R^{-\frac{1+n}{2}}J_{\frac{1+n}{2}}(R)\right] = \\ &= \frac{M^2}{8\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n\boldsymbol{\bar{\pi}_2}(n)}{(n-1)!!}T^n R^{-\frac{n+3}{2}}J_{\frac{n+3}{2}}(R) = \end{aligned} \tag{72}$$

$$D'_{R,T} = \frac{M^2}{8\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n\boldsymbol{\bar{\pi}_2}(n)}{(n+1)!!}T^{n+1}R^{-\frac{n+3}{2}}J_{\frac{n+3}{2}}(R) + f(R) \tag{73}$$

To find the function $f(R)$ which is a consequence of T integration consider the propagator $D'_{R,T}$ at $T=t=0$. We have

$$\begin{aligned}D'_x &= \frac{1}{8\pi r}\partial_r\left[\int_{-M}^{M}\frac{d|\boldsymbol{k}|}{2\pi}\frac{e^{i|\boldsymbol{k}|r-\sqrt{M^2-|\boldsymbol{k}|^2}ct}}{\sqrt{M^2-|\boldsymbol{k}|^2}} + (r\leftrightarrow -r)\right] \\ &= \frac{1}{16\pi^2 r}\partial_r\left[\int_{-M}^{M}dk\frac{e^{ikr}}{\sqrt{M^2-k^2}} + (r\leftrightarrow -r)\right]\end{aligned} \tag{74}$$

$$\begin{aligned} D'_{R,T}(T=0) &= \frac{M^2}{8\pi^2 R}\partial_R\left[\int_{-1}^{1} dz\frac{\cos(Rz)}{\sqrt{1-z^2}}\right] \\ &= \frac{M^2}{8\pi R}\frac{\partial}{\partial R}J_0(R) \\ &= -\frac{M^2}{8\pi R}J_1(R) \end{aligned} \quad (75)$$

Hence we see that

$$f(R) = -\frac{M^2}{8\pi R}J_1(R) \quad (76)$$

Therefore

$$\begin{aligned} D'_{R,T} &= -\frac{M^2}{8\pi R}J_1(R) + \frac{M^2}{8\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n\bar{\boldsymbol{\pi}}_{\mathbf{2}}(n)}{(n+1)!!}T^{n+1}R^{-\frac{n+3}{2}}J_{\frac{n+3}{2}}(R) \Longrightarrow \\ D'_x &= -\frac{M}{8\pi r}J_1(Mr) + \frac{1}{8\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n\bar{\boldsymbol{\pi}}_{\mathbf{2}}(n)}{(n+1)!!}M^{\frac{n+3}{2}}(ct)^{n+1}r^{-\frac{n+3}{2}}J_{\frac{n+3}{2}}(Mr) = \\ &= \frac{1}{8\pi^2}\sum_{n=-1}^{\infty}\frac{(-1)^n\bar{\boldsymbol{\pi}}_{\mathbf{2}}(n)}{(n+1)!!}M^{\frac{n+3}{2}}(ct)^{n+1}r^{-\frac{n+3}{2}}J_{\frac{n+3}{2}}(Mr) = \\ &= \frac{1}{8\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\bar{\boldsymbol{\pi}}_{\mathbf{2}}(n+1)}{n!!}M^{\frac{n+1}{2}}(ct)^{n}r^{-\frac{n+1}{2}}J_{\frac{n+1}{2}}(Mr) \end{aligned} \quad (77)$$

For this alternating series to be convergent the absolute value of series terms should be monotone decreasing. Figuring out the exact convergence conditions for ct seems hard. For r it is easy to see that each next term in the series is smaller and smaller. However, since we used the approximation of small k in the gradient expansion, our wave travels near the light-cone. Therefore ct cannot be much greater than r, and since Mr is much greater than 1, it damps terms, and so does the pure $r^{-(n+3)/2}$ dependence. We can suspect that series converges for ct and r that we will use. Therefore each next term gives smaller correction to radiation.

Numerical trial and error seems to imply quite convincingly that the series is convergent (monotone decreasing).

The full retarded propagator is

$$D_r(x) = -\Theta(t)\frac{\delta(ct-r)}{4\pi r} + \Theta(t)\frac{1}{8\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\bar{\boldsymbol{\pi}}_{\mathbf{2}}(n+1)}{n!!}M^{\frac{n+1}{2}}(ct)^{n}r^{-\frac{n+1}{2}}J_{\frac{n+1}{2}}(Mr) \quad (78)$$

Now we can calculate the Lienard-Wiechert potential $A^\mu(x)$. Since we seem to have lost the covariance of the propagator we can calculate the potential separately in its scalar and 3-vector potential $A^\mu = (\phi, \boldsymbol{A})$.

$$\phi = \phi_0 + \phi' \quad (79)$$

$$\begin{aligned} \phi' &= \Theta(t)\frac{e}{8\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\bar{\boldsymbol{\pi}}_{\mathbf{2}}(n+1)}{n!!}M^{\frac{n+1}{2}}c^n\int dt'\int d^3r'\delta(\boldsymbol{r'}-\boldsymbol{r}_0(t'))(t-t')^n(r-r')^{-\frac{n+1}{2}}J_{\frac{n+1}{2}}(M(r-r')) \\ &= \Theta(t)\frac{e}{8\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\bar{\boldsymbol{\pi}}_{\mathbf{2}}(n+1)}{n!!}M^{\frac{n+1}{2}}c^n\int dt'(t-t')^n(r-r_0(t'))^{-\frac{n+1}{2}}J_{\frac{n+1}{2}}(M(r-r_0(t'))) \end{aligned} \quad (80)$$

Integrate the above integral by parts.

$$
\begin{aligned}
&= \int dt'(t-t')^n (r-r_0(t'))^{-\frac{n+1}{2}} J_{\frac{n+1}{2}}(M(r-r_0(t'))) \\
&= -\frac{(t-t')^{n+1} J_{\frac{n+1}{2}}(M(r-r_0(t')))}{(n+1)(r-r_0(t'))^{\frac{n+1}{2}}} - M\int \frac{(t-t')^{n+1}}{(n+1)} \frac{J_{\frac{n+3}{2}}(M(r-r_0(t')))}{(r-r_0(t'))^{\frac{n+1}{2}}} \frac{dr_0(t')}{dt'} dt' \\
&= -\frac{(t-t')^{n+1} J_{\frac{n+1}{2}}(M(r-r_0(t')))}{(n+1)(r-r_0(t'))^{\frac{n+1}{2}}} + \frac{M(t-t')^{n+2} J_{\frac{n+3}{2}}(M(r-r_0(t')))}{(n+1)(n+2)(r-r_0(t'))^{\frac{n+1}{2}}} \frac{dr_0(t')}{dt'} \\
&\quad -M\int \frac{(t-t')^{n+1}}{(n+1)} \frac{d}{dt'} \left[\frac{J_{\frac{n+3}{2}}(M(r-r_0(t')))}{(r-r_0(t'))^{\frac{n+1}{2}}} \frac{dr_0(t')}{dt'} \right] dt'
\end{aligned} \tag{81}
$$

Write as an infinite series of derivative, since we can perform integration by parts infinite number of times, when we are left with infinite series and 1 integral that vanishes (probably. still need to get convergence more rigorous). If the series for the propagator converges, we can expect that this series converges too.

$$
\begin{aligned}
&= \left[\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(t-t')^{n+k+1} n!}{(n+k+1)!} \frac{d^k}{dt'^k} \left[\frac{J_{\frac{n+1}{2}}(M|\boldsymbol{r}-\boldsymbol{r_0}(t')|)}{|\boldsymbol{r}-\boldsymbol{r_0}(t')|^{\frac{n+1}{2}}} \right] \right]_{t'=0,\boldsymbol{r_0}(0)=0}^{t'=t,\boldsymbol{r_0}(t)} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k t^{n+k+1} n!}{(n+k+1)!} \frac{d^k}{dt'^k} \left[\frac{J_{\frac{n+1}{2}}(M|\boldsymbol{r}-\boldsymbol{r_0}(t')|)}{|\boldsymbol{r}-\boldsymbol{r_0}(t')|^{\frac{n+1}{2}}} \right]_{t'=0,\boldsymbol{r_0}(0)=0}
\end{aligned} \tag{82}
$$

So

$$
\begin{aligned}
\phi' &= \frac{e}{8\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\bar{\boldsymbol{\pi}}_{\boldsymbol{2}}(n+1)}{n!!} M^{\frac{n+1}{2}} c^n \sum_{k=0}^{\infty} \frac{(-1)^k t^{n+k+1} n!}{(n+k+1)!} \frac{d^k}{dt'^k} \left[\frac{J_{\frac{n+1}{2}}(M|\boldsymbol{r}-\boldsymbol{r_0}(t')|)}{|\boldsymbol{r}-\boldsymbol{r_0}(t')|^{\frac{n+1}{2}}} \right]_{t'=0,\boldsymbol{r_0}(0)=0} \\
&= \frac{e}{8\pi^2} \sum_{n=0}^{\infty} (n-1)!!\bar{\boldsymbol{\pi}}_{\boldsymbol{2}}(n+1) M^{\frac{n+1}{2}} c^n \sum_{k=0}^{\infty} \frac{(-1)^{n+k+1} t^{n+k+1}}{(n+k+1)!} \frac{d^k}{dt'^k} \left[\frac{J_{\frac{n+1}{2}}(M|\boldsymbol{r}-\boldsymbol{r_0}(t')|)}{|\boldsymbol{r}-\boldsymbol{r_0}(t')|^{\frac{n+1}{2}}} \right]_{t'=0,\boldsymbol{r_0}(0)=0}
\end{aligned} \tag{83}
$$

VII. EXAMPLE

Imagine a charge traveling with constant velocity and trajectory $\boldsymbol{r}_0(t) = Kct$ where $0 < K < 1$ is some constant.

$$
\phi' = \frac{e}{8\pi^2} \sum_{n=0}^{\infty} (n-1)!!\bar{\boldsymbol{\pi}}_{\boldsymbol{2}}(n+1) M^{\frac{n+1}{2}} c^n \sum_{k=0}^{\infty} \frac{(-1)^{n+k+1} t^{n+k+1}}{(n+k+1)!} \frac{d^k}{dt'^k} \left[\frac{J_{\frac{n+1}{2}}(M|r-Kct'|)}{|r-Kct'|^{\frac{n+1}{2}}} \right]_{t'=0,\boldsymbol{r_0}(0)=0} \tag{84}
$$

$M = 1$, $c = 1$, $K = 1/c$ just to see the shape of the graph. As t increases, the amplitude of graph increases.

Appendix A: CTP polarization tensor

Fourier integral:

$$
f_x = c\int_q f_q e^{ixq} = c\int_q f_q e^{itcq^0 - i\boldsymbol{x}\boldsymbol{q}} \tag{A1}
$$

$$
\tilde{G}_{(x\mu\sigma)(y\nu\tau)} = -i\hbar\sigma\tau\frac{1}{4}\sum_{\eta,\eta'=\pm 0^+} \mathrm{tr}[\gamma^\mu G^{\sigma\tau}_{x-\eta' e_0, y+\eta e_0} \gamma^\nu G^{\tau\sigma}_{y-\eta e_0, x+\eta' e_0}] \tag{A2}
$$

$$\begin{aligned}
\tilde{G}_{(x\mu\sigma)(y\nu\tau)} &= -i\hbar\sigma\tau\frac{1}{4}\sum_{\eta,\eta'}c^2\int_{p,q}\mathrm{tr}[\gamma^\mu G_q^{\sigma\tau}\gamma^\nu G_p^{\tau\sigma}]e^{-iq(x-y)-ip(y-x)+i\eta(q^0-p^0)+i\eta'(p^0-q^0)} \\
&= -i\hbar\sigma\tau\frac{1}{4}\sum_{\eta,\eta'}c^2\int_{p,q}\mathrm{tr}[\gamma^\mu G_q^{\sigma\tau}\gamma^\nu G_p^{\tau\sigma}]e^{-i(q-p)(x-y)+i(\eta-\eta')(q^0-p^0)} \\
&= -i\hbar\sigma\tau\frac{1}{4}\sum_{\eta,\eta'}c^2\int_{p,q}\mathrm{tr}[\gamma^\mu G_{q+p}^{\sigma\tau}\gamma^\nu G_p^{\tau\sigma}]e^{-iq(x-y)+i(\eta-\eta')q^0} \\
&= c\int_q \tilde{G}_q^{\mu\nu}e^{-i(x-y)q} = c\int_q \tilde{G}_q^{\mu\nu}e^{-i(t_x-t_y)cq^0+i(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{q}} \\
\tilde{G}_q^{\mu\nu} &= -i\hbar\sigma\tau\frac{1}{4}\sum_{\eta,\eta'}c\int_p\mathrm{tr}[\gamma^\mu G_{q+p}^{\sigma\tau}\gamma^\nu G_p^{\tau\sigma}]e^{i(\eta-\eta')q^0} \\
&= \left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\tilde{G}_q+\Delta\tilde{G}_q
\end{aligned} \tag{A3}$$

$c = 1$

$$\begin{aligned}
\hat{G}_k &= \begin{pmatrix} -G^n+iG_i & -G^f-iG_i \\ G^f-iG_i & G^n+iG_i \end{pmatrix} \\
&= (\not{k}+m)\left[\begin{pmatrix} \frac{1}{k^2-m^2+i\epsilon} & 2\pi i\delta(k^2-m^2)\Theta(-k^0) \\ 2\pi i\delta(k^2-m^2)\Theta(k^0) & -\frac{1}{k^2-m^2-i\epsilon} \end{pmatrix} + i\frac{2\pi\delta(k^2-m^2)}{e^{\beta\epsilon_k}+1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right] \\
&\to (\not{k}+m)\begin{pmatrix} \frac{1}{k^2-m^2+i\epsilon} & 2\pi i\delta(k^2-m^2)\Theta(-k^0) \\ 2\pi i\delta(k^2-m^2)\Theta(k^0) & -\frac{1}{k^2-m^2-i\epsilon} \end{pmatrix}
\end{aligned} \tag{A4}$$

$$\begin{aligned}
\hat{G}^r &= -G^{++}-G^{+-} \\
&= i\int_p \mathrm{tr}[G_{p+q}^{++}\gamma^{\mu'}G_p^{++}\gamma^\mu]e^{i\eta p^0} - i\int_p \mathrm{tr}[G_{p+q}^{+-}\gamma^{\mu'}G_p^{-+}\gamma^\mu]e^{i\eta p^0}
\end{aligned} \tag{A5}$$

++: Dirac matrix traces:

$$\begin{aligned}
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= -\mathrm{tr}(\gamma^\nu\gamma^\mu)+2g^{\mu\nu}\mathrm{tr}1 \\
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= g^{\mu\nu}\mathrm{tr}1 \to -\delta^{\mu\nu}\mathrm{tr}1 \\
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{tr}(\gamma^\nu\gamma^\mu)2g^{\rho\sigma}-\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) \\
&= 2g^{\mu\nu}g^{\rho\sigma}\mathrm{tr}1+\mathrm{tr}(\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho)-2g^{\sigma\nu}\mathrm{tr}(\gamma^\mu\gamma^\rho) \\
&= 2g^{\mu\nu}g^{\rho\sigma}\mathrm{tr}1-\mathrm{tr}(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho)+2g^{\mu\sigma}\mathrm{tr}(\gamma^\nu\gamma^\rho)-2g^{\sigma\nu}g^{\mu\rho}\mathrm{tr}1 \\
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= (g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\sigma\nu}g^{\mu\rho})\mathrm{tr}1
\end{aligned} \tag{A6}$$

Convention: $\mathrm{tr}1 = 4$

$$\begin{aligned}
N^{\mu\nu} &= \mathrm{tr}\gamma^\mu\left[(p_\alpha+q_\alpha)\gamma^\alpha+m\right]\gamma^\nu\left[p_\beta\gamma^\beta+m\right] \\
&= 4m^2g^{\mu\nu}+(p_\alpha+q_\alpha)p_\beta\mathrm{tr}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta \\
&= 4m^2g^{\mu\nu}+4(p^\mu+q^\mu)p^\nu+4(p^\nu+q^\nu)p^\mu-4g^{\mu\nu}p(p+q) \\
&= 4g^{\mu\nu}\left(m^2-p^2-pq\right)+8p^\mu p^\nu+4(p^\mu q^\nu+q^\mu p^\nu)
\end{aligned} \tag{A7}$$

$$\tilde{G}_q^{\mu\nu} = -i\hbar c\int_p \frac{4g^{\mu\nu}\left(\frac{m^2c^2}{\hbar^2}-p^2-pq\right)+8p^\mu p^\nu+4(p^\mu q^\nu+q^\mu p^\nu)}{(p^2-\frac{m^2c^2}{\hbar^2}+i\epsilon)[(p+q)^2-\frac{m^2c^2}{\hbar^2}+i\epsilon]} \tag{A8}$$

Feynman parametrization:

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[(1-x)a + xb]^2} \tag{A9}$$

$$\begin{aligned}
\tilde{G}_q^{\mu\nu} &= -i\hbar c \int dx \int_p \frac{4g^{\mu\nu}[\lambda_C^{-2} - p(p+q)] + 8p^\mu p^\nu + 4(p^\mu q^\nu + q^\mu p^\nu)}{[p^2 + 2xpq + xq^2 - \lambda_C^{-2} + i\epsilon]^2}, \qquad p \to p - qx \\
&= -i\hbar c \int dx \int_p \\
&\times \frac{4g^{\mu\nu}[\lambda_C^{-2} - (p-xq)(p+(1-x)q)] + 8(p^\mu - xq^\mu)(p^\nu - xq^\nu) + 4[(p^\mu - xq^\mu)q^\nu + q^\mu(p^\nu - xq^\nu)]}{[p^2 + x(1-x)q^2 - \lambda_C^{-2} + i\epsilon]^2} \\
&= -i\hbar c \int dx \int_p \frac{4g^{\mu\nu}[\lambda_C^{-2} - p^2 + x(1-x)q^2] + 8p^\mu p^\nu + 8x^2 q^\mu q^\nu - 8xq^\mu q^\nu}{[p^2 + x(1-x)q^2 - \lambda_C^{-2} + i\epsilon]^2} \\
&= -4i\hbar c \int dx \int_p \frac{g^{\mu\nu}[\lambda_C^{-2} - p^2 + x(1-x)q^2] + 2p^\mu p^\nu + 2x(x-1)q^\mu q^\nu}{[p^2 + x(1-x)q^2 - \lambda_C^{-2} + i\epsilon]^2}
\end{aligned} \tag{A10}$$

Euclidean integrals:

$$\begin{aligned}
\int \frac{d^d p}{(p^2 + M^2)^2} &= \int \frac{d^d p}{(p^2 + M^2)^2} = \pi^{d/2}(M^2)^{d/2-2}\frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Gamma(2)} \\
\int \frac{d^d p p^\mu p^\nu}{(p^2 + M^2)^2} &= \delta^{\mu\nu}\frac{M^2}{2-d} I_0
\end{aligned} \tag{A11}$$

Wick-rotation:

$$\tilde{G}_q^{\mu\nu} = 4\hbar c \int dx \int_p \frac{\delta^{\mu\nu}[p^2 - x(1-x)q^2 + \lambda_C^{-2}] - 2p^\mu p^\nu + 2(1-x)xq^\mu q^\nu}{[p^2 + x(1-x)q^2 + \lambda_C^{-2}]^2} \tag{A12}$$

Dimensional regularization:

$$\begin{aligned}
\tilde{G}_q^{\mu\nu} &= \mu^\epsilon 4\hbar c \int dx \int_p \frac{\delta^{\mu\nu}[p^2 - x(1-x)q^2 + \lambda_C^{-2}] - 2p^\mu p^\nu + 2(1-x)xq^\mu q^\nu}{[p^2 + x(1-x)q^2 + \lambda_C^{-2}]^2} \\
&= \mu^\epsilon \frac{4\hbar c}{(2\pi)^d} \int dx \left[\delta^{\mu\nu}\left(M^2 \frac{d}{2-d} - M^2 + 2\lambda_C^{-2}\right) - 2\delta^{\mu\nu} M^2 \frac{1}{2-d} + 2(1-x)xq^\mu q^\nu\right] \pi^{d/2}(M^2)^{d/2-2}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \\
&= 4\mu^\epsilon \hbar c \pi^{d/2} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)(2\pi)^d} \int dx \left[\delta^{\mu\nu}\left(M^2 \frac{2d-4}{2-d} + 2\lambda_C^{-2}\right) + 2(1-x)xq^\mu q^\nu\right] (M^2)^{d/2-2} \\
&= 8\mu^\epsilon \hbar c \pi^{d/2} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)(2\pi)^d} \int dx \left[\delta^{\mu\nu}(-M^2 + \lambda_C^{-2}) + (1-x)xq^\mu q^\nu\right] (M^2)^{d/2-2}
\end{aligned} \tag{A13}$$

$M^2 = x(1-x)q^2 + \lambda_C^{-2}$

$$\begin{aligned}
\tilde{G}_q^{\mu\nu} &= 8\mu^\epsilon \hbar c \pi^{d/2} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)(2\pi)^d} \int dx[-\delta^{\mu\nu}x(1-x)q^2 + (1-x)xq^\mu q^\nu][x(1-x)q^2 + \lambda_C^{-2}]^{d/2-2} \\
&= -q^2 T^{\mu\nu} 8\mu^\epsilon \hbar c \pi^{d/2} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)(2\pi)^d} \int dx x(1-x)[x(1-x)q^2 + \lambda_C^{-2}]^{d/2-2} \\
&= -q^2 T^{\mu\nu} 8\mu^\epsilon \hbar c \frac{\pi^{2-\frac{\epsilon}{2}}}{(2\pi)^{4-\epsilon}} \Gamma\left(\frac{\epsilon}{2}\right) \int dx x(1-x)[x(1-x)q^2 + \lambda_C^{-2}]^{-\frac{\epsilon}{2}} \\
&= -q^2 T^{\mu\nu} 8\mu^\epsilon \hbar c \frac{\pi^2(1-\frac{\epsilon}{2}\ln\pi)(1+\epsilon\ln 2\pi)(\frac{2}{\epsilon} - \gamma)}{(2\pi)^4} \int dx x(1-x)\left[1 - \frac{\epsilon}{2}\ln(x(1-x)q^2 + \lambda_C^{-2}\right] \\
&= -\frac{q^2}{\pi^2} T^{\mu\nu} \hbar c(1+\epsilon\ln\mu)(1-\frac{\epsilon}{2}\ln\pi)(1+\epsilon\ln 2\pi)(\frac{1}{\epsilon} - \frac{\gamma}{2}) \int dx x(1-x)\left[1 - \frac{\epsilon}{2}\ln(x(1-x)q^2 + \lambda_C^{-2}\right]
\end{aligned} \tag{A14}$$

$\tilde{G}^{\mu\nu} = T^{\mu\nu}\tilde{G}$:

$$\int dx x(1-x) = \frac{1}{6} \tag{A15}$$

$$\begin{aligned}\tilde{G}_q &= -\frac{q^2}{6\pi^2}\hbar c\left[\frac{1}{\epsilon} - \frac{\gamma}{2} + \ln\mu - \frac{1}{2}\ln\pi + \ln 2\pi\right] + \frac{q^2}{2\pi^2}\hbar c\int dx x(1-x)\ln(x(1-x)q^2 + \lambda_C^{-2})\\ &= -\frac{q^2}{6\pi^2}\hbar c\left[\frac{1}{\epsilon} - \frac{\gamma}{2} + \ln\mu - \frac{1}{2}\ln\pi + \ln 2\pi\right] + \frac{q^2}{6\pi^2}\hbar c\ln\mu + \frac{q^2}{2\pi^2}\hbar c\int dx x(1-x)\ln\frac{x(1-x)q^2 + \lambda_C^{-2}}{\mu^2}\end{aligned} \tag{A16}$$

Real time:

$$\tilde{G}_q = \frac{q^2}{6\pi^2}\hbar c\left[\frac{1}{\epsilon} - \frac{\gamma}{2} - \frac{1}{2}\ln\pi + \ln 2\pi\right] - \frac{q^2}{2\pi^2}\hbar c\int dx x(1-x)\ln\frac{\lambda_C^{-2} - x(1-x)q^2 - i\epsilon_F}{\mu^2} \tag{A17}$$

Gradient expansion:

$$\begin{aligned}\tilde{G}_q &= \frac{q^2}{6\pi^2}\hbar c\left[\frac{1}{\epsilon} - \frac{\gamma}{2} - \frac{1}{2}\ln\pi + \ln 2\pi\right] - \frac{q^2}{2\pi^2}\hbar c\int dx x(1-x)\ln\frac{\lambda_C^{-2}}{\mu^2}[1 - \lambda_C^2(x(1-x)q^2 + i\epsilon_F)]\\ &\approx \frac{q^2}{6\pi^2}\hbar c\left[\frac{1}{\epsilon} - \frac{\gamma}{2} - \frac{1}{2}\ln\pi + \ln 2\pi\right] - \frac{q^2}{12\pi^2}\hbar c\ln\frac{\lambda_C^{-2}}{\mu^2} + \frac{q^2\lambda_C^2 q^2}{2\pi^2}\hbar c\int dx x^2(1-x)^2\\ &= \frac{q^2}{6\pi^2}\hbar c\left[\frac{1}{\epsilon} - \frac{\gamma}{2} - \frac{1}{2}\ln\pi + \ln 2\pi - \frac{1}{2}\ln\frac{\lambda_C^{-2}}{\mu^2}\right] + \frac{q^2\lambda_C^2 q^2}{60\pi^2}\hbar c\end{aligned} \tag{A18}$$

$$\int dx x^2(1-x)^2 = \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} \to \frac{6 - 15 + 10}{30} = \frac{1}{30} \tag{A19}$$

On-shell subtraction, $\alpha = \frac{e^2}{4\pi\hbar c}$ inserted:

$$\tilde{G}_q^{++} = -\tilde{G}_q^{--} = \frac{q^2 e^2\hbar c}{60\pi^2\hbar^2 c^2}\lambda_C^2 q^2 = \frac{q^2\alpha}{15\pi}\lambda_C^2 q^2 \tag{A20}$$

+-:

$$\tilde{G}_q^{\mu\nu} = -i\hbar\sigma\tau\frac{1}{4}\sum_{\eta,\eta'} c\int_p \mathrm{tr}[\gamma^\mu G_{q+p}^{\sigma\tau}\gamma^\nu G_p^{\tau\sigma}]e^{i(\eta-\eta')q^0} \tag{A21}$$

$$\hat{G}_k = \hat{G}_k^{vac} + \hat{G}_k^{env} = (\not{k} + \frac{mc}{\hbar})\left[\begin{pmatrix}\frac{1}{k^2 - \frac{m^2c^2}{\hbar^2} + i\epsilon} & 2\pi i\delta(k^2 - \frac{m^2c^2}{\hbar^2})\Theta(-k^0)\\ 2\pi i\delta(k^2 - \frac{m^2c^2}{\hbar^2})\Theta(k^0) & -\frac{1}{k^2 - \frac{m^2c^2}{\hbar^2} - i\epsilon}\end{pmatrix} + i\frac{2\pi\delta(k^2 - \frac{m^2c^2}{\hbar^2})}{e^{\beta\epsilon_k} + 1}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right] \tag{A22}$$

Vacuum:

$$\begin{aligned}\tilde{G}_q^{+-\mu\nu} &= -i\hbar c\int_p \mathrm{tr}[\gamma^\mu(\not{p} + \not{q} + \lambda_C^{-1})2\pi\delta((p+q)^2 - \lambda_C^{-2})\Theta(-p^0 - k^0)\gamma^\nu(\not{p} + \lambda_C^{-1})2\pi\delta(p^2 - \lambda_C^{-2})\Theta(p^0)]\\ &= -i\hbar c\int_p N(p,q)2\pi\delta((p+q)^2 - \lambda_C^{-2})\Theta(-p^0 - q^0)2\pi\delta(p^2 - \lambda_C^{-2})\Theta(p^0)\end{aligned} \tag{A23}$$

$$N(p,q) = 4g^{\mu\nu}(\lambda_C^{-2} - p^2 - pq) + 8p^\mu p^\nu + 4(p^\mu q^\nu + q^\mu p^\nu) \to -4g^{\mu\nu}pq + 8p^\mu p^\nu + 4(p^\mu q^\nu + q^\mu p^\nu) \tag{A24}$$

$p \to p + aq$:

$$\begin{aligned}N(p,q) &\to -4g^{\mu\nu}(p + aq)q + 8(p^\mu + aq^\mu)(p^\nu + aq^\nu) + 4[(p^\mu + aq^\mu)q^\nu + q^\mu(p^\nu + aq^\nu)]\\ &= -4g^{\mu\nu}(q^2a + pq) + 8p^\mu p^\nu + 8a^2q^\mu q^\nu + 8a(q^\mu p^\nu + p^\mu q^\nu) + 4(p^\mu q^\nu + q^\mu p^\nu) + 8aq^\mu q^\nu\\ &= -4g^{\mu\nu}(q^2a + pq) + 8p^\mu p^\nu + 8a(a+1)q^\mu q^\nu + 4(2a+1)(q^\mu p^\nu + p^\mu q^\nu)\end{aligned} \tag{A25}$$

$a = -\frac{1}{2}$:

$$\begin{aligned} N'(p,q) &= 2g^{\mu\nu}(q^2 - 2pq) + 8p^\mu p^\nu - 2q^\mu q^\nu \\ &= 2q^2 T^{\mu\nu} - 4g^{\mu\nu} pq + 8p^\mu p^\nu \end{aligned} \tag{A26}$$

$$\begin{aligned} \tilde{G}_q^{+-\mu\nu} &= -i\hbar c \int_p (2q^2 T^{\mu\nu} - 4g^{\mu\nu} pq + 8p^\mu p^\nu) 2\pi\delta\left(\left(p + \frac{q}{2}\right)^2 - \lambda_C^{-2}\right)\Theta\left(-p^0 - \frac{q^0}{2}\right) \\ &\quad 2\pi\delta\left(\left(p - \frac{q}{2}\right)^2 - \lambda_C^{-2}\right)\Theta\left(p^0 - \frac{q^0}{2}\right) \\ &= -i\hbar c\Theta(-q^0)\frac{1}{4\pi^2}\int d^3p \int_{-\frac{|q^0|}{2}}^{\frac{|q^0|}{2}} dp^0 (2q^2 T^{\mu\nu} - 4g^{\mu\nu} pq + 8p^\mu p^\nu) \\ &\quad \times\delta\left(\left(p + \frac{q}{2}\right)^2 - \lambda_C^{-2}\right)\delta\left(\left(p - \frac{q}{2}\right)^2 - \lambda_C^{-2}\right) \\ &= -i\hbar c\Theta(-q^0)\frac{1}{4\pi^2}\int d^3p \int_{-\frac{|q^0|}{2}}^{\frac{|q^0|}{2}} dp^0 (2q^2 T^{\mu\nu} - 4g^{\mu\nu} pq + 8p^\mu p^\nu) \\ &\quad \times\frac{\delta(p^0 + \frac{q^0}{2} - \omega_{\boldsymbol{p}+\frac{\boldsymbol{q}}{2}}) + \delta(p^0 + \frac{q^0}{2} + \omega_{\boldsymbol{p}+\frac{\boldsymbol{q}}{2}})}{2\omega_{\boldsymbol{p}+\frac{\boldsymbol{q}}{2}}}\frac{\delta(p^0 - \frac{q^0}{2} - \omega_{\boldsymbol{p}-\frac{\boldsymbol{q}}{2}}) + \delta(p^0 - \frac{q^0}{2} + \omega_{\boldsymbol{p}-\frac{\boldsymbol{q}}{2}})}{2\omega_{\boldsymbol{p}-\frac{\boldsymbol{q}}{2}}} = 0 \end{aligned} \tag{A27}$$

for small q.