Cosmology/gravity

Matthew Stephenson

Stanford University

E-mail: matthewjstephenson@icloud.com

Contents

1 Boundary theory coupled to gravity

Bulk action

$$S = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R + 2\Lambda^{(5)}\right) - \int d^5x\mathcal{L}_m, \tag{1.1}$$

where $\Lambda^{(5)} = -d(d-1)/2L^2 = -6/L^2$. The stress-energy tensor is

$$T^{\mu\nu} = \frac{1}{8\pi G_5}\left[K^{\mu\nu} - K\gamma^{\mu\nu} - \frac{3}{L}\gamma^{\mu\nu} + \frac{L}{2}\left(R^{\mu\nu} - \frac{1}{2}R\gamma^{\mu\nu}\right) + \ldots\right]. \tag{1.2}$$

Introducing Λ as the four-dimensional cosmological constant, we find

$$R^{\mu\nu} - \frac{1}{2}R\gamma^{\mu\nu} - \Lambda\gamma^{\mu\nu} + \ldots - \frac{2}{L}8\pi G_5 T^{\mu\nu} = -\frac{2}{L}\left(K^{\mu\nu} - \gamma^{\mu\nu}K\right) + \left(\Lambda + \frac{6}{L^2}\right)\gamma^{\mu\nu}. \tag{1.3}$$

Setting the LHS Einstein's equation to zero, with an effective $G_4 = 2G_5/L$, we get the identity

$$K^{\mu\nu} = -\frac{1}{L}\left(1 + \frac{L^2\Lambda}{6}\right)\gamma^{\mu\nu}. \tag{1.4}$$

2 Dilaton gravity

Theory

$$S = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R - 2\partial_\mu\phi\partial^\mu\phi - 2\Lambda^{(5)}e^{\eta\phi}\right) \tag{2.1}$$

Equations of motion

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - \frac{1}{2}\eta\Lambda^{(5)}e^{\eta\phi} = 0 \tag{2.2}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda^{(5)}g_{\mu\nu}e^{\eta\phi} - 2\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\partial_\lambda\phi\partial^\lambda\phi = 0 \tag{2.3}$$

Solution with set $\Lambda^{(5)} = -6$ is

$$ds^2 = -f(r)dt^2 + \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}}\left(dx^2+dy^2+dz^2\right) + \frac{dr^2}{f(r)},$$
$$\text{where}\quad f(r) = \frac{\left(8+3\eta^2\right)^2 r_h^2}{64-6\eta^2}\left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}}\left[1-\left(\frac{r_h}{r}\right)^{\frac{32-3\eta^2}{8+3\eta^2}}\right] \tag{2.4}$$

$$\phi = -\frac{6\eta}{8+3\eta^2}\log\left(r/r_h\right) \tag{2.5}$$

Look for a brane embedding

$$t(r) \tag{2.6}$$

A set of normalised tangent vectors

$$T^\mu = \sqrt{\frac{f}{f^2\left(\partial_r t\right)^2 - 1}}\left(\frac{\partial t}{\partial r}, 0, 0, 0, 1\right) \tag{2.7}$$

$$X^\mu = r^{-\frac{8}{8+3\eta^2}}\left(0,1,0,0,0\right) \tag{2.8}$$

$$Y^\mu = r^{-\frac{8}{8+3\eta^2}}\left(0,0,1,0,0\right) \tag{2.9}$$

$$Z^\mu = r^{-\frac{8}{8+3\eta^2}}\left(0,0,0,1,0\right) \tag{2.10}$$

so that $T^2 = -1$ and $X^2 = Y^2 = Z^2 = 1$. A normal is

$$n_\mu = nf(r)\left(-1,0,0,0,\frac{\partial t}{\partial r}\right) \tag{2.11}$$

where

$$n^\mu n_\mu = 1 \tag{2.12}$$

implies

$$n = \frac{1}{\sqrt{f\left(f^2\left(\partial_r t\right)^2 - 1\right)}}, \tag{2.13}$$

hence

$$n_\mu = \sqrt{\frac{f}{f^2\left(\partial_r t\right)^2 - 1}}\left(-1, 0, 0, 0, \frac{\partial t}{\partial r}\right). \tag{2.14}$$

The full set of vectors is

$$T^\mu = \sqrt{\frac{f}{f^2\left(\partial_r t\right)^2 - 1}}\left(\frac{\partial t}{\partial r}, 0, 0, 0, 1\right) \tag{2.15}$$

$$\vec{X}^\mu = r^{-\frac{8}{8+3\eta^2}}\left(0, \vec{1}, 0\right) \tag{2.16}$$

$$n^\mu = \sqrt{\frac{f}{f^2\left(\partial_r t\right)^2 - 1}}\left(\frac{1}{f}, 0, 0, 0, f\frac{\partial t}{\partial r}\right) \tag{2.17}$$

Induced metric

$$g^{(\mathrm{ind})}_{\mu\nu} \equiv \gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \tag{2.18}$$

Extrinsic curvature

$$K_{\mu\nu} = -\left(\delta^\lambda_\mu - n_\mu n^\lambda\right)\nabla_\lambda n_\nu \tag{2.19}$$

Junction condition (check)

$$K_{\mu\nu} = -\gamma_{\mu\nu} \tag{2.20}$$

Solution

$$\frac{\partial t}{\partial r} = \pm\frac{\left(8 + 3\eta^2\right)r}{f\sqrt{\left(8 + 3\eta^2\right)^2 r^2 - 64f}} \tag{2.21}$$

The induced metric $g^{(\mathrm{ind})}_{\mu\nu} = \gamma_{\mu\nu}$ is given by the line element

$$ds^2_\gamma = -\frac{64}{\left(8 + 3\eta^2\right)^2 r^2 - 64f(r)}dr^2 + \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}}\left(dx^2 + dy^2 + dz^2\right) \tag{2.22}$$

Solve

$$\tau = C + 8\int\frac{dr}{\sqrt{\left(8 + 3\eta^2\right)^2 r^2 - 64f(r)}} \tag{2.23}$$

$$a(\tau)^2 = \left(\frac{r(\tau)}{r_h}\right)^{\frac{16}{8+3\eta^2}} \tag{2.24}$$

so that we find the induced FRW metric

$$ds^2_\gamma = -d\tau^2 + a(\tau)^2\left(dx^2 + dy^2 + dz^2\right) \tag{2.25}$$

At $\eta = 0$ we find the standard radiation-dominated (CFT) result of Gubser.
For $\eta > 0$ and at large r,

$$f(r) \to \frac{\left(8+3\eta^2\right)^2 r_h^2}{64-6\eta^2}\left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}} \tag{2.26}$$

and we find

$$r = \exp\left\{\frac{8+3\eta^2}{8}\left(\tau - const.\right)\right\} \tag{2.27}$$

and

$$a(\tau) = \mathcal{C}e^{\tau}. \tag{2.28}$$

At non-zero η, for $r \approx r_h$

$$a \tag{2.29}$$

3 Probe fields

Consider a probe scalar field ϕ with an action

$$S = -\frac{K}{2}\int_{\mathcal{M}} d^5x\sqrt{-g}\nabla_\mu\phi\nabla^\mu\phi + \ldots, \tag{3.1}$$

which satisfied the equation of motion in five dimensions. The boundary action is then

$$\begin{aligned} S &= -\frac{K}{2}\int_{\mathcal{M}} d^5x\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\phi\partial_\nu\phi\right) = -\frac{k}{2}\int_{\mathcal{M}} d^5x\sqrt{-g}\nabla_\mu\left(\phi\nabla^\mu\phi\right) \\ &= -\frac{K}{2}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}n_\mu\phi\partial^\mu\phi, \end{aligned} \tag{3.2}$$

since $\nabla_\mu\phi = \partial_\mu\phi$.

Using the foliation $t = t(r)$ and the normal $n_\mu = nf(r)\left(-1,0,0,0,\frac{\partial t}{\partial r}\right)$, the boundary action is then

$$S = -\frac{K}{2}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}\;\phi\; g^{\mu\nu}n_\mu\partial_\nu\phi \tag{3.3}$$

$$= -\frac{K}{2}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}\;n(r)f(r)\phi\left(-g^{tt}\frac{\partial}{\partial t} + g^{rr}\frac{\partial t}{\partial r}\frac{\partial}{\partial r}\right)\phi, \tag{3.4}$$

which for our theory gives

$$S = -K\int_{\partial\mathcal{M}} d^4x\frac{r^{\frac{24}{8+3\eta^2}}}{\left(8+3\eta^2\right)r}\frac{\left(8+3\eta^2\right)^2 r^2 - 32f}{\sqrt{\left(8+3\eta^2\right)^2 r^2 - 64f}}\phi\frac{\partial\phi}{\partial r} \tag{3.5}$$

We wish to impose the Dirichlet boundary condition on the hypersurface $t(r)$, i.e. $\phi = $ const., which means

$$\partial_i\phi\left(t,x^i,r\right)\Big|_{\partial\mathcal{M}} = 0, \tag{3.6}$$

and

$$\left[-f(r)^2\frac{\partial t}{\partial r}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right]\phi\left(t,x^i,r\right)\Big|_{\partial\mathcal{M}}=0 \tag{3.7}$$

$$\Longrightarrow\left[-\frac{\left(8+3\eta^2\right)rf}{\sqrt{\left(8+3\eta^2\right)^2r^2-64f}}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right]\phi\left(t,x^i,r\right)\Big|_{\partial\mathcal{M}}=0. \tag{3.8}$$

At $\eta=0$, this gives

$$\left[-\frac{r^4-r_h^4}{r_h^2}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right]\phi\left(t,x^i,r\right)\Big|_{\partial\mathcal{M}}=0 \tag{3.9}$$

Consider the bulk solution decomposed as

$$\phi(t,\vec{x},r)=\int\frac{d^4k}{(2\pi)^4}e^{i\omega t-i\vec{k}\cdot\vec{x}}\varphi_k(r) \tag{3.10}$$

$$\frac{\partial\phi}{\partial r}=\int\frac{d^4k}{(2\pi)^4}e^{i\omega t-i\vec{k}\cdot\vec{x}}\left(i\omega\frac{\partial t}{\partial r}\varphi_k+\frac{\partial\varphi_k}{\partial r}\right)=\int\frac{d^4k}{(2\pi)^4}e^{i\omega t-i\vec{k}\cdot\vec{x}}\left(\frac{i\omega\left(8+3\eta^2\right)r}{f\sqrt{\left(8+3\eta^2\right)^2r^2-64f}}\varphi_k+\frac{\partial\varphi_k}{\partial r}\right) \tag{3.11}$$

Hence

$$\begin{aligned}S=&-K\int\frac{d^4kd^4pd^3xdr}{(2\pi)^8}e^{i\left(k^0+p^0\right)t(r)-i\left(\vec{k}+\vec{p}\right)\cdot\vec{x}}\frac{r^{\frac{24}{8+3\eta^2}}}{\left(8+3\eta^2\right)r}\frac{\left(8+3\eta^2\right)^2r^2-32f}{\sqrt{\left(8+3\eta^2\right)^2r^2-64f}}\\&\times\left(\frac{ik^0\left(8+3\eta^2\right)r}{f\sqrt{\left(8+3\eta^2\right)^2r^2-64f}}\varphi_p\varphi_k+\varphi_p\frac{\partial\varphi_k}{\partial r}\right) \qquad (3.12)\\=&-K\int\frac{d^4kdp^0dr}{(2\pi)^5}e^{i\left(k^0+p^0\right)t(r)}\frac{r^{\frac{24}{8+3\eta^2}}}{\left(8+3\eta^2\right)r}\frac{\left(8+3\eta^2\right)^2r^2-32f}{\sqrt{\left(8+3\eta^2\right)^2r^2-64f}}\\&\times\left(\frac{ik^0\left(8+3\eta^2\right)r}{f\sqrt{\left(8+3\eta^2\right)^2r^2-64f}}\varphi_{p^0,-\vec{k}}\cdot\varphi_{k^0,\vec{k}}+\varphi_{p^0,-\vec{k}}\frac{\partial\varphi_{k^0,\vec{k}}}{\partial r}\right) \qquad (3.13)\end{aligned}$$

(3.14)

3.1 Conformal case

At $\eta=0$, we have

$$\frac{\partial t}{\partial r}=\frac{r}{f\sqrt{r^2-f}}=\frac{1}{r_h^2}\frac{1}{1-\left(r_h/r\right)^4} \tag{3.15}$$

we find

$$t=\frac{r_0+r}{r_h^2}+\frac{1}{4r_h}\sum_{n=0}^{3}\left(i^n\log\left[1-i^n\frac{r_h}{r}\right]\right) \tag{3.16}$$

Then

$$S=-K\int\frac{d^4kdp^0dr}{(2\pi)^5}e^{i\left(k^0+p^0\right)t(r)}\frac{r\left(r_h^4+r^4\right)}{2r_h^2}\left(\frac{ik^0r^4}{r_h^2\left(r^4-r_h^4\right)}\varphi_{p^0,-\vec{k}}\cdot\varphi_{k^0,\vec{k}}+\varphi_{p^0,-\vec{k}}\frac{\partial\varphi_{k^0,\vec{k}}}{\partial r}\right) \tag{3.17}$$

and furthermore

$$\begin{aligned}S=&-\frac{K}{2}\int\frac{d^4kdp^0dr}{(2\pi)^5}e^{\frac{i\left(k^0+p^0\right)}{r_h^2}(r_0+r)}\left(\frac{1-r_h/r}{1+r_h/r}\right)^{\frac{i\left(k^0+p^0\right)}{4r_h}}\left(\frac{1-ir_h/r}{1+ir_h/r}\right)^{-\frac{\left(k^0+p^0\right)}{4r_h}}\\&\times\frac{r^5\left(1+(r_h/r)^4\right)}{r_h^2}\left(\frac{ik^0}{r_h^2\left(1-(r_h/r)^4\right)}\varphi_{p^0,-\vec{k}}\cdot\varphi_{k^0,\vec{k}}+\varphi_{p^0,-\vec{k}}\frac{\partial\varphi_{k^0,\vec{k}}}{\partial r}\right)\end{aligned} \tag{3.18}$$

and using $z=r_h/r$, $z_0=r_h/r_0$ and $T=r_h/\pi$, as well as $\mathfrak{k}^0=k^0/(2\pi T)$, $\mathfrak{p}^0=p^0/(2\pi T)$,

$$\begin{aligned}S=&-\frac{\pi^3T^5K}{2}\int\frac{d^3k}{(2\pi)^3}\int_0^1 d\mathfrak{k}^0d\mathfrak{p}^0dz\;e^{2i\left(\mathfrak{k}^0+\mathfrak{p}^0\right)\frac{z_0+z}{z_0z}}\left(\frac{1-z}{1+z}\right)^{\frac{1}{2}i\left(\mathfrak{k}^0+\mathfrak{p}^0\right)}\\&\times\left(\frac{1-iz}{1+iz}\right)^{-\frac{1}{2}\left(\mathfrak{k}^0+\mathfrak{p}^0\right)}\frac{1+z^4}{z^5}\left(\frac{2i\mathfrak{k}^0}{z^2\left(1-z^4\right)}\varphi_{\mathfrak{p}^0,-\vec{k}}\cdot\varphi_{\mathfrak{k}^0,\vec{k}}-\varphi_{\mathfrak{p}^0,-\vec{k}}\cdot\frac{\partial\varphi_{\mathfrak{k}^0,\vec{k}}}{\partial z}\right)\end{aligned} \tag{3.19}$$

4 Fluid/gravity

Work out the foliation procedure in Eddington-Finkelstein coordinates! Consider the five-dimensional black brane metric

$$\begin{aligned}ds^2&=-r^2f(r)dt^2+\frac{dr^2}{r^2f(r)}+r^2\left(dx^2+dy^2+dz^2\right),\\&\text{where}\quad f(r)=1-\left(\frac{r_h}{r}\right)^4.\end{aligned} \tag{4.1}$$

Change coordinates to the Eddington-Finkelstein coordinate v,

$$t=v-\frac{1}{4r_h}\sum_{i=0}^{3}\left(i^k\log\left[1-i^k\frac{r}{r_h}\right]\right), \tag{4.2}$$

so that

$$ds^2=-r^2f(r)dv^2+2dvdr+r^2\left(dx^2+dy^2+dz^2\right). \tag{4.3}$$

We know the metric solution at first order. Perturb

$$n_\mu=n^\mu_{(0)}+\epsilon n^\mu_{(1)} \tag{4.4}$$

so

$$n^\mu n^\nu=n^\mu_{(0)}n^\nu_{(0)}+\epsilon\left(n^\mu_{(0)}n^\nu_{(1)}+n^\mu_{(1)}n^\nu_{(0)}\right) \tag{4.5}$$

and $K_{\mu\nu} = -\left(\delta_\mu^\lambda - n_\mu n^\lambda\right) \nabla_\lambda n_\nu$ leads to

$$K_{\mu\nu} = K_{(0)\mu\nu} + \epsilon K_{(1)\mu\nu}. \tag{4.6}$$

First-order metric takes the form

$$ds^2 = \sum_{n=1}^{6} \mathcal{A}_n, \tag{4.7}$$

where

$$\mathcal{A}_1 = -2u_a dx^a dr, \qquad \mathcal{A}_2 = -r^2 f_0(br) u_a u_b dx^a dx^b, \tag{4.8}$$

$$\mathcal{A}_3 = r^2 \Delta_{ab} dx^a dx^b, \qquad \mathcal{A}_4 = 2r^2 b F_0(br) \sigma_{ab} dx^a dx^b, \tag{4.9}$$

$$\mathcal{A}_5 = \frac{2}{3} r u_a u_b \partial_c u^c dx^a dx^b, \qquad \mathcal{A}_6 = -r u^c \partial_c \left(u_a u_b\right) dx^a dx^b \tag{4.10}$$

and f_0 and F_0 are expanded to first order in derivatives of b and u^μ.

Use the foliation

$$t(x^a, br) = t_0(r) + \epsilon\left(x^a \partial_a b_0 + b_1\right) r t_0'(r) + \epsilon t_1(r) \partial_a u^a + \epsilon t_2(r) u^a \partial_a b. \tag{4.11}$$

set of unnormalised tangent vectors

$$R^\mu = \left(\frac{\partial t}{\partial r}, 0, 0, 0, 1\right) \tag{4.12}$$

$$X^\mu = (0, 1, 0, 0, 0) \tag{4.13}$$

$$Y^\mu = (0, 0, 1, 0, 0) \tag{4.14}$$

$$Z^\mu = (0, 0, 0, 1, 0) \tag{4.15}$$

Thus

$$0 = g_{\mu\nu} R^\mu n^\nu = \frac{\partial t}{\partial r} n_0 + n_4 \Longrightarrow n_4 = -\frac{\partial t}{\partial r} n_0 \tag{4.16}$$

so

$$n_\mu = n\left(-1, 0, 0, 0, \frac{\partial t}{\partial r}\right) \tag{4.17}$$

5 Probe scalar in WKB approximation

5.1 Conformal case

Consider the conformal case with the metric

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2\left(dx^2 + dy^2 + dz^2\right),$$
$$\text{where} \quad f(r) = 1 - \left(\frac{r_h}{r}\right)^4. \tag{5.1}$$

The scalar two-point function in the large mass $m \gg 1$ approximation scales as

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim \exp\left\{-m\int d\tau\sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\right\} \equiv e^{-S}. \tag{5.2}$$

Let us compute an equal-time correlator, which implies that we are fixing the position of the brane $t(r)$ at some bulk position $\rho = r_h\sqrt{2\tau}$, in terms of the boundary time. Choosing the proper time $\tau = x$, the exponent is

$$S = m\int dx\sqrt{r^2 - r^2 f t'^2 + \frac{1}{r^2 f}r'^2}. \tag{5.3}$$

Since we want an equal time correlator we will set $t' = 0$. S possesses a conserved quantity

$$H = r'\frac{\partial L}{\partial r'} - L = -\frac{r^2}{\sqrt{r^2 + \frac{r'^2}{r^2 f}}}. \tag{5.4}$$

Let us focus only on late-time behaviour, so that $\rho \gg r_h$ and $f(r) \approx 1$. Looking for a geodesic between $x = \pm\ell/2$ at $r = \rho$ we find

$$x = \pm\sqrt{\frac{4+\ell^2\rho^2}{4\rho^2} - \frac{1}{r^2}} + \mathcal{O}(r_h^4). \tag{5.5}$$

The action the becomes

$$S = 2m\int_{2\rho/\sqrt{4+\ell^2\rho^2}}^{\rho}\frac{dr}{\sqrt{r^2 - \frac{4\rho^2}{2+\ell^2\rho^2}}} = \log\left[\frac{1}{4}\left(\ell\rho + \sqrt{4+\ell^2\rho^2}\right)^2\right], \tag{5.6}$$

hence for $\ell^2\rho^2 \gg 1$,

$$e^{-S} \sim \frac{1}{(\ell\rho)^{2m}} \tag{5.7}$$

Assuming $\Delta \sim m \gg 1$ and knowing that the scale factor scales as

$$a(\tau) \propto \sqrt{\tau}, \tag{5.8}$$

the equal time scalar correlator is

$$\langle \mathcal{O}(\tau,\vec{x})\mathcal{O}(\tau,\vec{y})\rangle \sim \frac{1}{|\vec{x}-\vec{y}|^{2\Delta}\, a(\tau)^{2\Delta}}. \tag{5.9}$$

5.2 Non-conformal case

The metric is

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}}\left(dx^2+dy^2+dz^2\right) + \frac{dr^2}{f(r)}, \\ \text{where} \quad f(r) &= \frac{\left(8+3\eta^2\right)^2 r_h^2}{64-6\eta^2}\left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}}\left[1-\left(\frac{r_h}{r}\right)^{\frac{32-3\eta^2}{8+3\eta^2}}\right]. \end{aligned} \tag{5.10}$$

Again we are interested in $r \gg r_h$, so

$$ds^2 = -f(r)dt^2 + \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}} \left(dx^2 + dy^2 + dz^2\right) + \frac{dr^2}{f(r)},$$
$$\text{where} \quad f(r) = \frac{\left(8+3\eta^2\right)^2 r_h^2}{64-6\eta^2} \left(\frac{r}{r_h}\right)^{\frac{16}{8+3\eta^2}} . \tag{5.11}$$

With $t' = 0$ we get with $\alpha = 8 + 3\eta^2$

$$S = m \int dx \sqrt{\left(\frac{r}{r_h}\right)^{16/\alpha} + \frac{64-6\eta^2}{\alpha r_h^2} \left(\frac{r_h}{r}\right)^{16/\alpha} r'^2} \tag{5.12}$$

and

$$H = -\frac{(r/r_h)^{16/\alpha}}{\sqrt{(r/r_h)^{16/\alpha} + \frac{64-6\eta^2}{\alpha r_h^2} (r_h/r)^{16/\alpha} r'^2}} \tag{5.13}$$

Hence

$$\frac{dr}{dx} = \frac{r_h \sqrt{\alpha}}{H\sqrt{64-6\eta^2}} \left(\frac{r}{r_h}\right)^{16/\alpha} \sqrt{\left(\frac{r}{r_h}\right)^{16/\alpha} - H^2} \tag{5.14}$$

We find with a new variable $u = r/r_h$,

$$x = \pm \frac{\sqrt{2\alpha\left(32-3\eta^2\right)}}{H(16-\alpha)} u^{-\frac{16-\alpha}{\alpha}} \sqrt{u^{16/\alpha} - H^2}\, {}_2F_1\left[1, -\frac{8-\alpha}{16}, \frac{\alpha}{16}, \frac{u^{16/\alpha}}{H^2}\right] \tag{5.15}$$

We need to fix H.....

Further

$$S = m\sqrt{\frac{64-6\eta^2}{8+3\eta^2}} \int_{u_{min}}^{u_\rho} \frac{du}{\sqrt{u^{16/(8+3\eta^2)} - H^2}}$$
$$= -\frac{m}{H^2}\sqrt{\frac{64-6\eta^2}{8+3\eta^2}} \left[u\sqrt{u^{16/(8+3\eta^2)} - H^2}\, {}_2F_1\left[1, \frac{16+3\eta^2}{16}, \frac{24+3\eta^2}{16}, \frac{u^{16/(8+3\eta^2)}}{H^2}\right]\right]_{u_{\min}}^{u_\rho} \tag{5.16}$$

6 Notes

The gravitational action has a restricted time domain because the brane moves outwards from the horizon or some radial position where the cosmological evolution in the model begins. Hence the action has the form

$$S_{bulk} = \int_{\mathcal{M}} d^5x\, \mathcal{L} = \int_{r_h}^{\infty} dr \int_{-\infty}^{\infty} d^3x \int_{t_0}^{\mathcal{T}(r)} dt\, \mathcal{L}. \tag{6.1}$$

This gives the usual bulk equations of motion which enable us to solve the hyper-surface embedding equation and find $t(r)$. In AdS-Schwarzschild this is

$$\mathcal{T}(r) = \frac{r}{r_h^2} + \frac{1}{4r_h}\sum_{n=0}^{3}\left(i^n \log\left[1 - i^n\frac{r_h}{r}\right]\right) - \mathcal{T}_0. \tag{6.2}$$

Notice that this expression diverges as $r \to r_h$. We can thus choose for the boundary to start at some $r_0 > r_h$ at time $t = 0$. Thus

$$\mathcal{T} = \frac{r_0}{r_h^2} + \frac{1}{4r_h}\sum_{n=0}^{3}\left(i^n \log\left[1 - i^n\frac{r_h}{r_0}\right]\right). \tag{6.3}$$

Consider a probe scalar field ϕ with an action

$$S = -\frac{K}{2}\int_{\mathcal{M}} d^5x\sqrt{-g}\nabla_\mu\phi\nabla^\mu\phi + \ldots, \tag{6.4}$$

which satisfied the equation of motion in five dimensions. The boundary action is then

$$\begin{aligned} S &= -\frac{K}{2}\int_{\mathcal{M}} d^5x\, \partial_\mu\left(\sqrt{-g}g^{\mu\nu}\phi\partial_\nu\phi\right) = -\frac{k}{2}\int_{\mathcal{M}} d^5x\,\sqrt{-g}\nabla_\mu\left(\phi\nabla^\mu\phi\right) \\ &= -\frac{K}{2}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}n_\mu\phi\partial^\mu\phi = -\frac{K}{2}\int_{r_0}^{\infty}dr\int d^3x\sqrt{-\gamma}n_\mu\phi\partial^\mu\phi, \end{aligned} \tag{6.5}$$

since $\nabla_\mu\phi = \partial_\mu\phi$ and having used $t = \mathcal{T}(r)$.