A new continued fraction approximation for the Gamma Function

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ABSTRACT

In this paper, by using Euler connection, we establish an accurate continued fraction approximation for the gamma function and determine all parameters of the continued fraction by Bernoulli numbers. Also new accurate continued fraction bounds for the gamma function are established.

Keywords: Gamma function, Continued fraction, Euler connection, Bernoulli number

1. Introduction

Today the Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ is one of the most well-known formulas for approximation of the factorial function by being widely applied in statistical physics, probability theory and number theory. Up to now, many researchers made great efforts in the area of establishing more accurate approximations for the factorial function and more precise inequalities, and had a lot of inspiring results.

The Stirling's series for the gamma function is presented (see [1, p.257, Eq. (7.1.40)]) by

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)x^{2n-1}}\right), \qquad x \to \infty. \tag{1.1}$$

where B_n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ denotes the Bernoulli numbers defined by the generating formula

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi .$$

Recently, some authors have focused on continued fractions in order to obtain new asymptotic formulas.

For example, on the one hand, Mortici [7] found Stieltjes' continued fraction

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp \cfrac{a_0}{x + \cfrac{a_1}{x + \cfrac{a_2}{x + \ddots}}}, \tag{1.2}$$

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where $a_0 = \frac{1}{12}$, $a_1 = \frac{1}{30}$, $a_2 = \frac{53}{210}$, $\cdots$.

Also Mortici [5] provided a new continued fraction approximation starting from the Nemes' formula as follows,

$$\Gamma(x+1) \approx \sqrt{2\pi x} e^{-x} \left(x + \cfrac{1}{12x - \cfrac{1}{10x + \cfrac{a}{x + \cfrac{b}{x + \cfrac{c}{\ddots}}}}}\right)^x, \tag{1.3}$$

where $a = -\frac{2369}{252}, b = \frac{2117009}{1193976}, c = \frac{393032191511}{1324011300744}, \cdots$.

On the other hand, Lu [4] provided a new continued fraction approximation based on the Burnside's formula as follows,

$$n! \approx \sqrt{2\pi} \left(\frac{n + \frac{1}{2}}{e}\right)^{n+\frac{1}{2}} \left(1 + \cfrac{a_1}{n^2 + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \ddots}}}\right)^{\frac{n-\frac{1}{2}}{k}}, \tag{1.4}$$

where $a_1 = -\frac{k}{24}$, $a_2 = \frac{k}{48} - \frac{23}{120}$, $a_3 = \frac{14}{5k-46}$, $\cdots$.

Also Lu [5] found two asymptotic formulas

$$\Gamma(x+1) \approx \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \cfrac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2} + \frac{1}{x^2} \cfrac{a_1}{x + \cfrac{a_2}{\ddots}}}\right)^{x^2 + \frac{53}{210}}, \tag{1.5}$$

where $a_1 = \frac{2117}{35280}, a_2 = \frac{188098}{116435}, a_3 = \frac{168152685438}{199300834733}, \cdots$, and

$$\Gamma(x+1) \approx \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2 + \frac{53}{210}} \left(1 + \frac{1}{x^6} \cfrac{b_1}{x + \cfrac{b_2}{x + \ddots}}\right), \tag{1.6}$$

where $b_1 = -\frac{2117}{5080320}, b_2 = \frac{1892069}{978054}, b_3 = \frac{40642696685}{92884530882}, \cdots$.

In this paper, we focus the continued fraction approximation for the gamma function.

Until now many continued fraction approximations for the gamma function were given, but the parameters of the continued fraction were not determined analytically, only determined by computer programs such as Mathematica and Maple system.([4]-[8])

By using Euler connection between series and continued fractions, we establish an accurate continued fraction approximation for the gamma function and determine all parameters of the continued fraction by Bernoulli numbers.

2. Lemmas

Lemma 2.1(The Euler connection [3, p.19, Eq. (1.7.1, 1.7.2)]). Let $\{c_k\}$ be a sequence in $\mathbb{C}\backslash\{0\}$ and

$$f_n = \sum_{k=0}^{n} c_k, \quad n \in \mathbb{N}_0. \tag{2.1}$$

Since $f_0 \neq \infty$, $f_n \neq f_{n-1}$, $n \in \mathbb{N}$, there exists a continued fraction $b_0 + K(a_m / b_m)$ with n^{th} approximant f_n for all n. This continued fraction is given by

$$c_0 + \frac{c_1}{1} \, \frac{-c_2/c_1}{+1+c_2/c_1} + \cdots \frac{-c_m/c_{m-1}}{+1+c_m/c_{m-1}} + \cdots. \tag{2.2}$$

Remark 2.1. The following lemma shows that transformation to the continued fraction from series. This lemma is very useful for research of the continued fraction approximation.
Lemma 2.2. For every $x > 0$,

$$\sum_{i=1}^{n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} = \mathop{K}_{i=1}^{n} \frac{a_i}{x + \frac{b_i}{x}}, \quad n \in \mathbb{N}, \tag{2.3}$$

where

$$a_1 = \frac{B_2}{2},\ b_1 = 0, a_i = -\frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}}, b_i = \frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}} = -a_i,\ i = 2, 3, \cdots n.$$

Lemma 2.3 ([2, Theorem 8]**).** Let $n \geq 0$ be an integer. The functions

$$F_n = \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln(2\pi) - \sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \tag{2.4}$$

and

$$G_n = -\ln\Gamma(x) + \left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \tag{2.5}$$

are strictly completely monotonic on (0, ∞).

3. Main results

Theorem 3.1. As $x \to \infty$, we have the continued fraction approximation of $\Gamma(x+1)$,

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\mathop{K}_{i=1}^{\infty} \frac{a_i}{x + \frac{b_i}{x}} \right), \tag{3.1}$$

where

$$a_1 = \frac{B_2}{2},\ b_1 = 0,\ a_i = -\frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}}, b_i = \frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}} = -a_i, \quad i = 2, 3, \cdots.$$

Proof. From Lemma 2.2, as $x \to \infty$,

$$\sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} = \mathop{K}_{i=1}^{\infty} \frac{a_i}{x + \frac{b_i}{x}} \Leftrightarrow \exp\left(\sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \right) = \exp\left(\mathop{K}_{i=1}^{\infty} \frac{a_i}{x + \frac{b_i}{x}} \right), \tag{3.2}$$

where

$$a_1 = \frac{B_2}{2},\ b_1 = 0,\ a_i = -\frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}}, b_i = \frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}} = -a_i, \quad i = 2, 3, \cdots.$$

According to the Stirling's series,

$$\Gamma(x+1) \approx \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)x^{2n-1}} \right) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\mathop{K}_{i=1}^{\infty} \frac{a_i}{x + \frac{b_i}{x}} \right) \tag{3.3}$$

Thus, our new continued fraction approximation can be obtained.

Remark 3.1. As you can see, our new continued fraction approximation for the gamma function is equal to the Stirling's series.

Theorem 3.2. *For every $x > 0$, we have continued fraction bounds for the gamma function:*

$$\exp\left(\mathop{K}_{i=1}^{2n} \frac{a_i}{x + \frac{b_i}{x}} \right) < \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} < \exp\left(\mathop{K}_{i=1}^{2n+1} \frac{a_i}{x + \frac{b_i}{x}} \right), \quad n \in \mathbb{N}_0, \tag{3.4}$$

where

$$a_1 = \frac{B_2}{2},\ b_1 = 0, a_i = -\frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}}, b_i = \frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}} = -a_i, \quad i = 2, 3, \cdots.$$

Proof. From Lemma 2.3, $F_n(x) > 0$ and $G_n(x) > 0$ for $x > 0$, so we obtain

$$\exp\left(\sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \right) < \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} < \exp\left(\sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} \right). \tag{3.5}$$

Then, from Lemma 2.2, for $x > 0$,

$$\sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} = \underset{i=1}{\overset{2n}{K}} \frac{a_i}{x + \frac{b_i}{x}} \Leftrightarrow \exp\left(\sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right) = \exp\left(\underset{i=1}{\overset{2n}{K}} \frac{a_i}{x + \frac{b_i}{x}}\right), \tag{3.6}$$

$$\sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}} = \underset{i=1}{\overset{2n+1}{K}} \frac{a_i}{x + \frac{b_i}{x}} \Leftrightarrow \exp\left(\sum_{i=1}^{2n+1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}\right) = \exp\left(\underset{i=1}{\overset{2n+1}{K}} \frac{a_i}{x + \frac{b_i}{x}}\right), \tag{3.7}$$

where

$$a_1 = \frac{B_2}{2},\ b_1 = 0, a_i = -\frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}}, b_i = \frac{(i-1)(2i-3)B_{2i}}{i(2i-1)B_{2(i-1)}} = -a_i, \quad i = 2, 3, \cdots.$$

Thus, our new continued fraction bounds for the gamma function are obtained.

Remark 3.2. Theorem 3.1 and **Theorem 3.2** show that all parameters of the continued fraction are determined analytically by Bernoulli numbers.

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