# A History of Light Deflection with

# Newtonian and General Relativity Perspectives

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Abstract

This work delves into three papers that address the topic of light being deflected by gravitating bodies, or alternately, light following geodesic paths through space-time. Einstein achieved public notoriety for his successful prediction that light would be deflected when passing close to the Sun, with his theory of General Relativity providing the foundation for an accurate estimate of the angle of deflection. However, 200 years prior to this Newton also predicted that light would be deflected by massive bodies such as our Sun. Newton did not attempt a calculation, but subsequently Johann Soldner and Henry Cavendish used Newtonian principles to study the degree of deflection.

In this paper we explore the three efforts, providing readers with a consolidated view of this topic, and a comprehensive analysis of the calculations that support the various approaches to estimating the deflection of a photon in the vicinity of a massive body. While it is clear that General Relativity provides the most accurate, and the overwhelmingly accepted solution, we provide details for the Newtonian efforts for historical perspective. We focus on Soldner and Cavendish first for this historic perspective, and then delve into the complexities of the Einstein General Relativity calculation.

We envision the target audience for this paper as undergraduate/early graduate students in math, physics, or astronomy, who have an interest in exploring the topic of light

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deflection. We also view this as a course outline for a professor that would like to offer a seminar on this topic.

Keywords: Cavendish, Einstein, General Relativity, light deflection, Soldner, space-time curvature.

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**INTRODUCTION:**

One of the more famous predictions made by Einstein [4] was that light would follow a geodesic path through curved space-time. He indicated that this would cause the altering of the apparent position of stars when their light passed by large gravitating bodies such as our Sun. He made this prediction, published in 1916 and based on his General Theory of Relativity, that light passing close to the Sun would be deflected by 1.75 arcseconds.

This prediction caught the attention of astronomers, since they saw a means for testing the prediction. During a solar eclipse, stars that are close to the horizon of the Sun might be observed. Since star positions were known with great precision, they could detect any apparent shift in the visual position of stars close to the edge of the Sun, allowing astronomers to confirm or deny Einstein's prediction.

In 1919, Eddington[2],[6] was able to conduct a successful test of Einstein's prediction and confirmed that a star's apparent position had shifted by an amount that was consistent with Einstein's prediction.

Prior to Einstein's prediction, however, Isaac Newton had suggested that light would be deflected by gravity. In 1704, when he was 62, he posited "Do not Bodies act upon Light at a distance, and by their action bend its rays, and is not this action strongest at the least distance?" [11].

Newton did not attempt to calculate the deflection, leaving this to others. Subsequently, several attempts were made to derive a means of calculating the deflection caused by a massive body. The most prominent was an effort by the German astronomer Johann Soldner (1776-1833). In 1801 he wrote a paper "On the Deflection of a Light Ray from its Rectilinear Motion". This paper develops a means of calculating the deflection of light, with Soldner predicting that the Sun would deflect a light ray passing close to the Sun by 0.84 arcseconds.

A second attempt at calculating the deflection of light was attributed to the English physicist Henry Cavendish (1731-1810). His work was only partially documented. There is uncertainty about when he did his work, with the time frame ranging between 1784 and 1804.

This paper pulls together information about the subject of light being deflected by gravity, and alternately, light following geodesic paths through curved space-time. We attempt to provide commentary and clarity to some of the available papers on this topic. Our target audience is those individuals with an interest in a historical view of this topic, individuals with basic understanding of general relativity that want a deeper understanding of light deflection, as well as those who want a better explanation of the mathematics behind the light deflection predictions. We have structured this paper as three independent sections, focused on:

- Section A - A translation of the Johann Soldner 1801 paper [13]. This paper uses Newtonian principles to calculate the deflection of a light ray due to gravity

- Section B - The Cavendish effort as described in the Lotze Simionato papers [8], [9]. Cavendish did his work at approximately the same time as Soldner, and also used Newtonian principles in his analysis
- Section C - A Mathpages.com paper that analyzes Einstein's calculation of light deflection associated with General Relativity and curved space-time [1]. In section C.2 we comment on this source for the calculations.

Soldner and Cavendish assumed that a light ray would be deflected by a gravitational force. Einstein, on the other hand, had introduced his General Relativity theory, which assumes that space-time is curved, and that light would follow a geodesic curve through space-time in the vicinity of a large gravitating body.

**<u>SECTION A.1</u> Soldner's approach**

Soldner[13] begins by focusing on creating an equation for the path that a light ray would follow as it moves under the influence of a gravitating body. In the scenario that he uses, a light beam is emitted from the surface of the Earth on a path tangential to the surface(see Figure 1 below). After a great deal of calculation he arrives at the following as the equation for this path.

Equation (IX) $y^2 = \frac{v^2}{g}x + \frac{v^2(v^2-4g)}{4g^2}x^2$.

The details for the derivation of this equation is in section A.2.

He notes that this is a hyperbola, since the value of the $\frac{v^2(v^2-4g)}{4g^2}$ is positive. Soldner also notes that a general form for a hyperbola is $y^2 = px + \frac{p}{2a}x^2$ where $p$ is a parameter of the hyperbola, known as the latus rectum. It is equal to $\frac{2b^2}{a}$ where the value $a$ is the distance from the vertex of the asymptotes to the transverse axis intersection with the hyperbola. This is AB in the Soldner diagram below. The value b is the length of a perpendicular to the transverse axis, at the intersection with the hyperbola, extending out to the asymptote. This is AD in the Figure 1 diagram.

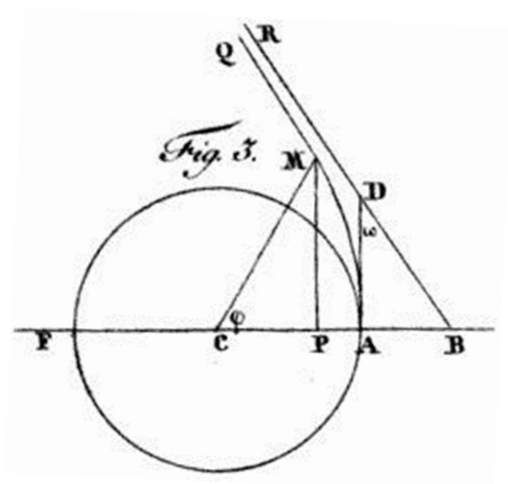


Figure 1. From Soldner's 1801 paper - On the deflection of a Light Ray from its Rectilinear Motion

In Figure 1, the circle represents the Earth with center at C. A beam of light travels from point A at the Earth's surface, on a path perpendicular to CA, the earth's radius. The curved line AMQ is the path of the ray.

Soldner focuses on Δ ADB and the ∠ ADB, labeling the angle ω. The sides of this triangle are the transverse axis of the hyperbola (AB), one of the asymptotes of the hyperbola (DB) and (AD), which Soldner refers to as the semi-lateral axis, and which is perpendicular to the transverse axis (CB)for the hyperbola. He does this since this triangle is similar to the triangle formed by the asymptote RB, the transverse axis CB and the latus-rectum of the hyperbola, which is the line through the center C and perpendicular to the transverse axis. The angle at the intersection of the latus-rectum and the asymptote will be the angle of deflection for the light path and is equal to ∠ ADB. Therefore, Soldner calculates ∠ ADB by deriving the lengths of sides AB and AD, and then computing ∠ ADB. Please note that this angle is ½ of the total deflection.

We see then that $tang\,\omega = \frac{AB}{AD}$ and thus we have $tang\,\omega = \frac{AB}{AD} = \frac{a}{b}$.

Some comments are in order. Soldner uses tang in place of the modern use of tan for tangent. In equation (IX) **v** is velocity, and **g** is gravitational acceleration, although both values are not traditional. In each case, Soldner divides velocity and gravitational acceleration by the Earth's radius. Velocity is stated as the number of Earth radii that light would travel in a decimal second.

Please note that he works with decimal seconds, which are a different time measurement than seconds. For example, Soldner states that light takes 564.8 decimal seconds to travel from the Sun to the Earth. Since a decimal second is equal to 0.864 seconds, this equates to 488 seconds. We will use 488 seconds in our calculation of Soldner's deflection, but a more accurate modern value is 499 seconds, and we will revisit that thought later.

To calculate $\boldsymbol{\omega}$ we need to derive values for a and b. We know that

$y^2 = \frac{v^2}{g}x + \frac{v^2(v^2-4g)}{4g^2}x^2$ and $y^2 = px + \frac{p}{2a}x^2$ where $p = \frac{2b^2}{a}$, and so $y^2 = \frac{2b^2}{a}x + \frac{b^2}{a^2}x^2$ .

Therefore, we have (Eq A) $\frac{2b^2}{a} = \frac{v^2}{g}$ and (Eq B) $\frac{b^2}{a^2} = \frac{v^2(v^2-4g)}{4g^2}$.

We can rearrange (Eq B) as

$b^2 = \frac{a^2v^2(v^2-4g)}{4g^2}$ and then insert this into (EqA) to get $\frac{2a^2v^2(v^2-4g)}{4ag^2} = \frac{v^2}{g}$ or $a = \frac{2g}{v^2-4g}$. We then insert this value for a into (EqA) giving

$\frac{2b^2(v^2-4g)}{2g} = \frac{v^2}{g}$ or $b = \frac{v}{\sqrt{v^2-4g}}$. Using these values for a and b we have

$\text{tang}\,\omega = \frac{a}{b} = \frac{2g\sqrt{v^2-4g}}{v(v^2-4g)} = \frac{2g}{v\sqrt{v^2-4g}}$ or $\text{tang}\,\omega = \frac{2g}{v\sqrt{v^2-4g}}$.

<u>Soldner's Value for Gravitational Acceleration:</u>

Soldner sources his value for gravitational acceleration at the surface of the Earth from the Laplace Celestial Mechanics Treatise [7]. Soldner states a gravitational acceleration value at the surface of the Earth as 3.66394 meters per decimal second squared. This is found on page 251 of the Celestial Mechanics treatise [7]. Please note that in the footnotes on page 251 of the Laplace work, there is a comment that this value is ½ of g.

We can compare this value to the accepted modern value. First, we convert from decimal seconds to standard seconds by dividing 3.66394 by $0.864^2$. We then double this value due to the Laplace comment. This gives us a gravitational acceleration value at the surface of the Earth of 9.8 $m/s^2$, which is the accepted modern value.

There is some amount of controversy about Soldner's use of 2g in his formula. We believe the above calculation illustrates the rationale behind 2g. The Tilman Sauer paper [12] that we reference provides support for this conclusion.

<u>Deflection value for the Earth:</u>

In his calculation for the deflection angle at the Earth, Soldner uses a value of v = 15.562085 Earth radii per decimal seconds, and a g value equal to 3.66394 $m/s^2$ divided by the Earth's radius (6369514 meters per Soldner) to get a g value of 5.75231 X $10^{-7}$. Soldner states that using these values for $v$ and $g$ leads to an ω value of 0.001 arcseconds of deflection at the Earth, and 0.84 arcseconds when calculating for the Sun. He provides details for the Earth, but not for the Sun. We used his values and showed that we get the deflection value he states for the Earth.

Deflection value for the Sun:

To perform the calculation for the Sun, we need **v** and **g** for the Sun, which we will then use in conjunction with his formula, to see if we can replicate his result.

To derive **v** for the Sun, we note that Soldner states that light takes 564.8 decimal seconds to travel from the Sun to Earth. 564.8 decimal seconds is equivalent to 488 standard seconds. The distance from Earth to the Sun is 1.496 X $10^8$ meters. To get a Soldner **v** value for the Sun we divide the distance by the time, and then divide this by the radius of the Sun $6.957\,x\,10^8$, to match his units. This gives us **v** = .4406 Sun radii per second.

To get a Soldner **g** for the Sun, we start with noting that the accepted value for gravitational acceleration at the surface of the Sun is 274 m/$s^2$. Again, Soldner would divide this by the radius of the Sun, and so, $g$ for the Sun in Soldner's units is $3.95233\,x\,10^{-7}$.

As an alternative to using the accepted modern value for gravitational acceleration at the surface of the Sun, we can derive that value by using a Kepler formula, shown below. This formula was known during Soldner's lifetime.

$g = 4\pi^2 \frac{r^3}{T^2R^2} = 4\pi^2 \frac{(1.496\,X\,10^{11})^3}{(3.156\,X\,10^7)^2} \frac{1}{(6.957\,X\,10^8)^2} = 274.2$ for the Sun where

$r = 1.496\,X\,10^{11}$ - the distance from Earth to the Sun in meters

$T = 3.156\,X\,10^7$ - the period of Earth's orbit in seconds

$R = 6.957\,X\,10^8$ - the radius of the Sun in meters.

Having derived values for $v$ and $g$, we plug them into his formula. We get tan ω = 0.00000407104 or ω = 0.00023325 degrees or 0.84 arcseconds.

Looking back at Soldner's Figure 1 diagram, ω is the deflection angle for one half of the hyperbola, and so the full deflection appears to be twice that or 1.68 arcseconds.

This calculation presents a problem, however. When he calculated the deflection at the Earth, he used a gravitational value that was just ½ of the accepted value for gravitational acceleration at the surface of the Earth. If we follow this same pattern for a calculation for the Sun, then we need to use ½ of the value of g (274) for the Sun. We have done this, and the resulting angle is 0.42 arcseconds of deflection. His diagram (Figure 1) is based on the angle for one half of the hyperbolic path, and so when we multiply this by 2, we get the predicted 0.84 arcsecond angle.

We commented that the value 488 seconds for the time for light to travel from the Sun to the Earth is not correct by modern standards. We suspect that the time value Soldner stated was the transit time for light when the earth was at perigee, since a modern calculation for this transit time is 490 seconds. The modern value for the transit time is usually stated as 499 seconds. Redoing the Soldner calculation for ω using 499 seconds gives us a deflection angle of 0.44 arcseconds for the ½ angle of deflection or .88 for the full angle of deflection. We will see that this value is consistent with the deflection we calculate using Cavendish's approach.

Comparing Soldner to Einstein

Soldner's equation for the 1/2 angle of deflection is $\mathrm{tang}\,\omega = \frac{2g}{v\sqrt{v^2-4g}}$. As we commented above $2g$ is equal to the modern value of gravitational acceleration at the Sun surface divided by the radius of the Sun or $2g = \frac{G_{Sun}}{R_{Sun}}$. We also note that $G_{Sun} = \frac{GM}{R^2_{Sun}}$ where $G$ is the gravitational constant, $M$ is the mass of the Sun, and $R_{Sun}$ is the radius of the Sun. We then note that $v\sqrt{v^2 - 4g} \approx$

$v^2$ since $4g$ is very small compared to $v^2$, *i.e.* $\sqrt{v^2 - 4g}$ = .440702, and $\sqrt{v^2}$ = .440704, a negligible difference. Assuming that $v$ is the speed of light, then Soldner's $v = \frac{c}{R_{Sun}}$. Combining these observations, we see that

$$\frac{2g}{v\sqrt{v^2-4g}} \approx \frac{2g}{v^2} = \frac{\frac{G_s}{R_{Sun}}}{\frac{c^2}{R_{Sun}^2}} = \frac{\frac{\frac{GM}{RSun^2}}{RSun}}{\frac{c^2}{RSun^2}} = \frac{GM}{c^2 R_{Sun}}$$ for ½ of the deflection, or

$\frac{2GM}{c^2 R_{Sun}}$ for the full deflection which compares to the Einstein value of $\frac{4GM}{c^2 R_{Sun}}$ for full deflection at the Sun.

## <u>SECTION A.2</u> Detailed mathematical derivations of equations (I) through (IX):

Details of the derivations for each of Soldner's equations (I) through (IX), that result in the key formula tang ω = $\frac{2g}{v\sqrt{(v^2-4g)}}$ are presented here. Other than basic algebraic steps a great amount of detail is included so the reader can move from equation to equation without having to fill in many gaps. A knowledge of basic calculus is assumed.

He sets up the proper (x,y) coordinate system for the gravitating body (in this case the Earth) and the light path by letting x = CP and y = MP in Figure 1. He then completes the following key steps where his overall strategy is to tie together the geometry of the light path with the physics (force, velocity). To do this Soldner performed a series of algebraic and calculus-based manipulations to develop the necessary equations. We have replaced the archaic 'dd' second differential notation with $d^2$.

A. Equates acceleration with vector components of the force experienced by light along its path, equations

(I) $\frac{d^2x}{dt^2} = -\frac{2g}{r^2}cos\varphi$ and (II) $\frac{d^2y}{dt^2} = -\frac{2g}{r^2}sin\varphi$. These equations include the radial angle φ, the radial distance r, and the velocity **v** of the light ray or "particle"[3]. Comments about Soldner's choice of dimensional units for his physical variables are given in the Step A details below.

B. Derives equation (VII) $d\varphi = \frac{vdt}{r^2}$ relating $d\varphi$ to $dt$ .

C. Derives an equation from (VII) relating $d\varphi$ to $dr$.

D. Derives equation (VIII)

$$r + \left(\frac{v^2-2g}{2g}\right) r cos\varphi \ = \frac{v^2}{2g}.$$

E. Transforms equation (VIII) to (x,y) coordinates as equation (IX) $y^2 = \frac{v^2}{g}x + \frac{v^2(v^2-4g)}{4g^2}x^2$ and shows that the conic section is indeed a hyperbola.

F. Defines the deflection angle and using the hyperbolic equation obtains $\text{tang}\,\omega = \frac{2g}{v\sqrt{v^2-4g}}$ which is used in the main discussion above to compute $\omega$.

We point out that Soldner's equations are not all created strictly in sequence so at first glance his development may seem somewhat circular. We provide this equation flowchart to aid the reader starting with the force $\frac{2g}{r^2}$ exerted by gravity on the light particle:

$$\begin{array}{llllll} \frac{2g}{r^2} \rightarrow & (I)\&(II) \rightarrow & (III) \rightarrow & (V) \rightarrow & (VII) & \\ & & \& & & \downarrow & \\ & & (IV) & \rightarrow & (VI) \rightarrow & (VIII) \rightarrow (IX) \rightarrow \text{tang}\ \omega \text{ deflection angle.} \end{array}$$

We note that in his original paper Soldner used roman numerals to label his key equations while Jaki [13] converted these for his translation. We have opted to retain Solder's original scheme.

**Step A.**

[3] Soldner was a supporter of the Newtonian corpuscular theory of light so we will sometimes refer to light as a particle, but not in the sense of the more modern theory of light 'quanta'.

To derive equations (I) and (II) he decomposes the radial vector force $\frac{2g}{r^2}$ into the $x$ and $y$ vector components $\frac{2g}{r^2}\cos\varphi$ and $\frac{2g}{r^2}\sin\varphi$ since addition of these orthogonal vectors is
$(\frac{2g}{r^2}\cos\varphi)^2 + (\frac{2g}{r^2}\sin\varphi)^2 = \frac{2g}{r^2}[(\cos\varphi)^2 + (\sin\varphi)^2] = \frac{2g}{r^2}$.

In equations (I) and (II) he equates these vector components to the corresponding accelerations terms $\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right)$ citing Laplace's Celestial Mechanics treatise [7] and includes a negative sign for each component since by convention a force toward the gravitating object is labeled with a negative sign.

$$\frac{d^2x}{dt^2} = -\frac{2g}{r^2}\cos\varphi \qquad \text{(I)}$$
$$\frac{d^2y}{dt^2} = -\frac{2g}{r^2}\sin\varphi \qquad \text{(II)}$$

We note that $\frac{2g}{r^2}$ does indeed have units of force working in geometric units which results in dimensional consistency in his equations.[4]

Before proceeding, we point out that Soldner's equations (I) and (II) and the time coordinate form of the geodesic equation used in Section C.2 (Deflection based on General Relativity) are similar. The geodesic equation is $(\frac{d^2x^\alpha}{dt^2}) + \Gamma^\alpha_{\gamma\beta}((\frac{dx^\beta}{dt})(\frac{dx^\gamma}{dt})) = 0$. Both (I) and (II) and this geodesic equation contain an acceleration term and a 'force' term, each based on

[4] A comment about Soldner's use of units which was further clarified by Karl-Heinz Lotze in an email: The dimensional units of 2gr^2 on the RHS of equations (I) and (II) is at first glance L(T^-2)(L^-2)=(L^-1)(T^-2) which conflicts with the acceleration units on the LHS which is L(T^-2). However, Solder works in so-called geometric units, i.e., he divides both g and length units by the radius of the gravitating body which has dimensional unit L. So for the dimensional analysis equations, where [z] signifies the units of a physical variable z, [g]=L(T^-2)(L^-1)=T^-2, and [r]=[x]=[y]=L(L^-1)=scalar. Then [-2g(r^-2)cosφ]=scalar(T^-2)(scalar^-2)scalar=T^-2. For an acceleration term, [(d^2)x)/dt^2]=scalar(T^-2)=T^-2) so both sides of the equations have the same units, namely, T^-2. Therefore we have dimensional consistency for equations (I) and (II). This consistency then carries over into his entire computation of the deflection angle and can easily be verified for each of the key equations.

gravitational acceleration but one formulated in a pre-relativity framework ($g$ vs the $g_{\mu\nu}$ tensor). Both approaches use their respective equations to find how the photon radial angle changes continuously with respect to time.

Continuing on with Soldner's derivation of the deflection angle, he performs basic algebraic manipulations on equations (I) and (II), and uses equations relating $x, y$ to $\cos\varphi$, $\sin\varphi$. All this yields equations (III) and (IV.) We do not provide these details since they are straight forward and outlined in his paper [13].

$$\frac{d^2y\,cos\varphi - d^2x\,sin\varphi}{dt^2} = 0 \qquad \text{(III)}$$

$$\frac{d^2x\,cos\varphi + d^2y\,sin\varphi}{dt^2} = - \frac{2g}{r^2} \qquad \text{(IV)}$$

He then arrives at equation (V) from (III) by computing $d^2x$ and $d^2y$ from the equations $x = r\,cos\varphi$ and $y = r\,sin\varphi$ to get

$$d^2x = cos\varphi\,d^2r - 2\,sin\varphi\,d\varphi dr - rsin\varphi\,d^2\varphi - rcos\varphi\,d\varphi^2$$
$$d^2y = sin\varphi\,d^2r + 2\,cos\varphi\,d\varphi dr + rcos\varphi\,d^2\varphi - rsin\varphi\,d\varphi^2.$$

Substituting these formulas for the second differentials into the numerator of (III) and some simple algebra yields

$$\frac{2d\varphi dr + rd^2\varphi}{dt^2} = 0. \qquad \text{(V)}$$

Soldner now derives equation (VI) from equation (IV). This can be done by showing that the numerator of (IV) $d^2x\,cos\varphi + d^2y\,sin\varphi$ equals the numerator of (VI) $d^2r - r\,d\varphi^2$ which we now do.

Substituting the above formulas for the second differentials into the numerator of (IV) we have

$$cos\varphi\left[cos\varphi\, d^2r - 2\, sin\varphi\, d\varphi dr - rsin\varphi\, dd\varphi - rcos\varphi\, d\varphi^2\right]$$
$$+\ sin\varphi[sin\varphi\, d^2r + 2\, cos\varphi\, d\varphi dr + rcos\varphi\, d^2\varphi - rsin\varphi\, d\varphi^2].$$

Distributing the $cos\varphi,\ sin\varphi$ factors and canceling out like terms of opposite sign shows that the numerator of (IV), namely, $d^2x\, cos\varphi + d^2y\, sin\varphi$, equals

$$d^2r\left(cos^2\varphi + sin^2\varphi\right) - rd\varphi^2\left(cos^2\varphi + sin^2\varphi\right)$$

which reduces to $d^2r - r\, d\varphi^2$.

So replacing the numerator with this expression yields

$$\frac{d^2r - r\, d\varphi^2}{dt^2} = -\frac{2g}{r^2}. \qquad \text{(VI)}$$

Notice that all differentials of x and y have been eliminated. He will also eventually eliminate $dt$ and arrive at equations relating $d\varphi$ and $dr$ as we indicated earlier.

**Step B.**

His next step is to take equation (V) and find an equation that relates the $d\varphi$ and $dt$ differentials, that is, how the angle changes with respect to time. This will be an intermediate step on the way to getting an equation relating $d\varphi$ to $dr$. In section 3 we will see this same strategy used but in the more complex setting of General Relativity.

He multiplies each side of (V) by the factor $rdt$ to put it into a form that can be integrated to find the antiderivative. This introduces a constant of integration C. Thus

$$\frac{2d\varphi dr + r\, d^2\varphi}{dt^2} = 0 \quad \rightarrow \quad \frac{2rdtd\varphi dr + r^2dt\, d^2\varphi}{dt^2} = 0 \rightarrow$$

$$\frac{2rd\varphi dr + r^2 d^2\varphi}{dt} = 0 \rightarrow \int \frac{2rd\varphi dr}{dt} + \int \frac{r^2 d^2\varphi}{dt} = \int 0 = C \rightarrow$$

$$\frac{2d\varphi}{dt}\int rdr + \frac{r^2}{dt}\int d(d\varphi) = C \rightarrow \frac{d\varphi}{dt}r^2 + \frac{r^2}{dt}d\varphi = C \rightarrow 2\frac{d\varphi}{dt}r^2 = C$$

$$\rightarrow 2r^2 d\varphi = Cdt$$

where throughout this derivation we have consolidated the various integration constants into one occurrence C. Since C is arbitrary, we can drop the factor of 2 to get $r^2 d\varphi = C\,dt.$ Once C is determined we have our sought-after equation relating $d\varphi$ and $d\varphi$.

Soldner then claims that C = v, the velocity of light. See appendix A for a proof.

So now we have $r^2 d\varphi = vdt$ or $d\varphi = \frac{vdt}{r^2}$. (VII)

This shows how the radial angle varies with time and this completes step B. Soldner now goes on to obtain an equation showing how the angle φ varies with the radius r, the distance of the light particle from the center of the gravitating body that is exerting a force on the particle. In other words, he shows how $d\varphi$ varies with $dr$.

**Step C.**

He substitutes the expression for $d\varphi$ in (VII) into (VI) and multiplying each side by $2dr$ he integrates the resulting equation with respect to r to get an antiderivative and integration constant D:

$$\int \frac{2dr\ d^2r}{dt^2} - \int \frac{2v^2 dr}{r^3} = -\int \frac{4gdr}{r^2} \rightarrow$$

$$\frac{2}{dt^2}\int dr\,d(dr) - 2v^2\int \frac{dr}{r^3} = -4g\int \frac{dr}{r^2} \rightarrow$$

$$\frac{dr^2}{dt^2} + \frac{v^2}{r^2} = \frac{4g}{r} + D.$$

This equation contains $dr$ but does not contain $d^2r$. In fact, all second order differentials have now been eliminated and will not appear going forward.

Solving the above equation for $dt$, substituting into equation (VII), and solving for $d\varphi$ he gets

$$d\varphi = \frac{v\,dr}{r^2\sqrt{D + \frac{4g}{r} - \frac{v^2}{r^2}}}.$$

This is the sought-after equation relating $d\varphi$ to $dr$ that tells us how a small change in the radial angle of the light particle is determined from a small change in the radial distance. Our next goal is to find an equation for just $\varphi$ itself, and from that show it is an equation of a hyperbola to determine the actual angle of deflection $\omega$.

In the original Soldner paper in German(see Note in [13]) there is an error(probably due to the typesetter) in one of the intermediate equations leading up to the above equation where $\frac{v^2}{r^2}$ term is instead expressed as $\frac{v^2}{r}$. This apparently was corrected by Jaki [13] during translation.

**Step D.**

Using the differential equation for $d\varphi$ in Step C Soldner finds the antiderivative. Notice that it already contains one unknown integration constant and integrating it will introduce another such constant. Fortunately, one of the constants will cancel out allowing Soldner to use a boundary condition to find the other constant.

So $\varphi = \int d\varphi = \int \frac{vdr}{r^2\sqrt{D+\frac{4g}{r}-\frac{v^2}{r^2}}} = \int \frac{1}{\sqrt{D+\frac{4g^2}{v^2}-(\frac{v}{r}-\frac{2g}{v})^2}}\frac{vdr}{r^2}$ and he then sets

$z = \frac{v}{r} - \frac{2g}{v}$. He puts the integral into this form and defines z as such to get a more tractable integral in standard form. This can be seen by noting that $\frac{v}{r} - \frac{2g}{v}$ contains only one term with r and taking the derivative of that term with respect to r will yield the $\frac{vdr}{r^2}$ factor in the integral with a negative sign: $\frac{dz}{dr} = -\frac{v}{r^2}$ or $- dz = \frac{vdr}{r^2}$. So the integral transforms to

$$\varphi = \int d\varphi = -\int \frac{dz}{\sqrt{D+\frac{4g^2}{v^2}-z^2}}.$$

This integral is in the standard form $\int \frac{Adx}{\sqrt{p^2-x^2}}$ where $A = -1, z = x$,

and $p = \sqrt{D + \frac{4g^2}{v^2}}$. We will see this same strategy of manipulating an integral in Section 3 but in the more complex setting of General Relativity.

Performing the integration $\varphi = -(-Arc\cos cos \frac{z}{\sqrt{D+\frac{4g^2}{v^2}}}) + a =$

$Arc\cos cos \frac{z}{\sqrt{D+\frac{4g^2}{v^2}}} + a$.

where $a$ is another integration constant. Moving the $a$ constant to the LHS and applying the inverse of arccos we get

$$\mathrm{Cos}\ (\varphi - a) = \frac{z}{\sqrt{D+\frac{4g^2}{v^2}}} = \frac{v^2-2gr}{r\sqrt{v^2D+4g^2}}$$

where we have substituted $z = \frac{v}{r} - \frac{2g}{v}$. There are now two constants of integration to somehow eliminate in order to prove the path of the photon is a conic section that is indeed a hyperbolic path.

Soldner first deals with the $a$ constant by using the boundary conditions of the coordinate system that he imposed on the problem where φ is the angle between the radial line CM and the x-axis. See Figure 1 in the section A.1 main discussion. These imposed conditions resulted in the above equation which now contains the angle $\varphi - \alpha$ instead of just φ. So α must be zero. Therefore

$$\text{Cos } \varphi = \frac{v^2 - 2gr}{r\sqrt{v^2 D + 4g^2}}.$$

This holds for all φ. In particular if $\varphi = 0$ then AC (the distance from the center of the gravitating body out to its surface) = 1 unit since he imposed this at the outset of the problem. Hence $AC = r = 1$ when $\varphi = 0$. Then since $\cos 0 = 1$

$$1 = \frac{v^2 - 2g}{\sqrt{v^2 D + 4g}} \quad \text{or} \quad \sqrt{v^2 D + 4g^2} = v^2 - 2g.$$

This implicitly solves for D in terms of v and g and can be substituted into the previous equation which conveniently cancels out the D yielding:

$$\text{Cos } \varphi = \frac{v^2 - 2gr}{r(v^2 - 2g)}.$$

One can see that this has all the ingredients to form a recognizable polar equation or use polar coordinates to transform it into an (x,y) equation that can be recognized as a conic section. Evidently he opted for the latter approach. The equation can be manipulated as follows:

$$\text{Cos } \varphi = \frac{v^2 - 2gr}{r(v^2 - 2g)} = \frac{v^2 - 2gr}{2g}\frac{2g}{v^2 - 2g}\frac{1}{r} \rightarrow \frac{v^2 - 2g}{2g} r\cos\varphi = \frac{v^2 - 2gr}{2g} = \frac{v^2}{2g} - r \rightarrow$$

$$r + \frac{v^2 - 2g}{2g} r\cos\varphi = \frac{v^2}{2g}. \qquad \text{(VIII)}$$

Soldner is now ready to write this in (x,y) coordinates.

**Step E.**

Next, Soldner derives a conic section equation for the light path. Using the line segments AP $= x$ and MP $= y$ he writes $x$, $y$, and $r$ in the form: $x = 1 - r\cos\varphi$, $y = r\sin\varphi$, $r = \sqrt{(1-x)^2 + y^2}$ and substitutes into (VIII).

As an aside, notice that he changes the definition of $x$ from the line segment CP to the segment AP. This has the effect of shifting the origin of his induced Cartesian coordinate system from point C to point A since point A is now where $x = y = 0$, not point C. As described below this eventually results in equation (IX). Soldner then makes a correspondence to a hyperbolic equation to arrive at the formula for $\tan\omega$, as described in Section A. The hyperbolic equation contains the key physical parameters $a$ and $b$ which correspond to the line segments AB and AD in Figure 1.

If he had kept the original definition of $x$ = CP he could have worked equation (VIII) into the standard form(if it was known to him)

$$\frac{(x-h)^2}{A^2} - \frac{(y-k)^2}{B^2} = 1$$

where, in this case, $k = 0$. However, the work needed to arrive at the formula for $\tan\omega$ is more complicated but which we do not show.

Using this new definition of $x$, there is some tedious algebra here that we include since it leads to showing another error in the paper, most likely due to the typesetter, but which does not affect the subsequent equations.

Using the above equations for $x$ and $y$ and (VIII) we get
$\sqrt{(1-x)^2 + y^2} + \left(\frac{v^2-2g}{2g}\right)(1-x) = \frac{v^2}{2g}$.
We solve for $y^2$ and do some algebra to get it equal to a quadratic in $x$. This will be our conic section equation.

$$[\sqrt{(1-x)^2 + y^2}\,]^2 = [\ \frac{v^2}{2g} - (\frac{v^2-2g}{2g})(1-x)]^2$$

$$(1-x)^2 + y^2 = \frac{v^4}{4g^2} - 2(\frac{v^2}{2g})\ (\frac{v^2-2g}{2g})(1-x) + (\frac{v^2-2g}{2g})^2(1-x)^2 \rightarrow$$

$$y^2 = (\frac{v^2-2g}{2g})^2(1-x)^2 - (1-x)^2 - \frac{v^2}{g}(\frac{v^2-2g}{2g})(1-x) + \frac{v^4}{4g^2} =$$

$$[(\frac{v^2-2g}{2g})^2 - 1]\ (1-x)^2 - \frac{v^2}{g}(\frac{v^2-2g}{2g})(1-x) + \frac{v^4}{4g^2} =$$

$$[\frac{v^4-4v^2g}{4g^2}](1-x)^2 - \frac{v^2}{g}(\frac{v^2-2g}{2g})(1-x) + \frac{v^4}{4g^2}.$$

$y^2$ is now expressed as a quadratic in (1-x). In the Jaki translation of the original paper [13] the last term is erroneously given as $\frac{v^2}{4g^2}$. We examined an image of the original paper in German and it also has this error. Since what follows in the document is mathematically correct, we can only surmise there was an error in typesetting or in preparation of the final draft submitted to the journal by Soldner and which has been passed along in the translation.

Proceeding with a little more algebra and carefully grouping like terms together we finally have
$y^2$ as a quadratic in x:

$$y^2 = [\frac{(v^4-4v^2g)}{4g^2}\ (1-2x+x^2)] - \frac{v^2}{g}(\frac{v^2-2g}{2g})(1-x) + \frac{v^4}{4g^2} =$$

$$\frac{v^4-4v^2g}{4g^2}\ x^2 + [\frac{v^2(v^2-2g)}{2g^2} - \frac{v^2(v^2-4g)}{2g^2}]\,x + [\frac{v^2(v^2-4g)}{4g^2} - \frac{v^2(v^2-2g)}{2g^2} + \frac{v^4}{4g^2}].$$

The <u>bracketed </u>coefficient of the x term reduces to $\frac{v^2}{g}$. Applying basic algebra to the right-most bracketed terms, which we do not supply here, shows that they cancel out to zero and we are left with $y^2 = \frac{v^2}{g}x + \frac{(v^4-4v^2g)}{4g^2}x^2$ which is <u>Soldner’s equation (IX)</u>. This is in the form of a conic section. Soldner then states that $v^2 > 4g$ since v is light speed (in geometrical units and large compared to g) and hence the term $x^2$ is positive which makes equation (IX) describe a hyperbolic path. It is worth noting that Soldner treats **v** (light speed) as a variable but in all cases the inequality holds due to its magnitude compared to g.

**Step F.**

Having shown that the path is a hyperbola Soldner goes on to derive the key equation $\text{tang}\,\omega = \frac{2g}{v\sqrt{v^2-4g}}$ using the coefficients of the hyperbola equation (IX). This has been previously explained in section A.1.

## <u>SECTION B:</u> Cavendish's approach

### Cavendish Analysis

Cavendish's analysis was more limited than Soldner's. He did not publish a result for his effort, and only left behind a pair of handwritten notes. Our comments here are based on these two notes. All of our comments about Cavendish are based on reading an excellent paper written by Lotze and Simionato, entitled "Henry Cavendish and the effect of gravity on propagation of light: a postscript" [8].

Cavendish assumed that a ray of light would be deflected by a gravitating body, and that the path would be a hyperbola. He does not attempt to prove that the path is hyperbolic.

In Figure 2 below, the ½ angle of deflection is the angle at the intersection of an asymptote of the hyperbola and the latus rectum. The latus rectum for a hyperbola is a line through a focus located at point C of the hyperbola, perpendicular to the transverse axis of the hyperbola, and extending out to the intersection with the two asymptotes.

Cavendish calls this angle $\frac{\delta}{2}$, since this is one branch of the hyperbola, and its value must be doubled to get the full deflection. The diagram below was created after looking at Cavendish's handwritten notes, and a Lotze interpretation of the notes [8].

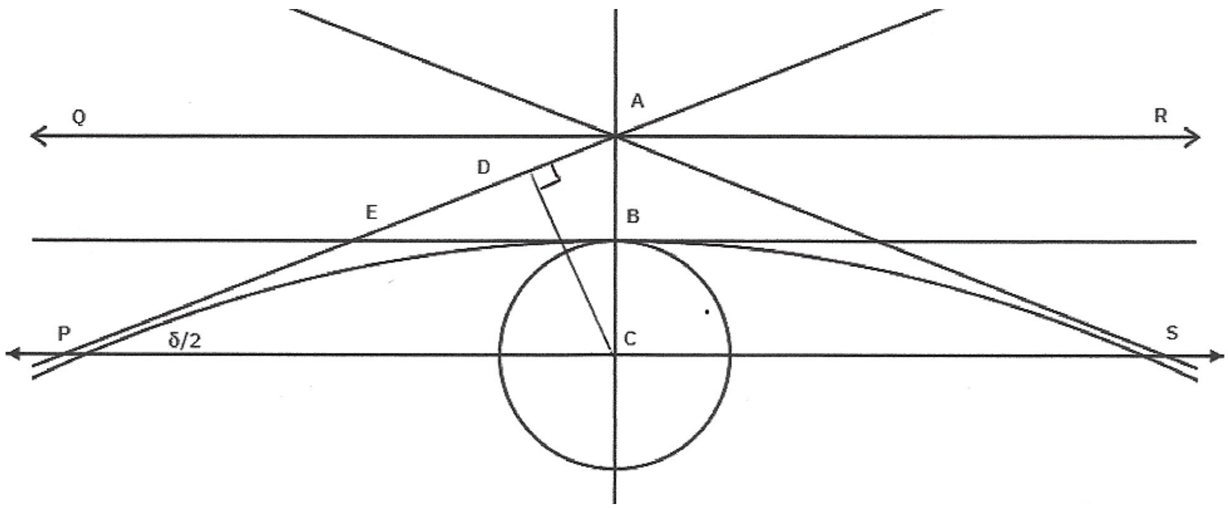


Figure 2. Our diagram, based on Cavendish's notes

While the half deflection $\angle\ APC = \frac{\delta}{2}$ is the angle we seek, several other angles on the diagram are equal to this angle, and make the calculation of the angle simpler.

$\angle\ APC = \angle\ QAP$ since the latus rectum, and the bisector of the angle of intersection of the asymptotes are parallel. Then $\angle\ QAP + \angle\ DAC = \angle\ DCA\ + \angle DAC = 90^{\circ}$, and therefore $\angle\ QAP = \angle\ DCA$. We also note that $\angle\ AEB = \angle$ QAP. So angles $\angle\ AEB$, $\angle$APC, $\angle$QAP, and $\angle$ACD are all equal to $\frac{\delta}{2}$.

The segment DC is the perpendicular from the asymptote to the foci at C. The length of this segment is referred to as the impact parameter of the hyperbola, and is usually designated by the value $b$. Segment EB is a perpendicular to the transverse axis AC, and intersects the asymptote. This segment has the same length as DC. Therefore $\Delta AEB$ and $\Delta ADC$ have three equal angles and a side of equal length, and so the two triangles are congruent. Consequently, segment DA = segment BA. Segment BA is usually referred to as the value *a*, the distance from the vertex of the asymptotes to the hyperbola's intersection with the transverse axis. Therefore $AD = AB = a$. Segment BC is the radius of the gravitating body, so we can designate this as R.

The length of $AC = a + R$. Since $\angle DCA = \frac{\delta}{2}$, so we can then state that $\sin\frac{\delta}{2} = \frac{AD}{AC}$ or $\sin\frac{\delta}{2} = \frac{a}{a+R}$. Cavendish states that $\sin\frac{\delta}{2} = \frac{1}{1+u^2}$ where $u = \frac{v_\infty}{v_{circ\,R}}$. For a photon $v_\infty = c$. Therefore $\sin\frac{\delta}{2} = \frac{1}{1+u^2} = \frac{a}{a+R}$. Simple manipulation then yields $u^2 = R/a$. Therefore $u = \sqrt{\frac{R}{a}} = \frac{c}{v_{circ\,R}}$.

The velocity of an object that is orbiting a mass M at distance R is given by $\sqrt{\frac{GM}{R}}$. Substituting this for $v_{circ\,R}$ yields $\sqrt{\frac{R}{a}} = \frac{c}{\sqrt{\frac{GM}{R}}}$.

We now calculate the value $a$. Solving $\sqrt{\frac{R}{a}} = \frac{c}{\sqrt{\frac{GM}{R}}}$ for $a$ we get $a = \frac{GM}{c^2}$, where

$$G = 6.67384\,X\,10^{-11}\frac{m^3}{kgs^2},\ M = 1.99\,X\,10^{30}kg,\ R = 6.957\,X\,10^{8}\,m,\ c = 3\,X\,10^{8}m/s.$$

Therefore $a = 1475\,\mathrm{m}$.

Using this in the formula $\sin\frac{\delta}{2} = \frac{a}{a+R}$ gives a value for $\frac{\delta}{2}$ of 0.44 arcseconds, or a total deflection of 0.88 arcseconds, which is very close to the Soldner value of 0.84 arcseconds.

If we substitute the modern value of 499 seconds for light traveling from the Sun to the Earth, replacing the Soldner stated value, the Soldner equation yields a deflection of 0.44 arcseconds, giving full agreement with the Cavendish approach.

We can assume that Cavendish understood Newtonian mechanics, and the characteristics of hyperbolas and hyperbolic orbits.

Cavendish postulated that for the ½ angle of deflection for a photon's hyperbolic path passing the Sun is sin $\frac{\delta}{2} = \frac{1}{1+u^2}$ where

u = $\frac{v_\infty}{v_{circ\,R}}$. $v_\infty$ is c for a photon and $v_{circ\,R}$ is the orbital velocity of an object orbiting at distance R, which is $\sqrt{\frac{GM}{R}}$. Therefore sin $\frac{\delta}{2} = \frac{1}{1+u^2} = \frac{1}{1+\frac{c^2}{\frac{GM}{R}}} = \frac{1}{\frac{GM+Rc^2}{GM}} = \frac{GM}{Rc^2+GM}$ for the ½ angle of deflection or or the full angle of deflection $\delta = 2\arcsin(\frac{GM}{Rc^2+GM})$.

To validate the Cavendish postulation we looked at a modern calculation for a hyperbolic orbit. The following equation can be used to calculate the angle of deflection for a hyperbolic orbit, which is the angle between the asymptotes of the hyperbola.

$\delta = 2arcsin\,(\frac{1}{e})$ where $e = \frac{r\,v_\infty^{\;2}}{GM} + 1.$ After substituting and simplifying this becomes $2\arcsin(\frac{GM}{R\,v_\infty^{\;2}+GM})$. For this case $v_\infty$ = c, and so $\delta = 2arcsin\,(\frac{GM}{R\,c^2+GM})$ . We see that this is identical to the Cavendish postulate.

**Cavendish Compared to Einstein**

We will compare the Cavendish formula to the Einstein format $tan\delta \approx \delta = \frac{4GM}{c^2 R_{sum}}$ for the total angle of deflection in general relativity. The Cavendish formula above is $sin\frac{\delta}{2} = \frac{a}{a+R_{Sun}}$, where $a$ can be derived from the statement $\frac{v_\infty}{v_{circ}} = \sqrt{\frac{R_{Sun}}{a}}$. If we assume $v_\infty = c$

and note that if $v_{circ} = \sqrt{\frac{GM}{R_{Sun}}}$ , then we have $\frac{c}{\sqrt{\frac{GM}{R_{Sun}}}} = \sqrt{\frac{R_{Sun}}{a}}$ or $a = \frac{GM}{c^2}$. This means that $sin\frac{\delta}{2} = \frac{a}{a+R_{Sun}} = \frac{\frac{GM}{c^2}}{(\frac{GM}{c^2}+R_{Sun})} = \frac{GM}{GM+c^2R_{Sun}}$.

Since $sin = \frac{opposite}{hypotenuse}$ we can treat $GM$ as the opposite and $GM + c^2R_{Sun}$ as the hypotenuse. We then need an "adjacent" term to allow us to derive an equation in the $tan\frac{\delta}{2}$ form. Since $h^2 - o^2 = a^2$ we have $(GM + c^2R_{Sun})^2 - (GM)^2 = a^2$ or $a = \sqrt{2GMc^2R_{Sun} + (c^2R_{Sun})^2}$. We can examine the radical since we have values for $G, M, c, and\ R_{Sun}$. The term $2GMc^2R_{Sun}$ has a value of 1.6634 X $10^{46}$. The term $(c^2R_{Sun})^2$ has a value of 3.9204 X $10^{51}$. Since the first term is significantly smaller than the second, we can ignore the first term and say that $a \approx \sqrt{(c^2R_{Sun})^2} = c^2R_{Sun}$ . Combining this with our work above we can say that $tan\frac{\delta}{2} = \frac{opposite}{adjacent} = \frac{GM}{c^2R_{Sun}}$, so for the full Cavendish angle of deflection $\delta = \frac{2GM}{c^2R_{Sun}}$ which is ½ the Einstein value of $\frac{4GM}{c^2R_{Sun}}$.

## **<u>SECTION C.1</u>** **Deflection based on General Relativity**

This section mainly provides the mathematical details for the calculations in section 6.3 (Bending Light) within the 'Reflection on Relativity' section of [1][5].

This writeup calculates the angle of deflection from a straight path for a photon passing near a spherical body of mass m,

[5] These calculations differ from those originally used by Einstein in 1911 [3] and 1916 [4] which were based on a relativistic application of Huygens principle of wave propagation. See [5] for a comparison between Einstein's original 1911 result and Soldner's work.

explicitly the Sun. This calculation is based on a General Relativity approach.

General Relativity is challenging, using Einstein's field equations, metrics, geodesics, Christoffel symbols and other complex notions. The calculation for deflection utilizes the Schwarzschild metric and the relativistic Geodesic equation for the path of a photon passing close to the Sun. The calculations are used to derive an equation for $\frac{d\theta}{dr}$ where θ is the angle of the radial distance from a given point along the photon path to the center of the body. The integral of $\frac{d\theta}{dr}$ is then used to calculate the total angle of deflection.

The $\frac{d\theta}{dr}$ relationship does not lend itself to a simple integral, and so the integral of $\frac{d\theta}{dr}$ is then stated as a power series, with the first 2 terms integrated over the range of r representing the sweep of the angle from a far distance (infinity) to the position at the surface of the Sun. The remaining power series terms are ignored since they are of second order and higher. Comparing this result to that for a straight-line path, taking the difference (in radians), and doubling gives the total deflection angle of a light ray passing by the surface of the Sun and continuing outbound. The resulting value is approximately 1.75 arcseconds, and is consistent with Einstein's work.

The detailed derivation of this result is given in these steps in section C.2. Mathpages.com [1] gives a full explanation for the reasoning behind each step.

A. Use the metric/line element to relate $\frac{dr}{dt}$ to $\frac{d\theta}{dt}$. The metric characterizes curvature of space-time through which the photon travels. This step is motivated by and provides input to step C.

B. Use the geodesic equation to find $\frac{d^2\theta}{dt^2}$. Any object (including light rays) moving under the influence of no other forces other than gravity moves along a path in space-time obeying the geodesic equation. This step is also motivated by and provides input to step C.

C. Then use the step A and B results to find $\frac{d\theta}{dr}$, the key quantity needed to find the angle $\varphi_D$ swept out by the light path as it approaches the Sun.

D. Express angle $\varphi_D$ as the integral $\int\left(\frac{d\theta}{dr}\right)dr$ evaluated starting at a far distance ($r = \infty$) and finishing at the point of closest approach to the Sun.

E. Transform this integral to a more tractable form using a power series to obtain $\varphi_D$. This angle is then used to compute the total deflection angle for a light ray arriving from $+\infty$ and departing to $-\infty$ after grazing the Sun.

Using this approach, a deflection angle of 1.75 arcseconds is calculated.

Note the similarity in steps A, B, and C to that of steps B and C in the Soldner approach in Section A.2.

Within the details of the derivation you will see the resulting key equations (1) through (5) contained in a box.

We emphasize that, as noted, in all that follows we are for the most part merely providing the mathematical details behind the derivation in [1]. Other than some basic algebraic steps a great amount of detail is included so the reader can move from equation to equation without having to fill in many gaps. In this section we assume a basic familiarity with calculus and general relativity and, in particular, the Schwarzschild metric.

## <u>SECTION C.2:</u> Detailed derivations

First, a comment on the source (mathapges.com [1]) for these derivations. The author's name(Kevin Brown) may be a pseudonym or a group of people since no further information seems to be available. It is not traditional to cite such a source. However, there are at least several different methods to determine relativistic light deflection. Of these, we have decided on his approach since it seems to be the one that requires less background knowledge of the required mathematical/physics tools and the resulting computations are comparatively straightforward if somewhat lengthy. Furthermore, we feel that these computations provided in our paper are detailed enough that they can stand on their own without reference to the general method provided in [1]. Nonetheless, it may be useful for the reader to peruse this source before following along with our calculations.

**Step A.**

We will derive the Einstein value for deflection, utilizing the Schwarzschild metric in the process.  The line element for the Schwarzschild metric can be expressed as

$$d\tau^2 = (1 - \frac{2GM}{c^2 r})c^2 dt^2 - \frac{1}{(1-\frac{2GM}{c^2 r})} dr^2 - r^2 d\theta^2 - r^2 (sin\theta)^2 d\varphi^2$$

Where *G*, *M*, and *c* are expressed in SI units which use kilograms, meters, seconds.

However, we are building on the outline given in [1] which uses the convention of geometric units where $G = c = 1$ (scalers set equal to 1) and mass of the Sun expressed as the gravitational radius m in meters[6] so $\frac{2GM}{c^2} = \frac{2\cdot 1\cdot m}{1^2} = 2m$ has replaced the upper-case *M* (mass in kilograms). The value of $m$ is 1475 meters.

This has the effect of replacing $\frac{GM}{c^2}$ with *m*, and the $c^2$ coefficient of $dt^2$ with 1.

This substitution cleans up the clutter in the equations, giving the line element as

$$d\tau^2 = (\frac{r-2m}{r})\, dt^2 - (\frac{r}{r-2m})dr^2 - r^2 d\theta^2 - r^2 (sin\theta)^2 d\varphi^2$$

where the resulting $1 - \frac{2m}{r}$ factors are re-expressed as $\frac{r-2m}{r}$ .  We now proceed with the detailed derivation of the deflection angle.

[6] Using geometric units allows both mass and radius to have units of length. This is done by multiplying mass M in kilograms by G/c^2, converting it to meters).

For the photon, the proper time $d\tau$ is zero. Now select the spherical coordinates plane $\varphi = 0$ so $d\varphi = 0$. This constrains the problem by eliminating the $\varphi$ variable in the metric equation since we are now working in one less dimension. This is allowable since our photon can be considered traveling in two physical dimensions (a plane) which simplifies the computations without affecting the accuracy of the result.

$$0 = \left(\frac{r-2m}{r}\right)dt^2 - \left(\frac{r}{r-2m}\right)dr^2 - r^2 d\theta^2$$

Solving for $dt^2$

$$dt^2 = \left(\frac{r}{r-2m}\right)^2 dr^2 + r^2\left(\frac{r}{r-2m}\right)d\theta^2$$

and dividing by $dt^2$

$$1 = \left(\frac{r}{r-2m}\right)^2 \left(\frac{dr}{dt}\right)^2 + \left(\frac{r^3}{r-2m}\right)\left(\frac{d\theta}{dt}\right)^2. \qquad (1)$$

Step A - Done

Equation (1) relates $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$ as desired.

**Step B.**

Now use the geodesic equation

$$\left(\frac{d^2x^\alpha}{d\lambda^2}\right) + \Gamma^\alpha_{\gamma\beta}\left(\frac{dx^\beta}{d\lambda}\right)\left(\frac{dx^\gamma}{d\lambda}\right) = 0$$

where

$$\Gamma^\alpha_{\gamma\beta} = \left(\frac{1}{2}\right) g^{\alpha\rho}\left(\frac{\partial g_{\rho\beta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\rho}}{\partial x^\beta} - \frac{\partial g_{\gamma\beta}}{\partial x^\rho}\right),$$

to find $d^2\theta/dt^2$. The standard Einstein summation convention is used so that summation over the repeated upper and lower indices within a term is understood. As in Step A, we will further constrain the relevant variables and make it easier to solve the problem.

As pointed out in Soldner's approach, section B.2, Step A, there is an interesting general similarity between his "acceleration" equations (I) and (II) and this geodesic equation since the Christoffel symbol term is sometimes called a 'force' term. This is in line with the frequently made observation that Newtonian based physics is a limiting case of General Relativity.

Continuing, recall that we now want to find an equation for $\frac{d^2\theta}{dt^2}$. Compute the $\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}$ terms to determine the $\Gamma^{\alpha}_{\gamma\beta}$ terms using the factors $g_{\mu\nu}$ of the Schwarzschild metric.

As found in Section A, $dt^2$ can be expressed as

$$dt^2 = (\frac{r}{r-2m})^2 dr^2 + r^2\left(\frac{r}{r-2m}\right)d\theta^2$$

which is a coordinate time metric line element for a surface in the two dimensional coordinate system $(r, \theta)$. As stated in [1],null paths satisfying the Schwarzschild metric with $d\tau = 0$ are stationary if and only if they are stationary with respect to the above metric where the line element is $dt$,not $ds$ . Using Fermat's principle of least time we apply the geodesic equation with this 'restricted' metric. The term stationery here means that the equations of motion derived from the geodesic equation make the variation of the corresponding action zero to first order. 'Action' here refers to the Least Action Principle in classical mechanics. For an explanation of this principle see for example [14].

The corresponding restricted metric tensor has the components:

$$g_{rr} = \left(\frac{r}{r-2m}\right)^2 \qquad g_{\theta\theta} = \frac{r^3}{r-2m}$$ for the covariant tensor

and $g^{rr} = \left(\frac{r-2m}{r}\right)^2$ $\quad g^{\theta\theta} = \frac{r-2m}{r^3}$ for the contravariant tensor.

The off-diagonal components are zero.

Then the only non-zero $\partial g_{\alpha\beta}/\partial x^{\gamma}$ components are

$$\frac{\partial g_{rr}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{r}{r-2m}\right)^2 = 2\left(\frac{r}{(r-2m)^2}\right) - 2\left(\frac{r^2}{(r-2m)^3}\right) = 2r\left(\frac{1}{r-2m}\right)^2 - 2r^2\left(\frac{1}{r-2m}\right)^3 = \frac{2r(r-2m)-2r^2}{(r-2m)^3} =$$

$$\frac{-4rm}{(r-2m)^3},$$

and $\frac{\partial g_{\theta\theta}}{\partial r} =$

$$\frac{\partial}{\partial r}\left(r^3\left(\frac{1}{r-2m}\right)\right) = 3r^2\left(\frac{1}{r-2m}\right) - r^3(-1)\left(\frac{1}{r-2m}\right)^2 = \frac{3r^2(r-2m)-r^3}{(r-2m)^2} = \frac{2r^3-6r^2m}{(r-2m)^2} = \frac{2r^2(r-3m)}{(r-2m)^2}.$$

And so $\frac{\partial g_{rr}}{\partial r} = \frac{-4rm}{(r-2m)^3}$ and $\frac{\partial g_{\theta\theta}}{\partial r} = \frac{2r^2(r-3m)}{(r-2m)^2}$.

Note: If we were to compute any other partial derivative, we would be either computing a derivative of a zero tensor component (and hence zero) or finding that the derivative of a non-zero tensor component equals zero.

It follows that the only non-zero Christoffel symbols are

$\Gamma^{r}_{rr}$, $\Gamma^{r}_{\theta\theta}$, and $\Gamma^{\theta}_{r\theta}$:

$$\Gamma^{r}_{rr} = \left(\frac{1}{2}\right)g^{r\rho}\left(\frac{\partial g_{\rho r}}{\partial r} + \frac{\partial g_{r\rho}}{\partial r} - \frac{\partial g_{rr}}{\partial x^{\rho}}\right) = \left(\frac{1}{2}\right)\left(g^{r\rho}\frac{\partial g_{\rho r}}{\partial r} + g^{r\rho}\frac{\partial g_{r\rho}}{\partial r} - g^{r\rho}\frac{\partial g_{rr}}{\partial x^{\rho}}\right) =$$

$$\left(\frac{1}{2}\right)\left[\left(\frac{r-2m}{r}\right)^2\left(\frac{-4mr}{(r-2m)^3}\right) + \left(\frac{r-2m}{r}\right)^2\left(\frac{-4mr}{(r-2m)^3}\right) - \left(\frac{r-2m}{r}\right)^2\left(\frac{-4mr}{(r-2m)^3}\right)\right] =$$

$$\left(\frac{1}{2}\right)\left[\left(\frac{r-2m}{r}\right)^2\left(\frac{-4mr}{(r-2m)^3}\right)\right] = \frac{-2m}{r(r-2m)}.$$

This computation was simplified by noting that any term where the dummy index $\rho \neq r$ equals zero since the corresponding tensor entry is zero.

$$\Gamma^{r}_{\theta\theta} = \frac{1}{2} g^{r\rho}\left(\frac{\partial g_{\rho\theta}}{\partial\theta} + \frac{\partial g_{\theta\rho}}{\partial\theta} - \frac{\partial g_{\theta\theta}}{\partial x^{\rho}}\right) = \frac{1}{2}\left(g^{r\rho}\frac{\partial g_{\rho\theta}}{\partial\theta} + g^{r\rho}\frac{\partial g_{\theta\rho}}{\partial\theta} - g^{r\rho}\frac{\partial g_{\theta\theta}}{\partial x^{\rho}}\right) =$$

$$\frac{1}{2}\left(0 + 0 - \left(\frac{r-2m}{r}\right)^{2} 2\,r^{2}\frac{(r-3m)}{(r-2m)^{2}}\right) = -(r-3m).$$

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{2} g^{\theta\rho}\left(\frac{\partial g_{\rho r}}{\partial\theta} + \frac{\partial g_{\theta\rho}}{\partial r} - \frac{\partial g_{\theta r}}{\partial x^{\rho}}\right) = \frac{1}{2}\left(g^{\theta\theta}\frac{\partial g_{\theta r}}{\partial\theta} + g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r} - g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial\theta}\right) = \frac{1}{2}\left(0 + \left(\frac{r-2m}{r^{3}}\right)2\,r^{2}\frac{(r-3m)}{(r-2m)^{2}} - 0\right)$$

Similar to above, any term is zero when $\rho \neq \theta$

and so $\Gamma^{r}_{rr} = \frac{-2m}{r(r-2m)}$, $\Gamma^{r}_{\theta\theta} = -(r-3m)$, and $\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \left(\frac{1}{r}\right)\frac{r-3m}{r-2m}$,

We are now ready to compute the relativistic equations of motion for the photon path. In the geodesic equation replace the space-time path parameter λ by the coordinate time t since an equation of motion requires derivatives of time. Then

$$\left(\frac{d^{2}x^{\alpha}}{dt^{2}}\right) + \Gamma^{\alpha}_{\gamma\beta}\left(\frac{dx^{\beta}}{dt}\right)\left(\frac{dx^{\gamma}}{dt}\right) = 0.$$

Using the non-zero Christoffel symbols we computed above and replacing the $dx^{\alpha}$ term, first by $dr$ and then by $d\theta$, the resulting two geodesic equations, both equal to zero, are:

$$\left(\frac{d^{2}r}{dt^{2}}\right) + \Gamma^{r}_{\gamma\beta}\left(\frac{dx^{\beta}}{dt}\right)\left(\frac{dx^{\gamma}}{dt}\right) = \left(\frac{d^{2}r}{dt^{2}}\right) + \Gamma^{r}_{rr}\left(\frac{dr}{dt}\right)\left(\frac{dr}{dt}\right) + \Gamma^{r}_{\theta\theta}\left(\frac{d\theta}{dt}\right)\left(\frac{d\theta}{dt}\right) =$$

$$\left(\frac{d^{2}r}{dt^{2}}\right) - \frac{2m}{r(r-2m)}\left(\frac{dr}{dt}\right)^{2} - (r-3m)\left(\frac{d\theta}{dt}\right)^{2} = 0$$

and

$$\left(\frac{d^{2}\theta}{dt^{2}}\right) + \Gamma^{\theta}_{r\theta}\left(\frac{dx^{\beta}}{dt}\right)\left(\frac{dx^{\gamma}}{dt}\right) = \left(\frac{d^{2}\theta}{dt^{2}}\right) + \Gamma^{\theta}_{r\theta}\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + \Gamma^{\theta}_{\theta r}\left(\frac{d\theta}{dt}\right)\left(\frac{dr}{dt}\right) =$$

$$\left(\frac{d^{2}\theta}{dt^{2}}\right) + \frac{r-3m}{r(r-2m)}\frac{dr}{dt}\frac{d\theta}{dt} + \frac{r-3m}{r(r-2m)}\frac{d\theta}{dt}\frac{dr}{dt} = 0$$

Solving for the second derivative term in each equation gives us the finalized form of the two equations of motion for the space-time path of light rays(photons) as derived from the geodesic equation:

$$\frac{d^2r}{dt^2} = \frac{2m}{r(r-2m)} \left(\frac{dr}{dt}\right)^2 + (r - 3m)\left(\frac{d\theta}{dt}\right)^2$$

and

$$\frac{d^2\theta}{dt^2} = -\ 2\ \frac{r-3m}{r(r-2m)}\ \frac{dr}{dt}\ \frac{d\theta}{dt} \qquad (2)$$

Step B - Done

**Step C.**

The two boxed equations (1) and (2) above will now be used to find $\varphi_D = \int_{\infty}^{r_0} \frac{d\theta}{dr} dr$ where $r_0$ is the point of closest approach to the center of the Sun for a light ray grazing the Sun's surface.

Let $u = \frac{d\theta}{dt}$. Then using equation (2)

$$\frac{d^2\theta}{dt^2} = \frac{du}{dt} = -\ 2\ \frac{r-3m}{r(r-2m)}\ \frac{dr}{dt}\ u\ \rightarrow\ \frac{1}{u}\ \frac{du}{dt} = -\ 2\ \frac{r-3m}{r(r-2m)}\ \frac{dr}{dt}\ \rightarrow$$

$$\frac{1}{u}\ \frac{du}{dt}\ \frac{dt}{dr} = -\ 2\ \frac{r-3m}{r(r-2m)}\ \ \rightarrow\ \ \frac{1}{u}\ du\ = -\ 2\ \frac{r-3m}{r(r-2m)} dr\ \ \rightarrow\ \int \frac{1}{u} du$$

$$= -\ 2\ \int \frac{r-3m}{r(r-2m)} dr\ \rightarrow \ln ln\, u\ +c_1\ =\ \ -\ 2\ \int \frac{1}{(r-2m)} dr\ +\ 6m \int \frac{1}{r(r-2m)} dr.$$

We have a natural logarithm expression on the LHS and the integrals on the RHS are in a form that will also yield logarithmic terms. This will help in finding the constant of integration.

Integrating these two RHS terms we get

$$\ln ln\, u\ +c_1\ =\ \ -\ 2 \ln ln\, (r\ -\ 2m)\ \ +c_2\ \ +\ 6m\left(\frac{1}{-2m}\right) ln \frac{r}{r-2m}\ +c_3.$$

Combining all the integration constants into one constant K we finally get

$$ln\, u\ = -\ 2ln\, (r\ -\ 2m) -\ 3ln(\frac{r}{r-2m})\ +\ K\ = -\ 2ln\, (r\ -\ 2m) -\ 3ln\, r\ +\ 3ln\, (r\ -\ 2m) +\ K\ =\ ln \frac{r-2m}{r^3}\ +\ K.$$

We now need to find the value of K by using some constraint (boundary) condition. One boundary condition is the point of closest approach of the photon to the Sun, namely $r_0$. We also know that this is a stationary point, i.e., $\frac{dr}{dt} = 0$, since this is where the rate of change of r with respect to t changes from decreasing to increasing so it must be zero at $r_0$. We will now leverage this information to determine K and in turn $\frac{d\theta}{dt}$.

$$ln\,u \;=\; ln\frac{r-2m}{r^3} + K \rightarrow e^{ln\,u} = e^{ln\frac{r-2m}{r^3}+K} \rightarrow u = \frac{d\theta}{dt} = \frac{r-2m}{r^3}e^K = \frac{r-2m}{r^3}K^{'}$$

Where $e^K \equiv K^{'}$ so $\frac{d\theta}{dt} = \frac{r-2m}{r^3}K^{'}$.

We know that $\frac{dr}{dt} = 0$ at $r = r_0$, we now know $\frac{d\theta}{dt}$ from the above, and we have equation (1)

$$1 \;=\; \left(\frac{r}{r-2m}\right)^2\left(\frac{dr}{dt}\right)^2 + \left(\frac{r^3}{r-2m}\right)\left(\frac{d\theta}{dt}\right)^2$$

which relates these two derivatives. So now let's evaluate equation (1) <u>at the stationary point</u>:

$$1 \;=\; \left(\frac{r_0}{r_0-2m}\right)^2(0)^2 + \left(\frac{r_0^{\;3}}{r_0-2m}\right)\left(\frac{d\theta}{dt}\right)^2 \rightarrow 1 \;=\; \left(\frac{r_0^{\;3}}{r_0-2m}\right)\left(\frac{d\theta}{dt}\right)^2 \rightarrow \frac{d\theta}{dt} = \sqrt{\frac{r_0-2m}{r_0^{\;3}}}\,.$$

So $\frac{r_0-2m}{r_0^{\;3}}K^{'} = \sqrt{\frac{r_0-2m}{r_0^{\;3}}}$ or $K^{'} = \frac{r_0^{\;3}}{r_0-2m}\sqrt{\frac{r_0-2m}{r_0^{\;3}}}$. Hence $K^{'} = \sqrt{\frac{r_0^{\;3}}{r_0-2m}}$ .

Therefore

$$\boxed{\frac{d\theta}{dt} = \frac{r-2m}{r^3}K^{'} = \frac{r-2m}{r^3}\sqrt{\frac{r_0^{\;3}}{r_0-2m}}}$$

for any value of r.

To compute $\varphi_D = \int_{r_0}^{\infty}\frac{d\theta}{dr}dr$ we need to derive an equation for $\frac{d\theta}{dr}$. By the chain rule

$\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \frac{r-2m}{r^3}\sqrt{\frac{r_0^{\;3}}{r_0-2m}}\,\frac{dt}{dr}$. We again use equation (1) to find $\frac{dr}{dt}$ since the inverse of this derivative is one of the above factors:

$$\left(\frac{dr}{dt}\right)^2 = \frac{\lceil 1-\left(\frac{r^3}{r-2m}\right)\left(\frac{d\theta}{dt}\right)^2 \rceil}{\left(\frac{r}{r-2m}\right)^2} =$$

$$\frac{\lceil 1-\left(\frac{r^3}{r-2m}\right)\left(\frac{r-2m}{r^3}\sqrt{\frac{r_0^{\ 3}}{r_0-2m}}\right)^2 \rceil}{\left(\frac{r}{r-2m}\right)^2} = \lceil 1 - \left(\frac{r_0^{\ 3}}{r_0-2m}\right)\frac{r-2m}{r^3}\rceil\left(\frac{r-2m}{r}\right)^2.$$

Therefore $\frac{dr}{dt} = \frac{r-2m}{r}\sqrt{1 - \left(\frac{r_0}{r}\right)^3 \frac{r-2m}{r_0-2m}}$ .

Inverting we get $\frac{dt}{dr} = \frac{1}{\left(\frac{r-2m}{r}\sqrt{1-\left(\frac{r_0}{r}\right)^3\frac{r-2m}{r_0-2m}}\right)}$ .

We finally arrive at $\frac{d\theta}{dr} = \frac{d\theta}{dt}\frac{dt}{dr} = \frac{\left[\frac{r-2m}{r^3}\sqrt{\frac{r_0^{\ 3}}{r_0-2m}}\right]}{\left[\left(\frac{r-2m}{r}\sqrt{1-\left(\frac{r_0}{r}\right)^3\frac{r-2m}{r_0-2m}}\right)\right]}$

which reduces to

$$\boxed{\frac{d\theta}{dr} = \frac{\left[\frac{1}{r^2}\sqrt{\frac{r_0^{\ 3}}{r_0-2m}}\right]}{\left[\left(\sqrt{1-\left(\frac{r_0}{r}\right)^3\frac{r-2m}{r_0-2m}}\right)\right]}. \qquad (3)\ \text{Step C - Done}}$$

**Step D.**

We now have a mathematical expression for how the infinitesimal angle $d\theta$ changes as the photon moves along a hyperbolic path grazing the gravitating body's surface (the stationary point).

$dr$ is the infinitesimal change in the distance of the photon from the center of the gravitating body. The angle is measured from the body's center out to the photon path and is subtended by the almost flat arc of the path. If gravity did not affect the path the arc would be a straight line.

$\varphi_D$ is the total angle swept out by the photon which is the "accumulation" of the infinitesimal angles $d\theta$ as r leaves the point of closest approach $r_0$ and moves out to a far distance $(r = \infty)$ along the path. See Figure 3 below. This is formally expressed as $\varphi_D = \int_{\infty}^{r_0} \frac{d\theta}{dr} dr$. The angle of deflection is then defined as $\delta = \varphi_D - \frac{\pi}{2}$. Note that for a straight path in flat space far from any gravitating source there is no deflection and we have $\delta = \frac{\pi}{2} - \frac{\pi}{2} = 0$. However, in our case $2\delta = 2\varphi_D - \pi$ then gives the total deflection of a photon arriving from a far distance $r = \infty$, grazing the Sun, and going back out to a far distance $r = -\infty$. This total deflection is then used to determine the apparent position of the photon source (a star) as viewed by an observer (on Earth in our case).

**Step D. (continued)**

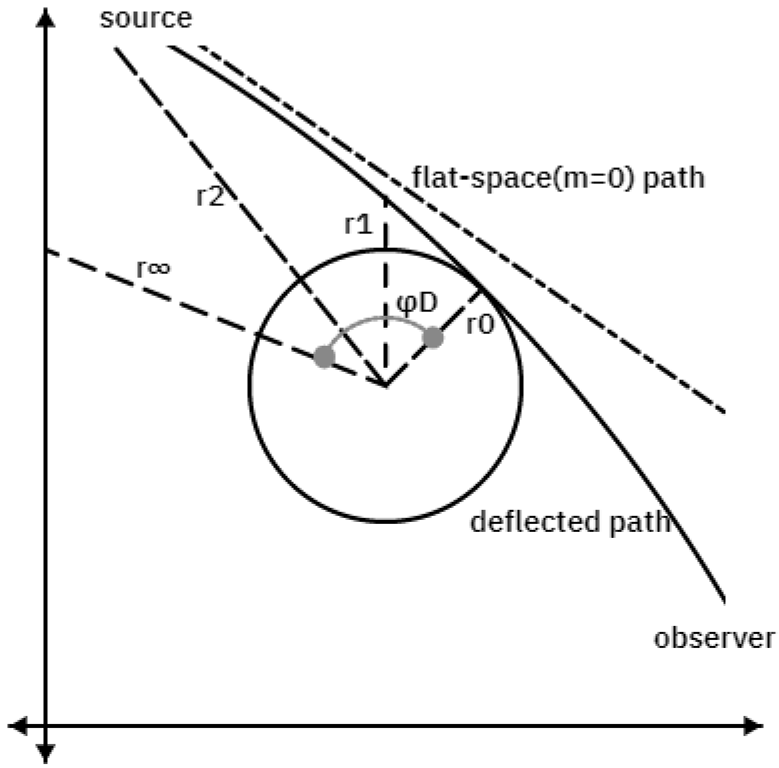


Figure 3. Angle $\varphi_D$ swept out by the light path showing several radial distances along the deflected path. $r_\infty$ is the radial distance for the light source arriving from a far distance $(r = \infty)$. The dotted straight line is the flat space $(m = 0)$ path for no deflection.

Then the integral of equation (3) is

$$\varphi_D = \int_{r_0}^{\infty} \frac{d\theta}{dr}\, dr = \int_{r_0}^{\infty} \left[ \frac{\frac{1}{r^2}\sqrt{\frac{r_0^3}{r_0 - 2m}}}{\sqrt{1 - \left(\frac{r_0}{r}\right)^3 \frac{r-2m}{r_0-2m}}} \right] dr \; . \qquad (4)$$

Step D - Done

**Step E.**

The integral (4) is not feasible to determine in closed form so a change of variables is done to reduce it to a simpler form which can then be expressed as a standard integrand form times a convergent power series. The higher order terms are then dropped giving an integrand of two terms which can be integrated and is a good approximation of $\varphi_D$ to first order[7].

First, the variable replacement $\rho = \frac{r_0}{r}$ is made so $d\rho = -\frac{r_0}{r^2}dr$ or $dr = -\frac{r^2}{r_0}d\rho$. Then $dr = -\frac{\left(\frac{r_0}{\rho}\right)^2}{r_0}d\rho = -\frac{r_0}{\rho^2}d\rho$. Substituting into (4), integrating over $\rho$, and noting that as $r$ ranges from $r_o$ to $\infty$, $\rho$ ranges from 1 to 0:

$$\varphi_D = \int_1^0 \left(\frac{\rho}{r_0}\right)^2 \sqrt{\frac{r_0^3}{r_0-2m}} \cdot \left[\sqrt{1-\rho^3\frac{\frac{r_0}{\rho}-2m}{r_0-2m}}\right]^{-1} \left(\frac{-r_0}{\rho^2}\right)d\rho. =$$

$$\int_0^1 \frac{1}{r_0}\sqrt{\frac{r_0^3}{r_0-2m}} \cdot \left[\sqrt{1-\rho^3\frac{\frac{r_0}{\rho}-2m}{r_0-2m}}\right]^{-1} d\rho.$$

We are not there yet so we need to keep manipulating this expression until, as stated above, it is the integral of two tractable factors, one of which is expressible as a power series. By absorbing the $\frac{1}{r_0}$ factor into the adjoining radical the above becomes

$$\int_0^1 \sqrt{\frac{r_0}{r_0-2m}} \bullet \left[\sqrt{1-\rho^3\frac{\frac{r_0}{\rho}-2m}{r_0-2m}}\right]^{-1} d\rho = \int_0^1 \left[\sqrt{r_0\cdot\left(r_0-2m-\rho^3\left(\frac{r_0}{\rho}-2m\right)\right)^{-1}}\right]d\rho$$

We will now further manipulate this expression into the general form $\int_0^1 \frac{1}{\sqrt{1-Q^2}}\frac{1}{\sqrt{P}}d\rho$ where $\frac{1}{\sqrt{P}}$ can be represented as a convergent power series with both Q and P functions of $\rho$.

---

[7] The 2007 paper by C. Magan [10] based on Taylor and Wheeler (Spacetime Physics,1965) has a similar development as the above but without the use of tensors. The result can be shown to be equivalent to the integral equation (4). He then uses a different approximation method than described here to transform it into a tractable form.

To achieve this goal, algebraic manipulation of the expression under the radical in the integrand is made:

$$\frac{r_0}{\left[\left(r_0 - 2m - \rho^3\left(\frac{r_0}{\rho} - 2m\right)\right)\right]} = \frac{r_0}{\left[\left(r_0 - 2m - \rho^2 r_0 + 2\rho^3 m\right)\right]} = \frac{r_0}{\left[\left(r_0\left(1-\rho^2\right) - 2m\left(1-\rho^3\right)\right)\right]} = \frac{1}{\left[\left(\left(1-\rho^2\right) - 2\left(1-\rho^3\right)\frac{m}{r_0}\right)\right]}$$

We now factor out $1-\rho^2$ in the denominator. Notice that $1-\rho^2$ is of the form $1-Q^2$ which is part of the sought-after general form of the integrand. The remaining term is P. Therefore. our integrand under the radical becomes

$$\frac{1}{\left[\left(\left(1-\rho^2\right)(1-2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}\right)\right]} = \frac{1}{1-\rho^2}\ \frac{1}{\left[\left(1-2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}\right)\right]}.$$

Then the final expression for our integrand is

$$\sqrt{\frac{1}{1-\rho^2}}\ \sqrt{\frac{1}{\left[\left(1-2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}\right)\right]}}.$$

The second radical factor is of the general form $\sqrt{\frac{1}{1-z}}$ which can be written as power series $1 + \frac{1}{2}z + \frac{3}{8}z^2 + \frac{5}{32}z^3 + \ldots\cdots + \frac{2n+1}{2^{2n+1}}z^{n+1} + \cdots$, for $n = 0, 1, 2, \ldots$ .

Each term in the series is generated from the calculus formula for a Maclauran power series, namely $\sum_0^\infty \frac{f^{[n]}}{n!}x^n$, where $f^{[n]}$ denotes the $n$th derivative of the function which in this case is $f(z) = \sqrt{\frac{1}{1-z}}$, $z = 2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}$ . The details of this calculation are not supplied here but computing the first few terms of the series should satisfy the reader of its validity. Before using this series we need to ensure it is convergent within the integration limits of $\rho$ which in turn is a function of $r$, the distance of the photon from the center of the Sun.

We can argue on physical grounds that this is so since $\varphi_D = \int_{\infty}^{r_0} \frac{d\theta}{dr} dr$ must be a finite value and therefore the value of the power series must itself be finite. The one case where this may be violated is if the photon could "wrap around" the Sun in an orbit yielding an infinite angle of deflection However its gravity is too weak for this to occur unlike a black hole where theoretically it can occur based on classical (not quantum mechanical) assumptions. A mathematical proof of convergence is detailed in Appendix B.

We are now ready to express the $\varphi_D$ in power series form.

$$\varphi_D = \int_{\infty}^{r_0} \frac{d\theta}{dr} dr = \int_0^1 \sqrt{\frac{1}{1-\rho^2}} \sqrt{\frac{1}{\left[\left(1-2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}\right)\right]}}\, d\rho =$$

$$\int_0^1 \sqrt{\frac{1}{1-\rho^2}} \left(1 + \frac{1}{2}z + \frac{3}{8}z^2 + \frac{5}{32}z^3 + \cdots + \frac{2n+1}{2^{2n+1}}z^{n+1} + \cdots\right) d\rho$$

where $n = 0, 1, 2, \cdots$ and, $z = 2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}$ , $\rho = \frac{r_0}{r}$.

Convergence guarantees that $\frac{1}{\sqrt{1-\rho^2}}$ can be distributed across each term of the series and that the definite integral is finite.

Then

$$\int_0^1 \left(\frac{1}{\sqrt{1-\rho^2}} + \frac{1}{2}\frac{1}{\sqrt{1-\rho^2}}z + \frac{3}{8}\frac{1}{\sqrt{1-\rho^2}}z^2 + \frac{5}{32}\frac{1}{\sqrt{1-\rho^2}}z^3 + \cdots\cdots\right) d\rho =$$

$$\int_0^1 \left(\frac{1}{\sqrt{1-\rho^2}} + \frac{1}{2}\frac{1}{\sqrt{1-\rho^2}}2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0} + \frac{3}{8}\frac{1}{\sqrt{1-\rho^2}}\left(2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}\right)^2 + \frac{5}{32}\frac{1}{\sqrt{1-\rho^2}}\left(2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}\right)^3 + \cdots\cdots\right) d\rho \approx$$

$$\boxed{\int_0^1 \left(\frac{1}{\sqrt{1-\rho^2}} + \frac{1}{\sqrt{1-\rho^2}}\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}\right) d\rho. \qquad (5)}$$

Step E - Done

In this last integral equation (5) we have dropped the terms of second order and higher since **$m/r_0$ <<1** and the integration value of these terms will not significantly contribute to our final result. Notice that the Sun's mass $m$ does not appear in the first term of the integrand. This corresponds to the light path when $m = 0$, i.e. the trajectory is a straight line. The presence of the second term with mass $m$ gives us the deflection. These terms in the integrand are of standard form and the integrals can be easily determined.

For the first one we have

$$\int_0^1 \frac{1}{\sqrt{1-\rho^2}} d\rho = [\sin^{-1}\rho]_0^1 = \frac{\pi}{2} - 0 = \frac{\pi}{2},$$

and for the second integral

$$\int_0^1 \frac{1}{\sqrt{1-\rho^2}} \left(\frac{1-\rho^3}{1-\rho^2}\right) \frac{m}{r_0}\, d\rho = \frac{m}{r_0} \int_0^1 \frac{1}{\sqrt{1-\rho^2}} \frac{(1+\rho+\rho^2)}{(1+\rho)}\, d\rho =$$

$$\frac{m}{r_0}\left[\int_0^1 \frac{1}{(1+\rho)\sqrt{1-\rho^2}}\, d\rho + \int_0^1 \frac{\rho}{\sqrt{1-\rho^2}}\, d\rho\right]$$

$$= \frac{m}{r_0}\left[-\frac{\sqrt{1-\rho^2}}{\rho+1} - \sqrt{1-\rho^2}\right]_0^1 = \frac{m}{r_0}[-0 - 0 - (-1 - 1)] = 2\frac{m}{r_0}$$

Adding these two results we finally obtain $\varphi_D = \int_\infty^{r_0} \frac{d\theta}{dr} dr \approx \frac{\pi}{2} + 2\frac{m}{r_0}$.

Per section D, the computed deflection over our integrating range for $r$ is $\delta = \varphi_D - \frac{\pi}{2}$.

Then the <u>total</u> deflection is $2\delta = 2\varphi_D - \pi = 2\left(\frac{\pi}{2} + 2\frac{m}{r_0}\right) - \pi = 4\frac{m}{r_0}$

$= 4\left(\frac{1475}{6.96x10^8}\right) = 8.492x10^{-6}$ radians which is $8.492x10^{-6}\, x\, \frac{180}{\pi}\, x\, 60\, x\, 60 = 1.75$ arcseconds.

To compare $4\,\frac{m}{r_0}$ to the Soldner and Cavendish results in section A and B it is best to convert from our geometric units, where $G = c = 1$ and $m$ is in meters, to SI units. Using the conversion equation $m = \frac{GM}{c^2}$ where $G$, $M$, and $c$ are in SI units, and setting $R_{Sun} \equiv r_0$, we get $\frac{4GM}{c^2 R_{Sun}}$ as the expression for the total deflection angle. We can then do the comparison which is done in Sections A and B.

If the third term of the series had been included in the deflection result, a purely numerical calculation shows that it would have contributed only $7.27\, x\, 10^{-6}$ additional arcseconds, which justifies ignoring the higher order terms.

This deflection angle prediction has been verified by ever more refined methods to very good accuracy since the first measurement was made in 1919 by the Eddington expedition. Some of these results are listed in [2]and [6].

**Summary/Conclusions:**

Drawing on the main sources referenced in this paper we compared the most prominent attempts to determine how a light ray passing close to the Sun is influenced by its gravitational field, that is, the deflection from a straight path. We also developed the detailed mathematics for each prediction. Our main goals were to

1. Compare the three approaches (Soldner, Cavendish, General Relativity)
2. Give sufficient computational mathematical details for the light deflection results without having the reader encounter major gaps.

Soldner (Section A) based his approach on assuming light itself experiences a force due to gravity based on Newtonian mechanics and he showed, based on calculus-based computations, that the path is a hyperbola. We provided most of the mathematical details resulting in the Soldner formula of $\mathrm{tang}\,\omega = \frac{2g}{v\sqrt{v^2 - 4g}}$ for the deflection. We used this formula and replicated his stated result with a deflection of 0.42 arcseconds for one half of the hyperbolic path for a light ray passing close to the Sun's surface.  Doubling it gives 0.84 arcseconds deflection for an Earth-based observer, his stated full deflection value.

To perform the computation for the Sun we had to develop values for $v$ and $g$ for the Sun, that was consistent with his approach, since he did not provide these values.

We noted, however, that Soldner used a time for the transit of light traveling from the Sun to the Earth that was slightly different from the modern value. Correcting for produced 0.88 arcseconds for the full deflection.  This correction for light speed showed that Soldner's approach yielded the same result as the Cavendish approach as stated below.

Cavendish's approach (Section B) used Kepler's and Newton's theories, extended to light, assuming light would follow a hyperbolic path near the Sun. He obtained the formula $sin\frac{\delta}{2} = \frac{a}{a+R}$ for light deflection. Using his formula and Newtonian based velocity equations, a deflection of 0.44 arcseconds was obtained. As with Soldner, it is doubled to give 0.88 arcseconds.

Our analysis in Section C is based on Einstein's General Theory of Relativity and the use of the Schwarzschild metric. The derived formula for the deflection angle equation cannot be put into a standard equation for a hyperbola as can Soldner's and Cavendish's equations. A closed formula is not obtained and a power series cut-off numerical approximation method (the $2\delta = 2\varphi_D - \pi$ calculation) is used to get a deflection value of 1.75 arcseconds. We supplied most of the mathematical details for this result.

We noted that the Soldner and Cavendish formulas for deflection can be related to the more conventional format of modern equations based on modern values for the gravitational constant, the speed of light and the mass of the Sun. This allows us to state that both the Soldner and Cavendish angles of deflection are $\approx \frac{2GM}{c^2 R_{Sun}}$.  Our Section C, in turn, shows that the Einstein approach results in $\approx \frac{4GM}{c^2 R_{Sun}}$. This verified that General Relativity yields a deflection that is twice the Newtonian approach to a first order approximation. Extensive astronomical experiments have shown the Einstein prediction for the angle of deflection is correct.

**Appendix A:**

Proof that $C = v$ in Step B of Section A.2 (Soldner approach)

To find C he uses the equation $r^2 d\varphi = Cdt$ and the fact that in an instant of time the light particle sweeps out an infinitesimal angle subtending an infinitesimal arc. He gives a very brief justification for his claim evidently using an argument from Laplace [7]. We provide a more detailed argument using current terminology.

First, we make the following observations:

a. Instead of arriving at point A (see Figure 1 in section A.1), light can be treated as emanating from that point. This makes no difference in Soldner's approach as he states in his opening remarks. Since we are dealing with an infinitesimal arc, $dx \ll dy$ and so the circular arc swept out by light can be treated as a straight line which then becomes side AM of the right triangle $\Delta CAM$ (see Figure 4 below) where CA, and M correspond to the same points as in Figure 1.
b. So in an instant of time light moves up $dy$ units (and inward $dx$ units along the infinitesimal path MA but we can ignore this since $dx \ll dy$.

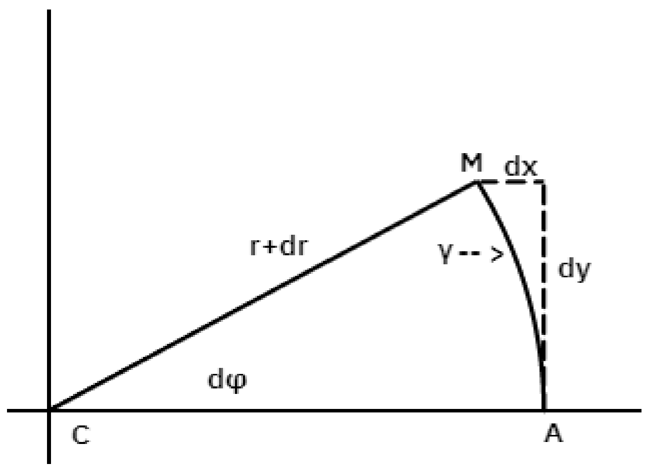


Figure 4.

*ɣ indicates the light path.*

We now proceed to show that $C = v$ .

1. Since path MA is infinitesimal and $dx \ll dy$ then MA = $\sqrt{dx^2 + dy^2} \approx dy$.
2. In general, for any small angle θ , $sin\theta \approx \theta$, so $sin\, d\varphi \approx d\varphi$.
3. $sin\, d\varphi = d\varphi = \frac{dy}{r+dr}$ or $rd\varphi + d\varphi dr = dy$. So to first order $rd\varphi$ = $dy$ since $d\varphi dr$ is of second order and therefore an order of magnitude smaller.
4. Area of ΔCAM = (1/2)$rdy = \left(\frac{1}{2}\right)r(rd\varphi) = \left(\frac{1}{2}\right)r^2 d\varphi$. Hence 2ΔCAM = $r^2 d\varphi$. Notice that the RHS is contained in the previously derived equation $r^2 d\varphi = Cdt$. We now need to bring the velocity of light v into the picture by noting that obviously MA, and hence y and $dy$, depend on time t. So $dy = vdt$ (distance = rate x time) since $dy$ ≈ MA.
5. Then the area of ΔCAM can be expressed as a function of light speed:

   ΔCAM = (1/2)$rdy = \left(\frac{1}{2}\right)r(vdt)$. Hence 2ΔCAM = $rvdt$.
6. Equating the results from steps 4 and 5 gives $r^2 d\varphi = rvdt$. But as shown above $r^2 d\varphi = Cdt$ so $Cdt = rvdt$. Also, $r$ = AC per Figure

4. Soldner has set AC $= 1$ so $\mathrm{r} = 1$ and canceling the infinitesimal $dt$ we finally get $\mathrm{C} = \mathrm{v}$.

To make this demonstration more rigorous we could have replaced all infinitesimals with an argument using limits.

**Appendix B:**

Mathematical proof that the power series in Step E of Section C.2 (General Relativity approach) is convergent.

The series is $1 + \frac{1}{2}z + \frac{3}{8}z^2 + \frac{5}{32}z^3 + \ldots\cdots + \frac{2n+1}{2^{2n+1}}z^{n+1} + \cdots$, for $\mathrm{n} = 0, 1, 2, \cdots$

where $z = 2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0}$.

To show convergence we need to show that an interval of convergence of the series exists. Since all the terms of the series are positive the ratio test for an infinite series can be used, i.e., show that $\frac{u_{n+1}}{u_n}$ exists and is less than 1.

$$\frac{2^{\frac{2(n+1)+1}{2(n+1)+1}} \quad z^{(n+1)+1}}{2^{\frac{2n+1}{2n+1}} \quad z^{n+1}} = \frac{(2n+3)2^{2n+1}}{(2n+1)2^{2n+3}}z = \left(\frac{1}{4}\right)\frac{\left(2+\left(\frac{3}{n}\right)\right)}{\left(2+\left(\frac{1}{n}\right)\right)}z = \left(\frac{1}{4}\right)z.$$

By the ratio test criterion the series converges if $\left(\frac{1}{4}\right)z < 1$ or $\left(\frac{1}{4}\right)2\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0} = \left(\frac{1}{2}\right)\left(\frac{1-\rho^3}{1-\rho^2}\right)\frac{m}{r_0} < 1$.

To show this, examine the range of $\frac{1-\rho^3}{1-\rho^2}$ by noting that $\frac{1-\rho^3}{1-\rho^2} = \frac{(1-\rho)\left(1+\rho+\rho^2\right)}{(1-\rho)(1+\rho)} = \frac{\left(1+\rho+\rho^2\right)}{(1+\rho)}$. The range of the integration variable $\mathrm{r}$ is $r_0$ to $\infty$ so the range of $\rho$ is 1 to 0 since $\rho = \frac{r_0}{r}$. Therefore $1 \leq \frac{\left(1+\rho+\rho^2\right)}{(1+\rho)} \leq \frac{3}{2}$. So $1 \leq \frac{1-\rho^3}{1-\rho^2} \leq \frac{3}{2}$.

Now use the fact that $r_0 = 6.93\, x\, 10^8$ meters $\gg m = 1475$ meters, where we recall, m is the mass of the Sun expressed in geometric units, so $\frac{m}{r_0} \ll 1$. Then $\left(\frac{1}{2}\right)\frac{1-\rho^3}{1-\rho^2}\ \frac{m}{r_0} \leq 1$ as required for convergence.

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