“The Beauty and The Beast”

(Standard Model and the Monster Group)

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Abstract

The Standard Model of Elementary Particle Physics is an amazing and beautiful achievement of theory, experiment and technology, explaining the foundations of Physics in terms of three out of four interactions, considered fundamental.
Classification of Finite Simple Groups is too, an amazing achievement in Mathematics.
Recent advancements in the understanding of SM lead to an unexpected "encounter" between the two: lepton masses are related to the Monster and VOAs, via j-invariant of elliptic curves ... under The Moonshine.
But there is a "contender": Platonic and Archimedean solids (models for baryons, like the proton and neutron) can be represented as Dessins d'Enfant, introduced by Grothendieck in the 1960s, using Belyi maps!
Who will win the heart of the Beauty?
The main concepts will be defined and pictures will help bring the subject to the understanding of a general audience.

Goals and Plan

Goals:

- Introduce new Math-Models for Elementary Particle Physicists
- State some "STEM" Problems for Grad. Students
- tell a nice "story" (news).

Plan:

- Standard Model and Physics news
- Why finite groups are important ...

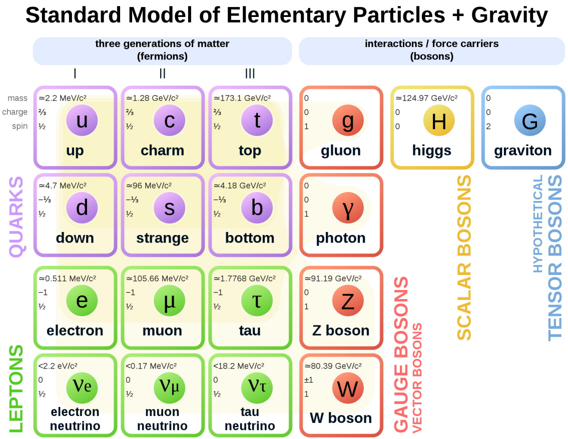
- Elliptic curves, modular group and j-invariant
- A) The Monstrous Moonshine ... and VOAs
- B) Belyi maps representing regular solids as dessins d'enfants
- The relation between A and B: modular curves and Belyi morphisms.

- Conclusions (The Universe IS Mathematical)

The Story ...

“The Beauty”

Standard Model of Particle Physics considers four interactions as fundamental, in terms of the *quark model* of protons and neutrons (1970s).



ElectroWeak Theory and Quantum Cromodynamics - No Gravity!

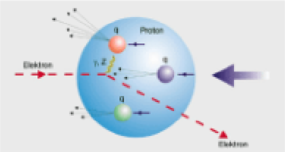
3rd Level of Elementary Structure of Matter: Quarks

1. Molecules & Atoms (Chemistry); 2. Electrons, neutrons & protons (Atomic Physics); 3. 3-quarks & interactions (Elem. Particle Physics).

Three Sources of Electric Field in Protons

- Under electron-proton scattering, a proton looks like a ball with three fractional charges
  - 2 positive (+⅔)
  - 1 negative (-⅓)
- Physicists *interpreted* these centers as particles called quarks; but they cannot be separated!

1. How to probe the quarks?



- Scatter high-energy electron off a proton:

  Deep-Inelastic Scattering (DIS)

- Highest energy *e-p* collider: HERA at DESY in Hamburg: ~ 300 GeV

- Relevant scales:

$$d_{probed} \propto \lambda = \frac{\hbar}{p} \approx 10^{-18}\ \text{m}$$

Physics News: Recent Developments

1) Quarks are not "particles", but part of the structure of a nucleon (baryon), defining a 3D-frame / basis of $SU(2)$, corresponding to quark colors and $SU(3)$ symmetry from which 3D-Space emerges.

2) Relativistic time emerges from $U(1)$-quantum phase.

3) Electroweak Theory and QCD are unified using Hopf fibration as a local model for Cartan Geometry: $SU(2)$-gauge theory of 3D-frames with $SU(3)$-symmetry group.

4) Gravity emerges from quark spin-spin interaction, part of the nuclear force and Electroweak Theory.

5) Gauge groups are FINITE subgroups: Z/n for $U(1)$, Platonic TOI (A4, S4, A5) for $SU(2)$ and four more for $SU(3)$;

5) The three fermion generations correspond to TOI symmetry groups, and quark flavors correspond to their dual Platonic Geometries.

... and The Monster

The Classification of Finite Simple Groups ("Leggo Problem") required "tens of thousands of pages in several hundred journal articles written by about 100 authors, published mostly between 1955 and 2004." (Wiki).

• Simple groups (no non-trivial quotient group) are the analog of primes, as building blocks for integers (Fundamental Theorem of Arithmetic).

Theorem

Every finite simple group is isomorphic to one from the following groups: A) Infinite "obvious" classes: a) Cyclic groups of prime order Z/p, b) Alternating groups A_n of degree at least 5, c) Groups of Lie type (finite coefficients, e.g. $PSL_n(F_p)$;
B) Sporadic groups: 26 of them ("Bonus Leggos").

• The largest sporadic finite group M is called *The Monster*; it's size is:

$$|M| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 5^9 \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 49 \cdot 71$$

The Monster and The "Monstrous Moonshine"

It came as a 1st surprise that *integral Mobius transformations* $SL_2(Z)$ (modular group) are related to the Monster:

Theorem

(Ogg, 1974) The modular curve $X_0^+(p) = \Gamma_0(p)/\mathcal{H}^*$ *is topologically a sphere exactly for the following primes:*

$$p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71$$

where $\mathcal{H}^*$ *is the upper C-plane and* $\Gamma_0(p)$ *is a modular subgroup [3].*

2nd surprise: the Fourier expansion of the j-invariant, classifying elliptic curves ("doughnuts") is related to the representations of the Monster:

$$j(q) = \frac{1}{q} + 744 + 19688q + \ldots$$

This "Monstrous Moonshine" conjecture was proved by Borcherds in 1992.

A Little Math-Physics ... and Computer Science (STEM)

Groups of transformations (Geometry and Dynamics):

- Euclidean Geometry: 3D-Rotations $SO(3)$ (preserve distance);
- Complex Analysis: Mobius transformations $PSL_2(C)$ (preserve angles);
- Lorentz Transformations: $SO(3,1)$ (preserve light speed), a.k.a. $PSL_2(C)$ (Mobius) ... Why!?

... because what we call Dynamics in Space-Time (Minkowski and Einstein 1900s) is in fact Quantum Computing (Paul Benioff, Feynman and Manin 1980) and thus theories are invariant under conformal transformations.

"Beautiful Moonshine" (Conjecture)

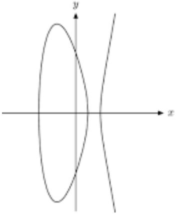
Neutron / Proton structure is described in terms of quark flavors, grouped in 3 generations, corresponding to Platonic symmetries and geometry.
From F. Potter's article: "Mass of leptons are related to the j-invariant and the Monster Group" (Monstrous Moonshine).

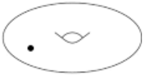
Leptons						Quarks					
group	order	family	N	Pred. Mass (MeV)	Emp. Mass (MeV)	**group**	order	family	N	Pred. Mass (GeV)	Emp. Mass (GeV)
					. . .	**[3, 3, 3]**	120	$d^{-1/3}$ $u^{+2/3}$	1/4	0.011 0.38	. 0.007 0.004
[3, 3, 2]	24	e^- ν_e	1	[1] 0?	0.511 0.0?	**[4, 3, 3]**	384	$s^{-1/3}$ $c^{+2/3}$	1	0.046 [1.5]	0.2 1.5
[4, 3, 2]	48	μ^- ν_μ	108	108 0?	103.5 0.0?	**[3, 4, 3]**	1152	$b^{-1/3}$ $t^{+2/3}$	108	[5] 160	5.0 171.4
[5, 3, 2]	120	τ^- ν_τ	1728	1728 0?	1771.0 0.0?	**[5, 3, 3]**	14400	$b'^{-1/3}$ $t'^{+2/3}$	1728	~ 80 ~ 2600	?.? ?.?

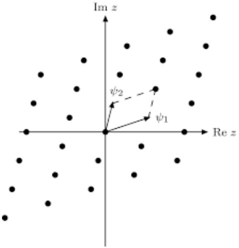
- Why lepton masses are j-invariant constants 1, 108, 1728? ("Beautiful Moonshine"?)

The Math ...

Elliptic Curves







- Algebraically (implicit), solutions of:

$$y^2 = x^3 + Ax + B, \quad x, y, A, B \in C.$$

- Differential Geometry: torus $S^1 \times S^1$, with a metric or *complex structure*, defining locally angles;

- Algebraic Geometry (explicit): C/L, Complex Plane modulo a 2D-lattice $L =< \omega_1, \omega_2 >$.

Weierstrass Elliptic Function and The Correspondence

It is useful to thin of EC / degree 3 / genus one as a 2D-Trigonometry, and compare with 1D: circle, cos and sin ($U(1)$-characters, Fourier Transform etc.) or sphere for complex coefficients.

• Weierstrass function $C = \mathcal{P}(z)$ and its derivative $S = \mathcal{P}(z)'$ provide a coordinate change allowing to relate the implicite form of the EC, in terms of A, B and the lattice presentation, via $q = exp(2\pi i\tau)$ (see Wiki).
• The coefficients A and B are given by Eisenstein series $c_2 = 60\ G_4$ and $c_4 = 120\ G_6$ for the lattice L and $\tau = \omega_1/\omega_2$, its generators (periods) [1].

One may think of C, S the analog of sine and cosine, and the EC as a relation between them (syzygy).

Equivalence / Isomorphic EC

• Isomorphic lattice, transformed via the modular group $SL_2(Z)$, define isomorphic elliptic curves: moduli space of EC.
• This moduli space can be parameterized by τ
• Alternatively, providing a classifying invariant would do.

Definition

Klein's j-invariant of an elliptic curve is [3]:

$$j(A, B) = 1728 \frac{4A^3}{4A^3 + 27B^2}.$$

• This j-invariant can be written in terms of the lattice L or parameter τ.

Note: The normalization coefficient 1728, appearing in the syzygy too, is independent of the EC ("universal") and appears also with Belyi functions over the sphere too ...

Fourier Expansion of j-invariant

The Fourier expansion of the j-invariant (periodic function), is:

$$j = q^{-1} + 744 + 196884q + 21493760q^2 \ldots, \quad q = exp(2\pi i\tau.)$$

... These are "numbers we used to know"[1]... The coefficients are related to the (characters) dimensions of representations of the Monster Group.

Monstrous Moonshine 1988

There is a naturally defined graded module V such that its *MacKay-Thomson series* $T_g(\tau)$ is a Hauptmodule for a discrete subgroup $\Gamma_g \subset SL_2(Z)$, for all $g \in M$.

• The j-invariant is an example of a *Hauptmodule* (*principal modular function*), as the (compactified) *modular curve* $Y(1) = SL_2(Z) \setminus \mathcal{H} \cong S^2$ is of genus zero.

• In brief: rational functions on such a "shere" are generated by such a "j"; algebra analogy: ideals in Z are principal.

[1] Not me, may be some of you ...

Intermezzo: Why SM and The Monster?

• Irreducible Reps of M (194 of them) are related to VOAs, hence to CFT.
• The Monster is related to other important "characters" in this "play": Leech lattice, quaternions / octonions, *exceptional Lie algebras* E_6, E_7, E_8 and their Weil groups, which are 2 : 1 to Platonic groups of symmetry, our TOIs and "quark flavors to spice baryons":

$$Platonic \to WeylGroups \to E6 - E8 \to Monster \ \& \ VOAs$$

• So, primes $p|size(M)$ correspond to spherical modular curves!! Is this (M and VOAS) the way to understand lepton masses? ... or there is a simpler route? ... after some $R\&G$[2]

$$Platonic \to Belyi \ maps \ \& \ Riemann \ Surfaces \to Invariants.$$

[2] Research and Google :)

M-Theory the "right way" (in hindsight)

The "contender" approach to the connection between SM and j-invariant is via Belyi maps associated to Platonic and Archimedean solids [4]. This is the direct approach to **model vibrations of baryons, with finite geometries, in the spirit of Membrane Theory** [5, 1].

• The "big picture" (in brief): Model SM interactions as RS (fat Feynman graphs) with paths (Quark Line Diagrams?) joining baryons as nodes. To make contact with SM and String Theory, consider these Network of Riemann Surfaces inside Space-Time (String Theory) but with particle fields (sections) in the frame bundle of the SM $SU(2)$-principal bundle over Space-Time.

• See also gauge theory and branes [4].

Belyi maps

We will proceed by example with pictures and references, following [4]; for further details see [2]; the EC case is presented in [3].

Definition

A *Belyi map* associated to the finite subgroup $G \subset Aut(P^1(C))$ is a function $\beta : P^1(C) \to P^1(C)$ satisfying the following [4]:
1) It is rational $\beta(z) = p(z)/q(z)$;
2) Has at most three critical points within $\{0, 1, \infty\}$;
3) It is invariant precisely under the G-action.

• $Aut(P^1(C))$ is the group of Mobius transformations of the sphere (conformal: preserve angles). This consists in fractional transformations $az + b)/(cz + d)$ of determinant one ($PSL_2(C)$; also physicists group of Lorentz transformations).

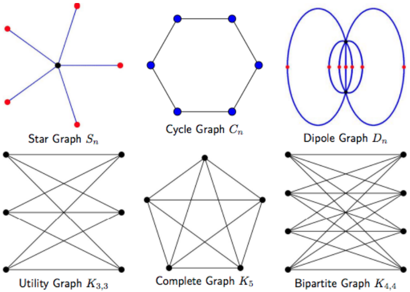
• The "integral part" of it $SL_2(Z)$ is the modular group! Here we focus on finite subgroups, not *congruence subgroups*, of finite index (dual?!).

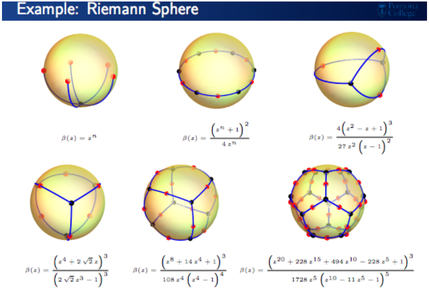
Dessins d'Enfants

Definition

Given a Belyi map $\beta : P^1(C) \to P^1(C)$, a Dessin d'Enfants is a connected, bipartite, planar graph $\Delta_\beta : B \to W$ with the following properties:

1) The "Black" vertices are $B = \beta^{-1}(0)$;
2) The "white" vertices are $W = \beta^{-1}(1)$;
3) The edges are $E = \beta^{-1}([0,1])$;
4) The midpoints of faces are $F = \beta^{-1}(\infty)$.





Belyi Theorem

General Belyi maps (Belyi functions) are defined on Riemann surfaces $\beta : X \to P^1(C)$ ("functionals" / "dual to R.S.).

Belyi Theorem (1979)

Any non-singular algebraic curve X, defined by algebraic number coefficients, represents a compact Riemann surface which is a ramified covering of the Riemann sphere, ramified at three points only.

Definition

Such a ramified cover is called a Belyi function and (X, f) a Belyi pair.

• A *morphism of Belyi pairs* is a morphism of ramified covers as fibrations, over the identity map of S^1.

• The 3 marked points can be interpreted as quarks, e.g. $\{1, \omega, \omega^2\}$ the cubic roots as "fractional charges".

Regular Solids as Dessins d'Enfant: The Tetrahedron

• Map the points of the tetrahedron B on the Riemann sphere $\sigma(B) = \{\infty, 1, \omega, \omega^2\}$ (cubic roots of 1), using the stereographic projection (see [4]).

• Define a *homogeneous polynomial* which vanishes on those points (projective coordinates (τ_1, τ_0)):

$$Y = \frac{X^3 - 1}{X} \quad \leftrightarrow \quad \delta(\tau_1, \tau_0) = 3\tau_0(\tau_1^3 - \tau_0^3).$$

• Use invariant theory to find 3 more homogeneous polynomials, and a relation between them:

$$c_4 = Hessian(\tau_1, \tau_0), c_6 = Jacobian(\delta, c_4), \Delta = Disc(\delta).$$

• The relation between them (syzygy):

$$c_4^3 - c_6^2 = 1728\Delta.$$

A few remarks

• Klein's approach essentially uses dessins d'enfant and yields a polynomial which looks like the inverse of the j-invariant for an EC:

$$\beta(z) = (c_4^3 - c_6^2)/c_4^3 \quad < - > \quad j(\tau) = 1/\beta(\tau).$$

It gives a way to relate Belyi pairs $\beta : EC \to P^1(C)$ and theory of elliptic curves (see [3]).

• The relation between discriminant, hessian and "cov" [4]:

a) Discriminant $\delta = \prod_{i<j}(r_i - r_j)^2$ can be written in terms of the coefficients (Viete's relation and Fundamental Theorem of Symmetric Functions), here:

$$Disc = -4A^3 - 3B^2.$$

b) The Hessian matrix is $Jacobian(grad(f))$ and $det(Hess(f)) = \Delta(f)$.

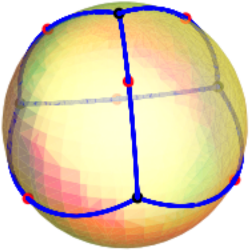
c) The covariant "cov" above is a Jacobian of $(f, Hess(f)) : C^2 \to C^2$.

• Problem: what is the syzygy of the three polynomials and why is it related with the j-invariant?

Pictures

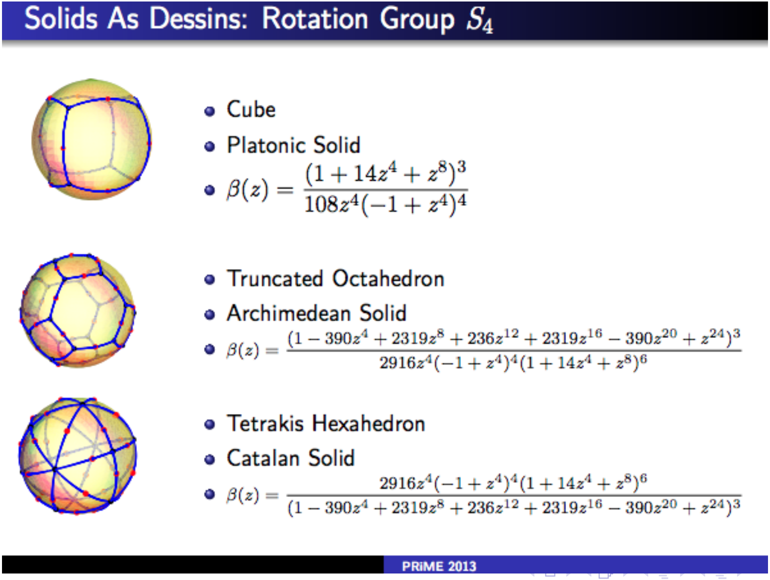
Exercise: check that the Belyi map for the tetrahedron is [4]:

$$\beta(z) = \frac{c_4(\tau_1, \tau_0)^3 - c_6(\tau_1, \tau_0)^2}{c_4(\tau_1, \tau_0)^3} = \frac{64(z^3 - 1)^3}{z^3(z^3 + 8)^3} \quad \text{where} \quad z = \frac{\tau_1}{\tau_0}$$



Exercise

- Compute the Belyi maps with symmetry group $O = S^4$ [4]:



Comparing with Lepton Masses

Pros

• The normalization coefficients include 108. Masses of course are not absolute, and both 108 and 1728 should be compared with mass ratios of leptons ($m_e \approx 1?$).

• Plotting both the coefficients $1, 108, 1728$ and the experimental data $0.511, 103.5, 1771$ gives a reasonable match (electron mass is small - exceptional?).

• Fun fact: $1729 = 1 + 1728$ is Ramanujan's taxi number [3], with many interesting properties (see Wiki: sum of cubes, Carmichael number, Loeschian norm of four 1st quadrant Eisenstein integers etc.)

... and **Cons**

• But why 1 is not there, $1/64$ occurs for the tetrahedron and 2916 for truncated octahedron? Also dual geometries, which in this author's opinion correspond to the same weak isospin (u/d-type per generation), have inverse rational functions, hence coefficients!?

[3] Thank you Sunil!

Modular Curves, Belyi Ramified Covers and Galois Group

With modular curves $X_0^+(p) = \Gamma_0(p)/\mathcal{H}^*$ part of Belyi pairs [2], e.g. sphere, elliptic curve (with additional structure) algebraic theory meets Topology; a few aspects need be better understood in Physics Applications (SM) ...

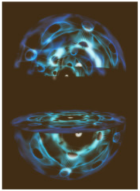
• A Belyi map / function is a ramified cover, with a group of covering transformations called *deck transformations*; when it is Galois, corresponding to the algebraic extension ("function fields") then the modular curve is called *quasi-Platonic* (regular dessin d'enfent), with a maximum number of automorphisms $84(g-1)$ [2], p.17.

• Question: is the Galois group the automorphisms group of the corresponding dessin d'enfant? (e.g. Platonic / Archemidian / Johnson polyhedron)

• How does this relate with tesselations of Riemann surfaces?

Graduate Math-Physics Project

More regular solids: There are many more such 3D-cymatics modes: Archimedian, Jonson solids etc. “upgrade” of String Theory:


Mass and Energy Levels: Think of these covering maps as *transitions between baryon states*, in analogy with electron transitions between orbitals: s,p,d, f ... (2D-cymatics / drums: see Wiki: atomic orbitals).

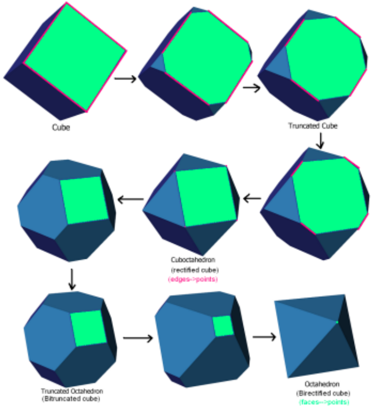
- Compute the Galois groups
- Compare with masses of baryons

Supplement:

Search for a relation between mesons (transition bonds between baryons) and morphisms of Belyi maps / functions, and their Galois groups.

Operations

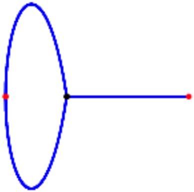
13 Archimedean solids and 13 Catalan solids can be obtained from Platonic solids using seven geometric operations; example:



These operations can be algebraically recognized as Belyi maps.

Algebraic-Geometric Morphisms

Associate hypermaps and Belyi maps to geometric operations [4, 2]:



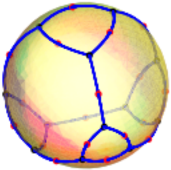
1. Hypermap of Truncation
2. Corresponding Belyĭ map

$$\phi_{\text{truncation}}(w) = \frac{(4\,w-1)^3}{27\,w}$$

3. Truncated Tetrahedron Belyĭ map

$$\beta = \phi_{\text{truncation}} \circ \beta_{\text{tetrahedron}}$$

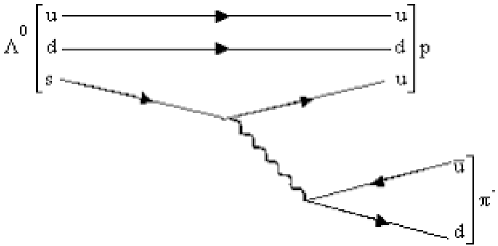
$$\beta(z) = \frac{(1-232z^3+960z^6-256z^9+256z^{12})^3}{1728z^3(z^3-1)^3(8z^3+1)^6}$$



Problems and Applications

- Math: Study the corresponding Category of Belyi pairs and morphisms.

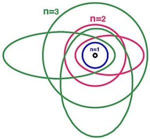
- Physics: Model Weak Force decays in this way.
  Example of weak force decay $\Lambda^0 \to p + \pi^-$, using Quark Line Diagrams:

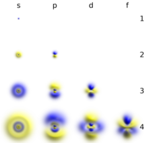


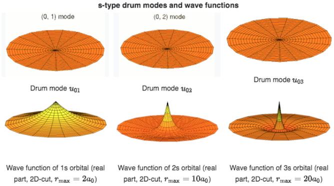
Lambda $\Lambda(uds)$ and proton $p = (udd)$ are baryons (3-points Riemann sphere) and π^- is a meson (quark-antiquark nuclear "bond").

Theory of Electron Orbitals vs. Theory of Nucleon States

The theory of baryons, from the **generic** $SU(3)$ symmetry to **specific finite groups** (Platonic, Galois etc.; 3D-Modes/cymatics), is similar to the theory of the Electronic Orbitals: after Bohr-Sommerfeld Model and BEFORE the Schrodinger Equation (2D-drum modes):







- **Rhombification**: Cuboctahedron → Rhombicuboctahedron

- **Snubification**: Cube → Snub Cube

Conclusions

(The END? No! just another beginning ...)

Physics related

The next level of refining the SM involves new mathematical tools: Belyi pairs, finite geometries etc.

• Modes of vibration of baryons ("discrete spheres") are modeled by Belyi pairs, similar with Bohr orbits corresponding to a discrete subgroup Z/n and the shape of an orbital to spin representation $SU(2)$ (EM-Connection);
• Weak decays are transitions between such geometries, and can be modeled as morphisms of Belyi functions: change of geometry corresponding to a change of group etc.
• Strong interactions correspond to pure "vibrations" (Klein geometries and reps).
• Macroscopic time emerges from quantum phase and magnetic field is holonomy of EM connection ... The measure of curvature (holonomy) is rest mass ($P = p + e/cA$ for the 3 quark fields).

Math related

• Study the Category of Belyi morphisms;
• Study how the absolute Galois group acts on it; this is the real "match" for the Beauty (soulmate), when the "spell" on the Beast fades away ...

How this relates with the Physics

• A Galois covering is a flat connection, with monodromy at ramification points; in Physics terms, the associated EM-connection has a quantized magnetic field $B = \nabla \times A$, with A the vector potential of the connection $d^A = d + A$ etc.
• Relate with masses and lifetime of particles; (see MacGregor [3]);
• Relate with String Theory (see [4]).

So, What is Mass?

• Mass is not an absolute invariant, like electric charge (or magnetic charge) which are quantized.
Mass is a symplectic conjugate to electric charge as generator of $U(1)$ (and conserved: Noether), corresponding to the RGB space generators of $SU(2)$, the quark fields of EM type A_R etc: $P = p_E + p_i = mv + \frac{e}{c}A$ ...

• The computation of mass for these modes and geometries of baryons needs a better understanding quark fields and Finite Geometries:
• Klein geometry: $H \to G$ acting on a space;
• Cartan Geometry: theory of parallel moving frames (vierbines), generalizing Levi-Civita metric connection (Einstein's way), with local model Klein Geometries $U(1) \to SU(2)$.
...
Mathematics is a designing language, when used in applications ...

Some Mathematics Problem for VOAs savvy

All these are related: modular curves of genus 0, Leech lattice, sporadic groups and Standard Model mass problem ...

Problems:
1) Understand and generalize:
"The sporadic group $Co3$ Hauptmodul and Belyĭ map"
https://www.arxiv-vanity.com/papers/1802.06923/
2) Is there a Tannaka-Krein duality here? A Tannakian category of Riemann cobordisms / Belyi morhisms represented by VOAs via M?

A Math Problem for Algebraic-Geometry Physicists

• String Theory embeds Riemann Surfaces in ST X Calabi-Yau; rephrase in terms of principal bundles (sheaves), with CY 6-dim as Cartan Geometry of connections (Hodge and Witten-Seiberg Duality etc.)

• Study how the discrete, quantized version: tessellations or Riemann surfaces and dessins d'Enfant, Belyi morphisms, Galois groups (connections) relate with the "known" String Theory, and get rid of the "Landscape" for the unique Cartan connection corresponding to the unique left and right bi-invariant connections for $SU(3)$, associated to $U(1) \to SU(2)$ (Hopf fibration / Cartan homogeneous space; $U(1) \times SU(2)$ from Electroweak Theory is the split version, after beta decay $n + \nu \to p + e$, and polarization of internal space as a Space-Time via a G-interaction).

Invitation

Standard Model & Absolute Galois Group
are getting married!

Invited: Galois, Riemann, Grothendieck, Belyi, Potter, Borcherds, Witten, Kontsevich and anybody wishing to represent them at this memorable event!

The End

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