Terms constructed by the "diagonal method" are unclosed terms

----Tranclosed logic princiole and its inference(4)

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Abstract: Nature of a proposition constructed by diagonal method of proof is a paradox, so it is an unclosed term and an extra-field proposition. There are two kinds of infinities, standard infinity and non-standard infinity, and we will explore the diagonal problem in each of the two kinds of infinities below.We conclude that: (1) In standard infinity, Cantor's diagonal number can metamorphose into real number and the contradiction vanishes. (2) In nonstandard infinity, Cantor's diagonal numbers become hyperreal number . Essentially both are unclosed terms of the calculation.

Therefore, Cantor's diagonal method proves that "the real numbers are not countable" is wrong.

Key words: Paradox, unclosed term (extra-field term), diagonal method of proof, uncountability, standard infinity, nonstandard infinity, hyperreal number

1 Cantor's diagonal method

As an important reduction to absurdity, the diagonal method, which was created by G. F. P. Cantor in 1874, was first used to prove that the set of real numbers is uncountable.

Note 1.1 Proof of that the set of real numbers is uncountable

To prove that R is uncountable, as long as $[0,\ 1]$ is proved uncountable, now reduction to absurdity is applied.

Let $[0,\ 1]$ be countable, then assume $[0,\ 1]=\left\{ x_1, x_2, x_3, \ldots, x_n, \ldots \right\}$. Since $x_i \in [0,\ 1]$, it can be represented by an infinite decimal fraction and these numbers are list in sequence:

$$x_1 = 0.a_{11}a_{12}a_{13}\ldots a_{1n}\ldots$$

$$x_2 = 0.a_{21}a_{22}a_{23}\ldots a_{2n}\ldots$$

$$x_3 = 0.a_{31}a_{32}a_{33}\ldots a_{3n}\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_n = 0.a_{n1}a_{n2}a_{n3}\ldots a_{nn}\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now define a number $T = 0.\overline{a_{11}}\,\overline{a_{22}}\,\overline{a_{33}}\ldots\overline{a_{nn}}\ldots$, thereinto,

$$\overline{a_{ii}} = \begin{cases} a_{ii}+1, & a_{ii}<9 \\ 0, & a_{ii}=9 \end{cases}$$,(namely $\overline{a_{ii}}$ is different from a_{ii})

It is obvious that $T \in [0,\ 1]$, but $T \neq x_i, i = 1,2,3,\cdots,n,\cdots$,which is contradictory with that $[0,\ 1]$ is countable, so that $[0,\ 1]$ is uncountable can be proved, and then R is uncountable.

With the diagonal method, Cantor (G. F. P. Cantor) also proves that "natural numbers do not have one-to-one correspondence with its power set", that is $|N|<|\wp(N)|$.

Note 1.2 Proof of that power set of natural numbers is uncountable

If $\wp(N)$ is the power set of N and one-to-one correspondence, namely bijective relation can be constructed between $N \sim \wp(N)$ and (N and $\wp(N)$, and then there is T , $T \subseteq N$, and $T \notin \wp(N)$.

For a N subset, if it can be determined that elements of N belong to the subset, then the subset of N is determined. We give a general criterion for determining whether any element x of N belongs to the subset.

Suppose that in $N \sim \wp(N)$, any $x \in N$ corresponds to $y \in \wp(N)$, namely $x \in N \leftrightarrow y \in \wp(N)$, y is one of subsets of N , $y \subseteq N$,

$x \in y$, or $x \notin y$; we provide that: $x \in y \leftrightarrow x \notin T$; $x \notin y \leftrightarrow x \in T$;

Assume that $T \in \wp(N), N \sim \wp(N)$, there is $x_0 \in N \leftrightarrow T \in \wp(N)$;

By rule: $x_0 \in T \leftrightarrow x_0 \notin T$; $x_0 \notin T \leftrightarrow x_0 \in T$;

It is contradictory, so $T \notin \wp(N)$.

Note: These proofs can be found in many books of "Mathematical Logic," such as "Introduction to Meta-Mathematics," by S. C. Klein.

2 Items constructed by the "diagonal method" are unclosed terms

Review inference rule of system S, classical predicate logic system K shall be modified to system ***SK*** and the axiom of the system is preceded by the “ $x \in U$ ” and all calculus of the system is carried out within U . With the inference rule of *SK* system, we can prove the following diagonal theorem further.

Theorem 2.1 Generalized Diagonal Theorem

Let the universal set $U=\{x_1,x_2,\cdots,x_i,\cdots\}$ be a defined set, P is a property defined on U and the proposition P is a partition related to $+\alpha$ and $-\alpha$, namely $+\alpha=\{x\,|\,P(x)\}$, $-\alpha=\{x\,|\,\neg P(x)\}$, construction term T satisfies the following relation $x\in+\alpha\leftrightarrow\neg P(T)$, $x\in-\alpha\leftrightarrow P(T)$.

（1）If $T\in U$, then $P(T)\leftrightarrow\neg P(T)$ .That is: $T\in U\vdash P(T)\leftrightarrow\neg P(T)$.

（2）If the calculation on U is consistent, then $T\notin U$ ， T is an extra-field term.

Proof:

(1) $\vdash x\in+\alpha\leftrightarrow P(x)$ -------------------------------definition of positive set,

(2) $\vdash x\in+\alpha\leftrightarrow\neg P(T)$ ------------------------------ definition of constructive term T

(3) $x\in+\alpha\vdash P(x)\leftrightarrow\neg P(T)$ --------------------------(1)(2)

(4) $\vdash x\in-\alpha\leftrightarrow\neg P(x)$ ----------------------------definition of inverse set,

(5) $x\in-\alpha\leftrightarrow P(T)$ ----------------------------definition of constructive term T ,

(6) $x\in-\alpha\vdash\neg P(x)\leftrightarrow P(T)$ ---------------------------- (4), (5)

(7) $x\in U\vdash P(x)\leftrightarrow\neg P(T)$ -------------------------(3), (6) $U=+\alpha\bigcup-\alpha$,

(8) $T\in U\vdash P(T)\leftrightarrow\neg P(T)$ ----------------------------（7）substitute $x=T$,

（9） $\vdash T\notin U$ --（8）.

The diagonal theorem can also be simply expressed as follows:

Let $U=\{x_1,x_2,\cdots,x_i,\cdots\}$ be a defined set, P is a property defined on U , construct a new term T **different from any term in** U **, that is** $x\in U\vdash P(x)\leftrightarrow\neg P(T)$, then

(1) If $T\in U$, then $P(T)\leftrightarrow\neg P(T)$, namely: $T\in U\vdash P(T)\leftrightarrow\neg P(T)$,

(2)If the calculus on U **is consistent, then** $T\notin U$, T is an extra-field term.

(Namely, if $T\in U$, then calculus on U is inconsistent, if the calculus on U is consistent, then $T\notin U$, T is the unclosed term of the calculus).

Example 2.1 Binary predicate representation of "Diagonal method of proof”

The “ diagonal method of proof ”can also be expressed as a binary predicate as follows:

Let $[0,1]=\{x_1,x_2,\cdots,x_i,\cdots\}$,

$$x_1=0.a_{11}a_{12}a_{13}\ldots a_{1n}\ldots$$

$$x_2=0.a_{21}a_{22}a_{23}\ldots a_{2n}\ldots$$

$$x_3=0.a_{31}a_{32}a_{33}\ldots a_{3n}\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_n=0.a_{n1}a_{n2}a_{n3}\ldots a_{nn}\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The nth term of the i th element x_i in the table is expressed as $P(x_i,n)$ by binary predicate, and above table can be arranged as follows:

$$\begin{aligned}
&x_1=0.P(x_1,1)P(x_1,2)P(x_1,3)\cdots P(x_1,n)\cdots\\
&x_2=0.P(x_2,1)P(x_2,2)P(x_2,3)\cdots P(x_2,n)\cdots\\
&x_3=0.P(x_3,1)P(x_3,2)P(x_3,3)\cdots P(x_3,n)\cdots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&x_n=0.P(x_n,1)P(x_n,2)P(x_n,3)\cdots P(x_n,n)\cdots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Now define a number $T=0.\overline{P(x_1,1)}\,\overline{P(x_2,2)}\,\overline{P(x_3,3)}\ldots\overline{P(x_n,n)}\ldots$, thereinto

$$\overline{P(x_i,i)}=\begin{cases}P(x_i,i)+1, & P(x_i,i)<9\\ 0, & P(x_i,i)=9\end{cases}$$, (that is $\overline{P(x_i,i)}$ is different from $P(x_i,i)$)

New number T, which satisfied the nth term of T is different from the nth term of x_n, that is

$$P(T,n)\leftrightarrow\neg P(x_n,n),$$

Is $T\in[0,1]$? Assume that $T\in[0,1]$ is true, and there is some x_n, meeting $x_n=T$, namely:

$$\vdash P(T,n)\leftrightarrow\neg P(T,n).$$

Inference 2.1 Cantor's diagonal number T is an extra-field term

(1) If T is Cantor's diagonal number at interval $[0,\ 1]$, then

$$T \in [0,1] \vdash P(T,n) \leftrightarrow \neg P(T,n);$$

(2)If the calculus on set of real numbers R is consistent, then $T \notin [0,\ 1], T \notin R$.

(It can be further proved that: Cantor's diagonal number T is a hyperreal number)

Example 2.2 Binary predicate representation of "Diagonal Proof Method"

Denote $x \in y$ with binary predicate $P(x, y)$, construct a set T satisfying:

$$x \in y \leftrightarrow x \notin T;\ x \notin y \leftrightarrow x \in T;$$

That is

$$P(x, y) \leftrightarrow \neg P(x,T); \neg P(x, y) \leftrightarrow P(x,T);$$

Assume that $T \in \wp(N), N \sim \wp(N)$, there exists $x_0 \in N \leftrightarrow T \in \wp(N)$;

There exists $x_0 \in y$, $y = T$, substitute x_0, T in above formula to get :

$$P(x_0,T) \leftrightarrow \neg P(x_0,T);$$

It is contradictory, so $T \notin \wp(N)$.

Inference 2.2 Diagonal set on set of natural numbers' power set is an extra-field term

(1) If T is the diagonal set on N, set of natural numbers' power set $\wp(N)$, then

$$T \in \wp(N) \vdash P(x_0,T) \leftrightarrow \neg P(x_0,T);$$

(2) If the calculus on N, set of real numbers is consistent, then $T \notin \wp(N), T \notin N$.

(Diagonal set T on $\wp(N)$, power set of natural numbers' set N, is an unclosed term in calculation, a undefined set)

In the proof process of diagonal theorem, we add a bijective relation $F: N \sim U$ between U and set of real numbers, N

$$F: N \sim U,\quad x \in U \ \vdash P(x) \leftrightarrow \neg P(T);$$

If $T \notin U$, the above formula is not contradictory;

If $T \in U$, there is $x = T$, if put $x = T$ into the above formula, then the contradiction is derived.

$$F: N \sim U,\ T \in U \vdash P(T) \leftrightarrow \neg P(T);$$

In above formula, the contradiction is generated by adding $T \in U$. The contradiction disappear without $T \in U$, $x \neq T$,

It is indicated that the contradiction is caused by $T \in U$ and is unrelated with the bijective relation $F: N \sim U$.

Namely the existence of fixed term contradiction is unrelated with the bijective relation $F: N \sim U$.

3 Review Cantor's diagonal numbers from the infinite perspective

The set of m-bit decimals of the interval $[0,1]$ is written as $[0,1]_m$, $x = 0.a_1a_2a_3\cdots a_m$, and since each bit can only be 0-9 of these ten numbers （$a_m = 0,1,2,\cdots,9$）, the complete arrangement of 10 numbers has a total of 10^m, that is, all m-bit decimals of the interval $[0,1]_m$ has a total of 10^m, so that the number of bits and the number is not equal, we can use the method of complementary 0 after the 10^m bits,

$$x = 0.a_1a_2a_3\cdots a_m00\cdots0_{10^m}$$

make the number of bits and the number of decimals are equal, and let $10^m = n$.

When $n \to \infty$, the complete arrangement of 10 numbers （$a_m = 0,1,2,\cdots,9$）, is all infinite decimal places of the interval $[0,1]$, and then construct Cantor's diagonal number.

Theorem 3.1 Cantor's diagonal number on a finite bit interval is an unclosed term

Let $T = 0.\overline{a_{11}}\,\overline{a_{22}}\,\overline{a_{33}}\cdots\overline{a_{nn}}$ be Cantor's diagonal number of all n-bit (n is a finite number) real numbers on the interval $[0,1]_n$, then,

$$T \notin [0,1]_n.$$

Proof: Let

$$x_1 = 0.a_{11}a_{12}a_{13}\cdots a_{1n}$$

$$x_2 = 0.a_{21}a_{22}a_{23}\cdots a_{2n}$$

$$x_3 = 0.a_{31}a_{32}a_{33}\cdots a_{3n}$$

$$\cdots\cdots$$

$$x_n = 0.a_{n1}a_{n2}a_{n3}\cdots a_{nn}$$

be a list of all real numbers in the interval $[0,1]_n$,

Define diagonal number $T=0.\overline{a_{11}}\,\overline{a_{22}}\,\overline{a_{33}}\cdots\overline{a_{nn}}$, ($\overline{a_{ii}}$ and a_{ii} are different numbers).

If $a_{ii}\neq 9$, then, definite $\overline{a_{ii}}=a_{ii}+1$;

If $a_{ii}=9$, then, definite $\overline{a_{ii}}=a_{ii}-1$ 。

$T\in[0,1]_n \Rightarrow \overline{a_{ii}}\neq a_{ii}$,（ $i=1,2,3,\cdots,n$ ）

That is, $T\notin[0,1]_n$.

That is, Cantor's diagonal number on a finite bit interval is an unclosed term.

Please consider the question: if the number of diagonals is arranged at infinity, $T\in[0,1]_n \Rightarrow \overline{a_{ii}}\neq a_{ii}$, does this contradiction still exist?

When $n\to\infty$,

$$x_1=0.a_{11}a_{12}a_{13}\ldots a_{1n}\ldots$$

$$x_2=0.a_{21}a_{22}a_{23}\ldots a_{2n}\ldots$$

…………………………

$$x_\infty=0.a_{\infty1}a_{\infty2}a_{\infty3}\ldots a_{\infty\infty}\ldots$$

$$T=0.\overline{a_{11}}\,\overline{a_{22}}\,\overline{a_{33}}\ldots\overline{a_{\infty\infty}}\ldots$$

$$T=\overline{a_{11}}\cdot10^{-1}+\overline{a_{22}}\cdot10^{-2}+\cdots+\overline{a_{nn}}\cdot10^{-n}+\cdots+\overline{a_{\infty\infty}}\cdot10^{-\infty}\text{ (Decimal expansion).}$$

$$T\in[0,\ 1]\Rightarrow\exists\infty(\overline{a_{\infty\infty}}\neq a_{\infty\infty}).$$

When $n\to\infty$, we have already explored in *Infinity and Infinite Induction* that there are two kinds of infinities, standard infinity and non-standard infinity, and we will explore the diagonal problem in each of the two kinds of infinities below.

Theorem 3.2 Cantor's diagonal number on infinite bit interval

Let $T=0.\overline{a_{11}}\,\overline{a_{22}}\,\overline{a_{33}}\cdots\overline{a_{\infty\infty}}$ be the Cantor's diagonal number of all ∞ bit real numbers on the interval $[0,1]$, then

(1) In standard infinity, the Cantor's diagonal numbers can be metamorphosed into real numbers and the contradiction vanishes, $T\in[0,1]$.

(2) In non-standard infinity, the contradiction still exists.

$T \in [0,1] \Rightarrow a_{ii} \neq a_{ii}$, $T \notin [0,1]$.The diagonal number is an unclosed term.

Proof: The finite representation of the Cantor's diagonal number is

$$T = \overline{a_{11}} \cdot 10^{-1} + \overline{a_{22}} \cdot 10^{-2} + \cdots + \overline{a_{nn}} \cdot 10^{-n} .$$

Cantor's diagonal number, when n is finite, $T \in [0,\ 1] \Rightarrow \exists n(a_{nn} \neq a_{nn})$. This is a contradiction.

When $n \to \infty$, according to the method of infinite induction, the identical equation holds for infinity (see *Infinity and Infinite Induction*),

(1) In standard infinity, $\infty + 1 = \infty$, $\frac{1}{\infty} = 0$, $10^{-\infty} = 0 \Rightarrow \overline{a_{\infty\infty}} \cdot 10^{-\infty} = 0$, the diagonal number

$$T = 0.\overline{a_{11}}\,\overline{a_{22}}\,\overline{a_{33}} \ldots \overline{a_{nn}} \ldots \overline{a_{\infty\infty}} \ldots ..$$

all become 0 at infinity,

$$T = 0.\overline{a_{11}}\,\overline{a_{22}}\,\overline{a_{33}} \ldots \overline{a_{nn}} \ldots 000 \ldots .$$

Since T is a real number, $T = 0.\overline{a_{11}}\,\overline{a_{22}}\,\overline{a_{33}} \ldots \overline{a_{nn}} \ldots$ we can only see the number $\overline{a_{11}}\,\overline{a_{22}}\,\overline{a_{33}} \ldots \overline{a_{nn}}$ at infinity, infinite bit $\overline{a_{\infty\infty}}$ of T can not be seen, or what can see only is the number 0, at this time the extra-domain term T metamorphoses into tthe real number of the domain term.

$T \in [0,\ 1] \Rightarrow \exists \infty(a_{\infty\infty} \neq a_{\infty\infty})$,This contradiction has actually disappeared because $\overline{a_{\infty\infty}}$ is not visible.

Cantor's diagonal number is contradicted only at finite, when T is arranged at infinity, this does not contradict.

(2) In non-standard infinity, $\infty + 1 \neq \infty$, $\frac{1}{\infty} \neq 0$ (has been rewritten as ϖ), $10^{-\infty} \neq 0 \Rightarrow \overline{a_{\infty\infty}} \cdot 10^{-\infty} \neq 0$, the diagonal number

$$T = 0.\overline{a_{11}}\,\overline{a_{22}}\,\overline{a_{33}} \ldots \overline{a_{nn}} \ldots \overline{a_{\infty\infty}} \ldots .$$

$$T \in [0,1] \Rightarrow a_{ii} \neq a_{ii} , \quad \Rightarrow T \notin [0,1] ;$$

The diagonal number is the unclosed term of the calculus, that is, the extra-field term.

(In non-standard infinity, $10^{-\infty} \neq 0$ has been rewritten as $10^{-\varpi} \neq 0$, $\overline{a_{\infty\infty}}$ has been rewritten as $\overline{a_{\varpi\varpi}}$, see *Infinity and Infinite Induction*)

$\overline{a_{\infty\infty}} \cdot 10^{-\infty} \neq 0$, the number at infinity is not 0, T is no longer a real number, and the hyperreal numbers have this property.

At this point Cantor's diagonal number T is a hyperreal number and T is arranged outside the domain, which is also not contradictory.

We conclude that: (1) In standard infinity, Cantor's diagonal number can metamorphose into real number and the contradiction vanishes. (2) In nonstandard infinity, Cantor's diagonal numbers become hyperreal number . Essentially both are unclosed terms of the calculation.

Therefore, Cantor's diagonal method proves that "the real numbers are not countable" is wrong.

Appendix References

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