An Inconsistent Hierarchy of Sets in [0, 1]

By Jim Rock

Abstract: Two contradictory arguments are developed from a hierarchy of sets in [*0, 1*]. One argument is a proof by contradiction and its conclusion is true. The other argument is an existence argument and while its conclusion is not true, it follows logically from the a valid assumption followed by three true statements that precede the conclusion.

Introduction. For all rational numbers $\boldsymbol{a}$ in the closed interval [*0, 1*] and {0}
define the collection of all $R_{\boldsymbol{a}}$ sets equal { y is a rational number | $0 \leq y < a$ } and {0}

The following four true statements characterize the collection of $R_{\boldsymbol{a}}$ sets and {0}.

a) The collection forms a hierarchy of sets with R_{1} at the top and {0} on the bottom.

b) Each $R_{\boldsymbol{a}}$ set contains all the elements in sets below it in the set hierarchy.

c) Each set is a proper subset of all the $R_{\boldsymbol{a}}$ sets above it in the set hierarchy.

d) *Used in Arg #1 step 4*. Each individual $R_{\boldsymbol{a}}$ set contains at least one element that is not in any of the sets below it in the hierarchy. Otherwise, the entire hierarchy would collapse.

Argument #1: R_{1} **contains a largest element.**
1) Let $\boldsymbol{c}$ and $\boldsymbol{d}$ be two elements of R_{1} with $\boldsymbol{c > d}$.
2) $\boldsymbol{d}$ is an element of R_{c}, which is a proper subset of R_{1}.
3) For any two elements in R_{1} the smaller element is contained in a proper subset of R_{1}.
4) **d)** R_{1} contains a largest element not contained in any set below it in the set hierarchy.

Argument #2: R_{1} **contains no largest element.**
1) Suppose there is a largest element a in R_{1}.
2) $a < (a+1)/2 < 1$.
3) Let $b = (a+1)/2$.
4) Then b is in R_{1} and $a < b$.

When a largest element is assumed in Argument #2 it leads to a contradiction so there is no largest element in R_{1}. A valid proof by contradiction.

The difference between the two arguments is no attempt is made to specify a largest element in argument #1. It is an existence argument only.

But in argument #1, step 1 is a valid assumption and statements 2, 3, and **d)** from the **Introduction** are true statements. Step 4 follows logically from steps 1, 2, 3, and **d).**

Thus, we have two contradictory arguments that can be developed in any formal system containing sets, arithmetic, and relations between the rational numbers.

Objection: For any $R_{\boldsymbol{a}}$ it is claimed that every element of $R_{\boldsymbol{a}}$ is in some proper subset below $R_{\boldsymbol{a}}$ in the hierarchy. Statement **d)** *Used in Arg #1 step 4* is claimed to be false. In **Argument #2** a is in $R_{\boldsymbol{b}}$ a proper subset below R_{1}. But, $R_{\boldsymbol{b}}$ is missing $b' = (b+1)/2$. So it is with every other subset $R_{\boldsymbol{x}}$ of R_{1}. They are all missing $x' = (x+1)/2$. Since the collection of $R_{\boldsymbol{a}}$ sets is in a nested hierarchy, there is one largest set element missing from all subsets of R_{1}. The same thing is true for each $R_{\boldsymbol{a}}$ set in the nested hierarchy. Each set has a largest element. Otherwise, the entire hierarchy would collapse.

If you wish, email comments to Jim Rock at **collatz3106@gmail.com**.