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Some Notes on Fermion Masses in the Tetron Model

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Abstract

Our universe according to the tetron model is a 3-dimensional elastic substrate expanding within some higher dimensional space. The elastic substrate is built from tiny invisible constituents, called tetrons, with bond length about the Planck length and binding energy the Planck energy[1]. Details of the approach provide a powerful unified picture for particle physics and cosmology. All physical properties in the universe can be derived from properties of the tetrons. This philosophy is applied here to the Standard Model mass and mixing parameters which are shown to be determined by the interactions among tetrons. The most general ansatz for these interactions leads to a Hamiltonian involving Dzyaloshinskii-Moriya (DM), Heisenberg and torsional isospin forces. While the masses of the third and second family arise from DM and Heisenberg type of isospin interactions, light family masses are related to torsional interactions among tetrons. Neutrino masses turn out to be special in that they are given in terms of tiny isospin non-conserving DM, Heisenberg and torsional couplings. Moving on to the CKM and PMNS mixing, some preliminary results are presented. They allow to trace the observed hierarchy in the CKM matrix to the dominance of the top mass and to attribute the large non-diagonal PMNS matrix elements to the approximate conservation of isospin. One important finding is the influence that the τ-lepton mass has on the CKM parameters.

I. Introduction

As suggested in [1] our universe consists of tiny constituents, called tetrons, which transform as the fundamental spinor representation of SO(6,1). This representation is 8-dimensional and sometimes called the octonion representation[2, 3].

The 24 known quarks and leptons arise as eigenmode excitations of a tetrahedral fiber structure, which is made up from 4 tetrons and extends into 3 extra 'internal' dimensions. While the laws of gravity are due to the elastic properties of the tetron bonds[5], particle physics interactions take place within the internal fibers, with the characteristic internal energy being the Fermi scale. All ordinary matter quarks and leptons are constructed as quasiparticle excitations of this internal fiber structure. Since the quasiparticles fulfill Lorentz covariant wave equations, they perceive the universe as a 3+1 dimensional spacetime continuum.

More in detail, the ground state of our universe looks like illustrated in Fig. 1. In this figure the tetrahedrons (='fibers') extend into the 3 extra dimensions. The picture is a little misleading because in the tetron model physical space and the extra ('internal') dimensions are assumed to be completely orthogonal. This means the whole game has to be played within a larger, at least 6 dimensional space, 3 physical dimensions and 3 internal ones. There are some indications that the system actually lives in 7+1 dimensions instead of 6+1; this however does not play a role in the calculations presented on the following pages.

Each tetrahedron in Fig. 1 is made up from 4 tetrons, depicted as dots. With respect to the decomposition of $SO(6,1) \rightarrow SO(3,1) \times SO(3)$ into the (3+1)-dimensional base space and the 3-dimensional internal space, a tetron Ψ possesses spin $\frac{1}{2}$ and isospin $\frac{1}{2}$. This means it can rotate both in physical space and in the extra dimensions, and corresponds to the fact that Ψ decomposes into an isospin doublet $\Psi = (U, D)$ of two ordinary SO(3,1) Dirac fields U and D.

$$8 \rightarrow (1,2,2) + (2,1,2) = ((1,2) + (2,1), 2) \tag{1}$$

Why the tetrahedral structure? This is needed in order to explain the observed quark and lepton spectrum, which means to get exactly 24 excitation states with

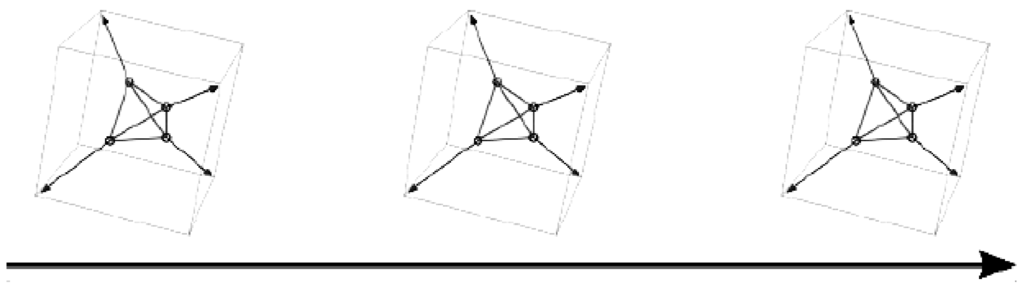

Figure 1: The global ground state of the universe after the electroweak symmetry breaking has occurred, considered at Planck scale distances. Before the symmetry breaking the isospin vectors are directed randomly, thus exhibiting a local SU(2) symmetry, but once the temperature drops below the Fermi scale Λ_F, they become ordered into a repetitive tetrahedral structure, thereby spontaneously breaking the initial SU(2).

the correct multiplet structure[1]. In fact, the tetrahedral symmetry is rather uniquely determined by this condition[21]. As shown below, under reasonable assumptions on the tetron dynamics, the numerical mass values of quarks and leptons can be correctly reproduced.

The arrows in Fig. 1 denote the isospins, i.e. internal spin vectors of the tetrons. More precisely, each arrow stands for two(!) vectors $\langle \vec{Q}_L \rangle = \langle \vec{Q}_R \rangle$ where[8]

$$\vec{Q}_L = \frac{1}{4}\Psi^\dagger(1-\gamma_5)\vec{\tau}\Psi \qquad\qquad \vec{Q}_R = \frac{1}{4}\Psi^\dagger(1+\gamma_5)\vec{\tau}\Psi \tag{2}$$

and $\langle\rangle$ denotes the ground state/vacuum expectation values. In other words, the ground state values $\langle \vec{Q}_{Li} \rangle$ and $\langle \vec{Q}_{Ri} \rangle$ are assumed to be equal on each tetrahedral site i and given by one of the arrows in Fig. 1 (after the SSB). $\vec{\tau}$ are the internal spin Pauli matrices.

According to (1) the tetron representation 8 contains both particle and antiparticle degrees of freedom. $\vec{Q}_L$ and $\vec{Q}_R$ cover 6 of its 8 dof[2]. Furthermore, $\vec{Q}_L$ and $\vec{Q}_R$

[1]The quark triplets are triplets under tetrahedral transformations at this point. For the question how to interpret them as color triplets one may consult [6].

[2]The remaining 2 dof correspond to the 'densities' $\Psi^\dagger\Psi$ and $\Psi^\dagger\gamma_5\Psi$ whose fluctuations actually are dark matter candidates[1].

are particularly useful to handle because quantum mechanically they commute with each other[8]. As turns out, the interactions of these internal spins play an essential role for particle physics and for electroweak symmetry breaking.

Due to the pseudovector property of the isospin vectors their tetrahedral symmetry group actually is a Shubnikov group [9, 10]. This means, while the coordinate symmetry is S_4, the arrangement of isospin vectors respects the tetrahedral Shubnikov symmetry

$$G_4 := A_4 + CPT(S_4 - A_4) \tag{3}$$

where $A_4(S_4)$ is the (full) tetrahedral symmetry group and CPT the usual CPT operation except that P is the parity transformation in physical space only. Since the elements of $S_4 - A_4$ contain an implicit factor of internal parity, the symmetry (3) certifies CPT invariance of the local ground state in the full of R^{6+1}.

Note that in the situation depicted in Fig. 1 the SU(2) symmetry breaking has already occurred, because the isospins are aligned between all the tetrahedrons. Before the symmetry breaking, which means above a certain temperature, isospins are distributed randomly, corresponding to a local $SU(2) \times U_1$ symmetry[3], but when the universe cools down, there is a phase transition, and the isospins freeze into the aligned structure, breaking the $SU(2)$ symmetry to the discrete 'family group' G_4. And the important point to note is, this temperature can be identified with the Fermi scale[21]. Moreover, the remaining symmetry $G_4 \times U(1)_{em}$ is valid down to the lowest energies.

As elaborated in the following sections, the mathematical treatment of the excitations arising from (7) and (25) is similar to that of magnons in ordinary magnetism. However, the physics is quite different, because in contrast to magnons the isospin excitations are pointlike, i.e. they can exist within one point of physical space, because they are vibrations of the isospin vectors of the tetrons within one internal tetrahedron. Note, that these internal vibrations are spin-$\frac{1}{2}$ because they inherit their fermion nature from the fermion property of the vibrating tetrons in their 3-dimensional physical 'base space'.

Similar to magnons, the vibrations can move in physical space[9] by hopping from

[3]Weak parity violation, vulgo the appearance of index L in $SU(2)_L$, is discussed in [1].

one tetrahedron to another (particle picture) or propagating as quasiparticle waves through physical space (wave picture). Thus, although it can exist at one point of physical space, when one tries to exactly measure its location, for example by scattering with another particle, the excitation will start to move on physical space, and this movement will follow a wave equation which naturally has an uncertainty in it according to Schwarz' inequality. Planck's constant enters this uncertainty because the whole process is taking place on a discrete system with Planck length 'lattice constant' and Planck energy 'response energy'.

II. Quark and Lepton Masses from the Interactions of Isospins

The SM SSB being realized by an alignment of the tetron isospins, it is not surprising that the masses of quarks and leptons, and thus the SM Yukawa couplings are determined by the interactions among those isospins. The simplest interaction Hamiltonian between isospin vectors of 2 tetrons i and j looks like

$$H = -J\,\vec{Q}_i\vec{Q}_j \tag{4}$$

So it has the form of a Heisenberg interaction - but for isospins, not for spins. The coupling J may be called an 'isomagnetic exchange coupling'. Note that the language of magnetism often is used in this paper, although interactions of isospins and not of spins are considered. Note further that isospin is not an abstract symmetry here, but corresponds to real rotations in the 3 extra dimensions.

In reality, the Hamiltonian H is more complicated than (4), for several reasons:

• There are inner- and inter-tetrahedral interactions of isospins, i.e. within the same and with a neighboring tetrahedron. The inner ones must have an energy minimum at the tetrahedral angle $\theta = \theta_{tet} = \arccos(-\frac{1}{3})$, while the inter ones correspond to a minimum at the collinear configuration $\theta = 0$, cf. Fig. 1.

• The appearance of antitetron degrees of freedom should be accounted for by using interactions both of $\vec{Q}_L$ and $\vec{Q}_R$ defined in (2) instead of $\vec{Q}$ in (4). The Heisenberg Hamiltonian for the interaction between 2 tetrons i and j then reads:

$$H_H = -J_{LL}\,\vec{Q}_{Li}\vec{Q}_{Lj} - J_{LR}\,\vec{Q}_{Li}\vec{Q}_{Rj} - J_{RR}\,\vec{Q}_{Ri}\vec{Q}_{Rj} \tag{5}$$

As shown later in Sect. IV, the three couplings J_{LL}, J_{LR} and J_{RR} can be roughly associated to the masses of the second family fermions, m_c, m_μ and m_s, respectively.

• In addition to the Heisenberg Hamiltonian (5) Dzyaloshinskii–Moriya interactions[11] are to be considered. They will be shown to give the dominant mass contributions to the heavy family. As well known, the form of the DM couplings $\vec{D}_{ab}$ in (25) is restricted by the ground state symmetry through the so-called Moriya rules[12]. Applying these rules to the given tetrahedral structure, the DM Hamiltonian can be shown to have the form (27).

• Heisenberg and DM terms do not contribute at all to the masses m_e, m_u and m_d of the first family. Therefore, small torsional interactions are introduced in Sect. V. They are characterized by the exerting torques $dQ_{L,R}/dt$ being proportional to the isospins $Q_{L,R}$ themselves, cf. Eq. (41).

• The masses of the neutrinos are yet another story. While the interactions discussed so far are isospin conserving, neutrino masses can arise only from isospin violation. Generation of these masses will be discussed in Sect. VI, and a physical explanation for the origin of the isospin violation will be given.

The DM-couplings K_{LL}, K_{LR} and K_{RR} introduced in (27) are much larger than both Heisenberg and torsional interactions and essentially determine the masses m_τ, m_b and m_t of the third family particles. K_{LL} will be shown to be particularly large. It gives the dominant contribution to the top mass as well as to inner- and inter-tetrahedral interactions, thus being the dominant source for the arrangement of isospins and the SU(2) SSB.

All the types of interaction mentioned above contribute to the angular dependence of the energy of 2 tetron isospins at angle θ which is basically of the form

$$E = A + B\ \cos(\theta) + C\ \cos^2(\theta) \tag{6}$$

where A, B and C are determined by the Heisenberg, torsional and DM-interactions. For example, the Heisenberg coupling J_{LL} in (5) concerns $\theta_{LL} = \sphericalangle(\vec{Q}_{Li}, \vec{Q}_{Lj})$ and gives a contribution to B_{LL} only. Altogether, they fix the *relative* directions of the ground state isospins at the energy minimum, both locally and globally in the way depicted in Fig. 1 (whereas the *absolute* arrangement of the tetrahedrons is spontaneous).

Furthermore, they give rise to the fermionic excitations which are interpreted as quarks and leptons. Masses can then be calculated using the Hamiltonians discussed above. Indeed, 24 eigen energies arise from the tetrahedral configuration by diagonalizing equations for the isospin torque which are generically of the form

$$\frac{d\vec{Q}}{dt} = i\,[H, \vec{Q}] \tag{7}$$

While the masses correspond to the eigenvalues, CKM and PMNS mixings can be deduced from the eigenvectors. This point will be discussed in Sect. VII.

More in detail, the quarks and leptons are vibrations δ of the isospin vectors $\vec{Q}_{Li}$ and $\vec{Q}_{Ri}$ of the tetrons i at sites $i = 1, 2, 3, 4$, i.e. fluctuations of the ground state values within one tetrahedron.

$$\vec{Q}_{Li} = \langle\vec{Q}_{Li}\rangle + \vec{\delta}_{Li} \qquad\qquad \vec{Q}_{Ri} = \langle\vec{Q}_{Ri}\rangle + \vec{\delta}_{Ri} \tag{8}$$

where

$$\langle\vec{Q}_{Li}\rangle = \frac{1}{4}\langle\Psi^\dagger(1-\gamma_5)\vec{\tau}\Psi\rangle \qquad\qquad \langle\vec{Q}_{Ri}\rangle = \frac{1}{4}\langle\Psi^\dagger(1+\gamma_5)\vec{\tau}\Psi\rangle \tag{9}$$

are the ground state radial isospin vectors of a tetrahedron in Fig. 1 assumed to be pointing outward

$$\langle\vec{Q}_{Li}\rangle = \langle\vec{Q}_{Ri}\rangle = \vec{e}_r \tag{10}$$

the corresponding radial isospinor $\langle\Psi\rangle$ defining the 'U'-direction[4][1].

Eq. (9) may be compared to the corresponding value for the ground state value of the SM Higgs field which is $\sim \langle\bar{\Psi}\Psi\rangle$. In that case, however, $\bar{\Psi}$ and Ψ are to be taken from different (which means neighboring) tetrahedrons. This is because gauge and Higgs bosons are constructed as excitations of tetron-antitetron pairs of aligned neighboring tetradedrons. Their masses are determined by the corresponding SM mass formulas - with the Higgs vev given by the tetron-antitetron ground state value

$$\langle\bar{\Psi}\Psi\rangle = \langle\bar{U}U\rangle = \langle\bar{U}_L U_L\rangle = \langle\bar{U}_R U_R\rangle \tag{11}$$

arising from the length of two parallel isospins in Figure 1.

[4]A radial isospin vector pointing inward would correspond to the 'D'-direction.

III. Physical Origin of the Isospin Interactions

This chapter is devoted to the question how the Heisenberg, DM and torsional interactions introduced in the last section can be understood from a more fundamental interaction among tetrons.

First of all, the interested reader should remember that Heisenberg used the Heitler-London results for the hydrogen molecule to understand the phenomenon of ferromagnetism. Heisenberg showed[13] that ferromagnetism is a quantum effect arising from the Pauli principle, more precisely, from the large exchange energies due to the overlap of the antisymmetrized electron wave functions.

The situation here is in principle similar - but in practice somewhat more complicated, because one deals with 6 dimensions with 2 types of rotations: spin and isospin.

In the non-relativistic limit $SO(6,1) \to SO(6)$ the tetron representation 8 of SO(6,1) reduces to

$$SO(6,1) \;\to\; SO(6) \tag{12}$$

$$8 \;\to\; 4 + \bar{4} \tag{13}$$

where 4 is the spinor representation of $SO(6)$ and $\bar{4}$ its complex conjugate. Since the universal covering of SO(6) is given by SU(4), the 4-representation actually is the fundamental representation of SU(4). This representation contains the spin $(\pm\frac{1}{2})$ and isospin $(\pm\frac{1}{2})$ of the tetron, while the $\bar{4}$-representation corresponds to the antitetron degrees of freedom.

Within a non-relativistic quantum mechanics the binding energy between a tetron and an (anti)tetron should generally be calculable from the expectation value

$$E_F = \int d^6x_i d^6x_j \Phi_F^*(x_i, x_j) U_F(|x_i - x_j|) \Phi_F(x_i, x_j) \tag{14}$$

of a non-relativistic potential U_F, where Φ is the complete wavefunction for the tetron-(anti)tetron system and F denotes its combined list of quantum numbers, i.e. spin, isospin, orbital angular momentum etc.

Φ_F may be approximated by a sum of products of two 1-tetron wave functions concentrated at the two tetrahedral sites x_i and x_j. Antisymmetrization of this sum will lead E_F to consist of two terms, the classical 'direct' integral D_F and the quantum mechanical exchange contribution J_F

$$E_F = D_F + J_F \tag{15}$$

While D_F determines the elastic binding among tetrons and thus the gravitational properties of the substrate, the exchange integral J_F can be used to understand the isospin interactions and thus the phenomena of particle physics. Actually, as seen below, J_F is directly related to the isomagnetic Heisenberg and DM couplings J and K defined in the last section.

If one assumes the single tetron wave functions to be fairly localized at their tetrahedral sites, there is a hierarchy $|J_F| \ll |D_F|$. This is different from ordinary 3-dimensional ferromagnetism and is even enhanced by the 12-dimensional integration in (14), through which any overlap contribution becomes strongly suppressed as compared to a direct one. In the extreme case of delta functions, D_F reflects the form of the potential, while J_F vanishes. In the general case, D_F will still be much larger than J_F. For example, assuming the single tetron wave function to fall off by a factor of 10 at half the distance between the 2 sites i and j, J_F will be smaller than D_F roughly by a factor of 10^{-12}. This, en passant, is the way the hierarchy between the Planck scale and the Fermi scale can be understood within the tetron approach. The item has been discussed more thoroughly in [5].

One may ask how the potential U_F transforms under SU(4). Since the energy must be a singlet, one has to have

$$(4 + \bar{4}) \times R_U \times (4 + \bar{4}) = 1 + ... \tag{16}$$

where R_U is the representation under which U_F transforms. Since $4 \times \bar{4} = 1 + 15$ and $4 \times 4 = 6 + 10$ and $15 \times 15 = 1 + ...$ and $6 \times 6 = 1 + ...$[2], it follows that U_F is either a scalar U_1, an adjoint $U_{15}^a \lambda^a$, a=1,...,15, or a vector $U_6^i e^i$, i=1,...,6, where λ^a are the generators of SU(4) and e^i are vectors which span 6-dimensional space. U_1 and U_{15} describe interactions among a tetron and an antitetron and U_6 is a tetron-tetron interaction.

In the present context, where the tetrahedrons are completely orthogonal to physical space, spin and isospin essentially decouple from each other, and the above analysis may be strongly simplified, in the following way: instead of (13) one may consider

$$SO(6,1) \rightarrow SO(3)_{spin} \times SO(3)_{isospin} \tag{17}$$

$$8 \rightarrow (1,2)+(1,2)+(2,1)+(2,1) \tag{18}$$

Assuming ordinary spin to be irrelevant for the internal interactions, it is enough to look for SO(3)$_{isospin}$ singlets in $2 \times R \times 2 = 1 + ...$, which implies $R=1$ or $R=3$, i.e. only an isospin singlet or a triplet potential V_1 or V_3 are allowed for the isomagnetic interactions among tetrons.

V_1 and V_3 may be considered as part of the above $SO(6)$ potentials U_1 and U_{15} and induced by them within the $SO(3)_{isospin}$ fibers. Alternatively, V_1 and V_3 can also be shown to arise in the relativistic framework, i.e. sticking to the original octonion representation 8 of SO(6,1) instead of using (13). Namely, a relativistic potential W_7 is allowed that transforms as 7 under SO(6,1), and the product[2]

$$8 \times 7 \times 8 = 1+7+7+21+21+27+35+35+105+189 \tag{19}$$

contains a singlet.

W_7 may well be a gauge potential and the basis for the fundamental tetron interaction. Furthermore, V_1 and V_3 are part of W_7 due to

$$SO(6,1) \rightarrow SO(3)_{spin} \times SO(3)_{isospin} \tag{20}$$

$$7 \rightarrow (1,1)+(1,3)+(3,1) \tag{21}$$

where V_1 transforms as (1,1) and

$$V_3 = V_3^a \tau^a \tag{22}$$

as the isospin triplet (1,3)[5].

While the Heisenberg interaction $\sim Q_i^a Q_j^a$ is associated to the singlet potential V_1 in the usual way[13], DM terms $\sim \epsilon^{abc} Q_i^b Q_j^c$ in (25) arise from the V_3 contributions. This can be shown by inserting the completeness relation for Pauli matrices

$$\delta_{sv}\delta_{ut} = 2\tau_{st}^a \tau_{uv}^a + \frac{1}{2}\delta_{st}\delta_{uv} \tag{23}$$

[5] In contrast to the suggestion in [21] the photon should *not* be assumed to be part of the gauge field W_7. As explained before, the photon as well as all the other SM gauge bosons are excitations of tetron-antitetron bonds of neighboring tetrahedrons.

into the V_3-exchange integral and afterwards noting that the factor of τ^a in (22) can be merged with one of the factors τ in (23) via

$$\tau^a \tau^b = i\epsilon^{abc}\tau^c + \delta^{ab} \tag{24}$$

The ϵ tensor part in (24) then directly yields the 'antisymmetric exchange'(=DM) contribution (25).

IV. Dzyaloshinskii Masses for the Heavy Family; Heisenberg Masses for the Second Family

My presentation of the mass calculations begins with the Dzyaloshinskii-Moriya (DM) coupling, firstly because it is the dominant isospin interaction and secondly it gives masses only to the third family, i.e. to top, bottom and τ, while leaving all other quarks and leptons massless.

Among all the fermion masses the top quark mass is by far the largest and is of the order of the Fermi scale. As turns out, this is no accident, but has to do with the largeness of the relevant DM coupling.

In the simplest version the isospin DM interaction[1, 11] is

$$H_{DM} = -K \sum_{a\neq b=1}^{4} \vec{D}_{ab}(\vec{Q}_a \times \vec{Q}_b) \tag{25}$$

to be compared to the Heisenberg interaction (4). The form of the vectors $\vec{D}_{ab}$ is dictated by the tetrahedral symmetry to be[12]

$$\vec{D}_{ab} = \vec{Q}_a \times \vec{Q}_b \tag{26}$$

As explained before, interactions among $\vec{Q}_L$ and $\vec{Q}_R$ have to be considered in order to cover all degrees of freedom. The complete DM Hamiltonian then reads

$$\begin{aligned} H_D \;\; = \;\; & -K_{LL} \sum_{a\neq b=1}^{4} (\vec{Q}_{Li} \times \vec{Q}_{Lj})^2 - K_{LR} \sum_{a\neq b=1}^{4} (\vec{Q}_{Li} \times \vec{Q}_{Rj})^2 \\ & -K_{RR} \sum_{a\neq b=1}^{4} (\vec{Q}_{Ri} \times \vec{Q}_{Rj})^2 \end{aligned} \tag{27}$$

with DM couplings (= V_3 exchange integrals) K_{LL}, K_{LR} and K_{RR}.

It is convenient to already include at this point the Heisenberg terms

$$H_H = -J_{LL} \sum_{a \neq b=1}^{4} \vec{Q}_{Li}\vec{Q}_{Lj} - J_{LR} \sum_{a \neq b=1}^{4} \vec{Q}_{Li}\vec{Q}_{Rj} - J_{RR} \sum_{a \neq b=1}^{4} \vec{Q}_{Ri}\vec{Q}_{Rj} \quad (28)$$

with V_1 exchange couplings J_{LL}, J_{LR} and J_{RR}. They are smaller than the DM interactions and turn out to give masses both to the second and third family (but not to the first one).

Phenomenologically, the Heisenberg couplings J are typically smaller than 1 GeV, while the DM couplings K are larger than 1 GeV. Altogether, Heisenberg and DM terms provide the most general isotropic and isospin conserving interactions within the internal space. Apart from that there will only be tiny torsional interactions responsible for the mass of the first family and the neutrinos, to be discussed in Sects. V and VI.

The masses m of the corresponding excitations δ defined in (8) arise from the exponents in the vibrations

$$\delta \sim \exp(imt) = \exp(iXt) \quad (29)$$

where X stands for the appropriate linear combination of the isospin couplings J and K introduced in (27) and (28). The X will be obtained from the torque equations (7), and using the angular momentum commutation relations for the isospin vectors[8]

$$[Q^a_{Ri}, Q^b_{Rj}] = i\delta_{ij}\epsilon^{abc}Q^c_{Ri} \qquad [Q^a_{Li}, Q^b_{Lj}] = i\delta_{ij}\epsilon^{abc}Q^c_{Li} \qquad [Q^a_{Ri}, Q^b_{Lj}] = 0 \quad (30)$$

where $i, j = 1, 2, 3, 4$ count the 4 tetrahedral edges and $a, b, c = 1, 2, 3$ the 3 internal directions(=extra dimensions).

It may be stressed that I have noch undertaken to calculate the couplings J and K in terms of the 12-dimensional V_1 and V_3 exchange integrals as defined in (15) and (14). What is done here, is to use the J and K as free parameters and calculate the masses of the excitations in terms of these couplings. This is the usual approach in magnetic theories, where it often turns out that calculation of integrals like (14) are plagued with large and uncertain corrections. Keeping the couplings as free parameters usually is more rewarding for physical applications.

When carrying out the calculation, care must be taken concerning the unique choice of a quantization axis $\vec{Q}_0$ [20], because this is the condition under which (30) holds.

One may choose one of the tetrahedral edges, e.g.

$$\vec{Q}_0 :\equiv \langle \vec{Q}_1 \rangle = \frac{1}{\sqrt{3}}(-1,-1,-1) \tag{31}$$

to define the axis of quantization and then has to rotate the other isospins to this system.

The 24 first order differential equations for dQ/dt arising from H_H and H_D are rather lengthy. In linear approximation they read

$$\begin{aligned}
\frac{d\vec{\delta}_{Li}}{dt} &= 2K_{LL}\{\vec{Q}_0 \times \vec{\Delta}_{LLi} + i[-\vec{\Delta}_{LLi} + (\vec{\Delta}_{LLi}.\vec{Q}_0)\,\vec{Q}_0]\} \\
&+ 2K_{LR}\{\vec{Q}_0 \times \vec{\Delta}_{LRi} + i[-\vec{\Delta}_{LRi} + (\vec{\Delta}_{LRi}.\vec{Q}_0)\,\vec{Q}_0]\} \\
&+ J_{LL}(\vec{Q}_0 \times \vec{\Delta}_{LLi}) + J_{LR}(\vec{Q}_0 \times \vec{\Delta}_{LLi}) \qquad (32) \\
\frac{d\vec{\delta}_{Ri}}{dt} &= 2K_{RR}\{\vec{Q}_0 \times \vec{\Delta}_{RRi} + i[-\vec{\Delta}_{RRi} + (\vec{\Delta}_{RRi}.\vec{Q}_0)\,\vec{Q}_0]\} \\
&+ 2K_{LR}\{\vec{Q}_0 \times \vec{\Delta}_{RLi} + i[-\vec{\Delta}_{RLi} + (\vec{\Delta}_{RLi}.\vec{Q}_0)\,\vec{Q}_0]\} \\
&+ J_{RR}(\vec{Q}_0 \times \vec{\Delta}_{RRi}) + J_{LR}(\vec{Q}_0 \times \vec{\Delta}_{RLi}) \qquad (33)
\end{aligned}$$

In these equations $\vec{\delta}_{Li} = \vec{Q}_{Li} - \langle \vec{Q}_{Li} \rangle$ and $\vec{\delta}_{Ri} = \vec{Q}_{Ri} - \langle \vec{Q}_{Ri} \rangle$, $a = 1, 2, 3, 4$, denote the isospin vibrations and the Δ's are certain linear combinations of them which will play an important role in discussing isospin conservation in Sect. VI:

$$\begin{aligned}
\vec{\Delta}_{LLi} &= -3\,\vec{\delta}_{Li} + \sum_{j\neq i} \vec{\delta}_{Lj} \\
\vec{\Delta}_{LRi} &= -3\,\vec{\delta}_{Li} + \sum_{j\neq i} \vec{\delta}_{Rj} \\
\vec{\Delta}_{RLi} &= -3\,\vec{\delta}_{Ri} + \sum_{j\neq i} \vec{\delta}_{Lj} \\
\vec{\Delta}_{RRi} &= -3\,\vec{\delta}_{Ri} + \sum_{j\neq i} \vec{\delta}_{Ri} \qquad (34)
\end{aligned}$$

Eqs. (32) and (33) are the basis of the mathematica program included in the Appendix and correspond to a 24×24 eigenvalue problem which - after the SSB - leads to 6 singlet and 6 triplet states of the Shubnikov group (3), the latter ones each consisting of 3 degenerate eigenstates.

After diagonalization one obtains the following results: the first family excitations are still massless at this point, but will get masses from the torsional interactions to be discussed in the next section. The DM exchange coupling K_{LL} is consistently of the order of the transition energy Λ_F and the DM and Heisenberg couplings can be accommodated to reproduce the third and second family masses.

Namely, assuming the DM couplings K to dominate over the Heisenberg couplings J, one can prove the following approximate relations

$$m_t = 4K_{LL} + O(J) \quad m_\tau = \frac{3}{2}K_{LR} + O(J) \quad m_b = 4K_{RR} + O(J) \tag{35}$$

$$m_c = 4J_{LL} \quad m_\mu = \frac{3}{2}J_{LR} \quad m_s = 4J_{RR} \tag{36}$$

In this approximation, the masses of quarks and leptons arise from different isospin interaction terms in (27) and (28), each mass associated essentially to one of the interactions.

Because of the DM dominance one may say that a single tetrahedron of isospin vectors is a ‘DM isomagnet’.

Due to the tetrahedral ’star’ configuration of the 4 isospin vectors pointing outward, it may also be called a ‘frustrated’ isomagnet[22] based on isospin interactions with ‘antiferromagnetic’ couplings.

There is, however, a different interpretation arising from (27) and (28), where one attains attraction among isospins instead of frustration, and furthermore both inner- and inter-tetrahedral interactions turn out to be of order Λ_F. Namely, there is a Hamiltonian for the interaction between 2 isospins $\vec{Q}_i$ and $\vec{Q}_j$ with minimum energy at the tetrahedral angle $\theta_{tet} = \arccos(-\frac{1}{3})$, thus stabilizing the tetrahedral ’star’ arrangement. As compared to (4) and (25) this Hamiltonian has the form

$$H \sim \sum_{i\neq j=1}^{4} \vec{Q}_i\vec{Q}_j - \frac{3}{2}\sum_{i\neq j=1}^{4} (\vec{Q}_i \times \vec{Q}_j)^2 \tag{37}$$

Since the Heisenberg term is $\sim \cos(\theta)$ and the DM-term involves $\sin(\theta)$, their linear combination (37) can be shown to have a minimum at θ_{tet}. One can then rewrite the top (K_{LL}) and charm (J_{LL}) mass part of the Hamiltonian $H_H + H_D$ eqs. (27) and (28) as a sum of 2 contributions

$$\frac{2}{3}K_{LL}[\sum_{i\neq j=1}^{4} \vec{Q}_{Li}\vec{Q}_{Lj} - \frac{3}{2}\sum_{i\neq j=1}^{4} (\vec{Q}_{Li} \times \vec{Q}_{Lj})^2\,] - (\frac{2}{3}K_{LL} + J_{LL})\sum_{i\neq j=1}^{4} \vec{Q}_{Li}\vec{Q}_{Lj} \tag{38}$$

where the first term is assumed to arise from the inner tetrahedral interactions, and the second from the inter ones. Both the inner and inter contributions now are of order Λ_F, the inner having a minimum at θ_{tet} thus stabilizing any tetrahedron of isospins, and the inter with coupling $J := \frac{2}{3}K_{LL}+J_{LL}$ being a 'ferromagnetic' Heisenberg interaction which supports the alignment of any 2 neighboring tetrahedrons of isospins.

V. Isospin Conserving Torsion and the Masses of the First Family

In the previous sections it was shown how the heaviness of the third family is related to large DM couplings. Afterwards masses of the quarks and leptons of the second family were obtained from Heisenberg exchange. In this section it will be seen how the small masses of the first family can be obtained from isospin conserving torsional interactions.

It turns out that torsional interactions give contributions to the masses of all families. However, since they are assumed to be small, the 2 heavy families remain dominated by DM and Heisenberg couplings, as given in (36).

The structure of torsional interactions is quite simple. They correspond to a generalization of Hooke's law to rotations, where instead of an exerting force which is proportional to the stretch x there is an exerting torque which is proportional to the stretch angle φ.

$$I\frac{d^2\varphi}{dt^2} = -C_T^2\varphi \tag{39}$$

with some constant C_T. The energy of the system is given by

$$E_T = \frac{1}{2}I(\frac{d\varphi}{dt})^2 + \frac{1}{2}C_T^2\varphi^2 \tag{40}$$

with I the moment of inertia.

By differentiation one can see that the second order differential equation (39) is equivalent to $d\varphi/dt = iC_T\varphi$ and thus to the first order equation

$$\frac{dQ}{dt} = iC_TQ \tag{41}$$

where $Q = Id\varphi/dt$ is the angular momentum and dQ/dt the torque.

In the present context (41) is more suitable than (39), because it can be immediately added to the system of differential equations for the $\vec{Q}_{Li}$ and $\vec{Q}_{Ri}$ which was obtained in (32) and (33) for the DM and Heisenberg interactions. Using the notation introduced in (34) one has

$$\frac{d\vec{\delta}_{Li}}{dt} = iC_{LL}\vec{\Delta}_{Li} + iC_{LR}\vec{\Delta}_{Ri} \tag{42}$$

$$\frac{d\vec{\delta}_{Ri}}{dt} = iC_{LR}\vec{\Delta}_{Li} + iC_{RR}\vec{\Delta}_{Ri} \tag{43}$$

where the couplings C_{LL}, C_{LR} and C_{RR} generalize C_T to $\vec{Q}_L$ and $\vec{Q}_R$.

In the formulation (43) care has been taken to maintain isospin conservation as defined in (50). This requirement leads to the appearance of the linear combinations Δ given in (34).

Since (42) and (43) give the only mass contributions to the first family, the C-couplings can be chosen to accommodate the mass of the up quark, down quark and electron, respectively. Namely, one arrives at the mass formulas

$$m_e = 6C_{LR} \tag{44}$$

$$m_u = 2C_{LL} + 3C_{LR} + 2C_{RR} - \sqrt{4(C_{LL} - C_{RR})^2 + C_{LR}^2} \tag{45}$$

$$m_d = 2C_{LL} + 3C_{LR} + 2C_{RR} + \sqrt{4(C_{LL} - C_{RR})^2 + C_{LR}^2} \tag{46}$$

Then, using the phenomenological values

$$m_e = 0.51\ MeV \qquad m_u = 2.2\ MeV \qquad m_d = 4.7\ MeV \tag{47}$$

one obtains

$$C_{LR} = 0.084\ MeV \qquad C_{LL} = 1.11\ MeV \qquad C_{RR} = 0.49 MeV \tag{48}$$

VI. Neutrino Masses and Isospin Nonconservation

In discussions of neutrino masses there is always the question whether they are of Dirac or Majorana type. Within the tetron model, neutrinos have the same spacetime properties as the other quarks and leptons, because all isospin excitations inherit their SO(3,1) transformation properties from the underlying octonion representation of SO(6,1) - which is Dirac.

This means, neutrinos are special only because of their small masses. In the tetron model small neutrino masses arise in the following way: among the 24 isospin excitations, which are the quarks and leptons, there are always 3 G_4-singlet modes which are approximately massless. This has to do with the conservation of total isospin. The 3 masses are suppressed because they correspond to the vibrations of the 3 components of the total internal angular momentum vector within one tetrahedron

$$\vec{\Sigma} := \sum_{a=1}^{4}(\vec{Q}_{Li} + \vec{Q}_{Ri}) = \sum_{a=1}^{4}\vec{Q}_a = \frac{1}{2}\sum_{a=1}^{4}\Psi_a^\dagger\vec{\tau}\Psi_a \tag{49}$$

Whenever this quantity is conserved

$$d\vec{\Sigma}/dt = 0 \tag{50}$$

the neutrino masses will strictly vanish. In fact, the combinations of Heisenberg, DM and torsional interactions (27), (28), (42) and (43) considered so far, conserve total isospin. They fulfill (50) and give no contribution to the neutrino masses. A signal for the conservation of isospin is the appearance in all those equations of the linear combinations

$$\vec{\Delta}_a = -3\vec{\delta}_a + \sum_{b\neq a}\vec{\delta}_b \tag{51}$$

Δ_a enters $d\vec{\Sigma}$ in the form of the sum $\sum_a \vec{\Delta}_a$ - and this sum trivially vanishes.

Nonvanishing contributions to the neutrino masses will be derived below in a systematic and comprehensive way. In order to enlighten the procedure, first consider as a simple example an isospin conserving torque of the form

$$\frac{d\vec{Q}_1}{dt} \sim (\vec{Q}_2 - \vec{Q}_1) + (\vec{Q}_3 - \vec{Q}_1) + (\vec{Q}_4 - \vec{Q}_1) = \vec{\Delta}_1 \tag{52}$$

and compare it with an isospin violating one

$$\frac{d\vec{Q}_1}{dt} = iN_T(\vec{Q}_1 - \vec{Q}_0) = N_T\vec{\delta}_1 \tag{53}$$

with some tiny new coupling N_T and $\vec{Q}_0 = \langle\vec{Q}_1\rangle$ denoting the ground state value of $\vec{Q}_1$. Similarly $dQ_j/dt = iN_T(\vec{Q}_j - \langle\vec{Q}_j\rangle)$ for $j = 2, 3, 4$.

What is the physical meaning of such an isospin violating contribution? After all, (53) does not exhibit any interaction of $\vec{Q}_1$ with the other $\vec{Q}_{2,3,4}$. It is an isospin non conserving reset torque towards $\vec{Q}_0$ and effects a mysterious steady gain or loss of isospin, which certainly needs understanding.

In my opinion there is only one plausible explanation: in order that isospin does not disappear into nirvana, the most straightforward assumption is the existence of some kind of nucleus sitting at the center of each tetrahedron and to whom isospin can be transferred, at least in tiny doses. There may be other explanations, but I find this one particularly appealing, because one may speculate that the nuclei are responsible for an additional stabilization of the substrate's skeleton structure in Fig. 1.

As seen below, in addition to giving neutrino masses, the coupling N_T also enters all the other quark and lepton mass formulas. Therefore, there is *always* this tiny exchange of isospins with the nucleus, whenever a tetrahedron of isospins gets excited to a quark or a lepton.

With contributions (53) alone, however, all 3 neutrinos ν_e, ν_μ and ν_τ get the same mass of order N_T. To obtain different masses it is instructive to remember how the different masses for the 3 families were obtained in the case of the other quarks and leptons, namely by use of isospin-preserving Heisenberg, torsional and DM interactions. Analogously, one may construct isospin violating DM and Heisenberg interactions by replacing $\Delta \to \delta$ in (32) and (33). One obtains

$$\begin{aligned} \frac{d\vec{Q}_{Li}}{dt} &= iN_T(\vec{Q}_{Li}-\vec{Q}_0)+N_H(\vec{Q}_{Li}\times\vec{Q}_0) \\ &\quad +2N_D\{-(\vec{Q}_{Li}\times\vec{Q}_0)+i(-(\vec{Q}_{Li}-\vec{Q}_0)+((\vec{Q}_{Li}-\vec{Q}_0)\vec{Q}_0)\vec{Q}_0]\} \\ &= iN_T\vec{\delta}_{Li}+N_H(\vec{\delta}_{Li}\times\vec{Q}_0)+2N_D\{-(\vec{\delta}_{Li}\times\vec{Q}_0)+i(-\vec{\delta}_{Li}+((\vec{\delta}_{Li}\vec{Q}_0)\vec{Q}_0)]\} \end{aligned} \tag{54}$$

and similarly for $\vec{Q}_{Ri}$. This procedure leads to different masses for the 3 neutrino mass eigenstates ν_1, ν_2 and ν_3 of the following form

$$m(\nu_1)=4N_T \qquad m(\nu_2)=N_T+N_H \qquad m(\nu_3)=4N_T-N_H-4N_D \tag{55}$$

As mentioned before, all other quarks and leptons get similar contributions to their masses from N_T, N_H and N_D. However, since the isospin violating couplings N are assumed to be tiny ($\leq$ 1eV), they can be neglected in the mass formulas which were presented in the preceeding sections.

One may accommodate (55) to the results from neutrino oscillation experiments. Consider first the case of the so called 'normal mass hierarchy' $m(\nu_1) < m(\nu_2) \ll m(\nu_3)$ where

$$m(\nu_1)/eV = 0.001 \qquad m(\nu_2)/eV = 0.0087 \qquad m(\nu_3)/eV = 0.048 \tag{56}$$

Lacking experimental informations on $m(\nu_1)$ I have guessed here a value of 0.001 eV. In the normal hierarchy limit $m(\nu_1) \ll m(\nu_2) \ll m(\nu_3)$ one sees that $m(\nu_1)$ is a measure of the torsional coupling N_T, $m(\nu_2)$ measures the strength N_H of the Heisenberg coupling and $m(\nu_3)$ of the DM coupling N_D. The situation is thus similar as for the other quarks and leptons, where the heavy family mass is dominated by DM interactions, the second family by Heisenberg and the light family by torsional couplings.

In the case of the so called 'inverted hierarchy' one has

$$m(\nu_1)/eV \approx m(\nu_2)/eV = 0.0245 \qquad m(\nu_3)/eV = 0.001 \tag{57}$$

where this time the assumption is made on the (unknown) mass $m(\nu_3)$. Trying to accommodate (57) with (55) one obtains $N_H \approx 0$. At the same time a small $m(\nu_1)$ leads to $N_D \approx N_T$, i.e. an accidental compensation between torsional und DM contribution is needed to occur.

In summary, it was found in this section, that the masses $m(\nu_1)$, $m(\nu_2)$ and $m(\nu_3)$ are a measure of the strength of the isospin-violating torsional, Heisenberg and DM interactions, respectively. This happens in a similar way, as the masses of the first, second and third family of quarks and (non-neutrino) leptons are determined by the strength of the isospin-conserving torsional, Heisenberg and DM interactions, cf. the discussion at the beginning of Sect. V.

VII. CKM and PMNS Mixing

This section deals with the mixing of families and the question to what extent it can be deduced from tetron ideas. Since as much as 8 of the 19 free SM parameters arise from the mixing, one would like to see their number reduced by BSM ideas. This is the reason why in the literature a lot of suggestions for relations among the CKM resp PMNS matrix elements can be found, mostly on the basis of ad hoc assumptions on additional discrete symmetries, from which such relations can then be derived.

In the tetron model, the transition from 'isomagnetic' to mass eigenstates is obtained directly via the diagonalization process described in the last sections. Since the isomagnetic eigenstates are to be identified with the weak interaction eigenstates, this offers a straightforward method to determine the CKM and PMNS matrix entries. Within the mathematica program presented in the appendix the physical mass eigenstates can be obtained from the isomagnetic eigenstates by simply changing the command 'eigenvalues' to 'eigensystem' in the last line. The results, which at first sight look rather cumbersome, will now be discussed and presented here in a condensed and compact form.

First of all, for symmetry reasons one need not write down the full 24×24 output. It is enough to give 6×6 results for quarks and 3×3 results for leptons. The reason is that for the leptons there is an identical contribution to the mass eigenstates from all 4 tetrons 1, 2, 3 and 4 on the tetrahedron, i.e. the structure of the eigenstate is always of the form of a sum '$1 + 2 + 3 + 4$', so that for the presentation it suffices to write down the contribution from tetron 1. Similarly, for the quarks there is a symmetry structure '$3 \times 1 - 2 - 3 - 4$' (for the first color, and '$3 \times 2 - 1 - 3 - 4$' and '$3 \times 3 - 1 - 2 - 4$' for the other two). Knowing this, it is again enough to present the result of tetron 1 of the first color.

Consider first the leptons and introduce an orthonormal system of unit vectors S_x, S_y and S_z, which describe the possible directions of $\delta\vec{Q}_L$ of tetron 1 in internal space. Similarly, T_x, T_y and T_z for $\delta\vec{Q}_R$. (x,y,z) is assumed to be a Cartesian coordinate system in internal space, with z corresponding to the axis of quantization. Using this notation, the result for the leptons is such that the 6×6 matrix for tetron 1 decouples to two separate 3×3 systems for $\delta\vec{Q}_L + \delta\vec{Q}_R$ and $\delta\vec{Q}_L - \delta\vec{Q}_R$, respectively.

By abuse of notation the neutrino mass eigenstates are obtained as

$$\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & -1 & -1 \\ 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & -\frac{1}{2}-\frac{\sqrt{3}}{2}i \\ -1 & \frac{1}{2}+\frac{\sqrt{3}}{2}i & \frac{1}{2}-\frac{\sqrt{3}}{2}i \end{bmatrix} \begin{bmatrix} S_z + T_z \\ S_y + T_y \\ S_x + T_x \end{bmatrix} \tag{58}$$

From the discussion of isospin conservation in section VI we know that neutrinos arise from variations of the total isospin. Therefore, the appearance of sums $\vec{S} + \vec{T}$ is not surprising. Thus, the neutrino mass eigenstates are 'far away' from the weak interaction eigenstates $\vec{S}$ and $\vec{T}$, and consequently the resulting PMNS mixing matrix will be 'far away' from the unit matrix. This is much in contrast to the case of quarks where it will be seen below that the mass eigenstates are small deviations $\vec{S}+\epsilon\vec{T}$ and $-\epsilon\vec{S}+\vec{T}$ from the weak eigenstates $\vec{S}$ and $\vec{T}$. The smallness of ϵ originates partly from the dominance of the top mass over the other fermion masses, and it leads to the well known hierarchy in the CKM matrix elements.

The matrix in (58) is similar to the PMNS matrix sometimes used in trimaximal mixing, the difference being that in the tetron model V_{PMNS} gets another contributing factor from e, μ and τ. Namely, the tetron model mass eigenstates for e, μ and τ are given by

$$\begin{bmatrix} e \\ \mu \\ \tau \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ -1 & \frac{1}{2}+\frac{\sqrt{3}}{2}i & \frac{1}{2}-\frac{\sqrt{3}}{2}i \\ 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}i & -\frac{1}{2}-\frac{\sqrt{3}}{2}i \end{bmatrix} \begin{bmatrix} S_z - T_z \\ S_y - T_y \\ S_x - T_x \end{bmatrix} \tag{59}$$

It is worth noting that all the eigenstates (58) and (59) are extremely stable under variations of isospin couplings. It only happens that some of the matrix elements change signs, when signs of DM, HH or torsional couplings are changed in the mathematica program.

In contrast to the lepton mixing there is a dependence of the quark mixing on the absolute magnitudes of DM, HH and torsional couplings - and therefore indirectly on the masses and Yukawa couplings. An interesting point is that there is even a rather strong dependence of the quark mixing matrix on the LR-couplings, which determine the lepton masses.

More in detail, the mass eigenstates for the up-type quarks are givrn by

$$\begin{aligned}
u &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_1^2}}[(S_x+\epsilon_1 T_x)+(S_y+\epsilon_1 T_y)+(S_z+\epsilon_1 T_z)] \qquad (60)\\
c &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_2^2}}[\frac{1+i\sqrt{3}}{2}(S_x+\epsilon_2 T_x)+\frac{1-i\sqrt{3}}{2}(S_y+\epsilon_2 T_y)-(S_z+\epsilon_2 T_z)]\\
t &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_3^2}}[-\frac{1-i\sqrt{3}}{2}(S_x+\epsilon_3 T_x)-\frac{1+i\sqrt{3}}{2}(S_y+\epsilon_3 T_y)-(S_z+\epsilon_3 T_z)]
\end{aligned}$$

and for the down quarks

$$\begin{aligned}
d &= -\frac{1}{\sqrt{3}\sqrt{1+\epsilon_1^2}}[(-\epsilon_1 S_x+T_x)+(-\epsilon_1 S_y+T_y)+(-\epsilon_1 S_z+T_z)] \qquad (61)\\
s &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_2^2}}[-\frac{1+i\sqrt{3}}{2}(-\epsilon_2 S_x+T_x)-\frac{1-i\sqrt{3}}{2}(-\epsilon_2 S_y+T_y)+(-\epsilon_2 S_z+T_z)]\\
b &= \frac{1}{\sqrt{3}\sqrt{1+\epsilon_3^2}}[\frac{1-i\sqrt{3}}{2}(-\epsilon_3 S_x+T_x)+\frac{1+i\sqrt{3}}{2}(-\epsilon_3 S_y+T_y)-(-\epsilon_3 S_z+T_z)]
\end{aligned}$$

Three small numerical coefficients appear in these equations

$$\epsilon_1 = 0.24 \qquad \epsilon_2 = 0.056 \qquad \epsilon_3 = 0.0072 \qquad (62)$$

proving that the quark mass eigenstates are indeed not far away from the weak eigenstates, and therefore the mixing is small.

One may compare to the corresponding measured entries of the CKM matrix

$$|V_{12}| = 0.22 \qquad |V_{23}| = 0.040 \qquad |V_{13}| \approx 0.005 \qquad (63)$$

in order to verify agreement by and large. More should not be expected from this calculation which is leading order with respect to many types of corrections. No effect from RG running is taken into account, for example, and next-to-leading effects from the large DM-couplings may overwhelm the tiny contributions from torsional interactions.

The numbers in (62) are obtained for the particular couplings used in the mathematica program in the Appendix corresponding to the phenomenological quark and lepton mass values. For other combinations of the couplings/masses the ϵ_i turn out to be different. In fact, they are the only quantities within the eigenvector formulas which - through the isospin couplings - depend on the fermion masses. For example, for smaller values of the top mass, they become larger and thus the CKM mixing - in particular ϵ_3 - becomes larger, too.

VIII. Summary and Discussion

In this paper it was shown how the observed spectrum of quarks and leptons can be derived from isospin interactions among tetrons. After clarifying the relations between SM Yukawa couplings and the isomagnetic couplings, the magnitude of the latter was adapted to the observed mass values. Furthermore, in Sect. III it was shown how these coupling parameters themselves can be calculated from exchange integrals involving the fundamental scalar and triplet potentials V_1 and V_3 among tetrons. The resulting optimized predictions for the masses can be found at the end of the mathematica program in Appendix A. Numbers are understood in GeV.

As turns out, Dzyaloshinskii-Moriya couplings are the largest, while Heisenberg interaction terms are smaller. It is a feature of the DM interaction to give masses only to the third family. In particular the top mass is the only excitation with mass of order Λ_F, because it corresponds to a minimum energy of the tetrahedral isospin Hamiltonian (37) and this is clearly linked to the SSB of the ordered isospins, i.e. to how the aligned tetrahedral 'stars' are oriented collectively in internal space. This ordering takes place below the transition temperature Λ_F, while isospins are distributed randomly (i.e. SU(2) symmetric) at temperatures above the Fermi scale.

All other quark and lepton masses naturally turn out to be much smaller. For example, the Heisenberg interactions characteristically give equal contributions to the masses of the seond and third family, keeping the first family massless. The first family then obtains its masses from still smaller torsional interactions, as explained in Sect. V.

As a byproduct of the calculations some suggestive solutions within the tetron model to several outstanding classical problems of particle physics have appeared:
-The hierarchy problem of why the Fermi scale is so small as compared to the Planck scale. In the tetron model this is due to the smallness of exchange integrals as compared to direct ones, see the discussion after (15) in Sect. III.
-The tinyness of neutrino masses arises from the conservation of tetron isospin. Tiny

isospin violating interactions have been introduced in Sect. VI in order to get the neutrino masses, and their physical origin has been clarified. Note that isospin in the tetron model is not an abstract concept but corresponds to real rotations in the 3 extra dimensions.
-Concerning the observed CKM und PMNS matrix elements, it was seen in the numerical results presented in Sect. VII, how the dominance of the top quark mass determines the CKM mixings to be small, while on the other hand the large mixings in the lepton sector are due to the conservation of isospin.

In the tetron model the internal dynamics of quarks and leptons is intertwined, and therefore it is not surprising that CKM mixing parameters also depend on the lepton masses/yukawa couplings. Details of these and other dependencies will be analyzed in a forthcoming publication.

Appendix: A Mathematica Program to calculate the Quark and Lepton Masses and Eigenstates

The following code allows to calculate quark and lepton masses and mixings, given the isospin couplings as defined in the main text. The resulting masses in GeV can be found at the bottom line of the program. Mathematica results for the mixings are not printed explicitly, but presented in more compact form in section VII.

```
s10:={-1,-1,-1}/√3
del1u:={d1x,d1y,d1z}*ef
del2u:={d2x,-d2y,-d2z}*ef
del3u:={-d3x,d3y,-d3z}*ef
del4u:={-d4x,-d4y,d4z}*ef

t10:=+s10
eel1u:={e1x,e1y,e1z}*ef
eel2u:={e2x,-e2y,-e2z}*ef
eel3u:={-e3x,e3y,-e3z}*ef
eel4u:={-e4x,-e4y,e4z}*ef
```

$\mathrm{dd1{:=}del2u + del3u + del4u - 3 * del1u}$

$\mathrm{dd2{:=}del1u + del3u + del4u - 3 * del2u}$

$\mathrm{dd3{:=}del1u + del2u + del4u - 3 * del3u}$

$\mathrm{dd4{:=}del1u + del2u + del3u - 3 * del4u}$

$\mathrm{ed1{:=}eel2u + eel3u + eel4u - 3 * del1u}$

$\mathrm{ed2{:=}eel1u + eel3u + eel4u - 3 * del2u}$

$\mathrm{ed3{:=}eel1u + eel2u + eel4u - 3 * del3u}$

$\mathrm{ed4{:=}eel1u + eel2u + eel3u - 3 * del4u}$

$\mathrm{de1{:=}del2u + del3u + del4u - 3 * eel1u}$

$\mathrm{de2{:=}del1u + del3u + del4u - 3 * eel2u}$

$\mathrm{de3{:=}del1u + del2u + del4u - 3 * eel3u}$

$\mathrm{de4{:=}del1u + del2u + del3u - 3 * eel4u}$

$\mathrm{ee1{:=}eel2u + eel3u + eel4u - 3 * eel1u}$

$\mathrm{ee2{:=}eel1u + eel3u + eel4u - 3 * eel2u}$

$\mathrm{ee3{:=}eel1u + eel2u + eel4u - 3 * eel3u}$

$\mathrm{ee4{:=}eel1u + eel2u + eel3u - 3 * eel4u}$

$\mathrm{vdd1{:=} - 2 * dd1 + 2 * dd1.s10 * s10}$

$\mathrm{vdd2{:=} - 2 * dd2 + 2 * dd2.s10 * s10}$

$\mathrm{vdd3{:=} - 2 * dd3 + 2 * dd3.s10 * s10}$

$\mathrm{vdd4{:=} - 2 * dd4 + 2 * dd4.s10 * s10}$

$\mathrm{ved1{:=} - 2 * ed1 + 2 * ed1.s10 * s10}$

$\mathrm{ved2{:=} - 2 * ed2 + 2 * ed2.s10 * s10}$

$\mathrm{ved3{:=} - 2 * ed3 + 2 * ed3.s10 * s10}$

$\mathrm{ved4{:=} - 2 * ed4 + 2 * ed4.s10 * s10}$

$\mathrm{vde1{:=} - 2 * de1 + 2 * de1.s10 * s10}$

$\mathrm{vde2{:=} - 2 * de2 + 2 * de2.s10 * s10}$

$\mathrm{vde3{:=} - 2 * de3 + 2 * de3.s10 * s10}$

$\text{vde4:}= -2*\text{de4}+2*\text{de4.s10}*\text{s10}$

$\text{vee1:}= -2*\text{ee1}+2*\text{ee1.s10}*\text{s10}$

$\text{vee2:}= -2*\text{ee2}+2*\text{ee2.s10}*\text{s10}$

$\text{vee3:}= -2*\text{ee3}+2*\text{ee3.s10}*\text{s10}$

$\text{vee4:}= -2*\text{ee4}+2*\text{ee4.s10}*\text{s10}$

$\text{ss:}= -10.70000000000000000$

$\text{st:}= -0.07700000000000000$

$\text{tt:}= -0.22000000000000000$

$\text{jss:}=0.32000000000000000$

$\text{jtt:}=0.01020000000000000$

$\text{jst:}=0.01750000000000000$

$\text{ff:}=0.00049000000000000$

$\text{gg:}=0.00113000000000000$

$\text{fg:}=0.00008500000000000$

$\text{ne:}= -0.00000000000103000$

$\text{nm:}= -0.00000000000790000$

$\text{nt:}=0.00000000001350000$

$\text{ndd1:}= -2*\text{del1u}+2*\text{del1u.s10}*\text{s10}$

$\text{ndd2:}= -2*\text{del2u}+2*\text{del2u.s10}*\text{s10}$

$\text{ndd3:}= -2*\text{del3u}+2*\text{del3u.s10}*\text{s10}$

$\text{ndd4:}= -2*\text{del4u}+2*\text{del4u.s10}*\text{s10}$

$\text{nee1:}= -2*\text{eel1u}+2*\text{eel1u.s10}*\text{s10}$

$\text{nee2:}= -2*\text{eel2u}+2*\text{eel2u.s10}*\text{s10}$

$\text{nee3:}= -2*\text{eel3u}+2*\text{eel3u.s10}*\text{s10}$

$\text{nee4:}= -2*\text{eel4u}+2*\text{eel4u.s10}*\text{s10}$

$\text{zx1:}=$

$\text{Coefficient}[\text{ss}*(2*\text{Cross}[\text{s10},\text{dd1}]+i*\text{vdd1})+$

$\text{nt}*(2*\text{Cross}[\text{s10},\text{del1u}]+i*\text{ndd1})$

$+\mathrm{st} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{ed1}] + i * \mathrm{ved1})$

$+\mathrm{jss} * \mathrm{Cross}[\mathrm{s10}, \mathrm{dd1}] + \mathrm{jst} * \mathrm{Cross}[\mathrm{s10}, \mathrm{ed1}] + \mathrm{nm} * \mathrm{Cross}[\mathrm{s10}, \mathrm{del1u}]$

$+i * \mathrm{ff} * \mathrm{dd1} + i * \mathrm{fg} * \mathrm{ed1} + i * \mathrm{ne} * \mathrm{del1u}, \mathrm{ef}, 1]$

zx2:=

$\mathrm{Coefficient}[\mathrm{ss} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{dd2}] + i * \mathrm{vdd2})+$

$\mathrm{nt} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{del2u}] + i * \mathrm{ndd2})$

$+\mathrm{st} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{ed2}] + i * \mathrm{ved2})$

$+\mathrm{jss} * \mathrm{Cross}[\mathrm{s10}, \mathrm{dd2}] + \mathrm{jst} * \mathrm{Cross}[\mathrm{s10}, \mathrm{ed2}] + \mathrm{nm} * \mathrm{Cross}[\mathrm{s10}, \mathrm{del2u}]$

$+i * \mathrm{ff} * \mathrm{dd2} + i * \mathrm{fg} * \mathrm{ed2} + i * \mathrm{ne} * \mathrm{del2u}, \mathrm{ef}, 1]$

zx3:=

$\mathrm{Coefficient}[\mathrm{ss} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{dd3}] + i * \mathrm{vdd3})+$

$\mathrm{nt} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{del3u}] + i * \mathrm{ndd3})$

$+\mathrm{st} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{ed3}] + i * \mathrm{ved3})$

$+\mathrm{jss} * \mathrm{Cross}[\mathrm{s10}, \mathrm{dd3}] + \mathrm{jst} * \mathrm{Cross}[\mathrm{s10}, \mathrm{ed3}] + \mathrm{nm} * \mathrm{Cross}[\mathrm{s10}, \mathrm{del3u}]$

$+i * \mathrm{ff} * \mathrm{dd3} + i * \mathrm{fg} * \mathrm{ed3} + i * \mathrm{ne} * \mathrm{del3u}, \mathrm{ef}, 1]$

zx4:=

$\mathrm{Coefficient}[\mathrm{ss} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{dd4}] + i * \mathrm{vdd4})+$

$\mathrm{nt} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{del4u}] + i * \mathrm{ndd4})$

$+\mathrm{st} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{ed4}] + i * \mathrm{ved4})$

$+\mathrm{jss} * \mathrm{Cross}[\mathrm{s10}, \mathrm{dd4}] + \mathrm{jst} * \mathrm{Cross}[\mathrm{s10}, \mathrm{ed4}] + \mathrm{nm} * \mathrm{Cross}[\mathrm{s10}, \mathrm{del4u}]$

$+i * \mathrm{ff} * \mathrm{dd4} + i * \mathrm{fg} * \mathrm{ed4} + i * \mathrm{ne} * \mathrm{del4u}, \mathrm{ef}, 1]$

zx5:=$\mathrm{Coefficient}[\mathrm{st} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{de1}] + i * \mathrm{vde1})$

$+\mathrm{tt} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{ee1}] + i * \mathrm{vee1}) + \mathrm{nt} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{eel1u}] + i * \mathrm{nee1})$

$+\mathrm{jst} * \mathrm{Cross}[\mathrm{s10}, \mathrm{de1}] + \mathrm{jtt} * \mathrm{Cross}[\mathrm{s10}, \mathrm{ee1}] + \mathrm{nm} * \mathrm{Cross}[\mathrm{s10}, \mathrm{eel1u}]$

$+i * \mathrm{gg} * \mathrm{ee1} + i * \mathrm{fg} * \mathrm{de1} + i * \mathrm{ne} * \mathrm{eel1u}, \mathrm{ef}, 1]$

zx6:=$\mathrm{Coefficient}[\mathrm{st} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{de2}] + i * \mathrm{vde2})$

$+\mathrm{tt} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{ee2}] + i * \mathrm{vee2}) + \mathrm{nt} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{eel2u}] + i * \mathrm{nee2})$

$+\mathrm{jst} * \mathrm{Cross}[\mathrm{s10}, \mathrm{de2}] + \mathrm{jtt} * \mathrm{Cross}[\mathrm{s10}, \mathrm{ee2}] + \mathrm{nm} * \mathrm{Cross}[\mathrm{s10}, \mathrm{eel2u}]$

$+i * \mathrm{gg} * \mathrm{ee2} + i * \mathrm{fg} * \mathrm{de2} + i * \mathrm{ne} * \mathrm{eel2u}, \mathrm{ef}, 1]$

zx7:=$\mathrm{Coefficient}[\mathrm{st} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{de3}] + i * \mathrm{vde3})$

$+\mathrm{tt} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{ee3}] + i * \mathrm{vee3}) + \mathrm{nt} * (2 * \mathrm{Cross}[\mathrm{s10}, \mathrm{eel3u}] + i * \mathrm{nee3})$

$+\mathrm{jst} * \mathrm{Cross}[\mathrm{s10}, \mathrm{de3}] + \mathrm{jtt} * \mathrm{Cross}[\mathrm{s10}, \mathrm{ee3}] + \mathrm{nm} * \mathrm{Cross}[\mathrm{s10}, \mathrm{eel3u}]$

$+i * \mathrm{gg} * \mathrm{ee3} + i * \mathrm{fg} * \mathrm{de3} + i * \mathrm{ne} * \mathrm{eel3u}, \mathrm{ef}, 1]$

zx8:=Coefficient[st $*$ (2 $*$ Cross[s10, de4] $+ i *$ vde4)

+tt $*$ (2 $*$ Cross[s10, ee4] $+ i *$ vee4) + nt $*$ (2 $*$ Cross[s10, eel4u] $+ i *$ nee4)

+jst $*$ Cross[s10, de4] + jtt $*$ Cross[s10, ee4] + nm $*$ Cross[s10, eel4u]

$+i * \mathrm{gg} * \mathrm{ee4} + i * \mathrm{fg} * \mathrm{de4} + i * \mathrm{ne} * \mathrm{eel4u}, \mathrm{ef}, 1]$

S535:=Flatten[i{zx1, zx2, zx3, zx4, zx5, zx6, zx7, zx8}]

Eigenvalues[

{

Coefficient[S535, d1x, 1],

Coefficient[S535, d1y, 1],

Coefficient[S535, d1z, 1],

Coefficient[S535, d2x, 1],

−Coefficient[S535, d2y, 1],

−Coefficient[S535, d2z, 1],

−Coefficient[S535, d3x, 1],

Coefficient[S535, d3y, 1],

−Coefficient[S535, d3z, 1],

−Coefficient[S535, d4x, 1],

−Coefficient[S535, d4y, 1],

Coefficient[S535, d4z, 1],

Coefficient[S535, e1x, 1],

Coefficient[S535, e1y, 1],

Coefficient[S535, e1z, 1],

Coefficient[S535, e2x, 1],

−Coefficient[S535, e2y, 1],

−Coefficient[S535, e2z, 1],

−Coefficient[S535, e3x, 1],

Coefficient[S535, e3y, 1],

−Coefficient[S535, e3z, 1],

−Coefficient[S535, e4x, 1],

−Coefficient[S535, e4y, 1],

Coefficient[S535, e4z, 1]

}

]

{170.794, 170.794, 170.794, 4.35497, 4.35497, 4.35497, 1.74351, 1.33497, 1.33497, 1.33497, 0.10551, 0.097825, 0.097825, 0.097825, 0.00477782, 0.00477782, 0.00477782, 0.00221218, 0.00221218, 0.00221218, 0.00051, 4.7123 * 10^ − 11, 8.92766 * 10^ − 12, 1.02624 * 10^ − 12}

References

[1] see the review B. Lampe, arXiv:1505.03477 [hep-ph] (2015).

[2] R. Slansky, Phys.Rept. 79 (1981) 1.

[3] G. Ross, Grand Unified Theories, Cummings Publishing Company, 1984.

[4] B. Lampe, Int. J. Mod. Phys. A30 (2015) 1550025.

[5] B. Lampe, Progr. of Phys. 69 (2021) 2000072.

[6] B. Lampe in Proc. of the MG16 Meeting on General Relativity, 5–10 July 2021, https://doi.org/10.1142/13149 (February 2023), 999.

[7] B. Lampe, Int. J. Mod. Phys. A30 (2015) 1550026.

[8] S. 0neda and K. Terasaki, Prog. Theor. Phys. Suppl. 82 (1985) 1.

[9] R.M. White, Quantum Theory of Magnetism, Springer Verlag, ISBN-10 3-540-65116-0.

[10] A.P. Cracknell, Progr. Theor. Phys. 35 (1966) 196.

[11] I. Dzyaloshinskii, J. Phys. Chem. Sol. 4 (1958) 241.

[12] T. Moriya, Phys. Rev. 120 (1960) 91.

[13] W. Heisenberg, Zeitschrift f. Phys. 38 (1926) 411.

[14] J. W. Moffat, Int. J. Mod, Phys. D2, 351 (1993).

[15] J. D. Barrow, Phys. Rev. Lett. 82 (1999) 884.

[16] J. D. Bekenstein, J. D. Phys.Rev. D25 (1982) 1527.

[17] H. Fritzsch and J. Sola, Adv. High Energy Phys. 2014 (2014) 361587.

[18] J. Magueijo, Rept. Prog. Phys. 66 (2003) 2025.

[19] J. K. Webb et al. Phys.Rev.Lett. 87 (2001) 091301.

[20] M. Elhajal, B. Canals, R.S. Borrell and C. Lacroix, Phys. Rev. B71 (2005) 094420.

[21] B. Lampe, Int. J. Mod. Phys. A30 (2015) 1550026.

[22] W. Zheng et al., Phys. Rev. Lett. 96 (2006) 057201.

[23] J. A. Belinchon et al., Int. J. Mod. Phys. D12 (2003) 1113.

[24] J. C. Carvalho et al. Phys. Rev. D46 (1992) 2404.

[25] T. Chiba, Prog. Theor. Phys. 126 (2011) 993.

[26] X. Calmet et al., Phys. Lett. B540 (2002) 173.

[27] C. L. Gardner, Phys. Rev. D68 (2003) 043513.

[28] D. F. Mota et al. Phys.Lett. B581 (2004) 141.

[29] D. Parkinson et al. Phys. Lett. B578 (2004) 235.

[30] E. J. Copeland, Phys. Rev. D69 (2004) 023501.

[31] A.S. Borovik-Romanov and H. Grimmer, International Tables for Crystallography D (2006) 105.

[32] Wolfram Research, Inc., Mathematica, Version 13.2, Champaign, IL (2022).