Semi-stable quiver bundles over Gauduchon manifolds

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Abstract: In this paper, we prove the existence of the approximate (σ, τ)-Hermitian Yang–Mills structure on the (σ, τ)-semi-stable quiver bundle $\mathcal{R} = (\mathcal{E}, \phi)$ over compact Gauduchon manifolds. An interesting aspect of this work is that the argument on the weakly L_1^2-subbundles is different from [Álvarez-Cónsul and García-Prada, Comm Math Phys, 2003] and [Hu–Huang, J Geom Anal, 2020].

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1. Introduction

The classical Hitchin–Kobayashi correspondence establishes a deep equivalent relation between the stability and the existence of the canonical metric (or connection) on holomorphic vector bundles. The study of Hitchin–Kobayashi correspondence has a huge story line, which can be trace back to the 1980s [5, 6, 11, 18, 19]. In the new century, this correspondence still attracted lots of researchers' attention (see [2–4, 8, 14, 17, 20–22] and references therein). And a lot of important and interesting applications of the correspondence come out. Takuro Mochizuki awarded the 2022 Breakthrough Prize in Mathematics, due to his excellent work in holonomic $\mathcal{D}$-modules. Among these excellent work is the complete proof of a stimulating conjecture of Masaki Kashiwara about an extension of the Hard Lefschetz Theorem and other nice properties from pure sheaves to semisimple $\mathcal{D}$-modules [15]. It is amazing that the Hitchin–Kobayashi correspondence on the filtered flat bundle plays a key role in the proof of Kashiwara's conjecture. In some sense, this reveals that the Hitchin–Kobayashi correspondence plays an important role in the development of modern mathematics.

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In an earlier paper, Álvarez-Cónsul and García-Prada [1] established a Hitchin–Kobayashi correspondence on quiver bunldes over the compact Kähler manifold. Recently, their result has been generalized by Hu–Huang [7] to a more general base manifold (generalized Kähler manifold). To be specific, they proved that the stability and the existence of Hermitian Yang–Mills metric on the quiver bundle are equivalent. The stability condition they considered is given by a strict inequality. When the inequality is not strict, such inequality condition is nothing but semi-stability. And we will consider such semi-stability case in the quiver bundle. In fact we can prove the following theorem.

Theorem 1.1. *Let $Q = (Q_0, Q_1)$ be a quiver, and $\mathcal{R} = (\mathcal{E}, \phi)$ be a holomorphic Q-bundle over a compact Gauduchon manifold (X, ω). Assume σ and τ are collections of positive real numbers σ_v and τ_v, where $v \in Q_0$. Assume that every $\mathcal{E}_v = \pi_v \circ \mathcal{E}$ admits non-positive mean curvature $\sqrt{-1}\Lambda_\omega F_{H_{0,v}}$. If $\mathcal{R} = (\mathcal{E}, \phi)$ is (σ, τ)-semi-stable, then it admits an approximate (σ, τ)-Hermitian Yang–Mills structure, i.e. the metrics satisfying the inequality* (2.1).

Remark 1.1. By the result of Nie–Zhang [16], every semi-stable holomorphic vector bundle $\mathcal{E}_v$ over compact Gauduchon manifold X must admit a Hermitian metric with negative mean curvature $\sqrt{-1}\Lambda_\omega F_{H_{0,v}}$ if the slope of $\mathcal{E}_v$ is negative. In a recently paper, Li–Zhang–Zhang [10] gave a brief characterization of mean curvature negativity of holomorphic vector bundles over compact Gauduchon manifold.

At first, we can not use Álvarez-Cónsul and García-Prada's techniques [1] to our setting directly. Since their proof is rather rely on the Donaldson's functional on Kähler manifold, but this functional is not well-defined on the Gauduchon manifold. Secondly, we can not use Hu–Huang's results [7] to our setting neither. This is because that they arrive at an inequality (not strictly) to get a contradiction with the strict inequality condition, and this is of course not valid to the semi-stable case. The proof of the main theorem will use the Uhlenbeck–Yau's continuity method [19]. It is also worth to mention that, the perturbed equation considered in this paper is also different to Hu–Huang [7]. In [7], the perturbed term is independent of the vertex numbers σ_v. We observe that, once we add the vertex numbers σ_v in the perturbed term, we can complete the proof of Theorem 1.1 by adapting with Simpson [18] and Nie–Zhang's [16] arguments.

An interesting aspect of this work is that the argument on the weakly L_1^2-subbundles is different from the previous quiver bundle case [1, 7]. In [7], they used Lübke–Teleman's argument [13] to run this step. To our best knowledge, we can not use this to our semi-stable setting. Hence, let us look back to the reference [1]. In [1], they construct a quantity χ [1, Page 22] by the eigenvalues λ_j of $u_\infty = \oplus_v u_{\infty,v}$, where $u_{\infty,v}$ is endomorphism on $\mathcal{E}_v$. In some sense, it is more natural to use eigenvalues $\lambda_{j,v}$ of $\mathcal{E}_v$ to construct the quantity χ. Once we began by doing this to start the argument, another difficulty came out. The eigenvalues $\lambda_{j,v}$, the real numbers σ_v, the rank of $\mathcal{E}_v$ and other quantities are intimately entangled, and these can not be seperated to run the next step. To fix this, we define the maximum of $\lambda_{j,v}$ and the minimum of $\sum_{j=1}^{l-1}(\lambda_{j+1,v} - \lambda_{j,v})$, then we are lucky to construct a new and useful quantity χ, which may be of independent interest.

2. Preliminaries

In this section we introduce the basic setup and notation that will be used throughout the paper. More detailed information on quiver bundles can be found in [1, 7].

2.1. *Gauduchon manifold*

Let X be an $n-$dimensional compact Hermitian manifold, and g be a Hermitian metric with associated Kähler form ω. g is called Gauduchon if ω satisfies $\partial\bar{\partial}\omega^{n-1} = 0$. Throughout the paper we assume (X, ω) is a Gauduchon manifold.

2.2. *Quiver bundle*

A quiver is a pair $Q = (Q_0, Q_1)$ together with two maps $h, t : Q_0 \to Q_1$. Elements of Q_0 (resp. Q_1) are called vertices (resp. arrows) of the quiver. For each $a \in Q_1$, ha (resp. ta) is called the head (resp. tail) of the arrow a.

A holomorphic Q-bundle over (X, ω) is a pair $\mathcal{R} = (\mathcal{E}, \phi)$, where $\mathcal{E}$ is a collection of holomorphic vector bundle $\mathcal{E}_v$ over (X, ω), for each $v \in Q_0$, and ϕ is a collection of morphisms $\phi_a : \mathcal{E}_{ta} \to \mathcal{E}_{ha}$, for each $a \in Q_0$, such that $\mathcal{E}_v = 0$ for all but finitely many $v \in Q_0$, and $\phi_a = 0$ for all but finitely many $a \in Q_1$.

2.3. (σ, τ)*-Hermitian Yang–Mills structure*

A Hermitian metric on holomorphic Q-bundle $\mathcal{R} = (\mathcal{E}, \phi)$ is a collection H of Hermitian metrics H_v on $\mathcal{E}_v$, for each $v \in Q_0$ with $\mathcal{E}_v \neq 0$. For each $v \in Q_0$, let σ and τ be collections of real numbers σ_v, τ_v with positive σ_v. A holomorphic Q-bundle $\mathcal{R} = (\mathcal{E}, \phi)$ is said to be admitting a (σ, τ)-Hermitian Yang–Mills structure if

$$\sigma_v \sqrt{-1}\Lambda_\omega F_{H_v} + \sum_{a \in h^{-1}(v)} \phi_a \circ \phi_a^{*H_v} - \sum_{a \in t^{-1}(v)} \phi_a^{*H_v} \circ \phi_a = \tau_v \cdot \mathrm{Id}_{\mathcal{E}_v},$$

for each $v \in Q_0$ such that $\mathcal{E}_v \neq 0$, where Λ_ω is the contraction with ω, F_{H_v} is the curvature of the Chern connection D_{H_v} with respect to the metric H_v on $\mathcal{E}_v$, for each $v \in Q_0$ with $\mathcal{E}_v \neq 0$.

Álvarez-Cónsul–García-Prada [1] and Hu–Huang [7] proved a holomorphic Q-bundle $\mathcal{R} = (\mathcal{E}, \phi)$ is said to be admitting a (σ, τ)-Hermitian Yang–Mills structure if and only if $\mathcal{R} = (\mathcal{E}, \phi)$ is poly-stable.

A holomorphic Q-bundle $\mathcal{R} = (\mathcal{E}, \phi)$ is said to be admitting an approximate (σ, τ)-Hermitian Yang–Mills structure if for every $\varepsilon > 0$, there exists a collection of Hermitian metrics $H_{\varepsilon,v}$ on each $\mathcal{E}_v$ such that

$$\max_X |\sigma_v \sqrt{-1}\Lambda_\omega F_{H_{\varepsilon,v}} + \sum_{a\in h^{-1}(v)} \phi_a \circ \phi_a^{*H_{\varepsilon,v}} - \sum_{a\in t^{-1}(v)} \phi_a^{*H_{\varepsilon,v}} \circ \phi_a - \tau_v \cdot \mathrm{Id}_{\mathcal{E}_v}|_{H_{\varepsilon,v}} < \varepsilon. \tag{2.1}$$

Kobayashi [9] introduced this notion for a holomorphic vector bundle ($\phi = 0$). When $|Q_0| = 1$, this notion has a strong relationship with the semi-stability of the bundle [9, 12, 16, 21, 22].

2.4. *Stability and semi-stability*

Given a holomorphic vector bundle $\mathcal{E}_v$ on X, by Chern–Weil theory [23], its degree is given by

$$\deg(\mathcal{E}_v) = \frac{1}{\mathrm{Vol}(X)} \int_X \mathrm{tr}(\sqrt{-1}\Lambda_\omega F_{H_v}),$$

where F_{H_v} is the curvature of the Chern connection D_{H_v} with respect to the metric H_v on $\mathcal{E}_v$. The (σ, τ)-degree and (σ, τ)-slope of holomorphic Q-bundle $\mathcal{R} = (\mathcal{E}, \phi)$ are given by

$$\deg_{\sigma,\tau}(\mathcal{R}) = \sum_{v\in Q_0} (\sigma_v \deg(\mathcal{E}_v) - \tau_v \mathrm{rk}(\mathcal{E}_v)), \qquad \mu_{\sigma,\tau}(\mathcal{R}) = \frac{\deg_{\sigma,\tau}(\mathcal{R})}{\sum_{v\in Q_0} \sigma_v \mathrm{rk}(\mathcal{E}_v)},$$

respectively. $\mathcal{R} = (\mathcal{E}, \phi)$ is called (σ, τ)-(semi)stable if for all proper Q-subsheaves $\mathcal{R}'$ of $\mathcal{R}$,

$$\mu_{\sigma,\tau}(\mathcal{R}') < (\leq) \mu_{\sigma,\tau}(\mathcal{R}).$$

3. Proof of Theorem 1.1

Fixing a proper background Hermitian metric H_0 on $\mathcal{R} = (\mathcal{E}, \phi)$, denote by $H_{\varepsilon,v} = H_{0,v} h_\varepsilon$. For each $v \in Q_0$, we consider the following perturbed equation

$$L^{\varepsilon}_{(\sigma,\tau)v}(h_\varepsilon) := \Phi(H_{\varepsilon,v}) + \varepsilon\sigma_v(\log h_{\varepsilon,v}) = 0, \tag{3.1}$$

where

$$\Phi(H_{\varepsilon,v}) = \sigma_v \sqrt{-1}\Lambda_\omega F_{H_{\varepsilon,v}} + \sum_{a\in h^{-1}(v)} \phi_a \circ \phi_a^{*H_{\varepsilon,v}} - \sum_{a\in t^{-1}(v)} \phi_a^{*H_{\varepsilon,v}} \circ \phi_a - \tau_v \cdot \mathrm{Id}_{\mathcal{E}_v}.$$

Following the techniques in [7, 13], it is not hard to show that (3.1) is solvable for all $\varepsilon \in (0,1]$. We omit this step here, since it is standard and tedious. Using the assumption of (σ,τ)-semi-stability, we can show that

$$\lim_{\varepsilon\to 0} \varepsilon\sigma_v \max_X |\log h_{\varepsilon,v}|_{H_{0,v}} = 0.$$

This implies that $\max_X |\Phi(H_{\varepsilon,v})|_{H_{\varepsilon,v}}$ converges to zero as $\varepsilon \to 0$.

By an appropriate conformal change, we can assume that H_0 satisfies

$$\sum_{v\in Q_0} \mathrm{tr}(\Phi(H_{0,v})) = 0.$$

Then using the maximum principle, we have

$$\sum_{v\in Q_0} \sigma_v \mathrm{tr}(\log h_{\varepsilon,v}) = 0.$$

By Moser's iteration method, it is not hard to prove the following lemma, which is similar to [13].

Lemma 3.1. *If $h_{\varepsilon,v} \in \mathrm{Herm}^+(\mathcal{E}_v, H_{0,v})$ satisfies $L^{\varepsilon}_{(\sigma,\tau)v}(h_\varepsilon) = 0$ for some $\varepsilon > 0$, then*

$$\sigma_v \max_X |\log h_{\varepsilon,v}|_{H_{0,v}} \le C_1\Big(\sum_{v\in Q_0} \sigma_v \|\log h_{\varepsilon,v}\|_{L^2} + \max_X \sum_{v\in Q_0} |\Phi(H_{0,v})|_{H_{0,v}}\Big),$$

where C_1 only depends on g and H_0.

Before giving the detailed proof, let's recall some notations. Fixing $\eta \in \mathrm{Herm}(\mathcal{E}, H_v)$, from [13, p. 237], we can choose an open dense subset $W \subseteq X$ satisfying at each $x \in X$ there exist an open neighbourhood U of x, a local unitary basis $\{e_i\}_{i=1}^r$ with respect to H_v and $\{\lambda_i \in C^\infty(U,\mathbb{R})\}_{i=1}^r$ such that

$$\eta(y) = \sum_{i=1}^r \lambda_i(y)\cdot e_i(y) \otimes e^i(y)$$

for all $y \in U$, where $\{e^i\}_{i=1}^r$ denotes the dual basis of $\mathcal{E}_v^*$. Let $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$, $\Psi \in C^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $A = \sum_{i,j=1}^r A^i_j e_i \otimes e^j \in \mathrm{End}(\mathcal{E}_v)$, where we also assume $\mathrm{rank}(\mathcal{E}_v) = r$. We denote $\varphi(\eta)$ and $\Psi(\eta)(A)$ by [18, p. 880]

$$\varphi(\eta)(y) = \sum_{i=1}^r \varphi(\lambda_i) e_i \otimes e^i$$

and

$$\Psi(\eta)(A)(y) = \sum_{i,j=1}^r \Psi(\lambda_j,\lambda_i) A^i_j e_i \otimes e^j.$$

Now, we are ready to prove the following identity.

Proposition 3.1. *If* $h_{\varepsilon,v} \in \mathrm{Herm}^+(\mathcal{E}_v, H_{0,v})$ *solves* (3.1) *for some* $\varepsilon > 0$ *and each* $v \in Q_0$*, then it holds*

$$\sum_{v\in Q_0}\Big(\int_X tr(\Phi(H_{0,v})s_{\varepsilon,v}) + \sigma_v\int_X \langle\Psi(s_{\varepsilon,v})(\overline{\partial}_{\mathcal{E}_v}s_{\varepsilon,v}), \overline{\partial}_{\mathcal{E}_v}s_{\varepsilon,v}\rangle_{H_{0,v}}\Big) \leq -\varepsilon\sum_{v\in Q_0}\sigma_v\|s_{\varepsilon,v}\|^2_{L^2},$$

where $s_{\varepsilon,v} = \log h_{\varepsilon,v}$ *and*

$$\Psi(x,y) = \begin{cases} \frac{e^{y-x}-1}{y-x}, & x \neq y; \\ 1, & x = y. \end{cases}$$

Proof. Direct calculations yield

$$\begin{aligned} &\sum_{v\in Q_0}\int_X \mathrm{tr}\Big((\Phi(H_{\varepsilon,v}) - \Phi(H_{0,v}))s_{\varepsilon,v}\Big) \\ &\geq \sum_{v\in Q_0}\int_X \sigma_v\Big\langle\sqrt{-1}\Lambda_\omega\overline{\partial}_{\mathcal{E}_v}(h^{-1}_{\varepsilon,v}\partial_{H_{0,v}}h_{\varepsilon,v}), s_{\varepsilon,v})\Big\rangle_{H_{0,v}} \\ &= \sum_{v\in Q_0}\sigma_v\int_X\langle\Psi(s_{\varepsilon,v})(\overline{\partial}_{\mathcal{E}_v}s_{\varepsilon,v}), \overline{\partial}_{\mathcal{E}_v}s_{\varepsilon,v}\rangle_{H_{0,v}}, \end{aligned} \tag{3.2}$$

in which the first inequality used [1, Lemma 3.5]

$$\begin{aligned} &\sum_{v\in Q_0}\langle\sum_{a\in h^{-1}(v)}\phi_a\circ\phi_a^{*H_{\varepsilon,v}} - \sum_{a\in t^{-1}(v)}\phi_a^{*H_{\varepsilon,v}}\circ\phi_a, s_{\varepsilon,v}\rangle \\ \geq &\sum_{v\in Q_0}\langle\sum_{a\in h^{-1}(v)}\phi_a\circ\phi_a^{*H_{0,v}} - \sum_{a\in t^{-1}(v)}\phi_a^{*H_{0,v}}\circ\phi_a, s_{\varepsilon,v}\rangle \end{aligned}$$

and the second equality derived from [16, Proposition 3.1].

Hence we complete the proof by combining (3.1) and (3.2). □

Now, we are ready to prove Theorem 1.1.

Let $\{h_{\varepsilon,v}\}_{0<\varepsilon\leq 1}$ be the solutions of equation (3.1) with the background metric $H_{0,v}$.

Case 1, There exists a constant $C_2 > 0$ such that $\sigma_v\|\log h_{\varepsilon,v}\|_{L^2} < C_2 < +\infty$ for each $v \in Q_0$. From Lemma 3.1, we have

$$\max_X|\Phi(H_{\varepsilon,v})|_{H_{\varepsilon,v}} = \varepsilon\sigma_v\max_X|\log h_{\varepsilon,v}|_{H_{\varepsilon,v}} < \varepsilon C_1(C_2|Q_0| + \max_X\sum_{v\in Q_0}|\Phi(H_{0,v})|_{H_{0,v}}).$$

Then it follows that $\max_M|\Phi(H_{\varepsilon,v})|_{H_{\varepsilon,v}} \to 0$ as $\varepsilon \to 0$.

Case 2, $\overline{\lim_{\varepsilon\to 0}}\sum_{v\in Q_0}\sigma_v\|\log h_{\varepsilon,v}\|_{L^2} \to \infty$.

Claim If $\mathcal{R} = (\mathcal{E}, \phi)$ is (σ, τ)-semi-stable, then for each $v \in Q_0$ it holds

$$\lim_{\varepsilon \to 0} \max_X |\Phi(H_{\varepsilon,v})|_{H_{\varepsilon,v}} = 0. \tag{3.3}$$

If the claim does not hold, then there exist $\delta > 0$ and a subsequence $\varepsilon_i \to 0,\ i \to +\infty$, such that

$$\sum_{v \in Q_0} \sigma_v \|\log h_{\varepsilon_i,v}\|_{L^2} \to +\infty$$

and

$$\max_X \sum_{v \in Q_0} |\Phi(H_{\varepsilon_i,v})|_{H_{\varepsilon_i,v}} = \varepsilon_i \max_X \sum_{v \in Q_0} \sigma_v |\log h_{\varepsilon_i,v}|_{H_{\varepsilon_i,v}} \geq \delta. \tag{3.4}$$

Setting

$$s_{\varepsilon_i,v} = \log h_{\varepsilon_i,v}, \quad l_{i,v} = \|s_{\varepsilon_i,v}\|_{L^2}, \quad u_{\varepsilon_i,v} = \frac{s_{\varepsilon_i,v}}{l_{i,v}},$$

it follows that $\sum_{v \in Q_0} \mathrm{tr}(\sigma_v u_{\varepsilon_i,v}) = 0$ and $\|u_{\varepsilon_i,v}\|_{L^2} = 1$. Then combining (3.4) with Lemma 3.1, we have

$$\sum_{v \in Q_0} \sigma_v l_{i,v} \geq \frac{\delta}{\varepsilon_i C_3} - \max_X \sum_{v \in Q_0} |\Phi(H_{0,v})|_{H_{0,v}} \tag{3.5}$$

and

$$\max_X |u_{\varepsilon_i,v}| \leq \frac{C_4}{l_{i,v}} (\sum_{v \in Q_0} \sigma_v l_{i,v} + \max_X \sum_{v \in Q_0} |\Phi(H_{0,v})|_{H_0}) < C_5 < +\infty. \tag{3.6}$$

Step 1 We will show that $\|u_{\varepsilon_i,v}\|_{L_1^2}$ are uniformly bounded. Since $\|u_{\varepsilon_i,v}\|_{L^2} = 1$, we only need to prove $\|\mathrm{d}u_{\varepsilon_i,v}\|_{L^2}$ are uniformly bounded.

By Proposition 3.1, for each h_{ε_i}, it holds

$$\begin{aligned} &\sum_{v \in Q_0} \Big(\int_X \mathrm{tr}(\Phi(H_{0,v}) u_{\varepsilon_i,v}) + \sigma_v \int_X l_{i,v} \langle \Psi(l_{i,v} u_{\varepsilon_i,v})(\overline{\partial}_{\mathcal{E}_v} u_{\varepsilon_i,v}), \overline{\partial}_{\mathcal{E}_v} u_{\varepsilon_i,v} \rangle_{H_{0,v}} \Big) \\ &\leq -\varepsilon_i \sum_{v \in Q_0} \sigma_v l_{i,v}, \end{aligned} \tag{3.7}$$

Substituting (3.5) into (3.7), we have

$$\begin{aligned} &\frac{\delta}{C_3} + \sum_{v \in Q_0} \Big(\int_X \mathrm{tr}(\Phi(H_{0,v}) u_{\varepsilon_i,v}) + \sigma_v \int_X l_{i,v} \langle \Psi(l_{i,v} u_{\varepsilon_i,v})(\overline{\partial}_{\mathcal{E}_v} u_{\varepsilon_i,v}), \overline{\partial}_{\mathcal{E}_v} u_{\varepsilon_i,v} \rangle_{H_{0,v}} \Big) \\ &\leq \varepsilon_i \sum_{v \in Q_0} \max_M |\Phi(H_{0,v})|_{H_{0,v}}. \end{aligned} \tag{3.8}$$

Consider the function

$$l\Psi(lx,ly)=\begin{cases} l, & x=y;\\ \frac{e^{l(y-x)}-1}{y-x}, & x\neq y.\end{cases}$$

From (3.6), we may assume that $(x,y)\in[-C_6,C_6]\times[-C_6,C_6]$. It is easy to check that

$$l\Psi(lx,ly)\to\begin{cases} (x-y)^{-1}, & x>y;\\ +\infty, & x\leq y,\end{cases} \tag{3.9}$$

increases monotonically as $l\to+\infty$. Let $\zeta\in C^{\infty}(\mathbb{R}\times\mathbb{R},\mathbb{R}^{+})$ satisfying $\zeta(x,y)<(x-y)^{-1}$ whenever $x>y$. From (3.8), (3.9) and the arguments in [18, Lemma 5.4], we have

$$\begin{aligned}&\frac{\delta}{C_3}+\sum_{v\in Q_0}\Big(\int_X \operatorname{tr}(\Phi(H_{0,v})u_{\varepsilon_i,v})+\sigma_v\int_X\langle\zeta(u_{\varepsilon_i,v})(\overline{\partial}_{\mathcal{E}_v}u_{\varepsilon_i,v}),\overline{\partial}_{\mathcal{E}_v}u_{\varepsilon_i,v}\rangle_{H_{0,v}}\Big)\\ &\leq\varepsilon_i\sum_{v\in Q_0}\max_X|\Phi(H_{0,v})|_{H_{0,v}}\end{aligned}\tag{3.10}$$

for $i\gg1$. In particular, we take $\zeta(x,y)=\frac{1}{3C_6}$. It is obvious that when $(x,y)\in[-C_6,C_6]\times[-C_6,C_6]$ and $x>y$, $\frac{1}{3C_6}<\frac{1}{x-y}$. This implies that

$$\begin{aligned}&\frac{\delta}{C_3}+\sum_{v\in Q_0}\Big(\int_X \operatorname{tr}(\Phi(H_{0,v})u_{\varepsilon_i,v})+\sigma_v\int_X\frac{1}{3C_6}|\overline{\partial}_{\mathcal{E}_v}u_{\varepsilon_i,v}|^2_{H_{0,v}}\Big)\\ &\leq\varepsilon_i\sum_{v\in Q_0}\max_X|\Phi(H_{0,v})|_{H_{0,v}}\end{aligned}$$

for $i\gg1$. Then we have

$$\sum_{v\in Q_0}\int_X|\overline{\partial}_{\mathcal{E}_v}u_{\varepsilon_i,v}|^2_{H_{0,v}}\frac{\omega^n}{n!}\leq C_7\sum_{v\in Q_0}\max_X|\Phi(H_{0,v})|_{H_{0,v}}\operatorname{Vol}(X).$$

Thus, $u_{\varepsilon_i,v}$ are bounded in L^2_1. Then we can choose a subsequence $\{u_{\varepsilon_{i_j},v}\}$ such that $u_{\varepsilon_{i_j},v}\rightharpoonup u_{\infty,v}$ weakly in L^2_1,. For simplicity, we still denoted by $\{u_{\varepsilon_i,v}\}$. Noting that $L^2_1\hookrightarrow L^2$, we have

$$1=\int_X|u_{\varepsilon_i,v}|^2_{H_{0,v}}\to\int_X|u_{\infty,v}|^2_{H_{0,v}}.$$

This indicates that $\|u_{\infty,v}\|_{L^2}=1$ and $u_{\infty,v}$ is non-trivial.

Using (3.10) and following a similar discussion as in [18, Lemma 5.4], it holds

$$\begin{aligned}&\frac{\delta}{C_3}+\sum_{v\in Q_0}\Big(\int_X \operatorname{tr}(\Phi(H_{0,v})u_{\infty,v})+\sigma_v\int_X\langle\zeta(u_{\infty,v})(\overline{\partial}_{\mathcal{E}_v}u_{\infty,v}),\overline{\partial}_{\mathcal{E}_v}u_{\infty,v}\rangle_{H_{0,v}}\Big)\\ &\leq 0\end{aligned}\tag{3.11}$$

Step 2 Using Uhlenbeck and Yau's trick from [19], we construct a subsheaf which contradicts the (σ,τ)-semi-stability of $\mathcal{R}=(\mathcal{E},\phi)$.

From (3.11) and the technique in [18, Lemma 5.5], we conclude that the eigenvalues of $u_{\infty,v}$ are constant almost everywhere. Let $\lambda_{1,v}<\lambda_{2,v}<\cdots<\lambda_{l,v}$ be the distinct eigenvalues of $u_{\infty,v}$. The facts that $\sum_{v\in Q_0}\mathrm{tr}(\sigma_v u_{\infty,v})=0$ and $\|u_{\infty,v}\|_{L^2}=1$ force $2\le l\le r$. For each $\lambda_{j,v}$ $(1\le j\le l-1)$, we construct a function

$$P_{j,v}:\mathbb{R}\to\mathbb{R}$$

such that

$$P_{j,v}=\begin{cases}1, & x\le\lambda_{j,v},\\ 0, & x\ge\lambda_{j+1,v}.\end{cases}$$

Setting $\pi_{j,v}=P_{j,v}(u_{\infty,v})$ and $\mathcal{E}_{j,v}=\pi_{j,v}(\mathcal{E}_v)$, from [18], we have

(1) $\pi_{j,v}\in L_1^2$;
(2) $\pi_{j,v}^2=\pi_{j,v}=\pi_{j,v}^{*H_{0,v}}$;
(3) $(\mathrm{Id}_{\mathcal{E}_{j,v}}-\pi_{j,v})\overline{\partial}_{\mathcal{E}_{j,v}}\pi_{j,v}=0$.

By Uhlenbeck and Yau's regularity statement of L_1^2-subbundle [19], $\{\pi_{j,v}\}_{j=1}^{l-1}$ determine $l-1$ coherent sub-sheaves of $\mathcal{E}_v$. Since

$$\sum_{v\in Q_0}\mathrm{tr}(\sigma_v u_{\infty,v})=0$$

and

$$u_{\infty,v}=\lambda_{l,v}\mathrm{id}_{\mathcal{E}_v}-\sum_{j=1}^{l-1}(\lambda_{j+1,v}-\lambda_{j,v})\pi_{j,v},$$

it holds

$$\sum_{v\in Q_0}\Big(\sigma_v\lambda_{l,v}\mathrm{rk}(\mathcal{E}_v)-\sum_{j=1}^{l-1}(\lambda_{j+1,v}-\lambda_{j,v})\sigma_v\mathrm{rk}(\mathcal{E}_{j,v})\Big)=0. \tag{3.12}$$

Denote by

$$\lambda_{l,\widetilde{v}}=\max_{v\in Q_0}\lambda_{l,v},\quad \sum_{j=1}^{l-1}(\lambda_{j+1,\widehat{v}}-\lambda_{j,\widehat{v}})=\min_{v\in Q_0}\sum_{j=1}^{l-1}(\lambda_{j+1,v}-\lambda_{j,v}).$$

Then from (3.12), we have

$$\sum_{v\in Q_0}\sigma_v\lambda_{l,\widetilde{v}}\mathrm{rk}(\mathcal{E}_v)\ge\sum_{v\in Q_0}\sum_{j=1}^{l-1}(\lambda_{j+1,\widehat{v}}-\lambda_{j,\widehat{v}})\sigma_v\mathrm{rk}(\mathcal{E}_{j,v}). \tag{3.13}$$

Construct

$$\chi = \mathrm{Vol}(X)\Big(\lambda_{l,\widetilde{v}} \deg_{\sigma,\tau}(\mathcal{R}) - \sum_{j=1}^{l-1}(\lambda_{j+1,\widehat{v}} - \lambda_{j,\widehat{v}}) \deg_{\sigma,\tau}(\mathcal{R}_j))\Big).$$

On one hand, substituting (3.13) into χ, we have

$$\chi \geq \mathrm{Vol}(X) \sum_{j=1}^{l-1}(\lambda_{j+1,\widehat{v}} - \lambda_{j,\widehat{v}}) \sum_{v\in Q_0} \sigma_v \mathrm{rk}(\mathcal{E}_{j,v})(\mu_{\sigma,\tau}(\mathcal{R}) - \mu_{\sigma,\tau}(\mathcal{R}_j)). \tag{3.14}$$

On the other hand, from [1, 18, 19], we have the following Chern–Weil formula

$$\mathrm{Vol}(X)\deg(\mathcal{E}_{j,v}) = \sum_{v\in Q_0} \Big(\int_X \langle \sqrt{-1}\Lambda_\omega F_{H_{0,v}}, \pi_{j,v}\rangle_{H_{0,v}} - \int_X |\overline{\partial}_{\mathcal{E}_v}\pi_{j,v}|^2_{H_{0,v}}\Big), \tag{3.15}$$

Substituting (3.15) into χ, we have

$$\begin{aligned}
\chi = &\sum_{v\in Q_0}\int_X \Big\langle \sigma_v\sqrt{-1}\Lambda_\omega F_{H_{0,v}}, \lambda_{l,\widetilde{v}}\mathrm{Id}_{\mathcal{E}_v} - \sum_{j=1}^{l-1}(\lambda_{j+1,\widehat{v}} - \lambda_{j,\widehat{v}})\pi_{j,v}\Big\rangle_{H_{0,v}} \\
&+ \sum_{v\in Q_0}\sigma_v \sum_{j=1}^{l-1}(\lambda_{j+1,\hat{v}} - \lambda_{j,\hat{v}})\|\overline{\partial}_{\mathcal{E}_v}\pi_{j,v}\|^2_{L^2} \\
&- \sum_{v\in Q_0}\tau_v \mathrm{Vol}(X)\Big(\lambda_{l,\widetilde{v}}\mathrm{rk}(\mathcal{E}_v) - \sum_{j=1}^{l-1}(\lambda_{j+1,\widehat{v}} - \lambda_{j,\widehat{v}})\mathrm{rk}(\mathcal{E}_{j,v}))\Big) \\
= &\sum_{v\in Q_0}\int_X \Big\langle \sigma_v\sqrt{-1}\Lambda_\omega F_{H_{0,v}}, \lambda_{l,v}\mathrm{Id}_{\mathcal{E}_v} - \sum_{j=1}^{l-1}(\lambda_{j+1,v} - \lambda_{j,v})\pi_{j,v}\Big\rangle_{H_{0,v}} + \sum_{v\in Q_0}\sigma_v\sum_{j=1}^{l-1}(\lambda_{j+1,v} \\
&- \lambda_{j,v})\|\overline{\partial}_{\mathcal{E}_v}\pi_{j,v}\|^2_{L^2} - \sum_{v\in Q_0}\tau_v\mathrm{Vol}(X)\Big(\lambda_{l,v}\mathrm{rk}(\mathcal{E}_v) - \sum_{j=1}^{l-1}(\lambda_{j+1,v} - \lambda_{j,v})\mathrm{rk}(\mathcal{E}_{j,v}))\Big) \\
&+ \sum_{v\in Q_0}\int_X\langle\sigma_v\sqrt{-1}\Lambda_\omega F_{H_{0,v}}, (\lambda_{l,\widetilde{v}} - \lambda_{l,v})\mathrm{Id}_{\mathcal{E}_v} + (\sum_{j=1}^{l-1}(\lambda_{j+1,v} - \lambda_{j,v}) - \sum_{j=1}^{l-1}(\lambda_{j+1,\widehat{v}} - \lambda_{j,\widehat{v}}))\pi_{j,v}\rangle_{H_{0,v}} \\
&+ \sum_{v\in Q_0}\sigma_v[\sum_{j=1}^{l-1}(\lambda_{j+1,\widehat{v}} - \lambda_{j,\widehat{v}}) - \sum_{j=1}^{l-1}(\lambda_{j+1,v} - \lambda_{j,v})]\|\overline{\partial}_{\mathcal{E}_v}\pi_{j,v}\|^2_{L^2} \\
&+ \sum_{v\in Q_0}\tau_v\mathrm{Vol}(X)\Big((\lambda_{l,v} - \lambda_{l,\widetilde{v}})\mathrm{rk}(\mathcal{E}_v) + (\sum_{j=1}^{l-1}(\lambda_{j+1,\widehat{v}} - \lambda_{j,\widehat{v}}) - \sum_{j=1}^{l-1}(\lambda_{j+1,v} - \lambda_{j,v}))\mathrm{rk}(\mathcal{E}_{j,v}))\Big) \\
\leq &\sum_{v\in Q_0}\int_X \Big(\langle\Phi(H_{0,v}), u_{\infty,v}\rangle_{H_{0,v}} + \langle\sigma_v\sum_{j=1}^{l-1}(\lambda_{j+1,v} - \lambda_{j,v})(\mathrm{d}P_{j,v})^2(u_{\infty,v})(\overline{\partial}_{\mathcal{E}_v}u_{\infty,v}), \overline{\partial}_{\mathcal{E}_v}u_{\infty,v}\rangle_{H_{0,v}}\Big)
\end{aligned}$$

where the function $\mathrm{d}P_{j,v} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathrm{d}P_{j,v}(x, y) = \begin{cases} \dfrac{P_{j,v}(x) - P_{j,v}(y)}{x - y}, & x \neq y; \\ P'_{j,v}(x), & x = y. \end{cases}$$

By (3.11) and the same arguments in [12, p. 793-794], it holds that

$$\chi \leq -\frac{\delta}{C_3}. \tag{3.16}$$

Combining (3.14) with (3.16), we arrive at a contradiction to (σ, τ)-semi-stability on the quiver bundle $\mathcal{R} = (\mathcal{E}, \phi)$. □

Declarations

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