# Exploration of the Law of Universal Gravitation with Modern Mathematical Tools

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February 10, 2023


## Abstract

To demonstrate the law of universal gravitation, Newton started from a system of two bodies, the sun and a planet. In addition, he considered the sun to be at rest, and yet, according to the third law, it must be accelerated by the force exerted on it by the planet. In this article, the acceleration of the planet is recalculated, considering the center of mass of the two bodies at rest, using today's math tools.


## 1 Introduction

Newton, to derive his universal law of gravitation, used mathematical tools much more complicated to understand and follow than we have today. Therefore, I will start from the same premises as Newton and, with them, I will arrive at a mathematical deduction of the formula with today's tools.

The premises to which I am referring are:

1. Newton's three laws of mechanics are fulfilled.
2. We will study the forces that two bodies are exerting on each other, assuming that no other forces are applied to them.
3. A body moves relative to another, fulfilling the first two planetary laws of Kepler. That is to say:
   (a) One of the bodies describes an ellipse, and the other body is at one of the two foci of the ellipse. To facilitate understanding, we will call the body at the foci, the Sun, and the body that describes the ellipse, the planet.
   (b) The planet revolves around the sun, describing angles such that the fraction of the area of the ellipse contained in that rotated angle will be the same at any two instants of its movement, as long as the time interval between those two instants is the same.

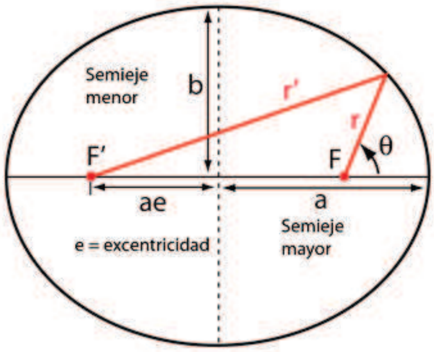
Given that Kepler's two planetary laws completely define the relative motion of the planet around the sun; using Newton's laws, we can obtain the force that the sun must exert on the planet for it to follow the aforementioned motion. In the next section, we will obtain the acceleration if we assume that the sun is in repose. Afterward, we will obtain it assuming the center of mass in repose.

In the demonstration, I will use the equation of the ellipse in polar coordinates. The justification of the ellipse equation in polar coordinates can be found on this website:
http://hyperphysics.phy-astr.gsu.edu/hbasees/Math/ellipse.html

$$r = |\vec{r}| = \frac{a(1-e^2)}{1+e\cos\theta} \quad ; \quad e = \sqrt{1-\frac{b^2}{a^2}}$$

In polar coordinates, unitary vectors are defined as:

$$\vec{u}_r = \frac{\vec{r}}{|\vec{r}|} \quad ; \quad \vec{u}_\theta = \frac{\mathrm{d}\vec{u}_r}{\mathrm{d}\theta} \quad ; \quad \frac{\mathrm{d}\vec{u}_\theta}{\mathrm{d}\theta} = -\vec{u}_r$$




To obtain the area within a figure defined in polar coordinates, the following integral is used:

$$A = \frac{1}{2}\int_{\theta_i}^{\theta_f} r^2\,\mathrm{d}\theta$$

And therefore:

$$\frac{\mathrm{dA}}{\mathrm{d}\theta} = \frac{1}{2}r^2 \quad ; \quad \frac{\mathrm{dA}}{\mathrm{dt}} = \frac{\mathrm{dA}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dt}} = \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{dt}} = \frac{\text{ellipse area}}{\text{orbital period}} \quad ; \quad \frac{\mathrm{d}\theta}{\mathrm{dt}} = 2\,\frac{\text{ellipse area}}{\text{orbital period}}\,\frac{1}{r^2}$$

$$\text{defining } k \text{ as: } 2\,\frac{\text{ellipse area}}{\text{orbital period}} \quad \Longrightarrow \quad \frac{\mathrm{d}\theta}{\mathrm{dt}} = \frac{k}{r^2}$$

## 2 Force exerted on the planet by a fixed sun

$\vec{r}(\theta(t))$vector function of the planet's position ; $\frac{\mathrm{d}\vec{r}}{\mathrm{dt}} = \frac{\mathrm{dr}}{\mathrm{dt}}\vec{u}_r + r\frac{\mathrm{d}\vec{u}_r}{\mathrm{dt}} = \frac{\mathrm{dr}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dt}}\vec{u}_r + r\frac{\mathrm{d}\vec{u}_r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dt}} = \left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_r + r\vec{u}_\theta\right)\frac{\mathrm{d}\theta}{\mathrm{dt}}$

$$\frac{\mathrm{d}\vec{r}}{\mathrm{dt}} = \left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_r + r\vec{u}_\theta\right)\frac{k}{r^2} \quad ; \quad \frac{\mathrm{d}^2\vec{r}}{\mathrm{dt}^2} = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_r + r\vec{u}_\theta\right)\frac{k}{r^2} + \left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_r + r\vec{u}_\theta\right)\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{k}{r^2}\right)$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{dt}^2} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_r + r\vec{u}_\theta\right)\frac{\mathrm{d}\theta}{\mathrm{dt}}\frac{k}{r^2} + \left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_r + r\vec{u}_\theta\right)\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{k}{r^2}\right)\frac{\mathrm{d}\theta}{\mathrm{dt}}$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{dt}^2} = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_r + r\vec{u}_\theta\right)\left(\frac{k}{r^2}\right)^2 + \left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_r + r\vec{u}_\theta\right)\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{k}{r^2}\right)\frac{k}{r^2}$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{dt}^2} = \left(\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2}\vec{u}_r + \frac{\mathrm{dr}}{\mathrm{d}\theta}\frac{\mathrm{d}\vec{u}_r}{\mathrm{d}\theta} + \frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_\theta + r\frac{\mathrm{d}\vec{u}_\theta}{\mathrm{d}\theta}\right)\left(\frac{k}{r^2}\right)^2 + \left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_r + r\vec{u}_\theta\right)\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{k}{r^2}\right)\frac{k}{r^2}$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{dt}^2} = \left(\left(\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} - r\right)\vec{u}_r + 2\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_\theta\right)\left(\frac{k}{r^2}\right)^2 - 2\left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_r + r\vec{u}_\theta\right)\left(\frac{k}{r^3}\right)\frac{\mathrm{dr}}{\mathrm{d}\theta}\frac{k}{r^2}$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{dt}^2} = \left(\left(\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} - r\right)\vec{u}_r + 2\frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_\theta\right)\left(\frac{k}{r^2}\right)^2 - 2\left(\frac{1}{r}\left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\right)^2\vec{u}_r + \frac{\mathrm{dr}}{\mathrm{d}\theta}\vec{u}_\theta\right)\left(\frac{k}{r^2}\right)^2$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{dt}^2} = \left(\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} - r - 2\frac{1}{r}\left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\right)^2\right)\left(\frac{k}{r^2}\right)^2\vec{u}_r$$

$$\frac{\mathrm{dr}}{\mathrm{d}\theta} = \frac{a(1-e^2)}{(1+e\cos\theta)^2}e\sin\theta = \frac{e\sin\theta}{1+e\cos\theta}r$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}\theta^2} = \left(\frac{e\sin\theta}{1+e\cos\theta}\right)^2 r + r\left(\frac{e\cos\theta}{1+e\cos\theta} + \left(\frac{e\sin\theta}{1+e\cos\theta}\right)^2\right) = \left(2\left(\frac{e\sin\theta}{1+e\cos\theta}\right)^2 + \frac{e\cos\theta}{1+e\cos\theta}\right)r$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \left(2\left(\frac{e\sin\theta}{1+e\cos\theta}\right)^2 + \frac{e\cos\theta}{1+e\cos\theta} - 1 - 2\left(\frac{e\sin\theta}{1+e\cos\theta}\right)^2\right) r \left(\frac{k}{r^2}\right)^2 \vec{u}_r$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \left(\frac{e\cos\theta}{1+e\cos\theta} - 1\right) r \left(\frac{k}{r^2}\right)^2 \vec{u}_r = -\frac{1}{1+e\cos\theta} r \left(\frac{k}{r^2}\right)^2 \vec{u}_r = -\frac{k^2}{r\,(1+e\cos\theta)} \frac{1}{r^2}\vec{u}_r = -\frac{k^2}{a(1-e^2)} \frac{1}{r^2}\vec{u}_r$$

$$\text{defining } T \text{ as orbital period} \quad \Longrightarrow \quad \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{ak^2}{b^2} \frac{1}{r^2}\vec{u}_r = -4\pi^2 \frac{a^3}{T^2} \frac{1}{r^2}\vec{u}_r$$

In the deduction, we have finally arrived at the fact that the constant of proportionality of the acceleration over the inverse of the square of the distance contains the ratio that is stated in Kepler's third planetary law.

However, we know experimentally that Kepler's third planetary law does not hold exactly and this implies that this constant depends on the planetary orbit. That is to say, according to what has been deduced, the universal gravitation law is slightly different for each planet that orbits around the sun. Outside the solar system, the third law also does not hold and according to the deduction, it should also hold.

The deduction made has assumed that the sun is at rest, however, if the sun exerts a force on the planet, the planet exerts the same force on the sun, according to Newton's third law. For this reason, the sun cannot be at rest if no other force is acting on it than the one exerted by the planet. Therefore, the acceleration of the planet must be obtained using the center of mass as the origin of the reference system.

## 3 Force exerted by the sun on the planet, when the center of mass is at rest

Defining $\vec{s}$ as the vector position of the sun and $\vec{p}$ as the vector position of the planet $\quad \Longrightarrow \quad \vec{p} = \vec{s} + \vec{r}$

The center of mass is at a point on the segment between the ends of the sun and the earth. The distance from this point to the sun $s$ is:

$$s = \frac{m_p}{m_s+m_p} r \quad \Longrightarrow \quad \vec{s} = -\frac{m_p}{m_s+m_p}\vec{r} \quad \Longrightarrow \quad \vec{p} = -\frac{m_p}{m_s+m_p}\vec{r} + \vec{r} = \frac{m_s}{m_s+m_p}\vec{r}$$

$$\frac{\mathrm{d}^2\vec{p}}{\mathrm{d}t^2} = \frac{m_s}{m_s+m_p}\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{k^2}{a(1-e^2)}\frac{m_s}{m_s+m_p}\frac{1}{r^2}\vec{u}_r = -4\pi^2\frac{a^3}{T^2}\frac{m_s}{m_s+m_p}\frac{1}{r^2}\vec{u}_r$$

As each planet has a different mass, each planetary orbit has a differential factor, for which Kepler's third law does not hold. Furthermore, this differential factor is the reason for the eccentricity of the ellipse. To obtain the proportional constant in which the force is inversely proportional to the square of the distance, we need to discriminate between the two sources that make different the acceleration. The two sources meant are the eccentricity of the ellipse and the mass of the planet.

To obtain a relation between the eccentricity and the mass of the two bodies, we need to find an equivalent circular movement. This circular movement must meet two conditions:

1. Its period must be the same as the orbital period.
2. Its acceleration must be equal to what is obtained in the expression $\frac{\mathrm{d}^2\vec{p}}{\mathrm{d}t^2} = -4\pi^2\frac{a^3}{T^2}\frac{m_s}{m_s+m_p}\frac{1}{r^2}\vec{u}_r$ in which the variable $r$ takes the value of the radius of the circle.

This means that we need to find a math condition to obtain the radio of this circular movement. A circular movement only has normal acceleration, as tangential acceleration is always null; if we equal its acceleration to the mean normal acceleration of the orbital movement, we can be sure that the circular movement will have enough speed to complete its movement in the same period.

Furthermore, because the acceleration's vector field is irrotational, using Stokes' theorem, we know that the curvilinear integral of the tangential acceleration along the elliptical path must be zero so that the average acceleration is equal to the average normal acceleration.

The mean acceleration of the planet is:
(How do you find the antiderivative of $(1+cosx)^2$)

$$\text{defining } a_m \text{ as the mean aceleration} \quad \Longrightarrow \quad a_m = -\frac{1}{2\pi}\int_0^{2\pi}\frac{k^2}{a(1-e^2)}\frac{m_s}{m_s+m_p}\frac{1}{r^2}\mathrm{d}\theta$$

$$a_m = -\frac{1}{2\pi}\frac{k^2}{a(1-e^2)}\frac{m_s}{m_s+m_p}\int_0^{2\pi}\left(\frac{1+e\cos\theta}{a(1-e^2)}\right)^2\mathrm{d}\theta = -\frac{1}{2\pi}\frac{k^2}{a^3\left(1-e^2\right)^3}\frac{m_s}{m_s+m_p}\int_0^{2\pi}\left(1+e\cos\theta\right)^2\mathrm{d}\theta$$

$$a_m = -\frac{1}{2\pi}\frac{k^2}{a^3\left(1-e^2\right)^3}\frac{m_s}{m_s+m_p}2\pi\left(1+\frac{e^2}{2}\right) = -\frac{k^2}{a^3\left(1-e^2\right)^3}\left(1+\frac{e^2}{2}\right)\frac{m_s}{m_s+m_p}$$

Equating the mean acceleration to the general expression of acceleration, we obtain:

$$-\frac{k^2}{a(1-e^2)}\frac{m_s}{m_s+m_p}\frac{1}{r^2} = -\frac{k^2}{a^3\left(1-e^2\right)^3}\left(1+\frac{e^2}{2}\right)\frac{m_s}{m_s+m_p}$$

$$\frac{1}{r^2} = \frac{1}{a^2\left(1-e^2\right)^2}\left(1+\frac{e^2}{2}\right) \quad \Longrightarrow \quad r^2 = \frac{a^2\left(1-e^2\right)^2}{\left(1+\frac{e^2}{2}\right)} \quad \Longrightarrow \quad r = \frac{a\left(1-e^2\right)}{\sqrt{1+\frac{e^2}{2}}}$$

As we stated, the equivalent circular motion must have the same period as the orbital period, therefore:

$$period = \frac{2\pi ab}{k} \quad \Longrightarrow \quad \omega = \frac{2\pi}{\frac{2\pi ab}{k}} = \frac{k}{ab} = \frac{k}{a^2\sqrt{1-e^2}}$$

Where $\omega$ is the angular velocity. With it and the radius of the circular movement, we can obtain the centripetal acceleration $a_c$ as follows:

$$a_c = -\omega^2 r = -\left(\frac{k}{a^2\sqrt{1-e^2}}\right)^2\frac{a\left(1-e^2\right)}{\sqrt{1+\frac{e^2}{2}}} = -\frac{k^2}{a^3\sqrt{1+\frac{e^2}{2}}}$$

$$\text{Equating } a_c \text{ to } a_m \quad \Longrightarrow \quad -\frac{k^2}{a^3\sqrt{1+\frac{e^2}{2}}} = -\frac{k^2}{a^3\left(1-e^2\right)^3}\left(1+\frac{e^2}{2}\right)\frac{m_s}{m_s+m_p}$$

$$\left(\frac{1-e^2}{\sqrt{1+\frac{e^2}{2}}}\right)^3 = \frac{m_s}{m_s+m_p}$$

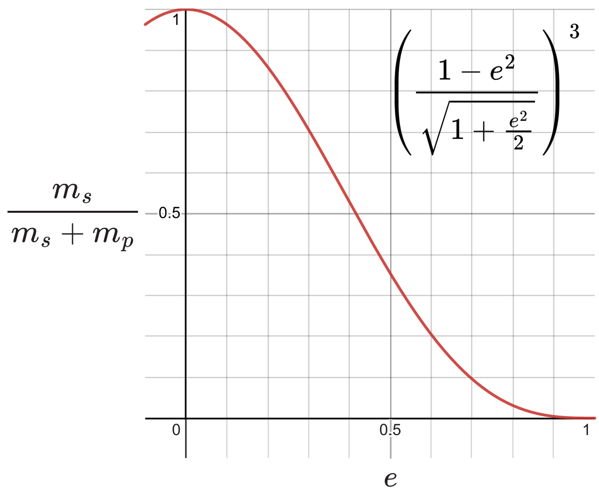
We have just obtained an equality that relates the eccentricity of the ellipse to a relationship between the mass at the focus and the mass of the planet. This is tremendously important because knowing the orbital data, we know the relationship between the masses of the two bodies. Furthermore, respecting the assumption of $m_s \geq m_p$ we have some limits for the relation between the masses and, therefore, for the eccentricity. If $m_s = m_p$, the two bodies will orbit in a circle, with the circle's center the two bodies' center of mass.

$$0.5 \leq \frac{m_s}{m_s+m_p} < 1 \quad \& \quad 0 < e \leq 0.415613$$

Substituting in $\frac{\mathrm{d}^2\vec{p}}{\mathrm{d}t^2} = -4\pi^2\frac{a^3}{T^2}\frac{m_s}{m_s+m_p}\frac{1}{r^2}\vec{u}_r$ the relation of the masses with the equality just obtained, we have:

$$\frac{\mathrm{d}^2\vec{p}}{\mathrm{d}t^2} = -4\pi^2\frac{a^3}{T^2}\left(\frac{1-e^2}{\sqrt{1+\frac{e^2}{2}}}\right)^3\frac{1}{r^2}\vec{u}_r = -\tau\frac{1}{r^2}\vec{u}_r \ ; \ \tau = 4\pi^2\frac{a^3}{T^2}\left(\frac{1-e^2}{\sqrt{1+\frac{e^2}{2}}}\right)^3$$

$$\frac{\mathrm{d}^2\vec{s}}{\mathrm{d}t^2} = -4\pi^2\frac{a^3}{T^2}\frac{m_p}{m_s+m_p}\frac{1}{r^2}\vec{u}_s = -\tau\frac{m_p}{m_s}\frac{1}{r^2}\vec{u}_s \ \left(\vec{u}_s = -\vec{u}_r\right)$$




Where $\tau$ is obtained only with the orbital information, and can be used to know the force that the sun is exerting on the planet at any moment of its orbit:

$$F = -\tau m_p \frac{1}{r^2}\vec{u}_r$$

Some clarifications regarding the force formula:

1. Newton's third law always holds.

   This means that regardless of whether we are studying the force that the sun exerts on the planet or vice versa, the formula's expression is always the same.

$$\text{In: } F = -\tau\, m_p \frac{1}{r^2}\vec{u}_r \quad \text{inequality } m_s \geq m_p \text{ always holds}$$

$$a_s = -\tau \frac{m_p}{m_s}\frac{1}{r^2}\vec{u}_r \quad ; \quad a_p = -\tau \frac{1}{r^2}\vec{u}_r$$

2. $\tau$ is dependent on the ratio of the masses. That is to say, if like the earth and mercury they have different masses, the eccentricity of their orbits around the Sun will be different and, therefore, $\tau$ does not have to be the same. Since there are more elements in the expression of $\tau$ that depend on mass, we could look at calculating the $\tau$ for mercury and for earth to make sure that they are not the same. However, it is not necessary because if it were, we would weigh the same on any planet, regardless of its mass.

Since $\tau$ is useless because it depends on mass, we need to find a way to extract from the expression what makes it dependent. To achieve this, we can use the technique of multiplying and dividing $\tau$ by the same amount to get the same result. If the result of dividing $\tau$ by that quantity is always the same regardless of the orbit, we will have found the gravitational constant necessary to have the gravitational field defined perfectly. My proposition is the following:

$$\tau = \frac{\tau}{\frac{m_p+m_s}{m_s}}\frac{m_p+m_s}{m_s} = \tau\frac{m_s}{m_p+m_s}\frac{m_p+m_s}{m_s} = \tau\left(\frac{1-e^2}{\sqrt{1+\frac{e^2}{2}}}\right)^3\frac{m_p+m_s}{m_s} = 4\pi^2\frac{a^3}{T^2}\left(\frac{1-e^2}{\sqrt{1+\frac{e^2}{2}}}\right)^6\frac{m_p+m_s}{m_s}$$

$$\text{defining } \kappa \text{ as: } 4\pi^2\frac{a^3}{T^2}\left(\frac{1-e^2}{\sqrt{1+\frac{e^2}{2}}}\right)^6 \implies \tau = \kappa\frac{m_p+m_s}{m_s} \qquad 1 < \frac{m_p+m_s}{m_s} \leq 2 \quad ; \quad m_p \leq m_s$$

$\kappa$ can still be obtained with the kinematic description of the orbit, so we could calculate it for all orbits and check if this value is the same in all of them. If it were, we would have a constant that is independent of the relationship between the masses and, furthermore, we would know that the gravitational field increases linearly with the sum of the masses. If it were not, the $\kappa$ strategy would not make sense, and it would be more appropriate to obtain $\tau$ values for different mass ratios according to the observations of the different orbits.

## 4 Conclusion

Newton made a mistake in his deduction that needed to be corrected. His error implied that the center of mass of the two-body system was accelerated, even though we considered the system isolated. Performing the deduction without making this mistake, we arrive at a universal gravitation law, where it can be seen that the accelerations of bodies depend on their mass.

The discrepancies between the expression of Newton's universal law of gravitation and the revised one are so small that we cannot detect them experimentally. Only astronomical observations allow discriminating between both observations.

Deductive error was also responsible for not finding Kepler's third planetary law constant. Which is essential to obtain the gravitational constant.

If this error had been corrected before the theory of general relativity was proposed, it would not have been proposed because it starts from the premise that the acceleration of falling bodies does not depend on their mass.

## 5 Warning

This article has been written expressly to show that the theory of general relativity is based on a wrong assumption. In Newton's deduction, there are more errors than those discussed here, however, the description of them is outside the purpose of the article.

The expression of the law of universal gravitation proposed in this article is incomplete, there are other kinematic aspects of the planets that must be observed to obtain the complete expression of universal gravitation. I am referring to acceleration due to the precision of orbits, this acceleration should be added to the expression exposed.

This article has been written expressly to show that the theory of general relativity is based on a wrong assumption. In Newton's deduction, there are more errors than those discussed here, however, the description of them is outside the purpose of the article.

The expression of the law of universal gravitation proposed in this article is incomplete, there are other kinematic aspects of the planets that must be observed to obtain the complete expression of universal gravitation. I am referring to acceleration due to the precession of orbits, this acceleration should be added to the expression exposed.

I suspended my analysis of Newton's Principia until I have time to continue it again, I found a multitude of errors not known during the 4 months of research. But my motivation to write this advanced report is to raise awareness of the need to change the scientific method.

My discoveries would not have been possible if I had not applied my new scientific method, which adds many restrictions to the way scientific knowledge is obtained. Neither the theory of relativity nor the quantum theory meets the requirements imposed by my scientific method.