

Some Deep Properties of the Green Function of Q . Guan on the line of Suita-Saitoh-Yamada's Conjectures

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Abstract: In this paper, we would like to refer to some deep results of the Green function of Q . Guan on the conjugate analytic Hardy H_2 norm and on the line of Oikawa-Sario's problems; Suita's conjecture, Saitoh's conjecture and Yamada's conjecture and to propose new related open problems. In particular, Q . Guan examined the deep properties of the inversion of the normal derivative of the Green function, the property of the level curve of the Green function and the magnitude of the logarithm capacity with his colleagues.

Key Words: Bergman kernel, Szegő kernel, Rudin kernel, Hardy reproducing kernel, weighted Szegő kernels, Dirichlet integral, weighted Bergman kernel, conjugate analytic Hardy H_2 norm, Green function, isoperimetric inequality, general theory of reproducing kernels, operator theory, positive definiteness, norm inequalities, Suita's conjecture, capacity, Saitoh's conjecture, Yamada's conjecture.

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1 Introduction

In order to state the contents in a self contained manner, first, we shall introduce the needed basic materials.

The Bergman kernel

Let D be a bounded regular domain on the complex $z = x + iy$ plane whose boundary is composed of a finite number N of disjoint analytic Jordan curves. Let $AL_2(D)$ be a Hilbert space (Bergman space) comprising analytic functions $f(z)$ on D and with finite norms $\|f\|_{AL_2(D)} = \left\{ \int_D |f(z)|^2 dx dy \right\}^{\frac{1}{2}}$. As we see simply, $f \rightarrow f(z)(z \in D)$ is a bounded linear functional on $AL_2(D)$. Therefore, there exists a reproducing kernel $K(z, u)$ such that for any $u \in D$ and for any function $f \in AL_2(D)$,

$$f(u) = \int \int_D f(z) \overline{K(z, u)} dx dy. \quad (1.1)$$

From $K(z, u) = \overline{K(u, z)}$, $K(z, u)$ is analytic in \bar{u} (complex conjugate). So, we shall denote it as $K(z, \bar{u})$. This is the Bergman kernel of D or on D .

Let $G(z, u)$ be the Green's function for the Laplace equation on D with pole at $u \in D$. Then, we have the identity

$$K(z, \bar{u}) = -\frac{2}{\pi} \frac{\partial^2 G(z, u)}{\partial z \partial \bar{u}}. \quad (1.2)$$

Here we shall introduce the adjoint L -kernel for $K(z, \bar{u})$ by

$$L(z, u) = -\frac{2}{\pi} \frac{\partial^2 G(z, u)}{\partial z \partial u}.$$

The adjoint L kernel $L(z, u)$ has one double pole at u on D such that

$$L(z, u) = \frac{1}{\pi(z-u)^2} + \text{regular terms}$$

of the second order. Furthermore, along the boundary ∂D we have the identity

$$\overline{K(z, \bar{u})} dz = -L(z, u) dz. \quad (1.3)$$

This important identity shows that the Bergman kernel is indeed an analytic differential $K(z, \bar{u}) dz$ and it is continued analytically to the double -

closed Riemann surface - of D , and at the symmetric point u of \bar{u} it has the same double pole as in the above. So, the Bergman kernel is a fundamental differential on the closed Riemann surface.

The Szegő kernel

Let $H_2(D)$ be the Hardy 2 analytic function space on D . A member of the class has nontangential boundary values belonging to $L_2(\partial D)$. Let $AL_2(\partial D)$ denote its closed subspace in $L_2(\partial D)$, and we introduce the norm in $H_2(D)$ by

$$\left\{ \int_{\partial D} |f(z)|^2 |dz| \right\}^{\frac{1}{2}} < \infty.$$

For a function $f \in AL_2(\partial D)$, as we see from the Cauchy integral formula, since $f \rightarrow f(u), u \in D$ is a bounded linear functional, there exists a reproducing kernel $\hat{K}(z, u)$ satisfying

$$f(u) = \int_{\partial D} f(z) \overline{\hat{K}(z, u)} |dz|.$$

As in the Bergman kernel, $\hat{K}(z, u)$ is analytic in \bar{u} , and so we shall denote it by $\hat{K}(z, \bar{u})$. This is the Szegő kernel of D or on D .

The important concept is the adjoint L kernel $\hat{L}(z, u)$ for $\hat{K}(z, \bar{u})$. That is, it is a meromorphic function on $D \cup \partial D$ and only at the point u , it has a simple pole

$$\hat{L}(z, u) = \frac{1}{2\pi(z - u)} + \text{regular terms}, \quad (1.4)$$

and along the boundary ∂D it satisfies the relation

$$\overline{\hat{K}(z, \bar{u})} |dz| = \frac{1}{i} \hat{L}(z, u) dz. \quad (1.5)$$

This adjoint L kernel is a uniquely determined by these properties on $D \cup \partial D$. From (1.5) we have the identity

$$\hat{L}(z, u) = -\hat{L}(u, z) \quad \text{on} \quad D \times D. \quad (1.6)$$

The adjoint L kernel $\hat{L}(z, u)$ is also characterized by the minimum property:

$$\min \int_{\partial D} |h(z, u)|^2 |dz| = \int_{\partial D} |\hat{L}(z, u)|^2 |dz|.$$

Here the minimum is considered on the meromorphic functions $\{h(z, u)\}$ on D with the singularity at u as in (1.4) and satisfying $\left\{\int_{\partial D} |h(z, u)|^2 |dz|\right\}^{\frac{1}{2}} < \infty$.

A deep and important property of the adjoint L kernel $\hat{L}(z, u)$ is that it does not have any zero point on $D \cup \partial D$. From this fact and (1.5),

$$f_0(z; u) = \frac{\hat{K}(z, \bar{u})}{\hat{L}(z, u)}$$

is the **Ahlfors function** of D with respect to u . That is, $f_0(z; u)$ is an analytic function on D , among the analytic functions satisfying the properties

$$f(u; u) = 0, f'(u; u) \geq 0 \quad \text{and} \quad |f(z; u)| \leq 1 \quad \text{on} \quad D,$$

it satisfies the extremal property $f'_0(u; u) = \max f'(u; u)$. - This may be related to the **analytic capacity**. Then, as we see from (1.5) and the principle of argument, the function $f_0(z; u)$ maps D onto the unit disc and the disc is covered N times. Here, we assume D is N -ply connected.

From (1.5) along ∂D we see

$$\overline{\hat{K}(z, \bar{u})^2 dz} = -\hat{L}(z, u)^2 dz. \quad (1.7)$$

From this relation, we see that the square of the Szegö kernel is an analytic differential, it is continued to the double of D analytically that is a closed Riemann surface as in the Bergman kernel. In this sense, the Szegö kernel is a half order differential on the closed Riemann surface and so its properties are very involved for the sake of multi-valuedness on the closed Riemann surface. The profound theory for these properties was done by D. A. Hejhal [22] and J. D. Fay [18] by using the **Riemann theta functions** and the **Klein prime forms**. As we see from (1.5), the Szegö kernel is a Cauchy kernel on the Riemann surface and so it is very important reproducing kernel.

The Hardy reproducing kernel

Let $H_2(D)$ ($\hat{H}_2(D)$, resp.) be the Hilbert space equipped the norm in the space $H_2(D)$:

$$\left\{ \frac{1}{2\pi} \int_{\partial D} |f(z)|^2 \frac{\partial G(z, t)}{\partial \nu_z} |dz| \right\}^{\frac{1}{2}} \quad (1.8)$$

$$\left(\left\{ \frac{1}{2\pi} \int_{\partial D} |f(z)|^2 \left(\frac{\partial G(z, t)}{\partial \nu_z} \right)^{-1} |dz| \right\}^{\frac{1}{2}}, \text{ resp.} \right). \quad (1.9)$$

Here, $\partial/\partial \nu_z$ is the inner normal derivative with respect to D . $\partial G(z, t)/\partial \nu_z$ is a positive continuous function on ∂D and so as in the Szegő space, we can consider the reproducing kernel $K_t(z, \bar{u})$ ($\hat{K}_t(z, \bar{u})$, resp.) satisfying

$$f(u) = \frac{1}{2\pi} \int_{\partial D} f(z) \overline{K_t(z, \bar{u})} \frac{\partial G(z, t)}{\partial \nu_z} |dz|,$$

$$\left(f(u) = \frac{1}{2\pi} \int_{\partial D} f(z) \overline{\hat{K}_t(z, \bar{u})} \left(\frac{\partial G(z, t)}{\partial \nu_z} \right)^{-1} |dz|, \text{ resp.} \right).$$

We call $K_t(z, \bar{u})$ and $\hat{K}_t(z, \bar{u})$ the Hardy reproducing kernel and its conjugate reproducing kernel of D .

For an arbitrary open Riemann surface S , we can introduce the space $H_2(S)$ similarly and the Hardy reproducing kernel. The reproducing kernel $K_t(z, \bar{u})$ is introduced by W. Rudin in 1955.

Let $G^*(z, t)$ be the conjugate harmonic function of $G(z, t)$ and when we form the multi-valued meromorphic function $W(z, t) = G(z, t) + iG^*(z, t)$, $idW(z, t)$ is a single-valued meromorphic differential, and it satisfies along ∂D ,

$$\frac{\partial G(z, t)}{\partial \nu_z} |dz| = idW(z, t).$$

Therefore, the integral (1.8) is represented by

$$\frac{1}{2\pi} \int_{\partial D} \frac{|f(z)dz|^2}{idW(z, t)}.$$

This will mean that $\hat{H}_2(D)$ is indeed comprised of analytic differentials $f(z)dz$. Therefore, the conjugate space may be considered on bordered Riemann surfaces with some good boundaries.

For $K_t(z, \bar{u})$ (resp. $\hat{K}_t(z, \bar{u})$), the adjoint L kernel $L_t(z, u)$ (resp. $\hat{L}_t(z, u)$), that is a meromorphic function on $D \cup \partial D$ with only one simple pole at u with residue 1, is characterized by the identity, along ∂D

$$\overline{K_t(z, \bar{u})} idW(z, t) = \frac{1}{i} L_t(z, u) dz \quad (1.10)$$

$$\left(\overline{\hat{K}_t(z, \bar{u})} dz = \frac{1}{i} \hat{L}_t(z, u) idW(z, t) \quad \text{along } \partial D, \text{ resp.} \right). \quad (1.11)$$

From (1.10) and (1.11), we have a very important property:

$$L_t(z, u) = -\hat{L}_t(u, z) \quad \text{on } D \times D. \quad (1.12)$$

The relations (1.10), (1.11) and (1.12) show that $L_t(z, u)dz$ is a meromorphic differential with respect to z , a meromorphic function in u , and along $z = u$ with one simple pole with residue 1; that is the Cauchy kernel of D . Therefore, these reproducing kernels are very important as in the Bergman and Szegö kernels as the third kind of reproducing kernels in one complex variable analysis ([17, 18]). These reproducing kernels were examined deeply on Riemann surfaces by J. D. Fay [4] and A. Yamada [21],[22] in terms of the Riemann theta functions and the Klein prime forms.

The norms of Bergman spaces are given by surface integrals on the domains and so, we can consider them on some very general domains as in on regular domains, meanwhile, for the Szegö spaces, their norms are given in terms of boundary integrals and so, the Szegö spaces may be considered for some very respective domains for their boundary integrals. In this viewpoint, it will be very interested in the H_2 spaces, because the norms are given by boundary integrals and however, for any open domains we can consider them.

For the conjugate Hardy norm

$$\frac{1}{2\pi} \int_{\partial D} |f(z)|^2 \left(\frac{\partial G(z, t)}{\partial \nu_z} \right)^{-1} |dz| = \frac{1}{2} \int_{\partial D} \frac{|f(z)dz|^2}{idW(z, t)},$$

we obtain the best possible norm inequality:

$$\int \int_D |f'(z)|^2 dx dy \leq \frac{1}{2} \int_{\partial D} \frac{|f'(z)dz|^2}{idW(z, t)}, \quad (1.13)$$

and from this we see the naturality of the norm looked curiously. This inequality is not so simple to derive and for its proof we must examine deeply the relations among the Hardy reproducing kernel, its conjugate kernel and the Bergman kernel ([16]).

For the Bergman kernel and the Szegö kernel on D , we have the basic and deep relation

$$K(z, \bar{u}) \gg 4\pi \hat{K}(z, \bar{u})^2$$

– the left minus the right is a positive definite quadratic form function – which was given by D. A. Hejhal [12]. This profound result was given on the long historical lines as in

G.F.B. Riemann (1826-1866); F. Klein (1849-1925); S. Bergman; G. Szegő; Z. Nehari; M.M. Schiffer; P.R. Garabedian (1949 published); D.A. Hejhal (1972 published).

It seems that any elementary proof is impossible, however, the result will, in particular, mean the fairly simple inequality:

For two functions φ and ψ of $H_2(D)$, we obtain the generalized isoperimetric inequality

$$\frac{1}{\pi} \int \int_D |\varphi(z)\psi(z)|^2 dx dy \leq \frac{1}{2\pi} \int_{\partial D} |\varphi(z)|^2 |dz| \frac{1}{2\pi} \int_{\partial D} |\psi(z)|^2 |dz|, \quad (1.14)$$

and we can determine completely the cases holding the equality here. In the thesis [15] of the author published in 1979 the result was given. The author realized the importance of the abstract and general theory of reproducing kernels by N. Aronszajn. In the paper, the core part was to determine the equality statement in the above inequality, surprisingly enough, some deep and general independent proof was appeared 26 years later in A. Yamada (see the appendix of [19]). A. Yamada was developed deeply equality problems for some general norm inequalities derived by the theory of reproducing kernels and it was published in the book appendix of [19]. Very recently his theory is developing much more in [23].

Of course, in the thesis we can find some fundamental idea for nonlinear transforms. In particular, for the special case $\varphi \equiv \psi \equiv 1$, for the plane measure $m(D)$ of D and the length ℓ of the boundary we have the isoperimetric inequality

$$4\pi m(D) \leq \ell^2.$$

Meanwhile, for the Hardy reproducing kernel we can give the conjecture in the converse direction, from the case of doubly connected domains

$$\pi K(z, \bar{u}) \ll K_t(z, \bar{u}) \hat{K}_t(z, \bar{u}) \quad (1.15)$$

([15]). From the relation we can derive various results, the conjecture will be, however, very difficult to solve.

The author attacked to this problem several years, it was, however, impossible to solve it and in those days, the author derived the above result

applying the result of Hejhal conversely to the general theory of reproducing kernels by Aronszajn. The author thinks for the conjecture we must use the deep theory of J. D. Fay, however, his profound theory seems to be too deep and great for the author. So, from the viewpoints of fundamental results and applicable analysis, the author turned his research interest for the general theory of reproducing kernels.

2 Basic open problems

For the space $H_2(D)$, by exchanging

$$\frac{1}{2\pi} \frac{\partial G(z, t)}{\partial \nu_z}$$

with a general positive continuous function ρ , we consider the weighed Szegő space $H_2^\rho(D)$ and the corresponding the weighted Szegő kernel $K_\rho(z, \bar{u})$ and the conjugate kernel $L_\rho(z, u)$; that is, for the space

$$\left\{ \int_{\partial D} |f(z)|^2 \rho(z) |dz| \right\}^{\frac{1}{2}} < \infty$$

with the weighted norm, we consider the reproducing kernel $K_\rho(z, \bar{u})$ satisfying

$$f(u) = \int_{\partial D} f(z) \overline{K_\rho(z, \bar{u})} \rho(z) |dz|.$$

and

$$\overline{K_\rho(z, \bar{u})} \rho(z) |dz| = \frac{1}{i} L_\rho(z, u) dz, \quad (2.1)$$

along ∂D . In particular, note that $L_\rho(z, u)$ has a similar singularity at $z = u$ as $\hat{L}(z, u)$.

For the weight ρ^{-1} , we see that the very important relations as in the Hardy case

$$L_\rho(z, u) = -L_{\rho^{-1}}(u, z)$$

and

$$\overline{K_\rho(z, \bar{u})} K_{\rho^{-1}}(z, \bar{u}) dz = -L_\rho(z, u) L_{\rho^{-1}}(u, z) dz. \quad (2.2)$$

From the boundary relations (2.1) and (2.2) on kernels, for any simply connected regular domain we have the identity

$$4\pi K(z, \bar{u}) \equiv K_\rho(z, \bar{u})K_{\rho-1}(z, \bar{u}). \quad (2.3)$$

We do not know whether the case (2.3) happens for $N > 1$ for some ρ . Note, in particular, that for this case we have the similar result in generalizations of the inequality (1.14) for the weighted Szegő norms.

Now, we can state the basic open problems:

Basic big open problems:

(A) Look for the condition of ρ such that

$$4\pi K(z, \bar{u}) \gg K_\rho(z, \bar{u})K_{\rho-1}(z, \bar{u}) \quad (2.4)$$

is valid.

(B) Look for the condition of ρ such that

$$4\pi K(z, \bar{u}) \ll K_\rho(z, \bar{u})K_{\rho-1}(z, \bar{u}) \quad (2.5)$$

is valid.

By the deep result of D. A Hejhal and our example, this open problems are valid, in general, as very difficult problems. We can state the partial open problems as in

Partial big open problems:

(A') Look for the condition of ρ such that

$$4\pi K(z, \bar{z}) > K_\rho(z, \bar{z})K_{\rho-1}(z, \bar{z}) \quad (2.6)$$

is valid.

(B') Look for the condition of ρ such that

$$4\pi K(z, \bar{z}) < K_\rho(z, \bar{z})K_{\rho-1}(z, \bar{z}) \quad (2.7)$$

is valid.

For doubly connected domains $N = 2$, since the differentials

$$4\pi K(z, \bar{u})dz\bar{d}u - K_\rho(z, \bar{u})K_{\rho-1}(z, \bar{u})dz\bar{d}u$$

is one dimensional first order differential that is complex Hermitian form of $dz\bar{d}u$, the above basic big open problems and partial big open problems are the same problems.

We do not know whether they are always the same problems or not.

3 Q. Guan's result

Surprisingly enough, since 40 years later after publication of [15], for some open question proposed there, an entirely unexpected partial solution was published in [6] that is an entirely new result.

For the conjecture (1.15), he proved that

$$\pi K(z, \bar{z}) \leq \hat{K}_z(z, \bar{z}) \quad (3.1)$$

and surprisingly enough, he completely determined the equality problem. Here note the important fact

$$K_t(z, \bar{t}) \equiv 1.$$

His interest is on the long and large topics on the Oikawa-Sario's problems and the Suita conjecture that determine the magnitudes of many conformal invariant quantities.

We recall the following solution of the famous conjecture posed by Suita [20].

Theorem: ([5]). *Let $c_\beta(z_0) = \lim_{z \rightarrow z_0} \exp(G(z, z_0) - \log |z - z_0|)$. Then $(c_\beta(z_0))^2 \leq \pi K(z_0, \bar{z}_0)$ and $(c_\beta(z_0))^2 = \pi K(z_0, \bar{z}_0)$ holds for some $z_0 \in D$ if and only if D conformally equivalent to the unit disc, i.e. $N = 1$.*

Note that

$$(c_\beta(z_0))^2 \leq \pi K(z_0, \bar{z}_0)$$

that was proved by Blocki in [3] for planar domains D . We can see the great impact of the Suita conjecture from many internet sites like Wikipedia; Suita conjecture, however, we see its frontier information in [14] and [21]. For a complete version of Yamada, see [22]. It seems that the introduction of the Suita conjecture by Ohsawa to several complex variables mathematicians was a big great monument as we see from its great impact.

Then, Guan derived the inequalities

$$(c_\beta(z_0))^2 \leq \pi K(z_0, \bar{z}_0) \leq \hat{K}_{z_0}(z_0, \bar{z}_0)$$

that means that the values $(c_\beta(z_0))^2$ and $\hat{K}_{z_0}(z_0, \bar{z}_0)$ are the extremals for the value $K(z_0, \bar{z}_0)$ and he solved completely the equality problems for these inequalities by an elementary means.

Surprisingly enough, he derived there the following identity:

For any fixed $t \in D$ and for any fixed analytic function f on D which is continuous on $D \cup \partial D$, the identity

$$\begin{aligned} & \lim_{r \rightarrow 1^-} \frac{1}{1-r} \int \int_{\{e^{-2G(z,t)} \geq r\}} |f(z)|^2 dx dy & (3.2) \\ & = \frac{1}{2} \int_{\partial D} |f(z)|^2 (\partial G(z,t) / \partial \nu)^{-1} |dz| \end{aligned}$$

holds.

Main Result

The following inequalities seem to be interesting on its own sense:

Theorem: For any given $\epsilon > 0$ and for any fixed analytic function $f(z)$ on $D \cup \partial D$, there exists $r : (0 < r < 1)$ satisfying the inequality

$$\int \int_D |f'(z)|^2 dx dy - \epsilon \leq \frac{1}{1-r} \int \int_{\{e^{-2G(z,t)} \geq r\}} |f'(z)|^2 dx dy.$$

This inequality may be looked as an isoperimetric inequality, because the Dirichlet integral on a domain is estimated (restricted) by the Dirichlet integral on some small boundary neighborhood of the domain. Here, the neighborhood size and estimation are stated by the level curve of the Green function, precisely.

Even the case of the identity function $f(z) = z$, we can enjoy the senses of the estimation and the result.

Out Line of Proof of the Main Result

The integral of the conjugate analytic Hardy $H^2(D)$ norm seems to be not popular, however, the norm will have a beautiful structure and the norm is conformally invariant. Furthermore, we obtain the inequality:

For any analytic function $f(z)$ on $D \cup \partial D$, we have the inequality

$$\int \int_D |f'(z)|^2 dx dy \leq \frac{1}{2} \int_{\partial D} |f'(z)|^2 (\partial G(z, t) / \partial \nu)^{-1} |dz|. \quad (3.3)$$

This result was derived from some complicated theory of reproducing kernels in ([16]) on some comparison of the magnitudes of the Bergman and the Rudin (Hardy H^2) reproducing kernels. The equality problem in the inequality was also established; that is, equality holds if and only if the domain is simply-connected and the function $f'(z)$ is expressible in the form $CK(z, \bar{t})$ for the Bergman reproducing kernel $K(z, \bar{u})$ on the domain D and for a constant C . Its source was given by [15] and then the topics was cited in the book [17] in detail.

We can obtain the main result by combining of the Guan identity and this inequality.

Remarks

We note the following interesting problems:

Problem 1 : *How will be some generalization of the Guan identity for a general weight for $(\partial G(z, t) / \partial \nu)^{-1}$?*

Problem 2 : *We can consider similar inequalities for various function spaces. For example, how will be the case for the harmonic Dirichlet integrals?*

Problem 3 : *The structure and proof of the inequality (3.2) are very complicated (involved) and the Guan identity is very unique. So, we are interested in some direct proof of the theorem.*

Problem 4 : *The theorem seems to be valid for a general domain D and for general analytic functions with finite Dirichlet integrals on D apart from the proof in this paper. However, in the theorem r is depending on the function f and therefore the generalization of the theorem is not simple.*

For the inequality (3.2), note the inequality

$$\begin{aligned} \left(\frac{1}{\pi} \int \int_D |f'(z)|^2 dx dy \right)^2 &= \left(\frac{1}{2\pi i} \int_{\partial D} \overline{f(z)} f'(z) dz \right)^2 \\ &\leq \frac{1}{2\pi} \int_{\partial D} |f(z)|^2 \frac{\partial G(z, t)}{\partial \nu} |dz| \frac{1}{2\pi} \int_{\partial D} |f'(z)|^2 (\partial G(z, t) / \partial \nu)^{-1} |dz|. \end{aligned}$$

Note that the relation of the Bergman norm and the weighted H^2 norm is very delicate. See [15, 16, 17].

4 Akira Yamada's results

A. Yamada gave the various interpretations for the deep result by D. A. Hejhal ([12]) from the viewpoint of the operator theory. In particular, he introduces the operator that corresponds the tensor product of two Szegő spaces to the Bergman space by the restriction to the diagonal set of $D \times D$ and its adjoint operator. He examines the integral operator realizations of the operators and their interesting properties in detail.

5 Addition with the new paper [9] by Q. Guan and Z. Yuan

The result of Q. Guan was very surprised from the line of D. A. Hejhal and J. D. Fay based on the representation (3.2). On this line Q. Guan and Z. Yuan got very deep and surprising results [9]. This direction was suggested by the conjecture of A. Yamada [21] that is a generalization of the conjecture of Suita:

Let u be a harmonic function on D , and let $\rho = e^{2u}$. For the weighted Bergman kernel $K_\rho^B(z, \bar{u})$

Yamada Conjecture:

$$(c_\beta(z))^2 \leq \pi \rho(z) K_\rho^B(z, \bar{z})$$

In [5], Guan-Zhou proved the Yamada conjecture and more general weighted versions were developed surprisingly and deeply in [7] and [8].

Following this line, Q. Guan and Z. Yuan is developing the corresponding result with the weighted Hardy H_2 reproducing kernels and their results are too deep and complicated to state them simply. A typical case, however, may be stated as follows as a definite result:

Let λ be any positive continuous function on ∂D . By solving the Dirichlet problem, there exists the continuous harmonic function u on the closed domain of D satisfying that $u = -\frac{1}{2} \log \lambda$.

Then, they obtain surprisingly:

$$K_{\lambda(\frac{\partial G(\cdot, z)}{\partial \nu})^{-1}}(z, \bar{z}) \geq K_{e^{-2u}}^B(z, \bar{z}).$$

In particular, they are developing a detail and deep analysis for the identity (3.2). However, not identities, but estimates.

Q. Guan and Z. Yuan gave a very delicate and profound property of

$$\frac{\partial G(z, t)}{\partial \nu}$$

for a very general situation for the Green function from the view point of the identity (3.2). For its complicated structure, it seems that the result is at this moment abstract in nature. However, as a typical case they were able to get the result for the general weighted Hardy H_2 norms.

This line was not an expected direction, however, we see that the results show more wide field for our mathematics. For example, how will be the values

$$K_{\left(\lambda\left(\frac{\partial G(\cdot, z)}{\partial \nu}\right)^{-1}\right)^{-1}}(z, \bar{z})$$

and

$$K_{\lambda\left(\frac{\partial G(\cdot, z)}{\partial \nu}\right)^{-1}}(z, \bar{z})K_{\left(\lambda\left(\frac{\partial G(\cdot, z)}{\partial \nu}\right)^{-1}\right)^{-1}}(z, \bar{z})$$

?

6 Moore and surprising addition with the new paper [10] by Q. Guan and Z. Yuan

They furthermore consider the complete version for the product space $D \times D \times D \times \cdots \times D \subset C^m$; indeed, they surprisingly formulate the complete version and complete results containing the equalities problems for the related estimates with the great paper.

See also [11] for a weighted version of Suita conjecture for higher derivatives.

In addition, S. R. Bell and B. Gustafsson [1] is discussing on the Hejhal's result and the Suita conjecture from a very interesting viewpoint.

7 Further open problems

The norm inequality (1.14) will not be proved (derived) by any elementary method without the theory of reproducing kernels, however, the inequality

$$\frac{1}{\pi} \int \int_D |\varphi(z)|^4 dx dy \leq \left(\frac{1}{2\pi} \int_{\partial D} |\varphi(z)|^2 |dz| \right)^2 \quad (7.1)$$

(or for (1.14) with $\psi(z) \equiv 1$) and the norm inequality (1.3) may be derived by means of elementary for their great power of the above results with further works ([9]), ([7]), ([8]), ([5]) and ([11]).

How will be these open problems?

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