CONNECTIONS BETWEEN THE PLASTIC CONSTANT, THE CIRCLE AND THE CUSPIDAL CUBIC

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Abstract: *The unit circle and the cuspidal cubic curve have been found to intersect at coordinates that can be defined by the Plastic constant (ρ), which is defined as the solution to the cubic function $x^3 = x + 1$. This report explores the connections between the algebraic properties of the Plastic constant and the geometric properties of the circle and this curve.*
Keywords: *Plastic; constant; circles; cuspidal; cubic*

1. INTRODUCTION

In this report, we will explore the connections between the algebraic properties of the Plastic constant and the geometric properties of the circle and the curve. The Plastic constant (ρ), originally studied in 1924 by Gérard Cordonnier, is defined as the unique real number that is the solution to the cubic function $x^3 = x + 1$ and is given by

$$\frac{\left(9-\sqrt{69}\right)^{\frac{1}{3}}+\left(9+\sqrt{69}\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}\cdot 3^{\frac{2}{3}}} = \left(\left(\frac{9-\sqrt{69}}{18}\right)^{\frac{1}{3}} + \left(\frac{9+\sqrt{69}}{18}\right)^{\frac{1}{3}}\right) = 1.32471795724...$$

It has been discovered that the circle $x^2 + y^2 = 1$ and the cubic curve $x^3 = y^2$ intersect at the coordinates $(\rho^{-1}, \pm\rho^{-\frac{3}{2}})$; this will be the topic for conversation and we will look at expanding on how these two figures are linked by ρ. Being a real number, the fact that ρ arises in both x and y (Cartesian) coordinates of the intersection points of the circle and the curve is interesting and may suggest a deep connection between the algebraic properties of ρ and the geometric properties of circles and cubic curves in general.

While there are already some interesting known connections between elliptic curves and circles, they are very different mathematical constructs with distinct characteristics that set them apart from one another: a circle is a conic section with a single center point and a constant radius, while an elliptic curve is a smooth curve that can take many different forms.

The circle and the the semi-cubical parabola are two fundamental geometric forms and the intersection between them is studied in this paper to gain a better understanding of the connections between the algebraic properties of the Plastic constant and the geometric properties of circles and the cubic curve, starting with the intersection $x^3 = x^2 | x^2 + y^2 = 1$.

2. LITERATURE REVIEW AND PRETEXT

While it is known that the Plastic constant does have connections to geometry and topology, it is mostly studied in the fields of complex dynamics [1], combinatorics [2 – 3], and arithmetic. Its connection to 2D geometric shapes, such as the unit circle and the cubic curve, is still a relatively new area of research. That said, there are some wonderful connections that have been shown between the Plastic constant and number theory, including how $\rho + 1 = \rho^3$ leads to the nested radical identity $\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1...}}}}}}}$ and how this relates to a 2D visual representation called the Padovan triangles. The Plastic constant is the limiting ratio of the successive terms of the Padovan sequence or Perrin sequence [4] which can be visualized by the side lengths of equilateral triangles in a fractal-like spiral [5]. The Padovan cuboid spiral is the spiral created by joining the diagonals of faces of successive cuboids added to a unit cube [6].

Dişkaya and Menken's [7] paper 'Some Properties of the Plastic constant' shows that Padovan numbers can also be obtained in spiral form with quadrangles, and this research goes on to show ρ's relationship with upright prisms, cylinders, cones, pyramids and spheres through connections with the Golden Ratio. While this and other works do touch on the fact that ρ satisfies the cubic equation $\rho^3 - \rho - 1 = 0$, they do not specifically focus on the intersection between circles and the cubic curve.

The Plastic constant is known to satisfy certain algebraic identities, specifically $P - 1 = P^{-4}$ and $P + 1 = P^3$. This places it in a unique class of numbers that can be represented by equations of the form $x + 1 = x^m$ and $x - 1 = x^{-n}$), where m and n are natural numbers. According to a study by Aarts et. al [8], the Plastic constant and the golden ratio ϕ are the only known numbers that possess this property.

The Plastic constant is known for its role in chaos theory and nonlinear dynamical systems and plays a key role as a parameter in the study of bifurcations of these systems, which describe how the system's behavior changes with parameter variations [9]. This makes the plastic constant an important concept in the field of nonlinear dynamics, and thus it is worth noting in the context of being a solution to certain polynomial equations as this is conceptually important in the field of chaos theory. In particular, it is related to the behavior of certain nonlinear systems known as logistic maps, which are used to model a wide range of phenomena, from population growth [10] to the fundamentals of electronics [11].

The expression $x^3 = y^2$ differs from the cubic formula as we may be used to seeing it, $y = ax^2 + bx + c$. This form of elliptic curve is important in number theory and cryptography [12 – 13]. The equation $x^3 - y^2 = 0$ defines a curve known as a cuspidal cubic or Neile's Semi-cubical Parabola [14]. Cuspidal cubics are a special type of algebraic curve, which means that it can be defined by polynomial equations [15]. In 1687, the Dutch mathematician Christiaan Huygens who patented the first pendulum clock, was able to show that the semi-cubical parabola is connected with physical science, that it is a solution of a particular physical problem, the motion of a particle under gravity [14].

A circle is a specific type of curve defined as the set of all points that are a fixed distance, called the radius, away from a central point. The equation for a circle with center (h, k) and radius r is: $(x-h)^2+(y-k)^2 = r^2$ [16]. Circles have a smooth curve that does not have any sharp turns or breaks. On the other hand, the cuspidal cubic does not have any central point. Cuspidal cubics are a type of algebraic curve that have a unique feature called a cusp. This is a singularity point on the graph of the curve where the curve has a sharp transition. At this point, the only tangent line that can be drawn is the one that is parallel to the Y-axis, Y=0 [15].

The fact that the intersection points of the circle and this cubic curve can be defined in terms of ρ suggests that the properties of ρ are somehow "encoded" in the geometric properties of the circle and cubic equations, and studying the intersection points could lead to a deeper understanding of this constant. Further to this, since any circle can be related to any other circle through a transformation such as scaling, rotation, or translation, and similarly, curves can also be transformed by changing the coefficients of the equation, or by applying transformations such as scaling, rotation or translation [17 – 18], it might be suggested that we can use the properties of the unit circle and the cubic curve, such as the plastic constant, to understand and solve problems involving circles and curves in general.

All these points together, can make the intersection of these shapes an interesting subject to study in mathematics and can give researchers more insight about algebraic and geometric properties of shapes. Given the Plastic constant's significance in the field of physics, it is anticipated that research in this area will continue to expand beyond the realm of mathematics.

3. DERIVATION AND ANALYSIS

The intersection in question thus far was discovered in an investigation into how the Fundamental Theorem of Calculus can be used with a regression in the Desmos Graphing Calculator [19] to find the Cartesian coordinates of the end-point of an arc length measured along smooth well-behaved curves, and how when calculated, these coordinates might be backed up using trigonometry and Pythagoras' theorem. Finding the intersection was a bit of a happy accident, however the methods used could not be considered rigorous and it then became important to the author to present a direct algebraic proof, for the reasons given in the previous sections.

To show that the circle $x^2 + y^2 = 1$ and the curve $x^3 = y^2$ intersect at the coordinates $(\rho^{-1}, \rho^{-3/2})$, we can substitute $x = \rho^{-1}$ into the equation of the circle and the equation of the cubic curve and see if they are satisfied simultaneously. We will use the letter P to denote the suggested factor of this intersection point instead of ρ until we have proved that the Plastic constant is a real solution. Since both the unit circle and the curve $x^3 = y^2$ are symmetrical about the x axis, proving this will logically prove that $(\rho^{-1}, -\rho^{-3/2})$ is also an intersection point. This author is not best-placed to discuss any complex solutions that may exist.

The equation of the circle with $x = P^{-1}$ is:

$$(P^{-1})^2 + y^2 = 1$$
$$y^2 = 1 - (P^{-1})^2$$

The equation of the cubic curve with $x = P^{-1}$ is:

$$(P^{-1})^3 = y^2$$
$$y^2 = P^{-3}$$

Now we can substitute the value of y^2 in the equation of the circle:

$P^{-3} = 1 - (P^{-1})^2$

$P^{-3} + (P^{-1})^2 = 1$

This is the equation we need to find true for values of P being equal to ρ. Before we do so, we can now rearrange the above equation and get

$(P^{-1})^2 = 1 - P^{-3}$

We then substitute this in the equation of the cubic curve:

$y^2 = (1 - P^{-3})$

Taking the square root of both sides we find that

$y = P^{-3/2}$

So, the coordinates of the point of intersection are $(P^{-1}, P^{-3/2})$.

To prove that when $P = \rho$, $p^{-3} + (p^{-1})^2 = 1$, we will substitute the value of ρ into the equation $P^{-3} + (P^{-1})^2 = 1$. We know that the value of ρ is

$$\frac{\left(9-\sqrt{69}\right)^{\frac{1}{3}}+\left(9+\sqrt{69}\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}\cdot 3^{\frac{2}{3}}} = \left(\left(\frac{9-\sqrt{69}}{18}\right)^{\frac{1}{3}} + \left(\frac{9+\sqrt{69}}{18}\right)^{\frac{1}{3}}\right) = 1.32471795724...$$

Plugging this value in to the equation $P^{-3} + (P^{-1})^2 = 1$ and evaluating it, we find the equation holds true, which confirms that the coordinates of the point of intersection are $(\rho^{-1}, \rho^{-3/2})$.

Looking for circles of the form $x^2+y^2 = \rho^A$ that intersect with the curve $x^3 = y^2$ at points (ρ^B, ρ^C) where A, B and C are integers, again using the graphing methods mentioned at the top of this section, it was first constant that $x^2 + y^2 = p^5$ appears to intersect $x^3 = y^2$ at coordinates $(p^1, \pm p^{3/2})$ and then that $x^2 + y^2 = p^{13}$ appears to intersect $x^3 = y^2$ at coordinates $(p^4, \pm p^6)$. Running through the process of proof employed above we'll look to prove these intersections:

To prove that the circle $x^2 + y^2 = p^5$ and the cubic curve $x^3 = y^2$ intersect at the coordinates $(\rho^1, \rho^{\frac{3}{2}})$, we can use similar algebraic techniques as the ones used to prove that the circle $x^2+y^2 = 1$ and the cubic curve $x^3 = y^2$ intersect at the coordinates $(\rho^{-1}, \pm\rho^{-\frac{3}{2}})$. Again we will use P as the suggested number until we can prove P=ρ.

First we substitute the coordinates into both equations and show that they are true.

Substituting $(P^1, P^{\frac{3}{2}})$ into the equation for the circle, we get:
$(P^1)^2 + (P^{\frac{3}{2}})^2 = P^5$

Simplifying this equation, we get:
$P^2 + P^3 = P^5$

Substituting $(P^1, P^{\frac{3}{2}})$ into the equation for the cubic curve, we get:
$(P^1)^3 = (P^{\frac{3}{2}})^2$

Simplifying this equation, we get:
$P^3 = P^3$

Since both equations $P^2 + P^3 = P^5$ and $P^3 = P^3$ are true when P is substituted for an evaluation of ρ, we can conclude that the circle $x^2 + y^2 = \rho^5$ and the cubic curve $x^3 = y^2$ intersect at the coordinates $(\rho^1, \rho^{\frac{3}{2}})$. Since the circle is on the origin point $(0, 0)$ and the curve is symmetrical about the x axis, we again can easily deduce that $(\rho^1, -\rho^{\frac{3}{2}})$ is also an intersection.

For the circle $x^2 + y^2 = \rho^{13}$, the two equations that we need to be true are $(P^4)^2 + (P^6)^2 = P^8 + P^{12} = P^{13}$ and $P^{12} = P^{12}$, which hold true when P$= \rho$.

The equations $x^2 + y^2 = p^{13}$ and $x^3 = y^2$ are both algebraic equations, and in general, such equations can have multiple solutions. Furthermore, since the exponents are integers, it is possible that there are infinitely many solutions which can be expressed in terms of integers powers of the variables.

The rule that this last intersection adheres to is quite simple: assuming x and y are real and positive, $x^y - x^{(y-1)} - x^{(y-5)} = 0$ can be reduced to $x^3 - x - 1 = 0$, the cubic function, and has a unique real solution of ρ.

4. CONCLUSION

In conclusion, this paper has explored the connections between the algebraic properties of the Plastic constant and the geometric properties of circles and cubic equations. Through the examination of the intersection between the unit circle $x^2 + y^2 = 1$ and the cubic curve $x^3 = y^2$, it was discovered that these two figures intersect at the coordinates $(\rho^{-1}, \pm\rho^{-\frac{3}{2}})$.

Looking for circles of the form $x^2 + y^2 = \rho^A$ that intersect with the curve

$x^3 = y^2$ at points (ρ^B, ρ^C) where A, B and C are integers, we find that $x^2 + y^2 = p^{13}$ intersects $x^3 = y^2$ at coordinates $(p^4, \pm p^6)$ and that this was due to how for real and positive values of x and y, $x^y - x^{(y-1)} - x^{(y-5)} = 0$ can be reduced to the the cubic function.

It is hoped that the findings of this research contribute to the understanding of the Plastic constant and its connections to geometry and topology, and that it adds further suggestion that the Plastic constant is not just a number that arises in complex dynamics, combinatorics, and arithmetic but it is also an important number in the geometric world.

Overall, this paper has demonstrated that the Plastic constant is a fascinating number with deep connections to both algebraic and geometric concepts.

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