Matrix Forms of Normal Linear Differential Equation Systems with Constant Coefficients

"Ageometretos medeis eisito"

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January 2023

Abstract

When the English physicist Isaac Newton and German mathematician Gottfried Leibniz constructed the differential equations, mathematics itself was reborn. After Leibniz and Newton there were still a lot of mathematicians trying to develop Calculus more and more. And here we are, still trying to develop mathematics itself. Even to this day we still try to construct a better solution for every field and subject of mathematics.

This paper will try to explain the fundamentals of matrix solutions for linear differential equation systems. Differential equation systems are commonly used, simplified, constructed, and notated for every field of science, engineering, and applications of technology.

1 Introduction

The strategy of this paper is to cover the subject that is "Linear Differential Equation Systems with Constant Coefficients" first, with proofs in an understandable manner. Thus the interested reader will be able to understand the main subject and use it as a reference, where other introductory papers mostly skip the elementary subject itself. After this we will move on to the main subject that is "Matrix Solutions for Normal Differential Equation Systems with Constant Coefficients", and we will focus more on the main subject itself. Therefore we will follow the same approach in both subjects for proofs, interpretations, and applications.

2 Linear Differential Equation Systems with Constant Coefficients

Linear system is a system that is linear with respect to unknown functions and their derivatives. In general terms, linear systems with constant coefficients are nothing but unknown functions and their derivatives with constant coefficients contained by the every single one of the equation in the system. This kind of a system is generally shown like this,

$$\begin{aligned} P_{11}(\mathscr{D})\psi_1 + P_{12}(\mathscr{D})\psi_2 + \ldots + P_{1n}(\mathscr{D})\psi_n &= f_1(x) \\ P_{21}(\mathscr{D})\psi_1 + P_{22}(\mathscr{D})\psi_2 + \ldots + P_{2n}(\mathscr{D})\psi_n &= f_2(x) \\ &\ldots \\ P_{n1}(\mathscr{D})\psi_1 + P_{n2}(\mathscr{D})\psi_2 + \ldots + P_{nn}(\mathscr{D})\psi_n &= f_n(x) \end{aligned} \tag{1}$$

In the equations $P_{ij}(\mathscr{D})$ is the function of the derivation operator $\mathscr{D} = \frac{d}{dx}$, Equations inside the system are linear but they are not supposed to be first order. As an example,

$$\begin{aligned} 2(\mathscr{D}-2)\psi_1 + (\mathscr{D}-1)\psi_2 &= e^x \\ (\mathscr{D}+3)\psi_1 + \psi_2 &= 0 \\ &\text{and} \\ (\mathscr{D}^2-2)\psi_1 - 3\psi_2 &= e^{2x} \\ \psi_1 + (\mathscr{D}^2+2)\psi_2 &= 0 \end{aligned}$$

Equations above are linear equation systems with constant coefficients.

We're not gonna do the proof but, take a look at the theorem below.

2.1 Theorem

$$\Delta = \begin{bmatrix} P_{11}(\mathscr{D}) & P_{12}(\mathscr{D}) & \ldots & P_{1n}(\mathscr{D}) \\ P_{21}(\mathscr{D}) & P_{22}(\mathscr{D}) & \ldots & P_{2n}(\mathscr{D}) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ P_{n1}(\mathscr{D}) & P_{n2}(\mathscr{D}) & \ldots & P_{nn}(\mathscr{D}) \end{bmatrix}$$

,

In the general solution for this system, we can get a arbitrary constant that is equal to the highest degree of $\mathscr{D}$'s that we can find by the expansion of the determinant.

Here we're gonna say that $\Delta \neq 0$. In the case where $\Delta = 0$, every equation in the system are connected and we are not going to cover this situation. As an example,

$$\Delta = \begin{bmatrix} 2(\mathscr{D}-1) & \mathscr{D}-1 \\ \mathscr{D}-3 & 1 \end{bmatrix} = -\mathscr{D}^2 + 1$$

can be constructed because,

$$\begin{aligned} &(2(\mathscr{D}-1)\cdot 1) - ((\mathscr{D}-1)\cdot(\mathscr{D}+3)) \\ &= 2\mathscr{D} + (-2) + (-\mathscr{D}^2) + (-2\mathscr{D}) + 3 \\ &= (-\mathscr{D}^2) + (2\mathscr{D} + (-2\mathscr{D})) + (-2+3) \\ &= -\mathscr{D}^2 + 1 \end{aligned}$$

Because of this we can see that the degree of this system is second degree with respect to the $\mathscr{D}$. In this situation we're going to find two arbitrary constants.

Even the solutions for linear equation systems with constant coefficients are constructed by derivation and elimination rule. However, in this case it is beneficial to derive operators. Take a look at the example below.

2.2 Example

$$\begin{aligned} 2(\mathscr{D}-2)\psi_1 + (\mathscr{D}-1)\psi_2 &= e^x \\ (\mathscr{D}+3)\psi_1 + \psi_2 &= 0 \end{aligned}$$

There will be three different solutions for this system. First of all we can write this system in this way,

$$\begin{aligned} 2\frac{d\psi_1}{dx} + \frac{d\psi_2}{dx} - 4\psi_1 - \psi_2 &= e^x \\ \frac{d\psi_1}{dx} + 3\psi_1 + \psi_2 &= 0 \end{aligned}$$

And if we derive the second equation,

$$\frac{d^2\psi_1}{dx^2} + 3\frac{d\psi_1}{dx} + \frac{d\psi_2}{dx} = 0$$

If we multiply this equation with -1 and after that add them together,

$$\frac{d^2\psi_1}{dx^2} + \psi_1 = e^x$$

Since this equation contains one unknown function ψ_1, it is a linear differential equation with constant coefficient and the general solution for this would be,

$$\psi_1 = c_1 \cos x + c_2 \sin x - \frac{e^x}{2}$$

If we put this relation in the second equation of the system,

$$\psi_2 = -\frac{d\psi_1}{dx} - 3\psi_1 = (c_1 - 3c_2)\sin x - (3c_1 + c_2)\cos x + 2e^x$$

Can be written, also these solutions contains two arbitrary constants. And because Δ is second degree, it is equal to the constant from the general solution. Thus we can see that these relations are the general solution for the system.

Now, lets move on to the second solution. We're going to make use of deriving operators in this solution. We're going to follow the way for algebraic equations with two unknowns. However, what we apply to letters in algebra, we're going to apply it to our derivation operator $\mathscr{D}$. Now, let's write the system above with our operator $\mathscr{D}$,

$$\begin{aligned} 2(\mathscr{D}-2)\psi_1 + (\mathscr{D}-1)\psi_2 &= e^2 \\ (\mathscr{D}+3)\psi_1 + \psi_2 &= 0 \end{aligned}$$

It is the system that we started with. Our goal is solving ψ_1 and ψ_2 from these equations and for this we have to multiply both sides of the second equation with $\mathscr{D}-1$ to equalize coefficients of ψ_2. In this case,

$$(\mathscr{D}-1)(\mathscr{D}+3)\psi_1 + (\mathscr{D}-1)\psi_2 = 0$$

can be written. And if we subtract the first equation of the system and this equation from each other we can write,

$$[(\mathscr{D}-1)(\mathscr{D}+3) - 2(\mathscr{D}-2)]\psi_1 = -e^x$$

or

$$(\mathscr{D}^2+1)\psi_1 = -e^x$$

This can be written as the equation that we found in the first solution,

$$\frac{d^2\psi_1}{dx^2} + \psi_1 = -e^x$$

And we can just do the same solution that we did in the first solution. This solution is not different from the first solution but it is much more systematic.

And lastly, lets move on to the third solution. This time we're going to make use of Cramer's rule with the help of determinants. Now, we have to remember the system once again,

$$\begin{aligned} 2(\mathscr{D}-2)\psi_1+(\mathscr{D}-1)\psi_2 &= e^x \\ (\mathscr{D}+3)\psi_1+\psi_2 &= 0 \end{aligned}$$

The determinant of the system that contains ψ_1 and ψ_2 as unknowns,

$$\Delta = -\mathscr{D}^2+1$$

And it is different than zero. According to this,

$$\begin{bmatrix} 2(\mathscr{D}-2) & \mathscr{D}-1 \\ \mathscr{D}+3 & 1 \end{bmatrix}\psi_1 = \begin{bmatrix} e^x & \mathscr{D}-1 \\ 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 2(\mathscr{D}-2) & \mathscr{D}-1 \\ \mathscr{D}+3 & 1 \end{bmatrix}\psi_2 = \begin{bmatrix} 2(\mathscr{D}-2) & e^x \\ \mathscr{D}+3 & 0 \end{bmatrix}$$

And the first relation will give us,

$$(\mathscr{D}^2+1)\psi_1 = -e^x$$

And the general solution for this is,

$$\psi_1 = c_1\cos x + c_2\sin x - \frac{e^x}{2}$$

And the second relation will give us,

$$\begin{aligned} -(\mathscr{D}^2+1)\psi_2 = -(\mathscr{D}+3)e^x &= -4e^x \\ (\mathscr{D}^2+1)\psi_2 &= 4e^x \end{aligned}$$

This equation is a linear differential equation with a constant coefficient that contains ψ_2 as the only unknown. The general solution for this system would be,

$$\psi_2 = c_3\cos x + c_4\sin x + 2e^x$$

However, since determinant of Δ is second degree with respect to $\mathscr{D}$, general solution for this system must contain two arbitrary constants. In that case c_3 and c_4 are connected to c_1 and c_2. To obtain these relations we must put ψ_1 and ψ_2 to their place in the system.

$$(3c_1 + c_2 + c_3)\cos x + (3c_2 - c_1 + c_4)\sin x \equiv 0$$

Can be written. This equality will be achieved regardless of x and because of this, equality can also be written like,

$$3_1 + c_2 + c_3 = 0 \ ; \ 3c_2 - c_1 + c_4 = 0$$
$$\text{or}$$
$$c_3 = -(3c_1 + c_2) \ ; \ c_4 = c_1 - 3c_2$$

If we put c_1 and c_2 instead of the place of c_3 and c_4 in the expression of ψ_2, we can construct the general solution.

$$\psi_1 = c_1 \cos x + c_2 \sin x - \frac{e^x}{2}$$
$$\psi_2 = -(3c_1 + c_2)\cos x + (c_1 - 3c_2)\sin x + 2e^x$$

And with this we have finished the three solutions for the example that we have been working on. Now, the elementary subject is done and we're moving on to the main subject. Thus, with this the reader must be able to understand the main subject.

3 Matrix Solutions for Normal Linear Differential Equation Systems with Constant Coefficients

$$\begin{aligned}
\frac{d\varphi_1}{dt} &= \alpha_{11}\varphi_1 + \alpha_{12}\varphi_2 + \ldots + \alpha_{1n}\varphi_n + f_1(t) \\
\frac{d\varphi_2}{dt} &= \alpha_{21}\varphi_1 + \alpha_{22}\varphi_2 + \ldots + \alpha_{2n}\varphi_n + f_2(t) \\
&\ldots \\
\frac{d\varphi_n}{dt} &= \alpha_{n1}\varphi_1 + \alpha_{n2}\varphi_2 + \ldots + \alpha_{nn}\varphi_n + f_n(t)
\end{aligned} \tag{2}$$

This kind of a system is a linear, inhomogeneous, and a normal system with constant coefficients.

In this system,

$$f_1(t) = f_2(t) = \dots\dots = f_n(t) = 0$$

Is a homogeneous system. To find the general solution for the inhomogeneous linear system, first we have to find the general solution for the homogeneous one. Then, we search for the special solution for the inhomogeneous system and add them together.

This system can be written with the help matrices like this,

$$\begin{bmatrix} \frac{d\varphi_1}{dt} \\ \frac{d\varphi_2}{dt} \\ . \\ . \\ . \\ \frac{d\varphi_n}{dt} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ . \\ . \\ . \\ \varphi_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ . \\ . \\ . \\ f_n(t) \end{bmatrix}$$

Or you can write this as a sum, to make it short.

$$\frac{d\varphi_i}{dt} = \sum_{j=1}^{n} \alpha_{ij}\varphi_j + f_i(t)$$

And we can divide this sum into three different parts,

$$\Phi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ . \\ . \\ . \\ \varphi_n \end{bmatrix}, \; \Lambda = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}, \; \Omega(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ . \\ . \\ . \\ f_n(t) \end{bmatrix}$$

And with this we can write the system in a new way,

$$\frac{d\Phi}{dt} = \Lambda\Phi + \Omega(t)$$

Also if the system is homogeneous, expressions above can be constructed like this,

$$\frac{d\varphi_i}{dt} - \sum_{j=1}^{n} \alpha_{ij}\varphi_j = 0$$

or

$$\frac{d\Phi}{dt} - \Lambda\Phi = 0$$

About the solutions of such a homogeneous system, we can say the following,

$$\Phi^{(1)}(t) = \begin{bmatrix} \varphi_1^{(1)}(t) \\ \varphi_2^{(1)}(t) \\ \cdot \\ \cdot \\ \cdot \\ \varphi_n^{(1)}(t) \end{bmatrix}, \ \dots, \ \Phi^{(m)}(t) = \begin{bmatrix} \varphi_1^{(m)}(t) \\ \varphi_2^{(m)}(t) \\ \cdot \\ \cdot \\ \cdot \\ \varphi_n^{(m)}(t) \end{bmatrix}$$

If this series of matrices are the solution for m then the linear solution can be written like this,

$$\sum_{k=1}^{m} c_k \Phi^{(k)}(t)$$

3.1 Theorem

Linear combination of solutions for a homogeneous system is also a solution for the system itself.

Proof.

$$\Phi^{(1)}(t)\ ,\dots,\ \Phi^{(m)}$$

$$\frac{d\Phi}{dt} - \Lambda\Phi = 0$$

or

$$\frac{d\varphi_i}{dt} - \sum_{j=1}^{n} \alpha_{ij}\varphi_j = 0$$

Let the above to be a solution for m. In this case,

$$\sum_{k=1}^{m} c_k \Phi^{(k)}(t)$$

Must be a solution as well. If this expression is a solution, it must satisfy the system. Now, let us put it in it’s place in the system,

$$\frac{d}{dt}\sum_{k=1}^{m} c_k \Phi^{(k)}(t) - \Lambda \sum_{k=1}^{m} c_k \Phi^{(k)}(t) = \sum_{k=1}^{m} c_k [\frac{d\Phi^{(k)}(t)}{dt} - \Lambda \Phi^{(k)}(t)]$$

And now, $\Phi^{(k)}(t)$ for every k must satisfy a system like this,

$$\frac{d\Phi}{dt} - \Lambda\Phi = 0$$

And the inside of square brackets on the second side will be like this,

$$\frac{d\Phi^{(k)}(t)}{dt} - \Lambda\Phi^{(k)}(t) = 0 \quad (k = 1, 2, ..., m)$$

According to this,

$$\frac{d}{dt}\sum_{k=1}^{m} c_k \Phi^{(k)}(t) - \Lambda \sum_{k=1}^{m} c_k \Phi^{(k)}(t) = 0$$

Can be written. This shows us $\sum_{k=1}^{m} c_k \Phi^{(k)}(t)$ relation does satisfy the homogeneous system and thus it is the general solution.

note

n-th solutions linear combination of a system with n unknowns, is the general solution of that system

3.2 Inhomogeneous Systems

3.2.1 Theorem

If $\mu(t)$ is a special solution for a inhomogeneous system and $\nu(t)$ is a solution for a homogeneous system. Then, $\Phi(t) = \mu(t) + \nu(t)$ must be a solution for a inhomogeneous system.

Proof

Let system be written like this,

$$\frac{d\Phi}{dt} = \Lambda\Phi + \Omega(t)$$

The condition to satisfy a system like this will be written like this,

$$\frac{d\Phi}{dt} = \frac{d\mu}{dt} + \frac{d\nu}{dt}$$

And also

$$\frac{d\mu}{dt} + \frac{d\nu}{dt} = \Lambda(\mu + \nu) + \Omega(t) \quad \text{or} \quad [\frac{d\mu}{dt} - \Lambda\mu] + [\frac{d\nu}{dt} - \Lambda\nu] = \Omega(t)$$

In this case we can clearly see that μ is a special solution for a inhomogeneous system and also ν is a solution for a homogeneous system. We can write them like this,

$$\frac{d\mu}{dt} - \Lambda\mu \equiv \Omega(t)$$

and for ν

$$\frac{d\nu}{dt} - \Lambda\nu \equiv \Omega(t)$$

And the condition above will take a form like this,

$$\Omega(t) + 0 \equiv \Omega(t)$$

And this means that $\Phi(t) = \mu(t) + \nu(t)$ is indeed a solution for a inhomogeneous system.

Now, according to this theorem and our note above, the general solution for a inhomogeneous system can be written like this,

$$\boxed{\Phi(t) = \mu(t) + \sum_{k=1}^{m} c_k \Phi^{(k)}(t)}$$

Summary

Solution for a inhomogeneous system, equals to the sum of the special solution for that system and the general solution for a homogeneous system.

In other words, to find the solution for a inhomogeneous system firstly we have to find the general solution for a homogeneous system and after that find the special solution for a inhomogeneous system and add them together.

To find the special solution, we make use of change of constant rule,

$$\frac{d\Phi}{dt} - \Lambda\Phi = 0$$

And lets say the general solution for this kind of a homogeneous system,

$$\Phi(t) = \sum_{k=1}^{n} c_k \Phi^{(k)}(t) \tag{3}$$

However, for this solution to satisfy a inhomogeneous system like this,

$$\frac{d\Phi}{dt} - \Lambda\Phi = \Omega(t) \tag{4}$$

Lets explore how c_k should be $c_k(t)$ functions. And for this lets write the condition for satisfying the system of (3) and (4).

$$\frac{d\Phi}{dt} = \sum_{k=1}^{n} \frac{dc_k}{dt}\Phi^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{d\Phi^{(k)}}{dt}$$

And if we put it in the system of,

$$\frac{d\Phi}{dt} - \Lambda\Phi = \Omega(t)$$

We can write,

$$\sum_{k=1}^{n} \frac{dc_k}{dt}\Phi^{(k)}(t) + \sum_{k=1}^{n} c_k \frac{d\Phi^{(k)}}{dt} - \sum_{k=1}^{n} \Lambda c_k \Phi^{(k)}(t) = \Omega(t)$$

or

$$\sum_{k=1}^{n} \frac{dc_k}{dt}\Phi^{(k)}(t) + \sum_{k=1}^{n} c_k [\frac{d\Phi^{(k)}}{dt} - \Lambda\Phi^{(k)}(t)] = \Omega(t)$$

Since, $\Phi^{(k)}(t)$ is a general solution and should be written like this,

$$\frac{d\Phi^{(k)}}{dt} - \Lambda\Phi^{(k)}(t) \equiv 0$$

Hence, we can write this equation,

$$\sum_{k=1}^{n} \frac{dc_k}{dt}\Phi^{(k)}(t) = \Omega(t)$$

According to this we can say that c_k's must satisfy the condition below,

$$\sum_{k=1}^{n} c'_k(t)\varphi_i^{(k)}(t) = f_i(t) \quad ; \quad (i = 1, 2, ..., n)$$

And now, because of this condition we can basically write,

$$c_k'(t) = \gamma_k(t)$$
$$c_k(t) = \int \gamma_k(t)\, dt = \vartheta_k(t)$$

Now, the special solution for the inhomogeneous system,

$$\varphi_p = \sum_{k=1}^{n} \vartheta_k(t) \Phi^{(k)}(t)$$

We can gain this result much more easier if we make us of matrices. Let us say that we have a system like this,

$$\frac{d\Phi}{dt} - \Lambda\Phi = \Omega(t)$$

And let us say that the general solution for the homogeneous system is this,

$$\varphi_i = \sum_{k=1}^{n} c_k \varphi_i^{(k)}$$

This solution can be shown like this,

$$\varphi_1 = c_1\varphi_1^{(1)} + c_2\varphi_1^{(2)} + \ldots + c_n\varphi_1^{(n)}$$
$$\varphi_2 = c_1\varphi_2^{(1)} + c_2\varphi_2^{(2)} + \ldots + c_n\varphi_2^{(n)}$$
$$\ldots$$
$$\varphi_n = c_1\varphi_n^{(1)} + c_2\varphi_n^{(2)} + \ldots + c_n\varphi_n^{n}$$

Or as matrices,

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ . \\ . \\ . \\ \varphi_n \end{bmatrix} = \begin{bmatrix} \varphi_1^{(1)} & \varphi_1^{(2)} & \ldots & \varphi_1^{(n)} \\ \varphi_2^{(1)} & \varphi_2^{(2)} & \ldots & \varphi_2^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ \varphi_n^{(1)} & \varphi_n^{(2)} & \ldots & \varphi_n^{(n)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix}$$

Or can be written even more short,

$$\Phi = \Phi^{(m)}\Gamma$$

In here Φ signifies the first matrix, $\Phi^{(m)}$ signifies the second matrix, and Γ signifies the third matrix.

Now, lets ask ourselves this question. The $\Phi^{(m)}$ in the expression of $\Phi = \Phi^{(m)}\Gamma$ must be such a function that the solution $\Phi = \Phi^{(m)}\Gamma$ is a solution for the system that is,

$$\frac{d\Phi}{dt} - \Lambda\Phi = \Omega(t)$$

To find an answer for this question we have to write the condition for satisfaction,

$$\frac{d\Phi}{dt} = \frac{d\Phi^{(m)}}{dt}\Gamma + \Phi^{(m)}\frac{d\Gamma}{dt}$$

And,

$$\frac{d\Phi^{(m)}}{dt}\Gamma + \Phi^{(m)}\frac{d\Gamma}{dt} - \Lambda\Phi^{(m)}\Gamma = \Omega(t)$$
$$(\frac{d\Phi^{(m)}}{dt} - \Lambda\Phi^{(m)})\Gamma + \Phi^{(m)}\frac{d\Gamma}{dt} = \Omega(t)$$

Since $\Phi^{(m)}$ is a solution for a homogeneous system, is

$$\frac{d\Phi^{(m)}}{dt} - \Lambda\Phi^{(m)} \equiv 0$$

And it is,

$$\Phi^{(m)}\frac{d\Gamma}{dt} = \Omega(t)$$

From here to obtain Γ, we have to solve $\frac{d\Gamma}{dt}$. For this, we have to multiply the left side of the relation with the $(\Phi^{(m)})^{-1}$ reverse matrix,

$$(\Phi^{(m)})^{-1}\Phi^{(m)}\frac{d\Gamma}{dt} = (\Phi^{(m)})^{-1}\Omega(t)$$

And,

$$\frac{d\Gamma}{dt} = (\Phi^{(m)})^{-1}\Omega(t)$$

And from above,

$$\boxed{\Gamma = \int (\Phi^{(m)})^{-1}\Omega(t)\, dt}$$

In that case the special solution of a inhomogeneous is,

$$\boxed{\varphi_p = \Phi_{(m)}\Gamma = \Phi^{(m)} \int (\Phi^{(m)})^{-1}\Omega(t)\, dt}$$

And the general solution,

$$\boxed{\Phi = \Phi^{(m)}\Gamma + \varphi_p}$$

3.3 Homogeneous System

$$\frac{d\varphi_1}{dt} = \alpha_{11}\varphi_1 + \alpha_{12}\varphi_2 + ... + \alpha_{1n}\varphi_n$$
$$\frac{d\varphi_2}{dt} = \alpha_{21}\varphi_1 + \alpha_{22}\varphi_2 + ... + \alpha_{2n}\varphi_n$$
$$...$$
$$\frac{d\varphi_n}{dt} = \alpha_{n1}\varphi_1 + \alpha_{n2}\varphi_2 + ... + \alpha_{nn}\varphi_n$$

α_{ij} coefficients on the equation are constants. t is independent variable and $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$ are unknown functions. Under these conditions the system above is a normal linear system with constant coefficients. In this system, as you can remember from Theorem 2.1, we can get a n-th order system with the method of derivation and elimination. However, here we're going to examine the system in an another way. This way will get us to solution without n-th order equation. This rule depends on the analysis of characters inside the solution. Let us search for special solutions like this,

$$\varphi_1 = \beta_1 e^{kt} \quad , \quad \varphi_2 = \beta_2 e^{kt} \quad ,..., \quad \varphi_n = \beta_n e^{kt}$$

Now, for this let's make use of condition for the functions below satisfying the equations inside the system,

$$\beta_1 e^{kt} \ , \ \beta_2 e^{kt} \ ,..., \ \beta_n e^{kt}$$

And lets define our constants,

$$\beta_1 \ , \ \beta_2 \ ,..., \ \beta_n \ , \ k$$

If we put these functions in to their place in the equations of the system we can write these relations,

$$
\begin{aligned}
k\beta_1 e^{kt} &= (\alpha_{11}\beta_1 + \alpha_{12}\beta_2 + \ldots + \alpha_{1n}\beta_n)e^{kt} \\
k\beta_2 e^{kt} &= (\alpha_{21}\beta_1 + \alpha_{22}\beta_2 + \ldots + \alpha_{2n}\beta_n)e^{kt} \\
&\ldots \\
k\beta_n e^{kt} &= (\alpha_{n1}\beta_1 + \alpha_{n2}\beta_2 + \ldots + \alpha_{nn}\beta_n)e^{kt}
\end{aligned}
$$

Let us shorten these relations with e^{kt} and move all the terms to the first side. Also, let us arrange them according to $\beta_1, \beta_2, \ldots, \beta_n$. In this case we can write a equation system like this,

$$
\begin{aligned}
(\alpha_{11} - k)\beta_1 + \alpha_{12}\beta_2 + \ldots + \alpha_{1n}\beta_n &= 0 \\
\alpha_{21}\beta_1 + (\alpha_{22} - k)\beta_2 + \ldots + \alpha_{2n}\beta_n &= 0 \\
&\ldots \\
\alpha_{1n}\beta_1 + \alpha_{n2}\beta_2 + \ldots + (\alpha_{nn} - k)\beta_n &= 0
\end{aligned}
$$

Now, from this system let us solve $\beta_1, \beta_2, \ldots, \beta_n$. This system is a linear and a homogeneous system with respect to $\beta_1, \beta_2, \ldots, \beta_n$. Coefficient determinant of this system would be,

$$
\Delta(k) = \begin{bmatrix}
\alpha_{11} - k & \alpha_{12} & . & . & . & . & . & . & . & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} - k & . & . & . & . & . & . & . & \alpha_{2n} \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
\alpha_{n1} & \alpha_{n2} & . & . & . & . & . & . & . & \alpha_{nn} - k
\end{bmatrix}
$$

If k does not make this determinant zero then we can surely say that this system has such obvious solutions like this,

$$\beta_1 = \beta_2 = \ldots\ldots = \beta_n = 0$$

In this case,

$$\varphi_1(t) = \varphi_2(t) = \ldots\ldots = \varphi_n(t) = 0$$

Expressions above will give us the obvious solutions.

To get the unambiguous solution we choose k as $\Delta(k) = 0$. In this case k,

$$\begin{bmatrix} \alpha_{11}-k & \alpha_{12} & . & . & . & . & . & . & . & \alpha_{1n} \\ \alpha_{21} & \alpha_{22}-k & . & . & . & . & . & . & . & \alpha_{2n} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ \alpha_{n1} & \alpha_{n2} & . & . & . & . & . & . & . & \alpha_{nn}-k \end{bmatrix} = 0$$

Will be the roots of the equation. This equation is called as the characteristic equation of the given differential equation system. And the roots of the equation is named as roots of the characteristic equation.

3.4 Application to the Differential Equation System with Three Unknowns

Now, finally we are at the very last title of the main subject. Do not worry this title is trivial and we are going to keep it short. Let us apply this rule to a system with three unknowns. This kind of a system can be written like this,

$$\frac{d\Phi}{dt} = \alpha_1\Phi + b_1\Upsilon + c_1\Sigma + \rho(t)$$
$$\frac{d\Upsilon}{dt} = \alpha_2\Phi + b_2\Upsilon + c_2\Sigma + \tau(t)$$
$$\frac{d\Sigma}{dt} = \alpha_3\Phi + b_3\Upsilon + c_3\Sigma + \eta(t)$$

And let us find the solution for the homogeneous system. Homogeneous system can be written like this,

$$\frac{d\Phi}{dt} = \alpha_1\Phi + b_1\Upsilon + c_1\Sigma$$
$$\frac{d\Upsilon}{dt} = \alpha_2\Phi + b_2\Upsilon + c_2\Sigma$$
$$\frac{d\Sigma}{dt} = \alpha_3\Phi + b_3\Upsilon + c_3\Sigma$$

And it defines three functions $\Phi(t), \Upsilon t, \Sigma(t)$. Let us look for the special solutions of the system such that,

$$\Phi = \beta_1 e^{kt} \quad , \quad \Upsilon = \beta_2 e^{kt} \quad , \quad \Sigma = \beta_3 e^{kt}$$

$\beta_1, \beta_2, \beta_3$ are constants here. Condition for satisfying the system for these solutions are,

$$(\alpha_1 - k)\beta_1 + b_1\beta_2 + c_1\beta_3 = 0$$
$$\alpha_2\beta_1 + (b_2 - k)\beta_2 + c_2\beta_3 = 0$$
$$\alpha_3\beta_1 + b_3\beta_2 + (c_3 - k)\beta_3 = 0$$

And this is a linear, homogeneous equation system that accepts $\beta_1, \beta_2, \beta_3$ as unknowns. This systems obvious set of solution,

$$\beta_1 = \beta_2 = \beta_3 = 0$$

Will give us these solutions,

$$\Phi = 0 \quad , \quad \Upsilon = 0 \quad , \quad \Sigma = 0$$

For this system to have an unambiguous solution set, coefficient determinant must be equal to zero. And this,

$$\begin{bmatrix} \alpha_1 - k & b_1 & c_1 \\ \alpha_2 & b_2 - k & c_2 \\ \alpha_3 & b_3 & c_3 - k \end{bmatrix} = 0$$

Is a third degree algebraic equation with respect to k. This equation, is the characteristic equation of the given differential equation system. If k_1 is a root for this equation, this corresponds to non-zero $\beta_1, \beta_2, \beta_3$ constants and for $k = k_1$,

$$\varphi_1 = \beta_1^{(1)} e^{k_1 t} \quad , \quad \upsilon_1 = \beta_2^{(1)} e^{k_1 t} \quad , \quad \sigma_1 = \beta_3^{(1)} e^{k_1 t}$$

Will be solution set for the differential equation system. If the characteristic equation has three different roots like this k_1, k_2, k_3, these will give us three different, special solution sets. Linear combinations of these,

$$\Phi = c_1\beta_1^{(1)} e^{k_1 t} + c_2\beta_1^{(2)} e^{k_2 t} + c_3\beta_1^{(3)} e^{k_3 t}$$
$$\Upsilon = c_1\beta_2^{(1)} e^{k_1 t} + c_2\beta_2^{(2)} e^{k_2 t} + c_3\beta_2^{(3)} e^{k_3 t}$$
$$\Sigma = c_1\beta_3^{(1)} e^{k_1 t} + c_2\beta_3^{(2)} e^{k_2 t} + c_3\beta_3^{(3)} e^{k_3 t}$$

Will be the general solution for the given system.

If the characteristic equation is, $k = \alpha \pm bi$ that is cojugate, complex, and has a root pair. The general solution above will turn into a exponential complex expression. In this case the solution can be written with sines and cosines.

If $k = k_1$, is double root. Solutions can be written like this,

$$\Phi = (\beta_1 + \Lambda t)e^{k_1 t} \quad , \quad \Upsilon = (\beta_2 + \mathscr{B} t)e^{k_2 t} \quad , \quad \Sigma = (\beta_3 + \Gamma t)e^{k_1 t}$$

$\beta_1, \beta_2, \beta_3, \Lambda, \mathscr{B}, \Gamma$ are variables that needs to be specified. If k is a root at λ value, we can give a fully polynomial coefficient with the degree of $\lambda - 1$ to $e^{k_1 t}$. Thus it is finished.

$\square$

Conclusion

This paper first shortly summarized the elementary subject "Linear Differential Equation Systems with Constant Coefficients", which is necessary to thoroughly develop Calculus itself further more and understand the main subject. Then, showed the main subject that is "Matrix Solutions for Normal Linear Differential Equation Systems with constant Coefficients" with making use of algebra, analysis, and of course matrices itself. While writing it I indeed tried to explain the elementary subject itself with an example but it is true that I kept it short, and I explained main subject itself with three different sub-titles with examples, summaries and theorems step by step just so that reader can understand the subject. The relationships between expressions and conditions to satisfy each other are explained and showed in an understandable manner. Hence, the reader should be capable of understanding the algebraic and analytical methods behind every expression in this paper thus it should be comprehensible. It may serve as an important reference paper for those who wants to study or work on this specific subject. More studies like this can be found in bibliography.

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