A New Formula for Ellipse Perimeter Approximation yielding Absolute Relative Error less than 1.83 ppm.

Author: K. Idicula Koshy

Abstract

In this article, the author presents a new formula for Ellipse Perimeter Approximation. This formula, with two parameters, is unique in form among all published formulae on Ellipse Perimeter Approximation. Of the two parameters, one is a constant and the other is a polynomial of the aspect ratio, which is dependent on the chosen constant. We were able to reduce the Absolute Relative Error to less than 1.83 parts per million (ppm) for any ellipse, by suitable choice of the parameters.

Keywords: Ellipse, Major and Minor Radii, Aspect Ratio, Eccentricity, Relative Error, ppm.

Introduction

The ellipse, whose rectangular cartesian equation is $(x/a)^2 + (y/b)^2 = 1$, is named here as the **standard ellipse**. '*a*' and '*b*' ($a \geq b \geq 0$) are the **major and minor radii** of the ellipse. **Its perimeter *P (a, b)*** is given by the formula:

$$P(a,b) = \int_0^{2\pi} \sqrt{(a^2 sin^2\theta + b^2 cos^2\theta)}\, d\theta$$

where ($a\cos\theta, b\sin\theta$), $0 \leq \theta < 2\pi$, is a parametric point on the ellipse. Due to the symmetry of the ellipse w. r. t. its axes, ***P (a, b) = 4* Q (a, b),*** where ***Q (a, b)*** is the perimeter of the standard ellipse in the first quadrant. Therefore**,**

$$Q(a,b) = \int_0^{\pi/2} \sqrt{(a^2 sin^2\theta + b^2 cos^2\theta)}\;\; d\theta$$

Obviously, $Q(a, 0) = a$, and, $Q(a, a) = \pi a/2$ and, hence, $P(a, 0) = 4a$ and $P(a, a) = 2\pi a$. The definite integral for *Q (a, b)* given above could not be evaluated so far by known direct integration methods. Therefore, Numerical Integration Methods are applied to approximate *Q (a, b)* to the desired degree of accuracy. Simpson's 1/3- Rule Method is very popular in this regard. The Quarter Perimeter values *Q (100, b; Sim)* used in Table 1 below are obtained by this method, by dividing the interval of integration [0, π /2] into 500 equal sub-intervals, so that the length of each sub-interval is h = $\pi/1000$. Then, the Absolute Relative Error is to the order of h^4 [1], which is to the order of 10^{-10}.

In an article published three years ago, the author[3] introduced a new Formula to approximate the ellipse perimeter. It was shown there that the Integral form of *Q (a, b)* follows Lagrange's first order linear partial differential equation in '*a*' and '*b*'. Therefore, any solution of *Q (a, b)*, has to be a function of $(a^p + b^p)^{(1/p)}$, $p \neq 0$ and $\sqrt{ab}$, which are two independent particular solutions of the partial differential equation [1], [3]. Further, it was shown there that the empirical formula: $\boldsymbol{Q(a, b) := (a^p + b^p)^{(1/p)} + \frac{k(ab)^2}{(a+b)^3}}$ ... (*),

where $k = 0.48251$ and $p = \ln(2) / \ln(\frac{\pi}{2} - \frac{k}{8}) = 1.6806453\ldots$ approximates the Quarter Perimeter with maximum Absolute Relative Error less than 60 ppm (that is: 6 cm per km).

In this article, we re-write the formula for *Q (a, b)* given in (*) above in a slightly different form, from which the form of the new formula introduced in this article is adopted (see Comments/Remarks).

Terminology and Notations

Conventional notations and terminologies related to the standard ellipse are used in this article. '*a*' and '*b*' denote the lengths of the semi-major axis (major radius) and the semi-minor axis (minor radius) of the standard ellipse, whose rectangular cartesian equation is: $(x/a)^2 + (y/b)^2 = 1$. The ratio (*b/a*) is called the **Aspect Ratio**. The eccentricity of the ellipse is the constant $e = \sqrt{1 - \frac{b^2}{a^2}}$. Both (*b/a*) and '*e*' take values in [0, 1]. The terms '**Quarter Perimeter'** and '**Absolute Relative Error'** are abbreviated as **QPM** and **ARE** respectively. Formulae for the Quarter Perimeter named after different authors are identified in this article by adding name-indicative characters after the parameter '*b*'. For example, *Q (a, b; Sim*) indicates the QPM formula/values by Simpson's (1/3) Rule and *Q (a, b; Kos)* denotes the QPM formula which the author presents in this article. GM and AM are the Geometric and Arithmetic Means of '*a*' and '*b*'.

Materials and Methods. As mentioned in the Introduction, *Q (a, b; Sim)* values used here are derived with step-width h = $\pi/1000$. Relative Error is computed taking *Q (a, b; Sim)* as basis. All computations are done in MS Excel.

Result (New Formula for Ellipse Perimeter Approximation)

$$\boldsymbol{Q\ (a,\ b;\ Kos)}: = (\ \boldsymbol{a^p + b^p})^{(\frac{1}{p})} + (\frac{\boldsymbol{\pi}}{\boldsymbol{2}} - (\boldsymbol{2})^{(\frac{1}{p})}\ ) * \sqrt{\boldsymbol{ab}} * (\boldsymbol{GM/AM})^{\boldsymbol{k}} \qquad \ldots \qquad (1)$$

Where 'p' and 'k' are parameters, which are suitably chosen to minimize the Absolute Relative Error.

Equation (1) gives exact values for the QPM for $(b/a) = 0$ (degenerate ellipse) and for $(b/a) = 1$ (circle). Further, it approximates the QPM with varying ARE, depending upon the choice of 'p' and 'k'.

To apply the formula, we choose 'p' at first. Although 'p' can be any number greater than or equal to 1, we consider only those p-vales greater than 2.0, as our objective is to reduce the ARE to the maximum extent possible,
It is found that for both 'p' = 2.1 and 'p' =2.11, and for appropriate choices of 'k', the ARE is less than 2.0 ppm for ellipses of all aspect ratios.

For p = 2.11, and k = 63957+1.6439*(b/a) - 2.1877*$(b/a)^2$ + 1.75*$(b/a)^3$ - 0.9370*$(b/a)^4$ + 0.2427*$(b/a)^5$, the ARE is less than 1.83 ppm (Table 1).
Further, in this case, the ARE is **less than one ppm** for all ellipses with Aspect Ratio greater than or equal to **0.35**

It may be verified that for p = 2.1, and k = 2.63595 + 1.6091*(b/a) – 2.1186*$(b/a)^2$ + 1.6448 *$(b/a)^3$ – 0.8366 * $(b/a)^4$ + 0.2094 * $(b/a)^5$, the ARE is less than 1.93 ppm.

Discussion/Remarks

The EPM Approximation Formula introduced in this article is independently developed by the author. It is purely empirical and is based on his discovery that EPM is a function of $\sqrt{ab}$ and $(a^p + b^p)^{(1/p)}$, for some $p \neq 0$ [3]. It gives the perimeter of any ellipse with very high accuracy, and, evaluation can be done on a scientific calculator. It is the lowest ARE yielding formula of its type known so far.

The form of the new formula is shaped out of the author's first formula for Ellipse Perimeter Approximation given in (*) above, which was published three years ago [3]. For, it is easy to verify that:

$(a^p+b^p)^{(1/p)}+\frac{k(ab)^2}{(a+b)^3}=(a^p+b^p)^{(1/p)}+(\frac{\pi}{2}-2^{\left(\frac{1}{p}\right)})*\sqrt{ab}*\left(\frac{GM}{AM}\right)^3$, where $2^{\left(\frac{1}{p}\right)}+\frac{k}{8}=\frac{\pi}{2}$.

Formula (1) is got by replacing the exponent of (GM/AM) by k.

The author has critically examined the EPM Approximation formulae named after several eminent Mathematicians: Kepler, Euler, Seki, Muir, Maertens (YNOT formula), Rivera, Lindner, Zafary, Cantrell etc. and, of course, the renowned formulae of the Great Indian Mathematical Genius Srinivasa Ramanujan. Their formulae all fail to give ARE less than 10 ppm across all ellipses [2], [5]. However, Ramanujan's Formula II:

$$\textbf{P (a, b; Ram): } = \boldsymbol{\pi}\,(\boldsymbol{a}+\boldsymbol{b})\{\mathbf{1}+\frac{\mathbf{3}\boldsymbol{h}^{\mathbf{2}}}{(\mathbf{10}+\sqrt{\mathbf{4}-\mathbf{3}\boldsymbol{h}^{\mathbf{2}}})}\},$$

where h = (a-b)/ (a + b), **is a Colossus** among such formulae. For, it generates only negligibly small ARE for ellipses of high and medium Aspect Ratios. However, ARE steadily increases to the order of 10^{-5}, 10^{-4} etc., for b: $0 < (b/a) \leq 0.1$; and P (a, 0; Ram) = 4a is true only if $\pi = 22/7$, which is incorrect.

Considering these facts, the author's new formula given in equation (1) gives much more accurate measure of the Ellipse Perimeter than all other known formulae of its category.

References

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Author's Profile

Dr. K. Idicula Koshy had his B.Sc. and M. Sc. Degrees in Mathematics from University of Kerala and Doctorate Degree, Dr. rer. nat., in Mathematics from Universitaet Dortmund, Germany. He had been teaching Pure and Applied Mathematics and Statistics in Science and Engineering Colleges in India and abroad for more than 40 years and was UGC Professor of Mathematics in Kerala Agricultural University for 17 years.

Table 1. Relative Error due to the author's new EPM Approx. Formula for $0 \leq (b/a) \leq 1$						
Notes: 1. $k = 2.63957 + 1.6439\ (b/a) - 2.1877\ (b/a)^2 + 1.75\ (b/a)^3 - 0.9307\ (b/a)^4 + 0.2427 (b/a)^5$ 2. Q (a, b; Kos): $(a^{2.11} + b^{2.11})^{(1/2.11)} + (\pi/2 - 2^{(1/2.11)})\ (ab)^{0.5}\ (GM / AM)^k$ 3. Relative Error of Q (a, b; Kos) is computed with Q (a, b; Sim) as basis.						
a	**b**	**b/a**	**Q (a, b; Sim)**	**k**	**Q (a b; Kos)**	**Relative Error**
100	100	1	157.0796326795	3.1577700000	157.0796326795	0.00000000E+00
100	99	0.99	156.2952211988	3.1576680586	156.2952211573	-2.65311040E-10
100	98	0.98	155.5128030354	3.1575444509	155.5128029324	-6.62375454E-10
100	97	0.97	154.7324086029	3.1573965498	154.7324084865	-7.52401108E-10
100	96	0.96	153.9540689771	3.1572217903	153.9540689334	-2.83784053E-10
100	95	0.95	153.1778159151	3.1570176667	153.1778160435	8.38375752E-10
100	94	0.94	152.4036818752	3.1567817292	152.4036822685	2.58108073E-09
100	93	0.93	151.6317000372	3.1565115815	151.6317007666	4.81048337E-09
100	92	0.92	150.8619043241	3.1562048778	150.8619054280	7.31713588E-09
100	91	0.91	150.0943294240	3.1558593196	150.0943309009	9.83938659E-09
100	90	0.9	149.3290108131	3.1554726530	149.3290126177	1.20848601E-08
100	89	0.89	148.5659847794	3.1550426658	148.5659868222	1.37499982E-08
100	88	0.88	147.8052884475	3.1545671847	147.8052905962	1.45376075E-08
100	87	0.87	147.0469598045	3.1540440718	147.0469618885	1.41723834E-08
100	86	0.86	146.2910377264	3.1534712227	146.2910395426	1.24143866E-08
100	85	0.85	145.5375620063	3.1528465625	145.5375633264	9.07044331E-09
100	84	0.84	144.7865733828	3.1521680437	144.7865739624	4.00347046E-09
100	83	0.83	144.0381135702	3.1514336429	144.0381131582	-2.86028294E-09
100	82	0.82	143.2922252901	3.1506413579	143.2922236385	-1.15260783E-08
100	81	0.81	142.5489523035	3.1497892051	142.5489491777	-2.19275848E-08
100	80	0.8	141.8083394449	3.1488752160	141.8083346339	-3.39258059E-08
100	79	0.79	141.0704326576	3.1478974349	141.0704259836	-4.73100010E-08
100	78	0.78	140.3352790307	3.1468539157	140.3352703579	-6.18005229E-08
100	77	0.77	139.6029268372	3.1457427189	139.6029160803	-7.70535368E-08
100	76	0.76	138.8734255741	3.1445619088	138.8734127051	-9.26674939E-08
100	75	0.75	138.1468260044	3.1433095508	138.1468110582	-1.08191292E-07
100	74	0.74	137.4231802006	3.1419837079	137.4231632791	-1.23133979E-07
100	73	0.73	136.7025415902	3.1405824386	136.7025228652	-1.36975880E-07
100	72	0.72	135.9849650034	3.1391037931	135.9849447171	-1.49180966E-07
100	71	0.71	135.2705067232	3.1375458113	135.2704851867	-1.59210324E-07
100	70	0.7	134.5592245368	3.1359065190	134.5592021278	-1.66536485E-07
100	69	0.69	133.8511777909	3.1341839257	133.8511549481	-1.70658452E-07
100	68	0.68	133.1464274484	3.1323760213	133.1464046647	-1.71117125E-07
100	67	0.67	132.4450361480	3.1304807732	132.4450139621	-1.67510924E-07
100	66	0.66	131.7470682676	3.1284961237	131.7470472524	-1.59511294E-07
100	65	0.65	131.0525899892	3.1264199866	131.0525707404	-1.46877823E-07
100	64	0.64	130.3616693686	3.1242502448	130.3616524903	-1.29472642E-07
100	63	0.63	129.6743764076	3.1219847469	129.6743624969	-1.07273776E-07
100	62	0.62	128.9907831301	3.1196213047	128.9907727609	-8.03871276E-08
100	61	0.61	128.3109636623	3.1171576900	128.3109573678	-4.90567048E-08
100	60	0.6	127.6349943170	3.1145916320	127.6349925719	-1.36727579E-08
100	59	0.59	126.9629536820	3.1119208139	126.9629568843	2.52225583E-08
100	58	0.58	126.2949227137	3.1091428705	126.2949311669	6.69323546E-08
100	57	0.57	125.6309848359	3.1062553849	125.6309987314	1.10605368E-07
100	56	0.56	124.9712260432	3.1032558859	124.9712454437	1.55239863E-07
100	55	0.55	124.3157350111	3.1001418449	124.3157598358	1.99691396E-07
100	54	0.54	123.6646032119	3.0969106729	123.6646332234	2.42684748E-07
100	53	0.53	123.0179250376	3.0935597179	123.0179598308	2.82830251E-07
100	52	0.52	122.3757979294	3.0900862617	122.3758369238	3.18644768E-07
100	51	0.51	121.7383225154	3.0864875171	121.7383649507	3.48577420E-07

Table 1 (contd…) Relative Error due to the author's new EPM Approx. Formula for 0 ≤ (b/a) ≤ 1						
a	**b**	**b/a**	**Q (a, b; Sim)**	**k**	**Q (a b; Kos)**	**Relative Error**
100	50	0.5	121.1056027568	3.0827606250	121.1056476919	3.71040152E-07
100	49	0.49	120.4777461026	3.0789026514	120.4777924194	3.84443117E-07
100	48	0.48	119.8548636541	3.0749105846	119.8549100660	3.87234748E-07
100	47	0.47	119.2370703402	3.0707813323	119.2371154055	3.77946261E-07
100	46	0.46	118.6244851042	3.0665117184	118.6245272444	3.55240193E-07
100	45	0.45	118.0172311018	3.0620984807	118.0172686269	3.17962436E-07
100	44	0.44	117.4154359143	3.0575382673	117.4154670525	2.65197005E-07
100	43	0.43	116.8192317748	3.0528276340	116.8192547090	1.96322646E-07
100	42	0.42	116.2287558112	3.0479630415	116.2287687208	1.11070067E-07
100	41	0.41	115.6441503067	3.0429408524	115.6441514144	9.57845299E-09
100	40	0.4	115.0655629783	3.0377573280	115.0655506030	-1.07550421E-07
100	39	0.39	114.4931472778	3.0324086259	114.4931198908	-2.39202405E-07
100	38	0.38	113.9270627145	3.0268907968	113.9270189998	-3.83707712E-07
100	37	0.37	113.3674752035	3.0211997814	113.3674141204	-5.38806701E-07
100	36	0.36	112.8145574428	3.0153314080	112.8144782890	-7.01627478E-07
100	35	0.35	112.2684893203	3.0092813890	112.2683917951	-8.68678402E-07
100	34	0.34	111.7294583560	3.0030453186	111.7293426201	-1.03585880E-06
100	33	0.33	111.1976601821	2.9966186692	111.1975269126	-1.19849146E-06
100	32	0.32	110.6732990664	2.9899967892	110.6731495046	-1.35138056E-06
100	31	0.31	110.1565884833	2.9831748995	110.1564244713	-1.48889880E-06
100	30	0.3	109.6477517392	2.9761480910	109.6475757428	-1.60510755E-06
100	29	0.29	109.1470226588	2.9689113213	109.1468377732	-1.69391328E-06
100	28	0.28	108.6546463399	2.9614594122	108.6544562743	-1.74926357E-06
100	27	0.27	108.1708799880	2.9537870466	108.1706890247	-1.76538491E-06
100	26	0.26	107.6959938396	2.9458887654	107.6958067648	-1.73706356E-06
100	25	0.25	107.2302721895	2.9377589648	107.2300941905	-1.65996924E-06
100	24	0.24	106.7740145365	2.9293918937	106.7738510635	-1.53101912E-06
100	23	0.23	106.3275368684	2.9207816499	106.3273934562	-1.34877695E-06
100	22	0.22	105.8911731067	2.9119221781	105.8910551569	-1.11387785E-06
100	21	0.21	105.4652767431	2.9028072665	105.4651892633	-8.29464898E-07
100	20	0.2	105.0502226984	2.8934305440	105.0501700035	-5.01616366E-07
100	19	0.19	104.6464094511	2.8837854774	104.6463948283	-1.39734661E-07
100	18	0.18	104.2542614858	2.8738653682	104.2542868345	2.43143055E-07
100	17	0.17	103.8742321348	2.8636633500	103.8742975928	6.30166040E-07
100	16	0.16	103.5068068971	2.8531723854	103.5069104756	1.00069335E-06
100	15	0.15	103.1525073527	2.8423852632	103.1526446086	1.33061101E-06
100	14	0.14	102.8118958245	2.8312945953	102.8120596100	1.59306018E-06
100	13	0.13	102.4855809909	2.8198928141	102.4857613373	1.75972434E-06
100	12	0.12	102.1742247323	2.8081721692	102.1744089380	**1.80285836E-06**
100	11	0.11	101.8785506041	2.7961247249	101.8787236223	1.69827983E-06
100	10	0.1	101.5993545025	2.7837423570	101.5994997464	1.42957445E-06
100	9	0.09	101.3375183618	2.7710167499	101.3376190688	9.93777635E-07
100	8	0.08	101.0940281651	2.7579393938	101.0940694864	4.08741077E-07
100	7	0.07	100.8699983194	2.7445015818	100.8699702959	-2.77817727E-07
100	6	0.06	100.6667058367	2.7306944069	100.6666073659	-9.78186144E-07
100	5	0.05	100.4856404786	2.7165087590	100.4854841602	-1.55563007E-06
100	4	0.04	100.3285828267	2.7019353223	100.3283999447	**-1.82283044E-06**
100	3	0.03	100.1977362407	2.6869645720	100.1975793448	-1.56586228E-06
100	2	0.02	100.0959790450	2.6715867719	100.0959140702	-6.49125581E-07
100	1	0.01	100.0274635978	2.6557919707	100.0275188749	5.52619516E-07
100	0	0	100.0000000000	2.6395700000	100.0000000000	4.26325641E-16