A Novel Formula for Ellipse Perimeter Approximation giving Absolute Relative Error less than 3.85 ppm.

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Abstract

In this article, the author presents a novel formula for Ellipse Perimeter Approximation. The algebraic form of the formula is unique in the sense that no other formula published so far has this form. It has achieved the objective of entering in the elite club of very few single-expression formulae yielding Absolute Relative Error less than 10 ppm for any ellipse. The Absolute Relative Error obtained with this novel formula is less than 3.85 ppm, (less than 3.85 millimeter per kilometer), for any ellipse, and, less than 2 ppm for about 80% of them.

Keywords: Ellipse, Major and Minor Radii, Aspect Ratio, Eccentricity, Relative Error.

Introduction

The ellipse, whose rectangular cartesian equation is $(x/a)^2 + (y/b)^2 = 1$, is named here as the **standard ellipse**. 'a' and 'b' ($a \geq b \geq 0$) are the **major and minor radii** of the ellipse. **Its perimeter P (a, b)** is given by the formula:

$$P(a,b) = \int_0^{2\pi} \sqrt{(a^2 sin^2\theta + b^2 cos^2\theta)}\, d\theta$$

where $(a \cos\theta, b \sin\theta)$, $0 \leq \theta < 2\pi$, is a parametric point on the ellipse. Due to the symmetry of the ellipse w. r. t. its axes, **P (a, b) = 4* Q (a, b),** where **Q (a, b)** is the perimeter of the standard ellipse in the first quadrant. Therefore**,** the **first-Quarter Perimeter** is given by the definite integral:

$$\boldsymbol{Q(a,b)} = \int_0^{\pi/2} \sqrt{(a^2 sin^2\theta + b^2 cos^2\theta)}\;\; d\theta$$

Obviously, Q (a, 0) = a, and, Q (a, a) = πa /2 and, hence, **P (a, 0) = 4 a** and **P (a, a) = 2πa.** The definite integral for **Q (a, b)** given above could not be evaluated so far by known direct integration methods. Therefore, Numerical Integration Methods are applied to approximate **Q (a, b)** to the desired degree of accuracy. **Simpson's 1/3- Rule** Method is very popular in this regard. The Quarter Perimeter values **Q (100, b; Sim)** used in Table 1 below are obtained by this method, by dividing the interval of integration [0, π /2] into 500 equal sub-intervals, so that the length of each sub-interval is h = π/1000. Then, the Absolute Relative Error is to the order of h^4 [1], which is to the order of 10^{-10}.

In an article published three years ago, the author [3] introduced a new Formula to approximate the ellipse perimeter. It was shown there that the Integral form of Q (a, b) follows Lagrange's first order linear partial differential equation in 'a' and 'b'. Therefore, any solution of the partial differential equation, in particular Q (a, b), has to be a function of $(a^p + b^p)^{(1/p)}$, $p \neq 0$ and $\sqrt{ab}$, which are two independent particular solutions of the partial differential equation [1], [3]. Further, it was shown there that the empirical formula:

$$\boldsymbol{Q(a,b) := (a^p + b^p)^{(1/p)} + \frac{k(ab)^2}{(a+b)^3}}$$, where

k = 0.48251 and p = ln (2) /ln $\left(\frac{\pi}{2} - \frac{k}{8}\right)$ = 1.6806453 approximates the Quarter Perimeter with maximum Absolute Relative Error **less than 60 ppm** (which is 6 cm per km).

In this article we construct another formula of the same type, but with different form.

Terminology and Notations

Conventional notations and terminologies related to the standard ellipse are used in this article. 'a' and 'b' denote the lengths of the semi-major axis (**major radius**) and the semi-minor axis (**minor radius**) of the **standard ellipse,** whose rectangular cartesian equation is: $\mathbf{(x/a)^2 + (y/b)^2 = 1.}$ The ratio (b/a) is called the **Aspect Ratio.** The **eccentricity** of the ellipse is the constant e = $\sqrt{1 - \frac{b^2}{a^2}}$. Both (b/a) and 'e' take values in [0, 1]. The terms '**Quarter Perimeter'** and '**Absolute Relative Error'** are abbreviated as **QPM** and **ARE** respectively. Formulae for the Quarter Perimeter named after different authors are identified in this article by adding name-indicative characters after the parameter 'b'. For example, **Q (a, b; Sim)** indicates the QPM formula/values by **Simpson's (1/3) Rule** and **Q (a, b; Kos) denotes the QPM formula which the author presents in this article. GM and AM** are the Geometric and Arithmetic Means of 'a' and 'b'.

Materials and Methods. As mentioned in the Introduction, Q (a, b; Sim) values used here are derived with step-width h = π/1000. Relative Error is computed taking **Q (a, b; Sim)** as basis. All computations are done in MS Excel.

<u>Result (New Formula for EPM Approximation)</u>

Q (a, b; Kos) := $\sqrt{(\boldsymbol{a^2+b^2})}$ + ($\boldsymbol{\frac{\pi}{2}-\sqrt{2}}$) $*$ $\sqrt{\boldsymbol{ab}}$ ***** $(\boldsymbol{GM/AM})^{\boldsymbol{k}}$ ……… (1)

where

k = 2.6037 + 1.2814* (b/a) – 1.5252 * $(b/a)^2$ + 0.86 * $(b/a)^3$ - 0.2022* $(b/a)^4$ … (2)

Equation (1) gives exact values of the QPM for (b/a) = 0 (degenerate ellipse) and for (b/a) =1 (circle) and approximates the QPM with ARE less than $3.85*10^{-6}$ for all (b/a) such that 0 < (b/a) < 1. (Table 1).

Discussion/Comments

The EPM Approximation Formula introduced in this article is independently developed by the author. It is purely empirical and is based on his discovery that EPM is a function of $\sqrt{ab}$ and $(a^p + b^p)^{(1/p)}$, for some p ≠ 0 [3]. The rhythm of the coefficients in the expression for 'k' makes it an 'easy to remember' formula. It gives the perimeter of any ellipse with very high accuracy, and, evaluation can be done on a scientific calculator. Though the Modified Koshy Formula # 1, referred to in the article [4], generates ARE less than 1.0 ppm for all ellipses with (b/a) falling in [0.37, 1], it is not so for the remaining (b/a). As the new formula is valid for all ellipses, it would enhance its acceptability as a simple and handy formula.

The author has critically examined the EPM Approximation formulae named after several eminent Mathematicians: Kepler, Euler, Seki, Muir, Maertens (YNOT formula), Rivera, Lindner, Zafary, Cantrell etc. and, of course, the renowned formulae of the Great Indian Mathematical Genius Sreenivasa Ramanujan. Their formulae all fail to give ARE less than 10 ppm across all ellipses [2], [5]. However, Ramanujan's Formula II:

$$\textbf{P (a, b; Ram) :=} \boldsymbol{\pi\,(a+b)\{1+\frac{3h^2}{(10+\sqrt{4-3h^2})}\}},$$

where h = (a-b)/ (a + b) **is a Colossus** among such formulae. For, it generates only negligibly small ARE for ellipses of high and medium Aspect Ratios. However, ARE steadily increases to the order of 10^{-5}, 10^{-4} etc., for b: 0 < (b/a) ≤ 0.1; and P (a, 0; Ram) = 4a is true only if $\pi = 22/7$, which is incorrect.

Another remarkable EPM Approximation Formula is Cantrell's Formula, given by $4\,(a + b) - 2*(4 - \pi) * a * b\, /H_p$, where $H_p = ((a^p + b^p\,)/2)^{1/p}$ is the Holder Mean of order 'p' [2], [5]. For optimal results, he considers different values for 'p'; The ARE is reported as 83 ppm for p = 0.825.

Considering these facts, the author's novel formula given in equation (1) gives much more accurate measure of the Ellipse Perimeter than all other known formulae of its category.

References

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Table 1. Relative Error due to the author's new EPM Approx. Formula for $0 \le (b/a) \le 1$						
Notes: 1. $k = 2.6037 + 1.2814\ (b/a) - 1.5252\ (b/a)^2 + 0.86\ (b/a)^3 - 0.2022\ (b/a)^4$. 2. Q (a, b; Kos): $(a^2 + b^2)^{0.5} + (\pi/2 - 2^{0.5})\ (ab)^{0.5}\ (GM / AM)^k$ 3. Relative Error of Q (a, b; Kos) is computed with Q (a, b; Sim) as basis.						
a	**b**	**b/a**	**Q (a, b; Sim)**	**k**	**Q (a b; Kos)**	**Relative Error**
100	100	1	157.0796326795	3.0177000000	157.0796326795	0.00000000E+00
100	99	0.99	156.2952211988	3.0176621068	156.2952208427	-2.27788963E-09
100	98	0.98	155.5128030354	3.0175921980	155.5128016263	-9.06087978E-09
100	97	0.97	154.7324086029	3.0174898938	154.7324054728	-2.02293036E-08
100	96	0.96	153.9540689771	3.0173547656	153.9540634945	-3.56123605E-08
100	95	0.95	153.1778159151	3.0171863363	153.1778074909	-5.49966520E-08
100	94	0.94	152.4036818752	3.0169840803	152.4036699672	-7.81343316E-08
100	93	0.93	151.6317000372	3.0167474236	151.6316841536	-1.04750858E-07
100	92	0.92	150.8619043241	3.0164757435	150.8618840253	-1.34552313E-07
100	91	0.91	150.0943294240	3.0161683689	150.0943043234	-1.67232281E-07
100	90	0.9	149.3290108131	3.0158245800	149.3289805772	-2.02478251E-07
100	89	0.89	148.5659847794	3.0154436087	148.5659491269	-2.39977536E-07
100	88	0.88	147.8052884475	3.0150246382	147.8052471473	-2.79422674E-07
100	87	0.87	147.0469598045	3.0145668033	147.0469126735	-3.20516314E-07
100	86	0.86	146.2910377264	3.0140691900	146.2909846264	-3.62975543E-07
100	85	0.85	145.5375620063	3.0135308363	145.5375028401	-4.06535670E-07
100	84	0.84	144.7865733828	3.0129507310	144.7865080908	-4.50953417E-07
100	83	0.83	144.0381135702	3.0123278149	144.0380421260	-4.96009532E-07
100	82	0.82	143.2922252901	3.0116609801	143.2921476958	-5.41510809E-07
100	81	0.81	142.5489523035	3.0109490701	142.5488685857	-5.87291490E-07
100	80	0.8	141.8083394449	3.0101908800	141.8082496498	-6.33214061E-07
100	79	0.79	141.0704326576	3.0093851562	141.0703368469	-6.79169440E-07
100	78	0.78	140.3352790307	3.0085305968	140.3351772769	-7.25076527E-07
100	77	0.77	139.6029268372	3.0076258511	139.6028192199	-7.70881163E-07
100	76	0.76	138.8734255741	3.0066695201	138.8733121764	-8.16554464E-07
100	75	0.75	138.1468260044	3.0056601563	138.1467069094	-8.62090566E-07
100	74	0.74	137.4231802006	3.0045962633	137.4230554885	-9.07503772E-07
100	73	0.73	136.7025415902	3.0034762967	136.7024113366	-9.52825145E-07
100	72	0.72	135.9849650034	3.0022986632	135.9848292770	-9.98098532E-07
100	71	0.71	135.2705067232	3.0010617210	135.2703655851	-1.04337609E-06
100	70	0.7	134.5592245368	2.9997637800	134.5590780404	-1.08871332E-06
100	69	0.69	133.8511777909	2.9984031013	133.8510259818	-1.13416359E-06
100	68	0.68	133.1464274484	2.9969778977	133.1462703659	-1.17977239E-06
100	67	0.67	132.4450361480	2.9954863333	132.4448738272	-1.22557103E-06
100	66	0.66	131.7470682676	2.9939265238	131.7469007419	-1.27157023E-06
100	65	0.65	131.0525899892	2.9922965363	131.0524172942	-1.31775332E-06
100	64	0.64	130.3616693686	2.9905943892	130.3614915462	-1.36406940E-06
100	63	0.63	129.6743764076	2.9888180529	129.6741935114	-1.41042625E-06
100	62	0.62	128.9907831301	2.9869654486	128.9905952313	-1.45668343E-06
100	61	0.61	128.3109636623	2.9850344495	128.3107708564	-1.50264535E-06
100	60	0.6	127.6349943170	2.9830228800	127.6347967311	-1.54805458E-06
100	59	0.59	126.9629536820	2.9809285161	126.9627514826	-1.59258550E-06
100	58	0.58	126.2949227137	2.9787490851	126.2947161156	-1.63583843E-06
100	57	0.57	125.6309848359	2.9764822660	125.6307741107	-1.67733427E-06
100	56	0.56	124.9712260432	2.9741256891	124.9710115288	-1.71651005E-06
100	55	0.55	124.3157350111	2.9716769363	124.3155171210	-1.75271525E-06
100	54	0.54	123.6646032119	2.9691335408	123.6643824447	-1.78520935E-06
100	53	0.53	123.0179250376	2.9664929874	123.0177019864	-1.81316063E-06
100	52	0.52	122.3757979294	2.9637527124	122.3755732907	-1.83564650E-06
100	51	0.51	121.7383225154	2.9609101036	121.7380970980	-1.85165557E-06

Table 1 (contd…) Relative Error due to the author's new EPM Approx. Formula for 0 ≤ (b/a) ≤ 1						
a	**b**	**b/a**	**Q (a, b; Sim)**	**k**	**Q (a b; Kos)**	**Relative Error**
100	50	0.5	121.1056027568	2.9579625000	121.1053774893	-1.86009178E-06
100	49	0.49	120.4777461026	2.9549071924	120.4775220404	-1.85978070E-06
100	48	0.48	119.8548636541	2.9517414228	119.8546419851	-1.84947839E-06
100	47	0.47	119.2370703402	2.9484623850	119.2368523888	-1.82788306E-06
100	46	0.46	118.6244851042	2.9450672240	118.6242723334	-1.79364987E-06
100	45	0.45	118.0172311018	2.9415530363	118.0170251135	-1.74540910E-06
100	44	0.44	117.4154359143	2.9379168699	117.4152384464	-1.68178805E-06
100	43	0.43	116.8192317748	2.9341557244	116.8190446961	-1.60143699E-06
100	42	0.42	116.2287558112	2.9302665507	116.2285811125	-1.50305939E-06
100	41	0.41	115.6441503067	2.9262462513	115.6439900879	-1.38544679E-06
100	40	0.4	115.0655629783	2.9220916800	115.0654194319	-1.24751853E-06
100	39	0.39	114.4931472778	2.9177996423	114.4930226673	-1.08836652E-06
100	38	0.38	113.9270627145	2.9133668950	113.9269593478	-9.07305234E-07
100	37	0.37	113.3674752035	2.9087901465	113.3673954011	-7.03926959E-07
100	36	0.36	112.8145574428	2.9040660564	112.8145034992	-4.78162249E-07
100	35	0.35	112.2684893203	2.8991912363	112.2684634598	-2.30345356E-07
100	34	0.34	111.7294583560	2.8941622486	111.7294626817	3.87157941E-08
100	33	0.33	111.1976601821	2.8889756077	111.1976966178	3.27665729E-07
100	32	0.32	110.6732990664	2.8836277793	110.6733692913	6.34524695E-07
100	31	0.31	110.1565884833	2.8781151805	110.1566938602	9.56610475E-07
100	30	0.3	109.6477517392	2.8724341800	109.6478932353	1.29046068E-06
100	29	0.29	109.1470226588	2.8665810978	109.1472007603	1.63175811E-06
100	28	0.28	108.6546463399	2.8605522056	108.6548609613	1.97526251E-06
100	27	0.27	108.1708799880	2.8543437263	108.1711303768	2.31475311E-06
100	26	0.26	107.6959938396	2.8479518345	107.6962784787	2.64298738E-06
100	25	0.25	107.2302721895	2.8413726563	107.2305886992	2.95168307E-06
100	24	0.24	106.7740145365	2.8346022689	106.7743595802	3.23153206E-06
100	23	0.23	106.3275368684	2.8276367015	106.3279060649	3.47225697E-06
100	22	0.22	105.8911731067	2.8204719344	105.8915609568	3.66272367E-06
100	21	0.21	105.4652767431	2.8131038994	105.4656765752	3.79112600E-06
100	20	0.2	105.0502226984	2.8055284800	105.0506266441	**3.84526224E-06**
100	19	0.19	104.6464094511	2.7977415109	104.6468084602	3.81292711E-06
100	18	0.18	104.2542614858	2.7897387785	104.2546453966	3.68244720E-06
100	17	0.17	103.8742321348	2.7815160205	103.8745898146	3.44339280E-06
100	16	0.16	103.5068068971	2.7730689262	103.5071264747	3.08750383E-06
100	15	0.15	103.1525073527	2.7643931363	103.1527765675	2.60987181E-06
100	14	0.14	102.8118958245	2.7554842428	102.8121025198	2.01042242E-06
100	13	0.13	102.4855809909	2.7463377897	102.4857137857	1.29574121E-06
100	12	0.12	102.1742247323	2.7369492718	102.1742739062	4.81274919E-07
100	11	0.11	101.8785506041	2.7273141359	101.8785092326	-4.06086351E-07
100	10	0.1	101.5993545025	2.7174277800	101.5992198732	-1.32510063E-06
100	9	0.09	101.3375183618	2.7072855537	101.3372936847	-2.21711685E-06
100	8	0.08	101.0940281651	2.6968827579	101.0937245499	-3.00329439E-06
100	7	0.07	100.8699983194	2.6862146452	100.8696368995	-3.58302658E-06
100	6	0.06	100.6667058367	2.6752764195	100.6663197262	**-3.83553359E-06**
100	5	0.05	100.4856404786	2.6640632363	100.4852758336	-3.62882758E-06
100	4	0.04	100.3285828267	2.6525702024	100.3282973541	-2.84537658E-06
100	3	0.03	100.1977362407	2.6407923762	100.1975912963	-1.44658297E-06
100	2	0.02	100.0959790450	2.6287247676	100.0960156102	3.65300886E-07
100	1	0.01	100.0274635978	2.6163623380	100.0276295794	1.65936070E-06
100	0	0	100.0000000000	2.6037000000	100.0000000000	0.00000000E+00