Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-)HyperAlgebra

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Abstract: We recall and improve our 2019 concepts of *n-Power Set of a Set, n-SuperHyperGraph, Plithogenic n-SuperHyperGraph,* and *n-ary HyperAlgebra, n-ary NeutroHyperAlgebra, n-ary AntiHyperAlgebra* respectively, and we present several properties and examples connected with the real world.

Keywords: n-Power Set of a Set, n-SuperHyperGraph (n-SHG), n-SHG-vertex, n-SHG-edge, Plithogenic n-SuperHyperGraph, n-ary HyperOperation, n-ary HyperAxiom, n-ary HyperAlgebra, n-ary NeutroHyperOperation, n-ary NeutroHyperAxiom, n-ary NeutroHyperAlgebra, n-ary AntiHyperOperation, n-ary AntiHyperAxiom, n-ary AntiHyperAlgebra

1. Introduction

In this paper, with respect to the classical HyperGraph (that contains HyperEdges), we add the SuperVertices (a group of vertices put all together form a SuperVertex), in order to form a SuperHyperGraph (SHG). Therefore, each SHG-vertex and each SHG-edge belong to $P(V)$, where V is the set of vertices, and $P(V)$ means the power set of V.

Further on, since in our world we encounter complex and sophisticated groups of individuals and complex and sophisticated connections between them, we extend the SuperHyperGraph to n-SuperHyperGraph, by extending $P(V)$ to $P^n(V)$ that is the n-power set of the set V (see below).
Therefore, the n-SuperHyperGraph, through its n-SHG-vertices and n-SHG-edges that belong to $P^n(V)$, can the best (so far) to model our complex and sophisticated reality.
In the second part of the paper, we extend the classical HyperAlgebra to n-ary HyperAlgebra and its alternatives n-ary NeutroHyperAlgebra and n-ary AntiHyperAlgebra.

2. n-Power Set of a Set

Let U be a universe of discourse, and a subset $V \subseteq U$. Let $n \geq 1$ be an integer.
Let $P(V)$ be the *Power Set of the Set V* (i.e. all subsets of V, including the empty set ϕ and the whole set V). This is the classical definition of power set.
For example, if $V = \{a, b\}$, then $P(V) = \{\phi, a, b, \{a, b\}\}$.
But we have extended the power set to *n-Power Set of a Set* [1].

For $n = 1$, one has the notation (identity): $P^1(V) \equiv P(V)$.

For $n = 2$, the 2-Power Set of the Set V is defined as follows:

$P^2(V) = P(P(V)$.

In our previous example, we get:

$P^2(V) = P(P(V) = P(\{\phi, a, b, \{a, b\}\}) = \{\phi, a, b, \{a, b\}; \{\phi, a\}, \{\phi, b\}, \{\phi, \{a, b\}\}, \{a, \{a, b\}\}, \{b, \{a, b\}\};$
$\{\phi, a, b\}, \{\phi, a, \{a, b\}\}, \{\phi, b, \{a, b\}\}, \{a, b, \{a, b\}\}; \{\phi, a, b, \{a, b\}\}\}$.

Definition of n-Power Set of a Set

In general, the **n-Power Set of a Set V** is defined as follows:

$$P^{n+1}(V) = P(P^n(V)), \text{ for integer } n \geq 1.$$

3. Definition of SuperHyperGraph (*SHG*)

A **SuperHyperGraph** (*SHG*) [1] is an ordered pair $SHG = (G \subseteq P(V), E \subseteq P(V))$, where

(i) $V = \{V_1, V_2, \ldots, V_m\}$ is a finite set of $m \geq 0$ vertices, or an infinite set.

(ii) *P*(V) is the power set of *V* (all subset of *V*). Therefore, an ***SHG-vertex*** may be a *single* (classical) *vertex*, or a *super-vertex* (a subset of many vertices) that represents a group (organization), or even an *indeterminate-vertex* (unclear, unknown vertex); ϕ represents the *null-vertex* (vertex that has no element).

(iii) $E = \{E_1, E_2, \ldots, E_m\}$, for $m \geq 1$, is a family of subsets of *V*, and each E_j is an SHG-edge, $E_i \in P(V)$. An ***SHG-edge*** may be a (classical) *edge*, or a *super-edge* (edge between super-vertices) that represents connections between two groups (organizations), or *hyper-super-edge*) that represents connections between three or more groups (organizations), *multi-edge*, or even *indeterminate-edge* (unclear, unknown edge); ϕ represents the *null-edge* (edge that means there is no connection between the given vertices).

4. **Characterization of the SuperHyperGraph**

Therefore, a **SuperHyperGraph** (*SHG*) may have any of the below:

- *SingleVertices* (V_i), as in classical graphs, such as: V_1, V_2, etc.;
- *SuperVertices (or SubsetVertices)* (SV_i), belonging to $P(V)$, for example: $SV_{1,3} = V_1V_3$, $SV_{2,57} = V_2V_{57}$, etc. that we introduce now for the first time. A super-vertex may represent a group (organization, team, club, city, country, etc.) of many individuals;

  The comma between indexes distinguishes the single vertexes assembled together into a single SuperVertex. For example $SV_{12,3}$ means the single vertex S_{12} and single vertex S_3 are put together to form a super-vertex. But $SV_{1,23}$ means the single vertices S_1 and S_{23} are put together; while $SV_{1,2,3}$ means S_1, S_2, S_3 as single vertices are put together as a super-vertex.

  In no comma in between indexes, i.e. SV_{123} means just a single vertex V_{123}, whose index is 123, or $SV_{123} \equiv V_{123}$.
- *IndeterminateVertices* (i.e. unclear, unknown vertices); we denote them as: IV_1, IV_2, etc. that we introduce now for the first time;
- *NullVertex* (i.e. vertex that has no elements, let's for example assume an abandoned house, whose all occupants left), denoted by ϕV.

- *SingleEdges*, as in classical graphs, i.e. edges connecting only two single-vertices, for example: $E_{1,5} = \{V_1, V_5\}$, $E_{2,3} = \{V_2, V_3\}$, etc.;
- *HyperEdges*, i.e. edges connecting three or more single-vertices, for example $HE_{1,4,6} = \{V_1, V_4, V_6\}$, $HE_{2,4,5,7,8,9} = \{V_2, V_4, V_5, V_7, V_8, V_9\}$, etc. as in hypergraphs;
- *SuperEdges* (or *SubsetEdges*), i.e. edges connecting only two SHG-vertices (and at least one vertex is SuperVertex), for example $SE_{(13,6),(45,79)} = \{SV_{13,6}, SV_{45,79}\}$ connecting two SuperVertices, $SE_{9,(2,345)} = \{V_9, SV_{2,345}\}$ connecting one SingleVertex V_9 with one SuperVertex, $SV_{2,345}$, etc. that we introduce now for the first time;
- *HyperSuperEdges* (or *HyperSubsetEdges*), i.e. edges connecting three or more vertices (and at least one vertex is SuperVertex, for example $HSE_{3,45,236} = \{V_3, V_{45}, V_{236}\}$, $HSE_{1234,456789,567,5679} = \{SV_{1234}, SV_{456789}, SV_{567}, SV_{5679}\}$, etc. that we introduce now for the first time;
- *MultiEdges*, i.e. two or more edges connecting the same (single-/super-/indeterminate-) vertices; each vertex is characterized by many attribute values, thus with respect to each attribute value there is an edge, the more attribute values the more edges (= multiedge) between the same vertices;
- *IndeterminateEdges* (i.e. unclear, unknown edges; either we do not know their value, or we do not know what vertices they might connect): IE_1, IE_2, etc. that we introduce now for the first time;
- *NullEdge* (i.e. edge that represents no connection between some given vertices; for example two people that have no connections between them whatsoever): denoted by ϕE .

5. Definition of the n-SuperHyperGraph (*n-SHG*)

A **n-SuperHyperGraph** (*n-SHG*) [1] is an ordered pair $n\text{-}SHG = (G_n \subseteq P^n(V), E_n \subseteq P^n(V))$, where $P^n(V)$ is the *n*-power set of the set *V*, for integer $n \geq 1$.

6. Examples of 2-SuperHyperGraph, SuperVertex, IndeterminateVertex, SingleEdge, Indeterminate Edge, HyperEdge, SuperEdge, MultiEdge, 2-SuperHyperEdge

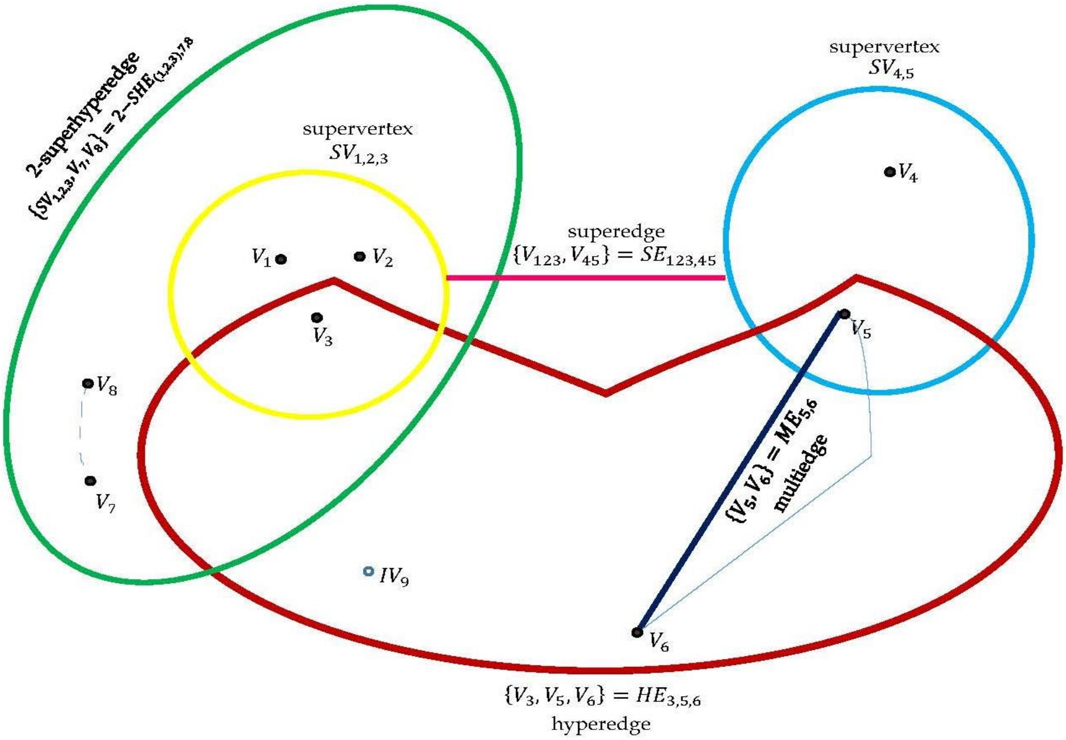

Figure 1. 2-SuperHyperGraph,
($IE_{7,8}$ = Indeterminate Edge between single vertices V_7 and V_8, since the connecting curve is dotted,
IV_9 is an Indeterminate Vertex (since the dot is not filled in),
while $ME_{5,6}$ is a MultiEdge (double edge in this case) between single vertices V_5 and V_6.

Let V_1 and V_2 be two single-vertices, characterized by the attributes a_1 = *size,* whose attribute values are *{short, medium, long}*, and a_2 = *color*, whose attribute values are *{red, yellow}*.

Thus we have the attributes values *(Size{short, medium, long}, Color{red, yellow}),* whence: $V_1(a_1\{s_1, m_1, l_1\}, a_2\{r_1, y_1\})$, where s_1 is the degree of short, m_1 degree of medium, l_1 degree of long, while r_1 is the degree of red and y_1 is the degree of yellow of the vertex V_1.

And similarly $V_2(a_1\{s_2, m_2, l_2\}, a_2\{r_2, y_2\})$.

The degrees may be fuzzy, neutrosophic etc.

Example of fuzzy degree:

$V_1(a_1\{0.8, 0.2, 0.1\}, a_2\{0.3, 0.5\})$.

Example of neutrosophic degree:

$V_1(a_1\{ (0.7,0.3,0.0), (0.4,0.2,0.1),(0.3,0.1,0.1) \}, a_2\{ (0.5,0.1,0.3), (0.0,0.2,0.7) \})$.

Examples of the SVG-edges connecting single vertices V_1 and V_2 are below:

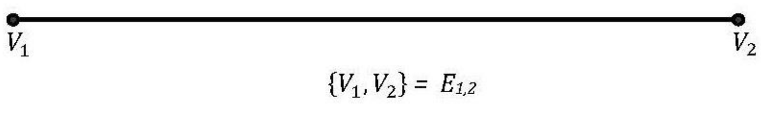

Figure 2. SingleEdge with respect to attributes a_1 and a_2 all together

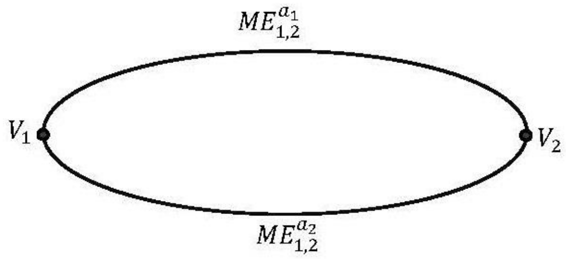

Figure 3. MultiEdge: top edge with respect to attribute a_1, and bottom edge with respect to attribute a_2

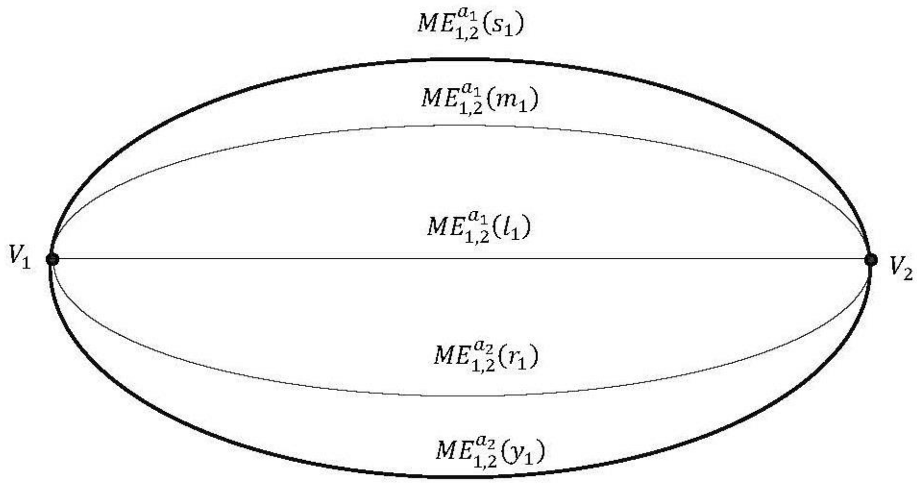

Figure 4. MultiEdge (= Refined MultiEdge from Figure 3):
the top edge from Figure 3, corresponding to the attribute a_1, is split into three sub-edges with respect to the attribute a_1 values s_1, m_1, and l_1;
while the bottom edge from Figure 3, corresponding to the attribute a_2, is split into two sub-edges with respect to the attribute a_2 values r_1, and y_1.

Depending on the application and on experts, one chooses amongst SingleEdge, MultiEdge, Refined-MultiEdge, Refined RefinedMultiEdge, etc.

7. Plithogenic n-SuperHyperGraph

As a consequence, we introduce for the first time the Plithogenic n-SuperHyperGraph.

A **Plithogenic n-SuperHyperGraph (n-PSHG)** is a n-SuperHyperGraph whose each *n-SHG-vertex* and each *n-SHG-edge* are characterized by many distinct attributes values $(a_1, a_2, \ldots, a_p, p \geq 1)$.

Therefore one gets *n-SHG-vertex*$(a_1, a_2, \ldots, a_p)$ and *n-SHG-edge*$(a_1, a_2, \ldots, a_p)$.

The attributes values degrees of appurtenance to the graph may be crisp / fuzzy / intuitionistic fuzzy / picture fuzzy / spherical fuzzy / etc. / neutrosophic / refined neutrosophic / degrees with respect to each *n-SHG-vertex* and each *n-SHG-edge* respectively.

For example, one has:

Fuzzy-*n-SHG-vertex*$(a_1(t_1), a_2(t_2), \ldots, a_p(t_p))$ and Fuzzy-*n-SHG-edge*$(a_1(t_1), a_2(t_2), \ldots, a_p(t_p))$;

Intuitionistic Fuzzy-*n-SHG-vertex*$(a_1(t_1, f_1), a_2(t_2, f_2), \ldots, a_p(t_p, f_p))$

and Intuitionistic Fuzzy-*n-SHG-edge*$(a_1(t_1, f_1), a_2(t_2, f_2), \ldots, a_p(t_p, f_p))$;

Neutrosophic-*n-SHG-vertex*$(a_1(t_1, i_1, f_1), a_2(t_2, i_2, f_2), \ldots, a_p(t_p, i_p, f_p))$

and Neutrosophic-*n-SHG-edge*$(a_1(t_1, i_1, f_1), a_2(t_2, i_2, f_2), \ldots, a_p(t_p, i_p, f_p))$;

etc.

Whence we get:

8. The Plithogenic (Crisp / Fuzzy / Intuitionistic Fuzzy / Picture Fuzzy / Spherical Fuzzy / etc. / Neutrosophic / Refined Neutrosophic) **n-SuperHyperGraph.**

9. Introduction to n-ary HyperAlgebra

Let U be a universe of discourse, a nonempty set $S \subset U$. Let $P(S)$ be the power set of S (i.e. all subsets of S, including the empty set ϕ and the whole set S), and an integer $n \geq 1$.

We formed [2] the following neutrosophic triplets, which are defined in below sections:

(n-ary HyperOperation, n-ary NeutroHyperOperation, n-ary AntiHyperOperation),

(n-ary HyperAxiom, n-ary NeutroHyperAxiom, n-ary AntiHyperAxiom), and

(n-ary HyperAlgebra, n-ary NeutroHyperAlgebra, n-ary AntiHyperAlgebra).

10. n-ary HyperOperation (n-ary HyperLaw)

A *n-ary HyperOperation (n-ary HyperLaw)* $*_n$ is defined as:

$*_n : S^n \to P(S)$, and

$\forall a_1, a_2, \ldots, a_n \in S$ one has $*_n(a_1, a_2, \ldots, a_n) \in P(S)$.

The n-ary HyperOperation (n-ary HyperLaw) is well-defined.

11. n-ary HyperAxiom

A *n-ary HyperAxiom* is an axiom defined of S, with respect the above *n-ary* operation $*_n$, that is true for all *n-plets* of S^n.

12. n-ary HyperAlgebra

A *n-ary HyperAlgebra* $(S, *_n)$, is the S endowed with the above n-ary well-defined HyperOperation $*_n$.

13. Types of n-ary HyperAlgebras

Adding one or more n-ary HyperAxioms to S we get different types of *n-ary* HyperAlgebras.

14. n-ary NeutroHyperOperation (n-ary NeutroHyperLaw)

A *n-ary NeutroHyperOperation* is a *n-ary* HyperOperation $*_n$ that is well-defined for some *n-plets* of S^n

[i.e. $\exists (a_1, a_2, \ldots, a_n) \in S^n, *_n(a_1, a_2, \ldots, a_n) \in P(S)$],

and indeterminate [i.e. $\exists (b_1, b_2, \ldots, b_n) \in S^n, *_n(b_1, b_2, \ldots, b_n) = indeterminate$]

or outer-defined [i.e. $\exists(c_1, c_2, ..., c_n) \in S^n, *_n(c_1, c_2, ..., c_n) \notin P(S)$] (or both), on other *n-plets* of S^n.

15. n-ary NeutroHyperAxiom

A *n-ary NeutroHyperAxiom* is an n-ary HyperAxiom defined of *S*, with respect the above *n-ary* operation $*_n$, that is true for some *n-plets* of S^n, and indeterminate or false (or both) for other *n-plets* of S^n.

16. n-ary NeutroHyperAlgebra is an n-ary HyperAlgebra that has some n-ary NeutroHyper-Operations or some n-ary NeutroHyperAxioms

17. n-ary AntiHyperOperation (n-ary AntiHyperLaw)

A *n-ary AntiHyperOperation* is a *n-ary* HyperOperation $*_n$ that is outer-defined for all *n-plets* of S^n [i.e. $\forall(s_1, s_2, ..., s_n) \in S^n, *_n(s_1, s_2, ..., s_n) \notin P(S)$].

18. n-ary AntiHyperAxiom

A *n-ary AntiHyperAxiom* is an n-ary HyperAxiom defined of *S*, with respect the above *n-ary* operation $*_n$ that is false for all *n-plets* of S^n.

19. n-ary AntiHyperAlgebra is an n-ary HyperAlgebra that has some n-ary AntiHyperOperations or some n-ary AntiHyperAxioms.

20. Conclusion

We have recalled our 2019 concepts of n-Power Set of a Set, n-SuperHyperGraph and Plithogenic n-SuperHyperGraph [1], afterwards the n-ary HyperAlgebra together with its alternatives n-ary NeutroHyperAlgebra and n-ary AntiHyperAlgebra [2], and we presented several properties, explanations, and examples inspired from the real world.

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