What is the factorization of $(x^n + y^n)$ when n is an even positive integer?

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Abstract: The main motivation behind this paper is the question 'What is the factorization of $(x^n + y^n)$ when *n* is an even positive integer ?' which *was* and *is* frequently asked on the Internet by many high school and university students and, to my knowledge, even the specialized textbooks and research articles have not yet answered the question, and with time the question itself transformed into a problem that needs to be solved. In the present article, the question is positively answered and the problem is solved through the detailed study of the factorization that leads directly to an apparently new type of indefinite irrational integrals.

Keywords: factorization, odd positive integer, even positive integer, algebraic expressions, indefinite irrational integrals

MSC (2020) : 12D05, 34A34, 44A45

1. Introduction

The title of this article is a question repeatedly asked on the Net by many high school and university students and it seems none got any satisfactory answer to the question which now transformed into a problem that needs attention and needs to be solved.

To my great surprise, until now, even the very specialized textbooks [1-10] and research articles have not tackled this problem.

Pedagogically and mathematically speaking, factorization simply means transforming an algebraic expression, usually a polynomial, into product of linear factors. For example, the factorization of the expression $(x^3 - 2x^2 - 5x + 6)$ is $(x-1)(x+2)(x-3)$. Without doubt, the factorization as a mathematical tool has always played an important role in algebra, analysis, number theory, numerical analysis and so on. For instance, how can we evaluate the following indefinite integral

$$\int \frac{dx}{x^3 - 2x^2 - 5x + 6}, \tag{1}$$

without performing factorization? Or how can we solve the first-order non-linear ODE

$$2zz' - \frac{1+z^4}{1+x^4} = 0, \tag{2}$$

without doing factorization? Frankly, we cannot since in these cases, the factorization is absolutely inevitable.

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It has long been known that, sometimes, it is quite impossible to factorize polynomial into linear factors employing only rational numbers, but it may be possible to factorize an algebraic expression containing terms with degree *n* into a product containing terms with degree *m* less than *n*. For example, the expression $\left(x^6 - x^4 + 2\sqrt{3}x^3 - \sqrt{3}x + 3\right)$ can, with some difficulty, be factorized as $\left(x^3 + \sqrt{3}\right)\left(x^3 - x + \sqrt{3}\right)$. The expression in each bracket cannot be further factorized using only rational numbers. In this sense, we usually say that these factors are irreducible over the rational numbers. However, since the set of irrational numbers is largest and dense than the set of rational numbers, there are a lot of important expressions that are irreducible over rational number, but which can be factorized if one, of course, allows irrational numbers or real numbers.

Since $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ (where $\mathbb{N}$ is the set of non-negative integers, *i.e.*, "natural numbers", and algebraically $\mathbb{N} = \mathbb{Z}_+$ that is why $\mathbb{N} \subset \mathbb{Z}$) hence over the real numbers every polynomial can be factorized into a product of linear factors and/or quadratic factors, and over the complex numbers every polynomial can be completely factorized into linear factors.

In passing, from the factorization of the expression $(x^n - 1)$ exclusively performed over the integer numbers occurred an interesting type of polynomials called 'cyclotomic polynomials' which were also studied by many mathematicians. For example, $(x^4 - 1) = (x - 1)(x^3 + x^2 + x + 1)$, the polynomial $(x^3 + x^2 + x + 1)$ is an illustrative example of a cyclotomic polynomial. In fact, the questions of factoring and developing methods of factoring, at a more advanced level, were an active part of an extensive work of renowned mathematicians such as Euler, Gauss, Galois, Abel, … etc.

2. Factorization of $(x^n - y^n)$

The factorization of the expression

$$f_n(x, y) = (x^n - y^n)\,, \qquad \forall x, y \in \mathbb{R},\ \ n \in \mathbb{N}. \tag{3}$$

We have the well-known result

$$(x^n - y^n) = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}), \tag{4}$$

or in compact form

$$(x^n - y^n) = (x - y)\sum_{k=0}^{n-1} x^{n-k-1}y^k = (x - y)\sum_{k=1}^{n} x^{n-k}y^{k-1}. \tag{5}$$

The first ten factorizations

$$f_1(x, y) = (x - y)$$

$$f_2(x, y) = (x - y)(x + y)$$

$$f_3(x, y) = (x - y)(x^2 + xy + y^2)$$

$$f_4(x,y) = (x-y)(x^3 + x^2y + xy^2 + y^3)$$

$$f_5(x,y) = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$$

$$f_6(x,y) = (x-y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5)$$

$$f_7(x,y) = (x-y)(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6)$$

$$f_8(x,y) = (x-y)(x^7 + x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5 + xy^6 + y^7)$$

$$f_9(x,y) = (x-y)(x^8 + x^7y + x^6y^2 + x^5y^3 + x^4y^4 + x^3y^5 + x^2y^6 + xy^7 + y^8)$$

$$f_{10}(x,y) = (x-y)(x^9 + x^8y + x^7y^2 + x^6y^3 + x^5y^4 + x^4y^5 + x^3y^6 + x^2y^7 + xy^8 + y^9)$$

As we can see, the factorization is valid when n is an odd or even positive integer, *i.e.*, $n = 2k$ or $n = 2k+1$ and $k \in \mathbb{N}$

2.1. Example

Now, let us illustrate the role and importance of the factorization (4) or (5) of the expression (3). To this end, we evaluate the following indefinite integral

$$I_n = \int \frac{x^n}{x-1} dx, \qquad \forall x \in \mathbb{R}\backslash\{1\}, \quad n \in \mathbb{N}. \tag{6}$$

Notice that $x^n = (x^n - 1) + 1$ hence after substitution into (6) we get

$$I_n = \int \frac{dx}{x-1} + \int \frac{x^n - 1}{x-1} dx. \tag{7}$$

If we put $y = 1$ into (5), we obtain

$$\frac{x^n - 1}{x-1} = \sum_{k=0}^{n-1} x^{n-k-1} \ . \tag{8}$$

Substituting (8) into (7) yields after integration

$$I_n = \ln|x-1| + \sum_{k=0}^{n-1} \frac{x^{n-k}}{n-k} + c. \tag{9}$$

3. Factorization of $(x^n + y^n)$

The well-known and commonly used factorization of the expression

$$g_n(x,y) = (x^n + y^n) \ , \qquad \forall x,y \in \mathbb{R}, \quad n \in \mathbb{N} \ , \tag{10}$$

is in general of the form

$$(x^n + y^n) = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + y^{n-1}), \tag{11}$$

or in compact form

$$(x^n + y^n) = (x + y) \sum_{k=0}^{n-1} (-1)^k x^{n-k-1} y^k = (x + y) \sum_{k=1}^{n} (-1)^{k+1} x^{n-k} y^{k-1}. \qquad (12)$$

The first five factorizations

$$g_1(x, y) = (x + y)$$

$$g_3(x, y) = (x + y)(x^2 - xy + y^2)$$

$$g_5(x, y) = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$$

$$g_7(x, y) = (x + y)(x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6)$$

$$g_9(x, y) = (x + y)(x^8 - x^7y + x^6y^2 - x^5y^3 + x^4y^4 - x^3y^5 + x^2y^6 - xy^7 + y^8).$$

It is clearly understandable that the factorization (11) or (12) is valid iff *n* is an odd positive integer, and for this reason many students asked the central question, the title of this article.

3.1. Example

Like before, let us exemplify the role and significance of the factorization (11) or (12) by solving the subsequent second-order linear ODE

$$z'' + \frac{2}{x} z' - \frac{x^{2k}}{x+1} = 0, \qquad (13)$$

$$\forall x \in \mathbb{R}_+ \backslash \{0\}, \quad k \in \mathbb{N}.$$

Eq.(13) can also be written in compact form as

$$\frac{1}{x} \frac{d^2}{dx^2}(xz) = \frac{x^{2k}}{x+1}, \qquad (14)$$

and from where we get

$$\frac{d^2}{dx^2}(xz) = \frac{x^{2k+1}}{x+1}, \qquad (15)$$

putting $w = xz$ and $n = 2k + 1$, Eq.(15) becomes after substitution

$$\frac{d^2 w}{dx^2} = \frac{x^n}{x+1}. \qquad (16)$$

Remark, $x^n = (x^n + 1) - 1$ hence Eq.(16) can be written in the form

$$\frac{d^2 w}{dx^2} = \frac{x^n + 1}{x+1} - \frac{1}{x+1}, \qquad (17)$$

or equivalently

$$d\left(\frac{dw}{dx}\right) = \left[\frac{x^n + 1}{x+1} - \frac{1}{x+1}\right] dx. \qquad (18)$$

If we put $y = 1$ into (12), we obtain

$$\frac{x^n+1}{x+1} = \sum_{k=0}^{n-1}(-1)^k x^{n-k-1}. \tag{19}$$

Substituting (19) into (18) yields

$$d\left(\frac{dw}{dx}\right) = \left[\sum_{k=0}^{n-1}(-1)^k x^{n-k-1} - \frac{1}{x+1}\right] dx. \tag{20}$$

Performing the first integration to get

$$\frac{dw}{dx} = \sum_{k=0}^{n-1}(-1)^k \frac{x^{n-k}}{n-k} - \ln(x+1) + c_1, \tag{21}$$

and the second integration gives

$$w = \sum_{k=0}^{n-1}(-1)^k \frac{x^{n-k+1}}{(n-k)(n-k+1)} - (x+1)\ln(x+1) + (x+1) + c_1 x + c_2, \tag{22}$$

finally, since $w = xz$, therefore, the desired solution is of the form

$$z = \sum_{k=0}^{n-1}(-1)^k \frac{x^{n-k}}{(n-k)(n-k+1)} - \frac{(x+1)}{x}\ln(x+1) + \frac{(x+1)}{x} + \frac{C_2}{x} + c_1\ . \tag{23}$$

4. What is the factorization of $(x^n + y^n)$ when *n* is an even positive integer?

At present, we arrive at the main subject of this article and our aim is not simply answering but also investigating, exploring and exploiting the result.

As we have already seen, the factorization (11) or (12) of the expression (10) is not valid when *n* is an even positive integer. However, there is another factorization valid when *n* is odd or even, and is of the form:

$$h_n(x,y) = (x^n + y^n) = \left(\sqrt{x^n} - \sqrt{2\sqrt{(xy)^n}} + \sqrt{y^n}\right)\left(\sqrt{x^n} + \sqrt{2\sqrt{(xy)^n}} + \sqrt{y^n}\right), \tag{24}$$

which is valid for the following cases

i) if $n = 2k + 1$; $\forall x, y \in \mathbb{R}_+$; $k \in \mathbb{N}$.
ii) if $n = 2k$; $\forall x, y \in \mathbb{R}_+$ or $\forall x, y \in \mathbb{R}_-$; $k \in \mathbb{N}$.
iii) if $n = 4k$; $\forall x, y \in \mathbb{R}$; $k \in \mathbb{N}$.

To my knowledge, this detailed result has never been mentioned in the textbooks otherwise the central question itself should not be frequently asked by students at all.

The first ten factorizations

$$h_1(x,y) = \left(\sqrt{x} - \sqrt{2\sqrt{xy}} + \sqrt{y}\right)\left(\sqrt{x} + \sqrt{2\sqrt{xy}} + \sqrt{y}\right); \qquad \forall x, y \in \mathbb{R}_+$$

$$h_2(x,y) = \left(x - \sqrt{2xy} + y\right)\left(x + \sqrt{2xy} + y\right); \qquad \forall x, y \in \mathbb{R}_+ \text{ or } \forall x, y \in \mathbb{R}_-$$

$$h_3(x,y) = \left(\sqrt{x^3} - \sqrt{2\sqrt{(xy)^3}} + \sqrt{y^3}\right)\left(\sqrt{x^3} + \sqrt{2\sqrt{(xy)^3}} + \sqrt{y^3}\right); \qquad \forall x,y \in \mathbb{R}_+$$

$$h_4(x,y) = \left(x^2 - \sqrt{2}xy + y^2\right)\left(x^2 + \sqrt{2}xy + y^2\right); \qquad \forall x,y \in \mathbb{R}$$

$$h_5(x,y) = \left(\sqrt{x^5} - \sqrt{2\sqrt{(xy)^5}} + \sqrt{y^5}\right)\left(\sqrt{x^5} + \sqrt{2\sqrt{(xy)^5}} + \sqrt{y^5}\right); \qquad \forall x,y \in \mathbb{R}_+$$

$$h_6(x,y) = \left(x^3 - \sqrt{2(xy)^3} + y^3\right)\left(x^3 + \sqrt{2(xy)^3} + y^3\right); \qquad \forall x,y \in \mathbb{R}_+ \text{ or } \forall x,y \in \mathbb{R}_-$$

$$h_7(x,y) = \left(\sqrt{x^7} - \sqrt{2\sqrt{(xy)^7}} + \sqrt{y^7}\right)\left(\sqrt{x^7} + \sqrt{2\sqrt{(xy)^7}} + \sqrt{y^7}\right); \qquad \forall x,y \in \mathbb{R}_+$$

$$h_8(x,y) = \left(x^4 - \sqrt{2}(xy)^2 + y^4\right)\left(x^4 + \sqrt{2}(xy)^2 + y^4\right); \qquad \forall x,y \in \mathbb{R}_+ \text{ or } \forall x,y \in \mathbb{R}_-$$

$$h_9(x,y) = \left(\sqrt{x^9} - \sqrt{2\sqrt{(xy)^9}} + \sqrt{y^9}\right)\left(\sqrt{x^9} + \sqrt{2\sqrt{(xy)^9}} + \sqrt{y^9}\right); \qquad \forall x,y \in \mathbb{R}_+$$

$$h_{10}(x,y) = \left(x^5 - \sqrt{2(xy)^5} + y^5\right)\left(x^5 + \sqrt{2(xy)^5} + y^5\right); \qquad \forall x,y \in \mathbb{R}_+ \text{ or } \forall x,y \in \mathbb{R}_-$$

Actually, the factorization (24) is also valid for the following important case, that is, when the exponent itself is of the form n/m. Hence, $\forall x,y \in \mathbb{R}_+$; $n \in \mathbb{N}$, $m \in \mathbb{N}\backslash\{0\}$:

$$\left(x^{\frac{n}{m}} + y^{\frac{n}{m}}\right) = \left(\sqrt{x^{n/m}} - \sqrt{2\sqrt{(xy)^{n/m}}} + \sqrt{y^{n/m}}\right)\left(\sqrt{x^{n/m}} + \sqrt{2\sqrt{(xy)^{n/m}}} + \sqrt{y^{n/m}}\right). \tag{25}$$

4.1. Example

Now, let us illustrate the role and importance of the factorization (24). Supposing a hypothetical physical phenomenon whose evolution depending quantitatively and qualitatively on $x \in [0,1]$ and $n \in \mathbb{N}$, respectively. This phenomenon is defined by the following second-order non-linear ODE

$$\frac{z'^2 - zz''}{x^n + 1} = -\frac{z^2}{\sqrt{x^n} - \sqrt{2\sqrt{x^n}} + 1}\,. \tag{26}$$

Our aim is to find a particular solution with the initial conditions:

$$z|_{x=0} = 1, \quad z'|_{x=0} = \frac{1}{4}\,.$$

Separation of variables yields

$$\frac{zz'' - z'^2}{z^2} = \frac{x^n + 1}{\sqrt{x^n} - \sqrt{2\sqrt{x^n}} + 1}, \tag{27}$$

furthermore, we have

$$\frac{zz''-z'^2}{z^2}=\frac{d}{dx}\left(\frac{z'}{z}\right), \tag{28}$$

and from (24) we can write

$$\frac{x^n+1}{\sqrt{x^n}-\sqrt{2\sqrt{x^n}+1}}=\sqrt{x^n}+\sqrt{2\sqrt{x^n}}+1. \tag{29}$$

Substituting (28) and (29) into (27) and separating the variables to get

$$d\left(\frac{z'}{z}\right)=\left[\sqrt{x^n}+\sqrt{2\sqrt{x^n}}+1\right]dx. \tag{30}$$

First integration gives

$$\frac{z'}{z}=\frac{2}{n+2}x^{(n+2)/2}+\frac{4\sqrt{2}}{n+4}x^{(n+4)/4}+x+c_1. \tag{31}$$

Again, separating and integrating to find the general solution

$$z=c_2e^{\varphi_n(x)}, \tag{32}$$

where

$$\varphi_n(x)=\frac{4}{(n+2)(n+4)}x^{(n+4)/2}+\frac{16\sqrt{2}}{(n+4)(n+8)}x^{(n+8)/4}+\frac{1}{2}x^2+c_1x. \tag{33}$$

Finally, by taking into account the initial conditions, we can deduce the desired particular solution

$$z=e^{\varphi_n(x)}, \tag{34}$$

where

$$c_1=\frac{1}{4},\ c_2=1,\ \varphi_n(x)=\frac{4}{(n+2)(n+4)}x^{(n+4)/2}+\frac{16\sqrt{2}}{(n+4)(n+8)}x^{(n+8)/4}+\frac{1}{2}x^2+\frac{1}{4}x. \tag{35}$$

4.2. Example

The role and significance of the factorization (25) can be exemplified by solving the following second-order non-linear ODE:

$$2z''z\left(\alpha^2z'^2+\frac{1}{2}z''z\right)+\alpha^4z'^4=x\left[\frac{K}{\sqrt{\sqrt{x}}-\sqrt{2\sqrt{\sqrt{x}}+1}}\right]^2, \tag{36}$$

$$\forall x\in\mathbb{R}_+;\ \alpha\in[-1,1];\ K\in\mathbb{R}.$$

I wish to leave this example as an open problem to the interested readers and I ask them to solve Eq.(36) for the interesting case when $\alpha=\pm1$ and $K=1$.

It is worthwhile to note that it is almost impossible or at least very difficult to integrate Eqs.(26) and (36) without using the factorizations (24) and (25), respectively. Hence, in this sense, some apparently new type of indefinite irrational integrals is originated as a direct consequence of (24) and (25).

5. Apparently new type of indefinite irrational integrals

I. $$I_n = \int \frac{dx}{\sqrt{x^n} - \sqrt{2\sqrt{x^n}} + 1}$$

II. $$I_n = \int \frac{dx}{\sqrt{x^n} + \sqrt{2\sqrt{x^n}} + 1}$$

III. $$I_n = \int \frac{x^n}{\sqrt{x^n} - \sqrt{2\sqrt{x^n}} + 1}\, dx$$

IV. $$I_n = \int \frac{x^n}{\sqrt{x^n} + \sqrt{2\sqrt{x^n}} + 1}\, dx$$

V. $$I_n = \int \frac{x^n + 1}{\sqrt{x^n} - \sqrt{2\sqrt{x^n}} + 1}\, dx$$

VI. $$I_n = \int \frac{x^n + 1}{\sqrt{x^n} + \sqrt{2\sqrt{x^n}} + 1}\, dx$$

VII. $$I_n = \int \frac{\sqrt{x^n} + \sqrt{2\sqrt{x^n}} + 1}{\sqrt{x^n} - \sqrt{2\sqrt{x^n}} + 1} dx$$

VIII. $$I_n = \int \frac{\sqrt{x^n} - \sqrt{2\sqrt{x^n}} + 1}{\sqrt{x^n} + \sqrt{2\sqrt{x^n}} + 1} dx$$

IX. $$I_n = \int \frac{dx}{\sqrt{x^{n/m}} - \sqrt{2\sqrt{x^{n/m}}} + 1}$$

X. $$I_n = \int \frac{dx}{\sqrt{x^{n/m}} + \sqrt{2\sqrt{x^{n/m}}} + 1}$$

XI. $$I_n = \int \frac{x^n}{\sqrt{x^{n/m}} - \sqrt{2\sqrt{x^{n/m}}} + 1}\, dx$$

XII. $$I_n = \int \frac{x^n}{\sqrt{x^{n/m}} + \sqrt{2\sqrt{x^{n/m}}} + 1}\, dx$$

XIII. $$I_n = \int \frac{x^n + 1}{\sqrt{x^{n/m}} - \sqrt{2\sqrt{x^{n/m}}} + 1}\, dx$$

XIV. $$I_n = \int \frac{x^n + 1}{\sqrt{x^{n/m}} + \sqrt{2\sqrt{x^{n/m}}} + 1}\, dx$$

XV. $$I_n = \int \frac{\sqrt{x^{n/m}} - \sqrt{2\sqrt{x^{n/m}}+1}}{\sqrt{x^{n/m}} + \sqrt{2\sqrt{x^{n/m}}+1}}\,dx$$

XVI. $$I_n = \int \frac{\sqrt{x^{n/m}} + \sqrt{2\sqrt{x^{n/m}}+1}}{\sqrt{x^{n/m}} - \sqrt{2\sqrt{x^{n/m}}+1}}\,dx$$

6. Conclusion

In this paper, the frequently asked question "What is the factorization of $(x^n + y^n)$ when n is an even positive integer?" by high school and university students on the Net has been answered, generalized and extended to the fractional exponents. Basing on this generalization and extension, an apparently new type of indefinite irrational integrals has been originated.

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